A Survey of Probabilistic Reasoning in Justification Logic

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ABSTRACT

In this thesis, we study the notion of *justification*, interpreted in a logical formalism. Specifically, we study the epistemic/doxastic interpretation of *justification logic*; i.e., an expansion of classical logic with formulae of the form $t{:}F$, which translate as "t is an evidence of the truth of F.". We present the basic semantics for justification logic, along with the corresponding theorems of soundness and completeness, and analyze how each one of them perceives the notion of justification.

Moreover, we examine the notion of justification in relation to the notion of uncertainty, by presenting the fundamental *probabilistic justification logics*. We present the corresponding semantics, accompanied with the corresponding soundness and (sort of) completeness and we investigate how each one of these perceives the uncertainty in the context of justification.

Last but not least, we define the *subset models*, a recent semantics for justification logic proposed and studied by E. Lehmann and T. Studer. We analyze the ontology of justification, as it is expressed in this framework, and we examine how subset models could probably combine with the notion of uncertainty, in a way that distinguishes between the suasiveness of the evidence t, the conclusiveness of evidence t over assertion F, and the certainty of F.

ΣΥΝΟΨΗ

Σε αυτήν τη διπλωματική εργασία μελετούμε την έννοια της επιχειρηματολογίας (justification), αναπαριστάμενη σε ένα λογικό φορμαλισμό. Μελετούμε την επιστημική / δοξαστική αναπαράσταση της *justification logic*, μίας επέκτασης της *κλασικής λογικής* (*classical logic*) με φόρμουλες της μορφής $t : F$, που μεταφράζονται ως "Το t είναι επιχείρημα που υποδεικνύει την αλήθεια της θέσης (ή την πίστη στη θέση) F.". Παρουσιάζουμε τις βασικές σημασιολογίες της justification logic, συνοδευόμενες από τα αντίστοιχα θεωρήματα ορθότητας και πληρότητας και αναλύουμε πώς εκλαμβάνει η κάθε μία την έννοια της επιχειρηματολογίας.

Επίσης, αναλύουμε την έννοια τις επιχειρηματολογίας συνυφασμένη με την έννοια της αβεβαιότητας, παρουσιάζοντας τις θεμελιώδεις *probabilistic justification logics*. Διατυπώνουμε τις αντίστοιχες σημασιολογίες, μαζί με τα αντίστοιχα θεωρήματα ορθότητας και πληρότητας και εξετάζουμε πώς η κάθε μία λογική αντιλαμβάνεται την αβεβαιότητα στο πλαίσιο της επιχειρηματολογίας.

Τέλος, μελετούμε μία νέα σημασιολογία που προτάθηκε και μελετήθηκε εκ των E. Lehmann και T. Studer τα τελευταία τρία χρόνια, ονόματι *subset models*. Ελέγχουμε πώς τα subset models θα μπορούσαν να συνδυαστούν με τη θεωρία πιθανοτήτων, στην προσπάθεια κατασκευής μίας πιθανοτικής λογικής που διαχωρίζει μεταξύ της αβεβαιότητας υπό το πρίσμα της πειστικότητας του επιχειρήματος, της αβεβαιότητας υπό το πρίσμα της αποδεικτικότητας της θέσης εκ του επιχειρήματος και τις αβεβαιότητας ισχύς της θέσης.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 History

Justification logic, despite its epistemic nature, was originally related with intuitionistic logic and specifically, with Brouwer's implicit interpretation, given in 1907-1908 ([38, 39]).

The most influential axiomatization of the corresponding intuitionistic propositional logic, known as Intuitionistic Propositional Calculus, IPC, was given by Arend Heyting in [40], in 1930.

In 1933, Gödel in [41], showed that there is an interpretation of the intuitionistic propositional logic IPC in the modal logic S4[1], i.e, he showed that there was a *translation function* **t** s.t. for any propositional formula F in the intuitionistic language

$$\mathsf{IPC} \vdash F \Leftrightarrow \mathsf{S4} \vdash \mathbf{t}(F).^2$$

Despite this result, in the same paper Gödel noted that S4 is not the proper formalization, for the description of provability, as the interpretation of $\Box F$ as "F is provable in formal system S", contradicts with his second incompleteness theorem (cf. [46]). Particularly, he states[3]

> It is to be noted that for the notion "provable in a certain formal system S" not all the formulas provable in $\mathfrak{S}$ hold. For example, $B(Bp \to p)$ never holds for that notion, that is, it holds for no system S that contains arithmetic. For otherwise, for example, $B(0 \neq 0) \to 0 \neq 0$ and therefore also ~$B(0 \neq 0)$ would be provable in S, that is, the consistency of S would be provable in S.

In this manner, in a lecture in Vienna on 29 January 1938 to a seminar organised by Edgar Zilsel (cf. [44]), Gödel replaced the modal operator $\Box$ (or B in his text), with a 3-ary relation zBp, q, which interprets the statement "z is a derivation of q from p." and then he formulated an axiomatic system which can be assumed as sort of the first

[1] viz. Definition 2.11.

[2] In fact, the "$\Leftarrow$" direction was proved in [43], by John McKinsey and Alfred Tarski.

[3] As transleted in [42].

justification logic. Unfortunately, his ideas where not published until 1995, in Volume III, of his collected works ([45]).

By this time, Sergei Artemov independently, from Gödel, rediscovered a similar operator to Gödel's zBp, q, in [1]. Artemov introduced the *logic of proofs*, LP, in which he defined the *proof letters* and the *proof axiom constants*, which correspond to the justification terms and justification constants of justification logic, respectively. He also defined the operation symbol $[\![\ ]\!]$ and interpret the formula $[\![t]\!]F$ to the statement "t is a code of the proof of F.". In the same paper, as also in [2], Artemov showed that S4 is embedded in LP, via a *realization theorem*. He also showed that LP is embedded into *classical proofs*, via showing that LP is *arithmetically complete* with respect to the intended provability semantics. In this way, Artemov bridged the gap between intuitionistic logic and classical proof in Peano Arithmetic, PA, which remained open for over a half century. As given schematically in [61], if we define by $\mathsf{X} \hookrightarrow \mathsf{Y}$ that "X is interpreted in Y.", we can write

$$\mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP} \hookrightarrow CLASSICAL\ PROOFS$$

The first non-arithmetical semantics were given by Mkrtychev in [6]. In this paper, which was published in 1997, he also gave a decidability theorem for LP.

The first *possible world semantics* for justification logic, and particularly LP was given by Melvin Fitting in [7], in 2005. This semantics associated justification logic with epistemic logic in a straight forward way. In the same paper Fitting gave a nonconstructive realization proof for LP, which turned out to generalize to a large number of justification logics for which no constructive version is currently found. Explicitly, Artemov's proof of realization theorem was based on cut-elimination for S4, which is not applicable in many other counterparts of modal logic. Fitting, through his new semantics, which are, now, known as *Fitting semantics*, gave a proof of the realization theorem that is not based on cut-elimination for S4.

In the same year, Eric Pacuit, in [9], gave soundness and completeness results for justification logics containing axiom scheme **J5**, via Fitting semantics. In the same paper, he also suggested the use of *tableau systems* for the realization theorems of such logics. It is worth mentioning that there are no sequent calculus for S5 and corresponding logics. A year after, a realization theorem for such logics was given, by Natalia Rubtsova in [10, 11].

The general definition of *justification logic* was first given in [5], by Artemov. In this paper, which was published in 2008, Artemov defined the most common justification logics and the corresponding realization axioms. This paper established the general setting of justification logic, as studied today.

In 2012, a new semantics for justification logic J_0 was defined by Artemov in [12]. Particularly, Artemov defined *basic model* and *modular model* semantics, which were suitable for the perception of the ontology of justification. In the same year, Roman Kuznets and Thomas Studer expand these semantics for the other most common justification logics, in [13].

In 2016, Fitting gave, in [8], a general method for the proof of realization of justification logics. He showed that the family of modal logics with justification counterparts is infinite. For instance, every *Geach logic* has a corresponding justification counterpart. His realiztion proof was non-constructive.

Last three years, Eveline Lehmann and Thomas Studer defined a new semantics for justification logic. These semantics, namely *subset models*, may regenerate the interest in justification logic.

From 2012, a new field of justification logic flourished, that of justification logics equipped with the notion of uncertainty. The first such logic was developed by Robert S. Milnikel, from 2012 until 2014, when his paper "The logic of uncertain justifications." ([21]) was published.

A year after, Ioannis Kokkinis et al. published his first ideas for the construction of the first justification logic, with a strong probability background (cf. [22]); i.e., the logic PJ. Until 2016, when Kokkinis finished his PhD he (et al.) published a stronger probability justification logic, the *probabilistic justification logic* PPJ, wherein the uncertainty was interpret not only on justification, but also on every kind of formula, and in addition wherein justification is also applicable on probability formulae.

Around the same time (2016), the first fuzzy justification logic was developed by Meghad Ghari.In his paper "Pavelka-style fuzzy justification logics." ([28]) he expand the fuzzy modal logic RPL to the justification setting. In this way he introduced the uncertainty on justification logic as a matter of vagueness.

From 2015-2017, Churn-Jung Liau et al. developed one more justification logic equipped with the notion of uncertainty; i.e., the logic PJL. This time the uncertainty did not arise from the evidence presented in the justification process, rather by the statement meant to be justified.

1.2 Motivation

1.2.1 Epistemic logic & Hyperintentionality

One of the most rapidly developing branches of philosophy and particularly epistemology, in our days, is the *epistemic logic*. Epistemic logic is concerned with the interpretation of knowledge in a logical setting. The flourishing of modern epistemic logic finds its roots in mid-20th century mainly by the work of von Wright, "*An essay in modal logic.*" ([19]) and Hintikka's "*Knowledge and Belief: An Introduction to the Logic of the Two Notions*" ([20]). Hintikka, based on ideas of von Wright, sets as basis of epistemic logic, the *modal logic*, by evaluating the truth on epistemic propositions through *possible world semantics*, i.e., *Kripke models*.

A *world* can be understand as a reality of our perception. It is a *complete* way how facts could hold. According to Wittgenstein's seminal work "*Tractatus Logico-Philosophicus*" ([48]) a *world* can be perceived as follows:

1 The world is all that is the case.

1.1 The world is the totality of facts, not of things.

1.11 The world is determined by the facts, and by their being all the facts.

1.12 For the totality of facts determines what is the case, and also whatever is not the case.

1.13 The facts in logical space are the world.

1.2 The world divides into facts.

The notion of a *possible* world can be defined as a maximal consistent world. By consistent we mean a world that the rules of logic apply; a world that the different facts that happen in it could coexist. In that manner, in Kripke semantics, any world is related to the set of facts that hold in it. By maximal world, we mean that this set of facts is also maximal; i.e., for every different fact, or this fact is happening, or the negation of it is happening.

Through the use of possible worlds we can defined the notion of *necessary equivalence*, that is the equivalence of the truth of some propositions in all the worlds. In contrast to the truth equivalence of two propositions F and G, i.e., the fact the two propositions have the same truth value; the necessary equivalence demands the truth equivalence of the propositions in all worlds, in a sense that in any possible world proposition F holds if and only if the proposition G also holds. For instance, the propositions

(A) Lionel Andrés Messi is from Argentina.

(B) Lionel Andrés Messi is the best football player in the world.

have the same truth value, as they are both true. But they are not necessary equivalent, as we could probably imagine a world that Messi was not from Argentina, or where he is not the best football player in the world[4]. On the other hand, the propositions

(A') Hjalmar Ekdal is a mallard.

(B') Hjalmar Ekdal is a wild duck.

are necessary equivalent, as mallard is a different name for wild duck.

This distinction between the two notions of equivalence, gave birth to a distinction between logical operators, that is the *extensional* and the *intensional* operators. An operator is called *extensional* iff we can substitute two sentences with the same truth value *salva veritate*, i.e., without changing their truth value. For instance $\neg$ is such an operator, as if we have that F and G have the same truth value, then we can exchange F in $\neg F$ with G and $\neg G$ will share the same truth value with $\neg F$. On the other hand, an operator is called *intensional* iff we can substitute two necessary equivalent sentences salva veritate. In fact, the *modal operator* $\Box$ of modal logic is an intensional operator. This operator, is used for expressing knowledge in epistemic logic.

For many years, the setting of modal logic was considered suitable for the perception of knowledge. However, on the modal setting for the interpretation of knowledge arose some philosophical drawbacks, known as *logical omniscience* problem. Particularly, we have that the following propositions hold for modal logic:

(LO1) If $F \to G$ holds, then $\Box F \to \Box G$ also holds.

(LO2) If F is valid, then $\Box F$ is also valid.

(LO3) If $\Box F$ and $\Box G$ hold, then $\Box(F \wedge G)$ also holds.

(LO4) It is not the case that $\Box F$ and $\Box\neg F$, concurrently. [5]

The interpretation of knowledge or belief by the modal operator $\Box$ is some kind of an overkill, in respect to human reasoning abilities. (LO1) states that if from a proposition F it logically follows that G, then if we know that F holds, .i.e., $\Box F$ holds, then we also know that G holds. A simple example that (LO1) should not hold for the perception of knowledge, is this:

[4]Oddly, there are some ignoramus people that doubt that Messi is the best football player in the world.

[5]This holds for modal logic, if we assume that axiom scheme **D** also holds (viz. Section 2.2).

Let that the proposition "If this butterfly flap its wings, a tornado in Halkidiki will occur the next month." holds and we see that the butterfly flap its wings and therefore we know that it did so. Then, it is unfortunately impossible for us to know that a tornado will occur in Halkidiki the next month.

(LO2) states that any valid proposition is also known. But even if $\mathbf{P} \neq \mathbf{NP}$ is valid, it is not known. For the decline of (LO3) we should probably assume the doxastic interpretation of the modal operator, i.e., $\Box F$ stands for "We believe that F holds.". An example where (LO3) seems to fail is the following:

Let me believe that it is possible to finish my thesis before going for my military services and also believe that a thesis, in order to be completed needs at least three months of work. Then, it does not mean also that I believe that I can finish my thesis before going for my military services and that a thesis needs at least three months to be complete, as the one contradicts the other.

Clearly, we can believe at contradictory statements, if we do not think them concurrently, as these are some common lies that we are telling to our selves. Last but not least, (LO4) can also be declined by the doxastic perception of the modal operator. For instance, we might believe that we love our parents the same, while also believe that we do not.

In this manner, it is probably better to define a non-intensional operator for the interpretation of knowledge and belief. These thoughts led Cresswell to define the notion of *hyperintensionaltiy*. In the first paragraph in [49], Cresswell says

> It is well known that it seems possible to have a situation in which there are two propositions p and q which are logically equivalent and yet are such that a person may believe the one but not the other. If we regard a proposition as a set of possible worlds then two logically equivalent propositions will be identical, and so if "x believes that" is a genuine sentential functor, the situation described in the opening sentence could not arise. I call this the paradox of hyperintensional contexts. Hyperintensional contexts are simply contexts which do not respect logical equivalence.

Based on Cresswell's view, by generalizing the notion of logical equivalence, to the one of necessary equivalence, a definition of an hyperintensional operator arises: An operator is called *hyperintensional* iff it is not intensional, i.e., we cannot substitute two necessary equivalent sentences salva veritate. That is, a hyperintensional operator distinguish necessary equivalent propositions. By considering a hyperintensional operator for the interpretation of knowledge/belief the logical omniscience problem vanishes. Thus, it seems logical to consider such an operator proper for the interpretation of knowledge/belief.

In that manner, justification logic seems a promising basis for the interpretation of epistmic/doxastic logic, even more prominent than modal logic. According to this interpretation, we can approach the notion of knowledge of a proposition interpreted from a formula F, by providing an evidence for its truth, i.e., a term t which justifies F, or syntactically $t{:}F$. In fact, it is proven that there is a realization theorem, which intuitively states that modal logic is expressible through justification logic(viz. Theorem 2.39)

One major advantage of the interpretation of epistemic/doxastic logic through justification logic versus through modal logic, is that, in contrast with modal logic, justification logic respects hyperintensionaltiy. As we will see (viz. Example 3.6), the justification operator distinguishes between necessary equivalent propositions, i.e., it is hyperintensional.

Another philosophical drawback for the modal interpretation of epistemic/doxastic logic is that it does not respect the *JTB* assumption of knowledge. *JTB* stands for *justified, true belief* and was first recorded as a perception of knowledge in Μένων (Διάλογος) and in Θεαίτητος (Διάλογος) ([58, 59]), from Plato, as apothegm of Socrates, even though the latter did not accept it as the correct assumption. According to this assumption, in order for a proposition to be considered known, it must be true, it must be believed and there must be a justification for believing in it. As we will see, the modal interpretation respects the truth[6] and belief requirements, but does nothing in respect to the justification concept. On the other hand, justification logic considers also the justification concept, as its name indicates.

1.2.2 Paraconsistency

In classical logic, as also in intuitionistic logic, it holds that *ex contradictione* (or *falso*) *quodlibet* (ECQ or EFQ), i.e., from a contradiction anything follows. This principle, also known as *principle of explosion*, sometimes seems problematic. For instance, in the legal system of some nation it is possible to belong contradictory, to each another, laws. It would be insane to consider that we can prove the guilt of a defendant based on this contradiction, as also her innocence. Therefore, it is reasonable to consider logics in which the principle of explosion fails. This kind of logics are called *paraconsistent logics*.

Justification logic, is capable of treating paraconsistency. Indeed, for every set of formulae Σ, even if this is inconsistent, we can reason about the logical consequences of Σ, without falling in the principle of explosion, by defining for each formula $F \in \Sigma$ a corresponding term t that justifies it. In this way, every logical consequence of Σ will be connected with a justification term which justifies it and which indicates the derivation steps which were applied, i.e., how the evidences were combined to prove this consequence.

This point of view of applying evidences for each proposition to deal with paraconsistency was extensively studied by Walter Carnielli and Abilio Rodrigues in [53, 54]. They explicate the acceptance of F concurrently with $\neg F$ as a result of *non-conclusive reasoning* and they named the evidence of such reasoning *conflicting evidence*. According to them, in [54]:

> ...There may be a proposition A such that it is not the case that both A and $\neg A$ are true, but in some sense both hold in a given context. In this case, the essential question ***Q*** is the following:
>
> ***Q:*** what property are we going to ascribe to a pair of accepted contradictory propositions such that it would be possible for a proposition to enjoy it without being true?
>
> Such a property has to be something weaker than truth. If we want to reject the principle of explosion together with dialetheism, without assuming a position of metaphysical neutrality, we have to give a convincing answer to ***Q***. Our proposal is that the notion of evidence is well suited to be such an answer.

[6] It respects the truth requirement, if we consider that the axiom scheme **T** holds. Similarly, for justification logic, the axiom scheme **JT**. (viz. Tables 2.3 and 2.5)

Melvin Fitting in [55], proved that the logic BLE introduced and studied by Carnielli and Rodriguez in [53, 54] can be embedded into KX4, which latter can be embedded into JX4, which is the justification logic J4 + **JX**, where J4 is defined in Definition 2.24 and axiom scheme **JX** is given in Table 1.1.

$s{:}t{:}F \to [s\,\mathsf{c}\,t]{:}F$	density	**JX**

Table 1.1: Axiom Scheme **JX**

1.2.3 Tracking Evidence

Besides the consideration of formulae of the form $t{:}F$ as "The evidence t justifies the proposition F.", with the definition of an *axiomatically appropriate constant specification* (viz. Definition 2.26), we can perceive the term t as an encoding of the proposition F. In this way we can keep track of which formulae took place in the reasoning process, i.e., the derivation, and how they were combined. Explicitly, a justification logic with an axiomatically appropriate constant specification has the *internalization property*, according to which for any theorem G of this justification logic, there is some term s that justifies it and which term internalizes the steps of the derivation.

In such a way, Artemov encountered the *Red Barn* example in [5] and the Russel's example in [56]. With such idea also, T. Studer developed techniques applied to data privacy in [57].

This internalization of the steps of derivation is also a reasonable measure of the complexity of the reasoning process. For instance, if we give a justification logic that interprets the Peano arithmetic[7], we expect that a term justifying the proposition that interprets the proposition "$1 + 1 = 2$" would be simpler than a term justifying *Fermat's last theorem*. Obviously, this once again accounts for the encounter of logical omniscience problem.

1.3 In this Thesis

In chapter 2, we give the groundwork of justification logic. Firstly (section 2.2), we make a brief introduction to modal logic by presenting the modal language, the basic axiomatic systems and several key theorems. We also present the corresponding semantics; i.e., the *Kripke models* accompanied with strong soundness and strong completeness theorems. Then (section 2.3), we introduce the justification logic by presenting its language and basic axiomatic system. We study how this logic expands with the introduction of new axiom schemes or constant specification and we define the main axiomatic systems. Finally (section 2.4), we provide the *realization theorem*, which establishes that a modal logic is embedded into its correlated justification logic.

In chapter 3, we present the main semantics for justification logic. Particularly, we define the *basic models* (section 3.1), the *Mkrtychev models* (section 3.2), the *Fitting models* (section 3.3), the *modular models* (subsection 3.4.1) and the *JYB-modular models* (subsection 3.4.2). Soundness and completeness theorems are given for each model. We also introduce in section 3.4 the *modal-justification logics* and show that that they

[7] Such a logic can be easily defined by adding in the language a corresponding predicate symbol $P_=$ for $=$ and corresponding function symbols f_S, f_+ and $f_\cdot$ for $S, +$ and $\cdot$, respectively, where S is the *successor function*. We can interpret some propositional atom p as the number 0. The axioms of Peano arithmetic with respect to equality, with their corresponding justifications, can be defined as premises of the derivation. We should also assume an axiomatically appropriate constant specification.

are sound and complete with a corresponding class of (JYB-)modular models. In section 3.5, we give the expressibility hierarchy of those semantics. Moreover, in section 3.6, we analyze which is the proper perception of justification, as well as how each semantics perceives the ontology of justification. Finally, in section 3.7, we demonstrate how justification logic falls into the *Gettier problem*.

In chapter 4, we study how uncertainty is inherent in justification by presenting the fundamental probabilistic justification logics. Particularly, we present the *logic of uncertain justifications*, UJ (section 4.1), the *probabilistic justification logic*, PPJ (section 4.2), the *Pavelka style fuzzy justification logic*, RPL(JL) (section 4.3) and the *possibilistic justification logic*, PJL (section 4.4). In section 4.5, we analyze the ontology of the uncertainty of justification in each formalism. Finally, in section 4.6 we present Artemov's *aggregated probabilistic evidence logic*. For each of these logics we provide some basic theorems as also soundness and (sort of) completeness theorems, for their corresponding semantics.

In chapter 5, we present the *subset model semantics* for justification logic. In section 5.2, we distinguish between normal and non-normal worlds and explain how the existence of impossible worlds is related to hyperintensionality. In section 5.4, we observe how we can adapt subset models so that they respect the aggregated probabilistic evidence logic. Finally, in section **??**, we investigate how we can adapt subset models in order to presume the notion of uncertainty of justification, by distinguishing between the uncertainty on the suasiveness of the evidence, the uncertainty on the conclusiveness of the evidence over the statement under justification, or the uncertainty on that statement.

The proofs of the majority of the theorems, lemmata and corollaries of this thesis can be found in the corresponding appendices.

CHAPTER 2

THE GROUNDWORK FOR JUSTIFICATION LOGIC

2.1 Preliminaries

In this section we give some basic definitions and notations, that will be common for any logic, which will be given. We also give the order in which any logic is defined. Particularly we will do as follows:

Firstly, we will define for every logic L, the corresponding language $\mathcal{L}$. The language will be given in BNF-notation. We will denote by Prop the set of *atomic propositions*. We assume that Prop is countably infinite and we use the symbols $p, q, r, \ldots, p_1, q_1, r_1, \ldots$ to represent its elements. We will also use the letters $F, G, H, \ldots, F_1, G_1, H_1, \ldots$ for the representation of *formulae*, of the language, of any logic. If a formula is considered an *a*xiom of some logic will denote it by $A, A_1, \ldots$, in order to distinguish it from the formulae that are not axioms.

Afterwards, we will define the logic per se. As logic L, we mean an *axiomatic system* (or *deductive system*), i.e., a system consisting of some *axioms* and *axiom schemes* and some *rules of inference*. By axiom we mean a particular formula of the corresponding language, which is assumed to hold. By axiom schemes we mean a particular formula of the corresponding metalanguage, for which by replacing each schematic variable by a corresponding formula of the language, we construct formulae which is assumed to hold. By rules of inference we mean some syntactical transform rules, which if are applied to formulae assumed as true, they will construct new true formulae.

We will define for each logic L the corresponding derivation, i.e., a method of proving the true formula in L. We will use the turnstile $\vdash_{\mathsf{L}}$ to denote the derivability by L and we will define as theorems of L the formulae that are derivable in L, i.e., the formulae $F \in \mathcal{L}$, such that $\vdash_{\mathsf{L}} F$. We will also define explicitly the derivability in L, by a set of premises Σ and we will denote it as $\Sigma \vdash_{\mathsf{L}} F$.

Definition 2.1 (Consistency). Let L is an arbitrary logic.

- We call the logic L inconsistent iff $\vdash_{\mathsf{L}} \perp$. Else, it is called *consistent*.
- We call a set of formulae Σ, in the language of L L-*inconsistent* iff $\Sigma \vdash_{\mathsf{L}} \perp$. Else, it is called L-*consistent*.

- We call a set of formulae Σ, in the language of L *maximal* iff for any formula F in the language of L it holds that $F \in \Sigma$ or $\neg F \in \Sigma$.

Definition 2.2 (Conservative Extension)**.** We call a logic L a *conservative extension* of *classical logic*, CL, iff every theorem of CL is a theorem of L and every theorem of L in the language of CL is a theorem of CL.

Moreover, for most logics we will give a deduction theorem.

Definition 2.3. For any logic L, we say that the *deduction theorem* holds for L iff for any set $\Sigma \cup \{F, G\}$ of formulae in the corresponding language, it holds that

$$\Sigma \cup \{F\} \vdash_{\mathsf{L}} G \Leftrightarrow \Sigma \vdash_{\mathsf{L}} F \to G.$$

2.2 Modal Logic

In this section we are making a brief introduction to modal logic. We firstly give the syntax of modal language. Then we define the simplest normal modal axiomatic system K which we expand with the addition of some axiomatic schemes, to corresponding axiomatic systems, according to the notion of modal interpretation that we want to presume. Finally, we give some useful semantics for modal logic and the corresponding theorems of soundness and completeness.

The language of modal logic is given in the following definition

Definition 2.4 (Modal Logic Language $\mathcal{L}_\Box$)**.** The language $\mathcal{L}_\Box$ of modal logic is defined by the following BNF-notation:

$$F \ ::= \ p \mid \bot \mid (F \to F) \mid \Box F,$$

where $p \in \mathsf{Prop}$.

The other propositional connectives are defined as abbreviations, in the standard way:

$$\neg F \equiv (F \to \bot) \qquad (F \vee G) \equiv (\neg F \to G)$$
$$(F \wedge G) \equiv \neg (F \to \neg G) \qquad (F \leftrightarrow G) \equiv ((F \to G) \wedge (G \to F))$$

Furthermore, we also use the following abbreviation

$$\Diamond F \equiv \neg \Box \neg F.$$

Definition 2.5 (Omitting Parentheses)**.** In order to omit parentheses we define the precedence and the associativity of the logical operators.

- $\Box$ and $\neg$ are granted the highest precedence. They are assumed right-associative.
- $\wedge$ and $\vee$ are granted the same precedence. They are also assumed right-associative, with respect to each other, i.e.,

  $$(F \wedge (G \vee H)) \equiv F \wedge G \vee H \qquad (F \vee (G \wedge H)) \equiv F \vee G \wedge H$$

  Moreover, they both have the associative property, i.e.,

  $$((F \wedge G) \wedge H) \equiv (F \wedge (G \wedge H)) \equiv F \wedge H \wedge G$$
  $$((F \vee G) \vee H) \equiv (F \vee (G \vee H)) \equiv F \vee G \vee H$$

- $\rightarrow$ and $\leftrightarrow$ are granted the lowest precedence. They are also assumed right-associative.

