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Singular sets of Sobolev functions

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Abstract We are interested in finding Sobolev functions with "large" singular sets. Given $N, k \in \mathbb{N}$, $1 < p < \infty$, $kp < N$, for any compact subset A of $\mathbb{R}^N$, such that its upper box dimension is less than $N - kp$, we construct a Sobolev function $u \in \mathrm{W}^{k,p}(\mathbb{R}^N)$ which is singular precisely on A. We introduce the notions of lower and upper singular dimensions of Sobolev space, and show that both are equal to $N - kp$. ***To cite this article: D. Žubrinić, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 539–544.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Ensembles singuliers des fonctions de Sobolev

Résumé Nous sommes intéressés à trouver des fonctions de Sobolev dont l'ensemble des singularités est « grand ». Étant donné $N, k \in \mathbb{N}$, $1 < p < \infty$, $kp < N$, pour chaque sous-ensemble A compact de $\mathbb{R}^N$, dont la « box-dimension » supérieure est plus petite que $N - kp$, nous construisons une fonction de Sobolev $u \in \mathrm{W}^{k,p}(\mathbb{R}^N)$ qui est singulière précisément sur A. Nous introduisons les notions de dimensions singulières inférieure et supérieure de l'espace de Sobolev, et montrons que ses valeurs sont $N - kp$. ***Pour citer cet article : D. Žubrinić, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 539–544.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soient $u : \mathbb{R}^N \to \overline{\mathbb{R}}$ une fonction mesurable et $\operatorname{Sing} u$ l'ensemble des singularités de u, c'est-à-dire $x_0 \in \operatorname{Sing} u$ si ils existant $\alpha > 0$, $R > 0$, $C > 0$, tels que $u(x) \geqslant C|x - x_0|^{-\alpha}$ p.p. sur $B_R(x_0)$. Nous introduisons la notion de *dimension singulière inférieure* de l'espace de Sobolev $\mathrm{W}^{k,p}(\mathbb{R}^N)$, $N, k \in \mathbb{N}$, par

$$\text{s-}\underline{\dim}\, \mathrm{W}^{k,p}\big(\mathbb{R}^N\big) = \sup\big\{\dim_{\mathrm{H}}(\operatorname{Sing} u) : u \in \mathrm{W}^{k,p}\big(\mathbb{R}^N\big)\big\},$$

où $\dim_{\mathrm{H}}$ est la dimension de Hausdorff. La *dimension singulière supérieure* est définie par

$$\text{s-}\overline{\dim}\, \mathrm{W}^{k,p}\big(\mathbb{R}^N\big) = \sup\big\{\dim_{\mathrm{H}}(\text{e-Sing}\, u) : u \in \mathrm{W}^{k,p}\big(\mathbb{R}^N\big)\big\},$$

où $\text{e-Sing}\, u$ est l'ensemble élargi des singularités de u, défini par (4). Il est clair que $\text{e-Sing}\, u$ contient $\operatorname{Sing} u$ et, par exemple, les singularités logarithmiques de u aussi. Le but de cet article est de prouver que si $1 < p < \infty$, k entier positif ou nul, $kp \leqslant N$, alors

$$\text{s-}\underline{\dim}\, \mathrm{W}^{k,p}\big(\mathbb{R}^N\big) = \text{s-}\overline{\dim}\, \mathrm{W}^{k,p}\big(\mathbb{R}^N\big) = N - kp.$$

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En particulier, si $kp = N$ alors $\dim_H(\text{e-Sing}\, u) = 0$ pour tout $u \in W^{k,p}(\mathbb{R}^N)$. Nous montrons aussi que pour chaque sous-ensemble A compact de $\mathbb{R}^N$, dont la « box-dimension » supérieure est plus petite que $N - kp$, on peut construire une fonction $u \in W^{k,p}(\mathbb{R}^N)$ qui est singulière précisément sur A. Il est intéressant de noter que $\text{s-}\underline{\dim}\, L^p(\mathbb{R}^N) = N$, à condition que $1 \leqslant p < \infty$. Pour $X = \bigcap_{1 \leqslant p < \infty} L^p(\mathbb{R}^N)$ on a $\text{s-}\underline{\dim}\, X = 0$, tandis que $\text{s-}\overline{\dim}\, X = N$.

