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Vibrations of elastic systems with a large number of tiny heavy inclusions

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Abstract We consider a spectral problem modeling natural vibrations of a complex medium that consists of an elastic medium and tiny rigid inclusions. We study the asymptotic behaviour of solutions of this problem when the total number of inclusions and their density tend to infinity. We obtain a limit problem being a spectral problem for a linear fractional operator pencil that describes the macroscopic behaviour of the system (global vibrations). ***To cite this article: V. Rybalko, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 245–250.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Vibrations de systèmes élastiques avec un grand nombre de petites inclusions à forte densité

Résumé On considère un problème spectral qui modélise les vibrations propres d'un milieu complexe constitué d'un milieu élastique et d'un grand nombre de petites inclusions rigides à forte densité. On étudie le comportement asymptotique de ce problème lorsque le nombre d'inclusions et leur densité tendent vers l'infini. On obtient un problème spectral limite pour une famille rationnelle fractionnaire d'opérateurs qui décrit le comportement macroscopique du système (vibrations globales). ***Pour citer cet article : V. Rybalko, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 245–250.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note, nous étudions les vibrations propres d'un système élastique comportant un grand nombre de masses concentrées constitué par des inclusions rigides.

On considère un ouvert borné G de $\mathbb{R}^3$, occupé par un milieu composite constitué d'un milieu élastique (la « matrice ») et d'un grand nombre n de petites inclusions rigides. Nous supposons que les inclusions sont des solides tous identiques occupant des ouverts $F^{jn} \subset G$ à frontières lisses $S^{jn} = \partial F^{jn}$. Le domaine $G^n = G \setminus \bigcup_j F^{jn}$ constitue la « matrice ». Les vibrations propres d'un tel milieu composite sont décrites par le problème spectral suivant : trouver $\omega^n \in \mathbb{R}$ et $\mathbf{u}^n \not\equiv 0$ tels que les conditions (2)–(6) soient vérifiées. Pour n fixé, ce problème a un spectre discret formé de valeurs propres $0 < \omega_1^n \leqslant \omega_2^n \leqslant \cdots \leqslant \omega_r^n \leqslant \cdots$. On

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étudie le comportement asymptotique de ces valeurs propres ω_r^n et des vecteurs propres $\mathbf{u}_r^n$ correspondants lorsque le nombre d'inclusions n et leur densité $\hat{\rho}^n$ tendent vers l'infini.

Nous définissons au moyen de (8)–(10) des nombres $\hat{\theta}_k^n$ et des distributions matricielles $\mathbf{B}^{kn}$ qui décrivent les caractéristiques locales du milieu. Nous supposons la géométrie du problème telle que la condition (I) soit satisfaite, i.e. les diamètres des inclusions sont au plus de l'ordre du cube des distances mutuelles. Nous supposons de plus que les caractéristiques locales $\hat{\theta}_k^n$ et $\mathbf{B}^{kn}$ ont des limites lorsque $n \to \infty$ (conditions (II) et (III)). Les termes principaux ω et $\mathbf{u}$ des développements asymptotiques de ω_r^n et $\mathbf{u}_r^n$ sont alors décrits par le problème suivant : trouver ω et $\mathbf{u} \not\equiv 0$ tels que

$$\begin{cases} -\mu\Delta\mathbf{u} - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \mathbf{\Phi}(x,\omega)\mathbf{u} - \omega\rho\mathbf{u} = 0 \quad \text{dans } G, \\ \mathbf{u} = 0 \quad \text{sur } \partial G, \end{cases} \tag{1}$$

où $\mathbf{\Phi}(x,\omega)$ est défini par (13), ce qui se traduit, en termes d'opérateurs, par $\mathbf{S}(\omega)\mathbf{u} = 0$, où $\mathbf{S}(\omega)$ est un faisceau d'opérateurs rationnel fractionnaire. Nous montrons, en particulier, que si la suite $(\omega_{r_n}^n, \mathbf{u}_{r_n}^n)$ converge dans $\mathbb{R} \times [\mathrm{L}^2(G)]^3$ vers $(\omega, \mathbf{u})$, alors :

(a) si ω n'est pas un pôle du faisceau $\mathbf{S}(\omega)$, $\mathbf{u}$ est solution de (1) ;

(b) si ω est un pôle du faisceau $\mathbf{S}(\omega)$, $\mathbf{u}$ appartient à l'image de l'opérateur $\mathbf{\Pi}_\omega = \lim_{\tau\downarrow 0} \tau \mathbf{S}^{-1}(\omega + i\tau)$, qui est un opérateur de rang fini.