Example 2.6. *For instance the formula*

$$(\neg F_1 \vee ((F_2 \wedge F_3) \wedge F_4)) \rightarrow (F_5 \rightarrow (F_6 \wedge (\Box F_7 \vee (F_8 \vee \Box\neg\Box\neg(F_9 \rightarrow F_{10})))))$$

can be written as

$$\neg F_1 \vee (F_2 \wedge F_3) \wedge F_4 \rightarrow F_5 \rightarrow F_6 \wedge \Box F_7 \vee F_8 \wedge \Box\neg\Box\neg(F_9 \rightarrow F_{10})$$

or even

$$(F_1 \rightarrow F_2 \wedge F_3 \wedge F_4) \rightarrow F_5 \rightarrow F_6 \wedge \Box F_7 \vee F_8 \wedge \Box\Diamond(F_9 \rightarrow F_{10})$$

by using the abbreviations

$$\neg F_1 \vee (F_2 \wedge F_3) \wedge F_4 \equiv F_1 \rightarrow (F_2 \wedge F_3) \wedge F_4$$

and

$$\Diamond F \equiv \neg\Box\neg F,$$

as also the associative property for $\wedge$.

Definition 2.7 (Polarity of Modal Operator)**.** We recursively define the polarity of an explicit modal operator $\Box$ in some modal formula $F \in \mathcal{L}_\Box$, as follows:

- the occurrence of $\Box$ is positive in $\Box F$,
- the occurrence of $\Box$ from F in $G \rightarrow F$, $G \wedge F$, $F \wedge G$, $F \vee G$, $G \vee F$ and $\Box F$ has the same polarity as the corresponding occurrence in F,
- the occurrence of $\Box$ from F in $F \rightarrow G$ and $\neg F$ has the opposite polarity as the corresponding occurrence in F.

Remark 2.8. *The modal language is usually defined using the Boolean connective* $\wedge$ *or* $\vee$*, in the place of* $\rightarrow$*. We used this Boolean connective as it is more useful for justification logic, in which the application axiom*[1] *has a central role. If we defined the modal language using either of* $\wedge$ *or* $\vee$*, the Definition 2.7 could be stated as:*

An occurrence of an explicit modal operator in some modal formula is positive if it is in the scope of even number of $\neg$'s. Otherwise, it is negative.

The most common interpretation of the modal operator $\Box$ is the *epistemic* one, i.e., "We know that F.", usually denoted also as $\mathbf{K}F$. But there are also many other interpretations such as, the *doxastic*, i.e., "We believe that F.", usually denoted also as $\mathbf{B}F$; the *deontic*, i.e., "It is obligatory that F.", usually denoted also as $\mathbf{O}F$; and the *boulomaic*, i.e., "We desire F.", usually denoted also as $\mathbf{D}F$. We will be mainly focused in this thesis on the epistemic and doxastic interpretation of the modal operator $\Box$.

[1] viz. Table 2.4

2.2.1 Modal Axiomatisation

The basic normal modal logic K is the groundwork for any other normal modal logic. It is in fact a minimal modal logic, on which all the other normal modal logics are constructed, by the addition of new axioms and axiomatic schemes.

Definition 2.9 (The logic K)**.** The *basic normal modal logic* K is given in Table 2.1.

Axiomatic Schemata		
all theorems of CL in $\mathcal{L}_\Box$		**P**
$\Box(F \to G) \to \Box F \to \Box G$	distribution of $\Box$ over $\to$	**K**
Rules of Inference		
From F and $F \to G$, infer G	modus ponens	**MP**
From F, infer $\Box F$	necessitation of $\Box$	**N**

Table 2.1: Axiomatic System K

What we mean by axiom scheme **P** is that we assume the propositional tautologies, where we can replace the same propositional symbols by the same formula in $\mathcal{L}_\Box$, e.g., for the propositional tautology

$$p \to p,$$

we can replace p by the modal formula $\Box\neg\Box(p \vee q)$ and get the formula

$$\Box\neg\Box(p \vee q) \to \Box\neg\Box(p \vee q),$$

as an axiom of modal logic. Instead of the whole set of propositional tautologies we could assume some axiomatic system of classical logic, CL. Such an example is the following axiomatic system introduced by Jan Łukasiewicz.

Definition 2.10. The Łukasiewicz's axiom system for classical logic is given in Table 2.2[2].

Axiomatic Schemata	
$F \to G \to F$	**Ł1**
$(F \to G \to H) \to (F \to G) \to (F \to H)$	**Ł2'**
$(\neg F \to \neg G) \to G \to F$	**Ł3**
Rules of Inference	
From F and $F \to G$, infer G	**MP**

Table 2.2: Łukasiewicz's Axiomatic System for CL

The basic normal modal logic K is getting extended by adding new axioms, accordingly to the notion that we want to presume. For instance, if we want to presume the notion of belief, it is common to equip K with the modal axioms **D**, **4** and **5** and construct the modal logic named KD45. Every such extension of the basic normal modal logic K is called a *normal* modal logic. Some additional, commonly used modal axiom schemes are given in Table 2.3. By adding some of these axiom schemes to K we define the following regularly used axiomatic systems.

[2] We have to add prime in **L2** so that we can distinguish it from Łukasiewicz's axiom scheme **L2**, as defined in Table 4.5.

Definition 2.11. We define the following modal axiomatic systems (or simpler modal logics):

$$\begin{aligned} \mathsf{T} &:= \mathsf{K} + \mathbf{T} \\ \mathsf{KD} &:= \mathsf{K} + \mathbf{D} \\ \mathsf{K4} &:= \mathsf{K} + \mathbf{4} \\ \mathsf{S4} &:= \mathsf{T} + \mathbf{4} \\ \mathsf{K5} &:= \mathsf{K} + \mathbf{5} \\ \mathsf{S5} &:= \mathsf{S4} + \mathbf{5} \\ \mathsf{KD45} &:= \mathsf{KD} + \mathbf{4} + \mathbf{5} \end{aligned}$$

where for any axiomatic system A and any axiom scheme **B**, we denote by $\mathsf{A} + \mathbf{B}$ the axiomatic system resulting from the addition of axiom scheme **B** to the axiomatic system A[3].

$\Box F \to F$	truth axiom	**T**
$\neg\Box\bot$	consistent beliefs	**D**
$\Box F \to \Box\Box F$	positive introspection	**4**
$\neg\Box F \to \Box\neg\Box F$	negative introspection	**5**

Table 2.3: Other Modal Axiom Schemes

All the modal logics defined until now are normal modal logics. A way of defining normal modal logics, other than as an extension of K, is by requiring their modal operator to be *normal*. A *normal modal operator* $\Box$ is any operator that respects the necessitation rule of inference and the axiom scheme **K**. From now on, all the modal logics that are gonna be assumed (in fact mainly the logics of Definition 2.11) are normal. Therefore, whenever we will use the term modal logic, we will mean normal modal logic.

Definition 2.12 (Modal Derivations [17])**.** Let ML be an arbitrary modal axiomatic system, with set of axioms **AX** and set of rules of inference **RI**, where each rule, of arity i, is of the form "From $F_1, \ldots, F_i$, infer F.". We define as a *modal derivation of F in* ML, any finite sequence $F_1, \ldots, F_n$ of modal formulae, s.t.:

- $F_n := F$,
- every F_i in the sequence is
  - either an instance of an axiom in **AX**,
  - or the result of the application of one of the rules in **RI**, to formulae of the subsequence $F_1, \ldots, F_{i-1}$.

If there is a derivation for F in ML, then we write $\vdash_{\mathsf{ML}} F$, and we say that F is a *theorem* of ML, or that ML proves F.

Let $\Sigma \cup \{F\}$ be a set of modal formulae. A *modal derivation of F, from Σ, in ML*, is a finite sequence $F_1, \ldots, F_n$ of modal formulae, s.t.:

[3]Where $+$ is assumed left-associative.

- $F_n := F$,
- every F_i in the sequence is
  - either a theorem of ML,
  - or a member of Σ,
  - or the result of the application of one of the rules in **RI**, different from the necessitation rule, to formulae of the subsequence $F_1, \ldots, F_{i-1}$.

If there is a derivation of F, from Σ, in ML, then we write $\Sigma \vdash_{\mathsf{ML}} F$, and we say that F is *derivable in* ML*, from the premises* Σ.

Theorem 2.13 (Conservativity of Modal Logic)**.**
Any modal logic ML is a *conservative extension* of classical logic, CL.

Corollary 2.14 (Consistency of Modal Logic)**.** Any normal modal logic ML is consistent.

Theorem 2.15 (Deduction Theorem for Modal Logic)**.**
The deduction theorem holds for any normal modal logic.

2.2.2 Modal Semantics

There are various semantics given for modal logic. The most common one, are the *Kripke semantics* (aka *relational semantics*), named after Saul Aaron Kripke, which are the one given in this subsection.

Definition 2.16 (Kripke Frames & Models)**.** A *Kripke frame* is a structure $F = \langle W, R \rangle$, where

- W is a set of objects called *worlds* (or *states*),
- R is a binary relation on W, called *accessibility relation*.

A *Kripke model* is a structure $M = \langle W, R, V \rangle$, where $\langle W, R \rangle$ is a Kripke frame and

- $V : \mathsf{Prop} \to \mathcal{P}(W)$, called *valuation function*.

Given a Kripke model $M = \langle W, R, V \rangle$, we say that M is *based on* the Kripke frame $F = \langle W, R \rangle$, or that F is the Kripke frame *underlying* M. We also call each pair (M, w), where $M = \langle W, R, V \rangle$ is a Kripke model and w is a world in W, an *epistemic world* (or *epistemic state*).

In order to presume the different modal logics, defined in previous subsection, we have to establish some restrictions in the class of Kripke frames. It comes out that those restrictions must be applied on the accessibility relation of Kripke frames.

Definition 2.17 (Classes of Frames)**.** According to the properties of their accessibility relation, we define the following classes of frames:

- The class of *Kripke frames* is denoted $\mathcal{K}$.
- The class of *reflexive Kripke frames* is denoted $\mathcal{T}$.

  - R is *reflexive* iff for every w, it holds that $R(w, w)$.

- The class of *serial Kripke frame*s is denoted $\mathcal{KD}$.
  - R is *serial* iff for every w, exists some u s.t. $R(w, u)$.

- The class of *transitive Kripke frames* is denoted $\mathcal{K}4$.
  - R is *transitive* iff for every w, u, v, it holds that $R(w, u)$ and $R(u, v)$ implies $R(w, v)$.

- The class of *reflexive transitive Kripke frames* is denoted $\mathcal{S}4$.

- The class of *euclidean Kripke frames* is denoted $\mathcal{K}5$.
  - R is *euclidean* iff for every w, u, v, it holds that $R(w, u)$ and $R(w, v)$ implies $R(u, v)$.

- The class of *serial, transitive and euclidean Kripke frames* is denoted $\mathcal{KD}45$.

- The class of Kripke frames with *equivalence* accessibility relations is denoted $\mathcal{S}5$.

Definition 2.18 (Truth in Kripke Models)**.** Truth of modal formulae in Kripke models is interpreted on pairs (M, w), where $M = \langle W, R, V \rangle$ is an arbitrary Kripke model and w is some world in W. Specifically, we define that a formula $F \in \mathcal{L}_\Box$ is *true* (or *satisfied*) *in* (M, w), denoted as $M, w \models F$, as follows:

$$
\begin{array}{lcr}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \not\models \bot & & \\
M, w \models F \to G & \text{iff} & M, w \not\models F \text{ or } M, w \models G \\
M, w \models \Box F & \text{iff} & \forall u \in R[w] \;\; M, u \models F
\end{array}
$$

where

$$R[w] := \{v \in W \mid R(w, v)\}\,.$$

- We say that a modal formula F is *satisfiable* iff there is some Kripke model M and some world w in this model, s.t. F is true in (M, w).

- We say that a formula F is *satisfiable* in a certain class $\mathcal{C}$ of Kripke frames iff F is true in some epistemic world (M, w), where M is based on a frame that belongs in $\mathcal{C}$.

- We say that a modal formula F is *true* in some Kripke model M, and we denote $M \models F$ iff it is true in all worlds of M.

- We say that a modal formula F is *valid* in a certain class $\mathcal{C}$ of Kripke frames, and we denote $\mathcal{C} \models F$ iff F is true in every Kripke model M based on some model F belonging in $\mathcal{C}$.

- We say that a set of modal formulae Σ is *true* in some world w, and we denote $M, w \models \Sigma$ iff all members of Σ are true in w. We define the *truth* of Σ in a Kripke model M, and the *satisfiability* and *validity* of Σ in a class of Kripke frames $\mathcal{C}$, in the obvious way.

- We say that a modal formula F is a *semantic consequence* of a set of modal formulae Σ in a class of Kripke frames $\mathcal{C}$, and we denote $\Sigma \models_{\mathcal{C}} F$ iff for every model M based on some frame in $\mathcal{C}$, $M \models \Sigma$ implies $M \models F$.

Theorem 2.19 (Soundness & completeness for modal logic)**.** Axiom system K is sound and complete with respect to the semantic class $\mathcal{K}$, i.e., for every formula $F \in \mathcal{L}_{\Box}$, we have

$$\vdash_{\mathsf{K}} F \iff \mathcal{K} \models F.$$

$$\begin{array}{ll} \vdash_{\mathsf{K}} F \Rightarrow \mathcal{K} \models F & \text{soundness} \\ \vdash_{\mathsf{K}} F \Leftarrow \mathcal{K} \models F & \text{completeness} \end{array}$$

The same holds for T w.r.t. $\mathcal{T}$, for KD w.r.t. $\mathcal{KD}$, for $\mathsf{K4}$ w.r.t. $\mathcal{K}4$, for $\mathsf{S4}$ w.r.t. $\mathcal{S}4$, for $\mathsf{K5}$ w.r.t. $\mathcal{K}5$, for $\mathsf{S5}$ w.r.t. $\mathcal{S}5$ and for $\mathsf{KD45}$ w.r.t. $\mathcal{KD}45$.

Theorem 2.20 (Strong Soundness & completeness for modal logic)**.** Axiom system K is strongly sound and complete with respect to the semantic class $\mathcal{K}$, i.e., for every set of formulae $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\Box}$, we have

$$\Sigma \vdash_{\mathsf{K}} F \iff \Sigma \models_{\mathcal{K}} F.$$

$$\begin{array}{ll} \Sigma \vdash_{\mathsf{K}} F \Rightarrow \Sigma \models_{\mathcal{K}} F & \text{strong soundness} \\ \Sigma \vdash_{\mathsf{K}} F \Leftarrow \Sigma \models_{\mathcal{K}} F & \text{strong completeness} \end{array}$$

The same holds for T w.r.t. $\mathcal{T}$, for KD w.r.t. $\mathcal{KD}$, for $\mathsf{K4}$ w.r.t. $\mathcal{K}4$, for $\mathsf{S4}$ w.r.t. $\mathcal{S}4$, for $\mathsf{K5}$ w.r.t. $\mathcal{K}5$, for $\mathsf{S5}$ w.r.t. $\mathcal{S}5$ and for $\mathsf{KD45}$ w.r.t. $\mathcal{KD}45$.

2.3 Justification Logic

In this section we make an introduction to justification logic. Specifically, we define its syntax and we give some basic axiomatic systems, similar to those given for modal logic. We postpone the introduction of semantics for justification logics, to the next chapters.

The syntactic representation of justification logic is similar to the one of modal logic, with only difference the replacement of the modal operator $\Box$, with $t:$, where t belongs to a set of new objects, called *terms*, that are gonna represent evidences for the truth of some propositions. Therefore, we have to explicitly define what a term is.

Definition 2.21 (Justification Terms)**.** The terms are gonna be constructed by a countably infinite set Con of objects, called *constant terms*, that will be denoted by $c, d, e, \ldots$, $c_1, d_1, e_1, \cdots$, a countably infinite set Var of objects, called *variable terms*, that will be denoted by $x, y, z, \ldots, x_1, y_1, z_1, \ldots$, two binary function symbols $+, \cdot$ and a countably infinite set of function symbols, denoted by $f, g, h, \ldots, f_1, g_1, h_1, \ldots$, of arbitrary arity.

The set Tm of *justification terms* or simply *terms* is defined by the following BNF-notation:

$$t \ ::= \ c \mid x \mid t + t \mid t \cdot t \mid f\underbrace{(t, \cdots, t)}_{n \text{ terms}}$$

where c is a constant term, x is a variable term, f is a function symbol and n is the arity of f.

We assume that $+, \cdot$ are right-associative and that $\cdot$ has higher precedence than $+$ and we might omit the parentheses, accordingly. We usually use the symbols $t, s, r, \ldots, t_1, s_1, r_1, \ldots,$ to represent an arbitrary term. By the inductive definition of justification terms, we can now inductively define the formulae that constitute the justification language.

Definition 2.22 (Justification Language $\mathcal{L}_\mathsf{J}$)**.** The *justification language* $\mathcal{L}_\mathsf{J}$ of an arbitrary justification logic is defined by the following BNF-notation:

$$F \ ::= \ p \mid \bot \mid (F \to F) \mid t{:}F$$

where $p \in \mathsf{Prop}$ and $t \in \mathsf{Tm}$.

The other propositional connectives are defined as abbreviations, in the standard way (see Definition 2.4). Moreover, precedence and the associativity of the logical operators is similar to the one for modal logic (see Definition 2.5) and we might omit the parentheses, accordingly.

2.3.1 The Basic Justification Logic J_0

In this subsection, we are giving the basic justification logic J_0. The *basic justification logic* J_0 is analogous to the *basic normal modal logic* K. As in the case of K for the other normal modal logics, J_0 is groundwork for the construction of any other justification logic, and thus in a sense is the simplest justification logic.

The corresponding language for J_0 does not contain any function symbols, other than $+$ and $\cdot$. This also adds up to the view of J_0 as the simplest justification logic.

Definition 2.23 (The J_0 Logic)**.** The *basic justification logic* J_0 is given in Table 2.4.

Axiomatic Schemata		
all theorems of CL in $\mathcal{L}_\mathsf{J}$		**P**
$s{:}(F \to G) \to t{:}F \to s \cdot t{:}G$	application	**J**
$s{:}F \to s + t{:}F$ & $t{:}F \to s + t{:}F$	sum	**+**
Rules of Inference		
From F and $F \to G$, infer G	modus ponens	**MP**

Table 2.4: Axiomatic System J_0

It is worth pointing up that in J_0 there is not the rule of inference of necessitation. In order to approximate such a rule, we will introduce in the next subsection the notion of a *constant specification*.

2.3.2 Arbitrary Justification Logic

The basic justification logic J_0 is indeed a very simple logic, but it is very useful as every other justification logic is constructed as an expansion of J_0. In particular, someone can expand J_0 by the addition of new axiom schemes, but also by the introduction of some constant specification.

The former approach is similar to the extension of K, to some other normal modal logic. We equip J_0 with some new additional axiom schemes, while concurrently we add in its language suitable function symbols. Some additional, commonly used justification axiom schemes are given in Table 2.5. It is easy to observe that those axiom schemes correspond to the additional modal axiom schemes we introduced in Table

2.3. Schematically, the justification axiom scheme **JA** corresponds to the modal axiom scheme **A**, where **A** an arbitrary modal axiom scheme. For any modal logic ML we call the justification logic that consists of the corresponding justification axioms of the modal axioms of ML, the *correlated* justification logic of ML.

$t{:}F \to F$	truth axiom	**JT**
$\neg t{:}\bot$	consistent justification	**JD**
$t{:}F \to\ !t{:}t{:}F$	positive introspection	**J4**
$\neg t{:}F \to\ ?t{:}\neg t{:}F$	negative introspection	**J5**

Table 2.5: Other Justification Axiom Schemes

We observe that with the addition of the axiom scheme **J4** or **J5** to J_0, we have to equip the language of J_0 with the unary function symbols ! or ?, respectively. This is very common for this kind of extensions and accounts for the existence of the function symbols for the justification terms.

Accordingly, to the previous axioms we define the following commonly used justification logics.

Definition 2.24. We define the following justification axiom systems (or simpler justification logics):

$$\begin{aligned} \mathsf{JT} &:= \mathsf{J}_0 + \mathbf{JT} \\ \mathsf{JD} &:= \mathsf{J}_0 + \mathbf{JD} \\ \mathsf{J4} &:= \mathsf{J}_0 + \mathbf{J4} \\ \mathsf{LP} &:= \mathsf{JT} + \mathbf{J4} \\ \mathsf{J5} &:= \mathsf{J}_0 + \mathbf{J5} \\ \mathsf{JD45} &:= \mathsf{JD} + \mathbf{J4} + \mathbf{J5} \end{aligned}$$

It is worth mentioning that accordingly, to the usual notation for the other justification logics, logic LP is actually the logic JT4. For historical reasons we use the LP notation.

We will denote by JL any arbitrary extension of J_0 that is constructed by the addition of action schemes and suitable function symbols. It is time to define the notion of constant specification.

Definition 2.25 (Constant Specification)**.** Given an arbitrary justification logic JL, a *constant specification* CS *for* JL is a set of formulae s.t.:

- its elements are of the form

$$F := c_n{:}c_{n-1}{:}\dots{:}c_1{:}A,$$

  where $n \geq 1$, $\{c_i\}_{i\in[n]} \subset \mathsf{Con}$ and A is an axiom of JL,

- if $c_n{:}c_{n-1}{:}\dots{:}c_1{:}A \in \mathsf{CS}$, then $c_{n-1}{:}\dots{:}c_1{:}A \in \mathsf{CS}$, where $n \geq 2$.

There are categories of constant specifications of special interest, respectively of the notion of reasoning that someone wants to work out. For our purposes, we will explicitly define some of those categories.

Definition 2.26 (Categories of Constant Specification)**.**

- A constant specification, CS, for some justification logic JL is the *empty constant specification* iff $\mathsf{CS} = \emptyset$.
- A constant specification, CS, for some justification logic JL is called *axiomatically appropriate*, if for any axiom A of JL and any $n > 0$ there are $c_1, \ldots, c_n \in$ Con s.t. $c_n{:}c_{n-1}{:} \ldots {:}c_1{:}A \in \mathsf{CS}$.
- A constant specification, CS, for some justification logic JL is called *total*, if for any axiom A of JL, any $n > 0$ and any $c_1, \ldots, c_n \in \mathsf{Con}$ it holds that $c_n{:}c_{n-1}{:}$ $\ldots{:}c_1{:}A \in \mathsf{CS}$.
- A constant specification, CS, for some justification logic JL is called *schematic*, if for any A, A' instances of the same axiom scheme, and for any constant term $c\mathsf{Con}$, it holds
$$c{:}A \in \mathsf{CS} \Leftrightarrow c{:}B \in \mathsf{CS}.$$

Note, that the empty constant specification is trivially a constant specification for any justification logic, JL.

We can now define the extension of an arbitrary justification logic JL, with some constant specification CS, i.e., the extension of J_0 with both axiom schemes (probably none) and constant specification (probably empty).

Definition 2.27. Let arbitrary justification logic JL (probably J_0). Let CS a constant specification for JL. We define by $\mathsf{JL}(\mathsf{CS})$ and say *the justification logic* JL *with constant specification* CS, the logic JL with the addition of all formulae in CS as axioms.

Note that if CS is the empty constant specification, then justification logic JL, without a constant specification and justification logic $\mathsf{JL}(\mathsf{CS}) = \mathsf{JL}(\emptyset)$ are essentially the same and therefore we may denote $\mathsf{JL}(\emptyset) := \mathsf{JL}$.

We have to give the definition of derivation in any justification logic $\mathsf{JL}(\mathsf{CS})$.

Definition 2.28 (Derivations in Justification Logic)**.** Let $\mathsf{JL}(\mathsf{CS})$ be an arbitrary justification logic JL with corresponding constant specification CS and set of axiom schemes **AX**. Let $\Sigma \cup \{F\}$ a set of justification formulae in the logic of JL. A *derivation of* F, *from* Σ, *in* $\mathsf{JL}(\mathsf{CS})$, is a finite sequence $F_1, \ldots, F_n$ of formulae, s.t.:

- $F_n := F$,
- every F_i in the sequence is
  - either an axiom in **AX**,
  - or a member of CS,
  - or a member of Σ,
  - or the result of the application of modus ponens, to formulae of the subsequence $F_1, \ldots, F_{i-1}$.

If there is a derivation of F, from Σ, in $\mathsf{JL}(\mathsf{CS})$, then we write $\Sigma \vdash_{\mathsf{JL}(\mathsf{CS})} F$, and we say that F is *derivable* in $\mathsf{JL}(\mathsf{CS})$, from the premises Σ. If Σ is the empty set, i.e., we do not assume any premises, then we write $\vdash_{\mathsf{JL}(\mathsf{CS})} F$. We also say that F is a theorem of $\mathsf{JL}(\mathsf{CS})$ is $\vdash_{\mathsf{JL}(\mathsf{CS})} F$, i.e., F is derivable in $\mathsf{JL}(\mathsf{CS})$, without premises.

We write $\vdash_{\mathsf{JL}} F$, if there is for JL an axiomatically appropriate constant specification CS s.t. $\vdash_{\mathsf{JL}(\mathsf{CS})} F$.

We have again to show that the justification logic is consistent. In this direction, we can once again give a conservativity theorem.

Theorem 2.29 (Conservativity of Justification Logic)**.**
Any justification logicJL is a *conservative extension* of classical logic, CL.

Corollary 2.30 (Consistency of Justification Logic)**.** Any justification logic $\mathsf{JL}(\mathsf{CS})$ is consistent.

Finally, we give the corresponding deduction theorem for justification logic.

Theorem 2.31 (Deduction Theorem for Justification Logic)**.**
The deduction theorem holds for any justification logic.

It is worth mentioning that the only rule of inference in justification logic is modus ponens. Therefore, for the justification derivation from premices, there is no need for the existence of restrictions, such as the prohibition of the necessitation rule in the modal derivation from premises, and thus no need for the distinction of the definition of justification derivation from premises from the one for justification derivation (without premises).

The reason for the existence of this prohibition in the modal case, is as in modal derivation it is meaningful and thus it holds that

$$\vdash_A F \Rightarrow \vdash_A \Box F,$$

while in modal derivation with premises, this property is meaningless, and thus does not hold, i.e., for a set of modal formulae $\Sigma \cup \{F\}$

$$\Sigma \vdash_A F \not\Rightarrow \Sigma \vdash_A \Box F.$$

In order to be convinced that this property should not hold for the modal derivation from premises, it suffices to observe that in such derivations we assume that the formulae in Σ are true, but not necessary known, as there is not given a a specific reason of why the premises are true, and therefore, we cannot apply the necessitation rule to them. Moreover, it is worth mentioning that if we allowed the necessitation rule in the modal derivation by premises, then the deduction theorem of modal logic, i.e., Theorem 2.15, should not hold. Indeed, if the deduction theorem was holding then we would prove that for any formula $F \in \mathcal{L}_\Box$, the formula $F \to \Box F$ is a theorem of the logic; i.e., we know everything that is true.

We have already mentioned that the role of the necessitation rule for justification logic is played by the constant specification, i.e., the constant specification determines on which justification axioms a sort of necessitation rule is applicable. Of course the necessitation rule for modal logic is not only applicable on modal axioms, but on any modal thoerem. Is there a corresponding property for necessitation rule of the modal derivation (without premises) for the justification logic given some constant specification? The answer to this question is given in the next definition.

Definition 2.32 (Internalization)**.** Let a justification logic JL with some corresponding constant specification CS.
JL has the *internalization property relative to* constant specification CS iff

$$\vdash_{\mathsf{JL}(\mathsf{CS})} F \Rightarrow (\exists t \in \mathsf{Tm}) \left[\vdash_{\mathsf{JL}(\mathsf{CS})} t{:}F\right].$$

JL has the *strong internalization property relative to* constant specification CS iff t is constructed only by justification constants or function symbol $\cdot$. Such terms are called *ground*.

It is straight forward to observe that the internalization property corresponds, at least in a syntactic way, to the necessitation axiom rule of modal logic. Accepting the epistemic interpretation of modal logic, the necessitation rule states that any tautology of modal logic is known to be true. This interpretation is probably distant from our intuition of knowledge, as we can imagine of some complex modal tautologies that are hard to be identified as true and thus as known. For instance, if the formulae of modal logic are considered to interpret mathematical propositions and therefore the modal tautologies interpret exactly the mathematical truths, it is unthinkable for us to consider any mathematical truth as known. On the other hand, the notion of internalization relates to the folklore belief of mathematicians that for any true proposition there is a proof that justifies it. From Gödel's incompleteness theorems [46] we know that this belief is not true at all. This accounts for the existence of justification logics, without the internalization property. For instance, if the justification formulae interpret the propositions of Peano arithmetic and the terms interpret the possible corresponding proofs, this justification logic should not have the internalization property.

