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On a class of anisotropic asymptotically periodic Hamiltonians

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Abstract We construct a C^*-algebra $\mathfrak{C}$ proper to an anisotropic asymptotically periodic quantum system and we compute its quotient by the algebra of compact operators. We describe then the self-adjoint operators affiliated to $\mathfrak{C}$ and their essential spectrum. ***To cite this article: O. Rodot, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 575–579.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Hamiltoniens anisotropes asymptotiquement périodiques

Résumé Nous construisons une C^*-algèbre $\mathfrak{C}$ adaptée au traitement des systèmes quantiques anisotropes asymptotiquement périodiques et nous calculons son quotient par l'algèbre des opérateurs compacts. Nous décrivons alors les opérateurs auto-adjoints affiliés à $\mathfrak{C}$ et leurs spectres essentiels. ***Pour citer cet article : O. Rodot, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 575–579.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Considérons l'opérateur auto-adjoint $H = -\Delta + V$ dans $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$, où V est l'opérateur de multiplication par une fonction asymptotiquement périodique avec des périodes différentes à $+\infty$ et $-\infty$. Les résultats obtenus par Georgescu et Iftimovici dans [6] nous ont suggéré l'étude, par des méthodes nouvelles, de cette classe particulière d'hamiltoniens. L'idée générale consiste à construire une C^*-algèbre $\mathfrak{C}$ dont le quotient par l'algèbre des opérateurs compacts puisse être calculé, et telle que les opérateurs que l'on veut étudier lui soient affiliés. Dans le cas présent, cette C^*-algèbre $\mathfrak{C}$ est obtenue grâce à la notion de produit croisé à partir d'une C^*-algèbre $\mathcal{C}$ suggérée par la classe de fonctions V. L'objectif est de préciser la classe la plus large d'hamiltoniens H affiliés à $\mathfrak{C}$. Ceci permet l'étude du spectre essentiel ou l'estimation de Mourre d'une manière unifiée. Introduisons à présent la C^*-algèbre commutative $\mathcal{C}$ suivante suggérée par une situation anisotrope :

$$\mathcal{C} = \left\{ f \in C_{bu}(\mathbb{R}) \mid \lim_{n\to\pm\infty} f(x + na_\pm) \text{ existent}, \ \forall x \in \mathbb{R} \right\},$$

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où $C_{bu}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f$ est bornée, uniformément continue$\}$, $n \in \mathbb{Z}$ et $(a_+, a_-) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. On montre que l'on a un plongement canonique : $\mathcal{C}/C_0(\mathbb{R}) \subset C(\mathbb{R}/a_+\mathbb{Z}) \oplus C(\mathbb{R}/a_-\mathbb{Z})$. Notons $\mathfrak{C}$ la C^*-algèbre $\mathcal{C} \rtimes \mathbb{R}$ (produit croisé de $\mathcal{C}$ par l'action du groupe additif $\mathbb{R}$). On a alors

$$\mathfrak{C}/K\big(\mathrm{L}^2(\mathbb{R})\big) \cong (\mathcal{C} \rtimes \mathbb{R})/\big(C_0(\mathbb{R}) \rtimes \mathbb{R}\big) \cong \big(\mathcal{C}/C_0(\mathbb{R})\big) \rtimes \mathbb{R}$$

si bien que $\mathfrak{C}/K(\mathrm{L}^2(\mathbb{R})) \subset \mathfrak{C}_+ \oplus \mathfrak{C}_-$ où $\mathfrak{C}_\pm = \mathcal{C}_\pm \rtimes \mathbb{R}$ et $\mathcal{C}_\pm = C(\mathbb{R}/a_\pm\mathbb{Z})$.

