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Équations différentielles/*Ordinary Differential Equations*

Invariant measures for dichotomous stochastic differential equations in Hilbert spaces *

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Abstract We study existence of invariant measures for semilinear stochastic differential equations in Hilbert spaces. We consider infinite dimensional noise that is white in time and colored in space and we assume that the nonlinearities are Lipschitz continuous. We show that if the equation is dichotomous in the sense that the semigroup generated by the linear part is hyperbolic and the Lipschitz constants of the nonlinearities are not too large, then existence of a solution with bounded mean squares implies existence of an invariant measure. ***To cite this article: O. Van Gaans, S. Verduyn Lunel, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1083–1088.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Mesures invariantes pour des équations différentielles stochastiques à dichotomies exponentielles dans les espaces de Hilbert

Résumé Nous étudions l'existence de mesures invariantes pour des équations différentielles stochastiques semilinéaires dans les espaces de Hilbert. Nous considérons des bruits de dimension infinie qui sont blancs en la variable du temps et colorés en les variables de l'espace et nous supposons que les nonlinéarités sont lipschitziennes. Supposons en outre que l'équation a une dichotomie au sens où le semigroupe engendré par la partie linéaire est hyperbolique et les constantes lipschitziennes ne sont pas trop grandes. Nous démontrons alors que l'existence d'une solution à variance bornée entraîne l'existence d'une mesure invariante. ***Pour citer cet article : O. Van Gaans, S. Verduyn Lunel, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1083–1088.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note nous étudions l'existence de mesures invariantes pour des équations différentielles stochastiques semi-linéaires dans un espace de Hilbert réel séparable H. Nous considérons des équations de la forme

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + \sum_{i=0}^{\infty} \Phi_i(X(t))\,\mathrm{d}W_i(t), & t \geqslant 0, \\ X(0) = X_0, \end{cases} \tag{1}$$

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où A désigne le générateur d'un semi-groupe $(S(t))_{t\geqslant 0}$ fortement continu sur H et $(W_i)_{i=0}^{\infty}$ désigne une suite de processus de Wiener (mouvements browniens) scalaires et mutuellement indépendents. Les applications F et $\Phi_i : H \to H$, $i \in \mathbb{N}$, satisfont aux conditions $\|F(x) - F(y)\| \leqslant L_F\|x - y\|$ et $\sum_{i=0}^{\infty} \|\Phi_i(x) - \Phi_i(y)\|^2 \leqslant L_\Phi^2\|x - y\|^2$ pour tout $x, y \in H$ et $\sum_{i=0}^{\infty}\int_0^t \|\Phi_i(s)\|^2\,\mathrm{d}s < \infty$ pour tout $t \geqslant 0$, pour certaines constantes $L_F, L_\Phi \geqslant 0$. Les équations de la forme (1) apparaissent, par exemple, dans l'étude de bruits gaussiens pour des équations différentielles avec retard, des équations fonctionnelles différentielles du type neutre et certaines classes d'équations aux dérivées partielles.

Nous supposons que tous les processus W_i, $i \in \mathbb{N}$, sont définis sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ et adaptés à une filtration $(\mathcal{F}_t)_{t\geqslant 0}$ dans $\mathcal{F}$. Nous appelons *solution* de (1) une fonction continue X sur $[0,\infty)$ à valeurs dans l'espace de Bochner $\mathrm{L}^2(\Omega; H)$, adaptée à $(\mathcal{F}_t)_{t\geqslant 0}$ et vérifiant

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F\big(X(s)\big)\,\mathrm{d}s + \sum_{i=0}^{\infty}\int_0^t S(t-s)\Phi_i\big(X(s)\big)\,\mathrm{d}W_i(s), \quad t \geqslant 0. \tag{2}$$

Il est bien connu que pour toute condition initiale $\mathcal{F}_0$-mesurable $X_0 \in \mathrm{L}^2(\Omega; H)$ il existe une solution de (1) qui est fortement et faiblement unique. En outre, l'équation (1) engendre un semi-groupe de transition sur l'espace des fonctions boréliennes bornées sur H. On appelle *mesure invariante* pour (1) une mesure borélienne de probabilité sur H qui reste invariante sous l'action de l'adjoint du semi-groupe de transition (*voir* [3, p. 303]).

