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Calcul des variations/*Calculus of Variations*

# Periodic unfolding and homogenization

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**Abstract** A novel approach to periodic homogenization is proposed, based on an unfolding method, which leads to a fixed domain problem (without singularly oscillating coefficients). This method is elementary in nature and applies to cases of periodic multi-scale problems in domains with or without holes (including truss-like structures). ***To cite this article: D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 99–104.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

**Éclatement périodique et homogénéisation**

**Résumé** Cette Note présente une approche originale des problèmes d'homogénéisation périodique. Basée sur une méthode d'éclatement périodique, elle conduit à un problème limite à coefficients non oscillants sur un domaine fixe. En comparaison avec les méthodes classiques, cette approche passe par des démonstrations relativement élementaires, et son champs d'application inclut le cas périodique multi-échelle ainsi que le cas des domaines perforés et des structures réticulées. ***Pour citer cet article : D. Cioranescu et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 99–104.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

La notion de convergence à deux échelles a été introduite dans Nguetseng [12] et précisée pour ses applications en homogénéisation périodique par Allaire [1]. Elle a été généralisée pour le cas multi-échelle par Ene et Saint Jean Paulin [8] et Allaire et Briane [2]. En 1990, Arbogast, Douglas et Hornung [3] ont introduit une technique de dilatation pour étudier l'homogénéisation de milieux à double porosité. Récemment cette technique a été reprise dans les travaux de Casado-Díaz et al. [4–6], en combinaison avec la convergence à deux échelles.

Cette Note a pour but de clarifier le lien entre ces deux approches tout en les simplifiant par l'introduction d'une méthode d'éclatement périodique. Elle est basée sur deux opérations : la première, l'opérateur

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d'éclatement $\mathcal{T}_\varepsilon$ (Section 2) qui, par la Proposition 2, montre qu'en fait la convergence à deux échelles d'une suite de fonctions est exactement la convergence faible de la suite des éclatées. Les Propositions 1 et 3 donnent les principales propriétés de l'opérateur $\mathcal{T}_\varepsilon$. Il possède en particulier la propriété suivante : si $\{w_\varepsilon\} \subset \mathrm{W}^{1,p}(\Omega)$, est une suite bornée dans $\mathrm{L}^p(\Omega)$, telle que $\varepsilon\|\nabla w_\varepsilon\|_{\mathrm{L}^p(\Omega)} \leqslant C$, et $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}$ dans $\mathrm{L}^p(\Omega \times Y)$, alors $\varepsilon\mathcal{T}_\varepsilon(\nabla_x w_\varepsilon) \rightharpoonup \nabla_y \widehat{w}$ in $\mathrm{L}^p(\Omega \times Y)$, où $\widehat{w}$ est une fonction $Y$-périodique, appartenant à $\mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{per}}(Y))$.

La seconde opération consiste à séparer les échelles par l'utilisation de deux opérateurs $\mathcal{Q}_\varepsilon$ et $\mathcal{R}_\varepsilon$. Chaque fonction $\varphi \in \mathrm{H}^1(\Omega)$ se décompose sous la forme $\varphi = \mathcal{Q}_\varepsilon\varphi + \mathcal{R}_\varepsilon\varphi$. La partie macroscopique $\mathcal{Q}_\varepsilon\varphi$, sans oscillations d'ordre $\varepsilon$, est construite par interpolation $Q_1$ (au sens des éléments finis) à partir des valeurs représentatives de $\varphi$ aux noeuds du $\varepsilon$-réseau considérée (par exemple en utilisant des moyennes). La partie microscopique $\mathcal{R}_\varepsilon\varphi$, capte les oscillations d'ordre $\varepsilon$. L'idée d'utiliser des interpolations apparaît déjà dans les travaux de Griso [10] et [11] dans le cas des structures réticulées périodiques. La Proposition 4 et le Théorème 1 donnent les principales propriétés de $\mathcal{Q}_\varepsilon$ et $\mathcal{R}_\varepsilon$ en liaison avec l'opérateur $\mathcal{T}_\varepsilon$. Mentionnons en particulier, les convergences (iii) et (iv) du Théorème 1.

La Section 3 montre que l'homogénéisation périodique classique est traitée de façon élementaire dans ce cadre. Partant du problème (1), par l'opération d'éclatement, on passe d'emblée au problème (5) posé sur $\Omega \times Y$ avec coefficients non-singuliers. Le passage à la limite est immédiat grâce au Théorème 1, et donne comme formulation limite (4). On y retrouve les équations classiques satisfaites par les correcteurs (*cf.* [7]).

