C. R. Acad. Sci. Paris, Ser. I 334 (2002) 457–462

Équations aux dérivées partielles/*Partial Differential Equations*

Normal solvability of linear elliptic problems

Vitaly Volpert [a], Aizik Volpert [b]

[a] Mathématiques appliquées, UMR 5585 CNRS, Université Lyon 1, 69622 Villeurbanne, France

[b] Department of Mathematics, Technion, 32000 Haifa, Israel

Received and accepted 4 January 2002

Note presented by Haïm Brezis.

Abstract The paper is devoted to general linear elliptic problems in Hölder spaces. We consider unbounded domains and define limiting problems at infinity. We give a necessary and sufficient condition of normal solvability through uniqueness of solutions of limiting problems. We study a structure of spaces dual to Hölder spaces and specify the subspace of functionals, which provide the condition of normal solvability. This allows us to prove that for Fredholm operators all limiting operators are invertible. ***To cite this article: V. Volpert, A. Volpert, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 457–462.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Résolubilité normale des problèmes elliptiques linéaires

Résumé L'article est consacré aux problèmes elliptiques généraux. Nous considérons des domaines non bornés et les espaces de Hölder. On définit les problèmes limites à l'infinie ce qui permet de formuler la condition nécessaire et suffisante de la résolubilité normale. Nous étudions la structure de l'espace dual et décrivons le sous-espace de fonctionnelles qui déterminent les conditions de resolubilité. Cela nous permet de démontrer que pour les opérateurs de Fredholm, tous les problèmes limites sont inversibles. ***Pour citer cet article : V. Volpert, A. Volpert, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 457–462.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Nous considérons le problème

$$A_i u = \sum_{k=1}^{p} \sum_{|\beta| \leqslant \beta_{ik}} a_{ik}^{\beta}(x) D^{\beta} u_k \quad (i = 1, \ldots, p),\ x \in \Omega \subset \mathbb{R}^n,$$

$$B_i u = \sum_{k=1}^{p} \sum_{|\beta| \leqslant \gamma_{ik}} b_{ik}^{\beta}(x) D^{\beta} u_k \quad (i = 1, \ldots, r),\ x \in \partial\Omega.$$

Nous supposons qu'il est uniformement elliptique dans le sense de [1]. Le domaine $\Omega \subset \mathbb{R}^n$ est non borné et les coefficients des opérateurs A_i et B_i, ainsi que la frontière $\partial\Omega$ du domaine satisfont les conditions de

E-mail addresses: volpert@maply.univ-lyon1.fr (V. Volpert); volp@techunix.technion.ac.il (A. Volpert).

S1631-073X(02)02286-0/FLA

regularité :

$$a_{ik}^{\beta} \in C^{l-s_i+\alpha}(\Omega), \quad b_{ik}^{\beta} \in C^{l-\sigma_i+\alpha}(\partial\Omega), \quad \partial\Omega \in C^{l+\lambda+\alpha},$$

où les entiers s_i, σ_i sont les mêmes que dans la condition d'ellipticité, $C^{m+\alpha}(\Omega)$ est un espace de Hölder. Notons $A = (A_1, \ldots, A_p)$, $B = (B_1, \ldots, B_r)$, $L = (A, B)$, $L : E_0 \to E$.

On définit les problèmes limites correspondants. Pour cela, il faut définir les domaines limites et les opérateurs limites. Pour définir les domaines limites on considère une suite $x_n \in \overline{\Omega}$ telle que $|x_n| \to \infty$. Soit $\chi(x)$ la fonction caractéristique de Ω. Notons Ω_n le domaine avec la fonction caractéristique $\chi(x + x_n)$. Nous disons que Ω_* est un domaine limite si Ω_n converge vers Ω_* localement dans la metrique de Hausdorff : $\Omega_n \to \Omega_*$ dans Ξ_{loc}.

