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Harnack inequality for symmetric stable processes on fractals

Krzysztof Bogdan, Andrzej Stós, Paweł Sztonyk

Institute of Mathematics, Wrocław University of Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

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Abstract We study nonnegative harmonic functions of symmetric α-stable processes on d-sets F. We prove the Harnack inequality for such functions when $\alpha \in (0, 2/d_w) \cup (d_s, 2)$. Furthermore, we investigate the decay rate of harmonic functions and the Carleson estimate near the boundary of a region in F. In the particular case of *natural cells* in the Sierpiński gasket we also prove the boundary Harnack principle. ***To cite this article: K. Bogdan et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 59–63.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

L'inégalité de Harnack pour les processus symétriques stables sur les fractals

Résumé Nous présentons l'inégalité de Harnack pour les fonctions α-harmoniques sur d-ensembles. En particulier cas de *cellule naturelle* du triangle de Sierpiński nous obtenons le principe de Harnack à la frontiére. Nous donnons aussi une estimation de la vitesse de decroissance des fonctions α-harmoniques près de la frontière ainsi que l'estimation de Carleson. ***Pour citer cet article : K. Bogdan et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 59–63.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Les processus symétriques stables sur les d-ensembles ont été introduits dans [18]. La théorie du potentiel de ces processus et des processus similaires sur les fractals est récemment intensement étudiée [13,16]. Soit F un fermé non-vide dans $\mathbb{R}^N$, $N \geqslant 1$, et soit $d \in (0, N]$. On dit que une mesure borélienne μ est une d-mesure sur F s'il existe des constantes $c_1, c_2 > 0$ telles que μ vérifie $c_1 r^d \leqslant \mu(B(x,r)) \leqslant c_2 r^d$, $x \in F$, $0 < r < \infty$. Nous appelons F un d-ensemble si $F = \mathrm{supp}(\mu)$ pour une d-mesure μ. Nous notons d_s la dimension spectrale et d_w la dimension «walk» de F. Dans toute cette Note on fixe $\alpha \in (0,2)$ et un d-ensemble F. Un processus symétrique stable X dans F est le processus obtenu par la subordination de la difusion fractale par rapport au subordinateur $\alpha/2$-stable.

Soit u une fonction borélienne sur F, minorée (ou majorée) par une constante. On dit que u est α-harmonique dans un ouvert $D \subseteq F$ lorsque $u(x) = E^x u(X(\tau_B))$, $x \in B$, pour tout ouvert borné B tel

E-mail addresses: bogdan@im.pwr.wroc.pl (K. Bogdan); stos@im.pwr.wroc.pl (A. Stós); sztonyk@im.pwr.wroc.pl (P. Sztonyk).

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que $\overline{B} \subseteq D$. On dit que u est régulière α-harmonique dans D si la derniére égalité à lieu avec D à la place de B.

Le résultat principal de cette Note est l'inégalité de Harnack. Nous montrons que pour tout $\alpha \in (0, 2/d_w) \cup (d_s, 2)$ et tout $\kappa > 1$ il existe $c_3 = c_3(\kappa)$ tel que pour tous $x_0 \in F$, $r > 0$ et toute fonction $h \geqslant 0$, α-harmonique dans $B(x_0, r)$ on a $h(x) \geqslant c_3 h(y)$, $x, y \in B(x_0, r/\kappa)$. Notre outil fondamental est la propriété faible d'échelle (3) qui remplace la propriété usuelle d'échelle du processus stable. Un autre outil important est la formule de Ikeda–Watanabe (*cf.* [14]). Leur utilisation permet d'obtenir les estimations (4) and (5) du noyau de Poisson d'une boule. C'est un point de départ pour démontrer l'inégalité de Harnack. Quand D est une cellule naturelle (un triangle élémentaire) du triangle de Sierpiński (non-borné) nous obtenons par la suite le principe de Harnack à la frontiére. Pour des fractals plus compliqués (p. ex. le tapis de Sierpiński) un argument plus fin devrait être apliqué. Nous donnons aussi une estimation de la vitesse decroissance des fonctions α-harmoniques près de la frontière et l'estimation de Carleson.

