

# Holomorphic vector bundles on non-algebraic surfaces

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Received 15 October 2001; accepted after revision 11 January 2002

Note presented by Jean-Pierre Demailly.

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## Abstract

The existence problem for holomorphic structures on vector bundles over non-algebraic surfaces is, in general, still open. We solve this problem in the case of rank 2 vector bundles over K3 surfaces and in the case of vector bundles of arbitrary rank over all known surfaces of class VII. Our methods, which are based on Donaldson theory and deformation theory, can be used to solve the existence problem of holomorphic vector bundles on further classes of non-algebraic surfaces. *To cite this article: A. Teleman, M. Toma, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 383–388.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## Fibrés vectoriels holomorphes sur les surfaces non algébriques

## Abstract

Le problème de l'existence des structures holomorphes sur les fibrés vectoriels au-dessus des surfaces non algébriques est en général encore ouvert. Nous résolvons ce problème pour les fibrés de rang 2 sur les surfaces K3 et pour les fibrés de rangs arbitraires sur toutes les surfaces connues de la classe VII. Nos méthodes, qui s'appuient sur la théorie de Donaldson et sur la théorie des déformations, peuvent être utilisées pour résoudre le problème de l'existence des fibrés vectoriels holomorphes sur d'autres classes de surfaces non algébriques. *To cite this article: A. Teleman, M. Toma, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 383–388.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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## Version française abrégée

Soit  $E$  un fibré vectoriel complexe sur une surface complexe compacte  $X$ . Nous posons

$$\Delta(E) := 2 \operatorname{rk}(E)c_2(E) - (\operatorname{rk}(E) - 1)c_1(E)^2.$$

Pour tout  $a \in \operatorname{NS}(X)$  et pour tout entier positif  $r$  soit

$$m(r, a) := r \inf \left\{ - \sum_{i=1}^r \left( \frac{a}{r} - \mu_i \right)^2 \mid \mu_1, \dots, \mu_r \in \operatorname{NS}(X) \text{ avec } \sum_{i=1}^r \mu_i = a \right\}.$$

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Ce nombre est un entier positif ou nul si  $X$  est non-algébrique [2]. Un fibré holomorphe  $\mathcal{E}$  est dit *filtrable* s'il existe une filtration  $0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \mathcal{E}$  avec des faisceaux cohérents  $\mathcal{E}_i$  tels que  $\text{rk}(\mathcal{E}_i) = i$  pour tous les  $i$ . Le problème de l'existence de structures holomorphes filtrables sur les surfaces complexes a été complètement résolu dans [2]. Des résultats sur les structures non filtrables ont été obtenus dans [1, 12–14]. Notre premier résultat est

**THÉORÈME 1.** – Soit  $E$  un fibré complexe de rang 2 différentiable sur une surface K3 non algébrique  $X$ . Alors, à l'exception du cas  $a(X) = 0$ ,  $\Delta(E) = 4$  et  $c_1(E) \in 2\text{NS}(X)$ , on a

(i)  $E$  admet une structure holomorphe filtrable si et seulement si

$$c_1(E) \in \text{NS}(X) \quad \text{et} \quad \Delta(E) \geq m(2, c_1(E)) ;$$

(ii)  $E$  admet une structure holomorphe si et seulement si

$$c_1(E) \in \text{NS}(X) \quad \text{et} \quad \Delta(E) \geq \min(6, m(2, c_1(E))).$$

Dans le cas écarté,  $E$  n'admet aucune structure holomorphe.

Pour démontrer l'existence des structures holomorphes nous utilisons la non trivialité de l'invariant polynomial de Donaldson  $q_P(X)$  associé au PU(2)-fibré  $P$  correspondant à une structure hermitienne sur  $E$ .

