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Mixed formulations for a class of variational inequalities

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Abstract This Note is an attempt to extend the mixed finite element method to a class of variational inequalities including the problems of Signorini and of unilateral contact in elasticity with or without friction. Existence and uniqueness for the continuous and the discrete problems as well as error estimates are established in a general abstract framework. As a result, the mixed approximation of the Signorini problem is proved to converge with an error bound in $h^{3/4}$. To cite this article: L. Slimane et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 87–92. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Formulations mixtes pour une classe d'inéquations variationnelles

Résumé Dans cette Note, on se propose d'étendre la méthode des éléments finis mixtes à une classe d'inéquations variationnelles comprenant les problèmes de Signorini et de contact unilatéral en élasticité avec ou sans frottement. L'existence, l'unicité pour les problèmes continu et discret ainsi que les estimations d'erreur sont établies dans un cadre général abstrait. L'application à l'approximation mixte du problème de Signorini permet alors de montrer une convergence d'ordre $h^{3/4}$. Pour citer cet article : L. Slimane et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 87–92. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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La méthode des éléments finis mixtes permet de remédier de façon efficace aux phénomènes de verrouillage numérique pouvant apparaître dans la résolution numérique d'équations variationnelles dépendant d'un petit paramètre : structures minces (plaques, coques), problèmes de transmission raide [3], problèmes d'élasticité presque incompressible [1], etc. ... Dans la présente Note, on se propose d'étendre le champ d'application de cette méthode aux inéquations variationnelles.

Soient X et M deux espaces de Hilbert. On note respectivement par $|\cdot|_X$ et $|\cdot|_M$ les normes de X et M et par X' et M' leurs espaces duaux topologiques. Soient a et b deux formes bilinéaires sur $X \times X$ et $X \times M$

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de constantes de continuité M_a et M_b. On se donne $L \in X'$ et $\chi \in M'$ et deux convexes fermés $K \subseteq X$ et $\Lambda \subseteq M$ contenant l'origine. On s'intéresse à l'étude du problème suivant

$$\begin{cases} (p,u) \in K \times \Lambda, \\ a(p,q-p) + b(q-p,u) \geqslant L(q-p), & \forall q \in K, \\ b(p,v-u) \leqslant \chi(v-u), & \forall v \in \Lambda. \end{cases} \tag{1}$$

Dans le cas où K coïncide avec X tout entier (la première inéquation est en fait une équation), où Λ est un cône de sommet l'origine et où a est coercive sur tout l'espace X, l'existence et l'unicité sont établies dans [2]. Lorsque la forme a est symétrique, le problème relève plutôt de techniques d'optimisation convexe [5]. Aucun des cadres précédents ne permet, cependant, de traiter les formulations mixtes pour lesquelles la forme a peut être non symétrique et n'est pas globalement coercive sur l'espace X.

Les problèmes concrets que nous avons en vue, formulation mixte duale du problème de Signorini (5) pour le laplacien et celle du problème de contact unilatéral en élasticité avec ou sans frottement (8), entrent dans le cadre abstrait suivant. La forme a est positive ($a(q,q) \geqslant 0,\ \forall q \in X$), et il existe deux sous-espaces W de X et Z de M contenus respectivement dans K et Λ pour lesquels on a les deux propriétés suivantes

- coercivité sur $V := \{q \in X \mid b(q,v) = 0,\ \forall v \in Z\}$:

$$\exists \alpha > 0, \quad a(q,q) \geqslant \alpha |q|_X^2, \quad \forall q \in V, \tag{2}$$

- condition inf-sup :

$$\exists \beta > 0, \quad \sup_{q \in W,\ |q|_X \leqslant 1} b(q,v) \geqslant \beta |v|_M, \quad \forall v \in M. \tag{3}$$

Ces conditions étendent en fait celles de Brezzi [1] au cas des inéquations.

THÉORÈME 1. – *Sous les conditions* (2) *et* (3), *le problème* (1) *possède une solution unique* (p,u) *vérifiant l'estimation suivante* :

$$|p|_X + |u|_M \leqslant C\big(|L|_{X'} + |\chi|_{M'}\big),$$

où C est une constante qui reste bornée si les constantes M_a, M_b, $1/\alpha$ et $1/\beta$ varient dans un borné.

La démonstration est obtenue en écrivant le problème (1) sous forme d'une inéquation variationnelle à un seul champ et en utilisant des techniques de perturbation.