THÉORÈME 1. – *Si le domaine Ω satisfait la Condition* D (voir *ci-dessous*)*, alors il exists une fonction $f(x)$ telle que* :
(1) $f(x) \in C^{k+\alpha}(\mathbb{R}^n)$;
(2) $f(x) > 0$ *ssi* $x \in \Omega$;
(3) $|\nabla f(x)| \geqslant 1$ *pour* $x \in \partial\Omega$;
(4) $\min(d(x), 1) \leqslant |f(x)|$, *où $d(x)$ est la distance de x à $\partial\Omega$.*

THÉORÈME 2. – *Soit Ω un domaine non borné qui satisfait la Condition* D, $x_m \in \overline{\Omega}$, $|x_m| \to \infty$, *et $f(x)$ la fonction construite dans le Théorème* 1. *Alors il existe une sous-suite x_{m_i} et la fonction $f_*(x)$ telle que $f_{m_i}(x) \equiv f(x + x_{m_i}) \to f_*(x)$ dans $C^k_{\text{loc}}(\mathbb{R}^n)$, et le domaine $\Omega_* = \{x : f_*(x) > 0\}$ satisfait la Condition* D, *où $\Omega_* = \mathbb{R}^n$.*

En plus, $\Omega_{m_i} \to \Omega_*$ dans Ξ_{loc}, où $\Omega_{m_i} = \{x : f_{m_i}(x) > 0\}$.

Pour définir les opérateurs limites, on considère les suites des coefficients $a_{ik}^{\beta}(x + x_n)$, $b_{ik}^{\beta}(x + x_n)$ et choisit les sous-suites convergentes vers certaines fonctions limites uniformement dans chaque ensemble borné. Ces fonctions limites sont les coefficients des opérateurs limites $\hat{L}$. On a donc défini les problèmes limites.

Condition NS. – Pour chaque domaine limite Ω_* et chaque opérateur limite $\hat{L}$ le problème limite $\hat{L} = 0$ a une seule solution $u = 0$.

THÉORÈME 3. – *Si la Condition* NS *est satisfaite, alors l'opérateur L est résoluble normalement et son noyau a une dimension finie.*

Le résultat inverse a lieu sous les conditions un peu plus fortes sur les coefficients et le domaine :

$$a_{ik}^{\beta} \in C^{l-s_i+\delta}(\Omega), \quad b_{ik}^{\beta} \in C^{l-\sigma_i+\delta}(\partial\Omega), \quad \partial\Omega \in C^{l+\lambda+\delta}$$

avec $\alpha < \delta < 1$.

THÉORÈME 4. – *Si l'opérateur L est normalement résoluble et son noyau a une dimension finie, alors la Condition* NS *est satisfaite.*

Notons E^0 le sous-espace de E qui contient les fonctions convergantes vers 0 à l'infinie. Soit E^* l'espace dual à E, $\widetilde{E}$ le sous-espace de fonctionnelles qui s'annulent sur E^0 et $\widehat{E}$ le sous-espace complémentaire dans E^*.

THÉORÈME 5. – *Soit l'opérateur $L : E_0 \to E$ normalement résoluble et son noyau a une dimension finie. Supposons que le problème $Lu = f$ et résoluble si et seulement si $\psi_i(f) = 0$, $i = 1, \ldots, N$, où $\psi_i \in E^*$ sont des fonctionnelles linéarement indépendantes. Alors, $\psi_i \in \widehat{E}$.*

Ce résultat nous permet de démontrer le théorème suivant.

To cite this article: V. Volpert, A. Volpert, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 457–462

THÉORÈME 6. – *Si L est un opérateur de Fredholm, alors chaque son opérateur limite est inversible.*

Normal solvability of some classes of elliptic operators in unbounded domains is studied in [5,9] for the whole $\mathbb{R}^n$, in [3,7] for conical and in [6,8] for cylindrical at infinity domains (*see* also [4]). In this work we consider general elliptic problems in arbitrary domains with some natural regularity conditions. We generalize the notion of limiting problems introduced already in the previous works, and formulate conditions of normal solvability through solvability of the limiting problems.

1. Operators and spaces

Let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index, β_i be nonnegative integers, $|\beta| = \beta_1 + \cdots + \beta_n$, $D^\beta = D_1^{\beta_1} \cdots D_n^{\beta_n}$, $D_i = \partial/\partial x_i$. We consider the following operators:

$$A_i u = \sum_{k=1}^{p} \sum_{|\beta| \leqslant \beta_{ik}} a_{ik}^{\beta}(x) D^\beta u_k \quad (i = 1, \ldots, p),\ x \in \Omega \subset \mathbb{R}^n,$$

$$B_i u = \sum_{k=1}^{p} \sum_{|\beta| \leqslant \gamma_{ik}} b_{ik}^{\beta}(x) D^\beta u_k \quad (i = 1, \ldots, r),\ x \in \partial\Omega.$$

Here β_{ik}, γ_{ik} are given numbers. As in [1] we suppose that there exist integers $s_1, \ldots, s_p$, $t_1, \ldots, t_p$, $\sigma_1, \ldots, \sigma_r$ such that

$$\beta_{ij} \leqslant s_i + t_j, \quad i, j = 1, \ldots, p; \quad \gamma_{ij} \leqslant \sigma_i + t_j, \quad i = 1, \ldots, r,\ j = 1, \ldots, p, \quad s_i \leqslant 0.$$

We suppose that the problem is elliptic in the sense of [2,1], i.e., the system is elliptic and supplementary and complementing (Shapiro–Lopatinskii) conditions are satisfied. Moreover we assume that the system is uniformly elliptic.

