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Graph-theoretical methods in general function theory

Amine El Sahili [a,b]

[a] Lebanese university I, El hadas, Beyrout, Lebanon

[b] El sahili Amine, BP 93, Tyr-Lebanon, Lebanon

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Abstract Consider two maps f and g from a set E into a set F such that $f(x) \neq g(x)$ for every x in E. What is the maximal cardinal of a subset A of E such that the images of the restriction of f and g to A are disjoint? Mekler, Pelletier and Taylor have shown that it is card(E) when the set E is infinite; in the finite case, we have proved that it is greater than or equal to card$(E)/4$. In this paper, using graph theoretical technics, we find these results as a direct application of a lemma of Erdös. Moreover, we show that if $E = F = \mathbb{R}$, then there exists a countable partition $\{E_n\}_{n\geqslant 1}$ of $\mathbb{R}$ such that $f(E_n) \cap g(E_n) = \phi$, for every $n \geqslant 1$. ***To cite this article: A. El Sahili, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 859–861.***

Théorie des graphes dans la théorie generale des fonctions

Résumé On considère deux applications f et g d'un ensemble E dans un ensemble F telles que $f(x) \neq g(x)$ pour tout x dans E. Quel est le cardinal maximal d'un sous-ensemble A de E tel que les images des restrictions de f et g à A soient disjointes ? Dans le cas où E est infini, la réponse est card(E), comme l'ont montré Mekler, Pelletier et Taylor ; dans le cas fini, nous avons prouvé que le cardinal en question est plus grand ou égale à card$(E)/4$. Dans cet article, en utilisant les outils de la théorie des graphes, nous retrouvons ces resultats comme application directe d'un lemme d'Erdös. Nous démontrons de plus que si $E = F = \mathbb{R}$, alors il existe une partition dénombrable $\{E_n\}_{n\geqslant 1}$ de $\mathbb{R}$ telle que $f(E_n) \cap g(E_n) = \phi$, pour tout $n \geqslant 1$. ***Pour citer cet article : A. El Sahili, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 859–861.***

1. Introduction

Multigraphs and multidigraphs considered here are obtained from graphs and digraphs by permitting multiple edges but no loops. When G is a multigraph, we denote by $e(G)$ the cardinal of the set of edges of G. If H is a submultigraph of G, $G - H$ denotes the multigraph obtained from G by deleting the edges of H. A subset A of $V(G)$ is said to be independent if the submultigraph induced by A has no edges. We denote by $G(D)$ the underlying multigraph of a multidigraph D. The chromatic number of a multidigraph D, denoted by $\chi(D)$, is the chromatic number of its underlying multigraph.

Consider two maps f and g from a set E into a set F, which satisfy the following property: for every element x in E, $f(x) \neq g(x)$.

E-mail addresses: sahili@jonas.univ-lyon1.fr; aminsahi@inco.com.lb (A. El Sahili).

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Pelletier, Mekler and Taylor announce in [4] the following theorem:

THEOREM 1 (Pelletier, Mekler and Taylor). – *Let f and g be two maps from a set E into a set F, such that $f(x) \neq g(x)$ for every x in E. If E is infinite, then there exists a subset A of E having the same cardinality as E such that $f(A) \cap g(A) = \phi$.*

In [1] we gave a simple proof of the above theorem and we proved in the finite case the following result:

THEOREM 2. – *Let f and g be two maps from a set E into a set F, such that $f(x) \neq g(x)$ for every x in E. If E contains at least $4m$ elements, then there exists a subset A of E with at least m elements such that $f(A) \cap g(A) = \phi$.*

Using graph-theoretical technics, we find the above results as an application of a lemma of Erdös [2], and we prove the following result:

THEOREM 3. – *Let f and g be two maps from $\mathbb{R}$ into $\mathbb{R}$ such that $f(x) \neq g(x)$ for every x in $\mathbb{R}$. Then there exists a countable partition $\{E_n\}_{n \geqslant 1}$ of $\mathbb{R}$ such that $f(E_n) \cap g(E_n) = \phi$, for every $n \geqslant 1$.*

2. Functions and multidigraphs

Let f and g be two maps from a set E into a set F which satisfy: $f(x) \neq g(x)$ for every x in E.

We define two multidigraphs D and H as follows:

$V(D) = F$, and for all $a, b \in V(D)$, we draw κ edges from a to b, where $\kappa = \mathrm{card}(g^{-1}(a) \cap f^{-1}(b))$. D contains no loops since $f(x) \neq g(x)$ for every x in E.

$V(H) = E$, and $(x, y) \in E(H)$ if $f(x) = g(y)$.

We remark that we may associate, in a bijective way, to each edge from a to b in D a vertex x of E such that $g(x) = a$ and $f(x) = b$. Then (x, y) is an edge in H if $h(x) = t(y)$. (The had of x is the tail of y viewed as edges in D.)

It is easy to see that an independent set in H is a subset A of E such that $f(A) \cap g(A) = \phi$.