THÉORÈME 1. – *Soit $\widetilde{\mathfrak{C}}$ l'ensemble des opérateurs $T \in B(\mathrm{L}^2(\mathbb{R}))$ tels que* :
(i) *il existe $T_+ \in \mathfrak{C}_+$ tel que $\|\chi(Q > r)(T - T_+)^{(*)}\| \to 0$ si $r \to \infty$* ;
(ii) *il existe $T_- \in \mathfrak{C}_-$ tel que $\|\chi(Q < -r)(T - T_-)^{(*)}\| \to 0$ si $r \to \infty$* ;
(iii) $\|(\mathrm{e}^{\mathrm{i}xP} - 1)T^{(*)}\| \to 0$ *si $x \to 0$.*
Alors $\widetilde{\mathfrak{C}}$ est une C^*-algèbre canoniquement isomorphe à $\mathfrak{C}$.

THÉORÈME 2. – *Soient H un opérateur auto-adjoint dans $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$ et $H_\pm$ un couple d'opérateurs auto-adjoints affiliés à $\mathfrak{C}_\pm$ tels que $D(H_\pm) = D(H)$. Alors H est affilié à $\mathfrak{C}$ si*

$$\big\|\theta_+(\varepsilon Q)(H - H_+)\big\|_{D(H)\to\mathcal{H}} \to 0 \quad et \quad \big\|\theta_-(\varepsilon Q)(H - H_-)\big\|_{D(H)\to\mathcal{H}} \to 0 \quad si\ \varepsilon \to 0.$$

The study of the Schrödinger operator with periodic potential is now a classical subject many times explored since the article of Bloch [1] published in 1928. This operator gives a description of the motion of a particle in a crystal. It is well known that the spectrum of this operator has a band structure. Gel'fand [4] and Titchmarsh [8] were among the first to study rigorously the periodic one dimensional Schrödinger operator $H = -\Delta + V$ in $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$ (where the Laplacian Δ, free Hamiltonian, is the quantization of the kinetic energy and V is the operator of multiplication by a periodic potential function). More recently Davies and Simon have studied in [3] the scattering theory for systems with asymptotic spatial behaviour different on the right and the left. Also Roberts develops in [7] the quantum scattering for impurities in potentials that tend to a periodic function in one direction and to a constant one in the other. The references on this topic are multiple.

The new results of Georgescu and Iftimovici in [6] suggested the study to us, with their new methods, of a class of anisotropic Hamiltonians: the periodic Schrödinger operator with different behaviors at $\pm\infty$. The general idea is to construct a C^*-algebra $\mathfrak{C}$, to study its properties and to point out a class H of Hamiltonians affiliated to $\mathfrak{C}$; this allows one to describe the essential spectrum. This C^*-algebra $\mathfrak{C}$ is obtained by the notion of crossed product from a C^*-algebra $\mathcal{C}$ suggested by the class of functions V. One of the main goals is to determine the largest class of Hamiltonians H affiliated to $\mathfrak{C}$.

Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator in $\mathcal{H}$. Recall that the formula $\sigma(H) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R})$ and $\varphi(\lambda) \neq 0 \Rightarrow \varphi(H) \neq 0\}$ gives a description of the spectrum of H. Here $C_0(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \varphi$ is continuous and convergent to zero at infinity$\}$. Recall also that the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of H is the set of $\lambda \in \sigma(H)$ such that either λ is not isolated from the rest of the spectrum or it is an eigenvalue of infinite multiplicity. We have:

$$\sigma_{\mathrm{ess}}(H) = \big\{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Rightarrow \varphi(H) \notin K(\mathcal{H})\big\},$$

where $K(\mathcal{H})$ is the C^*-algebra of compact operators in $\mathcal{H}$ (closed self-adjoint ideal in the C^*-algebra $B(\mathcal{H})$ of bounded linear operators in $\mathcal{H}$). Let $C(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$ be the Calkin algebra (it is also a C^*-algebra). Denote by $H \mapsto \widehat{H}$ the canonical surjection of $B(\mathcal{H})$ onto $C(\mathcal{H})$. Is is easy to see that $\sigma_{\mathrm{ess}}(H) = \sigma(\widehat{H})$. If $\mathfrak{C}$ is a C^*-subalgebra of $B(\mathcal{H})$, we say that H is affiliated to $\mathfrak{C}$ if its associated fonctional calculus is in $\mathfrak{C}$, i.e., if $\forall\varphi \in C_0(\mathbb{R})$, $\varphi(H) \in \mathfrak{C}$. In fact, it suffices to verify for a complex $z \notin \sigma(H)$ that $(H - z)^{-1} \in \mathfrak{C}$. In [2] and [6] several classes of operators related to interesting physical situations are shown to be affiliated to