We consider the space $E_0 = \mathrm{C}^{l+t_1+\alpha}(\Omega) \times \cdots \times \mathrm{C}^{l+t_p+\alpha}(\Omega)$ of vector valued functions $u(x) = (u_1(x), \ldots, u_p(x))$, $u_j \in \mathrm{C}^{l+t_j+\alpha}(\Omega)$, $j = 1, \ldots, p$, where l and α are given numbers, $l \geqslant \max(0, \sigma_i)$, $0 < \alpha < 1$. The space $\mathrm{C}^{l+t_j+\alpha}(\Omega)$ is a Hölder space of functions bounded together with their derivatives up to the order $l + t_j$, and the latter satisfy the Hölder condition uniformly in x.

The domain Ω is supposed to be of class $\mathrm{C}^{l+\lambda+\alpha}$, where $\lambda = \max(-s_i, -\sigma_i, t_j)$, $a_{ij}^{\beta} \in \mathrm{C}^{l-s_i+\alpha}(\Omega)$, $b_{ij}^{\beta} \in \mathrm{C}^{l-\sigma_i+\alpha}(\partial\Omega)$. We assume that $\lambda \geqslant 1$.

Operator A_i acts from E_0 into $\mathrm{C}^{l-s_i+\alpha}(\Omega)$. Denote $L_1 = (A_1, \ldots, A_p)$. Then L_1 acts from E_0 into $E_1 = \mathrm{C}^{l-s_1+\alpha}(\Omega) \times \cdots \times \mathrm{C}^{l-s_p+\alpha}(\Omega)$. Operator B_i acts from E_0 into $\mathrm{C}^{l-\sigma_i+\alpha}(\partial\Omega)$. Denote $L_2 = (B_1, \ldots, B_r)$. Then L_2 acts from E_0 into $E_2 = \mathrm{C}^{l-\sigma_1+\alpha}(\partial\Omega) \times \cdots \times \mathrm{C}^{l-\sigma_r+\alpha}(\partial\Omega)$. Let $E = E_1 \times E_2$. Then $L = (L_1, L_2)$ acts from E_0 into E.

2. Limiting domains

In this section we define limiting domains for unbounded domains in $\mathbb{R}^n$, show their existence and study some of their properties. We consider an unbounded domain $\Omega \subset \mathbb{R}^n$, which satisfies the following condition:

Condition D. – For each $x_0 \in \partial\Omega$ there exists a neighbourhood $U(x_0)$ such that

(1) $U(x_0)$ contains a sphere with the radius δ and the center x_0, where δ is independent of x_0;

(2) There exists a homeomorphism $\psi(x; x_0)$ of the neighbourhood $U(x_0)$ on the unit sphere $B = \{y : |y| < 1\}$ in $\mathbb{R}^n$ such that the images of $\Omega \cap U(x_0)$ and $\partial\Omega \cap U(x_0)$ coincide with $B_+ = \{y : y_n > 0, |y| < 1\}$ and $B_0 = \{y : y_n = 0,\ |y| < 1\}$ respectively;

(3) The function $\psi(x;x_0)$ and its inverse belong to the Hölder space $C^{k+\alpha}$ with $k=l+\lambda$. Their $\|\cdot\|_{k+\alpha}$-norms are bounded uniformly in x_0.

To define convergence of domains we use the following Hausdorff metric space. Let A and B denote two nonempty closed sets in $\mathbb{R}^n$. Denote $\varsigma(A,B)=\sup_{a\in A}\rho(a,B)$, $\varsigma(B,A)=\sup_{b\in B}\rho(b,A)$, where $\rho(a,B)$ denotes the distance from a point a to a set B, and let

$$\varrho(A,B)=\max\bigl(\varsigma(A,B),\varsigma(B,A)\bigr). \tag{1}$$

We denote Ξ a metric space of bounded nonempty sets in $\mathbb{R}^n$ with the distance given by (1). We say that a sequence of domains Ω_m converges to a domain Ω in Ξ_{loc} if

$$\varrho\bigl(\overline{\Omega}_m\cap\overline{B}_R,\overline{\Omega}\cap\overline{B}_R\bigr)\to 0,\quad m\to\infty,$$

for any $R>0$ and $B_R=\{x:|x|<R\}$. Here bar denotes the closure of domains.