Le Théorème 4 donne un résultat général sur la convergence des correcteurs sans aucune hypothèse de régularité supplémentaire, tant sur la solution limite que sur les correcteurs eux-mêmes. A cet effet, dans la Section 4, on définit un opérateur de moyennisation sur les fonctions définies sur le domaine éclaté $\Omega \times Y$ (cf. également [5,9]). Cette méthode s'étend directement ou cas multi-échelle, un exemple étant donné dans la Section 5 (Théorème 5).

Les démonstrations détaillées avec des applications comprenant les matériaux feuilletés, les matériaux perforés, les structures réticulées, les plaques et les poutres, en conduction ou en élasticité linéarisée, seront présentées dans des travaux ultérieurs.

## 1. Introduction

The notion of two-scale convergence was introduced in 1989 by Nguetseng [12] and further developed by Allaire [1] with applications to periodic homogenization. It was generalized to some multi-scale problems by Ene and Saint Jean Paulin [8] and Allaire and Briane in [2]. In 1990, Arbogast, Douglas and Hornung [3] defined a 'dilation' operation to study homogenization for a periodic medium with double porosity. This technique was not used again until the work of Casado-Díaz et al. [4–6] where it is combined with two-scale convergence to study perforated domains and thin structures.

It turns out that the dilation technique reduces two-scale convergence to weak convergence in an appropriate space. Combining this approach with ideas from Finite Element approximations, we give a very general and quite simple method ('periodic unfolding') to study homogenization of multi-scale periodic problems. It is a fixed-domain method (the dimension of the fixed domain depends on the number of scales) that applies as well to problems with holes and truss-like structures or in linearized elasticity.

## 2. The unfolding operator $\mathcal{T}_\varepsilon$

In $\mathbb{R}^n$, let $\Omega$ be an open set and $Y$ a reference cell (ex. $]0,1[^n$, or more generally a set having the paving property with respect to a basis $(b_1, \ldots, b_n)$ defining the periods). By analogy with the 1D notations, for $z \in \mathbb{R}^n$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^n k_j b_j$ of the periods such that $z - [z]_Y$ belongs to $Y$, and set $\{z\}_Y = z - [z]_Y \in Y$. Then for each $x \in \mathbb{R}^n$, one has $x = \varepsilon\big(\big[\frac{x}{\varepsilon}\big]_Y + \big\{\frac{x}{\varepsilon}\big\}_Y\big)$. For $w \in \mathrm{L}^p(\Omega)$

($p \in [1, \infty]$), extended by zero outside of $\Omega$, define $\mathcal{T}_\varepsilon(w) \in \mathrm{L}^p(\Omega \times Y)$ by

$$\mathcal{T}_\varepsilon(w)(x, y) = w\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) \quad \text{for } x \in \Omega \text{ and } y \in Y.$$

Obviously, for any $x \in \Omega$, $\mathcal{T}_\varepsilon(w)(x, \{\frac{x}{\varepsilon}\}_Y) = w(x)$, and $\mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\mathcal{T}_\varepsilon(w)$, $\forall v, w \in \mathrm{L}^2(\Omega)$.

PROPOSITION 1 (properties of $\mathcal{T}_\varepsilon$). – *One has the following integration formula*:

$$\int_\Omega w \,\mathrm{d}x = \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(w) \,\mathrm{d}x \,\mathrm{d}y \quad \forall w \in \mathrm{L}^1(\Omega).$$

*For* $\{w_\varepsilon\} \subset \mathrm{L}^p(\Omega)$, *if* $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}$ *in* $\mathrm{L}^p(\Omega \times Y)$, *then* $w_\varepsilon \rightharpoonup w$ *in* $\mathrm{L}^p(\Omega)$ *where* $w = \frac{1}{|Y|}\int_Y \widehat{w} \,\mathrm{d}y$.

PROPOSITION 2 (relation with two-scale convergence). – *Let* $\{w_\varepsilon\} \subset \mathrm{L}^p(\Omega)$, $p \in (1, \infty)$, *be a bounded sequence. The following are equivalent*:

(i) $\{\mathcal{T}_\varepsilon(w_\varepsilon)\}_\varepsilon$ *converges weakly to* $w$ *in* $\mathrm{L}^p(\Omega \times Y)$, (ii) $\{w_\varepsilon\}_\varepsilon$ *two-scale converges to* $w$.