1. Introduction

One of the earliest results related to the question of size of singular sets of Sobolev functions is stated in Reshetnyak [21, Theorem 1.8] (relying on Fuglede [9, Theorem 2]): if $f \in L^p(\mathbb{R}^N)$, $f \geqslant 0$, and G_α is the Bessel potential kernel, then the set of all x for which $(G_\alpha * f)(x) = \infty$, has (α, p)-Bessel capacity equal to zero. This implies that the Hausdorff dimension of this set is at most $N - \alpha p$, which is an immediate consequence of Reshetnyak [21, Corollary 2], or Adams and Hedberg [1, Theorem 5.1.13] (for the case $\alpha = 1$ see Heinonen, Kilpeläinen and Martio [13, Theorem 2.26], or Malý and Ziemer [18, Theorem 2.53]). The aim of this paper is to show that the upper bound $N - \alpha p$ for the Hausdorff dimension of singular sets of Sobolev functions cannot be improved. Our Theorems 1 and 2 are of a similar nature as Fuglede [9, Theorem 8], which characterizes subsets E of $\mathbb{R}^N$ for which the system $\mathbf{S}^k(E)$ of all k-dimensional Lipschitz surfaces intersecting the set is exceptional of order p. There, the "borderline" value $N - kp$ (with $kp \neq N$) appears analogous to our Theorem 1. As shown in Fuglede [9, Theorem 6], the system $\mathbf{S}^k(E)$ is exceptional iff there exists $f \in L^p(\mathbb{R}^N)$, $f \geqslant 0$, such that the corresponding Riesz potential is infinite on E, without being identically infinite. Here we deal with Bessel potentials.

Singularities of Sobolev functions have been studied in a monograph by Jaffard and Meyer [14, Chapter II] using wavelet methods, but with weaker type of singularities than we consider here. We deal with singularities in the classical sense. Our results complement those stated in [14, Theorem 2.1] and methods of proof are different. Among numerous contributions related to singular sets of Sobolev functions and quasilinear elliptic equations with singular solutions, we cite Deny [5], Deny and Lions [6], Fuglede [9, 10], Aronszajn and Smith [2], Serrin [22], Reshetnyak [21], Stein [23], Havin and Mazya [12], Bagby and Ziemer [3], Meyers [19] Veron [24], Mou [20], Grillot [11], Kilpeläinen [16], Korkut, Pašić and Žubrinić [17], Žubrinić [26], and the references therein.

Let $u : \mathbb{R}^N \to \overline{\mathbb{R}}$ be a measurable function. We say that u has singularity at least of order $\alpha > 0$ at $x_0 \in \mathbb{R}^N$ if there exist $R > 0$ and $C > 0$ such that $u(x) \geqslant C|x - x_0|^{-\alpha}$ for a.e. $x \in B_R(x_0)$, where $B_R(x_0)$ is an open ball of radius R centered at x_0. We say that u has singularity of order α on a nonempty subset A of $\mathbb{R}^N$, if there exist $R > 0$ and $C > 0$ such that $u(x) \geqslant Cd(x, A)^{-\alpha}$ for a.e. $x \in A_R$, where $d(x, A)$ is the Euclidean distance from x to A and A_R is R-neighbourhood of A.