DEFINITION. – Let $\Omega\subset\mathbb{R}^n$ be an unbounded domain, $x_m\in\overline{\Omega}$, $|x_m|\to\infty$ as $m\to\infty$; $\chi(x)$ be a characteristic function of Ω, and Ω_m be a shifted domain defined by the characteristic function $\chi_m(x)=\chi(x+x_m)$.

We say that Ω_* is a *limiting domain* of the domain Ω if $\Omega_m\to\Omega_*$ in Ξ_{loc} as $m\to\infty$.

THEOREM 2.1. – *If a domain Ω satisfies Condition* D, *then there exists a function $f(x)$ defined in $\mathbb{R}^n$ such that*:
(1) $f(x)\in C^{k+\alpha}(\mathbb{R}^n)$;
(2) $f(x)>0$ *iff* $x\in\Omega$;
(3) $|\nabla f(x)|\geqslant 1$ *for* $x\in\partial\Omega$;
(4) $\min(d(x),1)\leqslant|f(x)|$, *where $d(x)$ is the distance from x to $\partial\Omega$.*

THEOREM 2.2. – *Let Ω be an unbounded domain satisfying Condition* D, *$x_m\in\overline{\Omega}$, $|x_m|\to\infty$, and $f(x)$ be the function constructed in Theorem* 2.1. *Then there exists a subsequence x_{m_i} and a function $f_*(x)$ such that $f_{m_i}(x)\equiv f(x+x_{m_i})\to f_*(x)$ in $C^k_{\mathrm{loc}}(\mathbb{R}^n)$, and the domain $\Omega_*=\{x: f_*(x)>0\}$ satisfies Condition* D, *or $\Omega_*=\mathbb{R}^n$.*

Moreover $\Omega_{m_i}\to\Omega_*$ in Ξ_{loc}, where $\Omega_{m_i}=\{x: f_{m_i}(x)>0\}$.

3. Limiting problems

In the previous section we have introduced limiting domains. Here we will define the corresponding limiting problems. Let Ω be a domain satisfying Condition D and $\chi(x)$ be its characteristic function. Consider a sequence $x_m\in\overline{\Omega}$, $|x_m|\to\infty$ and the shifted domains Ω_m defined by the shifted characteristic functions $\chi_m(x)=\chi(x+x_m)$. We suppose that the sequence of domains Ω_m converge in Ξ_{loc} to some limiting domain Ω_*.

DEFINITION 1. – Let $u_m(x)\in C^k(\Omega_m)$, $m=1,2,\ldots$. We say that $u_m(x)$ converges to a limiting function $u_*(x)\in C^k(\Omega_*)$ in $C^k_{\mathrm{loc}}(\Omega_m\to\Omega_*)$ if there exists an extension $v_m(x)\in C^k(\mathbb{R}^n)$ of $u_m(x)$, $m=1,2,\ldots$, and an extension $v_*(x)\in C^k(\mathbb{R}^n)$ of $u_*(x)$ such that

$$v_m\to v_*\quad\text{in } C^k_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr).$$

DEFINITION 2. – Let $u_m(x)\in C^k(\partial\Omega_m)$, $m=1,2,\ldots$. We say that $u_m(x)$ converges to a limiting function $u_*(x)\in C^k(\partial\Omega_*)$ in $C^k_{\mathrm{loc}}(\partial\Omega_m\to\partial\Omega_*)$ if there exists an extension $v_m(x)\in C^k(\mathbb{R}^n)$ of $u_m(x)$, $m=1,2,\ldots$, and an extension $v_*(x)\in C^k(\mathbb{R}^n)$ of $u_*(x)$ such that

$$v_m\to v_*\quad\text{in } C^k_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr).$$

Remark. – In these definitions $u_*(x)$ does not depend on the choice of the extensions $v_m(x)$ and $v_*(x)$. Indeed, in Definition 1 for any point $x\in\overline{\Omega}_*$ there exists a sequence $\hat{x}_m\in\overline{\Omega}_m$ suh that $\hat{x}_m\to x$. Therefore

$$u_*(x)=v_*(x)=\lim_{m\to\infty}v_m(\hat{x}_m)=\lim_{m\to\infty}u_m(\hat{x}_m).$$

Similarly it can be checked for Definition 2.