C^*-algebras $\mathfrak{C}$ such that

$$K(\mathcal{H}) \subset \mathfrak{C} \subset B(\mathcal{H}) \quad \text{and} \quad \mathfrak{C}/K(\mathcal{H}) \subset \bigoplus_k \mathfrak{C}_k, \tag{1}$$

where $\mathfrak{C}_k$ are "simpler" C^*-algebras. So, if H is affiliated to a C^*-algebra $\mathfrak{C}$ which satisfies (1), then $\widehat{H} = \bigoplus_k H_k$ (with H_k affiliated to $\mathfrak{C}_k$) satisfies $\sigma(\widehat{H}) = \bigcup_k \sigma(H_k)$ with known $\sigma(H_k)$.

Let us consider the C^*-algebra $C_{bu}(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{C} \mid \varphi$ is bounded, uniformly continuous$\}$ and the following commutative C^*-algebra suggested by an anisotropic situation:

$$\mathcal{C} = \Big\{\varphi \in C_{bu}(\mathbb{R}) \mid \lim_{n\to\pm\infty} \tau_{na_\pm}\varphi(x) \equiv l_\pm(x) \text{ exist}, \ \forall x \in \mathbb{R}\Big\},$$

where $\tau_{na_\pm}\varphi(x) = \varphi(x + na_\pm)$, $n \in \mathbb{Z}$, and $(a_+, a_-) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. Then $\mathcal{C}$ is a C^*-subalgebra of $C_{bu}(\mathbb{R})$ which contains $C_0(\mathbb{R})$ and is stable by translations. The functions l_+ and l_- are continuous on $\mathbb{R}$ and clearly periodic of periods a_+ and a_-, respectively. We describe now the quotient C^*-algebra $\mathcal{C}/C_0(\mathbb{R})$.

PROPOSITION 1. – $\mathcal{C}/C_0(\mathbb{R}) \subset C(\mathbb{R}/a_+\mathbb{Z}) \oplus C(\mathbb{R}/a_-\mathbb{Z})$.

Recall that the self-adjoint operators of $\mathrm{L}^2(\mathbb{R})$, Q (position observable) and P (momentum observable), are defined by $(Qf)(x) = xf(x)$ and $(Pf)(x) = -\mathrm{i}\,\mathrm{d}f/\mathrm{d}x$. Then $\varphi(Q)$ is the operator of multiplication by the Borel function φ in $\mathrm{L}^2(\mathbb{R})$ and $\psi(P) = \mathcal{F}^*\psi(Q)\mathcal{F}$ where $\mathcal{F} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ with $\mathcal{F}f(y) = (2\pi)^{-1/2}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}xy} f(x)\,\mathrm{d}x$. An elementary computation gives $\mathrm{e}^{\mathrm{i}xP}\varphi(Q)\,\mathrm{e}^{-\mathrm{i}xP} = \varphi(Q + x)$ and $\mathrm{e}^{\mathrm{i}kQ}\psi(P)\,\mathrm{e}^{-\mathrm{i}kQ} = \psi(P - k)$. If $\mathfrak{A}$, $\mathfrak{B}$ are subspaces of an algebra $\mathfrak{D}$ then we denote by $\mathfrak{A} \cdot \mathfrak{B}$ the linear subspace of $\mathfrak{D}$ generated by the elements of the form AB with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. If $\mathfrak{D}$ is a C^*-algebra then $[\![\mathfrak{A} \cdot \mathfrak{B}]\!]$ is the norm closure of $\mathfrak{A} \cdot \mathfrak{B}$ in $\mathfrak{D}$. Since $\mathcal{C}$ is a C^*-subalgebra of $C_{bu}(\mathbb{R})$ stable under translations, $\mathcal{C}$ is provided with a continuous action of the additive group $\mathbb{R}$. Then the crossed product $\mathcal{C} \rtimes \mathbb{R}$ is well defined and denoted by $\mathfrak{C}$. It is shown in [6] that $\mathfrak{C}$ is isomorphic to the (norm) closed linear subspace of $B(\mathrm{L}^2(\mathbb{R}))$ generated by operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in \mathcal{C}$ and $\psi \in C_0(\mathbb{R})$. Below we denote $C_0(\mathbb{R}^*)$ the set of operators $\psi(P)$ with $\psi \in C_0(\mathbb{R})$. We thus may write $\mathfrak{C} = [\![\mathcal{C} \cdot C_0(\mathbb{R}^*)]\!]$. One also have $K(\mathrm{L}^2(\mathbb{R})) = [\![C_0(\mathbb{R}) \cdot C_0(\mathbb{R}^*)]\!]$ so $C_0(\mathbb{R}) \rtimes \mathbb{R}$ is isomorphic to $K(\mathrm{L}^2(\mathbb{R}))$. Then