LEMMA 1 ([2]). – *Any finite multigraph G contains a bipartite submultigraph $B = B(X, Y)$ such that $e(B) \geqslant e(G)/2$.*

We extend this lemma to infinite multigraphs as follows:

LEMMA 2. – *Any multigraph G contains a bipartite submultigraph $B = B(X, Y)$ such that $e(B) \geqslant e(G - B)$.*

Proof. – Enumerate $V(G)$ by an ordinal α and set $V(G) = \{v_\beta, \beta < \alpha\}$. The proof is by transfinite induction on α. If $\alpha = 0$, there is nothing to prove. Suppose that the lemma holds for all multigraphs G such that $V(G)$ can be enumerated by an ordinal $\beta < \alpha$. We consider two cases:

(1) α is a successor ordinal. Set $\alpha = \gamma + 1$. Since the lemma holds for the subgraph G_γ of G induced by $\{v_\beta,\ \beta < \gamma\}$, there exists a $B_\gamma = B_\gamma(X_\gamma, Y_\gamma)$ such that $e(B_\gamma) \geqslant e(G_\gamma - B_\gamma)$ and $V(G_\gamma) = X_\gamma \cup Y_\gamma$. Set

$$R_\alpha = \{e : e \text{ is an edge of } G \text{ incident with } v_\alpha \text{ and a vertex in } X_\gamma\},$$

$$T_\alpha = \{e : e \text{ is an edge of } G \text{ incident with } v_\alpha \text{ and a vertex in } Y_\gamma\}.$$

If $|R_\alpha| \geqslant |T_\alpha|$, we set $X_\alpha = X_\gamma$ and $Y_\alpha = Y_\gamma \cup \{v_\alpha\}$, otherwise we set $Y_\alpha = Y_\gamma$ and $X_\alpha = X_\gamma \cup \{v_\alpha\}$. We have $e(B_\alpha) \geqslant e(G - B_\alpha)$.

(2) α is a limit ordinal. By case 1, we may suppose that if $\beta < \gamma < \alpha$, we have $X_\beta \subseteq X_\gamma$ and $Y_\beta \subseteq Y_\gamma$. Put $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ and $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$. Then $e(B_\alpha) \geqslant e(G - B_\alpha)$. □

Proof of Theorem 1. – Let D and H be defined as above on F and E. We apply Lemma 2 to $G(D)$. It thus contains a bipartite submultigraph $B = B(X, Y)$ such that $e(B) \geqslant e(D - B)$. Since $|E| = e(D) = e(B) + e(D - B)$ and E is infinite, we have $e(B) = |E|$. We partition $E(B)$ into those edges whose tails lie in X and those whose tails lie in Y. One of these two subsets of $E(B)$ has the same cardinality as E. This subset corresponds to an independent set in H having the same cardinality as E. □

Proof of Theorem 2. – As in the above proof, we have $e(B) \geqslant e(D-B)$, so $e(B)+e(D-B)=E(D)=|E| \geqslant 4m$, and $e(B) \geqslant 2m$. We partition $E(B)$ into those edges whose tails lie in X and those whose tails lie in Y. One of these two subsets of $E(B)$ has at least m edges. This subset corresponds to an independent set in H having at least m elements. □

3. Application to real functions

Let f and g be two maps from $\mathbb{R}$ into $\mathbb{R}$ such that $f(x) \neq g(x)$ for every x in $\mathbb{R}$. We construct the digraphs D and H as in the above section. First we note that any vertex x of H can be viewed as a couple $(g(x), f(x))$ (two distinct vertices of H may have the same representation!). Since card($\mathbb{R}$) $= 2^{\aleph_0}$, then the elements of $\mathbb{R}$ can be replaced by the subsets of $\mathbb{N}$, and so the vertices of H by couples of distinct subsets of $\mathbb{N}$. Thus if $v=(A,B)$ and $v'=(A',B')$ are two vertices of H (A, B, A' and B' are subsets of $\mathbb{N}$), (v,v') is an edge of H if $B=A'$.

Proof of Theorem 3. – We shall prove that $\chi(H) \leqslant \aleph_0$, by considering the vertices of H as couples of distinct subsets of $\mathbb{N}$. For every $n \geqslant 1$, we define the following two sets:

$$F_n = \bigl\{(A,B) \in V(H);\ \inf(A-B)=n\bigr\},$$
$$F'_n = \bigl\{(A,B) \in V(H);\ \inf(B-A)=n\bigr\}.$$

These sets are independent in H. In fact, let $v=(A,B)$ and $v'=(A',B')$ be two vertices in F_n. If (v,v') is an edge of H then $B=A'$, but $(A,B) \in F_n$ means that $\inf(A-B)=n$ and so $n \notin B$ which contradicts the fact that $(B,B')=(A',B') \in F_n$. Similarly we show that F'_n is an independent set. In the other hand, let $v=(A,B)$ be any vertex of H. Since $A \neq B$, then $A \triangle B \neq \phi$ so $(A-B) \neq \phi$ or $(B-A) \neq \phi$. In the first case, $v \in F_s$ where $s=\inf(A-B)$, in the other case $v \in F'_t$ where $t=\inf(B-A)$. Thus $V(H)=\bigcup_{n \geqslant 1}(F_n \cup F'_n)$ and $\chi(H) \leqslant \aleph_0$. □

This fact directly proved on real functions can be obtained as a particular case of a result of Erdös and Hajnal on shift graphs.

If $D=(V,E)$ is a digraph, the shift-graph associated to D is by definition the digraph sh(D) $=(V',E')$ such that $V'=E$ and $E'=\{((i,j),(j,k)) : (i,j),(j,k) \in E\}$. If D is complete and if $V(D)$ is infinite of cardinal κ, the chromatic number $\chi(\mathrm{sh}(D))$ is calculated by Erdös and Hajnal [3] to be $\log_2(\kappa)$ (the smallest λ such that $\kappa \leqslant 2^{\lambda}$). Let D be the complete digraph defined on $\mathbb{R}$. It is clear that the digraph H defined above is a subdigraph of sh(D), then $\chi(H) \leqslant \chi(\mathrm{sh}(D)) = \log_2(|\mathbb{R}|) = \aleph_0$.

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