*Remarque.* – La théorie de Donaldson permet de résoudre le problème de l'existence des structures holomorphes dans le cas  $\text{rk}(E) = 2$  pour les surfaces kähleriennes  $X$  telles que  $H^0(K_X) \xrightarrow{\otimes^2} H^0(K_X^{\otimes 2})$  soit surjective. Pour une telle surface  $X$ , fixons  $c \in \text{NS}(X)$  et désignons par  $\bar{c} \in H^2(X, \mathbb{Z}_2)$  sa réduction mod 2. On peut montrer que

$$\begin{aligned} & \min\{\Delta(E) \mid c_1(E) = c, E \text{ admet des structures holomorphes}\} \\ & = \min\{-p_1(P) \mid w_2(P) = \bar{c}, q_P(X) \neq 0\}, \end{aligned}$$

lorsque la valeur minimale à droite est inférieure à  $m(c_1, 2)$ .

Ce résultat montre que, pour cette classe de surfaces kähleriennes, l'apparition des structures holomorphes dans le cas  $\Delta(E) < m(c_1(E), 2)$  est déterminée par le type topologique différentiable de la 4-variété différentiable sous-jacente.

Nous dirons qu'une surface complexe  $X$  a la propriété  $\mathfrak{P}_r$  si tout fibré vectoriel de rang  $r$  sur  $X$  qui admet une structure holomorphe, admet aussi une structure filtrable.

**PROPOSITION.** – Soit  $X$  une surface complexe,  $\pi : \tilde{X} \rightarrow X$  une modification propre et  $r \geq 2$  un entier. Alors  $\tilde{X}$  a la propriété  $\mathfrak{P}_r$  si et seulement si  $X$  a cette propriété.

L'idée de la démonstration est de montrer l'inégalité

$$\Delta(\mathcal{E}) - m(r, c_1(\mathcal{E})) \geq \Delta(\pi_*\mathcal{E}) - m(r, c_1(\pi_*\mathcal{E}))$$

pour tout fibré holomorphe  $\mathcal{E}$  de rang  $r$  sur  $\tilde{X}$ .

Notre deuxième résultat résout le problème de l'existence des structures holomorphes sur les fibrés vectoriels au-dessus de toutes les surfaces connues de la classe VII :

**THÉORÈME 2.** – Soit  $X$  une surface de la classe VII dont le modèle minimal est soit une surface à  $b_2 = 0$  soit une surface avec un cycle de courbes rationnelles. Soit  $E$  un fibré vectoriel différentiable de rang  $r$  sur  $X$ . Les propriétés suivantes sont équivalentes :

1.  $E$  admet une structure holomorphe,
2.  $E$  admet une structure holomorphe filtrable,
3.  $c_1(E) \in \text{NS}(X)$  et  $\Delta(E) \geq m(r, c_1(E))$ .

Lorsque le modèle minimal est une surface à  $b_2 = 0$ , on applique la proposition ci-dessus et le résultat fondamental de [2]. Dans le deuxième cas l'idée de la démonstration est de réduire le problème par petite déformation au cas d'une surface de Hopf éclatée, et d'appliquer de nouveau la même proposition.

### 1. Notation

Let  $E$  be a differentiable complex vector bundle over a complex surface  $X$ . The discriminant of  $E$  is defined by:

$$\Delta(E) := 2 \operatorname{rk}(E)c_2(E) - (\operatorname{rk}(E) - 1)c_1(E)^2.$$

For each  $a \in \operatorname{NS}(X)$  and any positive integer  $r$  we put

$$m(r, a) := r \inf \left\{ - \sum_{i=1}^r \left( \frac{a}{r} - \mu_i \right)^2 \mid \mu_1, \dots, \mu_r \in \operatorname{NS}(X) \text{ with } \sum_{i=1}^r \mu_i = a \right\}.$$

Note that  $m(r, a)$  equals  $-\infty$  if  $X$  is algebraic and is a non-negative integer if  $X$  is non-algebraic [2]. We recall that a holomorphic vector bundle  $\mathcal{E}$  is called *filtrable* if there exists a filtration  $0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \mathcal{E}$  with coherent sheaves  $\mathcal{E}_i$  such that  $\operatorname{rk}(\mathcal{E}_i) = i$  for all  $i$ . The existence problem for filtrable holomorphic structures on complex surfaces was completely solved in [2]. For non-filtrable structures the existence problem was solved over tori for rank 2 and over primary Kodaira surfaces for arbitrary rank (see [12–14,1]). We treat here the case of K3 surfaces and of class VII surfaces.

