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Weak hyperbolicity on periodic orbits for polynomials

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Abstract We prove that if the multipliers of the repelling periodic orbits of a complex polynomial grow at least like $n^{5+\varepsilon}$ with the period, for some $\varepsilon > 0$, then the Julia set of the polynomial is locally connected when it is connected. As a consequence for a polynomial the presence of a Cremer cycle implies the presence of a sequence of repelling periodic orbits with "small" multipliers. Somewhat surprisingly the proof is based on measure theorical considerations. ***To cite this article: J. Rivera-Letelier, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1113–1118.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Hyperbolicité faible sur les orbites périodiques pour les polynômes

Résumé On démontre que si les multiplicateurs des orbites périodiques répulsives d'un polynôme complexe croissent au moins comme $n^{5+\varepsilon}$ avec la période, où $\varepsilon > 0$, alors l'ensemble de Julia du polynôme est localement connexe quand il est connexe. Comme conséquence on obtient que pour un polynôme complexe l'existence d'un cycle de Cremer implique l'existence d'une suite de cycles répulsifs ayant des multiplicateurs « petits ». D'une façon un peu surprenante la démonstration utilise des arguments de la théorie de la mesure. ***Pour citer cet article : J. Rivera-Letelier, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1113–1118.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Considérons un polynôme P à coefficients complexes. Etant donné un point périodique $p \in \mathbb{C}$ de période minimale $n \geqslant 1$ on appelle $\lambda = (P^n)'(p)$ le *multiplicateur* du point périodique p. Alors on dit que p est *répulsif*, *indifférent* ou *attractif* selon que $|\lambda| > 1$, $|\lambda| = 1$ ou $|\lambda| < 1$, respectivement.

L'ensemble

$$K(P) = \left\{ z \in \mathbb{C} \mid \left\{ P^n(z) \right\}_{n \geqslant 1} \text{ est bornée} \right\}$$

est appelé *ensemble de Julia rempli* de P et sa frontière est l'*ensemble de Julia* de P, on le note $J(P)$.

Le théorème suivant est notre résultat principal.

THÉORÈME 1. – *Soit* $P \in \mathbb{C}[z]$ *un polynôme dont l'ensemble de Julia est connexe. Supposons qu'il existe des constantes* $C > 0$ *et* $\varepsilon > 0$ *telles que pour tout point périodique répulsif* $p \in \mathbb{C}$ *de période* $n \geqslant 1$ *on ait,*

$$\left| \left(P^n\right)'(p) \right| \geqslant C n^{5+\varepsilon}.$$

Alors l'ensemble de Julia de P *est localement connexe.*

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La classe de polynômes (et fonctions rationnelles) dont les multiplicateurs des points périodiques répulsifs croissent de façon exponentielle par rapport à la période a été étudiée dans [14] ; la démonstration du Théorème 1 est basée sur une variante du Lemme 3.1 de ce papier.

Rappelons qu'un *cycle de Cremer* est un cycle indifférent dont le multiplicateur de module égal à 1, n'est pas une racine de l'unité et tel que le polynôme n'est pas linéarisable en son voisinage.

D'après un théorème de A. Douady and D. Sullivan l'ensemble de Julia d'un polynôme ayant un cycle de Cremer n'est pas localement connexe, voir [18]. Donc le corollaire suivant est une conséquence immédiate du Théorème 1.

COROLLAIRE 1. – *Soit $P \in \mathbb{C}[z]$ un polynôme ayant un cycle de Cremer. Supposons que l'ensemble de Julia de P est connexe. Alors pour tout $\varepsilon > 0$ il existe un point périodique répulsif p, de période n arbitrairement grande, tel que*

$$\left|\left(P^n\right)'(p)\right| \leqslant n^{5+\varepsilon}.$$

Ce corollaire reste vrai si l'ensemble de Julia n'est pas connexe (*voir* Remarque 2) et il y a une assertion similaire pour des fonctions rationnelles en général, voir Remarque 3. Malheureusement ce résultat ne donne pas d'information sur la position de ces orbites périodiques. Sous certaines conditions un cycle de Cremer implique l'existence de « petits cycles » ; voir [19] et [11].