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1. Introduction

Systems with *concentrated masses* (CM), i.e., those containing small regions where the density is much higher than in the rest of the structure, have a significant interest in the framework of vibration theory. For instance, in [6–8] the vibrations of the media with a single concentrated mass are studied. In [1–3] the case of a large number of CM near the boundary is considered. This Note is devoted to the investigation of a model involving a large number of spatially distributed CM formed by rigid inclusions. The precise setting of the problem is as follows.

Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂G. We assume that G is occupied by a composite medium consisting of a homogeneous isotropic elastic medium ("matrix") with n homogeneous rigid inclusions. We assume that the inclusions are identical solids occupying nonintersecting simply connected subdomains $F^{jn} \subset G$ with smooth boundaries $S^{jn} = \partial F^{jn}$. We assume also that the total number n of inclusions and their density $\hat{\rho}^n$ are sufficiently large, so that the elastic medium and the set of inclusions form a strongly inhomogeneous composite medium with piecewise constant density $\rho^n(x)$, $\rho^n(x) = \hat{\rho}^n$ in $\bigcup_j F^{jn}$ and $\rho^n(x) = \rho$ in $G^n = G \setminus \bigcup_j F^{jn}$, where ρ is the density of the "matrix".

We consider small free harmonic vibrations of this composite medium. In the framework of linearized elasticity they are described by the following spectral problem: find $\omega^n \in \mathbb{R}$ and $\mathbf{u}^n(x) = (u_1^n(x), u_2^n(x), u_3^n(x)) \not\equiv 0$ such that

(i) $\mathbf{u}^n(x)$ satisfies the system of equations ($l = 1, 2, 3$)

$$-\sum_{m=1}^{3} \frac{\partial}{\partial x_m} \left\{ 2\mu\varepsilon_{lm}\left(\mathbf{u}^n\right) + \lambda\delta_{lm} \sum_{r=1}^{3} \varepsilon_{rr}\left(\mathbf{u}^n\right) \right\} = \omega^n \rho u_l^n \quad \text{in } G^n, \tag{2}$$

where

$$\varepsilon_{lm}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l} \right)$$

is the strain tensor, λ, μ are the Lamé constants of the "matrix", and δ_{lm} is the Kronecker symbol;

(ii) $\mathbf{u}^n$ belongs to the class $\mathcal{K}^n$ of rigid displacements on each F^{jn}, i.e.,

$$\varepsilon_{lm}\left(\mathbf{u}^n\right) = 0 \quad \text{in } \bigcup_j F^{jn} \ (l, m = 1, 2, 3); \tag{3}$$

(iii) $\mathbf{u}^n$ is continuous across the contact surfaces S^{jn},

$$\left(\mathbf{u}^n\right)^+ = \left(\mathbf{u}^n\right)^- \quad \text{on } S^{jn}\ (j = 1, \ldots, n); \tag{4}$$

(iv) for any $j = 1, \ldots, n$,

$$\begin{aligned} -\int_{S^{jn}} \mathbf{T_n}\left(\mathbf{u}^n\right) \mathrm{d}\sigma &= \omega^n \int_{F^{jn}} \mathbf{u}^n\, \hat{\rho}^n\, \mathrm{d}x, \\ -\int_{S^{jn}} \mathbf{x} \times \mathbf{T_n}\left(\mathbf{u}^n\right) \mathrm{d}\sigma &= \omega^n \int_{F^{jn}} \mathbf{x} \times \mathbf{u}^n\, \hat{\rho}^n\, \mathrm{d}x, \end{aligned} \tag{5}$$

where

$$\mathbf{T_n}(\mathbf{u}) = 2\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda\, \mathbf{n} \operatorname{div} \mathbf{u} + \mu\, \mathbf{n} \times \operatorname{rot} \mathbf{u}$$

is the stress vector evaluated on the exterior side $(S^{jn})^+$ of S^{jn}, and $\mathbf{n}$ is the unit outward normal vector to S^{jn};