La discrétisation repose de façon standard sur l'approximation des convexes K et Λ par des convexes K_h et Λ_h contenus dans des sous-espaces de dimension finie. Si K_h et Λ_h satisfont des propriétés de stabilité analogues à celles du cas continu, de façon uniforme par rapport à h (*cf.* (16) et (17) plus loin), on a le théorème suivant.

THÉORÈME 2. – *Le problème discret possède une solution unique* (p_h, u_h). *De plus, on a l'estimation*

$$\begin{aligned} |p - p_h|_X^2 \leqslant C_1\Big(&\inf_{q_h \in K_h} \big\{|p - q_h|_X^2 + A_1(q_h)\big\} + \inf_{q \in K} A_2(q) \\ &+ \inf_{v_h \in \Lambda_h} \big\{|u - v_h|_M^2 + B_1(v_h)\big\} + \inf_{v \in \Lambda} B_2(v)\Big), \\ |u - u_h|_M^2 \leqslant C_2\Big(&|p - p_h|_X^2 + \inf_{v_h \in \Lambda_h} |u - v_h|_M^2\Big), \end{aligned}$$

où C_1, C_2 sont des constantes positives indépendantes de h. Les formes $A_1(\cdot)$, $A_2(\cdot)$, $B_1(\cdot)$ et $B_2(\cdot)$ sont données plus loin (18).

L'application des résultats précédents à la résolution du problème de Signorini par les éléments finis de Raviart–Thomas de plus bas degré conduit à une méthode convergente d'ordre $h^{3/4}$.

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Introduction

Mixed finite element methods are generally used as conservative schemes in the approximation of elliptic boundary-value problems [1,6]. Their robusteness has been well-established in dealing with numerical locking effects occuring in the approximation of stiff problems including rapidly varying stiffness coeffcients, structural mechanics problems related to plates and shells or involving stiffeners [3]. These methods also give a correct way to construct stable numerical schemes for Stokes and nearly incompressible elasticity problems [1]. So it is desirable to extend this kind of numerical solutions to variational inequalities.

1. Motivation and examples

In the sequel, Ω denotes a given bounded domain of the plane. Its boundary $\partial\Omega$ is assumed to be at least Lipschitz and is decomposed as a non-overlapping union of three subsets $\Gamma_D,\ \Gamma_N,\ \Gamma_C$ with $\Gamma_D \neq \emptyset$. The outward unit normal to $\partial\Omega$ is denoted by $\boldsymbol{n}$.

1.1. Signorini problem

Let a matrix function $A = (A_{ij})_{1\leqslant i,j\leqslant 2}$ be given with $A_{ij} \in \mathrm{L}^\infty(\Omega)$, satisfying the usual ellipticity condition uniformly in Ω. Recall that the Laplace problem with the Signorini boundary condition consists in finding u such that

$$\begin{cases} -\,\mathbf{div}\, A\nabla u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma_D, \quad A\nabla u\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_N,\\ u \geqslant 0, \quad A\nabla u\cdot\boldsymbol{n} \geqslant 0, \quad (A\,\nabla u\cdot\boldsymbol{n})u = 0 \quad \text{on } \Gamma_C. \end{cases} \tag{4}$$

Setting $\boldsymbol{p} = A\,\nabla u$, the mixed dual formulation of (4) can be written as follows

$$\begin{cases} \boldsymbol{p}\in K,\ u\in \mathrm{L}^2(\Omega),\\ \displaystyle\int_\Omega A^{-1}\boldsymbol{p}\cdot(\boldsymbol{q}-p)\,\mathrm{d}\Omega + \int_\Omega u\,\mathbf{div}(\boldsymbol{q}-\boldsymbol{p})\,\mathrm{d}\Omega \geqslant 0, \quad \forall \boldsymbol{q}\in K,\\ \displaystyle\int_\Omega v\,\mathbf{div}\,\boldsymbol{p}\,\mathrm{d}\Omega = -\int_\Omega f v\,\mathrm{d}\Omega, \quad \forall v\in \mathrm{L}^2(\Omega). \end{cases} \tag{5}$$

Function f is assumed to be given in $\mathrm{L}^2(\Omega)$. Convex cone K consists of vector functions $\boldsymbol{q}$ in $H(\mathbf{div};\Omega)$ [6] satisfying the following condition

$$\int_\Omega \boldsymbol{q}\cdot\nabla v\,\mathrm{d}\Omega + \int_\Omega v\,\mathbf{div}\,\boldsymbol{q}\,\mathrm{d}\Omega \geqslant 0, \quad \forall v\in \mathrm{H}^1(\Omega),\ v=0 \text{ sur } \Gamma_D,\ v\geqslant 0 \text{ sur } \Gamma_C. \tag{6}$$