Pour l'existence d'une mesure invariante pour (1) il est nécessaire qu'il existe une solution bornée. Si la dimension de H est finie, l'existence d'une solution bornée est aussi suffisante. Si H est de dimension infinie, il y a plusieurs conditions sur $(S(t))_{t\geqslant 0}$ et F qui fournissent l'existence d'une mesure invariante dès qu'il existe une solution bornée. La plupart de ces résultats concerne des équations aux derivées partielles paraboliques (*voir* [4] et ses références). Par exemple, on suppose que les opérateurs $S(t)$, $t > 0$, sont compacts (*voir* [2] ou [3, Theorem 11.29]) ou que l'opérateur A est dissipatif (*voir*, e.g., [8,2,3]). Cependant, en général les équations différentielles avec retard ainsi que d'autres classes intéressantes, sont exclues par ces conditions. Notre but est de remplacer les conditions de régularité ou de dissipativité par une condition de dichotomie exponentielle et de présenter ainsi une théorie admettant des applications aux équations différentielles avec retard.

On dit que $(S(t))_{t\geqslant 0}$ est *hyperbolique* avec exposants $\alpha_1, \alpha_2 > 0$ si H admet une décomposition $H = H_1 \oplus H_2$ en deux sous-espaces fermés H_1 et H_2, tous les deux invariants par $S(t)$, $t \geqslant 0$, telle que $(S(t)|_{H_2})_{t\geqslant 0}$ s'étend à un groupe fortement continu sur H_2 et telle qu'on a pour une constante $M > 0$ et pour tout $t \geqslant 0$

$$\big\|S(t)x\big\| \leqslant M\,\mathrm{e}^{-\alpha_1 t}\|x\| \quad \text{pour tout } x \in H_1 \quad \text{et} \quad \big\|S(-t)x\big\| \leqslant M\,\mathrm{e}^{-\alpha_2 t}\|x\| \quad \text{pour tout } x \in H_2. \tag{3}$$

Nous désignons la projection sur H_1 le long de H_2 par P_1 et la projection sur H_2 le long de H_1 par P_2. Grâce aux théorèmes de Gearhart et Herbst (*voir* [6,7,9]) on peut vérifier la condition d'hyperbolicité de $(S(t))_{t\geqslant 0}$ en considérant seulement son générateur A qui est explicite dans l'équation (1). On dit que l'équation (1) a une *dichotomie exponentielle* si l'on peut choisir $(S(t))_{t\geqslant 0}$ et F tels que $(S(t))_{t\geqslant 0}$ soit hyperbolique et L_F soit plus petit que les exposants de l'hyperbolicité. Dans la situation présentée ci-dessus nous avons le résultat suivant.

THÉORÈME 0.1. – *Supposons que $(S(t))_{t\geqslant 0}$ soit hyperbolique et vérifie* (3) *et que L_F et L_Φ soient tels que $2L_1 < \alpha_1$, $2L_2 < \alpha_2$ et $2\alpha_1 L_2 + 2\alpha_2 L_1 < \alpha_1\alpha_2$, où $L_i = 3(L_\Phi^2 + L_F^2/\alpha_i)M^2\|P_i\|^2$, $i = 1, 2$. Soit $X_0 \in \mathrm{L}^2(\Omega; H)$ $\mathcal{F}_0$-mesurable et soit $X : [0,\infty) \to \mathrm{L}^2(\Omega; H)$ la solution de* (1). *Alors, si $t \mapsto \mathbb{E}\,\|X(t)\|^2$ est bornée sur $[0,\infty)$, on a les conclusions suivantes* :