THEOREM 3.1. – *Let* $u_m \in C^{k+\alpha}(\Omega_m)$, $\|u_m\|_{C^{k+\alpha}} \leqslant M$, *where the constant* M *is independent of* m, $0 \leqslant k \leqslant l + \lambda$. *Then there exists a function* $u_* \in C^{k+\alpha}(\Omega_*)$, $\|u_*\|_{C^{k+\alpha}} \leqslant M$ *and a subsequence* u_{m_j} *such that* $u_{m_j} \to u_*$ *in* $C^k_{\mathrm{loc}}(\Omega_{m_j} \to \Omega_*)$.

Let $u_m \in C^{k+\alpha}(\partial\Omega_m)$, $\|u_m\|_{C^{k+\alpha}} \leqslant M$. *Then there exists a function* $u_* \in C^{k+\alpha}(\partial\Omega_*)$, $\|u_*\|_{C^{k+\alpha}} \leqslant M$ *and a subsequence* u_{m_j} *such that* $u_{m_j} \to u_*$ *in* $C^k_{\mathrm{loc}}(\partial\Omega_{m_j} \to \partial\Omega_*)$.

We recall that the operator L consists of a pair of operators, $L = (L_1, L_2)$ where the operator L_1 acts inside the domain and L_2 is a boundary operator. So we can represent the boundary problem as

$$L_1 u = f_1, \qquad L_2 u = f_2, \tag{2}$$

where $u \in E_0(\Omega)$, $f_1 \in E_1(\Omega)$, $f_2 \in E_2(\partial\Omega)$, $E = E_1 \times E_2$. The coefficients $a_{ij}(x)$ of the operator L_1 are defined in $\overline{\Omega}$ and the coefficients $b_{ij}(x)$ of L_2 in $\partial\Omega$. The shifted coefficients $a_{ij}(x + x_m)$ and $b_{ij}(x + x_m)$ satisfy conditions of Theorem 3.1. Therefore we can define the *limiting problem*

$$\hat{L}_1 u = f, \qquad \hat{L}_2 u = g,$$

where $u \in E_0(\Omega_*)$, $f \in E_1(\Omega_*)$, $g \in E_2(\partial\Omega_*)$, $\hat{L}_1$ and $\hat{L}_2$ are operators with limiting coefficients $a^*_{ij}(x) \in C^{l-s_i+\alpha}(\Omega_*)$, $b^*_{ij}(x) \in C^{l-\sigma_i+\alpha}(\partial\Omega_*)$. The operator $\hat{L} = (\hat{L}_1, \hat{L}_2) : E_0(\Omega_*) \to E(\Omega_*)$ will be called *limiting operator.*

We note that for a given problem (2) there can exist a set of limiting problems corresponding to different sequences x_m and to different converging subsequences of coefficients of the operators.

4. Normal solvability

We introduce limiting domains and limiting operators defined above.

Condition NS. – For any limiting domain Ω_* and any limiting operator $\hat{L}$ the problem $\hat{L}u = 0$, $u \in E_0(\Omega_*)$ has only zero solution.

THEOREM 4.1. – *Let Condition* NS *be satisfied. Then the operator* L *is normally solvable and its kernel is finite dimensional.*

We suppose now that a little more smoothness of operators and domains is required:

$$a^{\beta}_{ik} \in C^{l-s_i+\delta}(\Omega),\ b^{\beta}_{ik} \in C^{l-\sigma_i+\delta}(\partial\Omega) \text{ and the domain } \Omega \text{ is of class } C^{l+\lambda+\delta}$$

with $\alpha < \delta < 1$.

THEOREM 4.2. – *Let the operator* L *be normally solvable and its kernel be finite dimensional. Then Condition* NS *is satisfied.*

5. Dual spaces; Invertibility of limiting operators

We consider now the space $E = E(\Omega)$ defined above and the space E^0, which consists of functions $u \in E$ converging to 0 at infinity in the norm E, i.e., $\|u\|_{E(\overline{\Omega}\cap\{|x|\geqslant N\})} \to 0$ as $N \to \infty$. We say that $u_n \to u_0$ in $E_{\mathrm{loc}}(\Omega)$ if this convergence holds in $\overline{\Omega} \cap \{|x| \leqslant N\}$ for any N.