### 2. The case of K3 surfaces

The following result answers completely the existence question for holomorphic structures on rank 2 vector bundles over K3 surfaces.

**THEOREM 1.** – *Let  $E$  be a rank 2 differentiable vector bundle over a non-algebraic K3 surface  $X$ . Excepting the case when  $a(X) = 0$ ,  $\Delta(E) = 4$  and  $c_1(E) \in 2 \operatorname{NS}(X)$ , one has*

(i)  *$E$  admits a filtrable holomorphic structure if and only if*

$$c_1(E) \in \operatorname{NS}(X) \quad \text{and} \quad \Delta(E) \geq m(2, c_1(E)).$$

(ii)  *$E$  admits a holomorphic structure if and only if*

$$c_1(E) \in \operatorname{NS}(X) \quad \text{and} \quad \Delta(E) \geq \min(6, m(2, c_1(E))),$$

*In the excepted case,  $E$  admits no holomorphic structure.*

*Proof.* – The first statement is a particular case of the main theorem in [2], where also the non-existence of holomorphic structures in the excepted case is proved. It remains to show:

A. If  $\Delta(E) < \min(6, m(2, c_1(E)))$ , then  $E$  admits no holomorphic structure;

B. If  $c_1(E) \in \operatorname{NS}(X)$ , and  $6 \leq \Delta(E) < m(2, c_1(E))$ , then  $E$  does admit holomorphic structures.

To prove A, notice first that, if  $\Delta(E) < m(2, c_1(E))$ , then any holomorphic structure on  $E$  is non-filtrable (by (i)), hence simple. Let  $\mathcal{E}$  be such a holomorphic structure. Since  $K_X = \mathcal{O}_X$ , one gets  $H^0(\operatorname{End}_0(\mathcal{E})) = H^2(\operatorname{End}_0(\mathcal{E})) = 0$ , and, by Riemann–Roch, this yields  $h^1(\operatorname{End}_0(\mathcal{E})) < 0$  when  $\Delta(E) < 6$ .

To prove B, let  $\delta$  be a fixed holomorphic connection on  $\det(E)$  and fix a Hermitian metric  $h$  on  $E$ . Denote also by  $P$  the  $\operatorname{PU}(2)$ -bundle associated with the  $\operatorname{U}(2)$ -bundle  $(E, h)$ . Its Pontrjagin class is  $p_1(P) = -\Delta(E)$ . The Kobayashi–Hitchin correspondence [3,7] gives an isomorphism of real analytic spaces

$$\mathcal{M}_g^{\operatorname{ASD}}(P) \simeq \mathcal{M}_{g,\delta}^{\operatorname{pst}}(E),$$

for every Gauduchon metric  $g$  on  $X$ . Here  $\mathcal{M}_g^{\text{ASD}}(P)$  denotes the moduli space of  $g$ -ASD connections on  $P$  and  $\mathcal{M}_{g,\delta}^{\text{pst}}(E)$  denotes the moduli space of  $g$ -poly-stable holomorphic structures on  $E$  which induce  $\delta$  on  $\det(E)$ . In our case we have  $w_2(P) \neq 0$ . Indeed, if  $w_2(P)$  vanished, then  $c_1(E) \in 2H^2(X, \mathbb{Z}) \cap \text{NS}(X) = 2\text{NS}(X)$ , which would imply  $m(2, c_1(E)) = 0$ .