Lemma 2.33 (Lifting Lemma)**.** Let JL be a justification logic that has the internalization property relative to some constant specification CS. Then it holds that

$$F_1, \ldots, F_n \vdash_{\mathsf{JL(CS)}} F \Rightarrow (\forall t_1, \ldots, t_n \in \mathsf{Tm})\,(\exists t \in \mathsf{Tm})\,\big[t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{JL(CS)}} t{:}F\big].$$

Theorem 2.34. If CS an axiomatically appropriate constant specification for JL then JL has the strong internalization property relative to constant specification CS

The name internalization comes from the fact that any justification logic $\mathsf{JL(CS)}$ that admits the internalization property, *internalizes* the steps of the proof of each theorem, i.e., it keeps track of all the application of **J** axiom, that is the steps that we applied modus ponens in our justifications. If it has the strong internalization property, and the constant specification has different terms for different axioms, then it makes clear exactly which formulae of the constant specification were used and how. Therefore, terms with different structure which justify the same justification formula, evince different derivations.

Example 2.35. *Let as assume* CS *an axiomatically appropriate constant specification for* J_0. *Then the ECQ is justified in* $\mathsf{J}_0(\mathsf{CS})$, *i.e., for any justification formula* $F \in \mathcal{L}_\mathsf{J}$, *there is some term* $t \in \mathsf{Tm}$, *s.t.*

$$t{:}(\bot \to F).$$

Proof. We know from Lemma 2.33 that $\mathsf{J}_0(\mathsf{CS})$ has the strong internalization property. We want to show that the tautology $\bot \to F$, where $F \in \mathcal{L}_\mathsf{J}$ is justified in $\mathsf{J}_0(\mathsf{CS})$.

$F_1 : c_1{:}((F \to \bot \to F) \to \bot \to F)$	CS, **P**
$F_2 : c_2{:}(F \to \bot \to F)$	CS, **P**
$F_3 : c_1{:}((F \to \bot \to F) \to \bot \to F) \to c_2{:}(F \to \bot \to F) \to c_1 \cdot c_2{:}(\bot \to F)$	**J**
$F_4 : c_1 \cdot c_2{:}(\bot \to F)$	**MP** 1, 2, 3

Where we assumed that : $c_1{:}((F \to \bot \to F) \to \bot \to F)\,, c_2{:}(F \to \bot \to F) \in \mathsf{CS}$, which makes sense as $(F \to \bot \to F) \to \bot \to F$ and $F \to \bot \to F$ are propositional tautologies and CS is axiomatically appropriate.

The term $c_1 \cdot c_2$ evinces that we applied one time the application axiom **J**, so that from $F \to \bot \to F$ and $(F \to \bot \to F) \to \bot \to F$, we concluded $\bot \to F$.

A different derivation could be by getting in one step $c{:}(\bot \to F)$, which belongs in CS as $\bot \to F$ is a propositional tautology and CS is axiomatically appropriate. In this derivation we did not apply the application axiom **J**. ☐

2.4 Realization

As we have stated multiple times until now, modal logic and justification logic are closely related to each other. For each modal logic ML, there is a corresponding justification logic JL, that seems to perceive similar notions. This loose and informal observation corresponds to major result in justification logic, namely, the *realization theorem*.before stating this theorem, it is necessary to give some definitions.

Definition 2.36 (Forgetful Functor)**.** We recursively define the *forgetful functor* $^\circ: \mathcal{L}_\mathsf{J} \to \mathcal{L}_\Box$, as follows: For any atomic proposition $p \in \mathsf{Prop}$, any justification formulae $F, G \in \mathcal{L}_\mathsf{J}$ and any term $t \in \mathsf{Tm}$

- $p^\circ := p$
- $(\neg F)^\circ := \neg F^\circ$
- $(F \to G)^\circ := F^\circ \to G^\circ$
- $(t{:}F)^\circ := \Box F$

where for any formula $H \in \mathcal{L}_\mathsf{J}$, we denote $H^\circ := {}^\circ(H)$.

As it is clear from Definition 2.36, the forgetful functor $^\circ$ translates each justification formula F,to the modal formula F°, in which all occurrences of justification term with the corresponding ":" are replaced by the modal operator $\Box$.

Definition 2.37 (Counterparts)**.** Let arbitrary normal modal logic ML. Let also arbitrary justification logic JL.
JL is a *counterpart* of ML if the following holds:

- if F is a theorem of JL then F° is a theorem of ML,
- if G is a theorem of ML, then there are a (generally axiomatically appropriate) constant specification CS for JL and a theorem F of JL(CS), s.t. $F^\circ := G$.

Definition 2.38 (Realization)**.** Let arbitrary modal language ML and arbitrary justification logic JL, with corresponding constant specification CS. Let also F arbitrary theorem of JL(CS) and G arbitrary theorem of ML.

- F is a *realization* of G iff $F^\circ = G$.
- F is a *normal realization* of G iff F is a realization of G and every negative occurrence of $\Box$ in G is replaced by a distinct justification variable in F; while every positive occurrence of $\Box$ in G is replaced by a justification term that need not be variable.

Theorem 2.39 (Realization Theorem)**.** Let arbitrary modal language ML, and JL its correlated justification logic. Let also CS an axiomatically appropriate constant specification for JL. Then,

- for every theorem G of ML there is a normal realization F of G in JL.
- JL is a counterpart of ML.

CHAPTER 3

SEMANTICS FOR JUSTIFICATION LOGIC

In the previous chapter, we defined the language of justification logic $\mathcal{L}_{\mathsf{J}}$ and gave the axiomatic systems for the predominant justification logics. Explicitly, we defined the basic justification logic J_0 and extended it to the other justification logics by the addition of some axiomatic schemes or the introduction of some constant specification.

In this chapter, we introduce corresponding semantics for these justification logics. In particular, we will introduce the *basic models*, the *Mkrtychev models*, the *Fitting models* and finally the *Modular Models*.

3.1 Basic Models

In this section we define the *basic model semantics*. They are considered the simplest semantics for justification logic. Even though, the simplicity of their structure, they are, in comparison with other semantics, capable of capturing the ontological perspective of the notion of justification. That is, they are giving a conception of what a justification for some statement is. Let us give the definition of this structure.

Definition 3.1 (Basic Models)**.** A *basic model*, usually denoted by $*$ is a function $*\colon \mathsf{Prop} \cup \mathsf{Tm} \to \{0,1\} \cup \mathcal{P}(\mathcal{L}_{\mathsf{J}})$, that interprets each atomic proposition in Prop, to a truth value in $\{0,1\}$ and each term in Tm to a subset of formulae in $\mathcal{L}_{\mathsf{J}}$, i.e.,

$$* \restriction_{\mathsf{Prop}} \colon \mathsf{Prop} \to \{0,1\}$$

and

$$* \restriction_{\mathsf{Tm}} \colon \mathsf{Tm} \to \mathcal{P}(\mathcal{L}_{\mathsf{J}}).$$

We usually, use the notation $x^* := *(x)$, where $x \in \mathsf{Prop} \cup \mathsf{Tm}$.

As basic models are meant to be a semantical object for justification logics, we have to define the notion of truth on any justification formula, in any basic model.

Definition 3.2 (Truth in Basic Models)**.** Let arbitrary basic model $*$. We say that the justification formula $F \in \mathcal{L}_{\mathsf{J}}$ is true (or satisfied) in the basic model $*$, denoted as $\models_* F$

or $* \models F$, as follows:

$$\begin{array}{lcr} \bot^* = 0 & & \\ * \models p & \text{iff} & p^* = 1 \\ * \models F \to G & \text{iff} & * \not\models F \text{ or } * \models G \\ * \models t{:}F & \text{iff} & F \in t^*. \end{array}$$

In this sense we expand the notation F^*, for arbitrary formula $F \in \mathcal{L}_{\mathsf{J}}$, as follows

$$\begin{array}{lcr} F^* = 1 & \iff & * \models F \\ F^* = 0 & \iff & * \not\models F. \end{array}$$

- We say that a justification formula F is *satisfiable* iff there is some basic model $*$, s.t. F is true in $*$.
- We say that a justification formula F is *satisfiable* in a certain class $\mathcal{C}$ of basic models iff F is true in some $* \in \mathcal{C}$.
- We say that a justification formula F is *valid* in a certain class $\mathcal{C}$ of basic models, and we denote it by $\mathcal{C} \models F$ iff F is true in every model $* \in \mathcal{C}$.
- For any set of justification formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{J}}$, we say that a basic model $*$, is a *basic model for* Σ, and we denote it by $* \models \Sigma$ or $\models_* \Sigma$ iff any formulae in Σ is true in $*$. We define the *satisfiability* and *validity* of Σ in a class of basic models $\mathcal{C}$, in the obvious way.
- We denote by $BM(\Sigma)$, the class of basic models for Σ, i.e.,

$$BM(\Sigma) := \{* \mid * \text{ is a basic model s.t. } * \models \Sigma\}.$$

- We say that a justification formula F is a *semantic consequence* of a set of formulae Σ in a class of basic models $\mathcal{C}$, and we denote it by $\Sigma \models_{\mathcal{C}} F$ iff for every basic model $* \in \mathcal{C}$, if $* \models \Sigma$ holds, then $* \models F$ also holds, or equivalently

$$BM(\Sigma) \subseteq BM(\{F\}).$$

By the definition above, it is clear how the basic models are focused on the ontological perspective of the justifications. In order for a formula of the form $t{:}F$ to be true, it is sufficient for the formula to belong to the set of formulae that are considered as justified by t. There is not any consideration about the truth of the formula F. The previous evinces that basic models are not attached with the truth of the statement that is considered to be justified. In contrast, basic models are concerned whether an argument t is indeed a justification of some statement F, regardless the truth of F.

Every justification logic is usually equipped with some constant specification[1]. Therefore, it is convenient to explicitly define the class of basic models that respect an arbitrary constant specification.

Definition 3.3. Let CS a constent specification. We call a basic model $*$, *basic* CS-*model* iff $*$ is a basic model for the set of formulae CS, i.e., $* \models \mathsf{CS}$, or equivalently $* \in BM(\mathsf{CS})$.

[1]Of course, this constant specification might be empty.

It is not hard to observe that an arbitrary basic model, as has been defined until now, does not respect necessary any of the justification axiom schemes that we have defined; not even axiom schemes **J** and $+$! Therefore, in order to define the corresponding basic model semantics for the different justification logics, we have to restrict the set of basic models, to those that respect the corresponding justification axiom schemes. We achieve this by introducing some closure conditions for the basic models, respectively to the justification axiom schemes that we want them to respect. In the following definition we give the closure conditions for each axiom scheme that we have already given.

Definition 3.4. We define, for any sets of justification formulae $X, Y \subseteq \mathcal{L}_{\mathsf{J}}$, by $X \triangleright Y$, the set

$$X \triangleright Y := \{G \mid (\exists F \in \mathcal{L}_{\mathsf{J}})\,[F \to G \in X \ \&\ F \in Y]\}\,.$$

The *closure conditions for basic models*, for the justification axiom schemes, which we have defined, are given in Table 3.1.

Axiom Schemes	Closure Conditions
J	$s^* \triangleright t^* \subseteq (s \cdot t)^*$
$+$	$s^* \cup t^* \subseteq (s + t)^*$
JT	$F \in t^* \Rightarrow * \models F$
JD	$\bot \notin t^*$
J4	$F \in t^* \Rightarrow t{:}F \in (!t)^*$
J5	$F \notin t^* \Rightarrow \neg t{:}F \in (?t)^*$

Table 3.1: Axiom Schemes & Closure Conditions for Basic Models

The first two closure conditions, i.e., the **J** and $+$ closure condition, are also called *minimum closure conditions* for basic models, as they are necessary for J_0 and thus for any other justification logic.

In order for the basic models to be proper semantics for justification logic, we have to show that all the formulae that are valid in some justification logic are true in the corresponding class of basic models and vice versa, i.e., we have to show the soundness and completeness of each justification logic, with respect to some class of basic models.

Theorem 3.5 (Soundness and Completeness for Basic Models)**.**

- Let CS a constant specification for J_0.

  $BM(\mathsf{J}_0(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J** and $+$ closure conditions, i.e., $* \in BM(\mathsf{J}_0(\mathsf{CS}))$ iff

  $$\forall s, t \in \mathsf{Tm} \begin{cases} s^* \triangleright t^* \subseteq (s \cdot t)^* \\ s^* \cup t^* \subseteq (s + t)^* \end{cases} \quad \& \quad * \models \mathsf{CS}.$$

  The basic justification logic $\mathsf{J}_0(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{J}_0(\mathsf{CS}))$, i.e.,

  $$\vdash_{\mathsf{J}_0(\mathsf{CS})} F \iff BM(\mathsf{J}_0(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JT.

  $BM(\mathsf{JT}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **JT** closure conditions, i.e., $* \in BM(\mathsf{JT}(\mathsf{CS}))$ iff

  $$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,(\forall F \in \mathcal{L}_{\mathsf{J}})\,[F \in t^* \Rightarrow * \models F]\,.$$

The justification logic $\mathsf{JT}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{JT}(\mathsf{CS}))$, i.e.,

$$\vdash_{\mathsf{JT}(\mathsf{CS})} F \iff BM(\mathsf{JT}(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JD.

  $BM(\mathsf{JD}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **JD** closure conditions, i.e., $* \in BM(\mathsf{JD}(\mathsf{CS}))$ iff

  $$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,[\bot \notin t^*]\,.$$

  The justification logic $\mathsf{JD}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{JD}(\mathsf{CS}))$, i.e.,

  $$\vdash_{\mathsf{JD}(\mathsf{CS})} F \iff BM(\mathsf{JD}(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JT.

  $BM(\mathsf{J4}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **J4** closure conditions, i.e. $* \in BM(\mathsf{J4}(\mathsf{CS}))$ iff

  $$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,(\forall F \in \mathcal{L}_{\mathsf{J}})\,\left[F \in t^* \Rightarrow t{:}F \in (!t)^*\right].$$

  The justification logic $\mathsf{J4}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{J4}(\mathsf{CS}))$, i.e.,

  $$\vdash_{\mathsf{J4}(\mathsf{CS})} F \iff BM(\mathsf{J4}(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JT.

  $BM(\mathsf{J5}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **J5** closure conditions, i.e., $* \in BM(\mathsf{J5}(\mathsf{CS}))$ iff

  $$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,(\forall F \in \mathcal{L}_{\mathsf{J}})\,\left[F \notin t^* \Rightarrow \neg t{:}F \in (?t)^*\right].$$

  The justification logic $\mathsf{J5}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{J5}(\mathsf{CS}))$, i.e.,

  $$\vdash_{\mathsf{J5}(\mathsf{CS})} F \iff BM(\mathsf{J5}(\mathsf{CS})) \models F.$$

The $BM(\mathsf{JL}(\mathsf{CS}))$ class, for arbitrary justification logic JL and corresponding constant specification CS is defined accordingly.

In the following example it will be clear that hyperintensionality holds for justification logic. The hyperinsionality of justification logic supports the justification logic as a proper basis for the epistemic/logic, as we have analyzed in subsection 1.2.1.

Example 3.6. *Justification is hyperintensional, i.e., there are $F, G \in \mathcal{L}_{\mathsf{J}}$ and $t \in \mathsf{Tm}$, s.t.*

$$\vdash_{\mathsf{JL}(\mathsf{CS})} F \leftrightarrow G \not\Rightarrow \vdash_{\mathsf{JL}(\mathsf{CS})} t{:}F \leftrightarrow t{:}G,$$

where CS *a constant specification for justification logic* JL.

Proof. We will prove it for J_0. Let some atomic proposition $p \in \mathsf{Prop}$ and some constant term $c \in \mathsf{Tm}$. We define a basic model $*$, s.t.

- $c^* := \{p\}$ and
- $t^* := \mathcal{L}_{\mathsf{J}}$, for any $t \in \mathsf{Tm} \setminus \{c\}$.

Clearly, $*$ is a basic model. We want to show that $*$ respects **J** and $+$ closure conditions. But, for any terms $t, s \in \mathsf{Tm}$, we have that $s+t, s\cdot t \neq c$, and thus $(s+t)^* = (s\cdot t)^* = \mathcal{L}_{\mathsf{J}}$. Therefore, the subset relation requirements trivially hold. Hence, $*$ is a J_0-basic model.

Moreover, from the definition of truth in basic models, we have that $* \models p \leftrightarrow p \wedge p$. But clearly, $p \in c^*$, hence $* \models c{:}p$, while $p \wedge p \notin c^*$, thus $* \not\models c{:}(p \wedge p)$. Therefore, $* \not\models c{:}p \leftrightarrow c{:}(p \wedge p)$. By completeness of justification logic in respect to basic models; i.e., Theorem 3.5, we have that

$$\vdash_{\mathsf{J}_0} (p \leftrightarrow p \wedge p) \quad \& \quad \nvdash_{\mathsf{J}_0} (c{:}p \leftrightarrow c{:}(p \wedge p))$$

□

3.2 Mkrtychev Models

All the closure conditions for basic models, that we have defined, other than the closure condition for **JT**, establish requirements only for the interpretation on Tm ($*\restriction_{\mathsf{Tm}}$). Therefore, we only have syntactic restrictions. On the other hand, the closure condition for **JT** simultaneously determines the interpretation on both Tm and Prop ($*\restriction_{\mathsf{Tm}}$, $*\restriction_{\mathsf{Prop}}$), by demanding

$$F \in t^* \Rightarrow * \models F.$$

This simultaneous restrictions on syntactic and semantic maters of the basic models breed problems, as there are potentially infinite things to check in order to ensure that the closure condition for **JT** holds. In particular, in order to ascertain that an arbitrary basic model $*$ belongs in $BM(\mathsf{JT})$, we have to assure that for any term $t \in \mathsf{Tm}$, any formula in t^* is also true in $*$. Therefore, the requirements for the definition of $*\restriction_{\mathsf{Tm}}$ is determined by the definition of $*\restriction_{\mathsf{Prop}}$.

Some fruitful semantic structures that help us overcome these difficulties are the *Mkrtychev models*. These were actually the first non-arithmetical semantics given for some justification logic. They were introduced by Mkrtychev, at 1997 in [6], in order to give semantics for the LP justification logic and establish decidability for infinite constant specifications.

Definition 3.7 (Mkrtychev models)**.** A *Mkrtychev model*[2] is a structure $*$ similar to $BM(\mathsf{J}_0)$-models, with only difference, that the truth assignment for formulae of the form $t{:}F$ is defined as

$$* \models t{:}F \iff F \in t^* \;\&\; * \models F.$$

It is not hard to observe that the Mkrtychev models, indeed, overcome the previous difficulties, as for any Mkrtychev model $*$, its restrictions $*\restriction_{\mathsf{Tm}}$ and $*\restriction_{\mathsf{Prop}}$ on Tm and Prop respectively, are independently defined. Specifically, there is not any use of the closure condition for **JT**, which is satisfied as a matter of the truth evaluation in Mkrtychev models and specifically, the truth on formulae of the form $t{:}F$. Thus, Mkrtychev models correspond to $BM(\mathsf{JT})$-models, without the need of a closure condition for **JT**. As expected, the following theorem gives this relation of Mkrtychev and $BM(\mathsf{JT})$ models.

[2] In fact this definition of Mkrtychev models is closely related with the definition of *pre-models* in [6]. In his paper, Mkrtychev gave two basic definitions, one of the *models*, which are in fact the $BM(\mathsf{LP})$-models; and one of the *pre-models*, which are exactly the Mkrtychev models, with the addition of closure condition for **J4**. He proved that those two definitions are equivalent, in the sense that they satisfy the same formulae (cf. Theorem 3.8).

Theorem 3.8. Every $BM(\mathsf{JT})$-model is a Mkrtychev model.

For every Mkrtychev model $*$, there is a basic model $*' \in BM(\mathsf{JT})$, s.t. for any formula $F \in \mathcal{L}_{\mathsf{J}}$

$$* \models F \iff *' \models F.$$

As a consequence of Theorems 3.5 and 3.8, soundness and completeness for JL justification logics, that contain the axiom scheme **JT**, with respect to Mkrtychev models comes for free.

Corollary 3.9 (Soundness and Completeness for Mkrtychev models)**.** Let CS a constant specification for JT.
JT(CS) is sound and complete with respect to Mkrtychev models.

Let CS a constant specification for LP.
JT(CS) is sound and complete with respect to Mkrtychev models that satisfies the **J4**-closure condition for basic models, i.e.,

$$F \in t^* \Rightarrow t{:}F \in (!t)^* \, .$$

The philosophical drawback of Mkrtychev models versus the basic models is that the truth of a formula of the form $t{:}F$, which is meant to express the fact that the argument t is a proof of the truth of proposition F, it demands also that the proposition F, is indeed true. Therefore, it is not possible to express the correctness of an argumentation, independently of the truth of the proposition that it is meant to be proven. As we will see in the next section, a similar drawback holds for the Fitting models, too.

3.3 Fitting models

As we have seen in the previous chapter, justification logic is closely related with modal logic. Each one of the modal axioms that we have defined, has a corresponding justification axiom. In fact there is the notion of *realization* which, in a sense, states that for each theorem of some of the basic modal logics, that we have defined, there is a corresponding justification theorem of the corresponding justification logic.

Nonetheless, the justification semantics that we have given, until now, seems to be irrelevant to the modal semantics we have given, i.e., the Kripke models. In this section, we define the *Fitting models*, some justification semantics, closely related with Kripke models.

Definition 3.10 (Fitting Models)**.** A *Fitting model* for justification logic is a structure $M = \langle W, R, V, E \rangle$, where $\langle W, R, V \rangle$ is a Kripke model and

- $E\colon \mathsf{Tm} \times \mathcal{L}_{\mathsf{J}} \to \mathcal{P}(W)$, called *evidence function*.

Given a Fitting model $M = \langle W, R, V, E \rangle$, we say that M is *based on* the Kripke frame $F = \langle W, R \rangle$, or that F is the frame *underlying* M.

The structure of the Fitting models is similar to the structure of Kripke models, with only difference the addition of the evidence function in it. We expect that definition of truth in Fitting models to be similar to the one for Kripke models. Of course, the definition of truth in Fitting models, is given on justification formulae, thus we have to modify properly the truth on modal formulae of the form $\Box F$ in Kripke models, to the truth of justification formulae of the form $t{:}F$, in Fitting models. The following definition explicitly shows how this is done.

Definition 3.11 (Truth in Fitting Models)**.** Truth of justification formulae in Fitting models is interpreted on pairs (M, w), where $M = \langle W, R, V, E\rangle$ is an arbitrary Fitting model and w is some world in W. Specifically, we define that a formula $F \in \mathcal{L}_\mathsf{J}$ is true (or satisfied) in (M, w), denoted as $M, w \Vdash F$, as follows:

$$
\begin{array}{lll}
M, w \Vdash p & \text{iff} & w \in V(p) \\
M, w \nVdash \bot & & \\
M, w \Vdash F \to G & \text{iff} & M, w \nVdash F \text{ or } M, w \Vdash G \\
M, w \Vdash t{:}F & \text{iff} & \begin{cases} \textit{Modal Condition}: & \forall u \in R[w]\ \ M, u \Vdash F \\ \textit{Evidence Condition}: & w \in E(t, F) \end{cases}
\end{array}
$$

- We say that a justification formula F is *satisfiable* iff there is some Fitting model M and some world w in this model, s.t. F is true in (M, s).
- We say that a justification formula F is *satisfiable* in a certain class $\mathcal{C}$ of Kripke frames iff F is true in some (M, w), where M is a Fitting model based on a frame that belongs in $\mathcal{C}$.
- We say that a justification formula F is *true* in a Fitting model M, and we denote $M \Vdash F$ iff it is true in all worlds of M.
- We say that a justification formula F is *valid* in a certain class $\mathcal{C}$ of Kripke frames, and we denote $\mathcal{C} \Vdash F$ iff F is true in every Fitting model M based on some model F belonging in $\mathcal{C}$.
- We say that a set of justification formulae Σ is *true* in some world w, and we denote $M, w \Vdash \Sigma$ iff all members of Σ are true in w. We define the *truth* of Σ in a Fitting model M, and the *satisfiability* and *validity* of Σ in a class of Kripke frames $\mathcal{C}$, in the obvious way.
- We say that a justification formula F is a *semantic consequence* of a set of formulae Σ in a class of Kripke frames $\mathcal{C}$, and we denote $\Sigma \Vdash_{\mathcal{C}} F$ iff for every Fitting model M based on some frame in $\mathcal{C}$, $M \Vdash \Sigma$ implies $M \Vdash F$.

We express the truth of a justification formula in Fitting models with $\Vdash$, in order to distinguish it from the truth in Kripke models, that is denoted by $\models$. From the definition above we have that in order for a justification formula of the form $t{:}F$ to be true, both *modal* and *evidence* condition must be fulfilled. Specifically, in order for an argumentation $t{:}F$ to be considered true, it must be the case that in every world that we consider possible (i.e., $u \in R[w]$) proposition F is true, but simultaneously, the argument t must be considered as evidence of F (i.e., $w \in E(t, F)$). This seems to comprehend the same philosophical drawback as Mkrtychev models. That is, we cannot consider an argumentation $t{:}F$ as true, unless we are convinced that F is true, regardless the argument t.

In Definition 3.12 we give two conditions for Fitting models, that are often considered useful in the literature.

Definition 3.12. Let arbitrary Fitting model $M = \langle W, R, V, E \rangle$.
We say that M is *fully explanatory* iff for any $w \in W$ and any justification formula $F \in \mathcal{L}_{\mathsf{J}}$, it holds that

$$(\forall u \in R[w])\,[M, u \Vdash F] \Rightarrow (\exists t \in \mathsf{Tm})\,[M, w \Vdash t{:}F]\,.$$

We say that M has a *strong evidence function* iff for any $w \in W$ and any justification formula $F \in \mathcal{L}_{\mathsf{J}}$, it holds that

$$(\exists t \in \mathsf{Tm})\,[w \in E(t, F)] \Rightarrow (\forall u \in R[w])\,[M, u \Vdash F]\,.$$

From the definition above it is clear that if a Fitting model is fully explanatory, then in any world $w \in W$ that we are in, if a proposition F is true in any world that we consider possible, then there must be an argument t, that is a proof of this proposition. This has a clear philosophical aspect as, in a sense, it says that for any statement that we consider true, there is (even if we do not know it), a proof of this statement. On the other hand, if a Fitting model has a strong evidence function then any argument t, that justifies a statement F, is sufficient to evince the belief on F. If a Fitting model has a strong evidence function then its evidence function is "strong" enough to completely determine the truth of any formula of the form $t{:}F$.

Likewise the basic models, we can easily observe that a Fitting model does not necessarily respect the justification axiom schemes that we have defined, not even **J** and $+$! As the truth on justification formulae of the form $t : F$ requires both a modal and an evidence condition, we expect that in order for Fitting models to respect some of the given justification axiom schemes, we have to justify both the class of Kripke frames that they are based on, and their evidence function. In Definition 3.13, we give the justification axiom schemes, that we have defined, and the corresponding modal and evidence conditions that they have to fulfill.

Definition 3.13. The *modal* and *evidence conditions* of Fitting models, for the justification axiom schemes, which we have defined, are given in the following table.

Axiom Schemes	Modal Conditions	Evidence Conditions
J	-	$E(s, F \to G) \cap E(t, F) \subseteq E(s \cdot t, G)$
$+$	-	$E(s, F) \cup E(t, F) \subseteq E(s + t, F)$
JT	based on $\mathcal{T}$	-
JD	based on $\mathcal{KD}$	-
J4	based on $\mathcal{K}4$	*Monotonicity Condition* if wRu and $w \in E(t, F)$ then $u \in E(t, F)$ *!-Condition* $E(t, F) \subseteq E(!t, t{:}F)$
J5	based on $\mathcal{K}5$	*Strong Evidence Function* *?-Condition* if $w \notin E(t, F)$ then then $w \in E(?t, t{:}F)$

The first two evidence conditions, i.e., the **J** and $+$ evidence conditions, are usually called *minimum evidence conditions*, as they are introduced to Fitting models in order for them to respect the J_0's axiom schemes. The rest modal and evidence conditions are usually called by the justification axiom scheme that they are meant to presume, e.g., if a Fitting model respects the **JD** axiom scheme, we want it to be based on a frame in $\mathcal{KD}$, or equivalently to fulfil the **JD** modal condition.

As usual, we have to introduce some additional conditions to the Fitting models, so that they respect a given constant specification CS.

Definition 3.14. Let CS a constant specification.
We say that a Fitting model $M = \langle W, R, V, E\rangle$ *meets constant specification* iff for each $c{:}F \in \mathsf{CS}$, it holds that $E(c, F) := W$.