Periodic unfolding appears to be equivalent to two-scale convergence. However, it is both simpler and more efficient (in particular, it is well-suited for multiscales of many types).

PROPOSITION 3 ($\mathcal{T}_\varepsilon$ and gradients). – *For every* $w \in \mathrm{W}^{1,p}(\Omega)$ *one has* $\nabla_y(\mathcal{T}_\varepsilon(w)) = \varepsilon\mathcal{T}_\varepsilon(\nabla_x w)$. *If* $\{w_\varepsilon\} \subset \mathrm{W}^{1,p}(\Omega)$, *is a bounded sequence in* $\mathrm{L}^p(\Omega)$ *such that* $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}$ *in* $\mathrm{L}^p(\Omega \times Y)$ *with* $\varepsilon\|\nabla w_\varepsilon\|_{\mathrm{L}^p(\Omega)} \leqslant C$, *then*

$$\varepsilon\mathcal{T}_\varepsilon(\nabla_x w_\varepsilon) \rightharpoonup \nabla_y \widehat{w} \quad \text{in } \mathrm{L}^p(\Omega \times Y).$$

*Furthermore, the limit function* $\widehat{w}$ *is* $Y$*-periodic, namely belongs to* $\mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{per}}(Y))$.

## 3. Macro–micro decomposition of functions: the scale-splitting operators $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$

For $k \in \mathbb{Z}^n$, set $\xi_k = \sum_{j=1}^n k_j b_j$ and $\widetilde{\Omega^\varepsilon} = \bigcup_{\varepsilon\xi_k + \varepsilon\overline{Y} \cap \overline{\Omega} \neq \emptyset} \{\varepsilon\xi_k + \varepsilon\overline{Y}\}$. One assumes that $\partial\Omega$ is bounded and Lipschitz. Then, it is well known that there exists a continuous extension operator $\mathcal{P} : \mathrm{H}^1(\Omega) \mapsto \mathrm{H}^1(\mathbb{R}^n)$ such that $\forall\varphi \in \mathrm{H}^1(\Omega)$, $\|\mathcal{P}(\varphi)\|_{\mathrm{L}^2(\mathbb{R}^n)} + \|\nabla\mathcal{P}(\varphi)\|_{\mathrm{L}^2(\mathbb{R}^n)} \leqslant C\,\|\varphi\|_{\mathrm{H}^1(\Omega)}$, where $C$ is a constant depending only upon $\partial\Omega$.

Every $\varphi \in \mathrm{H}^1(\Omega)$ can be split as $\varphi = \mathcal{Q}_\varepsilon(\varphi) + \mathcal{R}_\varepsilon(\varphi)$ as follows:

DEFINITION OF $\mathcal{Q}_\varepsilon$. – For a given function $\varphi$ on $\Omega$, one starts by defining $\mathcal{Q}_\varepsilon(\varphi)$ at the nodes $\varepsilon\xi_k$ included in $\widetilde{\Omega^\varepsilon}$, using any reasonable averaging method. For example (this is enough here),

$$\mathcal{Q}_\varepsilon(\varphi)(\varepsilon\xi_k) = \frac{1}{|Y|}\int_Y \mathcal{P}(\varphi)(\varepsilon\xi_k + x)\,\mathrm{d}x.$$

Then, $\mathcal{Q}_\varepsilon(\varphi)$ is the restriction to $\Omega$ of $Q_1$-interpolate of the discrete function $\mathcal{Q}_\varepsilon(\varphi)(\varepsilon\xi_k)$ as customary in the Finite Element Method. The ides of using these type of interpolates was already present in Griso [10,11], for the study of truss-like structures.

DEFINITION OF $\mathcal{R}_\varepsilon$. – It is clear that $\mathcal{Q}_\varepsilon$ is designed not to capture any oscillations of order $\varepsilon$. What captures these oscillations is therefore the remainder $\mathcal{R}_\varepsilon(\varphi) = \varphi - \mathcal{Q}_\varepsilon(\varphi)$.