Using a weak form for the unilateral condition on Γ_C, we get the following mixed formulation

$$\begin{cases} \boldsymbol{p}\in H(\mathbf{div};\Omega),\ (u,\lambda)\in\Lambda,\\ \displaystyle\int_\Omega A^{-1}\boldsymbol{p}\cdot\boldsymbol{q}\,\mathrm{d}\Omega + \int_\Omega u\,\mathbf{div}\,\boldsymbol{q}\,\mathrm{d}\Omega + \langle\lambda,\boldsymbol{q}\cdot\boldsymbol{n}\rangle = 0, \quad \forall \boldsymbol{q}\in H(\mathbf{div};\Omega),\\ \displaystyle\int_\Omega (v-u)\,\mathbf{div}\,\boldsymbol{p}\,\mathrm{d}\Omega + \langle\mu-\lambda,\boldsymbol{p}\cdot\boldsymbol{n}\rangle \leqslant -\int_\Omega f(v-u)\,\mathrm{d}\Omega, \quad \forall (v,\mu)\in\Lambda. \end{cases} \tag{7}$$

The convex set Λ consists of elements (v,μ) in $\mathrm{L}^2(\Omega)\times \mathrm{H}^{1/2}_{00}(\Gamma_{NC})$ such that $\mu\leqslant 0$ on Γ_C, where Γ_{NC} is the interior of $\overline{\Gamma_N\cup\Gamma_C}$. Brackets $\langle\cdot,\cdot\rangle$ stand for the duality pairing between $\mathrm{H}^{1/2}_{00}(\Gamma_{NC})$ and its dual $(\mathrm{H}^{1/2}_{00}(\Gamma_{NC}))'$.

1.2. Contact problem

Let us now consider the following unilateral contact problem in elasticity [5]

$$\boldsymbol{\sigma} = A\boldsymbol{\varepsilon}(\boldsymbol{u}), \quad -\mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u} = 0 \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma}\boldsymbol{n} = 0 \quad \text{on } \Gamma_N. \tag{8}$$

On Γ_C, using standard decomposition of stress and displacement vectors on $\partial\Omega$, we can have either a frictionless condition

$$u_N \leqslant 0,\ \sigma_N \leqslant 0, \quad u_N\sigma_N = 0, \quad \boldsymbol{\sigma}_T(\boldsymbol{u}) = 0, \tag{9}$$

or a Tresca friction condition

$$\begin{aligned} &u_N \leqslant 0,\ \sigma_N \leqslant 0, \quad u_N\sigma_N = 0, \quad |\boldsymbol{\sigma}_T| \leqslant s \\ &|\boldsymbol{\sigma}_T| < s \Longrightarrow \boldsymbol{u}_T = 0 \\ &|\boldsymbol{\sigma}_T| = s \Longrightarrow \exists\lambda \geqslant 0 \text{ such that } \boldsymbol{u}_T = -\lambda\boldsymbol{\sigma}_T. \end{aligned} \tag{10}$$

The threshold s, assumed to be known a priori, is such that $s \in \mathrm{L}^\infty(\Gamma_C)$ and $s \geqslant 0$.

Let us denote by $\boldsymbol{K}$ the convex hull set of that $\boldsymbol{\tau}$ in $H(\mathbf{div};\Omega)$ [1] satisfying

$$\langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{\varphi}\rangle \leqslant 0, \quad \forall\boldsymbol{\varphi} \in \mathrm{H}^{1/2}_{00}\big(\Gamma_{NC};\mathbb{R}^2\big),\ \varphi_N \geqslant 0 \text{ sur } \Gamma_C,$$

in frictionless case (9), or such that

$$\langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{\varphi}\rangle \leqslant \int_{\Gamma_C} s|\boldsymbol{\varphi}|\,\mathrm{d}\Gamma_C, \quad \forall\boldsymbol{\varphi} \in \mathrm{H}^{1/2}_{00}\big(\Gamma_{NC};\mathbb{R}^2\big),\ \varphi_N \geqslant 0 \text{ sur } \Gamma_C,$$

for the Tresca condition (10). We are directly led to the following dual mixed formulation for the contact problem