By the previous definition it is easy to observe that for any $c{:}A \in \mathsf{CS}$, for some constant specification CS, if an axiom A is true in all worlds that we consider possible in some world w, then the argument c justifies the truth of A. Therefore, in contrast with basic models, the fact that a formula $c{:}A$ belongs in some constant specification CS, it is not, at first glance, necessary that $c{:}A$ holds. In Fitting models we must also believe that A is true, i.e., in every world that we consider possible A holds. By proving soundness of justification logics, with respect to Fitting models, the fact that axiom A is true in every world comes for free and therefore any formula of the form $c{:}A$ in CS is true in any world in W, as wanted. Trivially, this expands for any $c{:}F$, i.e., even F is not an axiom but it is a formula of the form $c_n{:}\cdots{:}c_1{:}A \in \mathsf{CS}$, where A is an axiom.

It is time to show that Fitting models are indeed proper semantics for the justification logics that we have defined.

Theorem 3.15 (Soundness and Completeness for Fitting Models)**.** Let CS an arbitrary constant specification for the corresponding justification logic JL. Then,

- $\mathsf{J}_0(\mathsf{CS})$ is sound and complete with respect to the class of Fitting models that meet CS and respect the minimum evidence conditions, i.e., they respect the **J** and $+$ Evidence conditions.

- JT(CS) is sound and complete with respect to the class of Fitting models that meet CS and respect the **JT** modal condition and the minimum evidence conditions.

- J4(CS) is sound and complete with respect to the class of Fitting models that meet CS and respect the **J4** modal condition, the minimum evidence conditions, the monotonicity condition and the !-condition.

- J5(CS) is sound and complete with respect to the class of Fitting models that meet CS and respect the **J5** modal condition, the minimum evidence conditions and the ?-condition and they have a strong evidence function.

Let JD has the internalization property relative to constant specification CS, e.g., CS is axiomatically appropriate. Then,

- JD(CS) is sound and complete with respect to the class of Fitting models that meet CS and respect the **JD** modal condition and the minimum evidence conditions.

We have the corresponding soundness and completeness theorems for the other justification logics. For instance, LP(CS) is sound and complete with respect to the class of Fitting models based on a reflexive and transitive Kripke frame, which fulfill the monotonicity and ! evidence condition and which meet CS.

Remark 3.16. *The completeness of* JD *in respect with the class of Fitting models based on frames in $\mathcal{KD}$, requires to* JD *having the internalization property relative to the corresponding constant specification. We can omit this requirement by replacing the modal condition with the following evidence condition:*

For any term $t \in \mathsf{Tm}$, it holds that $E(t, \bot) = \emptyset$.

*We gave the modal requirement as it was the original requirement, defined for justification logics containing axiom scheme **JD**.*

Theorem 3.17. Let CS arbitrary constant specification for JD. Then,

- JD(CS) is sound and complete with respect to the class of Fitting models that meet CS and respect the minimum and **JD** evidence conditions.

We have the corresponding soundness and completeness theorems for the other justification logics, containing the axiom scheme **JD**.

3.4 Modular Models

In the previous section we spoke about the weakness of Fitting models to presume the ontological concept of the justification. In this philosophical manner Fitting models seems inferior to basic models. On the other hand, Fitting models are closely related with modal logic, as their structure is based on Kripke frames. Therefore they are superior than basic models on an epistemic (deontic, etc) conception.

3.4.1 General Modular Models

A structure that seems to encapsulate the advantages of both basic and Fitting models are the *modular models*. They were first introduced by Artemov in [12]

Definition 3.18 (Modular Models)**.** A *modular model* for justification logic is a structure $M = \langle W, R, * \rangle$, where $\langle W, R \rangle$ is a Kripke frame and

- $*\colon W \rightarrow BM(\emptyset)$, called *basic model function*, that defines for each world a corresponding basic model.

We usually, use the notation $*_w := *(w)$, where $w \in W$, to denote the corresponding basic model.

Note that $BM(\emptyset)$ is the set of all basic models, as

$$\begin{aligned} BM(\emptyset) &:= \{* \mid * \text{ is a basic model s.t. } * \models \emptyset\} \\ &= \{* \mid * \text{ is a basic model s.t. } (\forall F \in \emptyset)\,[* \models F]\} \\ &= \{* \mid * \text{ is a basic model}\} \end{aligned}$$

Definition 3.19 (Truth in Modular Models)**.** Truth of justification formulae in modular models is interpreted on pairs (M, w), where $M = \langle W, R, * \rangle$ is an arbitrary modular model and w is some world in W. Specifically, we define that a formula $F \in \mathcal{L}_{\mathsf{J}}$ is true (or satisfied) in (M, w), denoted as $M, w \Vvdash F$, as follows:

$$M, w \Vvdash F \Leftrightarrow *_w \models F,$$

i.e., F is true in (M, w) if and only if it is true in the basic model defined for w.

For an arbitrary modular model $M = \langle W, R, * \rangle$, the basic model function $*$ and the accessibility relation R are defined independently and serve different porpoises. The basic model function $*$ corresponds to each world $w \in W$, a basic model $*_w$, that defines the truth on any justification formula in w. Therefore, as the truth of the justification formulae is completely defined by the basic models, according to our previous discussion, the modular models perceive the ontology of the justification. On the other hand, the accessibility relation R abstain from truth valuation of the justification formulae. Its role is purely modal (e.g., epistemic, doxastic). In fact, we will be mainly involved with the epistemic/doxastic notion. Specifically, given a modular model M and the fact that we are in a world w of this model, we will say that

"We know/believe that F is true."

iff

$$(\forall v \in R[w])\,[M, v \Vdash F]$$

and we will denote it, as usual, by $\Box F$. Of course, formulae of the form $\Box F$ do not belong in $\mathcal{L}_{\mathsf{J}}$. It is probably worth enriching the language $\mathcal{L}_{\mathsf{J}}$ in a way that such formulae may belong.

It is trivial to observe that modular models have similar structure as the Kripke models. Clearly, the basic model function $*$ corresponds to the valuation function V of some Kripke model $M = \langle W, R, V\rangle$. Of course, the basic model function $*$ determines the truth not only on the atomic propositions Prop, but also on any justification formula in some world. Therefore, we can imagine of a modular model as a "stronger" Kripke model capable of defining the truth for any justification formula. It is worth mentioning that in fact a valuation function V determines on its own, the truth on any modal formula that the modal operator $\Box$ does not appear, and not only the truth of the atomic propositions. With this last observation in mind, we enrich our justification language $\mathcal{L}_{\mathsf{J}}$, as it is made clear in Definition 3.20.

Definition 3.20 (Modal-Justification Language)**.** The *modal-justification language* $\mathcal{L}_{\Box\mathsf{J}}$ is defined by the following BNF-notation:

$$\begin{aligned} F &::= \ p \mid \bot \mid (F \to F) \mid t{:}F \\ G &::= \ F \mid (G \to G) \mid \Box G \end{aligned}$$

where $p \in \mathsf{Prop}$ and $t \in \mathsf{Tm}$.

The other propositional connectives are defined as abbreviations, in the standard way (see Definition 2.4).

The language $\mathcal{L}_{\Box\mathsf{J}}$ enables the doxastic interpretation even on justification formulae. On the other hand, it does not allow the justification interpretation on any formula that includes the modal operator $\Box$. Clearly, there is not any formula of the form $t{:}\cdots\Box\cdots F$. If we wanted to allow such interpretations we should enrich not only the language $\mathcal{L}_{\mathsf{J}}$, to some language $\mathscr{L}$ in which such formulae belong, but also to expand the notion of the basic models in such a way that $* \restriction_{\mathsf{Tm}} : \mathsf{Tm} \to \mathcal{P}(\mathscr{L})$.

We will define some classes of models accordingly to the class of basic models (Theorem 3.5) that the basic model function defines. Those classes will also correspond to the appropriate semantics for the different justification logics.

Definition 3.21 (Classes of Modular Models)**.** Let $M = \langle W, R, *\rangle$ be a modular model. Let also CS be a constant specification for JL, where JL some of the already defined justification logics. We say that M is a $\mathsf{JL(CS)}$-model iff $*: W \to BM(\mathsf{JL(CS)})$.

E.g., if CS is a constant specification for $\mathsf{J4}$, then M is a $\mathsf{J4(CS)}$-model if and only if $*: W \to BM(\mathsf{J4(CS)})$.

Theorem 3.22 (Soundness and Completeness for Modular Models)**.** Let CS be an arbitrary constant specification for justification logic JL, where JL one of the defined justification logics.
The justification logic JL(CS) is sound and complete with respect to the class of JL(CS)-models.

E.g. the justification logic J4(CS) is sound and complete with respect to the class of J4(CS)-models.

With the enrichment of the language $\mathcal{L}_{\mathsf{J}}$ to the language $\mathcal{L}_{\Box\mathsf{J}}$, it is plausible to define logics that combine the modal axiom schemes with the justification axiom schemes. Such logics, called *modal-justification logics*, are defined in the straight forward way that is given in Definition 3.23

Definition 3.23 (Modal-Justification Logics)**.** Let ML, one of the given modal logics and JL, one of the given justification logics. We denote by MLJL the axiomatic system that consists from the union of the axiomatic schemes of ML and JL, and the corresponding rules of inferences. E.g., the axiomatic system K4JD is given in Table 3.2.

If an arbitrary constant specification CS for JL is also given, then we define by MLJL(CS), the axiomatic system MLJL with the addition of the formulae in CS, as axioms.

Axiomatic Schemata		
all theorems of CL in $\mathcal{L}_{\Box\mathsf{J}}$		**P**
$\Box(F \to G) \to \Box F \to \Box G$	distribution of $\Box$ over $\to$	**K**
$\Box F \to \Box\Box F$	positive introspection	**4**
$s{:}(F \to G) \to t{:}F \to s \cdot t{:}G$	application	**J**
$s{:}F \to s+t{:}F$ & $t{:}F \to s+t{:}F$	sum	**+**
$\neg t{:}F \to ?t{:}\neg t{:}F$	negative introspection	**J5**
Rules of Inference		
From F and $F \to G$, infer G	modus ponens	**MP**
From F, infer $\Box F$	necessitation of $\Box$	**N**

Table 3.2: Axiomatic System K4J5

The derivation in modal-justification logics is exactly as expected by Definitions 2.12 and 2.28.

Definition 3.24 (Derivations in Modal-Justification Logic)**.** Let MLJL(CS) be an arbitrary modal-justification logic, where CS a constant specification for justification logic JL and **AX** the set of axiom schemes of MLJL. Let $F \in \mathcal{L}_{\Box\mathsf{J}}$ a modal-justification formula. A *derivation of F in* MLJL(CS), is a finite sequence $F_1, \ldots, F_n$ of formulae, s.t.:

- $F_n := F$,
- every F_i in the sequence is
  - either an axiom in **AX**,
  - or a member of CS,
  - or the result of the application of modus ponens or necessitation rule, to formulae of the subsequence $F_1, \ldots, F_{i-1}$.

If there is a derivation for F in $\mathsf{MLJL(CS)}$, then we write $\mathsf{MLJL(CS)} \vdash F$ or $\vdash_{\mathsf{MLJL(CS)}} F$, and we say that F is a *theorem* of $\mathsf{MLJL(CS)}$, or that $\mathsf{MLJL(CS)}$ proves F.

Let $\Sigma \cup \{F\}$ a set of modal-justification formulae in the logic of MLJL. A *derivation of* F, *from* Σ, *in* $\mathsf{MLJL(CS)}$, is a finite sequence $F_1, \ldots, F_n$ of formulae, s.t.:

- $F_n \coloneqq F$,
- every F_i in the sequence is
  - either a theorem of $\mathsf{MLJL(CS)}$,
  - or a member of Σ,
  - or the result of the application of modus ponens, to formulae of the subsequence $F_1, \ldots, F_{i-1}$.

If there is a derivation of F, from Σ, in $\mathsf{MLJL(CS)}$, then we write $\Sigma \vdash_{\mathsf{MLJL(CS)}} F$ and we say that F is *derivable in* $\mathsf{MLJL(CS)}$, *from the premices* Σ.

The consistency of this logic follows straight forward from the consistency of modal and justification logic, i.e., Theorems 2.13 and 2.29. Moreover, we have that the deduction theorem for modal-justification logic also holds as a consequence of the deduction theorems for modal and justification case. Finally, it seems obvious that the corresponding soundness and completeness theorems of these logics also hold.

Theorem 3.25 (Soundness & Completeness of Modal-Justification Logics for Modular Models)**.** Each logic MLJL of the modal-justification logics is sound and complete with respect to the class of modular models that

- are based on Kripke frames in the class of Kripke frames which corresponds to the modal counterpart of the logic;
- and whose basic model function respects the requirements for the justification logic counterpart, i.e., they are JL-models.

If an arbitrary constant specification CS for JL is also given, then $\mathsf{MLJL(CS)}$ is sound and complete with respect to the class of $\mathsf{JL(CS)}$-models based on frames in the class of Kripke frames that corresponds to ML.

E.g., the modal-justification logic $\mathsf{K4J5(CS)}$ is sound and complete with respect to the class of $\mathsf{J5(CS)}$-models that are based on Kripke frames in $\mathcal{K}4$, where CS is some constant specification for $\mathsf{J5}$.

The proof of Theorem 3.25 is an immediate consequence of the proofs of Theorems 2.19 and 3.22. It is almost a trivial proof, as the truth of the modal and the justification counterpart are in a sense irrelevant. Specifically, the truth on the justification formulae is independent from the accessibility relation, whereas for the truth on modal formulae we perceive the basic model function as an evaluation function on justification formulae.

3.4.2 JYB-Modular Models

So far, we have not seen an essential correlation between the justification and epistemic/doxastic notion of the modular models. In other words, there is no correlation between the existence of an argument justifying some state F and the belief on this statement. A useful property that we would probably want to hold is to believe any statement for which there is a justification, i.e., we would want in the modular models to hold that $t{:}F \to \Box F$, where $t \in \mathsf{Tm}$ and $F \in \mathcal{L}_\mathsf{J}$. This property is known as JYB, which stands for "Justification Yields Belief". Thanks to this property, there is a class of modular models that is defined.

Definition 3.26 (JYB)**.** A modular model $M = \langle W, R, * \rangle$ is a *JYB-modular model* iff for any world $w \in W$ and any term $t \in \mathsf{Tm}$

$$t^{*_w} \subseteq \Box^w,$$

where

$$\Box^w := \{F \in \mathcal{L}_\mathsf{J} \mid M, w \Vdash \Box F\}.$$

Derivation in JYB-modular models is exactly like in modular models. Soundness and completeness for justification logic $\mathsf{JL}(\mathsf{CS})$, where JL some of the already defined justification logics and CS some constant specification for JL, with respect to the class of JYB-$\mathsf{JL}(\mathsf{CS})$-models trivially holds. The soundness holds from the soundness with respect to $\mathsf{JL}(\mathsf{CS})$-models (Theorem 3.22), while completeness follows from the completeness of $\mathsf{JL}(\mathsf{CS})$-models with respect to the the class $BM(\mathsf{JL}(\mathsf{CS}))$, as each basic model in $BM(\mathsf{JL}(\mathsf{CS}))$ is a JYB-$\mathsf{JL}(\mathsf{CS})$-model with unit set of worlds and empty accessibility relation.

The more interesting cases of soundness and completeness are those for modal-justification logics. The derivation in this case assumes the existence of axiom scheme **C** in the set of axiom schemes **AX**, where **C** the axiom scheme of *connection*, given in Table 3.3. For instance let us give the following example of derivation.

Example 3.27. *Let* $\mathsf{MLJL}+\boldsymbol{C}$ *the modal-justification logic that arises from the addition of axiom* **C** *in the modal-justification logic* MLJL*. Let also* CS *an arbitrary constant specification for* JL*. For finite set of modal-justification formulae* $\Sigma \cup \{F\} \subset \mathcal{L}_{\Box\mathsf{J}}$ *it holds that*

$$\mathsf{MLJL}(\mathsf{CS}) + \boldsymbol{C} \vdash \bigwedge \Sigma \to F \Leftrightarrow \mathsf{MLJL}(\mathsf{CS}) + \boldsymbol{C} \vdash \bigwedge \Box\Sigma \to \Box F,$$

where $\Box\Sigma := \{\Box H \mid H \in \Sigma\}$*.*

Proof. We will prove it in the case where Σ has two elements. Then by induction, we can prove it for any $n \in \mathbb{N}$ greater than 2, while the cases for $n = 0$ or $n = 1$ are trivial. Let $F, G, H \in \mathcal{L}_{\Box\mathsf{J}}$, s.t. $\mathsf{MLJL}(\mathsf{CS}) + \mathbf{C} \vdash F \wedge G \to H$. We want to show that $\mathsf{MLJL}(\mathsf{CS}) + \mathbf{C} \vdash \Box F \wedge \Box G \to \Box H$. We observe that

$$\begin{aligned} &F_1 : F \wedge G \to H && \\ &F_2 : \Box(F \wedge G \to H) && \mathbf{N}, 1 \\ &F_3 : \Box(F \wedge G) \to \Box H && \mathbf{K}, \mathbf{MP}, 2 \end{aligned}$$

Therefore, it suffices to show that $\Box F \wedge \Box G \to \Box(F \wedge G)$ and then by propositional reasoning we have the requested property.

$F_1 : F \to G \to F \wedge G$	**P**
$F_2 : \Box(F \to G \to F \wedge G)$	**N**, 1
$F_3 : \Box F \to \Box(G \to F \wedge G)$	**K**, **MP**, 2
$F_4 : \Box(G \to F \wedge G) \to \Box G \to \Box(F \wedge G)$	**K**
$F_5 : \Box F \to \Box G \to \Box(F \wedge G)$	**P**, 3, 4
$F_6 : \Box F \wedge \Box G \to \Box(F \wedge G)$	**P**, 5

By the above derivation, we have the requested property. □

$t{:}F \to \Box F$	connection	**C**

Table 3.3: Axiom Scheme **C**

First of all, we have to prove the consistency of $\mathsf{MLJL(CS)} + $ **C**. In this direction, once again we will give the conservativity of $\mathsf{MLJL(CS)} + $ **C** in respect to CL.

Theorem 3.28 (Conservativity of Modal-Justification Logic with Connection)**.** Let CS a constant specification for justification logic JL.
$\mathsf{MLJL(CS)} + $ **C** is a conservative extension of CL.

Corollary 3.29 (Consistency of Modal-Justification Logic with Connection)**.** Any modal justification logic with connection is consistent.

In order to prove the soundess and completeness for the modal-justification logic with connection, we have also to give the corresponding deduction theorem

Theorem 3.30 (Deduction Theorem for Modal-Justification Logic with Connection)**.** The deduction theorem holds for any modal-justification logic with connection.

Theorem 3.31 (Soundness & Completeness for Modal-Justification Logic with Connection)**.** Let MLJL be an arbitrary modal-justification logic.
$\mathsf{MLJL} + $ **C** is sound and complete with respect to the class of JYB-modular models that

- are based on Kripke frames in the class of Kripke frames which corresponds to the modal counterpart of the logic;
- and whose basic model function respects the requirements for the justification logic counterpart, i.e., they are JYB-JL-models.

If an arbitrary constant specification CS for JL is also given, then $\mathsf{MLJL(CS)} + $ **C** is sound and complete with respect to the class of JYB-$\mathsf{JL(CS)}$-models based on frames in the class of Kripke frames that corresponds to ML.

Note that the constant specification depends only on the justification axioms that we assume, and does not depend to neither modal axioms, nor the axiom **C**. This was expected if we consider the form of formulae in $\mathcal{L}_{\Box\mathsf{J}}$.

Even more interesting are the cases that we have the interrelated modal and justification axiom schemes in some modal-justification logic, e.g., axiom schemes **4** and **J4**. For some of those cases the justification restrictions are unnecessary.

Lemma 3.32. The following propositions hold:

- Axiom scheme **JT** is a theorem of $\mathsf{TJ}_0 + \mathbf{C}$, or equivalently,
$$\mathsf{TJ}_0 + \mathbf{C} = \mathsf{TJT}$$
.

- Axiom scheme **JD** is a theorem of $\mathsf{KDJ}_0 + \mathbf{C}$, or equivalently,
$$\mathsf{KDJ}_0 + \mathbf{C} = \mathsf{KDJD}$$
.

As a trivial result of the previous lemma we have the following corollary.

Corollary 3.33. The following propositions hold:

- TJT is sound and complete with respect to the class of JYB-modular models based on reflexive Kripke frames.
- KDJD is sound and complete with respect to the class of JYB-modular models based on serial Kripke frames.

3.5 Hierarchy of Justification Logic Semantics

For each of the given semantics for justification logic, there was a corresponding soundness and completeness theorem. Thus the definition of one of them would be sufficient for working with justification logic. The definition of the different semantics aims on different epistemic and ontological concepts that we wanted to presume, as also different applications.

Despite the conceptual differences of the given semantics, there is a hierarchy between them, defined by the expressibility of each model of one semantics to some model of some other semantics. This hierarchy is given on Figure 3.1.

Theorem 3.34 (Hiearchy of Justification Logic Semantics)**.** For each model in some level of the pyramid, there is a model of the next level of the pyramid that expresses that model, i.e., satisfies exactly the same justification formulae.

3.6 The Ontology of Justification

The truth of a sentence of the form $t{:}F$ could probably be perceived in different manners, while all of them might respect our axiomatization. Specifically, $t:F$ could be understand as follows:

i The statement F is true and the argument t is an evidence of its truth.

ii The argument t is an evidence of the truth of statement F.

iii The argument t, which is true, is an evidence of the truth of statement F.

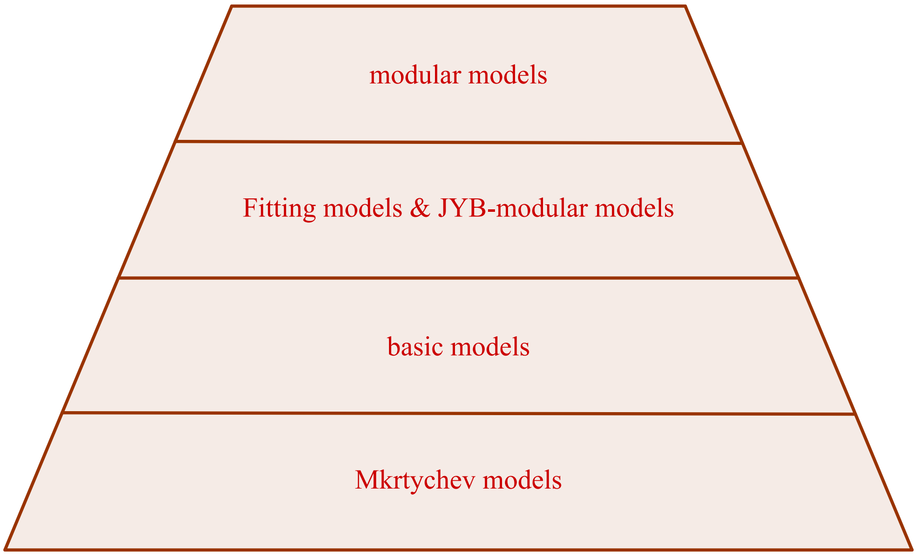

Figure 3.1: Hierarchy Pyramid for Justification Semantics

iv The argument t, which is convincing, is an evidence of the truth of statement F.

Let us state two examples that will help us perceive the ontology of the justification.

1. In the courthouse, the judge hears the evidence, denoted by t presented by the defence attorney, who is trying to prove the innocence of the defendant, which is denoted by F. Then the judge issues a ruling on whether the evidence confirms the innocence, i.e., $t{:}F$ holds, or it does not confirm it, i.e., $\neg t{:}F$ holds. If the judge, in order to issue a ruling that the evidence presented confirms the innocence, she had to know concurrently that the defendant was indeed innocent; i.e., F holds, then the trial would be meaningless. Therefore, the truth of F should not be required in order to accept a defense. In a perfect world though, we would love to hold that, if a defense $t{:}F$ is accepted, then the defendant is indeed innocent. In the justification axiomatization, that is we would love axiom scheme **JT** to hold; i.e., $t{:}F \rightarrow F$. Needless to say that this is unfortunately not the case. On the other hand, a successful defence of the defendant, by the defence attorney, should persuade the judge that the defendant is innocent; i.e. the JYB axiom scheme $t{:}F \rightarrow \Box F$ should hold. Of course, for the judge to be convinced that the evidence yields the defendant's innocence, i.e.; for $t{:}F$ to hold, it does not obligate she believes that the defendant is innocent *a priori*. Therefore, the notions of justification and belief should be distinct. Furthermore, in order for the judge to accept a defense of innocence $t{:}F$, it is reasonable to assume that two things must hold:

   - the evidence given is convincing for the judge,
   - the evidence given evinces the innocence of the defendant.

   In a perfect world, we would love the evidence to be actually true, but in real life this is probably an idealized, if not only, strict requirement. Thus we should be content with the evidence to be convincing.

2. In science (i.e., natural, social and applied sciences, but not formal sciences) a scientific truth, denoted by F, is verified by the content of some scientific publication (e.g., a paper), denoted by t. Therefore, if the content of some paper t verifies the scientific truth F, we would write $t{:}F$. Clearly, the acceptability $t{:}F$ of the paper t as a verification of the scientific truth, must be separated from the truth of F, as else the existence of the paper would be superfluous. Moreover, after many years it is possible, with the growth of science, for some new publication s to verify the truth of the negation of F; i.e., $s{:}\neg F$. That is, some scientific truths might be proven wrong after some new discoveries[3]. This evinces that the justification $t{:}F$, should not yield the truth of the statement F. The contradictory of such scientific results could probably lie on the following two reasons:

   - the paper contained some mistake that was not found,
   - the content of the paper was falsely assumed that was proving the scientific truth.

   The later case could hopefully arise due to misunderstanding of the interpretation of the content, or unhopefully to fault of the scientific method. It is probably scary to think of the notion of belief in sciences, as the belief changes with the development of science. But for sure, for a paper t to justify some scientific truth F; i.e., $t{:}F$ to hold, it is not obligated to believe a priori in F[4].

From the previous examples it is clear that a justification should respect the following properties:

- The ontology of a justification should be separated from the a priori truth of the statement meant to be justified.
- The existence of some justification for some statement should not necessary yield the truth of the justified statement.
- The ontology of a justification should be separated from the a priori belief in the statement meant to be justified.
- The existence of some justification for some statement should probably, but not necessary yield the belief in the justified statement.
- The argument that is meant to justify a statement should be convincing.
- The argument should evince the statement meant to be justified.

The last two bullets might need some more analysis.

The argument that is meant to justify a statement should be convincing.
This could be separated in two different manners.

- The argument must be reliable. For instance, a scientific paper or an article of a trustworthy magazine should be more reliable than a random post on the internet.

[3]For instance, last year researchers have discovered that mammalian cells can convert RNA sequences back into DNA using Polθ, something that contradicts *Central Dogma* as defined by James Watson in [37]. Luckily, Covid-19 anti-vaxxers have not yet discovered the existence of this paper.

[4]How many could believe in the incompleteness of Peano Arithmetic, before Gödel's famous paper [46] (even though mathematics are not considered a science)?!

- The argument must be thought as possible. For instance, if I state that my thesis is not decent because my dog ate the chapter which contained a huge breakthrough on algebraic topology, the argument given would not be assumed as possible, as the rest of the thesis is not related with algebraic topology, but also my dog (which in fact does not exist) could not eat a chapter of my thesis, as it is written with XeLaTeX.

Therefore, for an argument to be assumed convincing we do not consider it in relation with the statement meant to be proven, but we are only focused on the argument, per se.

The argument should evince the statement meant to be justified.
That is the argument which is given must indicate the truth of the statement meant to be proven. For instance, if I provide as an argument for the fact that I can sing beautifully, that I have scored 10 goals on football, last season, no mater the reliability of my argument, my argumentation would not be accepted as the argument does not evince the truth of the statement. Otherwise, if my evidence was that I have won *Operalia*[5] of 2021, then my evidence indubitably indicates the fact that I can sing beautifully, despite if my evidence is true or convincing.

Clearly, this time the argument is examined correlated with the statement meant to be proven.

Therefore, the most suitable interpretation of formulae of the form $t{:}F$, should probably be the fourth one. Let us see which notion of justification each of our semantics is able to perceive.

3.6.1 The Ontology of Justification in Basic Models

The truth of a formula of the form $t{:}F$ in some basic model $*$ is completely determined from whether F belongs in the set of formulae justified by the term t; i.e., $F \in t^*$. It is not clear whether the evidence t is convincing or not, but we are assured that t evinces the truth of statement F, or equivalently that argument t is evidence for the truth of statement F.