PROPOSITION 4 (properties of $\mathcal{Q}_\varepsilon$ and $\mathcal{R}_\varepsilon$). – *For every* $\varphi \in \mathrm{H}^1(\Omega)$,

(i) $\|\mathcal{Q}_\varepsilon(\varphi)\|_{\mathrm{H}^1(\Omega)} \leqslant C\|\varphi\|_{\mathrm{H}^1(\Omega)}$, (ii) $\|\varphi - \mathcal{Q}_\varepsilon(\varphi)\|_{\mathrm{L}^2(\Omega)} \leqslant \varepsilon C\|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}$

(iii) $\|\mathcal{R}_\varepsilon(\varphi)\|_{\mathrm{L}^2(\Omega)} \leqslant \varepsilon C\|\varphi\|_{\mathrm{H}^1(\Omega)}$, (iv) $\|\mathcal{R}_\varepsilon(\varphi)\|_{\mathrm{H}^1(\Omega)} \leqslant C\|\varphi\|_{\mathrm{H}^1(\Omega)}$.

THEOREM 1 (convergence for sequences in $\mathrm{W}^{1,p}(\Omega)$, $p \in [1, \infty)$). – *Let* $\{w_\varepsilon\}$ *converge weakly to* $w$ *in* $\mathrm{W}^{1,p}(\Omega)$. *There exist a subsequence* (*still denoted* $\varepsilon$) *and a* $\widehat{w} \in \mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{per}}(Y))$ *such that*

(i) $\mathcal{Q}_\varepsilon(w_\varepsilon) \rightharpoonup w$ *in* $\mathrm{W}^{1,p}(\Omega)$, (ii) $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w$ *in* $\mathrm{L}^p\big(\Omega; \mathrm{W}^{1,p}(Y)\big)$,

(iii) $\dfrac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_\varepsilon(w_\varepsilon)\big) \rightharpoonup \widehat{w}$ *in* $\mathrm{L}^p\big(\Omega; \mathrm{W}^{1,p}(Y)\big)$, (iv) $\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla_x w + \nabla_y \widehat{w}$ *in* $\mathrm{L}^p(\Omega \times Y)$.

## 4. Periodic unfolding and homogenization

One considers the limit behavior as $\varepsilon$ goes to $0^+$ of the solutions of the $\varepsilon$-problem:

$$\int_\Omega A^\varepsilon(x)\nabla u_\varepsilon \nabla v = \int_\Omega f v, \quad \forall v \in \mathrm{H}^1_0(\Omega), \tag{1}$$

where, for each $\varepsilon$ (in a sequence going to 0), $A^\varepsilon$ is assumed measurable and bounded in $\mathrm{L}^\infty(\Omega)$ One also assumes uniform ellipticity $c|\xi|^2 \leqslant A^\varepsilon(x)\xi\cdot\xi \leqslant C|\xi|^2$ a.e. $x \in \Omega$, with strictly positive constants $c$ and $C$. Traditionally, $A^\varepsilon(x)$ is derived as $A(x, \frac{x}{\varepsilon})$ from a $A(x,y)$ which is assumed $Y$-periodic as a function of its second variable. With $f$ in $\mathrm{H}^{-1}(\Omega)$, $\{u_\varepsilon\}$ is bounded in $\mathrm{H}^1_0(\Omega)$ so that there is a subsequence(still denoted $\varepsilon$), and some $u_0$ with $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_0(\Omega)$.

THEOREM 2 (standard periodic homogenization). – *Suppose that $A^\varepsilon$ and $f$ satisfy the above hypotheses. Suppose furthermore that*

$$B^\varepsilon(x,y) \doteq \mathcal{T}_\varepsilon\big(A^\varepsilon\big)(x,y) \to B(x,y) \quad \textit{a.e. in } \Omega\times Y. \tag{2}$$

*Then there exists $\widehat{u} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ such that*

$$\text{(i) } \mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0 \quad \textit{in } \mathrm{L}^2\big(\Omega; \mathrm{H}^1(Y)\big), \qquad \text{(ii) } \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla_x u_0 + \nabla_y \widehat{u} \quad \textit{in } \mathrm{L}^2(\Omega\times Y). \tag{3}$$

*The pair $(u_0, \widehat{u})$ is the unique solution of the problem*: $\forall \Psi \in \mathrm{H}^1_0(\Omega)$, $\forall \Phi \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$,

$$\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\big(\nabla_x \Psi(x) + \nabla_y \Phi(x,y)\big) = \int_\Omega f\Psi. \tag{4}$$

*Remarks*. – (1) Problem (4) is of standard variational form on $\mathcal{H} = \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R})$.