$$\mathfrak{C}/K\big(\mathrm{L}^2(\mathbb{R})\big) \cong (\mathcal{C} \rtimes \mathbb{R})/\big(C_0(\mathbb{R}) \rtimes \mathbb{R}\big) \cong \big(\mathcal{C}/C_0(\mathbb{R})\big) \rtimes \mathbb{R}$$

so that $\mathfrak{C}/K(\mathrm{L}^2(\mathbb{R})) \subset \mathfrak{C}_+ \oplus \mathfrak{C}_-$ where $\mathfrak{C}_\pm = \mathcal{C}_\pm \rtimes \mathbb{R}$ and $\mathcal{C}_\pm = C(\mathbb{R}/a_\pm\mathbb{Z})$. Recall again that $\mathfrak{C}_\pm = [\![\mathcal{C}_\pm \cdot C_0(\mathbb{R}^*)]\!]$.

Now we give a new characterization of $\mathfrak{C}$. If a symbol like $T^{(*)}$ is used then the relation must hold both for the operator T and for its adjoint T^*. We denote $\chi(A > r)$ the spectral projection of a self-adjoint operator A associated to the interval (r, ∞). The symbol $\chi(A < r)$ has a similar meaning.

THEOREM 2. – *$\mathfrak{C}$ coincides with the set of the operators $T \in B(\mathrm{L}^2(\mathbb{R}))$ such that*
(i) *there is some $T_+ \in \mathfrak{C}_+$ such that $\|\chi(Q > r)(T - T_+)^{(*)}\| \to 0$ if $r \to \infty$;*
(ii) *there is some $T_- \in \mathfrak{C}_-$ such that $\|\chi(Q < -r)(T - T_-)^{(*)}\| \to 0$ if $r \to \infty$;*
(iii) *$\|(\mathrm{e}^{\mathrm{i}xP} - 1)T^{(*)}\| \to 0$ if $x \to 0$.*

Proof. – Let $\widetilde{\mathfrak{C}}$ be the set of operators T satisfying (i)–(iii); we first show that $\mathfrak{C} \subset \widetilde{\mathfrak{C}}$. Since $\mathfrak{C}$ is isomorphic to the (norm) closed linear subspace of $B(\mathrm{L}^2(\mathbb{R}))$ generated by operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in \mathcal{C}$ and $\psi \in C_0(\mathbb{R}^*)$, our task reduces to show that these operators belong to $\widetilde{\mathfrak{C}}$. So let $T = \varphi(Q)\psi(P)$ and set $T_\pm = l_\pm(Q)\psi(P)$. It suffices to show (i) and (ii) for $r = na_+$.