If we assume that we do not presume that t is also convincing then we could probably denote that some justification t is a lie or unconvincing by interpreting it, through $*$ to the empty set; i.e., $t^* := \emptyset$. In this way, we can implicitly define that a argument is a lie or unconvincing, by prohibiting to justify any formula. But it is comprehensible from the fact that the **J** and $+$ closure conditions demand only subset restrictions and not equality, that it is possible some unconvincing or bogus arguments, to be used as part of a justification. Therefore, this interpretation of the bogus arguments is ineffective.

At least, we are certain that the truth of formulae of the form $t{:}F$ is separated from the truth of the formula F, which is meant to be justified. Moreover, if the basic model $*$ does not belong in the class $BM(\mathsf{JT})$, then it does not obligate for a justification of a statement to yield the truth of the statement.

The correlation of justification, as presumed in basic models, with the notion of belief is also vague. The non-existence of possible worlds may make any correlation of justification and belief assailable.

Summarizing, the most representative interpretation of a formula of the form $t{:}F$ in basic models is probably interpretation ii, i.e., "The argument t is an evidence of the truth of statement F.".

[5] A.k.a. *Plácido Domingo's Operalia, The World Opera Competition*.

3.6.2 The Ontology of Justification in Mkrtychev Models

The Mkrtychev models are basic models where the truth on formulae of the form $t{:}F$, demands concurrently with $F \in t^*$, that F is true. That is, in order to accept a justification $t{:}F$, except of the fact that the argument t should evince the truth of statement F, it must be also the case that the statement F is true. Therefore, the Mkrtychev models does not perceive the ontology of justification as they involve the truth of the statement meant to be proven in the truth evaluation on formulae of the form $t{:}F$. As a minor result, the interpretation of $t{:}F$, yields the truth on F.

Again, as Mkrtychev models differ from basic models only in the interpretation of truth on $t:F$ formulae, it is obvious that both the correlation of justification with belief and the reliability of the arguments t are not strictly defined. This time, the most representative interpretation of a formula of the form $t{:}F$ is probably the interpretation i, i.e., "The statement F is true and the argument t is an evidence of its truth.".

3.6.3 The Ontology of Justification in Fitting models

In Fitting models, a justification $t{:}F$ is true in some world w, iff the argument t is an evidence of F in w; i.e., $w \in E(t, F)$, but also we believe in F; i.e., in any world $u \in R[w]$ that we consider possible, the statement F is true. Therefore, the ontology of justification is separated from truth of the statement meant to be justified, but it is correlated with the belief in this statement. As a minor consequence, we have that in any Fitting model justification of a statement might not yield the truth of this statement, but it definitely yield the belief in this statement, as the belief inherently correlates with the justification.

As we have mentioned, the perception of justification in Fitting models is not suitable for the ontological concept of justification. However, the notion of belief is clearly interpreted in this semantics. Yet again, there is no talk about the persuasiveness or the reliability of the arguments. Probably the most suitable read of a formula of the form $t{:}F$ is "The statement F is believed due to evidence t."

3.6.4 The Ontology of Justification in Modular models

The truth of formulae of the form $t{:}F$ is the same with basic models, only this time the truth is interpreted in worlds. As the modular models are based on Kripke frames, there is a natural interpretation of the notion of belief in them. Of course, the notion of justification and the notion of belief are irrelevant. The requirement for justification of a statement to yield belief is only existent in JYB-modular models. From the above modular models seems to be the most convenient semantics for the notion of justification. But even in this semantics we cannot define the persuasiveness or the reliability of the arguments of each justification.

From the above observations it is made clear that there is not a unique reading of formulae of the form $t{:}F$. The interpretation as "The evidence t justifies proposition F.", does not make clear whether F and t are true, convincing or believed. As certain demand, the term t has to evince the truth of the proposition F, but nothing more can be safely considered.

3.7 Gettier Problem

Even though, the first justification logic, i.e., LP, was introduced to give classical provability semantics for intutionistic logic, justification logic was naturally connected with epistemic logic and particularly with the JTB assumption of knowledge. This assumption of knowledge was firstly defined in Plato's dialogues *Meno* and *Theaetetus*, still not accepted by Socrates [58, 59]. According to JTB notion of knowledge, in order for a proposition F to be considered known, it is not sufficient to be true and believed, but the belief in F must also be satisfied by some evidence. Therefore, it must be the case that justification yields belief, i.e., the JYB axiom scheme $t{:}F \to \mathbf{B}F$, but also that

$$\mathbf{K}F \leftrightarrow F \wedge \mathbf{B}F \wedge t{:}F,$$

for some evidence t.

In JTB definition of knowledge it was not explicitly considered the truth of the evidence given as a justification of the proposition meant to be known, but the evidence was tacitly perceived as true. In this margin, Gettier in [60] (as also others before him, e.g. Bernard Russel et al.) challenged this definition of knowledge by constructing two examples in which both the truth of the proposition and the belief in it hold, but the belief in the proposition stands on wrong evidence. There are various approaches trying to overcome this problem, known as *Gettier problem*, which approaches are out of the scope of this thesis. What we are interested in is that, in general, justification logic falls into the Gettier problem, as it does not take into account neither the truth, nor the reliability and the persuasiveness of the evidence. In justification logic we are assured that we consider if the evidence yields the belief the proposition meant to be justified, but not assured if we considered the truthiness of evidence.

Let us take the first case of Gettier problem. Particularly, quoted from [60]

> Suppose that Smith and Jones have applied for a certain job. And suppose that Smith has strong evidence for the fol1owing conjunctive proposition: (d) Jones is the man who will get the job, and Jones has ten coins in his pocket.
>
> Smith's evidence for (d) might be that the president of the company assured him that Jones would in the end be selected, and that he, Smith, had counted the coins in Jones's pocket ten minutes ago. Proposition (d) entails:
>
> (e) The man who will get the job has ten coins in his pocket.
>
> Let us suppose that Smith sees the entailment from (d) to (e), and accepts (e) on the grounds of (d), for which he has strong evidence. In this case, Smith is clearly justified in believing that (e) is true.
>
> But imagine, further, that unknown to Smith, he himself, not Jones, will get the job. And, also, unknown to Smith, he himself has ten coins in his pocket. Proposition (e) is then true, though proposition (d), from which Smith inferred (e), is false.

We define a modal-justification logic such that the corresponding language contains a modal operator $\mathbf{B}$ for belief and a modal operator $\mathbf{K}$ for knowledge. We assume that both modal operators respects the axioms of K. We also assume that the JYB axiom scheme $t{:}F \to \mathbf{B}F$ and the JTB axiom scheme

$$\mathbf{K}F \leftrightarrow F \wedge \mathbf{B}F \wedge t{:}F,$$

hold. Let us also equip J_0 with an axiomatically appropriate constant specification CS. We assume that Smith is our agent. We will interpret the following formulae as bellow

J: "Jones will take the job."

J_{10}: "Jones has ten coins in his pocket."

S: "Smith will take the job."

S_{10}: "Smith has ten coins in his pocket."

E: "The man who will get the job has ten coins in his pocket."

Let j states the fact that the president of the company informed Smith that Jones will take the job, thus, $j : J$ states "The talk with the president of the company, evinces that Jones will take the job.". Let also j_{10} states the fact that Smith counted the coins in Jones pocket, hence, $j_{10}{:}J_{10}$ stands for the fact that this counting justifies that Jones has ten coins in his pocket. By the description of Gettier's story, we have that the following formulae hold:

$$\begin{array}{rrr} S & S_{10} & s_e{:}(S \wedge S_{10} \to E) \\ \neg J & J_{10} & j_e{:}(J \wedge J_{10} \to E) \\ j{:}J & j_{10}{:}J_{10} & E \end{array}$$

By assuming those formulae as premises of our derivation, we have

$$\begin{array}{lr} F_1: c{:}(J \to J_{10} \to J \wedge J_{10}) & \mathsf{CS} \\ F_2: c \cdot j{:}(J_{10} \to J \wedge J_{10}) & \mathbf{J},\ 1,\ j{:}J \\ F_3: (c \cdot j) \cdot j_{10}{:}(J \wedge J_{10}) & \mathbf{J},\ 2,\ j_{10}{:}J_{10} \\ F_4: j_e \cdot (c \cdot j) \cdot j_{10}{:}E & \mathbf{J},\ 3,\ j_e{:}(J \wedge J_{10} \to E) \\ F_5: \mathbf{B}E & \text{JYB},\ 4 \\ F_6: \mathbf{K}E & \text{JTB},\ E,\ 4,\ 5 \end{array}$$

As we observe justification logic falls into the Gettier problem. Clearly, the definition knowledge in the JTB sense seems wrong. In the derivation, we concluded that Smith knows that the man who will get the job has ten coins in his pocket, but he grounded his knowledge on wrong evidence. Thus, this should not be considered as knowledge. In order to overcome, such obstacles inside justification logic we can do it at a metalogical level. One attempt, is to reject that $j{:}J$ holds, by interpret formulae of the form $t : F$ as the 3rd assumption of knowledge in section 3.6. But this seems problematic, as we have analyzed in section 3.6. Moreover, with this interpretation it is not clear why formulae of the form $\neg t{:}F$ hold, i.e., is it because t is false, or is it because t does not evince F? A second attempt is as in [56], where Artemov defined some logic that there was an external observer, who was reasoning in tandem with the actual agent.

CHAPTER 4

UNCERTAINTY IN JUSTIFICATION LOGIC

The main application of justification logic, as its name indicates, is that of an epistemic doxastic logic of justification; i.e., an evidential logic. Nevertheless, as the two examples of section 3.6 denote, it is rarely the case that a justification yields the truth/belief of a statement without doubt. Therefore, it seems essential to equip justification logic with the notion of uncertainty. Of course, this can be done by introducing a probability setting in justification logic. The whole chapter, except the last section, is devoted in this idea. In the last section, we introduce the aggregated probabilistic evidence logic, PE, which again combines the probability with justification setting, by equipping, this time, the probability theory with the justification logic.

4.1 The Logic of Uncertain Justifications

Historically, the first justification logic equiped with the notion of uncertainty is the logic of uncertain justifications, UJ. It was first presented by Robert S. Milnikel in preliminary form, at a symposium in honor of Sergei Artemov's birthday in May 2012. The paper was published at 2014.

In order to define the logic of uncertain justifications, UJ, we have to replace the justification operator : in $\mathcal{L}_{\mathsf{J}}$, with a family of operators $\{:_{\mathtt{p}}\}_{\mathtt{p}\in\mathtt{S}_{>0}}$, where we denote by $\mathtt{S}$ the set of rational numbers in $[0,1]$, i.e, $\mathtt{S} := \mathbb{Q} \cap [0,1]$.

Definition 4.1 (Language of Uncertain Justifications $\mathcal{L}_{\mathsf{UJ}}$)**.** The *language of uncertain justifications* $\mathcal{L}_{\mathsf{UJ}}$ is defined by the following BNF-notation:

$$F \ ::= \ p \mid \bot \mid (F \to F) \mid t{:}_{\mathtt{p}}F$$

where $p \in \mathsf{Prop}$, $t \in \mathsf{Tm}$ and $\mathtt{p} \in \mathtt{S}_{>0}$.

The other propositional connectives are defined as abbreviations, in the standard way (viz. Definition 2.4). Moreover, precedence and the associativity of the logical operators is similar to the one for modal logic (viz. Definition 2.5) and we might omit the parentheses, accordingly. The precedence of the uncertain justification operator $:_{\mathtt{p}}$ is assumed the same as the one of the justification operator : in $\mathcal{L}_{\mathsf{J}}$.

The intended meaning of formulae of the form $t{:}_{\mathtt{p}}F$, as stated in [21], is

"We have at least degree $\mathtt{p}$ of confidence in the reliability of t as evidence for belief in F.".

4.1.1 Axiomatization of Logic of Uncertain Justifications

Definition 4.2 (The UJ Logic)**.** The *logic of uncertain justifications* UJ is given in Table 4.1, where $F, G \in \mathcal{L}_{\mathsf{UJ}}$, $s, t \in \mathsf{Tm}$ and $\mathtt{p}, \mathtt{q}, \mathtt{q}' \in \mathtt{S}_{>0}$ s.t. $\mathtt{p} \geq \mathtt{q}'$.

Axiomatic Schemata		
all theorems of $\mathtt{CL}$ in $\mathcal{L}_{\mathsf{UJ}}$		**P**
$s{:}_{\mathtt{p}}(F \to G) \to t{:}_{\mathtt{q}}F \to s \cdot t{:}_{\mathtt{p}\cdot\mathtt{q}}G$	UJ-application	**UJJ**
$s{:}_{\mathtt{p}}F \to s + t{:}_{\mathtt{p}}F$ & $t{:}_{\mathtt{p}}F \to s + t{:}_{\mathtt{p}}F$	UJ-sum	**UJ+**
$t{:}_{\mathtt{p}}F \to t{:}_{\mathtt{q}'}F$	confidence weakening	**CW**
Rules of Inference		
From F and $F \to G$, infer G	modus ponens	**MP**

Table 4.1: Axiomatic System UJ

We are gonna give the corresponding notion of constant specification for UJ.

Definition 4.3 (Constant Specification for UJ)**.** A *constant specification* CS *for* UJ is a set of formulae s.t.:

- its elements are of the form

$$F \coloneqq c_n{:}_1 c_{n-1}{:}_1 \ \ldots {:}_1 c_1{:}_1 A,$$

  where $n \geq 1$, $\{c_i\}_{i\in[n]} \subset \mathsf{Con}$ and A is an axiom of UJ,

- if $c_n{:}_1 c_{n-1}{:}_1 \ \ldots {:}_1 c_1{:}_1 A \in \mathsf{CS}$, then $c_{n-1}{:}_1 \ \ldots {:}_1 c_1{:}_1 A \in \mathsf{CS}$, where $n \geq 2$.

The notions of empty, axiomatically appropriate and total constant specification is defined in the obvious way. We also denote, as always, by $\mathsf{UJ}(\mathsf{CS})$ the axiomatic system UJ with the addition of formulae in CS as axioms.

In [21], where the logic of uncertain justifications was originally defined, the constant specification is assumed to be total. In fact, instead of the introduction of constant specification, an additional rule of inference was defined. That is, the *internalization rule of inference*, which explicitly states that for any $n \in \mathbb{N}_{\geq 1}$, for any constant terms $c_1, \ldots, c_n \in \mathsf{Con}$ and for any axiom A of UJ, infer

$$c_n{:}_1 \ \ldots {:}_1 c_1{:}_1 A.$$

Trivially, that is the same as assuming the logic $\mathsf{UJ}(\mathsf{CS})$, where CS is the total constant specification for UJ.

As per usual, let us define the notion of derivation in UJ.

Definition 4.4 (Derivations in Logic of Uncertain Justifications)**.** Let CS be a constant specification for UJ. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{UJ}}$. A *derivation of F, from Σ in* $\mathsf{UJ}(\mathsf{CS})F$, is a finite sequence $F_1, \ldots, F_n$ of formulae, s.t.:

- $F_n \coloneqq F$,

- every F_i in the sequence is
  - either an axiom of UJ,
  - or a member of CS,
  - or a member of Σ,
  - or the result of the application of modus ponens, to formulae of the subsequence $F_1, \dots, F_{i-1}$.

If there is a derivation of F, from Σ, in $\mathsf{UJ}(\mathsf{CS})$, then we write $\Sigma \vdash_{\mathsf{UJ}(\mathsf{CS})} F$, and we say that F is *derivable* in $\mathsf{UJ}(\mathsf{CS})$, from the premises Σ. If Σ is the empty set, i.e., we do not assume any premises, then we write $\vdash_{\mathsf{UJ}(\mathsf{CS})} F$ and we say that F is a theorem of $\mathsf{UJ}(\mathsf{CS})$.

Example 4.5. *Let* CS *an axiomatically appropriate constant specification for* UJ*. For any formulae* $F, G \in \mathcal{L}_{\mathsf{UJ}}$*, any terms* $s, t \in \mathsf{Tm}$ *and any* $\mathsf{p}, \mathsf{q} \in \mathsf{S}_{>0}$*, there exists some term* $\tilde{t}$ *s.t.*

$$\vdash_{\mathsf{UJ}(\mathsf{CS})} s{:}_{\mathsf{p}}F \vee t{:}_{\mathsf{q}}G \to \tilde{t}{:}_{\min\{\mathsf{p},\mathsf{q}\}}(F \vee G)].$$

Proof. W.l.o.g. let $\mathsf{q} \leq \mathsf{p}$. Clearly, there are constant terms $c, c' \in \mathsf{Con}$ s.t. $c{:}_1(F \to F \vee G)$ and $c'{:}_1(G \to F \vee G)$ belong in CS. Then, we have

$F_1:$	$s{:}_{\mathsf{p}}F \to s{:}_{\mathsf{q}}F$	**CW**
$F_2:$	$c{:}_1\,(F \to F \vee G) \to s{:}_{\mathsf{q}}F \to c \cdot s{:}_{\mathsf{q}}(F \vee G)$	**UJP**
$F_3:$	$s{:}_{\mathsf{q}}F \to c \cdot s{:}_{\mathsf{q}}(F \vee G)$	CS, **MP** 2
$F_4:$	$s{:}_{\mathsf{p}}F \to c \cdot s{:}_{\mathsf{q}}(F \vee G)$	**P** 1, 3
$F_5:$	$c'{:}_1\,(G \to F \vee G) \to t{:}_{\mathsf{q}}G \to c' \cdot t{:}_{\mathsf{q}}(F \vee G)$	**UJP**
$F_6:$	$t{:}_{\mathsf{q}}G \to c' \cdot t{:}_{\mathsf{q}}(F \vee G)$	CS, **MP** 5
$F_7:$	$s{:}_{\mathsf{p}}F \vee t{:}_{\mathsf{q}}G \to c \cdot s + c' \cdot t{:}_{\mathsf{q}}(F \vee G)]$	**P** 4, 6

□

From the definition of UJ axiomatic system and the corresponding language, $\mathcal{L}_{\mathsf{UJ}}$, it is almost clear that UJ is a conservative extension of J_0.

Theorem 4.6 (Conservativity of Logic of Uncertain Justifications)**.**
UJ is a *conservative extension* of the basic justification logic, J_0.

Corollary 4.7 (Consistency of Logic of Uncertain Justifications)**.** Let CS an arbitrary constant specification for UJ. Then, $\mathsf{UJ}(\mathsf{CS})$ is consistent.

Finally, we give the corresponding deduction theorem for UJ.

Theorem 4.8 (Deduction Theorem for Logic of Uncertain Justifications)**.** Let CS an arbitrary constant specification for UJ. Then, the deduction theorem holds for $\mathsf{UJ}(\mathsf{CS})$.

We also have a corresponding Lifting Lemma, in the sense of certain justification.

Lemma 4.9 (UJ-Internalization Property & Lifting Lemma for UJ)**.**
Let CS an axiomatically appropriate constant specification for UJ. Then, $\mathsf{UJ}(\mathsf{CS})$ has the UJ-*internalization property*, i.e., for any formula $F \in \mathcal{L}_{\mathsf{UJ}}$ it holds that

$$\vdash_{\mathsf{UJ}(\mathsf{CS})} F \Rightarrow (\exists t \in \mathsf{Tm})\big[\vdash_{\mathsf{UJ}(\mathsf{CS})} t{:}_1F\big].$$

Let UJ has the UJ-internalization property relative to some constant specification CS (e.g. axiomatically appropriate). Then, if

$$F_1, \ldots, F_n \vdash_{\mathsf{UJ(CS)}} F$$

it holds that for every $t_1, \ldots, t_n \in \mathsf{Tm}$, there exists some $t \in \mathsf{Tm}$ s.t.

$$t_1{:}_1 F_1, \ldots, t_n{:}_1 F_n \vdash_{\mathsf{UJ(CS)}} t{:}_1 F.$$

4.1.2 Semantics for Logic of Uncertain Justifications

Definition 4.10 (UJ-Fitting model)**.** A UJ-*Fitting model* is a structure $M = \langle W, R, V, E \rangle$, where $\langle W, R, V \rangle$ is a Kripke model and

- $E \colon W \times \mathsf{Tm} \times \mathcal{L}_{\mathsf{UJ}} \to \mathcal{P}([0,1])$, called *uncertain evidence function*, s.t. for every world $w \in W$, every term $t \in \mathsf{Tm}$ and every formula $F \in \mathcal{L}_{\mathsf{UJ}}$

  $$E(w,t,F) = [0,r) \text{ or } [0,r]\,,$$

  for some $r \in \mathsf{S}$.

Given a UJ-Fitting model $M = \langle W, R, V, E \rangle$, we say that M is *based on* the Kripke frame $F = \langle W, R \rangle$, or that F is the frame *underlying* M.

Note that the structure of UJ-Fitting models is similar to that of Fitting models, with only difference the replacement of the evidence function in Fitting models, from the uncertain evidence function. We remind the reader that an evidence function in Fitting models is a function E with domain $\mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$ and codomain $\mathcal{P}(W)$, which for any term $t \in \mathsf{Tm}$ and any formula $F \in \mathcal{L}_{\mathsf{J}}$ it defines the set $E(t,F)$ of worlds that t is an evidence of the truth of F, i.e., an arbitrary world w belongs in $E(t,F)$ iff the term t evinces the truth of F in the world w. This last one, could also be written as $E'(w,t,F) = 1$, and correspondingly we could wright $E'(w,t,F) = 0$, iff t does not evince the truth of F in the world w. With this observation in mind, the definition of uncertain evidence function as $E \colon W \times \mathsf{Tm} \times \mathcal{L}_{\mathsf{UJ}} \to \mathcal{P}([0,1])$ comes natural. That is, we replace the codomain from $\{0,1\}$ to the set of downward-closed non-empty subsets of $[0,1]$, with rational supreme. Then, $E(w,t,F) = [0,r)$ represents the fact that the degree of confidence in the reliability of t as evidence for belief in F, in world w, belongs in the set $[0,r)$ and accordingly for $E(w,t,F) = [0,r]$.

Definition 4.11 (Truth in UJ-Fitting Models)**.** Truth of $\mathcal{L}_{\mathsf{UJ}}$-formulae in UJ-Fitting models is interpreted on pairs (M,w), where $M = \langle W, R, V, E \rangle$ is an arbitrary UJ-Fitting model and w is some world in W. Specifically, we define that a formula $F \in \mathcal{L}_{\mathsf{UJ}}$ is true (or satisfied) in (M,w), denoted as $M, w \Vdash F$, as in Fitting models, where for any term $t \in \mathsf{Tm}$, any $\mathtt{p} \in \mathsf{S}^*$ and any formula $F \in \mathcal{L}_{\mathsf{UJ}}$

$$M, w \Vdash t{:}_{\mathtt{p}} F \quad \text{iff} \quad \begin{cases} \textit{Modal Condition}: & \forall u \in R[w] \;\; M, u \Vdash F \\ \textit{Evidence Condition}: & \mathtt{p} \in E(w,t,F) \end{cases}$$

- We say that a formula $F \in \mathcal{L}_{\mathsf{UJ}}$ is *satisfiable* iff there is some UJ-Fitting model M and some world w in this model, s.t. F is true in (M,s).

Axiom Schemes	Evidence Conditions
UJJ	$\{\mathtt{p} \cdot \mathtt{q} \mid \mathtt{p} \in E(w, s, F \rightarrow G)\ \&\ \mathtt{q} \in E(w, s, F)\} \subseteq E(w, s \cdot t, G)$
UJ+	$E(w, s, F) \cup E(w, t, F) \subseteq E(w, s + t, F)$

Table 4.2: UJ-Minimum Evidence Conditions

- We say that a formula $F \in \mathcal{L}_{\mathsf{UJ}}$ is *satisfiable* in a certain class $\mathcal{C}$ of Kripke frames iff F is true in some (M, w), where M is a UJ-Fitting model based on a frame that belongs in $\mathcal{C}$.
- We say that a formula $F \in \mathcal{L}_{\mathsf{UJ}}$ is *true* in a UJ-Fitting model M, and we denote $M \Vdash F$ iff it is true in all worlds of M.
- We say that a formula F is *valid* in a certain class $\mathcal{C}$ of Kripke frames, and we denote $\mathcal{C} \Vdash F$ iff F is true in every UJ-Fitting model M based on some model F belonging in $\mathcal{C}$.
- We say that a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{UJ}}$ is *true* in some world w, and we denote $M, w \Vdash \Sigma$ iff all members of Σ are true in w. We define the *truth* of Σ in a UJ-Fitting model M, and the *satisfiability* and *validity* of Σ in a class of Kripke frames $\mathcal{C}$, in the obvious way.
- We say that a formula $F \in \mathcal{L}_{\mathsf{UJ}}$ is a *semantic consequence* of a set of formulae Σ in a class of Kripke frames $\mathcal{C}$, and we denote $\Sigma \Vdash_{\mathcal{C}} F$ iff for every UJ-Fitting model M based on some frame in $\mathcal{C}$, $M \Vdash \Sigma$ implies $M \Vdash F$.

Yet again, we give the minimum evidence conditions for the UJ-Fitting models. Note, that these corresponds exactly to those for Fitting models.

Definition 4.12. The minimum evidence conditions for the UJ-Fitting models are given in Table 4.2.

As in Fitting models, we have to introduce some additional conditions to the evidence function, so that the UJ-Fitting models respect a given constant specification CS.

Definition 4.13. Let CS a constant specification for UJ. We say that a UJ-Fitting model $M = \langle W, R, V, E \rangle$ *meets constant specification* CS iff for each $c{:}F \in \mathsf{CS}$, it holds that $E(w, c, F) := [0, 1]$.

We are ready to define the corresponding soundness and completeness theorem.

Theorem 4.14 (Soundness and Completeness for UJ)**.** Let CS an arbitrary constant specification for UJ.
UJ(CS) is sound and complete in respect with the class of UJ-Fitting models that meat constant specification CS and respect the UJ-minimum evidence conditions.

4.2 Probabilistic Justification Logic

In order to define the PPJ axiomatic system we have to expand $\mathcal{L}_{\mathsf{J}}$ with formulae of the form $P_{\geq \mathrm{p}}F$. According to [22, 23, 24], the intended meaning of such formula is

"The probability of truthfulness for the PPJ-formula F is at least p.".

Definition 4.15 (Probabilistic Justification Language $\mathcal{L}_{\mathsf{PPJ}}$)**.** The *probabilistic justification language* $\mathcal{L}_{\mathsf{PPJ}}$ is defined by the following BNF-notation:

$$F \ ::= \ p \mid \bot \mid (F \to F) \mid t{:}F \mid P_{\geq \mathsf{p}}F$$

where $p \in \mathsf{Prop}$, $t \in \mathsf{Tm}$ and $\mathsf{p} \in \mathsf{S}$.

The other propositional connectives are defined as abbreviations, in the standard way (viz. Definition 2.4). Moreover, precedence and the associativity of the logical operators is similar to the one for modal logic (viz. Definition 2.5) and we might omit the parentheses, accordingly. The precedence of the probability operator is assumed the same as $\neg$.

We will also use the following abbreviations for the probability operator

$$P_{<\mathsf{p}}F \equiv \neg P_{\geq \mathsf{p}}F \qquad P_{\leq \mathsf{p}}F \equiv P_{\geq 1-\mathsf{p}}\neg F$$
$$P_{>\mathsf{p}}F \equiv \neg P_{\leq \mathsf{p}}F \qquad P_{=\mathsf{p}}F \equiv P_{\geq \mathsf{p}}F \wedge P_{\leq \mathsf{p}}F$$

It is worth mentioning that $\mathcal{L}_{\mathsf{PPJ}}$ allows the iteration of the probability operator, e.g., formula $P_{\geq \mathsf{p}}P_{\geq \mathsf{q}}F \in \mathcal{L}_{\mathsf{PPJ}}$, while this was not allowed in the first probabilistic logic, PJ, defined also by Kokkinis et al., in [22]. That is what the double P in PPJ stands for. Moreover, in contrast to PJ, justification over probability formulae is also allowed, e.g., $t{:}P_{\geq \mathsf{p}}F \in \mathcal{L}_{\mathsf{PPJ}}$. We are ready to define the axiomatic system PPJ.

4.2.1 Axiomatization of Probabilistic Justification Logic

Definition 4.16 (The PPJ Logic)**.** The axiomatic system PPJ constitutes from the axiomatic system J_0 defined on $\mathcal{L}_{\mathsf{PPJ}}$, expanded by the axiomatic schemes and rules of inference of Table 4.3, where $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ and $\mathsf{p}, \mathsf{q} \in \mathsf{S}$ and $\mathsf{p}' \in \mathsf{S}_{>0}$.