LEMMA 5.1. – *Let* ϕ *be a functional in the dual space* $(E^0)^*$, $u \in E$ *and* $u \notin E^0$, $u_n \in E^0$, $\|u_n\|_E \leqslant 1$, *and* $u_n \to u$ *in* E_{loc}. *Then there exists a limit*

$$\hat{\phi} = \lim_{n\to\infty} \phi(u_n).$$

It does not depend on the sequence u_n. *If* $u_n \to 0$ *in* E_{loc}, *then* $\phi(u_n) \to 0$.

We can extend now the functional ϕ to the space $E(\Omega)$. For any $u \in E(\Omega)$ we put $\hat{\phi}(u) = \phi(u)$ if $u \in E^0(\Omega)$ and $\hat{\phi}(u) = \lim_{n\to\infty} \phi(u_n)$, where $u_n \in E^0(\Omega)$ is an arbitrary sequence converging to u in E_{loc}. This is a linear bounded functional on $E(\Omega)$.

Denote all such functionals $\widehat{E}$. It is a linear subspace in E^*. Suppose that $\widehat{E} \neq E^*$. We take a functional $\psi \in E^*$, which does not belong to $\widehat{E}$. Let ψ_0 be a restriction of ψ on E^0. Then $\psi_0 \in (E^0)^*$. As above we can define the functional $\hat{\psi}_0 \in (E)^*$. By assumption $\psi \neq \hat{\psi}_0$. Denote $\tilde{\psi} = \psi - \hat{\psi}_0$. Then

$$\tilde{\psi} = 0, \quad \forall u \in E^0. \tag{3}$$

Thus we have the following theorem.

THEOREM 5.2. – *The dual space E^* is a direct sum of the extension $\widehat{E}$ of $(E^0)^*$ on E and of the subspace $\widetilde{E}$ consisting of all functionals satisfying* (3).

LEMMA 5.3. – *Suppose that the operator $L : E_0 \to E$ is normally solvable with a finite dimensional kernel, and the problem $Lu = f$, $f \in E$, is solvable if and only if $\psi_i(f) = 0$, $i = 1, \ldots, N$, where ψ_i are linearly independent functionals in E^*. Then $\psi_i \in \widehat{E}$.*

These results allow us to prove invertibility of limiting operators (*cf.* Theorem 4.2).

THEOREM 5.4. – *If the operator L is Fredholm, then any its limiting operator is invertible.*

COROLLARY 5.5. – *If an operator L coincides with its limiting operator, and it is Fredholm, then it is invertible.*

The last result shows in particular that the spectrum of operators with constant, periodic or quasi-periodic coefficients in unbounded cylinders does not contain eigenvalues and consists only of points of the essential spectrum.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial diffrential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 12 (1959) 623–727, 17 (1964) 35–92.

[2] A. Douglis, L. Nirenberg, Interior estimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math. 8 (1955) 503–538.

[3] V.A. Kondratiev, Boundary value problems for elliptic equations in domains with conical or angular points, Trudy Moskov. Mat. Obshch. 16 (1967) 209–292. English transl.: Trans. Math. Soc. 16 (1967).

[4] R.B. Lockhart, Fredholm property of a class of elliptic operators on non-compact manifolds, Duke Math. J. 48 (1981) 289–312.

[5] E.M. Mukhamadiev, Normal solvability and the noethericity of elliptic operators in spaces of functions on $\mathbb{R}^n$, Part I, Zap. Nauch Sem. LOMI 110 (1981) 120–140. English transl.: J. Soviet Math. 25 (1) (1984) 884–901.

[6] S.A. Nazarov, K. Pileckas, On the Fredholm property of the Stokes operator in a layer-like domain, J. Anal. Appl. 20 (1) (2001) 155–182.

[7] V.S. Rabinovich, The Fredholm property of general boundary value problems on noncompact manifolds, and limit operators, Dokl. Akad. Nauk 325 (2) (1992) 237–241 (in Russian). English transl.: Russian Acad. Sci. Dokl. Math. 46 (1) (1993) 53–58.

[8] V. Volpert, A. Volpert, J.F. Collet, Topological degree for elliptic operators in unbounded cylinders, Adv. Differential Equations 4 (6) (1999) 777–812.

[9] H.F. Walker, A Fredholm theory for a class of first-order elliptic partial differential operators in $\mathbb{R}^n$, Trans. Amer. Math. Soc. 165 (1972) 75–86.