Dans les preuves nous developpons des analogues fractals des méthodes données dans [7] pour les domaines de Lipschitz et les processus α-stables dans $\mathbb{R}^N$ invariants par rotation. Nous présentons les idées des preuves de nos principaux résultats. Les preuves complètes trop longues et trop techniques pour être incluses ici, seront données dans une publication ultérieure [10].

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1. Preliminaries

Symmetric stable processes on d-sets were introduced in [18] (*see* also [12,13,16]). The aim of this Note is to present the research which is a continuation of that paper. Let F be a nonempty closed subset of $\mathbb{R}^N$, $N \geqslant 1$. Set $d \in (0, N]$. We say that a (positive) Borel measure μ is a *d-measure* on F if for some constant c it satisfies $c^{-1} r^d \leqslant \mu(\{y : |y - x| < r\}) \leqslant c r^d$, $0 < r < \infty$, $x \in F$. We call F a *d-set* if $F = \operatorname{supp}(\mu)$ for some d-measure μ. Note that such a set is necessarily unbounded. For more information on d-sets we refer to [11] and [15]. A Markov process $Z = (P^x, Z_t)_{x \in F, t \geqslant 0}$ is called a *fractional diffusion* on F if Z is a diffusion with state space F, and Z has a symmetric transition density $q(t, x, y) = q(t, y, x)$, $t > 0$, $x, y \in F$, which is jointly continuous for each $t > 0$ and satisfies for a constant c and all $x, y \in F$ and $t \in (0, \infty)$

$$c^{-1} t^{-d_s/2} \exp\left(-c\left(\frac{|x-y|}{t^{1/d_w}}\right)^{\gamma}\right) \leqslant q(t, x, y) \leqslant c t^{-d_s/2} \exp\left(-c^{-1}\left(\frac{|x-y|}{t^{1/d_w}}\right)^{\gamma}\right). \tag{1}$$

Here $d_s = 2d/d_w$, $\gamma = d_w/(d_w - 1)$. For a survey on fractional diffusions see [1]. (We note that our definition is slightly different from that given in [1].) Following [17] and [18] we use the Euclidean distance instead of the shortest path metric (*see* [1]). Since the well known fractal diffusions were constructed in the shortest path metric setting (e.g. [1]), (1) is virtually tantamount to the assumption that the two metrics are equivalent. For the rest of the Note fix a d-set F and $\alpha \in (0, 2)$. By a symmetric stable process X on F we mean the process obtained by the subordination of a fractional diffusion Z to the $\alpha/2$-stable subordinator Y_t. In particular, the transition densities $p(t, x, y)$ of X are given by

$$p(t, x, y) = \int_0^{\infty} q(u, x, y) \eta_t(u) \, \mathrm{d}u, \quad x, y \in F,$$

where $\eta_t(\cdot)$ is the density function of Y_t (*see* [18,5] or [6] for more details).

Let u be a Borel measurable function u on F, which is bounded from below (above). We say that u is *α-harmonic* in a relatively open set $D \subseteq F$ if $u(x) = E^x u(X(\tau_B))$, $x \in B$, for every bounded (relatively) open set B with the closure $\overline{B}$ contained in D. We say that u is *regular α-harmonic* in D if the above equality holds with D instead of B.

In what follows we denote $B(x,r)=\{y\in F: |y-x|<r\}$. Also, $B(x,r)^c=F\setminus B(x,r)$ and $\mathrm{int}(B(x_0,r)^c)=F\setminus\overline{B(x,r)}$.

All constants c are positive and depend on F, μ, α and Z. Dependence of constants on additional quantities will be explicitly expressed.

2. Statement of results

Denote $d_\alpha=d+\alpha d_w/2$. The main result can be stated as follows.

THEOREM 1 (Harnack inequality). – *Let $0<\alpha<2/d_w$ or $d_s<\alpha<2$. For each $\kappa>1$ there exists constant $c=c(\kappa)$ such that for every $x_0\in F$, $r>0$ and function $h\geqslant 0$, regular α-harmonic in $B(x_0,r)$, we have*

$$c^{-1}h(y)\leqslant h(x)\leqslant ch(y),\quad x,y\in B(x_0,r/\kappa). \tag{2}$$

The idea of the proof is given at the end of the Note. The next two results are important in the proof.