Axiomatic Schemata	
$P_{\geq 0}F$	**PI**
$P_{\leq \mathsf{p}}F \to P_{<\mathsf{q}}F$, where $\mathsf{p} < \mathsf{q}$	**WE**
$P_{<\mathsf{p}}F \to P_{\leq \mathsf{p}}F$	**LE**
$P_{\geq \mathsf{p}}F \wedge P_{\geq \mathsf{q}}G \wedge P_{\geq 1}\neg(F \wedge G) \to P_{\geq \min\{1,\mathsf{p}+\mathsf{q}\}}(F \vee G)$	**DIS**
$P_{\leq \mathsf{p}}F \wedge P_{<\mathsf{q}}G \to P_{<\mathsf{p}+\mathsf{q}}(F \vee G)$, where $\mathsf{p}+\mathsf{q} \leq 1$	**UN**
Rules of Inference	
From F, infer $P_{\geq 1}F$	**CE**
From $F \to P_{\geq \mathsf{p}'-\frac{1}{\mathsf{k}}}G$ for every $\mathsf{k} \in \mathbb{N}_{\geq \frac{1}{\mathsf{p}'}}$, infer $F \to P_{\geq \mathsf{p}'}G$	**ST**

Table 4.3: Axiomatic System PPJ

We expand the notion of constant specification for PPJ.

Definition 4.17 (Constant Specification for PPJ)**.** A *constant specification* CS *for* PPJ is a set of formulae s.t.:

- its elements are of the form

$$F := c_n{:}c_{n-1}{:}\ldots{:}c_1{:}A,$$

  where $n \geq 1$, $\{c_i\}_{i\in[n]} \subset \mathsf{Con}$ and A is an axiom of PPJ,

- if $c_n{:}c_{n-1}{:}\dots{:}c_1{:}A \in \mathsf{CS}$, then $c_{n-1}{:}\dots{:}c_1{:}A \in \mathsf{CS}$, where $n \geq 2$.

The notions of empty, axiomatically appropriate and total constant specification are defined in the obvious way. We also denote, as always, by $\mathsf{PPJ}(\mathsf{CS})$ the axiomatic system PPJ with the addition of formulae in CS as axioms.

Once again, we have to define derivation in PPJ. As in the modal case, we will separately define the derivation and the derivation from premises.

Definition 4.18 (Derivations in PPJ)**.** Let $F \in \mathcal{L}_{\mathsf{PPJ}}$ and CS a constant specification for PPJ. We define as a *derivation of F in* $\mathsf{PPJ}(\mathsf{CS})$, any countable sequence $F_1, F_2, \dots, F$ of formulae in $\mathcal{L}_{\mathsf{PPJ}}$, s.t. every F' in the sequence is:

- either an instance of an axiom scheme of PPJ,
- or a member of CS,
- or the result of the application of one of the rules of inference, to formulae of the subsequence before F'.

If there is a derivation for F in $\mathsf{PPJ}(\mathsf{CS})$, then we write $\vdash_{\mathsf{PPJ}(\mathsf{CS})} F$, and we say that F is a *theorem* of $\mathsf{PPJ}(\mathsf{CS})$, or that $\mathsf{PPJ}(\mathsf{CS})$ proves F.

Let $\Sigma \cup \{F\}$ be a set of formulae in $\mathcal{L}_{\mathsf{PPJ}}$. A *derivation of F from Σ, in* $\mathsf{PPJ}(\mathsf{CS})$, is a countable sequence $F_1, F_2, \dots, F$ of formulae in $\mathcal{L}_{\mathsf{PPJ}}$, s.t. every F' in the sequence is:

- either a theorem of $\mathsf{PPJ}(\mathsf{CS})$,
- or a member of Σ,
- or the result of the application of one of the rules of inference, other than **CE**, to formulae of the subsequence before F'.

If there is a derivation of F, from Σ, in $\mathsf{PPJ}(\mathsf{CS})$, then we write $\Sigma \vdash_{\mathsf{PPJ}(\mathsf{CS})} F$, and we say that F is *derivable in* $\mathsf{PPJ}(\mathsf{CS})$*, from the premises* Σ.

Note that the sequence of derivation could probably be denumerable, as **ST** is an infinite rule of inference, i.e., it has infinite number of premises. Let us give an example of some PPJ derivations.

Example 4.19. *For any* $\mathtt{p} \in \mathtt{S}$ *and any constant specification* CS *for* PPJ*, it holds that*

1. *if* $\vdash_{\mathsf{PPJ}(\mathsf{CS})} F$*, then* $\vdash_{\mathsf{PPJ}(\mathsf{CS})} P_{\geq \mathtt{p}} F$*;*
2. $\vdash_{\mathsf{PPJ}(\mathsf{CS})} P_{\geq 1}(F \to G) \to P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G$.

Note that the operator $P_{\geq 1}$ is normal*, while for arbitrary* $\mathtt{p} \in \mathtt{S}$ *the operator* $P_{\geq \mathtt{p}}$ *might not be normal.*

Proof.

1. If $\mathtt{p} = 1$ it follows by the rule of infernce **CE**. Else if $\mathtt{p} = 0$ it follows by axiom scheme **PI**. Elsewhere, if $\mathtt{p} \in \mathtt{S} \cap (0,1)$, we have for an arbitrary theorem $F \in \mathcal{L}_{\mathsf{PPJ}}$ of $\mathsf{PPJ(CS)}$:

$$
\begin{array}{lr}
F_1 : F & \\
F_2 : P_{\geq 1} F & \textbf{CE} \\
F_3 : P_{\geq 1-\mathtt{p}} \neg F \to \neg P_{\geq 1} F & \mathtt{p} < 1,\ \textbf{WE} \\
F_4 : \neg P_{\geq \mathtt{p}} F \to P_{\geq 1-\mathtt{p}} \neg F & \textbf{LE} \\
F_5 : \neg P_{\geq \mathtt{p}} F \to \neg P_{\geq 1} F & \textbf{P}\ 3,4 \\
F_6 : P_{\geq 1} F \to P_{\geq \mathtt{p}} F & \textbf{P}\ 5 \\
F_7 : P_{\geq \mathtt{p}} F & \textbf{MP}\ 2,6
\end{array}
$$

2. We will first unriddle two abbreviations. We observe that

$$
\begin{aligned}
\neg(P_{\leq 1-\mathtt{p}} \neg F \wedge \neg\neg P_{<\mathtt{p}} G) &\equiv \neg(P_{\geq \mathtt{p}} F \wedge \neg P_{\geq \mathtt{p}} G) && \text{abbr. for } P_{\leq \mathtt{p}} \text{ and } P_{<\mathtt{p}} \\
&\equiv P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G && \text{abbr. for } \wedge
\end{aligned}
$$

and

$$\neg P_{<1}(\neg F \vee G) \equiv P_{\geq 1}(F \to G),$$

where we applied the abbreviations for $\vee$ and $P_{<1}$.

We are ready to continue with the derivation.

$$
\begin{array}{lr}
F_1 : \neg((P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G) \wedge \neg(P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G)) & \textbf{P} \\
F_2 : \neg(P_{\leq 1-\mathtt{p}} \neg F \wedge \neg\neg P_{<\mathtt{p}} G) \to P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G & \text{1st abbr., abbr. for } \wedge \\
F_3 : \neg(P_{\leq 1-\mathtt{p}} \neg F \wedge P_{<\mathtt{p}} G) \to P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G & \textbf{P}\ 2 \\
F_4 : \neg P_{<1}(\neg F \vee G) \to P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G & \textbf{UN, P}\ 3 \\
F_5 : P_{\geq 1}(F \to G) \to P_{\geq \mathtt{p}} F \to P_{\geq \mathtt{p}} G & \text{2nd abbr}
\end{array}
$$

□

As always, we will give a conservativity result of PPJ in respect with CL, as also a consistency result and the applicability of deduction theorem in PPJ.

Theorem 4.20 (Conservativity of PPJ)**.**
PPJ is a *conservative extension* of classical logic, CL.

Corollary 4.21 (Consistency of PPJ)**.** Let CS an arbitrary constant specification for PPJ. Then, PPJ(CS) is consistent.

Theorem 4.22 (Deduction Theorem for PPJ)**.** Let CS an arbitrary constant specification for PPJ. Then, the deduction theorem holds for PPJ(CS).

4.2.2 Semantics for Probabilistic Justification Logic

Before continuing with the definition of the corresponding models, we have to give some auxiliary definitions for the construction of the probability setting. Let us start with the definition of an algebra.

Definition 4.23 (Algebra Over a Set)**.** Let W a non-empty set and H a non-empty subset of $\mathcal{P}(W)$. H is called an *algebra over* W iff the following hold:

- $W \in H$
- H is closed under finite unions, i.e.,

$$U, V \in H \Rightarrow U \cup V \in H,$$

- H is closed under complementation in W, i.e.,

$$U \in H \Rightarrow W \setminus U \in H.$$

Definition 4.24 (Finitely Additive Measure)**.** Let H be an algebra over W.
$\mu\colon H \to [0, 1]$ is called a *finitely additive measure* iff the following hold:

- $\mu(W) = 1$,
- μ is finitely additive, i.e., for any $U, V \in H$ it holds

$$U \cap V = \emptyset \Rightarrow \mu(U \cup V) = \mu(U) + \mu(V).$$

Definition 4.25 (Finitely Additive Probability Space)**.** $\langle W, H, \mu\rangle$ is a *finitely additive probability space* iff the following hold:

- W is a non-empty set,
- H is an algebra over W,
- $\mu\colon H \to [0, 1]$ is a finitely additive measure.

Definition 4.26 (PPJ-Model)**.** Let CS be an arbitrary constant specification for PPJ. A $\mathsf{PPJ}(\mathsf{CS})$-*model* $\mathcal{M} = \langle U, W, H, \mu, *\rangle$ is defined as follows:

- U is a non-empty set of worlds,
- W, H and μ are functions over U, s.t. for each $w \in U$ $\langle W(w), H(w), \mu(w)\rangle$ is a finitely additive probability space,
- $*$ is a $\mathsf{PPJ}(\mathsf{CS})$-*modular model* over U, i.e., for any $w \in U$, $*(w)$ is the expansion of a basic $\mathsf{J}_0(\mathsf{CS})$-model to the language $\mathcal{L}_{\mathsf{PPJ}}$, that is

$$*(w)\restriction_{\mathsf{Prop}}\colon \mathsf{Prop} \to \{0, 1\}$$

and

$$*(w)\restriction_{\mathsf{Tm}}\colon \mathsf{Tm} \to \mathcal{P}(\mathcal{L}_{\mathsf{PPJ}})$$

and $*(w)$ satisfies the **J**, $+$ and CS closure conditions, for the language $\mathcal{L}_{\mathsf{PPJ}}$.

Note that PPJ-modular models is the expected expansion of modular models to the language $\mathcal{L}_{\mathsf{PPJ}}$. As in modular models, we will some times write $*_w$ instead of $*(w)$. We will also use the notation W_w, H_w and μ_w for $W(w)$, $H(w)$ and $\mu(w)$, respectively.

Definition 4.27 (Truth in PPJ-Models)**.** Let CS an arbitrary constant specification for PPJ. Truth of formulae in $\mathsf{PPJ(CS)}$-models is interpreted on pairs $(\mathcal{M}, w)$, where $\mathcal{M} = \langle U, W, H, \mu, * \rangle$ is an arbitrary $\mathsf{PPJ(CS)}$-model and w is some world in U. Specifically, we recursively define that a formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ is true (or satisfied) in $(\mathcal{M}, w)$, denoted as $\mathcal{M}, w \Vdash F$, as follows:

$$\begin{array}{lll}
\mathcal{M}, w \nVdash \bot & & \\
\mathcal{M}, w \Vdash p & \text{iff} & p^{*_w} = 1 \\
\mathcal{M}, w \Vdash F \to G & \text{iff} & \mathcal{M}, w \nVdash F \text{ or } \mathcal{M}, w \Vdash G \\
\mathcal{M}, w \Vdash t{:}F & \text{iff} & F \in t^{*_w} \\
\mathcal{M}, w \Vdash P_{\geq \mathtt{p}} F & \text{iff} & [F]_{\mathcal{M},w} \in H_w \text{ and } \mu_w\Big([F]_{\mathcal{M},w}\Big) \geq \mathtt{p}
\end{array}$$

where $p \in \mathsf{Prop}$, $F, G \in \mathcal{L}_{\mathsf{PPJ}}$, $\mathtt{p} \in \mathtt{S}$ and for any $F' \in \mathcal{L}_{\mathsf{PPJ}}$

$$[F']_{\mathcal{M},w} := \{u \in W_w \mid \mathcal{M}, u \Vdash F'\}.$$

- We say that a formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ is *true* in (or *satisfied* by) some $\mathsf{PPJ(CS)}$-model $\mathcal{M}$, and we denote $\mathcal{M} \Vdash F$ iff it is true in all worlds of $\mathcal{M}$.
- We say that a set Σ of formulae in $\mathcal{L}_{\mathsf{PPJ}}$ is *true* in some world w of some $\mathsf{PPJ(CS)}$-model $\mathcal{M}$, and we denote $\mathcal{M}, w \Vdash \Sigma$ iff all members of Σ are true in w. We write $\mathcal{M} \Vdash \Sigma$ iff Σ is true in all worlds of $\mathcal{M}$.

As it is obvious by the definition of truth in PPJ-models, the reason why a formula of the form $P_{\geq \mathtt{p}} F$ might be false in some world w of some model $\mathcal{M}$ might be due to two different reasons:

- it is probable that $[F]_{\mathcal{M},w} \notin H_w$, i.e., the set of worlds that satisfy F is not measurable in $\langle W_w, H_w, \mu_w \rangle$;
- or it might hold that $\mu_w\Big([F]_{\mathcal{M},w}\Big) < \mathtt{p}$, i.e., the measure of the set of worlds that satisfy F is less than $\mathtt{p}$.

The first reason of why $P_{\geq \mathtt{p}} F$ is not true in w seems to be away of our intuition. In a sense, it states that in w we can not speak about the probability of F to be true, and it does not state, as it might probably expected, the fact that in w, F has less probability than $\mathtt{p}$ to be true. In order to eliminate this case we define the class of *measurable* PPJ*-models*.

Definition 4.28 (Measurable PPJ-Models)**.** Let CS be an arbitrary constant specification for PPJ and $\mathcal{M} = \langle U, W, H, \mu, * \rangle$ be an arbitrary $\mathsf{PPJ(CS)}$-model.
$\mathcal{M}$ is called *measurable* iff for every $w \in U$ and for every $F \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that

$$[F]_{\mathcal{M},w} \in H_w.$$

We denote by $\mathsf{PPJ(CS)}_{\mathsf{Meas}}$ the class of measurable $\mathsf{PPJ(CS)}$-models.

Clearly, for any measurable PPJ-model $\mathcal{M} = \langle U, W, H, \mu, * \rangle$, a formula of the form $P_{\geq \mathtt{p}} F$ is false in some world w iff $\mu_w\Big([F]_{\mathcal{M},w}\Big) < \mathtt{p}$, i.e., if the probability of F to be true in w is less than $\mathtt{p}$, as wanted.

Definition 4.29. Let CS an arbitrary constant specification for PPJ.

- We say that a formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ is CS-*satisfiable* iff there is some $\mathsf{PPJ}(\mathsf{CS})$-measurable model $\mathcal{M} \in \mathsf{PPJ}(\mathsf{CS})_{\mathsf{Meas}}$ and some world w in this model, s.t. F is true in $(\mathcal{M}, w)$.
- We say that a formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ is a CS-*consequence* of a set Σ of formulae in $\mathcal{L}_{\mathsf{PPJ}}$, and we write $\Sigma \Vdash_{\mathsf{CS}} F$ iff for any $\mathcal{M} \in \mathsf{PPJ}(\mathsf{CS})_{\mathsf{Meas}}$, we have that

$$\mathcal{M} \Vdash \Sigma \Rightarrow \mathcal{M} \Vdash F.$$

Theorem 4.30 (Soundness and Completeness for PPJ)**.** Let CS an arbitrary constant specification for PPJ.
$\mathsf{PPJ}(\mathsf{CS})$ is sound and complete in respect with the class $\mathsf{PPJ}(\mathsf{CS})_{\mathsf{Meas}}$.

Theorem 4.31 (Strong Soundness and Completeness for PPJ)**.** Let CS an arbitrary constant specification for PPJ.
$\mathsf{PPJ}(\mathsf{CS})$ is strongly sound and strongly complete in respect with the class $\mathsf{PPJ}(\mathsf{CS})_{\mathsf{Meas}}$, i.e., for any set of formulae $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PPJ}}$ it holds that

$$\Sigma \vdash_{\mathsf{PPJ(CS)}} F \Rightarrow \Sigma \Vdash_{\mathsf{CS}} F \qquad \text{strong soundness}$$
$$\Sigma \vdash_{\mathsf{PPJ(CS)}} F \Leftarrow \Sigma \Vdash_{\mathsf{CS}} F \qquad \text{strong completeness}$$

4.3 Pavelka Style Fuzzy Justification Logic

In this section whenever we assume some justification logic JL, we mean any justification logic constructed as an extension of J_0 with the addition of some axiom scheme in Table 2.5 other than **J5**. The language of $\mathsf{RPL}(\mathsf{JL})$, $\mathcal{L}_{\mathsf{RPL(JL)}}$, is constructed from the language of justification logic, $\mathcal{L}_{\mathsf{J}}$, with the addition of a constant $\overline{\mathtt{p}}$ for every $\mathtt{p} \in \mathtt{S}$, where $\perp$ is replaces by $\overline{0}$.

Definition 4.32 (Fuzzy Justification Language $\mathcal{L}_{\mathsf{RPL(JL)}}$)**.** The *fuzzy justification language*, $\mathcal{L}_{\mathsf{RPL(JL)}}$, is defined by the following BNF-notation:

$$F \ ::= \ p \mid \overline{\mathtt{p}} \mid (F \to F) \mid t{:_{\mathtt{p}}}F$$

where $p \in \mathsf{Prop}$, $t \in \mathsf{Tm}$ and $\mathtt{p} \in \mathtt{S}$.

We also define the following propositional connectives:

$$\neg F \equiv \big(F \to \overline{0}\big)$$
$$(F \& G) \equiv \neg\,(F \to \neg G) \qquad (F \wedge G) \equiv F \& (F \to G)$$
$$(F \veebar G) \equiv (\neg F \to G) \qquad (F \vee G) \equiv (F \to G) \to G$$
$$(F \leftrightsquigarrow G) \equiv ((F \to G)\,\&(G \to F)) \qquad (F \leftrightarrow G) \equiv ((F \to G) \wedge (G \to F))$$

Moreover, we define, as abbreviations, the following operators:

$$t{:_{\mathtt{p}}}F \equiv \overline{\mathtt{p}} \to t{:}F \qquad t{:^{\mathtt{p}}}F \equiv t{:}F \to \overline{\mathtt{p}} \qquad t \overset{\mathtt{p}}{:} F \equiv \overline{\mathtt{p}} \leftrightarrow t{:}F$$

It is worth mentioning we have no more the usual Boolean connectives of classical logic, rather some new connectives defined for *Łukasiewicz logic*. The corresponding names for each connective are given in Table 4.3.

Name	Propositional Connective
Negation	$\neg$
Strong Conjuction	$\&$
Weak Conjuction	$\wedge$
Strong Disjunction	$\veebar$
Weak Disjunction	$\vee$
Strong Equivalence	$\Bumpeq$
Weak Equivalence	$\leftrightarrow$

Table 4.4: $\mathcal{L}_{\mathsf{RPL(JL)}}$ Propositional Connectives

Definition 4.33 (Omitting Parentheses in $\mathcal{L}_{\mathsf{RPL(JL)}}$)**.** In order to omit parentheses in $\mathcal{L}_{\mathsf{RPL(JL)}}$ we define the precedence and the associativity of the logical operators.

- $:$, $:_{\mathtt{p}}$, $:^{\mathtt{p}}$, $\overset{\mathtt{p}}{:}$ and $\neg$ are granted the highest precedence. They are assumed right-associative.
- $\veebar$, $\&$, $\vee$ and $\wedge$ are granted the same precedence. They are also assumed right-associative, with respect to each other. Moreover, they all have the associative property, e.g.,

$$((F\&G)\,\&H) \equiv (F\&(G\&H)) \equiv F\&H\&G.$$

- $\rightarrow$, $\Bumpeq$ and $\leftrightarrow$ are granted the lowest precedence. They are also assumed right-associative.

Before defining the axiomatic system $\mathsf{RPL(JL)}$ we have also to define Łukasiewicz t-norm.

Definition 4.34. *Łukasiewicz t-norm*, $*_{\text{Ł}} \colon [0,1] \times [0,1] \to [0,1]$, is defined as

$$\mathtt{p} *_{\text{Ł}} \mathtt{q} := *_{\text{Ł}}(\mathtt{p},\mathtt{q}) = \max\{0, \mathtt{p}+\mathtt{q}-1\}.$$

Łukasiewicz implication, $\Rightarrow_{\text{Ł}} \colon [0,1] \times [0,1] \to [0,1]$, is defined as

$$\mathtt{p} \Rightarrow_{\text{Ł}} \mathtt{q} := \Rightarrow_{\text{Ł}}(\mathtt{p},\mathtt{q}) = \min\{1, 1-\mathtt{p}+\mathtt{q}\}.$$

Lemma 4.35. Łukasiewicz t-norm is a *continuous t-norm*, i.e., for any $\mathtt{p},\mathtt{q},\mathtt{r} \in [0,1]$

- $\mathtt{p} *_{\text{Ł}} \mathtt{q} = \mathtt{q} *_{\text{Ł}} \mathtt{p}$;
- $(\mathtt{p} *_{\text{Ł}} \mathtt{q}) *_{\text{Ł}} \mathtt{r} = \mathtt{p} *_{\text{Ł}} (\mathtt{q} *_{\text{Ł}} \mathtt{r})$;
- if $\mathtt{p} \leq \mathtt{q}$, then $\mathtt{p} *_{\text{Ł}} \mathtt{r} \leq \mathtt{q} *_{\text{Ł}} \mathtt{r}$;
- $1 *_{\text{Ł}} \mathtt{p} = \mathtt{p}$;
- $*_{\text{Ł}}$ is continuous function.

Moreover, Łukasiewicz implication is the corresponding *residuum*, i.e., for any $\mathtt{p},\mathtt{q} \in [0,1]$

$$\mathtt{p} \Rightarrow_{\text{Ł}} \mathtt{q} = \max\{\mathtt{r} \in [0,1] \mid \mathtt{p} *_{\text{Ł}} \mathtt{r} \leq \mathtt{q}\}.$$

It easy to observe that for every $\mathtt{p},\mathtt{q} \in \mathtt{S}$ all of $\mathtt{p} *_{\text{Ł}} \mathtt{q}$, $\mathtt{p} \Rightarrow_{\text{Ł}} \mathtt{q}$, $\min\{\mathtt{p},\mathtt{q}\}$ and $\max\{\mathtt{p},\mathtt{q}\}$ belong in $\mathtt{S}$. Thus, all of $\overline{\mathtt{p} *_{\text{Ł}} \mathtt{q}}$, $\overline{\mathtt{p} \Rightarrow_{\text{Ł}} \mathtt{q}}$, $\overline{\min\{\mathtt{p},\mathtt{q}\}}$ and $\overline{\max\{\mathtt{p},\mathtt{q}\}}$ belong in $\mathcal{L}_{\mathsf{RPL(JL)}}$.

4.3.1 Axiomatization of Fuzzy Justification Logic

Definition 4.36 (The $\mathsf{RPL}(\mathsf{JL})$ Logic)**.** Let JL arbitrary justification logic. The axiomatic system $\mathsf{RPL}(\mathsf{JL})$ constitutes from the axiomatic system JL, without axiom schemes **P**, defined on $\mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$, expanded by the axiomatic schemes and rules of inference of Table 4.5, where $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ and $\mathtt{p}, \mathtt{q} \in \mathtt{S}$.

Axiomatic Schemata	
$F \to G \to F$	**Ł1**
$(F \to G) \to (G \to H) \to (F \to H)$	**Ł2**
$(\neg F \to \neg G) \to G \to F$	**Ł3**
$((F \to G) \to G) \to (G \to F) \to F$	**Ł4**
$(\overline{\mathtt{p}} \to \overline{\mathtt{q}}) \eqcirc \overline{\mathtt{p} \Rightarrow_{Ł} \mathtt{q}}$	**TC**
Rules of Inference	
From F and $F \to G$, infer G	**MP**

Table 4.5: Axiomatic System RPL

The notion of constant specification is also present in $\mathsf{RPL}(\mathsf{JL})$.

Definition 4.37 (Constant Specification for $\mathsf{RPL}(\mathsf{JL})$)**.** Let JL an arbitrary justification logic. A *constant specification* CS *for* $\mathsf{RPL}(\mathsf{JL})$ is a set of formulae s.t.:

- its elements are of the form

$$F \coloneqq c_n{:_1}c_{n-1}{:_1}\ \ldots{:_1}c_1{:_1}A,$$

  where $n \geq 1$, $\{c_i\}_{i\in[n]} \subset \mathsf{Con}$ and A is an axiom of $\mathsf{RPL}(\mathsf{JL})$,

- if $c_n{:_1}c_{n-1}{:_1}\ \ldots{:_1}c_1{:_1}A \in \mathsf{CS}$, then $c_{n-1}{:_1}\ \ldots{:_1}c_1{:_1}A \in \mathsf{CS}$, where $n \geq 2$.

The notions of empty, axiomatically appropriate and total constant specification is defined in the obvious way. We also denote by $\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))$ the axiomatic system $\mathsf{RPL}(\mathsf{JL})$ with the addition of formulae in CS as axioms.

Definition 4.38 (Derivations in Fuzzy Justification Logic)**.** Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL}(\mathsf{JL})$. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$. A *derivation of* F, *from* Σ, *in* $\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))$, is a finite sequence $F_1, \ldots, F_n$ of formulae, s.t.:

- $F_n \coloneqq F$,
- every F_i in the sequence is
  - either an axiom of $\mathsf{RPL}(\mathsf{JL})$,
  - or a member of CS,
  - or a member of Σ,
  - or the result of the application of modus ponens, to formulae of the subsequence $F_1, \ldots, F_{i-1}$.

If there is a derivation of F, from Σ, in $\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))$, then we write $\Sigma \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F$, and we say that F is *derivable* in $\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))$, from the premises Σ. If Σ is the empty set, i.e., we do not assume any premises, then we write $\vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F$ and we say that F is a theorem of $\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))$.

There exists a special case of deduction theorem in $\mathsf{RPL}(\mathsf{JL})$.

Theorem 4.39 (Deduction Theorem for Fuzzy Justification Logic)**.** Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL}(\mathsf{JL})$. Let also arbitrary set of formulae $\Sigma \cup \{F, G\} \subseteq \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$ s.t. $\Sigma \cup \{F\} \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} G$. Then, there is some $n \in \mathbb{N}_{>0}$ s.t.

$$\Sigma \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F^n \to G,$$

where

$$F^n := \underbrace{F\& \cdots \&F}_{n}.$$

We also have a corresponding Lifting Lemma.

Lemma 4.40 ($\mathsf{RPL}(\mathsf{JL})$-Internalization Property & Lifting Lemma for $\mathsf{RPL}(\mathsf{JL})$)**.**
Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL}(\mathsf{JL})$. Then, $\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))$ has the $\mathsf{RPL}(\mathsf{JL})$-*internalization property*, i.e., for any formula $F \in \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$ it holds that

$$\vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F \Rightarrow (\exists t \in \mathsf{Tm})\big[\vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} t{:}F\big].$$

Let $\mathsf{RPL}(\mathsf{JL})$ has the $\mathsf{RPL}(\mathsf{JL})$-internalization property relative to some constant specification CS (e.g. axiomatically appropriate). Then if

$$F_1, \ldots, F_n \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F,$$

then it holds that for every $t_1, \ldots, t_n \in \mathsf{Tm}$, there exists some $t \in \mathsf{Tm}$ s.t.