$$\begin{aligned}\big\|\chi(Q > na_+)(T - T_+)\big\| &= \big\|\mathrm{e}^{-\mathrm{i}na_+P}\chi(Q > 0)\,\mathrm{e}^{\mathrm{i}na_+P}\big(\varphi(Q)\psi(P) - l_+(Q)\psi(P)\big)\big\| \\ &= \big\|\chi(Q > 0)\,\mathrm{e}^{\mathrm{i}na_+P}\big(\varphi(Q)\psi(P) - l_+(Q)\psi(P)\big)\,\mathrm{e}^{-\mathrm{i}na_+P}\big\|.\end{aligned}$$

But $\mathrm{e}^{\mathrm{i}na_+P}l_+(Q)\psi(P)\,\mathrm{e}^{-\mathrm{i}na_+P}l_+(Q + na_+)\psi(P) = l_+(Q)\psi(P)$ since l_+ is periodic of period a_+. Similarly $\mathrm{e}^{\mathrm{i}na_+P}\varphi(Q)\psi(P)\,\mathrm{e}^{-\mathrm{i}na_+P} = \varphi(Q + na_+)\psi(P)$.

$$\big\|\chi(Q>na_+)(T-T_+)\big\| = \big\|\chi(Q>0)\big(\varphi(Q+na_+)-l_+(Q)\big)\psi(P)\big\| \leqslant \sup_{x>0}\big|\varphi(x+na_+)-l_+(x)\big|\|\psi\|_\infty \to 0 \quad \text{as } n\to\infty.$$

The assertion (iii) is a consequence of $\psi \in C_0(\mathbb{R}^*)$ (*see* [2] and [5]).

Reciprocally, let $T \in \widetilde{\mathfrak{C}}$. Let $\theta_+ \in \mathrm{C}^\infty(\mathbb{R})$, $0 \leqslant \theta_+ \leqslant 1$, such that $\theta_+(x) = 0$ if $x < 1$ and $\theta_+(x) = 1$ if $x > 2$. Set $\theta_-(x) = \theta_+(-x)$, $\theta_0 = 1-\theta_- - \theta_+$, $\theta_\pm^\varepsilon = \theta_\pm(\varepsilon Q)$ and $\theta_0^\varepsilon = \theta_0(\varepsilon Q)$. (i) and (ii) are equivalent to $\|\theta_\pm^\varepsilon(T-T_\pm)^{(*)}\| \to 0$ if $\varepsilon \to 0$ respectively. Since $T = \theta_0^\varepsilon T + \theta_+^\varepsilon T_+ + \theta_-^\varepsilon T_- + \theta_+^\varepsilon(T-T_+) + \theta_-^\varepsilon(T-T_-)$, the last two terms tend to zero in norm when $\varepsilon \to 0$. Also, $\theta_\pm^\varepsilon \in \mathcal{C}$ yelds $\theta_\pm^\varepsilon T_\pm \in \mathfrak{C}$. Finally we have to prove that $\theta_0^\varepsilon T \in K(\mathrm{L}^2(\mathbb{R})) \subset \mathfrak{C}$. By a characterization of $K(\mathrm{L}^2(\mathbb{R}))$ given in [6], $\theta_0^\varepsilon T \in K(\mathrm{L}^2(\mathbb{R}))$ iff $\|(\mathrm{e}^{\mathrm{i}xP}-1)\theta_0^\varepsilon T\| \to 0$ as $x \to 0$ and $\|(\mathrm{e}^{\mathrm{i}kQ}-1)\theta_0^\varepsilon T\| \to 0$ as $k \to 0$. But $\|(\mathrm{e}^{\mathrm{i}xP}-1)\theta_0^\varepsilon T\| \leqslant \|[\mathrm{e}^{\mathrm{i}xP}, \theta_0^\varepsilon]\|\|T\| + \|\theta_0^\varepsilon\|\|(\mathrm{e}^{\mathrm{i}xP}-1)T\|$. Since $\|(\mathrm{e}^{\mathrm{i}xP}-1)T\| \to 0$ by (iii), and

$$\big\|\big[\mathrm{e}^{\mathrm{i}xP}, \theta_0^\varepsilon\big]\big\| = \big\|\mathrm{e}^{\mathrm{i}xP}\theta_0^\varepsilon \mathrm{e}^{-\mathrm{i}xP} - \theta_0^\varepsilon\big\| = \big\|\theta_0^\varepsilon(\cdot + x) - \theta_0^\varepsilon\big\|_\infty \to 0 \quad \text{as } x \to 0.$$