Remarque 1. – Comme conséquence de [2] (Corollary 1.1) l'hypothèse du Théorème 1 est satisfaite par les polynômes tels que pour toute valeur critique $v \in J(P)$ la suite $\{|(P^n)'(v)|\}_{n\geqslant 1}$ croît au moins comme n^α, où $\alpha > 1$ dépend seulement du degré de P. Ces polynômes satisfont une condition de sommabilité, voir [10,13,5,16] et [17].

Remarque 2. – Pour montrer que le Corollaire 1 reste vrai pour les polynômes dont l'ensemble de Julia n'est pas connexe, on remarque d'abord que la démonstration du Théorème 1 s'applique aux applications à allure polynômiale au sens de [3]. Alors si $P \in \mathbb{C}[z]$ est un polynôme ayant un point périodique de Cremer $p \in \mathbb{C}$ de période $n \geqslant 1$, la restriction de P^n à un voisinage convenable de p est une application à allure polynômiale dont l'ensemble de Julia est connexe.

Remarque 3. – Avec une méthode similaire on obtient que si $R \in \mathbb{C}(z)$ est une fonction rationnelle ayant un cycle de Cremer, alors il existe une suite de points périodiques $\{p_k\}$ de période n_k dont le multiplicateur est majoré par $\exp(C\sqrt{n_k}(\ln n_k)^{3/2+2/h})$, où $C > 0$ est une constante universelle et $0 < h < HD_{\mathrm{hyp}}(R)$, où $HD_{\mathrm{hyp}}(R)$ est le supremum de la dimension de Hausdorff des ensembles hyperboliques de R.

Consider a polynomial P with complex coefficients. Given a periodic point $p \in \mathbb{C}$ of P of minimal period $n \geqslant 1$, we call $\lambda = (P^n)'(p)$ the *multiplier* of p. We say that p is *repelling*, *indifferent* or *attracting* if $|\lambda| > 1$, $|\lambda| = 1$ or $|\lambda| < 1$, respectively.

Recall that the set,

$$K(P) = \left\{z \in \mathbb{C} \mid \left\{P^n(z)\right\}_{n\geqslant 1} \text{ is bounded}\right\},$$

is called the *filled-in Julia set* of P and its boundary is called the *Julia set* of P, which is denoted by $J(P)$.

Our main result is the following.

THEOREM 1. – *Let $P \in \mathbb{C}[z]$ be a polynomial with connected Julia set. Suppose that there are constants $C > 0$ and $\varepsilon > 0$ such that for every repelling periodic point $p \in \mathbb{C}$ of period n,*

$$\left|\left(P^n\right)'(p)\right| \geqslant C n^{5+\varepsilon}.$$

Then the Julia set of P is locally connected.

The class of polynomial (and rational) maps for which the multipliers grow exponentially fast with the period is studied in [14] and the proof of Theorem 1 is based on a variant of Lemma 3.1 of that paper.

Recall that a *Cremer cycle* is a cycle for which the polynomial is not locally linearizable and whose multiplier of modulus equal to 1, is not a root of unity. By a theorem of A. Douady and D. Sullivan the Julia set of a polynomial with a Cremer cycle is not locally connected, see [18]. So the following corollary follows directly from Theorem 1.

COROLLARY 1. – *Let $P \in \mathbb{C}[z]$ be a polynomial with connected Julia set and with a Cremer cycle. Then for every $\varepsilon > 0$ there is a periodic point p, with period n arbitrarily big, satisfying*

$$\left|\left(P^n\right)'(p)\right| \leqslant n^{5+\varepsilon}.$$

The corollary remains true if the Julia set is disconnected (see Remark 2) and there is an analogous statement for general rational maps, see Remark 3. Unfortunately this result does not give information about the location of these periodic points. Under certain conditions Cremer cycles imply the existence of the so called "small" cycles; see [19] and [11].