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} t{:}F.$$

4.3.2 Semantics for Fuzzy Justification Logic

Definition 4.41 (Fuzzy Fitting Model)**.** A *fuzzy Fitting model* is a structure $M = \langle W, R, V, E\rangle$, where $\langle W, R\rangle$ is a Kripke frame and

- $V\colon W \times \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})} \to [0,1]$, called *fuzzy valuation function* s.t. for any world $w \in W$, any $\mathtt{p} \in \mathsf{S}$, any formulae $F, G \in \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$ and any term $t \in \mathsf{Tm}$ it holds
  - $V(w, \overline{\mathtt{p}}) := \mathtt{p}$;
  - $V(w, F \to G) := V(w, F) \Rightarrow_{Ł} V(w, G)$;
  - $V(w, t{:}F) := \min\big\{E(w, t, F), V_w^{\square}(F)\big\}$, where

$$V_w^{\square}(F) := \inf_{v \in R[w]} \{V(v, F)\}.$$

- $E\colon W \times \mathsf{Tm} \times \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})} \to [0,1]$, called *fuzzy evidence function*.

Given a fuzzy Fitting model $M = \langle W, R, V, E\rangle$, we say that M is *based on* the Kripke frame $F = \langle W, R\rangle$, or that F is the frame *underlying* M.

We will also, for brevity, denote for any $w \in W$, any term $t \in \mathsf{Tm}$ and any formula $F \in \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$

$$V(w, F) := V_w(F) \qquad\qquad W(w, t, F) := W_w(t, F).$$

Lemma 4.42. Let $M = \langle W, R, V, E \rangle$ an arbitrary fuzzy Fitting model. Then, for any world $w \in W$, any formulae $F, G \in \mathcal{L}_{\mathsf{RPL(JL)}}$, any term $t \in \mathsf{Tm}$ and any $\mathtt{p} \in \mathtt{S}$, it holds:

- $V_w(\neg F) := 1 - V_w(F)$,
- $V_w(F \&, G) := V_w(F) *_{Ł} V_w(G)$
- $V_w(F \wedge G) := \min\{V_w(F), V_w(G)\}$,
- $V_w(F \veebar G) := \min\{1, V_w(F) + V_w(G)\}$,
- $V_w(F \vee G) := \max\{V_w(F), V_w(G)\}$,
- $V_w(F \leftrightharpoons G) := 1 - |V_w(F) - V_w(G)|$,
- $V_w(F \leftrightarrow G) := 1 - |V_w(F) - V_w(G)|$,
- $V_w(t{:}_{\mathtt{p}} F) := \mathtt{p} \Rightarrow_{Ł} V_w(t{:}F)$,
- $V_w(t{:}^{\mathtt{p}} F) := V_w(t{:}F) \Rightarrow_{Ł} \mathtt{p}$,
- $V_w\left(t \overset{\mathtt{p}}{:} F\right) := \min\{V_w(t{:}_{\mathtt{p}} F), V_w(t{:}^{\mathtt{p}} F)\}$.

Definition 4.43 (Truth in Fuzzy Fitting Models)**.** Truth of $\mathcal{L}_{\mathsf{RPL(JL)}}$-formulae in fuzzy Fitting models is interpreted on pairs (M, w), where $M = \langle W, R, V, E \rangle$ is an arbitrary fuzzy Fitting model and w is some world in W. Specifically, we define that a formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ is true (or satisfied) in (M, w), denoted as $M, w \Vdash F$, as follows

$$M, w \Vdash F \Leftrightarrow V_w(F) = 1.$$

- We say that a formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ is *satisfiable* iff there is some fuzzy Fitting model M and some world w in this model, s.t. F is true in (M, s).
- We say that a formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ is *satisfiable* in a certain class $\mathcal{C}$ of fuzzy Fitting models iff F is true in some (M, w), where M belongs in $\mathcal{C}$.
- We say that a formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ is *true* in a fuzzy Fitting model M, and we denote $M \Vdash F$ iff it is true in all worlds of M.
- We say that a formula F is *valid* in a certain class $\mathcal{C}$ of fuzzy Fitting models, and we denote $\mathcal{C} \Vdash F$ or $\Vdash_{\mathcal{C}} F$ iff F is true in every fuzzy Fitting model M belonging in $\mathcal{C}$.
- We say that a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{RPL(JL)}}$ is *true* in some world w, and we denote $M, w \Vdash \Sigma$ iff all members of Σ are true in w. We define the *truth* of Σ in a fuzzy Fitting model M, and the *satisfiability* and *validity* of Σ in a class of fuzzy Fitting models, $\mathcal{C}$, in the obvious way.
- We say that a formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ is a *semantic consequence* of a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{RPL(JL)}}$ in a class of fuzzy Fitting models $\mathcal{C}$, and we denote $\Sigma \Vdash_{\mathcal{C}} F$ iff for every fuzzy Fitting model M based on some frame in $\mathcal{C}$, $M \Vdash \Sigma$ implies $M \Vdash F$.

Lemma 4.44. Let $M = \langle W, R, V, E\rangle$ an arbitrary fuzzy Fitting model. Then, for any world $w \in W$, any formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$, any term $t \in \mathsf{Tm}$ and any $\mathtt{p} \in \mathtt{S}$, it holds that

- $M, w \Vdash t{:}_{\mathtt{p}}F \Leftrightarrow V_w(t{:}F) \geq \mathtt{p}$,
- $M, w \Vdash t{:}^{\mathtt{p}}F \Leftrightarrow V_w(t{:}F) \leq \mathtt{p}$,
- $M, w \Vdash t \overset{\mathtt{p}}{:} F \Leftrightarrow V_w(t{:}F) = \mathtt{p}$.

The previous lemma denotes the intended meaning of $t{:}_{\mathtt{p}}F$, $t{:}^{\mathtt{p}}F$ and $t \overset{\mathtt{p}}{:} F$, i.e.,

- t is a justification for believing F at least to the certainty degree $\mathtt{p}$,
- t is a justification for believing F at most to the certainty degree $\mathtt{p}$,
- t is a justification for believing F with certainty degree $\mathtt{p}$,

respectively.

Once again, for every justification logic JL, we have to introduce to the accessibility relation and the fuzzy evidence function, conditions corresponding to the axiomatic system $\mathsf{RPL(JL)}$ that we want to presume.

Definition 4.45. The *modal* and *fuzzy evidence conditions* of fuzzy Fitting models, for the justification axiom schemes, which we have defined, are given in the following table.

Axiom Schemes	Modal Conditions	Fuzzy Evidence Conditions
J	-	$\min\{E_w(s, F \to G), E_w t, F\} \leq E_w(s \cdot t, G)$
$+$	-	$E_w(s, F) \leq E_w(s + t, F) \quad E_w(t, F) \leq E_w(s + t, F)$
JT	based on $\mathcal{T}$	-
JD	-	$E_w(t, \overline{0}) = 0$
J4	based on $\mathcal{K}4$	*Monotonicity Condition* if $u \in R[w]$, then $E_w(t, F) \leq E_v(t, F)$ *!-Condition* $E_w(t, F) \leq E_w(!t, t{:}F)$

Once more, the first two evidence conditions, i.e., the **J** and $+$ evidence conditions, are called *minimum fuzzy evidence conditions*. We have also to introduce some additional conditions to the Fitting models, so that they respect a given constant specification CS.

Definition 4.46. Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL(JL)}$. We say that a fuzzy Fitting model $M = \langle W, R, V, E\rangle$ *meets constant specification* iff for each $c{:}F \in \mathsf{CS}$, it holds that $E(c, F) := W$.

Moreover, we say that a fuzzy Fitting model $M = \langle W, R, V, E\rangle$ is a $\mathsf{RPL(JL(CS))}$-*model* (or a $\mathsf{RPL(JL(CS))}$-*fuzzy Fitting model*) iff it meats constant specification CS and also respects the corresponding modal and fuzzy evidence conditions.

- We write $\Vdash_{\mathsf{RPL(JL(CS))}} F$, if $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ is true in all $\mathsf{RPL(JL(CS))}$-models and we say that F is $\mathsf{RPL(JL(CS))}$-*valid*.
- We write $\Sigma \Vdash_{\mathsf{RPL(JL(CS))}} F$, if $\Sigma \Vdash_{\mathcal{C}} F$ where $\mathcal{C}$ the class of $\mathsf{RPL(JL(CS))}$-models and we say that F is a *semantic consequence of* Σ *in* $\mathsf{RPL(JL(CS))}$.

Before continuing with the corresponding proof of soundness and completeness, we have to give some additional definitions.

Definition 4.47 (Truth & Probability Degrees)**.** Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL}(\mathsf{JL})$. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$.

- The *truth degree* of F over Σ respecting CS in $\mathsf{RPL}(\mathsf{JL})$ is defined as follows:

$$\|F\|_{\Sigma}^{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} := \inf\{V_w(F) \mid \langle W, R, V, E\rangle \text{ is a } \mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))\text{-model and } w \in W\}.$$

- The *probability degree* of F over Σ respecting CS in $\mathsf{RPL}(\mathsf{JL})$ is defined as follows:

$$|F|_{\Sigma}^{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} := \sup\left\{\mathtt{p} \in \mathtt{S} \mid \Sigma \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} \overline{\mathtt{p}} \to F\right\}.$$

The intended interpretation of $\|F\|_{\Sigma}^{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))}$ is the degree to which F is a semantic consequence of Σ. On the other hand, the intended interpretation of $|F|_{\Sigma}^{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))}$ is the degree to which F is provable by Σ. The expected notion of completeness is those two degrees to be equal. This is exactly what the Pavelka-style completeness theorem states.

Theorem 4.48 (Strong Soundness & Pavelka-Style Completeness for $\mathsf{RPL}(\mathsf{JL})$)**.** Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL}(\mathsf{JL})$. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{RPL}(\mathsf{JL})}$.

$$\Sigma \vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F \Rightarrow \Sigma \Vdash_{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} F \qquad \text{strong soundness}$$

$$\|F\|_{\Sigma}^{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} = |F|_{\Sigma}^{\mathsf{RPL}(\mathsf{JL}(\mathsf{CS}))} \qquad \text{Pavelka-style completeness}$$

4.4 Possibilistic Justification Logic

As in the other probability-justification logics, in order to define the logic PJL we have to expand $\mathcal{L}_{\mathsf{J}}$ to a corresponding language $\mathcal{L}_{\mathsf{PJL}}$.

Definition 4.49 (Possibilistic Justification Language $\mathcal{L}_{\mathsf{PJL}}$)**.** The *possibilistic justification language*, $\mathcal{L}_{\mathsf{PJL}}$, is defined by the following BNF-notation:

$$F \ ::= \ p \mid \bot \mid (F \to F) \mid t{:_{\mathtt{p}}}F \mid t{:^{+}_{\mathtt{p}}}F$$

where $p \in \mathsf{Prop}$, $t \in \mathsf{Tm}$ and $\mathtt{p} \in \mathtt{S}$.

The other propositional connectives are defined as abbreviations, in the standard way (viz. Definition 2.4). Moreover, precedence and the associativity of the logical operators is similar to the one for modal logic (viz. Definition 2.5) and we might omit the parentheses, accordingly. The precedence of the possibilistic operators $:_{\mathtt{p}}$ and $:^{+}_{\mathtt{p}}$ is assumed the same as the one of the justification operator $:$ in $\mathcal{L}_{\mathsf{J}}$.

As it is obvious, from the previous definition, $\mathcal{L}_{\mathsf{PJL}}$ is the extension of the logic of uncertain justifications, $\mathcal{L}_{\mathsf{UJ}}$, with the addition of the possibilistic operator $:^{+}_{\mathtt{p}}$, where this time $\mathtt{p} \in \mathtt{S}$, i.e., could probably be equal to 0. The intended meaning of formulae of the form $t{:_{\mathtt{p}}}F$, as stated in [31, 32] is that

"According to the evidence t, F is believed with certainty at least p.",

while the corresponding for formulae of the form $t{:}^{+}_{\mathsf{p}}F$ is that "According to the evidence t, F is believed with certainty greater than p.".

Definition 4.50. Let $F \in \mathcal{L}_{\mathsf{PJL}}$.

- Any $\mathsf{p} \in \mathsf{S}$ that appears in F is called a *grade* (or a *degree*) of F.
- We denote by $\mathcal{G}(F)$ the set of grades appearing in F.

Let $\Sigma \subseteq \mathcal{L}_{\mathsf{PJL}}$. Then, we denote

$$\mathcal{G}(\Sigma) \coloneqq \bigcup_{F \in \Sigma} \mathcal{G}(F).$$

Let $X \subseteq \mathsf{S}$, e.g. $X = \mathcal{G}(\Sigma)$, where $\Sigma \subseteq \mathcal{L}_{\mathsf{PJL}}$. Then, we denote by $\mathcal{L}_{\mathsf{PJL}}(X)$ the fragment of $\mathcal{L}_{\mathsf{PJL}}$ in which only grades in X occur.

4.4.1 Axiomatization of Possibilistic Justification Logic

Definition 4.51 (The PJL Logic)**.** The *possibilistic justification logic* PJL is given in Table 4.6, where $F, G \in \mathcal{L}_{\mathsf{PJL}}$, $s, t \in \mathsf{Tm}$ and $\mathsf{p}, \mathsf{q} \in \mathsf{S}$.

Axiomatic Schemata		
all theorems of CL in $\mathcal{L}_{\mathsf{PJL}}$		**P**
$s{:}_{\mathsf{p}}(F \to G) \to t{:}_{\mathsf{p}}F \to s \cdot t{:}_{\mathsf{p}}G$	PJL-application	**PJ**
$s{:}^{+}_{\mathsf{p}}(F \to G) \to t{:}^{+}_{\mathsf{p}}F \to s \cdot t{:}^{+}_{\mathsf{p}}G$	PJL^{+}-application	**PJ^{+}**
$s{:}_{\mathsf{p}}F \to s + t{:}_{\mathsf{p}}F$ & $t{:}_{\mathsf{p}}F \to s + t{:}_{\mathsf{p}}F$	PJL-sum	**PJ+**
$s{:}^{+}_{\mathsf{p}}F \to s + t{:}^{+}_{\mathsf{p}}F$ & $t{:}{+}\mathsf{p}^{+}F \to s + t{:}^{+}_{\mathsf{p}}F$	PJL^{+}-sum	**PJ^{+}+**
$s{:}_{0}F \wedge t{:}_{\mathsf{p}}F \to s{:}_{\mathsf{p}}F$	strongest justification	**SJ**
$s{:}^{+}_{0}F \wedge t{:}^{+}_{\mathsf{p}}F \to s{:}^{+}_{\mathsf{p}}F$	strongest justification^{+}	**SJ^{+}**
$t{:}_{\mathsf{p}}F \to t{:}^{+}_{\mathsf{q}}F$, where $\mathsf{p} > \mathsf{q}$	greater inequality 1	**G1**
$t{:}^{+}_{\mathsf{p}}F \to t{:}_{\mathsf{p}}F$	greater inequality 2	**G2**
$\neg t{:}^{+}_{1}F$	not greater	**NG**
Rules of Inference		
From F and $F \to G$, infer G	modus ponens	**MP**

Table 4.6: Axiomatic System PJL

As in PPJ and UJ, the notion of constant specification expands for PJL.

Definition 4.52 (Constant Specification for PJL)**.** A *constant specification* CS *for* PJL is a set of formulae s.t.:

- its elements are of the form

$$F \coloneqq c_n{:}_1 c_{n-1}{:}_1 \ \ldots {:}_1 c_1{:}_1 A,$$

  where $n \geq 1$, $\{c_i\}_{i \in [n]} \subset \mathsf{Con}$ and A is an axiom of PJL,
- if $c_n{:}_1 c_{n-1}{:}_1 \ \ldots {:}_1 c_1{:}_1 A \in \mathsf{CS}$, then $c_{n-1}{:}_1 \ \ldots {:}_1 c_1{:}_1 A \in \mathsf{CS}$, where $n \geq 2$.

The notions of empty, axiomatically appropriate and total constant specification are defined in the obvious way. We also denote, as always, by $\mathsf{PJL(CS)}$ the axiomatic system PJL with the addition of formulae in CS as axioms.

Similarly to the case of UJ, in [31, 32], where the possibilistic justification logic, PJL, was originally defined, the corresponding *internalization rule of inference* was assumed. That is for us, the constant specification is total. It is time to define the notion of derivation in PJL.

Definition 4.53 (Derivations in Possibilistic Justification Logic)**.** Let CS be a constant specification CS for PJL. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PJL}}$. A *derivation of F, from Σ, in* $\mathsf{PJL(CS)}$, is a finite sequence $F_1, \dots, F_n$ of formulae, s.t.:

- $F_n := F$,
- every F_i in the sequence is
  - either an axiom of PJL,
  - or a member of CS,
  - or a member of Σ,
  - or the result of the application of modus ponens, to formulae of the subsequence $F_1, \dots, F_{i-1}$.

If there is a derivation of F, from Σ, in $\mathsf{PJL(CS)}$, then we write $\Sigma \vdash_{\mathsf{PJL(CS)}} F$, and we say that F is *derivable* in $\mathsf{PJL(CS)}$, from the premises Σ. If Σ is the empty set, i.e., we do not assume any premises, then we write $\vdash_{\mathsf{PJL(CS)}} F$ and we say that F is a theorem of $\mathsf{PJL(CS)}$.

As expected by the syntactic and logical similarities of PJL and UJ, as also by Theorem 4.6, PJL is also a conservative extension of $\mathsf{J_0}$.

Theorem 4.54 (Conservativity of Possibilistic Justification Logic)**.**
PJL is a *conservative extension* of the basic justification logic, $\mathsf{J_0}$.

Corollary 4.55 (Consistency of Possibilistic Justification Logic)**.** Let CS an arbitrary constant specification for PJL. Then, $\mathsf{PJL(CS)}$ is consistent.

Let us give the corresponding deduction theorem for PJL.

Theorem 4.56 (Deduction Theorem for Possibilistic Justification Logic)**.** Let CS an arbitrary constant specification for PJL. Then, the deduction theorem holds for PJL.

We finally give a corresponding lifting lemma.

Lemma 4.57 (PJL-Internalization Property & Lifting Lemma for PJL)**.**
Let CS an axiomatically appropriate constant specification for PJL. Then, $\mathsf{PJL(CS)}$ has the PJL-*internalization property*, i.e., for any formula $F \in \mathcal{L}_{\mathsf{PJL}}$ it holds that

$$\vdash_{\mathsf{PJL(CS)}} F \Rightarrow (\exists t \in \mathsf{Tm})\big[\vdash_{\mathsf{PJL(CS)}} t{:}_1 F\big].$$

Let PJL has the PJL-internalization property relative to some constant specification CS (e.g. axiomatically appropriate). Then it holds that, if

$$F_1, \dots, F_n \vdash_{\mathsf{PJL(CS)}} F$$

then for any $t_1, \dots, t_n \in \mathsf{Tm}$, for any $\mathrm{p}_1, \dots, \mathrm{p}_n \in \mathsf{S}$ and any $I \subseteq \{1, \dots, n\}$ there exist some $t \in \mathsf{Tm}$ s.t.

$$\{t_i{:}_{\mathrm{p}_i} F_i \mid i \in I\} \cup \big\{t_i{:}^{+}_{\mathrm{p}_i} F_i \mid i \notin I\big\} \vdash_{\mathsf{PJL(CS)}} t{:}_{\mathrm{p}} F,$$

where $\mathrm{p} := \min_{i \in \{1,\dots,n\}} \{\mathrm{p}_i\}$.

4.4.2 Semantics for Possibilistic Justification Logic

Before continuing with the definition of the corresponding models for PJL,we have to give some auxiliary definitions.

Definition 4.58 (Possibility & Necessity Measures)**.** Let W a non-empty set. A *possibility distribution on* W is a function $\pi\colon W \to [0,1]$. Then, we can define the corresponding

- *possibility measure* $\Pi\colon \mathcal{P}(W) \to [0,1]$ s.t.

$$\Pi(X) \coloneqq \sup_{w\in X} \{\pi(w)\},$$

- *necessity measure* $N\colon \mathcal{P}(W) \to [0,1]$ s.t

$$N(X) \coloneqq 1 - \Pi(W \setminus X).$$

Let us give now, the PJL-semantics.

Definition 4.59 (PJL-Model)**.** A PJL-*model* is a structure $M = \langle W, R, V, E\rangle$, where

- W is a non-empty set of worlds;
- $R\colon W \times W \to [0,1]$ is a *fuzzy accesibility relation on* W;
- $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ is a *valuation function*;
- $E\colon \mathsf{Tm} \times \mathcal{L}_{\mathsf{PJL}} \to \mathcal{P}(W)$ is an evidence function.

We also define for each $w \in W$, the possibility distribution $\pi_w\colon W \to [0,1]$ s.t. for each $u \in W$

$$\pi_w(u) \coloneqq R(w,u)$$

and the corresponding possibility measure Π_w and necessity measure N_w.

From the Definition 4.59, above, it is probably obvious to the reader that a PJL-model is in fact a Fitting model for which the accessibility relation is replaced by a fuzzy accessibility relation. Let us now define the truth in such models.

Definition 4.60 (Truth in PJL-Models)**.** Truth of $\mathcal{L}_{\mathsf{PJL}}$-formulae in PJL-models is interpreted on pairs (M,w), where $M = \langle W, R, V, E\rangle$ is an arbitrary PJL-model and w is some world in M. Specifically, we define that a formula $F \in \mathcal{L}_{\mathsf{PJL}}$ is true (or satisfied) in (M,w), denoted as $M,w \Vdash F$, as in Fitting models, where for any term $t \in \mathsf{Tm}$, any $\mathtt{p} \in \mathtt{S}$ and any formula $F \in \mathcal{L}_{\mathsf{UJ}}$

$$M,w \Vdash t{:}_{\mathtt{p}}F \qquad \text{iff} \qquad \begin{cases} \textit{Modal Condition}: & N_w([F]_M) \geq \mathtt{p} \\ \textit{Evidence Condition}: & w \in E(t,F) \end{cases}$$

$$M,w \Vdash t{:}^{+}_{\mathtt{p}}F \qquad \text{iff} \qquad \begin{cases} \textit{Modal Condition}: & N_w([F]_M) > \mathtt{p} \\ \textit{Evidence Condition}: & w \in E(t,F) \end{cases}$$

- We say that a formula $F \in \mathcal{L}_{\mathsf{PJL}}$ is *satisfiable* iff there is some PJL-model M and some world w in this model, s.t. F is true in (M,s).

- We say that a formula $F \in \mathcal{L}_{\mathsf{PJL}}$ is *satisfiable* in a certain class $\mathcal{C}$ of PJL-models iff F is true in some (M, w), where M belongs in $\mathcal{C}$.
- We say that a formula $F \in \mathcal{L}_{\mathsf{PJL}}$ is *true* in a PJL-Fitting model M, and we denote $M \Vdash F$ iff it is true in all worlds of M.
- We say that a formula F is *valid* in a certain class $\mathcal{C}$ of PJL-models, and we denote $\mathcal{C} \Vdash F$ iff F is true in every PJL-model M belonging in $\mathcal{C}$.
- We say that a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{PJL}}$ is *true* in some world w, and we denote $M, w \Vdash \Sigma$ iff all members of Σ are true in w. We define the *truth* of Σ in a PJL-Fitting model M, and the *satisfiability* and *validity* of Σ in a class of PJL-models, $\mathcal{C}$, in the obvious way.
- We say that a formula $F \in \mathcal{L}_{\mathsf{PJL}}$ is a *semantic consequence* of a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{PJL}}$ in a class of PJL-models $\mathcal{C}$, and we denote $\Sigma \Vdash_{\mathcal{C}} F$ iff for every PJL-Fitting model M in $\mathcal{C}$, $M \Vdash \Sigma$ implies $M \Vdash F$.

Of course, we have once again to define suitable evidence conditions s.t. the PJL-models respect the axiom schemes of Table 4.6 and meat the given constant specification.

Definition 4.61. The minimum evidence conditions for the PJL-Fitting models are given in Table 4.7.

Axiom Schemes	Evidence Conditions
PJ, **PJ**$^{+}$	$E(s, F \rightarrow G) \cap E(t, F) \subseteq E(s \cdot t, G)$
PJ+, **PJ**$^{+}$**+**	$E(s, F) \cup E(t, F) \subseteq E(s + t, F)$

Table 4.7: PJL-Minimum Evidence Conditions

Definition 4.62. Let CS a constant specification for PJL. We say that a PJL-model $M = \langle W, R, V, E \rangle$ *meets constant specification* CS iff for each $c{:}F \in \mathsf{CS}$, it holds that $E(c, F) := W$.

We are ready to define the corresponding soundness and completeness theorem.

Theorem 4.63 (Soundness and Completeness for PJL)**.** Let CS an arbitrary constant specification for PJL.
PJL(CS) is sound and complete in respect with the class of PJL-models that meat constant specification CS and respect the PJL-minimum evidence conditions.

Theorem 4.64 (Finitely-Strong Soundness and Completeness for PJL)**.** Let CS an arbitrary constant specification for PJL. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PJL}}$ s.t. $\mathcal{G}(\Sigma \cup \{F\})$ is finite. Then, it holds that

$$\Sigma \vdash_{\mathsf{PJL(CS)}} F \Leftrightarrow \Sigma \Vdash_{\mathcal{C}} F,$$

where $\mathcal{C}$ the class of PJL-models that meat constant specification CS and respect the PJL-minimum evidence conditions.

4.5 The Ontology of Uncertainty in Justification Logic

In section 3.6, we studied the ontology of justification in respect with the various semantics that we have given. We concluded that in order for some evidence t to justify a statement F, it must be the case that

- t is convincing and
- t indicates the truth of F.

In this chapter, we studied various logics where the uncertainty was introduced in the epistemic setting. It is reasonable to expect the uncertainty of the justification to arise as a result of these two concepts. Another reason why uncertainty my arise is due to the uncertainty of the truth of the statement meant to be justified. Of course, if the uncertainty also arise from the later case, we are not presuming the ontology of uncertainty of the justification process, as we are concerned with the truth of the justified statement, which as analyzed in section 3.6 should not be the case. For clarification we enumerate the reasons why the uncertainty on a justification might arise:

I Does it arise as uncertainty whether t is convincing?

II Does it arise as uncertainty whether t indicates F?

III Does it arise as uncertainty whether F is true?

According to each logic it is probable that this uncertainty arises for different reasons. Let us analyze how this uncertainty arouse in each logic.

4.5.1 The Ontology of Uncertainty in UJ

According to Milnikel his logic of uncertain justification, as introduced in [21], interprets formulae of the form $t{:}_{\mathtt{p}}F$ to the statement

> "I have at least degree $\mathtt{p}$ of confidence in the reliability of t as evidence for belief in F.".

He also states, in the introduction of his paper, that his logic does not reflect the degree of belief in F. Explicitly, he states

> I might have heard the same thing from both the irate caller on the radio and from the *Times* and the fact that one wouldn't consider the caller a reliable source on this matter doesn't cause one to believe it any less. This distinguishes both our intention and approach from that of logics dealing with the probability that certain propositions are true.

Let us examine the corresponding semantics, i.e., the UJ-Fitting models, to observe how the uncertainty actually arises. As we have explained, UJ-Fitting models, are Fitting models for which the evidence function is replaced by an uncertainty evidence function, which interprets any triplet $\langle w, t, F\rangle$ of world w, term t and formula F to the set of degrees of confidence in the reliability of t as evidence for belief in F. i.e., a set of the form $[0, \mathtt{p})$ or of the form $[0, \mathtt{p}]$, where $\mathtt{p} \in \mathtt{S}_{>0}$.

In order for a formula of the form $t{:}_{\mathtt{p}}F$ to be true in a pair (M, w), where $M = \langle W, R, V, E\rangle$ is a UJ-Fitting model and w some world in M, it must be the case that the following conditions hold

Modal Condition: $(\forall u \in R[w])\,[M, u \Vdash F]$, i.e., we believe in F, when we are in world w;

Evidence Condition: $\mathsf{p} \in E(w, t, F)$, which seems it was wanted to represent the fact that we have at least degree p of confidence in the reliability of t as evidence for belief in F, in the world w.

We immediately observe that, as in Fitting models (viz. 3.6.3), this semantics does not presume the ontology of justification, as we also require the a priori belief in F. Note, that the belief in F is completely determined by the accessibility relation R which has nothing to do with the evidence t that is given.

In that manner, it is not completely determined what the set $E(w, t, F)$ stands for.

- We can not assume that $E(w, t, F)$ explicitly represents the degree of which t indicates F. In order to observe that, we give the next example.

  Example 4.65. *For any term* $t \in \mathsf{Tm}$*, any degree* $\mathsf{p} \in \mathsf{S}_{>0.5}$ *and any formula* $F \in \mathcal{L}_{\mathsf{UJ}}$*, the formula*

  $$t{:}_{\mathsf{p}}F \wedge t{:}_{\mathsf{p}}\neg F$$

  is satisfiable, or equivalently

  $$t{:}_{\mathsf{p}}F \wedge t{:}_{\mathsf{p}}\neg F$$

  is not a theorem of UJ.