THEOREM 2 (Weak scaling). – *For $t>0$, $x,y\in F$ we have*

$$c^{-1}\min\bigl(t|x-y|^{-d_\alpha},\ t^{-d_s/\alpha}\bigr)\leqslant p(t,x,y)\leqslant c\min\bigl(t|x-y|^{-d_\alpha},\ t^{-d_s/\alpha}\bigr). \tag{3}$$

(3) is a replacement for the usual scaling of the symmetric stable processes in $\mathbb{R}^N$ [7].

By Ikeda–Watanabe formula [14] we obtain the following estimates for the Poisson kernel $P_{B(x,r)}(\cdot,\cdot)$ of the ball.

PROPOSITION 3 (Poisson kernel). – *For each $\kappa>1$ there exists $c=c(\kappa)$ such that for all $x\in F$ and $r>0$*

$$P_{B(x,r)}(y,z)\leqslant cr^{\alpha d_w/2}|y-z|^{-d_\alpha},\quad y\in B(x,r),\ z\in B(x,\kappa r)^c, \tag{4}$$

$$P_{B(x,r)}(x,z)\geqslant c^{-1}r^{\alpha d_w/2}|y-z|^{-d_\alpha},\quad y\in B(x,r/\kappa),\ z\in \mathrm{int}\bigl(B(x,r)^c\bigr). \tag{5}$$

Below we provide "fractal" analogues of the results obtained in [7] for Lipschitz domains and the rotation invariant α-stable process in $\mathbb{R}^N$. Consider a (relatively) open subset $D\subseteq F$. Assume that there are constants $R_0=R_0(D)$, $c=c(D)$ and $\theta=\theta(D)\in(0,1)$ such that for all $Q\in\partial D$ and $r\in(0,R_0)$ we have

$$\mu\bigl(D^c\cap B(Q,r)\bigr)\geqslant cr^d, \tag{6}$$

and there is a point A such that

$$B(A,\theta r)\subseteq D\cap B(Q,r). \tag{7}$$

Under these assumptions we obtain the following results.

THEOREM 4. – *There exist constants $\beta=\beta(D)$, $r_0=r_0(D)$ and $c=c(D)$ such that for all $Q\in\partial D$ and $r\in(0,r_0)$, and functions $u\geqslant 0$, regular α-harmonic in $D\cap B(Q,r)$ and satisfying $u(x)=0$ on $D^c\cap B(Q,r)$, we have*

$$u(x)\leqslant c\bigl(|x-Q|/r\bigr)^\beta \sup\bigl\{u(y): y\in D\cap B(Q,r)\bigr\},\qquad x\in D\cap B(Q,r).$$

The above estimate of the rate of decay of harmonic functions near the boundary corresponds to Lemma 3 in [7]. However, [7] makes an essential use of the exact formula for the Poisson kernel for the ball, which is not available in our case. We use Proposition 3 instead. We also note that Theorem 4 holds for all $\alpha\in(0,2)$.

Proposition 5 below is an analogue of the Carleson estimate. We adapt the proof from [8] (*see* also [7]). Our main contribution is the application of Theorem 2 instead of (unavailable) scaling.

PROPOSITION 5 (Carleson estimate). – *Let $\alpha < 2/d_w$. There exist a constant $c = c(\theta)$ such that for all $Q \in \partial D$ and $r \in (0, R_0/2)$, and functions $u \geqslant 0$, regular α-harmonic in $D \cap B(Q, 2r)$ and satisfying $u(x) = 0$ on $D^c \cap B(Q, 2r)$, we have*

$$u(x) \leqslant cu(A), \quad x \in D \cap B(Q, r),$$

where A satisfies (7).

We consider the following example. Let F_0 be the closed convex planar triangle with vertices at $(0, 0)$, $(1, 0)$ and $(1/2, \sqrt{3}/2)$. Let A be the interior of the triangle whose vertices are the midpoints of the edges of F_0. Let $F_1 = F_0 \setminus A$. Then F_1 consists of three closed triangles of side $1/2$. To obtain F_2 we apply subsequently the above procedure to these triangles in F_1. Then we obtain F_3 and so on. Let $F_\infty = \bigcap_{n=0}^{\infty} F_n$, $\widetilde{F} = \bigcup_{n=0}^{\infty} 2^n F_\infty$. $\widetilde{F}$ is the (unbounded) Sierpiński gasket (*see* e.g. [3]). By a *natural cell* (or simply a cell) we mean the intersection of $\widetilde{F}$ with a triangle from F_n (some $n \in \mathbb{N}$).