Remark 1. – It follows by [2] (Corollary 1.1) that the hypothesis of Theorem 1 are satisfied for polynomials such that for every critical value $v \in J(P)$ the derivatives $|(P^n)'(v)|$ grow at least as n^α, for some $\alpha > 1$ only depending on the degree of P. Such polynomials satisfy a summability condition, see [10,13,5,16] and [17].

Remark 2. – To see that Corollary 1 applies for polynomials with disconnected Julia set we first observe that the proof of the theorem applies for polynomial like maps in the sense of [3]. Then if $P \in \mathbb{C}[z]$ is a polynomial with a Cremer periodic point $p \in \mathbb{C}$ of period $n \geqslant 1$, then the restriction of P^n to a suitable neighborhood of p is a polynomial like map with connected Julia set.

Remark 3. – A similar method allows us to prove that if a rational map $R \in \mathbb{C}(z)$ has a Cremer cycle, then there is a sequence of periodic points $\{p_k\}$ of period n_k, whose multiplier is bounded by $\exp(C\sqrt{n_k}(\ln n_k)^{3/2+2/h})$, where $C > 0$ is a universal constant and $0 < h < HD_{\text{hyp}}(R)$, where $HD_{\text{hyp}}(R)$ is the supremum over the Hausdorff dimensions of hyperbolic sets of R.

1. Proof of the theorem

Fix a polynomial $P \in \mathbb{C}[z]$ of degree $d > 1$ and with connected Julia set. Consider a base point $w_0 \in \mathbb{C} - K(P)$ to be chosen in Lemma 2 below.

Theorem 1 will be reduced to the following lemma, see [5] for its proof.

LEMMA 1. – *Let $\{\omega_n\}_{n\geqslant 1}$ be an increasing sequence such that $\sum_{n\geqslant 1} \omega_n^{-1} < \infty$. If for every $n \geqslant 1$ and every $z \in P^{-n}(w_0)$ we have $|(P^n)'(z)| \geqslant \omega_n$, then $J(P)$ is locally connected.*

Let $\{\lambda_n\}_{n\geqslant 1}$ be an increasing sequence and suppose that for every $n \geqslant 1$ the repelling periodic points of period n have multipliers of norm at least $\lambda_n > 1$.

The following lemma estimates the derivative at a large number of preimages of a base point $w_0 \in \mathbb{C} - K(P)$. Lemmas 3 and 4 are distortion lemmas that will allow us to estimate derivatives at *all* preimages of w_0, as required by Lemma 1.

LEMMA 2. – *Let μ be an invariant probability measure with positive entropy $h_\mu > 0$ and supported on a hyperbolic set $K \subset J(P)$. Then for every $\varepsilon_0 > 0$ there is a base point $w_0 \in \mathbb{C} - K(P)$ and $C_0 > 0$ such that the following property holds.*

For every every integer $n \geqslant 1$ there is an integer $\ell = \ell(n)$, satisfying $0 \leqslant \ell \leqslant (h_\mu^{-1} + \varepsilon_0) \cdot \ln n$, and there is $x \in P^{-\ell}(w_0)$ such that for every $w \in P^{\ell - n}(x)$

$$\left|\left(P^n\right)'(w)\right| \geqslant C_0 \lambda_n.$$

Proof. – Here a *pull-back* of a set $W \subset \mathbb{C}$ by P means a connected component of a preimage of W by some iterate of P.

Fix a bounded neighborhood U of $K(P)$. Let $\rho > 0$ be such that for $z_0 \in K(P)$ and $0 < r < \rho$, any pull-back of $B(z_0, r)$ by P is contained in U.

1. Since K is a hyperbolic set the following well-known univalent pull-back property holds. There is $\delta > 0$, fixed in the following, such that for every $z_0, \zeta_0 \in K$ and $r > 0$ small enough there is a univalent pull-back $V \subset B(z_0, r)$ of $B(\zeta_0, 3\delta)$ by P, whose diameter is comparable to r.