(i) *l'ensemble $\{\mu_{X(t)} : t \geqslant 0\}$ des distributions de $X(t)$, $t \geqslant 0$, satisfait à la condition de Prohorov, c'est-à-dire, pour tout $\varepsilon > 0$ il existe une ensemble compact $K_\varepsilon \subset H$ tel que $\mu_{X(t)}(K_\varepsilon) \geqslant 1 - \varepsilon$ pour tout $t \geqslant 0$, ou, de façon équivalente, $\mathbb{P}(X(t) \in K_\varepsilon) \geqslant 1 - \varepsilon$ pour tout $t \geqslant 0$;*

(ii) *il existe une mesure borélienne de probabilité μ sur H telle que* $\int_H \|x\|^2\,\mathrm{d}\mu(x) < \infty$ *et* $\mu_{X(t)} \to \mu$, $t \to \infty$, *au sens de la convergence faible des mesures*;
(iii) *la mesure μ de* (ii) *est une mesure invariante de* (1).

Afin d'établir ce résultat, nous montrons d'abord que la solution bornée X satisfait à
(*) pour tout $\tau \geqslant 0$ il existe une fonction continue $Y_\tau : [0,\infty) \to \mathrm{L}^2(\Omega; H)$ telle que $Y_\tau(t)$ et $X(t)$ ont la même distribution pour tout $t, \tau \geqslant 0$ et il existe $C, \gamma > 0$ tels que

$$\mathbb{E}\left\| X(t+\tau) - Y_\tau(t)\right\|^2 \leqslant C\,\mathrm{e}^{-\gamma t} \quad \text{pour tout } t, \tau \geqslant 0. \tag{4}$$

Les fonctions Y_τ sont obtenues comme solutions de l'équation (1) où nous remplaçons les processus de Wiener $(W_i(t))_{t\geqslant 0}$, $i \in \mathbb{N}$, par les processus translatés $(W_i(\tau + t))_{t\geqslant 0}$. L'inégalité (4) se déduit de la dichotomie et de lemmes semblables au lemme de Gronwall. Ensuite, nous utilisons la théorie des mesures de probabilités sur les espaces métriques séparables pour démontrer (i) et (ii) du Théorème 0.1. Alors, il est connu que (iii) peut être déduit de (ii) (*voir* aussi [3, Proposition 11.2]).

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1. Introduction

In this paper we are interested in the existence of invariant measures for semilinear stochastic evolution equations in a real separable Hilbert space H. We study equations of the form

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + \sum_{i=0}^{\infty} \Phi_i(X(t))\,\mathrm{d}W_i(t), & t \geqslant 0, \\ X(0) = X_0, \end{cases} \tag{5}$$

where A is the generator of a C_0-semigroup $(S(t))_{t\geqslant 0}$ on H, the maps F and Φ_i, $i \in \mathbb{N}$, are Lipschitz continuous from H to H, and $(W_i)_{i=0}^{\infty}$ is a sequence of independent scalar Wiener processes (Brownian motions). We consider mild solutions that are functions from $[0,\infty)$ to the Bochner space $\mathrm{L}^2(\Omega; H)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space. Equations of the form (5) arise, for instance, in the study of stochastic delay differential equations, neutral functional differential equations and certain classes of partial differential equations.

Eq. (5) induces a *transition semigroup* acting on the space of bounded Borel functions on H. A Borel probability measure on H that is invariant under the dual of the transition semigroup is called an *invariant measure* for (5) (*see* [3, p. 303]). A necessary condition for existence of an invariant measure for (5) is the existence of a bounded solution. If H is finite dimensional, it is also sufficient. For the infinite dimensional case, there are various conditions on the coefficients of (5) that provide that the existence of a bounded solution implies existence of an invariant measure. The existing results are mainly tailored for parabolic partial differential equations (*see* [4] for a recent account and references). They are, for instance, based on compactness of $S(t)$ for $t > 0$ (*see* [2] or [3, Theorem 11.29]) or dissipativity of A (*see* [8,2,3] and references therein). Delay equations and several other interesting applications, however, do not fit in these frameworks.