Observe that the natural cells (and their finite unions of) in the Sierpiński gasket satisfy the assumptions (6) and (7). The same is true for the natural cells of the Sierpiński carpet (with a similar the definition of the natural cell; see also [2]).

THEOREM 6 (Boundary Harnack principle). – *Assume that D is the interior of a sum of a finite number of natural cells* (*possibly of different sizes*) *of the Sierpiński gasket. Let $0 < \alpha < 2/d_w$ or $d_s < \alpha < 2$. There exist constants $c = c(D)$ and $\rho_0 = \rho_0(D)$ such that for every $Q \in \partial D$, $r \in (0, \rho_0)$, and functions $u, v \geqslant 0$, regular α-harmonic in $D \cap B(Q, 2r)$, which vanish on $D^c \cap B(Q, 2r)$, and satisfy $u(x) = v(x)$ for some $x \in D \cap B(Q, r/4)$, we have*

$$c^{-1}v(y) \leqslant u(y) \leqslant cv(y), \quad y \in D \cap B(Q, r/4). \tag{8}$$

Our proof of the result hinges on the simple geometry of the Sierpiński gasket (∂D is totally disconnected) and is analogous to the proof of the boundary Harnack principle for intervals on the real line, e.g. for Brownian motion or symmetric stable processes. It turns out that in this case the boundary Harnack principle follows easily from the usual the Harnack inequality.

We note that for more complicated fractals (e.g. the Sierpiński carpet) more advanced methods need to be applied.

3. Idea of proof of Theorem 1

The method of our proof depends on whether the process is point-recurrent ($\alpha > d_s$) or transient ($\alpha < 2/d_w$). In the former case (2) yields the estimates for the Green function of a ball:

$$G_{B(x,r)}(x, y) \leqslant cr^{-d+\alpha d_w/2}, \quad G_{B(x,a|x-y|)}(x, y) \geqslant c^{-1}|x - y|^{-d+\alpha d_w/2}, \quad x, y \in F, \tag{9}$$

for a constant $a > 1$ and all $x, y \in F$, $r > 0$. Moreover, by the strong Markov property and the continuity of the Green function

$$P^x[T_y < \tau_D] = G_D(x, y)/G_D(y, y),$$

so that the estimates (9) imply that with positive probability the process starting from x hits y before leaving $B(x, a|x - y|)$. By the mean value property this yields the Harnack inequality.

In the more interesting transient case we apply the method of regularization of the Poisson kernel, which proved to be useful in the case of symmetric stable processes (*see* [7,9]). This method yields comparability of values of a given harmonic function with its integral over a neighborhood of a region of harmonicity. However, possible geometrical irregularities of our d-set F make the present proof more difficult. In particular, in the present setting it is not clear whether or not $X_{\tau_D} \in \partial D$ with positive probability. Our approach shows that this event has probability 0 (comp. [7]) and provides an upper bound for the Poisson kernel, which is crucial for the method of regularization. The key step is an estimate for the dimension of the Euclidean sphere intersected with F, which yields relevant capacitary estimates. We note that the application of the capacity argument imposes the restriction $\alpha \in (0, 2/d_w)$. (When $F = \mathbb{R}^N$, $d_w = 2$, this translates into $\alpha < 1$.) This seems to be merely a technical restriction and we conjecture that Theorem 1 holds for all $\alpha \in (0, 2)$.

Remark. – It is interesting whether similar results hold for other type of stable processes on d-sets [16,12]. There are some hints that the answer, at least partially, is positive (*see* e.g. [16] where the equivalence relation for the Dirichlet forms is provided). Our method, however, is closely related to the weak scaling properties so that a new approach is needed.

For details of the proofs and more complete results we refer the reader to the forthcoming paper [10]. We also note that similar considerations apply to symmetric (not necessarily rotation invariant) stable processes on $\mathbb{R}^N$ [4,9].

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