2. Let $R = R(\delta) > 1$ be such that for any $z_0 \in K(P)$ and $r > 0$ satisfying $Rr < \rho$, we have that for any univalent pull-back of $B(z_0, Rr)$, the corresponding pull-back of $B(z_0, r)$ has diameter at most δ.

Moreover let $m_0 \geqslant 1$ be such that for every $\zeta_0 \in J(P)$ the set $\cup_{0 \leqslant m \leqslant m_0} P^m(B(\zeta_0, \delta))$ contains U.

3. Since $h_\mu > 0$, Ruelle's inequality $\chi_\mu = \int \ln|P'| \,\mathrm{d}\mu \geqslant \frac{1}{2} h_\mu$ implies that χ_μ is positive.

By Birkhoff Ergodic Theorem that there is a set of μ positive measure of points in K whose Lyapunov exponent is at least $\chi_\mu > 0$, cf. [15].

4. Note that $HD(\mu) = h_\mu / \chi_\mu$ is positive, where $HD(\mu)$ denotes the infimum of the Hausdorff dimensions of sets $X \subset K$ such that $\mu(X) = 1$; see [9] or [15].

5. Choose $\alpha > HD(\mu)^{-1}$. By a Borel-Cantelli argument for every $z_0 \in K$, outside an exceptional set of Hausdorff dimension at most $\alpha^{-1} < HD(\mu)$, there is $n_0 = n_0(z_0)$ such that for all $n \geqslant n_0$ the ball $B(z_0, Rn^{-\alpha})$ is disjoint form $\bigcup_{0 \leqslant i \leqslant n+m_0} P^i(\mathrm{Crit})$; here $\mathrm{Crit} \subset \mathbb{C}$ denotes the set of critical points of P.

So for big values of n all pull-backs of $B(z_0, Rn^{-\alpha})$ by P^{n+m_0} are univalent.

6. Since $\alpha^{-1} < HD(\mu)$ it follows that the set of points $z_0 \in K$ satisfying property 5 has full μ measure. Hence there is such a point $z_0 \in K$, also having Lyapunov exponent at least χ_μ (*cf.* 3).

7. Assume α close enough to $HD(K)^{-1}$, so that there is $\lambda \in (0, \chi_\mu)$ satisfying

$$\frac{\alpha}{\ln \lambda} \leqslant \frac{HD(\mu)^{-1}}{\chi_\mu} + \varepsilon_0 = h_\mu^{-1} + \varepsilon_0.$$

8. Fix $\zeta_0 \in K$ and choose any point $w_0 \in B(\zeta_0, \delta) - K(P)$ as a base point.

Fix a big integer $n \geqslant 1$ and let $\ell = \ell(n)$ be an integer such that there is an univalent pull-back $V \subset B(z_0, n^{-\alpha})$ of $B(\zeta_0, 3\delta)$ by P^ℓ with $\mathrm{diam}(V)$ comparable to $n^{-\alpha}$.

Notice that $\mathrm{diam}(V)$ is comparable to $|(P^\ell)'(z_0)|^{-1}$, so if n is large enough we have

$$\ell \leqslant \frac{\alpha}{\ln \lambda} \cdot \ln n \leqslant \left(h_\mu^{-1} + \varepsilon_0\right) \cdot \ln n.$$

9. Let $x \in V$ be the preimage of $w_0 \in B(\zeta_0, \delta)$ by P^ℓ. Let $w \in P^{\ell - n}(x)$ and let $w' \in B(\zeta_0, \delta)$ a preimage of w by P^m, for some $0 \leqslant m \leqslant m_0$ (*cf.* 2). Let W and W' be the pull-backs of V by $P^{n-\ell}$ and $P^{n-\ell+m}$ containing w and w', respectively.