(v) $\mathbf{u}^n$ satisfies the Dirichlet boundary condition

$$\mathbf{u}^n = 0 \quad \text{on } \partial G. \tag{6}$$

This last condition means that the medium is fixed on ∂G.

As will be shown below, the spectrum Θ^n of problem (2)–(6) consists of a discrete set of positive real eigenvalues ω_r^n. Our main goal is to study the asymptotic behaviour of the eigenvalues ω_r^n and their corresponding eigenvectors $\mathbf{u}_r^n$ when the total number of inclusions n and their density $\hat{\rho}^n$ tend to infinity while inclusions are not too close to one another and form a fine-grained structure in G.

2. Variational reformulation of the problem. Discreteness of the spectrum

Let us give the notion of variational solution of problem (2)–(6). To this end, we introduce the following spaces: $V = \mathrm{L}^2(G; \mathbb{R}^3)$ and $V^n = V \cap \mathcal{K}^n$ are Hilbert spaces with the scalar products

$$(\mathbf{u}, \mathbf{v})_V = \int_G \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}x, \qquad (\mathbf{u}, \mathbf{v})_{V^n} = \int_G \mathbf{u} \cdot \mathbf{v}\, \rho^n(x)\, \mathrm{d}x; \tag{7}$$

$W = \{\mathbf{u} \in V : \partial\mathbf{u}/\partial x_l \in V\ (l = 1, 2, 3),\ \mathbf{u}|_{\partial G} = 0\}$ is a Sobolev space endowed with a standard scalar product, and $W^n = W \cap \mathcal{K}^n$.

Let $\vartheta[\mathbf{u}, \mathbf{v}]$ be the bilinear form on W defined by

$$\vartheta[\mathbf{u}, \mathbf{v}] = \int_G E(\mathbf{u}, \mathbf{v})\, \mathrm{d}x,$$

where

$$E(\mathbf{u}, \mathbf{v}) = 2\mu \sum_{l,m=1}^{3} \varepsilon_{lm}(\mathbf{u})\varepsilon_{lm}(\mathbf{v}) + \lambda \sum_{l,m=1}^{3} \varepsilon_{ll}(\mathbf{u})\varepsilon_{mm}(\mathbf{v}).$$

This is clearly a bounded Hermitian form and the associated quadratic form $\vartheta[\mathbf{u}, \mathbf{u}]$ is coercive.

DEFINITION 1. – A function $\mathbf{u}^n$ is said to be a *variational solution* of problem (2)–(6) if $\mathbf{u}^n \in W^n$ and $\vartheta[\mathbf{u}^n, \mathbf{v}] - \omega^n (\mathbf{u}^n, \mathbf{v})_{V^n} = 0$ holds for any test function $\mathbf{v} \in W^n$.

Standard regularity arguments (*see*, e.g., [4]) ensure that the notions of variational and classical solutions of problem (2)–(6) are equivalent. At the same time, it follows from the coerciveness of the form $\vartheta[\mathbf{u}, \mathbf{u}]$ and compactness of the embedding of W^n in V^n that eigenvalues of problem (2)–(6), enumerated in non-decreasing order and repeated according to their finite multiplicities, define a sequence $0 < \omega_1^n \leqslant \omega_2^n \leqslant \cdots \leqslant \omega_r^n \leqslant \cdots$ such that $\omega_r^n \to \infty$ as $r \to \infty$. Corresponding eigenvectors $\mathbf{u}_r^n$ can be chosen to form an orthonormal basis of V^n.