Our aim is to replace smoothness or dissipativity assumptions by an exponential dichotomy. Many delay equations are exponentially dichotomous and this property is relatively easy to check (*see* for example [9]). We will establish the following result (see for a more detailed version Theorem 2.1).

THEOREM 1.1. – *If* (5) *has an exponential dichotomy in the sense that* $(S(t))_{t\geqslant 0}$ *is hyperbolic and the Lipschitz constants of F and Φ_i, $i \in \mathbb{N}$, are not too large, then the existence of a bounded solution of* (5) *implies the existence of an invariant measure for* (5).

The main idea of the proof of Theorem 1.1 is to derive an asymptotic property of the bounded solution X that resembles a Cauchy condition, namely:

(*) for every $\tau \geqslant 0$ there is a continuous function $Y_\tau : [0, \infty) \to \mathrm{L}^2(\Omega; H)$ such that $Y_\tau(t)$ and $X(t)$ have the same distribution for all $t, \tau \geqslant 0$ and such that there are $C, \gamma > 0$ with

$$\mathbb{E}\big\|X(t+\tau) - Y_\tau(t)\big\|^2 \leqslant C\,\mathrm{e}^{-\gamma t} \quad \text{for all } t, \tau \geqslant 0. \tag{6}$$

The functions Y_τ are obtained as solutions of the same Eq. (5), but with the Wiener processes shifted over a time interval τ. The proof consists of two steps. In the first step, the exponential dichotomy of the linear part is used by means of Gronwall-type lemmas to derive (*). The second step uses measure theoretic arguments in order to deduce from (*) that the set of distributions of $X(t)$, $t \geqslant 0$, is tight. Since the tightness of the distributions of $X(t)$, $t \geqslant 0$, implies the existence of an invariant measure, Theorem 1.1 follows.

2. Problem and main result

We interpret Eq. (5) in the following integrated form:

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F\big(X(s)\big)\,\mathrm{d}s + \sum_{i=0}^{\infty}\int_0^t S(t-s)\Phi_i\big(X(s)\big)\,\mathrm{d}W_i(s), \quad t \geqslant 0. \tag{7}$$

We denote the Lipschitz constants of F and Φ_i by L_F and L_{Φ_i} and assume that $L_\Phi^2 := \sum_{i=0}^{\infty} L_{\Phi_i}^2 < \infty$ and $\sum_{i=0}^{\infty}\int_0^t \|\Phi_i(s)\|^2\,\mathrm{d}s < \infty$ in order to have proper convergence of the noise term. Here $\|\cdot\|$ denotes the norm of H. We assume furthermore that $(W_i)_{i=0}^{\infty}$ are Wiener processes with respect to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ of $\mathcal{F}$ and that they are normalized. The initial condition X_0 is an $\mathcal{F}_0$-measurable function from Ω to H.

For each t, $X(t)$ will be an H-valued random variable on Ω and we assume it to be a member of the Bochner space $\mathrm{L}^2(\Omega; H)$ of square integrable random variables. This space is a Hilbert space with the norm $\|Y\|_{\mathrm{L}^2} = (\mathbb{E}\|Y\|^2)^{1/2}$. We denote the *distribution* or *law* of $Y \in \mathrm{L}^2(\Omega; H)$ by μ_Y, which is the Borel probability measure on H given by $\mu_Y(B) = \mathbb{P}(Y \in B)$, $B \subset H$ a Borel set.

A (*mild*) *solution* of (7) is a function $X : [0, \infty) \to \mathrm{L}^2(\Omega; H)$ that is continuous, adapted to $(\mathcal{F}_t)_{t\geqslant 0}$, and satisfies (7). For such a function X the series in the right-hand side of (7) converges in $\mathrm{L}^2(\Omega; H)$ and it is shown in [5] that the series is equivalent to the stochastic integral presented in [3]. For each $\mathcal{F}_0$-measurable initial condition $X_0 \in \mathrm{L}^2(\Omega; H)$, Eq. (7) has a unique mild solution, which is also *weakly unique*. This means that if $(\Omega', \mathcal{F}', \mathbb{P}')$ is another probability space with another sequence $(W_i')_{i=1}^{\infty}$ of independent normalized scalar Wiener processes with respect to a filtration $(\mathcal{F}_t')_{t\geqslant 0}$, and if $X_0' \in \mathrm{L}^2(\Omega'; H)$ is $\mathcal{F}_0'$-measurable with $\mu_{X_0'} = \mu_{X_0}$, then the solution X' of (7) with W_i replaced by W_i' satisfies $\mu_{X'(t)} = \mu_{X(t)}$ for every $t \geqslant 0$.