Since the corresponding pull-backs of $B(z_0, Rn^{-\alpha})$ are univalent (*cf.* 5), it follows that $\mathrm{diam}(W') \leqslant \delta$ (*cf.* 2) and therefore $W' \subset B(\zeta_0, 2\delta)$. Thus W' contains a repelling periodic point of period $n+m$. By Koebe Distortion Theorem there is a universal constant $K_1 > 0$ such that $|(P^{n+m})'(w')| \geqslant K_1 \lambda_{n+m}$. Thus letting $M = \sup_U |P'|$ we have,

$$\left|\left(P^n\right)'(w)\right| \geqslant M^{-m_0} \left|\left(P^{n+m}\right)'(w')\right| \geqslant K_1 M^{-m_0} \lambda_{n+m} \geqslant K_1 M^{-m_0} \lambda_n. \qquad \square$$

Let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane and consider a covering map $\psi : \mathbb{H} \to \mathbb{C} - K(P)$ with deck transformation $z \to z + 1$ and such that $P(\psi(z)) = \psi(dz)$. In particular $\psi(z+1) = \psi(z)$, for $z \in \mathbb{H}$, and for any $r \in \mathbb{R}$, ψ is injective in $\{z \in \mathbb{H} \mid r < \mathrm{Re}(z) < r+1\}$.

The following lemma is an easy consequence of Koebe Distortion Theorem.

LEMMA 3. – *There is a constant $D > 1$ such that for all $\widetilde{w}$ and $\widetilde{w}' \in \mathbb{H}$ satisfying $\mathrm{Im}(\widetilde{w}) = \mathrm{Im}(\widetilde{w}') \leqslant 1/2$ and $|\widetilde{w} - \widetilde{w}'| \leqslant 1/2$ we have*

$$\left|\psi'(\widetilde{w})\right| \leqslant D\left(\frac{|\widetilde{w} - \widetilde{w}'|}{\mathrm{Im}(\widetilde{w})}\right)^4 \left|\psi'(\widetilde{w}')\right|,$$

if $\mathrm{Im}(\widetilde{w}) \leqslant |\widetilde{w} - \widetilde{w}'|$, and $|\psi'(\widetilde{w})| \leqslant D|\psi'(\widetilde{w}')|$ otherwise.

Proof. – Put $\rho = \mathrm{Re}((\widetilde{w} + \widetilde{w}')/2)$ and note that ψ is univalent in the square

$$S = \bigl\{z \in \mathbb{H} \mid |\mathrm{Re}(z) - \rho| < 1/2 \text{ and } \mathrm{Im}(z) < 1\bigr\} \subset \bigl\{z \in \mathbb{H} \mid |\mathrm{Re}(z) - \rho| < 1/2\bigr\}.$$

Moreover by hypothesis $\widetilde{w}, \widetilde{w}' \in S_0 = \{z \in \mathbb{H} \mid |\mathrm{Re}(z) - \rho| \leqslant 1/4 \text{ and } \mathrm{Im}(z) \leqslant 1/2\}$.

Then the lemma is an immediate consequence of Koebe Distortion Theorem, that can be stated as follows. There is a universal constant $D_0 > 1$ such that for any conformal map $\varphi : \mathbb{D} \to \mathbb{C}$ and every $w, w' \in \mathbb{D}$ we have $|\varphi'(w)| \leqslant D_0 \eta^{-4} |\varphi'(w')|$, where $\eta = 1 - \max\{|w|, |w'|\}$ (this statement can be obtained by combining both inequalities of (15), in Theorem 1.3 of [12]). □

LEMMA 4. – *Fix $w_0 \in \mathbb{C} - K(P)$. Then there is a constant $C_1 > 1$ such that the following property holds. Consider an integer $\ell \geqslant 1$ and $x \in P^{-\ell}(w_0)$. Then for every integer $n \geqslant \ell$ and every $w \in P^{-n}(w_0)$ there is $w' \in P^{\ell - n}(x)$ such that,*

$$\left|\bigl(P^n\bigr)'(w)\right| \geqslant C_1 d^{-4\ell} \left|\bigl(P^n\bigr)'(w')\right|.$$