3. Statement of main results

In order to state the main results we introduce local characteristics of the composite medium. Namely, we denote by $\hat{a}^n$ and $\hat{x}^{jn}$ the radius and the center of the minimal ball containing F^{jn}. Then we set $\hat{d}^n = \min_{1\leqslant j\leqslant n} d^{jn}$, where d^{jn} is the distance from $\hat{x}^{jn}$ to $\{\hat{x}^{j'n} : j' \neq j\} \cup \partial G$. Let us also introduce the space $\mathcal{R}^{jn}$ of functions $\mathbf{u}$ such that $\mathbf{u} = \mathbf{U}_1 + \mathbf{U}_2 \times \mathbf{x}$ in F^{jn}, with $\mathbf{U}_1,\ \mathbf{U}_2 = \text{const}$, and $\mathbf{u} = \mathbf{w}$ in $\mathbb{R}^3 \setminus F^{jn}$, where $\mathbf{w}$ is the unique solution of the boundary value problem

$$\begin{cases} \mu\Delta\mathbf{w} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{w} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{F^{jn}}, \\ \mathbf{w} = \mathbf{U}_1 + \mathbf{U}_2 \times \mathbf{x} \quad \text{on } S^{jn}, \\ |\mathbf{w}| = \mathrm{O}\big(|x|^{-1}\big), \quad E(\mathbf{w},\mathbf{w}) = \mathrm{O}\big(|x|^{-4}\big), \quad |x| \to \infty. \end{cases}$$

Endowing $\mathcal{R}^{jn}$ with the scalar product $(\mathbf{u},\mathbf{v})_{\mathcal{R}^{jn}} = \int_{F^{jn}} \mathbf{u}\cdot\mathbf{v}\ \hat{\rho}^n\,\mathrm{d}x$, we obtain a 6-dimensional Hilbert space. Moreover, one can choose an orthonormal basis $\widehat{\mathbf{w}}_1^{jn}, \dots, \widehat{\mathbf{w}}_6^{jn}$ of $\mathcal{R}^{jn}$ in such a way that

$$\begin{aligned} &\int_{\mathbb{R}^3} E\big(\widehat{\mathbf{w}}_k^{jn}, \widehat{\mathbf{w}}_{k'}^{jn}\big)\,\mathrm{d}x = \hat{\theta}_k^n\big(\widehat{\mathbf{w}}_k^{jn}, \hat{\mathbf{w}}_{k'}^{jn}\big)_{\mathcal{R}^{jn}} = \hat{\theta}_k^n\delta_{kk'} \\ &\text{with } 0 < \hat{\theta}_1^n \leqslant \hat{\theta}_2^n \leqslant \cdots \leqslant \hat{\theta}_6^n, \end{aligned} \tag{8}$$

the $\hat{\theta}_k^n$ being independent of j. Bearing this in mind we associate to the j-th inclusion 6 matrices $\mathbf{B}^{1jn}, \dots, \mathbf{B}^{6jn}$ whose entries are

$$B_{lm}^{kjn} = \frac{\hat{\theta}_k^n}{\hat{\theta}_k^n + 1}\int_{F^{jn}} \widehat{\mathbf{w}}_k^{jn}\cdot\mathbf{e}^l\,\hat{\rho}^n\,\mathrm{d}x \int_{F^{jn}} \widehat{\mathbf{w}}_k^{jn}\cdot\mathbf{e}^m\,\hat{\rho}^n\,\mathrm{d}x, \tag{9}$$

where $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ is the canonical basis of $\mathbb{R}^3$. Then we define the matrix-valued distributions

$$\mathbf{B}^{kn}(x) = \sum_{j=1}^{n} \mathbf{B}^{kjn}\delta\big(x - \hat{x}^{jn}\big). \tag{10}$$

Notice that the entries of $\mathbf{B}^{kjn}$ defined by (9) satisfy

$$\big|B_{lm}^{kjn}\big| \leqslant C\hat{a}^n\frac{\hat{\theta}_k^n + \tau}{\tau(\hat{\theta}_k^n + 1)}, \tag{11}$$

where C is independent of j, n and $\tau > 0$.