The set of all singular points of a given measurable function $u : \mathbb{R}^N \to \overline{\mathbb{R}}$, having order at least $\alpha > 0$, will be denoted by $\text{Sing}_\alpha u$. We define the *singular set of u* by

$$\text{Sing}\, u = \bigcup_{\alpha > 0} \text{Sing}_\alpha u. \tag{1}$$

Let X be an arbitrary Banach space (or just a nonempty set) of measurable functions $u : \mathbb{R}^N \to \overline{\mathbb{R}}$. We define *lower and upper singular dimension* of X by

$$\text{s-}\underline{\dim}\, X = \sup\bigl\{\dim_H(\text{Sing}\, u) : u \in X\bigr\}, \tag{2}$$

$$\text{s-}\overline{\dim}\, X = \sup\bigl\{\dim_H(\text{e-Sing}\, u) : u \in X\bigr\}, \tag{3}$$

respectively, where $\text{e-Sing}\, u$ is defined by

$$\text{e-Sing}\, u = \left\{x_0 \in \mathbb{R}^N : \limsup_{r \to 0} \frac{1}{r^N} \int_{B_r(x_0)} u(x)\, \mathrm{d}x = +\infty\right\}. \tag{4}$$

We call it extended singular set of u. As we see, e-Sing u is contained in the complement of the set of Lebesgue points of u. Note that e-Sing u contains among others also iterated logarithmic singularities of u. Since Sing $u \subseteq$ e-Sing u, we have s-$\underline{\dim}\, X \leqslant$ s-$\overline{\dim}\, X \leqslant N$. If s-$\underline{\dim}\, X =$ s-$\overline{\dim}\, X$, the common value is called *singular dimension* of X and denoted by s-dim X. In the sequel we shall need the notion of upper box dimension of a subset A of $\mathbb{R}^N$ (also known as the upper Minkowski dimension), that we denote by $\overline{\dim}_B A$, see, e.g., Falconer [7]. Recall that $\dim_H A \leqslant \overline{\dim}_B A$. Here is the main result of this paper.

THEOREM 1. –
(a) *If* $1 < p < \infty$, $k \in \mathbb{N}$, $kp < N$, *and* A *is a compact subset of* $\mathbb{R}^N$ *such that*

$$\overline{\dim}_B A < N - kp, \tag{5}$$

then there exists a Sobolev function $u \in \mathrm{W}^{k,p}(\mathbb{R}^N)$ *which is singular precisely on* A. *Furthermore,*

$$\text{s-}\dim \mathrm{W}^{k,p}\big(\mathbb{R}^N\big) = N - kp. \tag{6}$$

(b) *If* $kp = N$, *then* $\dim_H(\text{e-Sing}\, u) = 0$ *for any* $u \in \mathrm{W}^{k,p}(\mathbb{R}^N)$. *In other words,* s-$\overline{\dim}\, \mathrm{W}^{k,p}(\mathbb{R}^N) = 0$, *which is* (6) *for* $kp = N$.
(c) *If* $1 \leqslant p < \infty$, *then* s-$\underline{\dim}\, \mathrm{L}^p(\mathbb{R}^N) = N$, *which is* (6) *for* $k = 0$.
(d) *For* $X = \bigcap_{1 \leqslant p < \infty} \mathrm{L}^p(\mathbb{R}^N)$ *we have* s-$\underline{\dim}\, X = 0$, *while* s-$\overline{\dim}\, X = N$.

Of course, the same result is true for the corresponding Sobolev and Lebesgue spaces modelled on arbitrary open domain $\mathcal{O}$ in $\mathbb{R}^N$.

Example 1. – Antoine's necklace A in $\mathbb{R}^3$ (for its definition see, e.g., [15]) is clearly compact and 3-rectifiable, so that its three dimensional Minkowski content exists and is finite, and equals to its Lebesgue measure (*see* Federer [8, Theorem 3.2.39]). Using Theorem 1(a) we obtain that there exists a Sobolev function $u \in \mathrm{H}^1(\mathbb{R}^6)$ which is singular precisely on $A \subset \mathbb{R}^3 \subseteq \mathbb{R}^6$. We do not know if $N = 6$ is the smallest possible number with the above property.