We get $\|(\mathrm{e}^{\mathrm{i}kQ}-1)\theta_0^\varepsilon T\| \leqslant \|(\mathrm{e}^{\mathrm{i}kQ}-1)\theta_0^\varepsilon\|\|T\| = \|(\mathrm{e}^{\mathrm{i}k\cdot}-1)\theta_0^\varepsilon\|_\infty\|T\| \to 0$ if $k \to 0$, thus $\theta_0^\varepsilon T \in K(\mathrm{L}^2(\mathbb{R}))$, which finishes the proof of the theorem. □

We shall point out now an affiliation criterion to $\mathfrak{C}$ for a self-adjoint operator. We first introduce some useful definitions (*see* [5]).

DEFINITION 3. – An operator $T \in B(\mathrm{L}^2(\mathbb{R}))$ will be called *semi-compact* if for all $\theta \in C_0(\mathbb{R})$ the operators $\theta(Q)T$ and $T\theta(Q)$ are compact. The self-adjoint operators affiliated to $SK(\mathrm{L}^2(\mathbb{R}))$ are called *locally compact*.

Remark that the set $SK(\mathrm{L}^2(\mathbb{R}))$ of semi-compact operators is a C^*-subalgebra of $B(\mathrm{L}^2(\mathbb{R}))$ and that $\mathfrak{C}_\pm \subset HK(\mathrm{L}^2(\mathbb{R})) \subset SK(\mathrm{L}^2(\mathbb{R}))$ where

$$HK\big(\mathrm{L}^2(\mathbb{R})\big) = \Big\{T \in B\big(\mathrm{L}^2(\mathbb{R})\big) \mid \lim_{r\to\infty}\big(\big\|\chi\big(|P|>r\big)T\big\| + \big\|T\chi\big(|P|>r\big)\big\|\big) = 0\Big\}.$$

DEFINITION 4. – We say that $T \in B(\mathrm{L}^2(\mathbb{R}))$ has limit $T_\pm$ at $Q = \pm\infty$ (and set $\lim_{Q\to\pm\infty} T = T_\pm$) if $\lim_{\varepsilon\to 0}\{\|\theta_\pm(\varepsilon Q)(T-T_\pm)\| + \|(T-T_\pm)\theta_\pm(\varepsilon Q)\|\} = 0$.

PROPOSITION 5. – $\mathfrak{C} = \{T \in SK(\mathrm{L}^2(\mathbb{R})) \mid \lim_{Q\to\pm\infty} T$ *exist and belong to* $\mathfrak{C}_\pm\}$.

COROLLARY 6. – *The following assertions are equivalent*:
(i) H *self-adjoint operator affiliated to* $\mathfrak{C}$.
(ii) H *is locally compact and* $\exists z \in \mathbb{C}\setminus\mathbb{R}$ *such that* $\lim_{Q\to\pm\infty}(H-z)^{-1}$ *exist and are in* $\mathfrak{C}_\pm$.

The next proposition gives a method for checking local compactness.