Proof. – Let $\widetilde{w}_0$ and $\widetilde{w} \in \mathbb{H}$ be such that $\psi(\widetilde{w}_0) = w_0$ and $\psi(\widetilde{w}) = w$. We may choose $\tilde{x} \in \mathbb{H}$ such that $\psi(\tilde{x}) = x$ and $|\tilde{x} - d^{n-\ell}\widetilde{w}| \leqslant 1/2$. Put $\widetilde{w}' = d^{\ell - n}\tilde{x}$ and $w' = \psi(\widetilde{w}') \in P^{\ell - n}(x)$. Since $\mathrm{Im}(\widetilde{w}') = d^{-n}\mathrm{Im}(\widetilde{w}_0) = \mathrm{Im}(\widetilde{w})$ the previous lemma implies

$$\left|\psi'(\widetilde{w})\right| \leqslant D_1 d^{4\ell} \left|\psi'(\widetilde{w}')\right|, \quad \text{where } D_1 = D \cdot \max\bigl\{\mathrm{Im}(\widetilde{w}_0)^{-4}, 1\bigr\}.$$

On the other hand from the equation $P^n(\psi(z)) = \psi(d^n z)$ it follows that for every $\widehat{w} \in \mathbb{H}$ satisfying $P^n(\psi(\widehat{w})) = w_0$, we have

$$\bigl(P^n\bigr)'\bigl(\psi(\widehat{w})\bigr)\psi'(\widehat{w}) = d^n \psi'\bigl(d^n \widehat{w}\bigr) = d^n \psi'(\widetilde{w}_0).$$

So the lemma follows with $C_1 = D_1^{-1}$. □

Note that as the invariant measure μ is chosen with bigger entropy h_μ, the (asymptotic) estimate of Lemma 2 is better. The topological entropy of P is equal to $\ln d$ (*see* [6] and [8]), so by the variational principle $h_\mu \leqslant \ln d$; cf. [15]. On the other hand the harmonic measure of P is invariant under P and it has metric entropy equal to $\ln d$, see [1] and also [4] and [7]. It follows by Pesin theory that we can choose an invariant measure μ supported on a hyperbolic set and such that h_μ is as close to $\ln d$ as wanted, see [15].

Proof of the theorem. – Put $\lambda_n = Cn^{5+\varepsilon}$ and let $\varepsilon_0 > 0$ be such that $(4\ln d)\varepsilon_0 < \varepsilon/3$. Let μ be an invariant probability measure supported on a hyperbolic set and whose metric entropy h_μ is close enough to $\ln d$ so that $(4\ln d)h_\mu^{-1} < 4 + \varepsilon/3$.

Let $w_0 \in \mathbb{C} - K(P)$ and $C_0 > 0$ as in Lemma 2. Moreover given $n \geqslant 1$ and $w \in P^{-n}(w_0)$ consider $\ell = \ell(n)$ and $x \in P^{-\ell}(w_0)$ as in Lemma 2.

By Lemma 4 there is $w' \in P^{-k}(x)$ so that,

$$\left|\left(P^n\right)'(w)\right| \geqslant C_1 d^{-4\ell}\left|\left(P^n\right)'(w')\right| \geqslant C_0 C_1 d^{-4\ell}\lambda_n \geqslant C_0 C_1 d^{-4\ell} n^{5+\varepsilon}.$$

Since by Lemma 2, $\ell = \ell(n) \leqslant (h_\mu^{-1} + \varepsilon_0) \cdot \ln n$, we have $d^{4\ell} \leqslant n^{4 \ln d (h_\mu^{-1} + \varepsilon_0)} \leqslant n^{4+2\varepsilon/3}$.

Thus the hypothesis of Lemma 1 is satisfied with $\omega_n = C_0 C_1 n^{1+\varepsilon/3}$. $\square$

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