2. Singular sets of Bessel potentials of L^p-functions

Let $\alpha > 0$, and let $G_\alpha : \mathbb{R}^N \to \mathbb{R}$ be the Bessel kernel, which is defined by its Fourier transform: $\widehat{G}_\alpha(x) = (2\pi)^{-N/2}(1 + |x|^2)^{-\alpha/2}$. It is well known that $G_\alpha(x) > 0$ for all $x \in \mathbb{R}^N$, see, e.g., Ziemer [25]. We shall need the following asymptotic properties of the Bessel kernel, which follow immediately from [25, p. 65]. Assuming that $0 < \alpha < N$, then for any $R > 0$ there exist positive constants C_1, C_2 and D, such that

$$\frac{C_1}{|x|^{N-\alpha}} \leqslant G_\alpha(x) \leqslant \frac{C_2}{|x|^{N-\alpha}} \quad \text{if } |x| \leqslant R, \tag{7}$$

and $G_\alpha(x) \leqslant D \exp(-|x|)$ if $|x| \geqslant R$. Let us introduce Bessel potential spaces $\mathrm{L}^{\alpha,p}(\mathbb{R}^N) = \{G_\alpha * f : f \in \mathrm{L}^p(\mathbb{R}^N)\}$, where $*$ is the usual convolution operator. For $\alpha = 0$ we define $\mathrm{L}^{0,p}(\mathbb{R}^N) = \mathrm{L}^p(\mathbb{R}^N)$. Now we formulate a result about singular sets of Bessel potentials.

THEOREM 2. – *Assume that* $1 < p < \infty$, $0 < \alpha < N/p$. *Then for any compact subset* A *of* $\mathbb{R}^N$ *such that*

$$\overline{\dim}_B A < N - \alpha p, \tag{8}$$

there exists a function $u \in \mathrm{L}^{\alpha,p}(\mathbb{R}^N)$ *which is singular precisely on* A. *Furthermore,*

$$\text{s-}\dim \mathrm{L}^{\alpha,p}\big(\mathbb{R}^N\big) = N - \alpha p. \tag{9}$$

If $\alpha p = N$ *then* $\dim_H(\text{e-Sing}\, v) = 0$ *for any* $v \in \mathrm{L}^{\alpha,p}(\mathbb{R}^N)$, *that is* s-$\overline{\dim}\, \mathrm{L}^{\alpha,p}(\mathbb{R}^N) = 0$.

For the proof of Theorem 1 we shall need the following result due to A.P. Calderón, see, e.g., [25, Theorem 2.6.1], or the original paper of Calderón [4].

THEOREM 3 (A.P. Calderón). – *Assume that k is a positive integer, and* $1 < p < \infty$. *Then* $\mathrm{L}^{k,p}(\mathbb{R}^N) = \mathrm{W}^{k,p}(\mathbb{R}^N)$.

Next, we shall make use of an interesting result about integrability of the function $d(x, A)^{-\gamma}$ on R-neighbourhood A_R of A, due to Hardt and Mou. See Mou [20, Lemma 3.6], where it was formulated for the case when A has finite s-dimensional Minkowski content, but the same proof holds when A has finite s-dimensional *upper* Minkowski content as well.

LEMMA 1 (Hardt, Mou). – *Assume that* $0 \leqslant s < N$ *and A is a compact subset of* $\mathbb{R}^N$ *such that its s-dimensional upper Minkowski content of A is finite. If* $0 < \gamma < N - s$, *then* $\int_{A_R} d(x, A)^{-\gamma}\,\mathrm{d}x < \infty$ *for any* $R > 0$.

In the proof of Theorem 2, step (b), the following result will be essential, which seems to be of interest in itself. Theorem 4 implies seemingly obvious inclusion $\operatorname{Sing} v \subseteq \{v = +\infty\}$ for a class of Bessel potentials v (and also for Riesz potentials, provided $\alpha p < N$). We state it without proof.