Since  $w_2(P) \neq 0$  and  $X$  is simply connected, the Donaldson polynomial invariant associated with  $P$  is well-defined [3].<sup>1</sup> On the other hand, by the same argument as above,  $h^0(\text{End}_0(\mathcal{E})) = h^2(\text{End}_0(\mathcal{E})) = 0$  for any bundle  $\mathcal{E}$  with  $\Delta(\mathcal{E}) < m(2, c_1(\mathcal{E}))$ , hence the moduli space  $\mathcal{M}_g^{\text{ASD}}(P)$  and all similar moduli spaces corresponding to the lower Uhlenbeck strata, are regular of expected dimension. Therefore, one can compute the Donaldson polynomial invariant corresponding to  $P$  using the Uhlenbeck compactification  $\overline{\mathcal{M}}_g^{\text{ASD}}(P)$ . But, by classical results in gauge theory [3,6,9] the Donaldson polynomial invariant  $q_P(X)$  associated to any  $\text{PU}(2)$  bundle  $P$  with  $-p_1(P) \geq 6$  is non-trivial. This shows that  $\mathcal{M}_g^{\text{ASD}}(P)$  (hence also  $\mathcal{M}_{g,\delta}^{\text{st}}(E)$ ) cannot be empty.  $\square$

*Remark.* – Donaldson theory can be used to solve the existence problem for  $\text{rk}(E) = 2$  and all Kählerian surfaces  $X$  with  $H^0(K_X) \xrightarrow{\otimes^2} H^0(K_X^{\otimes 2})$  surjective. This condition ensures that any non-filtrable holomorphic bundle defines a smooth point in the corresponding moduli space. For such a surface  $X$ , let us fix  $c \in \text{NS}(X)$  and denote by  $\bar{c} \in H^2(X, \mathbb{Z}_2)$  its reduction mod 2. One can show that

$$\begin{aligned} & \min\{\Delta(E) \mid c_1(E) = c, E \text{ admits holomorphic structures}\} \\ & = \min\{-p_1(P) \mid w_2(P) = \bar{c}, q_P(X) \neq 0\}, \end{aligned}$$

when the minimal value on the right is smaller than  $m(c_1, 2)$ .

This statement shows that, for this class of surfaces, the existence of holomorphic structures in the range  $\Delta(E) < m(c_1(E), 2)$  can be decided in terms of the *differential topological type* of the underlying differentiable 4-manifold. The proof is based on Donaldson’s non-vanishing result for Kählerian surfaces ([3], p. 378).

### 3. Blow up inequalities

We say that a complex surface  $X$  has the property  $\mathfrak{F}_r$  if every rank  $r$  vector bundle on  $X$  which admits a holomorphic structure also admits a filtrable one.

By [2] a complex surface  $X$  has the property  $\mathfrak{F}_r$  if and only if any torsion free sheaf  $\mathcal{F}$  on  $X$  has  $\Delta(\mathcal{F}) \geq m(r, c_1(\mathcal{F}))$ .

**PROPOSITION.** – *Let  $X$  be a compact complex surface,  $\pi : \tilde{X} \rightarrow X$  a proper modification and  $r \geq 2$  an integer. Then  $\tilde{X}$  has the property  $\mathfrak{F}_r$  if and only if  $X$  does.*

*Proof.* – Let  $X$  be non-algebraic, otherwise the statement is trivial. We may suppose that  $\pi$  consists of a single blow-up, and let  $D$  be the exceptional divisor. Suppose that  $\mathfrak{F}_r$  holds for  $X$ . By [2] any rank  $r$  holomorphic bundle  $\mathcal{F}$  on  $X$  has  $\Delta(\mathcal{F}) \geq m(r, c_1(\mathcal{F}))$ .

We will show that every holomorphic vector bundle  $\mathcal{E}$  of rank  $r$  over  $\tilde{X}$  satisfies  $\Delta(\mathcal{E}) \geq m(r, c_1(\mathcal{E}))$ , which will imply the property  $\mathfrak{F}_r$  for  $\tilde{X}$ , again by [2].