PROPOSITION 7. – *Assume that* $\mathcal{K}$ *is a Banach space continuously embedded in* $\mathcal{H}$ *and such that* $\theta(Q) \in K(\mathcal{K}, \mathcal{H})$ *for each* $\theta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$. *If* H *is a self-adjoint operator in* $\mathcal{H}$ *and* $(H+\mathrm{i})^{-1}\mathcal{H} \subset \mathcal{K}$, *then* H *is locally compact. In particular, if the domain of* H *is included in the form domain of a locally compact operator, then* H *is locally compact.*

Our second important result states as follows:

THEOREM 8. – *Let* H *be a self-adjoint operator in* $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$ *and* $H_\pm$ *a pair of self-adjoint operators affiliated to* $\mathfrak{C}_\pm$ *respectively such that* $D(H_\pm) = D(H)$. *Assume that*

$$\big\|\theta_+(\varepsilon Q)(H-H_+)\big\|_{D(H)\to\mathcal{H}} \to 0 \quad \textit{and} \quad \big\|\theta_-(\varepsilon Q)(H-H_-)\big\|_{D(H)\to\mathcal{H}} \to 0 \quad \textit{if } \varepsilon \to 0.$$

Then H *is affiliated to* $\mathfrak{C}$. *In particular* $\sigma_{\mathrm{ess}}(H) = \sigma(H_+) \cup \sigma(H_-)$.

Proof. – Since $H_\pm$ are affiliated to $\mathfrak{C}_\pm \subset SK(\mathcal{H})$, $H_\pm$ are locally compact. H is locally compact because of $D(H_\pm) = D(H)$ and of Proposition 7. So, by Corollary 6, it suffices to prove that $\lim_{Q\to\pm\infty}(H-z)^{-1} = (H_\pm - z)^{-1}$. We denote $R(z) = (H-z)^{-1}$ and $R_\pm(z) = (H_\pm - z)^{-1}$. Then

$$\begin{aligned}\left\|\theta_\pm(\varepsilon Q)(R-R_\pm)\right\| &= \left\|\theta_\pm(\varepsilon Q)R_\pm(H-H_\pm)R\right\| \\ &\leqslant \left\|\left[\theta_\pm(\varepsilon Q), R_\pm\right]\right\| \cdot \left\|(H-H_\pm)R\right\| + \|R_\pm\| \cdot \left\|\theta_\pm(\varepsilon Q)(H-H_\pm)R\right\|.\end{aligned}$$

But $\|\theta_\pm(\varepsilon Q)\,(H-H_\pm)R\,\| \to 0$ when $\varepsilon \to 0$ because of the hypothesis. On the other hand, $\|[\theta_\pm(\varepsilon Q), R_\pm]\| \leqslant (\varepsilon/\sqrt{2\pi}) \cdot \|\widehat{\theta'_\pm}\|_{\mathrm{L}^1(\mathbb{R})} \cdot \left\| R'_\pm\right\|_{\mathrm{L}^\infty(\mathbb{R})}$, thus $\|\theta_\pm(\varepsilon Q)\,(R-R_\pm)\,\| \to 0$ when $\varepsilon \to 0$. $\square$

We give now an example. Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $c^{-1}|x|^s \leqslant |h(x)| \leqslant c|x|^s$ for some real $s>0$, a constant $c>0$ and large x. Set $H_0 = h(P)$, its domain being the Sobolev space $\mathcal{H}^s(\mathbb{R})$. Let V be a real function such that $V(Q) \in B(\mathcal{H}^s, \mathcal{H})$ with H_0-bound less than 1 and assume that there are $a_\pm$-periodic functions $V_\pm$ such that $\|\theta_\pm(\varepsilon Q)\,(V-V_\pm)\|_{\mathcal{H}^s\to\mathcal{H}} \to 0$ if $\varepsilon \to 0$. Then $H = H_0 + V(Q)$ is affiliated to $\mathfrak{C}$ and $H_\pm = H_0 + V_\pm(Q)$, so

$$\sigma_{\mathrm{ess}}(H) = \sigma(H_+) \cup \sigma(H_-).$$

Remark. – If E is a Hilbert space, then one can replace above $\mathfrak{C}$ by $\mathfrak{C} \otimes K(\mathrm{L}^2(E))$. Thus our results cover the case of one-dimensional Dirac operators for example (then E is finite dimensional).

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