We impose the following conditions on the local characteristics of the medium. Let n take values in an increasing sequence $\mathcal{N}$ of positive integers and let

(I) $\hat{a}^n \leqslant C(\hat{d}^n)^3$ with a constant C independent of $n \in \mathcal{N}$.

Then it follows from (11) that the sequences $\{\mathbf{B}^{kn}\}$ are relatively compact sets in the weak topology of distributions on G. We assume that there exist weak limits

(II) $\text{w-}\lim_{n\to\infty} \mathbf{B}^{kn}(x) = \mathbf{B}^k(x)$ $(k = 1, \dots, 6)$.

According to (11) and condition (I) the limits $\mathbf{B}^k(x)$ are essentially bounded in G. We also assume that $\hat{\theta}_k^n$ have limits (finite or infinite) as $n \to \infty$:

(III) $\lim_{n\to\infty} \hat{\theta}_k^n = \theta_k$ if $k \leqslant K$, for some $0 \leqslant K \leqslant 6$, and $\lim_{n\to\infty} \hat{\theta}_k^n = \infty$ if $k > K$.

Notice that in view of (11) and condition (I), we have $\mathbf{B}^k \equiv 0$ for $k > K$.

Before stating the main results let us remark first that we describe the asymptotic behaviour of the eigenvalues ω_r^n by means of the counting function $N^n(\alpha,\beta) = \#\{\omega_r^n \in (\alpha,\beta)\}$. Furthermore, it turns out that although the original spectral problem (2)–(6) has a discrete spectrum for all n, the eigenvalues may asymptotically accumulate at some finite points as $n \to \infty$. It is important then to describe all possible limits of normalized linear combinations of the corresponding eigenvectors. Namely, we introduce the following definition.

DEFINITION 2. – A sequence $\{\mathbf{u}^n\}$ is called a *sequence of normalized quasi-eigenvectors* (or Q-*sequence*) corresponding to the *quasi-eigenvalue* ω, if $\|\mathbf{u}^n\|_{V^n} = 1$ and for any $\delta > 0$ there exists s such that $\mathbf{u}^n \in \mathcal{E}^n_{(\omega-\delta,\omega+\delta)} V^n$ whenever $n > s$. Here $\mathcal{E}^n_{(\alpha,\beta)}$ denotes the projection operator defined by

$$\mathcal{E}^n_{(\alpha,\beta)}\mathbf{u} = \sum_{r:\omega^n_r \in (\alpha,\beta)} \mathbf{u}^n_r \left(\mathbf{u}, \mathbf{u}^n_r\right)_{V^n}.$$

Now we can state the main results. Consider in the space V the following pencil of operators depending on the spectral parameter $\omega \notin \{\theta_k : \mathbf{B}^k \not\equiv 0\}$:

$$\mathbf{S}(\omega) = \mathbf{A} + \mathbf{\Phi}(\omega) - \omega\rho\mathbf{I}, \tag{12}$$

where $\mathbf{I}$ is the identity operator, $\mathbf{A}$ is a selfadjoint extension of the operator $-\mu\Delta - (\lambda + \mu)\,\mathrm{grad\,div}$ in G with the Dirichlet condition on ∂G, and $\mathbf{\Phi}(\omega)$ is the operator of multiplication by the matrix-valued function

$$\mathbf{\Phi}(x,\omega) = \sum_{k=1}^{K} \frac{\omega(\theta_k + 1)}{\omega - \theta_k}\mathbf{B}^k(x). \tag{13}$$

We say that ω belongs to the spectrum $\Theta(\mathbf{S})$ of the pencil (12) if $\mathbf{S}(\omega)$ has no bounded inverse. Correspondingly, ω is an eigenvalue of multiplicity ν of the pencil (12) if zero is an eigenvalue of multiplicity ν of the operator $\mathbf{S}(\omega)$.