We have

$$c_1(\mathcal{E}) = \pi^*c_1(\pi_*(\mathcal{E})) + k[D], \tag{1}$$

where  $k := -[D] \cdot c_1(\mathcal{E}) \in \mathbb{Z}$ . By tensorizing  $\mathcal{E}$  by  $\mathcal{O}(lD)$ ,  $l \in \mathbb{Z}$ , if necessary, we may suppose that  $0 \leq k < r$ . The Riemann–Roch formula yields:

$$\chi(\mathcal{E}) = r \left[ \chi(\mathcal{O}_{\tilde{X}}) + \frac{c_1(\mathcal{E}) \cdot c_1(\tilde{X})}{2r} + \frac{c_1(\mathcal{E})^2 - \Delta(\mathcal{E})}{2r^2} \right].$$

Since  $\chi(\mathcal{E}) = \chi(\pi_*\mathcal{E}) - \chi(R^1\pi_*\mathcal{E})$  we have  $\chi(\mathcal{E}) \leq \chi(\pi_*\mathcal{E})$ , which, by (1), is equivalent to

$$\Delta(\mathcal{E}) \geq \Delta(\pi_*\mathcal{E}) + k(r - k). \tag{2}$$

On the other hand

$$\begin{aligned} m(r, c_1(\pi_*\mathcal{E})) &= -r \max \left\{ \sum_{i=1}^r \left( \frac{c_1(\pi_*\mathcal{E})}{r} - \mu_i \right)^2 \mid \mu_1, \dots, \mu_r \in \text{NS}(X), \sum_{i=1}^r \mu_i = c_1(\pi_*\mathcal{E}) \right\} \\ &= -r \sum_{i=1}^r \left( \frac{c_1(\pi_*\mathcal{E})}{r} - \mu_i^0 \right)^2, \end{aligned}$$

where the last equality holds for those  $\mu_i^0$  for which the maximum is attained.

But

$$\begin{aligned} m(r, c_1(\mathcal{E})) &= -r \max \left\{ \sum_{i=1}^r \left( \frac{c_1(\mathcal{E})}{r} - v_i \right)^2 \mid v_1, \dots, v_r \in \text{NS}(\tilde{X}) \text{ with } \sum_{i=1}^r v_i = c_1(\mathcal{E}) \right\} \\ &\leq -r \sum_{i=1}^k \left( \frac{\pi^*c_1(\pi_*\mathcal{E}) + k[D]}{r} - \pi^*\mu_i^0 - [D] \right)^2 - r \sum_{i=k+1}^r \left( \frac{\pi^*c_1(\pi_*\mathcal{E}) + k[D]}{r} - \pi^*\mu_i^0 \right)^2 \\ &= -r \sum_{i=1}^r \left( \frac{c_1(\pi_*\mathcal{E})}{r} - \mu_i^0 \right)^2 + k(r - k). \end{aligned}$$

Combining this inequality with (2) we obtain

$$\Delta(\mathcal{E}) - m(r, c_1(\mathcal{E})) \geq \Delta(\pi_*\mathcal{E}) - m(r, c_1(\pi_*\mathcal{E})).$$

Therefore, since  $\mathfrak{F}_r$  holds for  $X$ ,

$$\Delta(\mathcal{E}) \geq m(r, c_1(\mathcal{E})),$$

which proves one implication of the proposition.

The converse is easier; if for a locally free sheaf  $\mathcal{F}$  of rank  $r$  on  $X$  we had

$$\Delta(\mathcal{F}) < m(r, c_1(\mathcal{F})),$$

then

$$\Delta(\pi^*\mathcal{F}) = \Delta(\mathcal{F}) < m(r, c_1(\mathcal{F})) = m(r, c_1(\pi^*\mathcal{F}))$$

proving our claim.  $\square$

#### 4. The case of class VII surfaces

A compact complex surface is said to be in *class VII* if its first Betti number  $b_1$  equals one. The subclass of such surfaces with  $b_2 = 0$  is completely understood; cf. [5,11]. Further surfaces in class VII can be obtained by an explicit construction due to M. Kato. All known minimal surfaces in class VII with  $b_2 > 0$  contain one or two cycles of rational curves. On the other hand it was shown by Nakamura, [8], that any surface in class VII containing a cycle of rational curves is a deformation of a blown-up Hopf surface. Using this result we are able to prove