(2) The only situations for which (2) is known to hold, are sums of the following four cases where $B$ always equals $A$: $A(x,y) = A(y)$, $A(x,y) = A_1(x)A_2(y)$, $A \in \mathrm{L}^1(Y; \mathrm{C}(\Omega))$, $A \in \mathrm{L}^1(\Omega; \mathrm{C}(Y))$.

*Proof*. – The proof with periodic unfolding is elementary! Convergence (i) and (ii) are given by Propositions 1 and 3 respectively. For a sequence of test functions $v^\varepsilon$ the integration formula gives

$$\frac{1}{|Y|}\int_{\Omega\times Y} B^\varepsilon \mathcal{T}_\varepsilon(\nabla_x u_\varepsilon)\mathcal{T}_\varepsilon\big(\nabla_x v^\varepsilon\big) = \int_\Omega f v^\varepsilon. \tag{5}$$

Choosing the test-function $v^\varepsilon \doteq \Psi(x)$, with $\Psi \in \mathcal{D}(\Omega)$ gives at the limit

$$\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\nabla_x \Psi(x) = \int_\Omega f\Psi \tag{6}$$

which still holds for every $\Psi \in \mathrm{H}^1_0(\Omega)$.

Now, taking for test function $v^\varepsilon(x) = \varepsilon\Psi(x)\psi(\frac{x}{\varepsilon})$, $\Psi \in \mathcal{D}(\Omega)$, $\psi \in \mathrm{H}^1_{\mathrm{per}}(Y)$, one has successively $v^\varepsilon \rightharpoonup 0$ in $\mathrm{H}^1_0(\Omega)$, and $\mathcal{T}_\varepsilon(\nabla_x v^\varepsilon) \to \Psi(x)\nabla\psi(y)$ (uniformly over $\Omega\times Y$), so that

$$\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)(\nabla_x u_0 + \nabla_y \widehat{u})\Psi(x)\nabla\psi(y) = 0 \tag{7}$$

and by density of the tensor product, for all $\Phi \in \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{per}}(Y))$. □

*Remarks*. – (1) As in the two-scale method, (7) gives $\widehat{u}$ in terms of $\nabla u_0$ which, carried over to (6), yields the standard form of the homogenized equation.

(2) The energy convergence is obtained in the standard way. It implies the strong convergence

$$\mathcal{T}_\varepsilon(\nabla_x u_\varepsilon) \to \nabla_x u_0 + \nabla_y \widehat{u} \quad \text{in } \mathrm{L}^2(\Omega\times Y). \tag{8}$$

## 5. Relationship between convergence and unfolding: correctors

DEFINITION (of the averaging operator $\mathcal{U}_\varepsilon$). – For $\Psi$ in $\mathrm{L}^2(\Omega \times Y)$, set

$$\mathcal{U}_\varepsilon(\Phi)(x) = \frac{1}{|Y|}\int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right)\mathrm{d}z, \quad \mathcal{U}_\varepsilon(\Phi) \in \mathrm{L}^2(\Omega).$$

PROPOSITION 5 (properties of $\mathcal{U}_\varepsilon$). – *For given $\varphi \in \mathrm{L}^2(\Omega)$ and $\Phi$ in $\mathrm{L}^2(\Omega \times Y)$,*

(i) $\mathcal{U}_\varepsilon\big(\mathcal{T}_\varepsilon(\varphi)\big) = \varphi$, *and* $\mathcal{T}_\varepsilon\big(\mathcal{U}_\varepsilon(\Phi)\big)(x,y) = \frac{1}{|Y|}\int_Y \Phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, y\right)\mathrm{d}z,$

(ii) $\displaystyle\int_\Omega \mathcal{U}_\varepsilon(\Phi)(x)\,\mathrm{d}x = \frac{1}{|Y|}\int_{\Omega\times Y}\Phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$, *and* $\|\mathcal{U}_\varepsilon(\Phi)\|_{\mathrm{L}^2(\Omega)} \leqslant \dfrac{1}{\sqrt{|Y|}}\|\Phi\|_{\mathrm{L}^2(\Omega\times Y)},$

(iii) $\mathcal{T}_\varepsilon(\mathcal{U}_\varepsilon(\Phi)) \to \Phi$ *in* $\mathrm{L}^2(\Omega \times Y)$, *and* $\mathcal{U}_\varepsilon(\Phi) \rightharpoonup \dfrac{1}{|Y|}\displaystyle\int_Y \Phi(x,y)\,\mathrm{d}y$ *in* $\mathrm{L}^2(\Omega)$.