THEOREM 4. – *Assume that* $1 < p < \infty$, $0 < \alpha < N$, *and let* $G : \mathbb{R}^N \times \mathbb{R}^N \to \overline{\mathbb{R}}$ *be a nonnegative potential kernel, such that* $G(x, y)$ *is lower semicontinuous in x for a.e. y, and measurable in y for all x. We assume that there exist* $C_1, C_2, R > 0$ *such that for any x,*

$$\frac{C_1}{|x-y|^{N-\alpha}} \leqslant G(x, y) \leqslant \frac{C_2}{|x-y|^{N-\alpha}} \quad \text{for a.e. } y \in B_R(x), \tag{10}$$

and there exists a bounded, nonnegative, nonincreasing function $g \in \mathrm{L}^{p'}((R, \infty); r^{N-1})$ *such that for all x we have* $G(x, y) \leqslant g(|x - y|)$ *for a.e.* $y \in \mathbb{R}^N \setminus B_R(x)$. *Let* $v = G * f$, *where* $f \in \mathrm{L}^p(\mathbb{R}^N)$, $f \geqslant 0$. *Then* $\text{e-Sing}\, v \subseteq \{v = \infty\}$.

Proof of Theorem 2. – (a) Let us choose any $s \in (\overline{\dim}_{\mathrm{B}} A, N - \alpha p)$. Since $s > \overline{\dim}_{\mathrm{B}} A$, then the s-dimensional upper Minkowski content of A (*see*, e.g., Federer [8] for its definition) is equal to 0. Let us define an auxilliary function $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ by

$$f(x) = \begin{cases} d(x, A)^{-\gamma} & \text{for } x \in A_R, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus A_R, \end{cases} \tag{11}$$

where we take γ such that $\alpha < \gamma < \frac{N-s}{p}$, which is possible due to $s < N - \alpha p$. By Lemma 1 we have that $f \in \mathrm{L}^p(\mathbb{R}^N)$. Let x be any point contained in $A_R \setminus A$ and choose any $x_0 \in A$ such that $|x - x_0| = d(x, A)$. We assume without loss of generality that $x_0 = 0$. Hence, $d(y, A) \leqslant |y|$ and using (7) we obtain:

$$\begin{aligned} u(x) = (G_\alpha * f)(x) &= \int_{A_R} G_\alpha(x-y)\, d(y, A)^{-\gamma}\,\mathrm{d}y \\ &\geqslant \int_{B_{|x|/2}(0)} \frac{C_1}{|x-y|^{N-\alpha}} \cdot |y|^{-\gamma}\,\mathrm{d}y. \end{aligned}$$

It is clear that for $y \in B_{|x|/2}(0)$ we have $|x - y| \leqslant \frac{3}{2}|x|$, so that:

$$u(x) \geqslant \int_{B_{|x|/2}(0)} C_1 \left(\frac{3}{2}|x|\right)^{\alpha - N} \cdot |y|^{-\gamma}\,\mathrm{d}y = \frac{C}{|x|^{\gamma-\alpha}} = \frac{C}{d(x, A)^{\gamma-\alpha}}, \tag{12}$$

where C is a positive constant. Since $\gamma > \alpha$, we obtain that A is a singular set of u of order $\gamma - \alpha$, that is $A \subseteq \operatorname{Sing}_{\gamma-\alpha} u$.

(b) Inequality $\text{s-}\underline{\dim}\, \mathrm{L}^{\alpha,p}(\mathbb{R}^N) \geqslant N - \alpha p$ in (9) follows immediately from (a) and the fact that for any $\lambda \in (0, N)$ there exists a set A in $\mathbb{R}^N$ whose upper box dimension and Hausdorff dimension are both equal to λ. For example, if λ is noninteger, we write $\lambda = s + \lfloor\lambda\rfloor$ with $s \in (0, 1)$ and set $A = C \times [0, 1]^{\lfloor\lambda\rfloor}$, where C is a generalized Cantor set such that $\dim_{\mathrm{H}} C = s$, see Falconer [7, Example 4.5 or 4.7].