If X_0 is such that $\mu_{X(t)} = \mu_{X_0}$ for all $t \geqslant 0$, then we call μ_{X_0} a *stationary distribution*. Every stationary distribution μ is an invariant measure and the converse is true if in addition $\int_H \|x\|^2\,\mathrm{d}\mu(x) < \infty$. It is known (*see* [3, Proposition 11.28]) that existence of an invariant measure for (7) comes down to existence of an initial condition X_0 such that the set of distributions $\{\mu_{X(t)} : t \geqslant 0\}$ of the solution X is *tight*, which means that for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset H$ with $\mu_{X(t)}(K_\varepsilon) \geqslant 1 - \varepsilon$ for all $t \geqslant 0$, or, in other words, $\mathbb{P}(X(t) \in K_\varepsilon) \geqslant 1 - \varepsilon$ for all $t \geqslant 0$. Since we are interested in existence of invariant measures, we will seek a solution with a tight set of distributions.

The class of equations that we consider is determined by the assumption of exponential dichotomy (*see* [1]). The semigroup $(S(t))_{t\geqslant 0}$ is called *hyperbolic* with exponents $\alpha_1, \alpha_2 > 0$ if there exist closed linear subspaces H_1 and H_2 of H and an $M > 0$ such that $H = H_1 \oplus H_2$, H_1 and H_2 are invariant under $S(t)$ for all $t \geqslant 0$, $(S(t)|_{H_2})_{t\geqslant 0}$ extends to a strongly continuous group on H_2, and if for all $t \geqslant 0$

$$\big\|S(t)x\big\| \leqslant M\,\mathrm{e}^{-\alpha_1 t}\|x\| \quad \text{for all } x \in H_1 \quad \text{and} \quad \big\|S(-t)x\big\| \leqslant M\,\mathrm{e}^{-\alpha_2 t}\|x\| \quad \text{for all } x \in H_2. \tag{8}$$

We will denote by P_1 the projection on H_1 along H_2 and by P_2 the projection on H_2 along H_1. We say that (7) has an *exponential dichotomy* if $(S(t))_{t\geqslant 0}$ and F can be chosen such that $(S(t))_{t\geqslant 0}$ is hyperbolic and the exponents of hyperbolicity are greater than the Lipschitz constant of F.

Hyperbolicity of the semigroup can be formulated directly in terms of the spectrum and resolvent of the generator A, which appears explicitly in the differential equation (5) (*see* [6,7], or [9]). Notice that every exponentially stable semigroup is hyperbolic and therefore in our class. With the above notations and assumptions we have:

THEOREM 2.1. – *Assume that the semigroup* $(S(t))_{t\geqslant 0}$ *is hyperbolic with the estimates* (8) *and assume that* L_F *and* L_Φ *are so small that* $2L_1 < \alpha_1$, $2L_2 < \alpha_2$, *and* $2\alpha_1 L_2 + 2\alpha_2 L_1 < \alpha_1\alpha_2$, *where* $L_i = 3(L_\Phi^2 + L_F^2/\alpha_i)M^2\|P_i\|^2$, $i = 1, 2$. *Let* $X : [0,\infty) \to \mathrm{L}^2(\Omega; H)$ *be the solution of* (7). *If* $t \mapsto \mathbb{E}\|X(t)\|^2$ *is bounded on* $[0,\infty)$, *then*

(i) *the set of distributions* $\{\mu_{X(t)}\}_{t\geqslant 0}$ *is tight,*

(ii) *there exists a Borel probability measure* μ *on* H *with* $\int_H \|x\|^2\,\mathrm{d}\mu(x) < \infty$ *such that* $\mu_{X(t)} \to \mu$ *in the sense of weak convergence of measures as* $t \to \infty$, *and*

(iii) *the measure* μ *from* (ii) *is an invariant measure for* (7).