THEOREM. – *Let* $\hat{\rho}^n \to \infty$ *as* $n \to \infty$ *and let conditions* (I)–(III) *be fulfilled. Then the following statements hold true.*

(A) *The spectrum* $\Theta(\mathbf{S})$ *of the family* (12) *is a countable set of positive eigenvalues which may accumulate only at points* $\theta_1, \ldots, \theta_K$ *and at infinity. For any* $\omega \in M = \Theta(\mathbf{S}) \cup \{\theta_1, \ldots, \theta_K\}$, *there exists a strong operator limit* $\mathbf{\Pi}_\omega = \lim_{\tau\downarrow 0} \tau\mathbf{S}^{-1}(\omega + i\tau)$ *which is a finite rank operator. If* $\omega \in M \setminus \{\theta_k : \mathbf{B}^k \not\equiv 0\}$ *then the range of* $\mathbf{\Pi}_\omega$ *coincides with the linear hull spanned by the solutions* $\mathbf{g}$ *of the equation* $\mathbf{S}(\omega)\mathbf{g} = 0$.

(B) *For any* $\alpha, \beta \in \mathbb{R} \setminus M$, $\alpha < \beta$, $N^n(\alpha,\beta)$ *converges to*

$$N^\infty(\alpha,\beta) = \begin{cases} \sum_{\omega\in\Theta(\mathbf{S})} \chi_{(\alpha,\beta)}(\omega)\nu(\omega), & (\alpha,\beta) \cap \{\theta_1, \ldots, \theta_K\} = \emptyset, \\ \infty, & (\alpha,\beta) \cap \{\theta_1, \ldots, \theta_K\} \neq \emptyset, \end{cases}$$

where $\chi_{(\alpha,\beta)}(\omega)$ *is the indicator of the interval* (α,β) *and* $\nu(\omega)$ *is the multiplicity of the eigenvalue* ω *of the pencil of operators* (12).

(C1) *If* $\{\mathbf{u}^n,\ n \in \mathcal{N}' \subset \mathcal{N}\}$ *is a* Q-*sequence corresponding to a quasi-eigenvalue* ω, *then* $\omega \in M$ *and there exists a subsequence converging strongly in* V *to a vector* $\mathbf{u}$ *that belongs to the range of the operator* $\mathbf{\Pi}_\omega$.

(C2) *Conversely, for any* $\omega \in M$ *and any* $\mathbf{u} \not\equiv 0$ *from the range of* $\mathbf{\Pi}_\omega$, *there exist a* Q-*sequence* $\{\mathbf{u}^n,\ n \in \mathcal{N}\}$ *corresponding to the quasi-eigenvalue* ω *and a sequence* $\{\gamma^n\}$ *of positive numbers bounded away from zero, such that* $\|\mathbf{u}^n - \gamma^n\mathbf{u}\|_V \to 0$ *as* $n \to \infty$.

Remark. – It is easy to derive from (11) that $\mathbf{\Phi}$ in (12) is nonzero only if $\hat{a}^n \sim (\hat{d}^n)^3$ (critical size of inclusions). In this (most interesting) case, the pencil (12) has in general nontrivial residues at points θ_k. It is worth to be noticed also that although $\mathbf{\Phi} \equiv 0$ in the non-critical case ($\hat{a}^n \ll (\hat{d}^n)^3$), nevertheless each θ_k ($k = 1, \ldots, K$) is still an accumulation point of the eigenvalues ω^n_r as $n \to \infty$.

Example. – Let Ψ be a diffeomorphism mapping G onto a domain G', and let $J = J(x)$ be the Jacobian of Ψ. Suppose that all inclusions are identical balls with radius $\hat{a}^n = Rs^3$ and centers at $x^{jn} \in \mathcal{Q}^s = \{x : \Psi(x) \in s\mathbb{Z}^3 \cap G',\ \mathrm{dist}(\Psi(x), \partial G') \geqslant s\}$, where $n(= n(s)) = \mathrm{card}\,\mathcal{Q}^s$, and s takes values in a sequence $s_i \downarrow 0$ such that $n(s_{i+1}) > n(s_i)$. Then the local characteristics $\hat{\theta}^n_k$ and B^{kjn}_{lm} are calculated in an explicit form. Namely, setting $\alpha_1 = \frac{9\mu(\lambda+2\mu)}{5\mu+2\lambda}$, $\alpha_2 = 15\mu$ and assuming for definiteness that $\alpha_1 \leqslant \alpha_2$, we have