Now we prove that s-$\overline{\dim}\,\mathrm{L}^{\alpha,p}(\mathbb{R}^N) \leqslant N - \alpha p$. For Bessel potentials $v = G_\alpha * f$, where $f \in \mathrm{L}^p(\mathbb{R}^N)$, we consider the set $\{G_\alpha * f = \infty\}$. Since we are interested in generating singularities, we can assume without loss of generality that $f(x) \geqslant 0$ (note that $G_\alpha * f \leqslant G_\alpha * f^+$). Due to Theorem 4 we have e-Sing$(G_\alpha * f) \subseteq \{G_\alpha * f = \infty\}$. This implies,

$$\dim_{\mathrm{H}}\bigl(\text{e-Sing}(G_\alpha * f)\bigr) \leqslant \dim_{\mathrm{H}}\{G_\alpha * f = \infty\}. \tag{13}$$

Using Reshetnyak [21, Theorem 1.8], or Adams and Hedberg [1, Proposition 2.3.7], we have that $\mathrm{Cap}_{\alpha,p}\{G_\alpha * f = \infty\} = 0$, where $\mathrm{Cap}_{\alpha,p}$ is (α, p)-capacity. By Reshetnyak [21, Corollary 2], or Adams and Hedberg [1, Theorem 5.1.13] (for the case of $\alpha = 1$ see Heinonen, Kilepläinen and Martio [13, Theorem 2.26], or Malý and Ziemer [18, Theorem 2.53]), it follows that $\dim_{\mathrm{H}}\{G_\alpha * f = \infty\} \leqslant N - \alpha p$. Taking supremum in (13) over all $f \geqslant 0$ we obtain that s-$\overline{\dim}\,\mathrm{L}^{\alpha,p}(\mathbb{R}^N) \leqslant N - \alpha p$. This completes the proof of (9).

(c) It is easy to see that the function $G_\alpha * f$ is continuous on $\mathbb{R}^N \setminus \overline{A_R}$ (an easy consequence of the Lebesgue Dominated Convergence Theorem) and dominated by a continuous function on $\overline{A_R} \setminus A$. Hence, A is precisely the set of singularities of $G_\alpha * f$. □

Proof of Theorem 1. – Claims (a) and (b) follow immediately from Theorems 2 and 3. (c) From Lemma 1, using functions of the form (11), we conclude that s-$\underline{\dim}\,\mathbb{R}^N = N$. (d) The proof of s-$\underline{\dim}\, X = 0$ is trivial, since $\operatorname{Sing} u = \emptyset$ for any $u \in X$. To prove that s-$\overline{\dim}\, X = N$, take any $s < N$. We define $u(x) = \log 1/d(x, A)$ on A_R for fixed $R > 0$, and $u(x) = 0$ otherwise, where A is any given compact set with $s = \dim_{\mathrm{H}} A = \dim_{\mathrm{B}} A < N$ (*see* step (b) in the proof of Theorem 2). By noting that for any $\gamma > 0$ there exists $C > 0$ such that $u(x) \leqslant C \cdot d(x, A)^{-\gamma}$ on A_R, and using Lemma 1, we easily derive that $u \in \mathrm{L}^p(\mathbb{R}^N)$ by taking $\gamma < \frac{1}{p}(N - s)$. Hence $u \in X$. Note that e-Sing$\,u = A$. □

Modifying slightly the proof of Theorem 2, it is easy to see that we can generate singularities of Sobolev functions having prescribed positive order β or larger, on a given set.

THEOREM 5. – *For any positive real numbers α and β, $(\alpha + \beta)p < N$, and any compact subset A of $\mathbb{R}^N$ such that*

$$\overline{\dim}_{\mathrm{B}} A < N - (\alpha + \beta)p, \tag{14}$$

there exists a Sobolev function $u \in \mathrm{L}^{\alpha,p}(\mathbb{R}^N)$ having singularity at least of order β on A, that is $u(x) \geqslant C \cdot d(x, A)^{-\beta}$ a.e. on A_R, for some $C > 0$ and $R > 0$.

Remark 1. – It is easy to extend the notions of lower and upper singular dimension from sets of measurable functions to any nonempty subset X of the space of Schwarz distributions $\mathcal{D}'(\mathbb{R}^N)$.

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