3. Step 1: exploiting the exponential dichotomy

The first step of the proof of Theorem 2.1 is to show that the bounded solution X has property (*). Let $\tau \geqslant 0$ and let Y_τ be the solution of Eq. (7) with $\mathrm{d}W_i(s)$ replaced by $\mathrm{d}W_i(\tau + s)$. Then $Y_\tau : [0,\infty) \to \mathrm{L}^2(\Omega; H)$ is continuous and weak uniqueness of solutions yields that $\mu_{Y_\tau(t)} = \mu_{X(t)}$ for all $t \geqslant 0$. By subtracting the equations for $X(t+\tau)$ and $Y_\tau(t)$ we obtain

$$X(t+\tau) - Y(t) = S(t)\big(X(\tau) - X_0\big) + \int_0^t S(t-s)\Big(F\big(X(\tau+s)\big) - F\big(Y_\tau(s)\big)\Big)\,\mathrm{d}s$$
$$+ \sum_{i=0}^{\infty}\int_0^t S(t-s)\Big(\Phi_i\big(X(\tau+s)\big) - \Phi_i\big(Y_\tau(s)\big)\Big)\,\mathrm{d}W_i(\tau+s),$$

$t \geqslant 0$. We split this equation into two equations by applying P_1 and P_2, respectively. We consider the first equation in forward time and the second one in backward time. If we denote

$$z_i(t) = \mathbb{E}\Big\|P_i\big(X(t+\tau) - Y_\tau(t)\big)\Big\|^2, \quad t \geqslant 0,\ i = 1, 2,$$

then application of the Itô isometry and elementary inequalities as Hölder and Cauchy–Schwarz lead to

$$z_1(t) \leqslant D\,\mathrm{e}^{-2\alpha_1 t} z_1(0) + 2L_1\int_0^t \mathrm{e}^{-\alpha_1(t-s)}\big(z_1(s) + z_2(s)\big)\,\mathrm{d}s, \tag{9}$$

$$z_2(t) \leqslant 2L_2\int_t^\infty \mathrm{e}^{-\alpha_2(s-t)}\big(z_1(s) + z_2(s)\big)\,\mathrm{d}s, \tag{10}$$

for every $t \geqslant 0$ and a certain positive constant D. By mutual substitution of (9) and (10) and use of Gronwall's lemma we are able to derive the explicit estimate (6) for certain constants $C, \gamma > 0$ independent of τ. Here we also use an extension of Gronwall's lemma to deal with the integral in (10) running from t to ∞. Thus, we have established that it follows from the conditions of Theorem 2.1 that X has property (*).

4. Step 2: from property (*) to tightness

The second step of the proof of Theorem 2.1 is to show that it follows from property (*) that the set of distributions $\{\mu_{X(t)} : t \geqslant 0\}$ is tight. This fact is independent of X being the solution of (7). Moreover, we do not need an exponential decay rate as in (6). Indeed, we have the following result.

THEOREM 4.1. – *Let* $X : [0,\infty) \to \mathrm{L}^2(\Omega; H)$ *be an arbitrary continuous function such that for every* $\tau \geqslant 0$ *there is a continuous function* $Y_\tau : [0,\infty) \to \mathrm{L}^2(\Omega; H)$ *with* $\mu_{Y_\tau(t)} = \mu_{X(t)}$ *for all* $t, \tau \geqslant 0$ *and such that there are* $C, \alpha > 0$ *with*

$$\mathbb{E}\big\|X(t+\tau) - Y_\tau(t)\big\|^2 \leqslant C(t+1)^{-(3+\alpha)} \quad \textit{for all } t, \tau \geqslant 0. \tag{11}$$

Then $\{\mu_{X(t)} : t \geqslant 0\}$ *is tight and there exists a Borel probability measure* μ *on* H *with* $\int_H \|x\|^2 \,\mathrm{d}\mu(x) < \infty$ *such that* $\mu_{X(t)} \to \mu$ *in the sense of weak convergence of measures as* $t \to \infty$.