$$\hat{\theta}_k^n = \begin{cases} \alpha_1/((\hat{a}^n)^2\hat{\rho}^n), & k=1,2,3, \\ \alpha_2/((\hat{a}^n)^2\hat{\rho}^n), & k=4,5,6, \end{cases} \qquad B_{lm}^{kjn} = \begin{cases} \frac{4\pi\alpha_1(\hat{a}^n)^3\hat{\rho}^n}{3(\alpha_1+(\hat{a}^n)^2\hat{\rho}^n)}\delta_{kl}\delta_{km}, & k=1,2,3, \\ 0, & k=4,5,6. \end{cases}$$

Let us denote by $\mathcal{M}^n = \frac{4}{3}\pi n\hat{\rho}^n(\hat{a}^n)^3$ the total mass of inclusions and consider the following three cases: (H_1) $\mathcal{M}^n \to 0$, (H_2) $\mathcal{M}^n \to \gamma > 0$, ($H_3$) $\mathcal{M}^n \to \infty$, as $n \to \infty$. In all these cases the conditions of the theorem are fulfilled. We have, under hypothesis (H_1) $K = 0$, $\mathbf{\Phi} \equiv 0$; under hypothesis (H_2) $K = 6$, $\theta_k = \frac{4\pi}{3\gamma}\alpha_i R \operatorname{meas} G'$ ($i = 1$ if $k \leqslant 3$ and $i = 2$ otherwise),

$$\mathbf{\Phi}(x,\omega) = \frac{4\pi\alpha_1 R\,\omega\gamma}{3\omega\gamma - 4\pi\alpha_1 R \operatorname{meas} G'}\big|J(x)\big|\mathbf{I};$$

under hypothesis (H_3) $K = 6$, $\theta_k = 0$ ($1 \leqslant k \leqslant 6$), $\mathbf{\Phi}(x,\omega) = \frac{4\pi}{3}\alpha_1 R|J(x)|\mathbf{I}$.

Proof of the theorem (*sketch*). – The proof is carried out in two main steps. We first rewrite the spectral problem (2)–(6) in the operator form: $(\mathbf{A}^n - \omega^n\mathbf{I})\,\mathbf{u}^n = 0$, where $\mathbf{A}^n$ is a selfadjoint operator in V^n whose spectrum is $\Theta^n = \bigcup_r\{\omega_r^n\}$. By means of the Energy Method (*see*, e.g., [5]) we establish the limiting localization of the spectra ($\lim_{n\to\infty}\Theta^n \subset M$) and prove that, for any $\omega \notin M$ and $\mathbf{v} \in V$,

$$\mathbf{Q}^n\left(\mathbf{A}^n - \omega\mathbf{I}\right)^{-1}\mathbf{Q}^n\mathbf{v} \to \rho\mathbf{S}^{-1}(\omega)\mathbf{v} \quad \text{strongly in } V \text{ as } n \to \infty,$$

where $\mathbf{Q}^n$ is the projection operator given by $\mathbf{Q}^n\mathbf{u} = \mathbf{u}$ in G^n and $\mathbf{Q}^n\mathbf{u} = 0$ otherwise. Then, the analyticity of $\mathbf{Q}^n(\mathbf{A}^n - \omega\mathbf{I})^{-1}\mathbf{Q}^n$ and $\rho\mathbf{S}^{-1}(\omega)$ in ω allows us to translate the above result on the convergence of singularities of $\mathbf{Q}^n(\mathbf{A}^n - \omega\mathbf{I})^{-1}\mathbf{Q}^n$ to those of $\rho\mathbf{S}^{-1}(\omega)$. This provides an information on the asymptotic behaviour of the spectral family of $\mathbf{A}^n$ needed to complete the proof. □

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