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{u}) \in \boldsymbol{K} \times \mathrm{L}^2\big(\Omega;\mathbb{R}^2\big), \\ \displaystyle\int_\Omega A^{-1}\boldsymbol{\sigma} : (\boldsymbol{\tau} - \boldsymbol{\sigma})\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau} - \boldsymbol{\sigma})\,\mathrm{d}\Omega \geqslant 0, & \forall\boldsymbol{\tau} \in \boldsymbol{K}, \\ \displaystyle\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}\,\mathrm{d}\Omega = -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\Omega, & \forall\boldsymbol{v} \in \mathrm{L}^2\big(\Omega;\mathbb{R}^2\big) \end{cases} \tag{11}$$

($\boldsymbol{\sigma} : \boldsymbol{\tau}$ denotes the usual inner product of two tensors). All the above formulations can be studied within the following general framework.

2. Abstract framework

Let X and M be two Hilbert spaces. We denote by $|\cdot|_X$ and $|\cdot|_M$ the norms in X and M respectively, and by X' and M' their dual topological spaces. Let a and b be two bilinear forms on $X \times X$ and $X \times M$. Their continuity constants are denoted by M_a and M_b. Let $L \in X'$, $\chi \in M'$ be given along with two closed convex subsets $K \subseteq X$ and $\Lambda \subseteq M$ containing the origin. The problem to be solved can be stated as follows

$$\begin{cases} (p, u) \in K \times \Lambda, \\ a(p, q - p) + b(q - p, u) \geqslant L(q - p), & \forall q \in K, \\ b(p, v - u) \leqslant \chi(v - u), & \forall v \in \Lambda. \end{cases} \tag{12}$$

In the case where K is equal to X, where Λ is a cone and where bilinear form a is coercive on the whole space X, the existence and the uniqueness of (p, u) are well-known [4]. Observe that for $K = X$ the first inequation actually reduces to an equation. When the form a is symmetric, the problem can be handled by standard convex optimization techniques [5]. However, none of these approches can be used for the above mixed formulations since the form a can be both nonsymmetric and coercive on a strict subspace of X only.

We are thus led to consider the following framework. The bilinear form a is assumed to be nonnegative (that is, $a(q,q) \geqslant 0,\ \forall q \in X$). There exist two subspaces W of X and Z of M, contained in K and Λ respectively, with the following properties

- coercivity on $V := \{q \in X \mid b(q,v) = 0,\ \forall v \in Z\}$:

$$\exists \alpha > 0, \quad a(q,q) \geqslant \alpha |q|_X^2, \quad \forall q \in V, \tag{13}$$

- inf-sup condition:

$$\exists \beta > 0, \quad \sup_{q \in W,\ |q|_X \leqslant 1} b(q,v) \geqslant \beta |v|_M, \quad \forall v \in M. \tag{14}$$

This framework extends the well-known Brezzi's conditions to the inequalities.

THEOREM 1. – *Under conditions* (13) *and* (14), *problem* (12) *has one and only one solution satisfying the following estimate*

$$|p|_X + |u|_M \leqslant C\big(|L|_{X'} + |\chi|_{M'}\big),$$

where constant C remains bounded on bounded subsets of M_a, M_b, $1/\alpha$ and $1/\beta$.

The proof is obtained by rewriting problem (12) in the form of a single variational inequality and using perturbation techniques.

As a result, we can readily prove that the above mixed formulations are well posed.

COROLLARY 1. – *Each of problems* (5), (7) *and* (11) *has one and only one solution continuously depending on the data.*

3. Discrete problem and error estimates

Let X_h and M_h be two finite dimensional subspaces of X and M, and K_h, Λ_h two closed convex subsets of X_h and M_h respectively, assumed to contain the origin. As usual h denotes the discretization parameter. We consider the following discrete problem:

$$\begin{cases} (p_h, u_h) \in K_h \times \Lambda_h, \\ a(p_h, q_h - p_h) + b(q_h - p_h, u_h) \geqslant L(q_h - p_h), & \forall q_h \in K_h, \\ b(p_h, v_h - u_h) \leqslant \chi(v_h - u_h), & \forall v_h \in \Lambda_h. \end{cases} \tag{15}$$

The following stability conditions are analogous to that of the exact problem: there exist two subspaces W_h in X_h and Z_h in M_h, contained respectively in K_h and Λ_h, such that

- conforming conditions:

$$V_h := \{q_h \in X_h \mid b(q_h, v_h) = 0,\ \forall v_h \in Z_h\} \subseteq V, \quad W_h \subseteq W \text{ and } Z_h \subseteq Z, \tag{16}$$

- uniform discrete inf-sup condition:

$$\exists \tilde\beta > 0, \quad \sup_{q_h \in W_h,\ |q_h|_X \leqslant 1} b(q_h, v_h) \geqslant \tilde\beta |v_h|_M, \quad \forall v_h \in M_h. \tag{17}$$

Theorem 1 applies in the discret context too and readily gives that problem (15) is well-posed. Observe that the conforming condition on V_h ensures that the bilinear form a is uniformely coercive on V_h.