Let us explain the main steps of the proof. We denote by $\mathcal{P}(H)$ the set of all Borel probability measures on H and endow it with the topology corresponding to weak convergence of measures. It is known that $\mathcal{P}(H)$ is metrizable (e.g., [10, Theorem 6.2, p. 43] or [11, Theorem 1.1.2, p. 9]) and that a subset Γ of $\mathcal{P}(H)$ is pre-compact if and only if it is tight (e.g., [10, Theorem 6.7, p. 47] or [11, Theorems 1.1.3–4, p. 9–10]). We want to show that the set of distributions $\{\mu_{X(t)} : t \geqslant 0\}$ on H is tight or, equivalently, pre-compact in $\mathcal{P}(H)$.

It follows from the continuity of X that for each $0 \leqslant a < b$ the set $\{\mu_{X(t)} : t \in [a, b]\}$ is compact in $\mathcal{P}(H)$. Consequently, if we are given an $\varepsilon > 0$, we can find for each $n \geqslant 1$ a pre-compact set A_n in H such that $\mu_{X(t)}(A_n) \geqslant 1 - \varepsilon$ for all $t \in [n-1, n]$. Of course, the union of all A_n need not be pre-compact. However, condition (11) allows us to choose the sets A_n closer and closer to one another as n increases. In fact we can accomplish that there is a summable sequence $r_1, r_2, \ldots$ in $[0, \infty)$ with $A_{n+1} \subset A_n + B(0, r_n)$ for all n, and that suffices to conclude that $\cup_n A_n$ is pre-compact. Here $B(0, r)$ denotes the ball $\{x \in H : \|x\| < r\}$, $r > 0$.

Let us explain how we choose such A_n. Let $\varepsilon > 0$ and let $r_n = cn^{-(1+\alpha/3)}$ and $\varepsilon_n = (C/c^2)n^{-(1+\alpha/3)}$, $n \geqslant 1$, where we choose c so large that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon/2$. By Markov's inequality we find from (11) that $\mathbb{P}(\|X(t) - Y_1(t-1)\| \geqslant r_n) \leqslant \varepsilon_n$ for all $t \in [n, n+1]$ and $n \geqslant 1$. For each n we take a compact $C_n \subset H$ with $\mathbb{P}(X(t) \in C_n) \geqslant 1 - \varepsilon_n$ for all $t \in [n-1, n]$, and we define

$$A_1 := C_1, \quad A_{n+1} := C_{n+1} \cap \big(A_n + B(0, r_n)\big), \quad n = 1, 2, \ldots.$$

Then we have for every $n \geqslant 1$ and $t \in [n, n+1]$ that $\mathbb{P}(X(t) \in A_{n+1}) \geqslant \mathbb{P}(X(t-1) \in A_n) - \varepsilon_n - \varepsilon_{n+1}$, so that $\mu_{X(t)}(\bigcup_n A_n) \geqslant 1 - \varepsilon$ for all $t \geqslant 0$. Moreover, $\bigcup_n A_n$ is pre-compact and it follows that $\{\mu_{X(t)} : t \geqslant 0\}$ is tight.

From the pre-compactness of the set $\{\mu_{X(t)} : t \geqslant 0\}$ and the fact that the estimate in (11) is uniform in τ, we can deduce that $\mu_{X(t)}$ converges in $\mathcal{P}(H)$ to a measure μ as $t \to \infty$. It is straightforward to check that $\int_H \|x\|^2 \,\mathrm{d}\mu(x) < \infty$.

Finally, observe that the arguments of Section 3 and Theorem 4.1 together yield (i) and (ii) of Theorem 2.1. The proof of (iii) is standard (*see* also [3, Proposition 11.2]).

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