THEOREM 3. – *Let* $\{\varphi_\varepsilon\} \subset \mathrm{L}^2(\Omega)$. *The following weak convergences are equivalent*:

$$\text{(i)}\quad \mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \Phi \quad \textit{in } \mathrm{L}^2(\Omega\times Y), \qquad \text{(ii)}\quad \varphi_\varepsilon - \mathcal{U}_\varepsilon(\Phi) \rightharpoonup 0 \quad \textit{in } \mathrm{L}^2(\Omega).$$

*A similar equivalence holds for strong convergences.*

THEOREM 4 (correctors). – *One has* $\nabla_x u_\varepsilon - \nabla_x u_0 - \mathcal{U}_\varepsilon(\nabla_y \widehat{u}) \to 0$ *in* $\mathrm{L}^2(\Omega \times Y)$.

*Proof.* – Using convergence (8) and Theorem 3, $\nabla_x u_\varepsilon - \mathcal{U}_\varepsilon(\nabla_x u_0) - \mathcal{U}_\varepsilon(\nabla_y \widehat{u}) \to 0$ in $\mathrm{L}^2(\Omega)$. Then, Proposition 5(iii) gives $\mathcal{U}_\varepsilon(\nabla_x u_0) \to \nabla_x u_0$ in $\mathrm{L}^2(\Omega)$. □

## 6. Periodic unfolding and multiscales

Consider a 'partition' of $Y$ in two non-empty disjoint open subsets $\omega_1$ and $\omega_2$, such that $\overline{Y} = \overline{\omega}_1 \cup \overline{\omega}_2$ and let $u_{\varepsilon\delta} \in \mathrm{H}^1_0(\Omega)$ be the solution of

$$\int_\Omega A^{\varepsilon\delta}\nabla u_{\varepsilon\delta}\nabla w = \int_\Omega f w \quad \forall w \in \mathrm{H}^1_0(\Omega),$$

where $A^{\varepsilon\delta}$ is defined by $A^{\varepsilon\delta}(x) = \begin{cases} A_1\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right) & \text{for } \left\{\frac{x}{\varepsilon}\right\}_Y \in \omega_1, \\ A_2\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y, \left\{\frac{x}{\varepsilon\delta}\right\}_Z\right) & \text{for } \left\{\frac{x}{\varepsilon}\right\}_Y \in \omega_2. \end{cases}$

Suppose that $A_1$ is in $\mathrm{L}^\infty(\Omega \times Y)$ and $A_2$ in $\mathrm{L}^\infty(\Omega \times Y \times Z)$. Here $Y$ and $Z$ are two reference cells, $Y$ associated with the scale $\varepsilon$, $Z$ with the scale $\varepsilon\delta$.

With standard ellipticity hypotheses, there is a subsequence such that $u_{\varepsilon\delta} \rightharpoonup u_0$ in $\mathrm{H}^1_0(\Omega)$). Using the unfolding method for scale $\varepsilon$, as before we have

$$\mathcal{Q}_\varepsilon(u_{\varepsilon\delta}) \rightharpoonup u_0 \quad \text{in } \mathrm{H}^1_0(\Omega), \qquad \mathcal{T}_\varepsilon(u_{\varepsilon\delta}) \rightharpoonup u_0 \quad \text{in } \mathrm{L}^2\big(\Omega; \mathrm{H}^1(Y)\big)$$

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_\varepsilon(u_{\varepsilon\delta})\big) \rightharpoonup \widehat{u} \quad \text{in } \mathrm{L}^2\big(\Omega; \mathrm{H}^1(Y)\big), \qquad \mathcal{T}_\varepsilon\big(\nabla_x u_{\varepsilon\delta}\big) \rightharpoonup \nabla_x u_0 + \nabla_y \widehat{u} \quad \text{in } \mathrm{L}^2(\Omega\times Y).$$

These convergences do not see the oscillations at the scale $\varepsilon\delta$. In order to capture them, one considers the restrictions to the set $\Omega \times \omega_2$ defined by

$$v_{\varepsilon\delta}(x,y) \doteq \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\mathcal{R}_\varepsilon(u_{\varepsilon\delta})\big)|_{\omega_2}.$$