Our aim now is to obtain an error estimate. The linear case corresponding to $K = X$ and $\Lambda = M$ is well-known [1]. For the case where the first inequality reduces to an equation, that is, $K = X$, where Λ is a cone and where a is coercive on the whole space X, an error estimate can be found in [4].

THEOREM 2. – *Under conditions* (16) *and* (17), *there exist two positive constants independent of h such that*

$$|p - p_h|_X^2 \leqslant C_1\Big(\inf_{q_h \in K_h}\big\{|p - q_h|_X^2 + A_1(q_h)\big\} + \inf_{q \in K} A_2(q)$$

$$+ \inf_{v_h \in \Lambda_h} \left\{ |u - v_h|_M^2 + B_1(v_h) \right\} + \inf_{v \in \Lambda} B_2(v) \Big),$$

$$|u - u_h|_X^2 \leqslant C_2 \Big(|p - p_h|_X^2 + \inf_{v_h \in \Lambda_h} |u - v_h|_M^2 \Big),$$

where

$$\begin{aligned} A_1(q_h) &= a(p, q_h - p) + b(q_h - p, u) - L(q_h - p), \qquad & B_1(v_h) &= b(p, u - v_h) - \chi(u - v_h), \\ A_2(q) &= a(p, q - p_h) + b(q - p_h, u) - L(q - p_h), \qquad & B_2(v) &= b(p, u_h - v) - \chi(u_h - v). \end{aligned} \tag{18}$$

Two kinds of terms are involved in the estimates: approximation and consistency errors bounds. The consistency errors come from nonconforming approximations of the convex sets and are not invoked when $K_h \subseteq K$ and $\Lambda_h \subseteq \Lambda$. Note that terms $A_1(q_h)$ and $B_1(v_h)$ are related to the nonlinearities in the problem.

Now, we apply this abstract framework to the dual mixed formulation of Signorini problem (5). For simplicity, Ω is assumed to be polygonal shaped. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of meshes of Ω in triangles. Raviart–Thomas finite elements of the lowest degree are used to construct discrete approximations of spaces $X = H(\mathbf{div}; \Omega)$ and $M = \mathrm{L}^2(\Omega)$ [6]

$$X_h = \left\{ \boldsymbol{q}_h \in H(\mathbf{div}; \Omega) \mid \forall T \in \mathcal{T}_h, \boldsymbol{q}_h|_T \in RT_0 \right\}, \qquad M_h = \left\{ v_h \in \mathrm{L}^2(\Omega) \mid \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_0 \right\},$$

and next the discrete convex cone

$$K_h = \{ \boldsymbol{q}_h \in X_h \mid \boldsymbol{q}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_N \text{ and } \boldsymbol{q}_h \cdot \boldsymbol{n} \geqslant 0 \text{ on } \Gamma_C \}.$$

THEOREM 3. – *The discrete approximation of Signorini problem* (5) *associated to the above mixed finite element method is well posed. Moreover, if* $f \in \mathrm{H}^1(\Omega)$ *and* $u \in \mathrm{H}^2(\Omega)$ *the following estimate holds*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{H(\mathbf{div};\Omega)} + \|u - u_h\|_{\mathrm{L}^2(\Omega)} \leqslant C h^{3/4},$$

where $(\boldsymbol{p}, u)$ *is the unique solution of* (5) *and* C *is a constant depending only on* $\|u\|_{2,\Omega}$.

The error bound on $\boldsymbol{p}$

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{H(\mathbf{div};\Omega)}^2 \leqslant C \inf_{\boldsymbol{q}_h \in K_h} \left(\|\boldsymbol{p} - \boldsymbol{q}_h\|_{H(\mathbf{div};\Omega)}^2 + \left| \left\langle (\boldsymbol{p} - \boldsymbol{q}_h) \cdot \boldsymbol{n}, u \right\rangle \right| \right)$$

involves an additional term preventing the rate of convergence to be in h as for the linear case.

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