**THEOREM 2.** – *Let  $X$  be a class VII surface whose minimal model either has  $b_2 = 0$  or contains a cycle of rational curves, and let  $E$  be a rank  $r$  differentiable vector bundle over  $X$ . Then the following are equivalent*

1.  $E$  admits a holomorphic structure;
2.  $E$  admits a filtrable holomorphic structure;
3.  $c_1(E) \in \text{NS}(X)$  and  $\Delta(E) \geq m(r, c_1(E))$ .

*Proof.* – We prove only  $1. \Rightarrow 2.$  The rest follows from [2]. By the previous proposition, we may suppose that  $X$  is minimal. If  $b_2(X) = 0$ , the statement follows again from [2]. In particular it holds for Hopf surfaces, hence also for blown-up Hopf surfaces.

Suppose that  $X$  contains a cycle of rational curves. Let  $\mathcal{E}$  be a non-filtrable holomorphic structure on  $E$ . We may suppose that  $\mathcal{E}$  does not admit any non-trivial coherent subsheaf  $\mathcal{F}$  of lower rank (if not, we replace  $\mathcal{E}$  by  $\mathcal{F}^{\vee\vee}$  and we argue by induction).

Denote by  $\Theta_X$  the tangent sheaf of  $X$ . Consider the sheaf  $\Sigma$  arising as middle term of the Atiyah sequence:

$$0 \rightarrow \mathcal{E}nd(\mathcal{E}) \rightarrow \Sigma \rightarrow \Theta_X \rightarrow 0.$$

For the deformation theory of the pair  $(X, \mathcal{E})$ , the spaces  $H^0(X, \Sigma)$ ,  $H^1(X, \Sigma)$ ,  $H^2(X, \Sigma)$  play the roles of tangent space to the group of automorphisms, tangent space to the versal deformation germ and space of obstructions to deformation respectively (see [10]). Moreover, the natural map  $H^1(X, \Sigma) \rightarrow H^1(X, \Theta_X)$  is the tangent map at  $(X, \mathcal{E})$  of the natural morphism between the versal deformation spaces of  $(X, \mathcal{E})$  and of  $X$ .

Note that  $H^2(X, \mathcal{E}nd(\mathcal{E})) = H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes K_X)^* = 0$ . Indeed, if  $f \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes K_X)$ , then  $\det(f) \in H^0(X, K_X^{\otimes 2})$  vanishes because  $\text{kod}(X) = -\infty$ , hence  $\ker f$  would contradict our assumption on  $\mathcal{E}$ .

We also have  $H^2(X, \Theta_X) = 0$  (see [8]), hence  $H^2(X, \Sigma) = 0$ . Therefore the versal deformation space is smooth at  $(X, \mathcal{E})$  and the germ of the map to the versal deformation of  $X$  is submersive at  $(X, \mathcal{E})$ . In particular, over any deformation  $X_s$  of  $X$  sufficiently close to  $X$ , there exists a holomorphic vector bundle  $\mathcal{E}_s$  which is a deformation of  $\mathcal{E}$ . Choosing  $X_s$  to be a blown-up Hopf surface and remembering that  $\text{NS}(X) = H^2(X, \mathbb{Z}) = H^2(X_s, \mathbb{Z}) = \text{NS}(X_s)$  we see that  $E_s$  and hence  $E$  satisfies the third condition of our theorem, hence also the second by [2].  $\square$

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<sup>1</sup> The Donaldson polynomial invariants associated with PU(2)-bundles with  $w_2 = 0$  on simply connected manifolds are defined only for sufficiently large instanton number (the stable range). A more refined theory [4], which uses the thickened ASD moduli space leads to well defined invariants for any value of  $c_2$ , but this theory is not useful for our purposes.

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