Obviously, $v_{\varepsilon\delta} \rightharpoonup \widehat{u}|_{\omega_2}$ in $\mathrm{L}^2(\Omega; \mathrm{H}^1(\omega_2))$. Now, we apply to $v_{\varepsilon\delta}$, a similar unfolding operation for the variable $y$, thus adding a new variable $z \in Z$ ($x$ being here a mere parameter),

$$\mathcal{T}_\delta(v_{\varepsilon\delta})(x,y,z) = v_{\varepsilon\delta}\left(x, \delta\left[\frac{y}{\delta}\right]_Z + \delta z\right) \quad \text{for } x \in \Omega,\ y \in Y \text{ and } z \in Z.$$

As before, we use the macro–micro decomposition from Section 2, $v_{\varepsilon\delta} = \mathcal{R}_\delta(v_{\varepsilon\delta}) + \mathcal{Q}_\delta(v_{\varepsilon\delta})$. Then, Propositions 3 and 4, and Theorem 1 imply the convergences

$$\mathcal{Q}_\delta(v_{\varepsilon\delta}) \rightharpoonup \widehat{u}|_{\omega_2} \quad \text{in } \mathrm{L}^2\big(\Omega; \mathrm{H}^1(\omega_2)\big), \ \mathcal{T}_\delta(v_{\varepsilon\delta}) \rightharpoonup \widehat{u}|_{\omega_2} \quad \text{in } \mathrm{L}^2\big(\Omega \times \omega_2; \mathrm{H}^1(Z)\big),$$
$$\frac{1}{\delta}\mathcal{T}_\delta\big(\mathcal{R}_\delta(v_{\varepsilon\delta})\big) \rightharpoonup \widetilde{u} \quad \text{in } \mathrm{L}^2\big(\Omega \times \omega_2; \mathrm{H}^1(Z)\big), \qquad \mathcal{T}_\delta(\nabla_x u_{\varepsilon\delta}) \rightharpoonup \nabla_y \widehat{u}|_{\omega_2} + \nabla_z \widetilde{u} \quad \text{in } \mathrm{L}^2(\Omega \times \omega_2 \times Z),$$
$$\mathcal{T}_\delta\big(\mathcal{T}_\varepsilon(\nabla_x u_{\varepsilon\delta})\big) \rightharpoonup \nabla_x u_0 + \nabla_y \widehat{u} + \nabla_z \widetilde{u} \quad \text{in } \mathrm{L}^2(\Omega \times \omega_2 \times Z).$$

THEOREM 5. – *The functions* $u_0 \in \mathrm{H}^1_0(\Omega)$, $\widehat{u} \in \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R})$ *and* $\widetilde{u} \in \mathrm{L}^2(\Omega \times \omega_2, \mathrm{H}^1_{\mathrm{per}}(Z)/\mathbb{R})$ *are the unique solutions of the following variational problem*

$$\frac{1}{|Y||Z|}\int_\Omega\int_{\omega_2}\int_Z A_2(x,y,z)\{\nabla_x u_0 + \nabla_y \widehat{u} + \nabla_z \widetilde{u}\}\{\nabla_x \Psi + \nabla_y \Phi + \nabla_z \Theta\}$$
$$+ \frac{1}{|Y|}\int_\Omega\int_{\omega_1} A_1(x,y)\{\nabla_x u_0 + \nabla_y \widehat{u}\}\{\nabla_x \Psi + \nabla_y \Phi\} = \int_\Omega f\Psi$$
$$\forall \Psi \in \mathrm{H}^1_0(\Omega),\ \forall \Phi \in \mathrm{L}^2\big(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R}\big),\ \forall \Theta \in \mathrm{L}^2\big(\Omega \times \omega_2, \mathrm{H}^1_{\mathrm{per}}(Z)/\mathbb{R}\big).$$

The proof uses test functions of the form $\Psi(x) + \varepsilon\Psi_1(x)\Phi_1(x/\varepsilon) + \varepsilon\delta\Psi_2(x)\Phi_2(x/\varepsilon)\Theta_2(x/\varepsilon\delta)$, where $\Psi$, $\Psi_1$, $\Psi_2$ are in $\mathcal{D}(\Omega)$, $\Phi_1$, $\Phi_2$ in $\mathrm{H}^1_{\mathrm{per}}(Y)$ and $\Theta_2 \in \mathrm{H}^1_{\mathrm{per}}(Z)$.

Detailed proofs together with examples of applications including multi-layered materials, perforated materials, truss-like structures, plates and beams in conduction and linearized elasticity, will be presented in forthcoming publications.

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