  Proof. We define the model $M = \langle W, R, V, E\rangle$ s.t.

  - $W = \{w\}$ is a singleton;
  - $R = \emptyset$ is the empty accessibility relation;
  - $V\colon \mathsf{Prop} \to \mathcal{P}(W)$ arbitrary
  - $E\colon W \times \mathsf{Tm} \times \mathcal{L}_{\mathsf{UJ}} \to \mathcal{P}([0,1])$ s.t. for any term $t \in \mathsf{Tm}$ and any formula $F \in \mathcal{L}_{\mathsf{UJ}}$

    $$E(w, t, F) = [0,1]\,.$$

  It is trivial to observe that M is an UJ-Fitting model, particularly that it satisfies the minimum evidence conditions for the UJ-Fitting models. Moreover, it is easy to observe that

  $$M, w \Vdash t{:}_{\mathsf{p}}F \wedge t{:}_{\mathsf{p}}\neg F,$$

  for any term $t \in \mathsf{Tm}$, any degree $\mathsf{p} \in [0,1]$ and any formula $F \in \mathcal{L}_{\mathsf{UJ}}$. Hence,

  $$M \Vdash t{:}_{\mathsf{p}}F \wedge t{:}_{\mathsf{p}}\neg F,$$

  i.e., $t{:}_{\mathsf{p}}F \wedge t{:}_{\mathsf{p}}\neg F$ is satisfiable.

  □

  From the previous example we have that it is possible for some evidence to justify both a formula F and its negation $\neg F$ with probability greater that 0.5, something that must be an evidence for us that $E(w, t, F)$ cannot represent the degree of which t indicates F and similarly for $E(w, t, \neg F)$, as those two events should be complementary.

- We can not assume that $E(w,t,F)$ explicitly represents that t is a convincing evidence in world w, as by the definition of the domain of E we also take F under consideration.

From the previous deliberation it must be clear that there is not a certain read of $E(w,t,F)$, thus neither for $M, w \Vdash t{:}_{\mathtt{p}}F$.

One more weakness of UJ is that it lacks of the probability setting. It seems that the uncertainty evidence function (which is the only part where the uncertainty arises from) does not have sufficient requirements to represent a proper probability setting, but instead it is defined arbitrarily. It only have to satisfy the UJ-minimum evidence conditions which has nothing to do with the notion of probability.

But even from the definition of the axiomatic system UJ, the only requirements that we assume for the probabilities is in fact

- the degrees to belong in $\mathtt{S}_{>0}$,

- the UJ-application axiom scheme **UJJ**, which represents the application of modus ponens in independent evidences,

- the UJ-sum axiom scheme **UJ+**, which a monotonicity axiom equivalent to the sum axiom of justification logic, which is in fact not related with the probability setting and

- the confidence weakening axiom **CW**, which states the if we have at least degree $\mathtt{p}$ of confidence in the reliability of t as evidence for belief in F, then we also have at least degree $\mathtt{q}$ of confidence in the reliability of t as evidence for belief in F, for any $\mathtt{q} \leq \mathtt{p}$.

Those requirements are not sufficient to express a logic which respects the probability theory. First of all as the axiom scheme **UJJ** indicates we assume that all evidences are independent, something that is a really strict requirement. Moreover, as shown in Example 4.65 not even propositions $t{:}_{\mathtt{p}}F \wedge t{:}_{\mathtt{p}}\neg F$ are excluded from UJ, which seem impossible for a probability logic.

On the other hand, UJ is a really important logic. First of all, was the first justification logic that was combined with the notion of probability. It was the prelude of many new probability justification logics that followed. Moreover, it is a really simple logic closely related with the basic justification logic J_0. Milnikel managed to combine the notion of probability and justification in a straightforward way, far more uncomplicated than the other logics we introduced. Finally, the fact that the notion of $E(w,t,F)$ was not distinctly specified may give us the freedom to apply it in different manners.

4.5.2 The Ontology of Uncertainty in PPJ

As stated by Kokkinis in his PhD thesis [24] the intended meaning of formula $P_{\geq \mathtt{p}}F$, as paraphrased for PPJ[1] is

> "The probability of truthfulness for the PPJ-formula F is at least p.".

[1] The original statement was given in [24] for PJ and stated that the intended meaning of formula of the form $P_s\alpha$ is that "the probability of truthfulness for the justification formula α is at least s.", where α this time is a justification formula in $\mathcal{L}_{\mathsf{J}}$.

This makes clear that PPJ represents the uncertainty not only on justification but also on every kind of formula in $\mathcal{L}_{\mathsf{PPJ}}$. By this observation, we already observe that PPJ has wider expressibility than UJ. Moreover, by the above quote we can translate a formula of the form $P_{\geq \mathrm{p}} t{:}F$ to the statement

> "The probability of truthfulness for t to be an evidence of the belief in F is at least p.".

By this translation it is not made clear where the uncertainty of this justification arises; e.g., from which of the cases I, II and III

Once again we have to analyze the corresponding semantics to understand the notion of uncertainty of justification in PPJ. As described in Definition 4.26, for any PPJ(CS)-model $\mathcal{M} = \langle U, W, H, \mu, * \rangle$ and any world w in it, we have a finitely additive probability space, which arises from perception of reality in this world. Then in any measurable PPJ(CS)-model, a formula $P_{\geq \mathrm{p}} F$ states the fact that the set of worlds that are assumed possible in w, i.e., belong in W_w, and F is true, i.e., the set $[F]_{\mathcal{M},w}$, has probability at least p, i.e.,

$$\mu_w\Big([F]_{\mathcal{M},w}\Big) \geq \mathtt{p}.$$

Moreover, the formula $P_{\geq \mathrm{p}} t{:}F$ states that the set of worlds that are assumed possible in w and for which t is an evidence of F, i.e., the set of the worlds $u \in W_w$ s.t. $F \in t^{*_u}$, have probability at least p.

Note, if a term t is an evidence of a formula F in some world w has nothing to do with the truth of F in w. Thus, the truth of $P_{\geq \mathrm{p}} t{:}F$ is completely independent from the truth of $P_{\geq \mathrm{p}} F$. Hence, the uncertainty of a justification $t{:}F$ cannot arise as a result of the uncertainty of F, i.e., it is not the case III.

On the other hand, by the Definition 4.26, the relation of PPJ-models with the modular models of justification logic should be obvious. The PPJ(CS)-modular model is exactly the expected expansion of modular models to the language $\mathcal{L}_{\mathsf{PPJ}}$. Thus, by the analysis of subsection 3.6.4 (and as a result also of subsection 3.6.1), it should be comprehensible that we cannot assure whether the uncertainty arise as a result of both cases I and II or just of case II, as we are not assure whether $F \in t^{*_w}$ presumes that t is also convincing or not[2].

It should be clear that PPJ is strongly based in probability theory. Indeed, the semantics are equipped for each world with a finitely additive probability space and every formula where the probability operator appears is determined by this probability spaces.

There is a twofold reason why a *finitely additive* probability space was selected and not a probability space, where the countable addition necessary holds:

1. The *small model property* holds for PPJ (cf. [25], section 3.2); i.e., for any formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that F is satisfiable iff there is a finite[3] measurable PPJ-model which satisfies F. Therefore, trivially for any such model the finite additive probability space should be sufficient for our work.

2. The length of any formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ is finite. Therefore, any PPJ-model $\mathcal{M} = \langle U, W, H, \mu, * \rangle$ s.t. for any world $w \in U$, any $p \in \mathsf{Prop}$ and any term $t \in \mathsf{Tm}$ it holds that

$$[p]_{\mathcal{M},w} \in H_w \qquad \text{and} \qquad t^{*_w} \in H_w$$

[2] Of course, it is convincing to believe that the case II is assumed, as F is also taken under consideration.

[3] A PPJ-model is finite iff the corresponding set of worlds is finite.

is measurable.

Indeed, for any such PPJ-model $\mathcal{M} = \langle U, W, H, \mu, * \rangle$, any world $w \in U$ and any $F \in \mathcal{L}_{\mathsf{PPJ}}$ the set of worlds $[F]_{\mathcal{M},w}$ can be written as a finite intersection and union of the corresponding sets of its subformulae, as the Lemma C.3 indicates. The basis of these subformulae are clearly the the propositional atoms, the constant $\perp$ and the justifications terms whose sets $[\;]_{\mathcal{M},w}$ where all assumed measurable in $\langle W_w, H_w, \mu_w \rangle$. Hence, the finite addition is sufficient for those models to be measurable.

Moreover, in contrast with UJ, in PPJ the different events, as represented by the formulae, are not necessary assumed independent; fact that is convenient for the actual correlation of justification logic with probability theory. Besides, for any measurable PPJ-model $\mathcal{M} = \langle U, W, H, \mu, * \rangle$, if we assume that for any $w \in U$ and any formulae $F \to G, F \in \mathcal{L}_{\mathsf{PPJ}}$ the events $[F \to G]_{\mathcal{M},w}$ and $[F]_{\mathcal{M},w}$ are independent; i.e.,

$$\mu_w\Big([F \to G]_{\mathcal{M},w} \cap [F]_{\mathcal{M},w}\Big) = \mu_w\Big([F \to G]_{\mathcal{M},w}\Big) \cdot \mu_w\Big([F]_{\mathcal{M},w}\Big),$$

then PPJ respects the axioms of UJ by translating $t{:}_{\mathtt{p}}F$ as $P_{\geq \mathtt{p}} t{:}F$. The details of this statement can be found in [22, 24] for PJ, which is a counterpart of PPJ.

As it is, hopefully, clear from the above, as also will be understandable from the two subsections that follow, PPJ appears to be the properest probability justification logic. Yet again it unable to discern from where the uncertainty of the justification arise, i.e., as a result of cases I and II, or just as a result of case II.

4.5.3 The Ontology of Uncertainty in RPL(JL)

The groundwork of all the probability-justification logics, other than RPL(JL), which were defined in this chapter is the classical logic CL, i.e., a two valued logic. In contrast, the basis of RPL(JL) is a fuzzy infinite value logic, particularly the *rational Pavelka logic* RPL (cf. [27, 28]). Similarly with PPJ, the uncertainty in RPL(JL) is interpreted on all kind of formulae and not only on justification. But, this time the uncertainty arises due to vagueness and not as a matter of possibilities, as in the literature, fuzzy logic is meant to perceive the former notion. In order to distinguish the two notions let us give an example.

Let a pharmaceutical corporation developed a pill against SARS-CoV-2. After many testings the pharmaceutical corporation found that 60% of the participants that got the virus, did not developed any severe symptoms of SARS-CoV-2. This may make someone to believe that this pill is effective against SARS-CoV-2, and someone else that it is not. The uncertainty arise due to vagueness of the word "effective".

On the other hand, let the same pharmaceutical corporation developed another pill that, after many testings found out that all the participants did not get the virus even if they had a close contact with some diseased person. Let also someone has this pill and a placebo and ask us to chose one of these pills, without telling us which one is which. After our choice we believe that it is possible we have the actual pill with probability 50%. The uncertainty this time arises as a result of the two different possibilities; i.e., choosing the right pill or choosing the placebo; and not as a result of vagueness.

This example should make clear the differences in the perception of $\mathsf{RPL}(\mathsf{JL})$ of the rest probability-justification logics we have defined.

Meghdad Ghari stated in [28], that the intended meaning of formulae of the form $t{:}_{\mathtt{p}}F$ is

> "t is a justification for believing F at least to certainty degree $\mathtt{p}$.".

By this statement it is not clear if this certainty degree is determined by the vagueness of F, or just the uncertainty arise from the vagueness of t as an evidence of F.

Let us, once again, analyze the corresponding semantics. For any fuzzy Fitting model $M = \langle W, R, V, E\rangle$, any world $w \in W$, any term $t \in \mathsf{Tm}$ and any $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ we have

$$\begin{aligned} M, w \Vdash t{:}_{\mathtt{p}}F & \quad\Leftrightarrow \text{ Lem. 4.44} \\ V_w(t{:}F) \geq \mathtt{p} & \quad\Leftrightarrow \\ \min\left\{E(w,t,F), V_w^{\Box}(F)\right\} \geq \mathtt{p} & \end{aligned}$$

Therefore, it must be the case that

$$E(w,t,F) \geq \mathtt{p} \qquad \text{and} \qquad V_w^{\Box}(F) \geq \mathtt{p}.$$

The first inequality is probably meant to represent the vagueness of whether t is an evidence of F in world w. Once again, it is not clear if the vagueness arises as a result of the vagueness whether t is convincing, or only as a result of whether t evinces F. The second inequality ensures that in all worlds that are considered possible from w we have certainty degree of at least p. That is, the vagueness of F is also taken under consideration. Therefore, it seems the uncertainty -which this time is translated as a result of vagueness- could arise from all three cases I-III.

We conclude that $\mathsf{RPL}(\mathsf{JL})$ is a really important logic, as it was the first fuzzy justification logic. A combination of $\mathsf{RPL}(\mathsf{JL})$ with PPJ could be an interesting subject for future work.

4.5.4 The Ontology of Uncertainty in PJL

Churn-Jung Liau et al. introduced in [31, 32] the *possibilistic justification logic* PJL (or POJ in [32]), which is the expansion of his logic QML (cf. [29, 30]) from modal logic to justification logic. As quoted from [31, 32],

> The intuitive interpretation of $t{:}_{\alpha}\varphi$ is that, according to the evidence t, φ is believed with certainty at least α, and $t{:}_{\alpha}^{+}\varphi$ can be interpreted analogously.

Let us inspect how the corresponding semantics interpret such formulae. Let an arbitrary PJL-model $M = \langle W, R, V, E\rangle$. It is obvious that the uncertainty in PJL, as also in UJ, is correlated only on justification formulae. But this time the uncertainty arises only from the fuzziness of the accessibility relation. Indeed, let us analyze when the formula $t{:}_{\mathtt{p}}F$ is true in (M, w), where $w \in W$:

$$M, w \Vdash t{:}_{\mathtt{p}}F \qquad \text{iff} \qquad \begin{cases} \textit{Modal Condition}: & N_w([F]_M) \geq \mathtt{p} \\ \textit{Evidence Condition}: & w \in E(t,F) \end{cases}$$

The evidence condition assures us that t is an evidence of F in w. Once again, it does not specify if t is also convincing. The evidence condition does not have any uncertainty attached. The uncertainty arise from the modal condition. It is easy to observe that

$$\begin{aligned} N_w([F]_M) \geq \mathtt{p} & \quad \Leftrightarrow \\ 1 - \Pi_w(W \setminus [F]_M) \geq \mathtt{p} & \quad \Leftrightarrow \\ 1 - \sup_{u \in W \setminus [F]_M} \{\Pi_w(u)\} \geq \mathtt{p} & \quad \Leftrightarrow \\ 1 - \sup_{u \in W \setminus [F]_M} \{R(w,u)\} \geq \mathtt{p}; & \end{aligned}$$

i.e., the modal condition states “We believe in F with certainty at least p.”, in w. From the previous analysis it should be clear that the only possible case for the uncertainty is case III.

An additional reason why the uncertainty does not arise as from cases I and II, can be found in the following example.

Example 4.66. *In* PJL *all the evidences share the same certainty; i.e., for any terms* $s, t \in \mathsf{Tm}$, *any formula* $F \in \mathcal{L}_{\mathsf{PJL}}$ *and any* $\mathtt{p}, \mathtt{q} \in \mathsf{S}$

$$\vdash_{\mathsf{PJL}} s{:}_{\mathtt{p}}F \wedge t{:}_{\mathtt{q}}F \rightarrow s{:}_{\max\{\mathtt{p},\mathtt{q}\}}F.$$

Proof. If either of p, q is equal to 0 it follows trivially from axiom scheme **SJ**. Let us assume w.l.o.g. that $\mathtt{q} \geq \mathtt{p} > 0$. We have

$$\begin{array}{lr} F_1: s{:}_{\mathtt{p}}F \rightarrow s{:}_0^+F & \textbf{G1} \\ F_2: s{:}_0^+F \rightarrow s{:}_0F & \textbf{G2} \\ F_3: s{:}_{\mathtt{p}}F \rightarrow s{:}_0F & \textbf{P}\ 1,2 \\ F_4: s{:}_{\mathtt{p}}F \wedge t{:}_{\mathtt{q}}F \rightarrow s{:}_0F \wedge t{:}_{\mathtt{q}}F & \textbf{P}\ 3 \\ F_5: s{:}_0F \wedge t{:}_{\mathtt{q}}F \rightarrow s{:}_{\mathtt{q}}F & \textbf{SJ} \\ F_6: s{:}_{\mathtt{p}}F \wedge t{:}_{\mathtt{q}}F \rightarrow s{:}_{\mathtt{q}}F & \textbf{P}\ 4,5 \end{array}$$

□

From the above, we conclude that PJL is also a usefull logic, as it presumes a different notion of uncertainty of justification, this that arise from the uncertainty of the under justification statement; i.e., the case III.

4.6 Aggregated Probabilistic Evidence Logic

In this section we will consider another correlation of justification logic with probability theory, i.e., the aggregated probabilistic evidence logic, PE. Nevertheless, this time we will not equip justification logic with some formalism that defines the probability of its formulae, but in contrast, we will define in the probability theory and explicitly in the probability aggregation part of it, some formalism based on the formalism of justification logic, in order to develop some technique of constructing aggregated evidence for some event, i.e. constructing some evidence whose probability is a good measure of the probability of the corresponding event. The work of this section was given by Artemov in [33].

In order to give the setting of the probability aggregation we have to provide some definitions.

Definition 4.67 (σ-Algebra Over a Set)**.** Let W a non-empty set and H a non-empty subset of $\mathcal{P}(W)$. H is called a *σ-algebra over* W iff the following hold:

- $W \in H$
- H is closed under countable unions, i.e.,

$$U_1, U_2, \ldots \in H \Rightarrow \bigcup_{i \in \mathbb{N}} U_i \in H,$$

- H is closed under complementation in W, i.e.,

$$U \in H \Rightarrow W \setminus U \in H.$$

Definition 4.68 (Probability Measure)**.** Let H be a σ-algebra over W.
$\mu\colon H \to [0,1]$ is called a *probability measure* iff the following hold:

- $\mu(W) = 1$,
- μ is countable additive, i.e., for any countable family $\{U_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets in H it holds

$$\mu\left(\bigcup_{i \in \mathbb{N}} U_i\right) = \sum_{i \in \mathbb{N}} \mu(U_i).$$

Definition 4.69 (Probability Space)**.** $\langle W, H, \mu \rangle$ is a *probability space* iff the following hold:

- W is a non-empty set, called the *sample space*,
- H is a σ-algebra over W, called the *event space*,
- $\mu\colon H \to [0,1]$ is a probability measure.

The sample space of a probability space is meant to represent the set of the different outcomes of the examined experiment. The event space, on the other hand, represents the set of the different *specified* events that could happened, i.e., contains subsets of the sample space.

For instance, if consider the experiment of tossing a single six-sided die, the sample space is $W = \{1, \ldots, 6\}$, where i represents the fact that the die lands with the side of the number i facing up. Then, a specified event could be the statement "The number of the die is a prime number.", which is explicitly the set

$$\{2, 3, 5\}.$$

Thus, the event space can be assumed to be $H = \mathcal{P}(W)$. But, with this event space we can state some events that can not be specified by a set in H. As an example, we can state

- "The die lands to a different number than the previous toss."
- "The die lands to a number smaller than the previous toss."

Clearly, both of these events can not be explicitly represent by a set in H, as we do not know a priori the number of the previous toss. Such events are called *unspecified events*. Of course, we can easily calculate the probability of the first unspecified event to be equal to $5/6$. But, this does not hold for the second unspecified event.

Probability aggregation is meant to provide a reasonable method for the computation of the probability of each unspecified event. In PE logic, we try to evaluate the probability of such unspecified event F by providing a proper *evidence* of that event, i.e. by providing a specified event t, whose outcomes belong in F. As expected, such statements are interpreted as $t{:}F$.

Let us give the language $\mathcal{L}_{\mathsf{PE}}$ for PE. We will have first to redefine the notion of term.

Definition 4.70 (Aggregated Probabilistic Evidence Language)**.** The set $\mathsf{Tm}^{\mathsf{PE}}$ of *terms* for PE is defined by the following BNF-notation:

$$t \ ::= \ 0 \mid 1 \mid x \mid t \cup t \mid t \cdot t,$$

where 0, 1 are distinguished constant terms and x is a variable term.

The *aggregated probabilistic evidence language*, $\mathcal{L}_{\mathsf{PE}}$, is defined by the following BNF-notation:

$$\begin{aligned} F \ &::= \ p \mid \bot \mid (F \to F) \\ G \ &::= \ F \mid t{:}F \end{aligned}$$

where $p \in \mathsf{Prop}$ and $t \in \mathsf{Tm}^{\mathsf{PE}}$.

The other propositional connectives are defined as abbreviations, in the standard way (see Definition 2.4). Moreover, precedence and the associativity of the logical operators is similar to the one for modal logic (see Definition 2.5) and we might omit the parentheses, accordingly.

From the definition above it must be clear that the justification operator, or this time the *evidence operator*, is applicable only on *purely propositional formulae*, i.e., formulae that there are no occurrences of the evidence operator in them.

Definition 4.71 (Evidence Lattice)**.** A *lattice* is a poset, for any two objects x and y of which, there is the least upper bound, $x \cup y$ called the *join*, but also there is the greatest lower bound, $x \cdot y$ called the *meet*.

A lattice is *distributive* is a lattice in which join $\cup$ and meet $\cdot$ distribute over each other, i.e.,

- $x \cup (y \cdot z) = (x \cup y) \cdot (y \cup z)$
- $x \cdot (y \cup z) = (x \cdot y) \cup (y \cdot z)$

We define as the *evidence lattice over* $t_1, \ldots, t_n$, the free distributed lattice generated by $t_1, \ldots, t_n$, with minimum element 0 and maximum element 1. We will usually interpret such a lattice by a fixed canonical representative $\mathfrak{L}_n$ and the corresponding ordering by $\preceq_n$. We will also assume that $t_1, \ldots, t_n$ are fixed for any $n \in \mathbb{N}$.

We will also define the *evidence lattice* $\mathfrak{L}$, as

$$\mathfrak{L} := \biguplus_{n \in \mathbb{N}} \mathfrak{L}_n,$$

which indicates the direct sum of the corresponding lattices and we denote evidence lattice's ordering by $\preceq$. Clearly, for each term $t \in \mathsf{Tm}^{\mathsf{PE}}$ there is some $n \in \mathbb{N}$ s.t. $t \in \mathfrak{L}_n$ and of course, $t \in \mathfrak{L}$.

4.6.1 Axiomatization of Aggregated Probabilistic Evidence Logic

We are ready to define the PE logic.

Definition 4.72 (The PE Logic)**.** The axiomatic system PE is given in Table 4.8, where F, G are purely propositional formulae, $F', G' \in \mathcal{L}_{\mathsf{PE}}$, $s, t \in \mathsf{Tm}$ s.t. $t \preceq s$ and A is a propositional tautology, i.e., a theorem of CL, where A is purely propositional.

Axiomatic Schemata		
all theorems of CL in $\mathcal{L}_{\mathsf{PE}}$		**P**
$s{:}(F \to G) \to t{:}F \to s \cdot t{:}G$	application in PE	**J**
$s{:}F \wedge t{:}F \to s \cup t{:}F$	union	**U**
$1{:}A$	sample space	**SS**
$0{:}F$	empty set	**ES**
$s{:}F \to t{:}F$	greater inequality	**G**
Rules of Inference		
From F' and $F' \to G'$, infer G'	modus ponens	**MP**

Table 4.8: Axiomatic System PE

This is the first sort of justification logic that we give that does not contain a notion of constant specification. This should be expected as the justification operator : is only applicable on purely propositional formulae.

Definition 4.73 (Derivations in Aggregated Probabilistic Evidence Logic)**.**
Let $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PE}}$. A *derivation of F in* PE, is a finite sequence $F_1, \dots, F_n$ of formulae, s.t.:

- $F_n := F$,
- every F_i in the sequence is
  - either an axiom of PE,
  - or a member of Σ,
  - or the result of the application of modus ponens, to formulae of the subsequence $F_1, \dots, F_{i-1}$.

If there is a derivation of F, from Σ, in PE, then we write $\Sigma \vdash_{\mathsf{PE}} F$, and we say that F is *derivable* in PE, from the premises Σ. If Σ is the empty set, i.e., we do not assume any premises, then we write $\vdash_{\mathsf{PE}} F$ and we say that F is a theorem of PE.

A form of Lifting Lemma is clearly applicable in PE, as the next lemma indicates.

Lemma 4.74 (Lifting Lemma for PE)**.** If $\{F_1, \dots, F_n, F\}$ is a set of purely propositional formulae s.t.

$$F_1, \dots, F_n \vdash_{\mathsf{CL}} F,$$

then for every $t_1, \dots, t_n \in \mathsf{Tm}^{\mathsf{PE}}$ it holds that

$$t_1{:}F_1, \dots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \dots \cdot t_n{:}F.$$

4.6.2 Semantics for Aggregated Probabilistic Evidence Logic

Definition 4.75 (PE-Model)**.** A PE-model is a structure $M = \langle W, H, \mu, *\rangle$, where

- $\langle W, H, \mu\rangle$ is a probability space;
- $*\colon \mathcal{L}_{\mathsf{PE}} \cup \mathsf{Tm}^{\mathsf{PE}} \to \mathcal{P}(W)$ is an interpretation s.t.

$$* \restriction_{\mathsf{Tm}} \colon \mathsf{Tm} \to \mathcal{P}(\mathcal{L}_{\mathsf{J}}).$$

  is recursively defined as

$$*(0) := \emptyset \qquad\qquad *(1) := W$$
$$*(s \cdot t) := *(s) \cap *(t) \qquad\qquad *(s \cup t) := *(s) \cup *(t)$$

  and

$$* \restriction_{\mathsf{Prop}} \colon \mathsf{Prop} \to \{\{x\} \mid x \in W\},$$

  i.e., interprets each atomic proposition to a singleton subset of W and

$$* \restriction_{\mathcal{L}_{\mathsf{PE}}} \colon \mathcal{L}_{\mathsf{PE}} \to \mathcal{P}(W),$$

  is recursively defined as

$$*(\bot) := \emptyset$$
$$*(F \to G) := (W \setminus *(F)) \cup *(G)$$
$$*(t{:}F) := (W \setminus *(t)) \cup *(F)$$

For brevity, we will denote $F^* := *(F)$ and $t^* := *(t)$, for each formula $F \in \mathcal{L}_{\mathsf{PE}}$ and term $t \in \mathsf{Tm}^{\mathsf{PE}}$, but also $\overline{X} := W \setminus X$, for each $X \subseteq W$.

For any set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{PE}}$ we will also denote

$$\Sigma^* := \bigcap_{F \in \Sigma} F^*.$$

Lemma 4.76. For any PE-model $M = \langle W, H, \mu, *\rangle$ and any formulae $F, G \in \mathcal{L}_{\mathsf{PE}}$ it holds

- $(\neg F)^* = \overline{F^*}$,
- $(F \wedge G)^* = F^* \cap G^*$,
- $(F \vee G)^* = F^* \cup G^*$.

Definition 4.77 (Truth in PE-Models)**.** Truth of $\mathcal{L}_{\mathsf{PE}}$-formulae in PE-models is interpreted on the whole PE-model $M = \langle W, H, \mu, *\rangle$. Specifically, we define that a formula $F \in \mathcal{L}_{\mathsf{PE}}$ is true (or satisfied) in M, denoted as $M \Vdash F$, as follows

$$M \Vdash F \Leftrightarrow F^* = W.$$

- We say that a formula $F \in \mathcal{L}_{\mathsf{PE}}$ is *satisfiable* iff there is some PE-model M, s.t. F is true in M.
- We say that a formula $F \in \mathcal{L}_{\mathsf{PE}}$ is *valid*, and we denote $\Vdash F$ iff F is true in every PE-model M.

- We say that a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{PE}}$ is *true* in some PE-model M, and we denote $M \Vdash \Sigma$ iff all members of Σ are true in M, i.e. $\Sigma^* = W$. We define the *satisfiability* and *validity* of Σ, in the obvious way.
- We say that a formula $F \in \mathcal{L}_{\mathsf{PE}}$ is a *semantic consequence* of a set of formulae $\Sigma \subseteq \mathcal{L}_{\mathsf{PE}}$, and we denote $\Sigma \Vdash F$ iff for every PE-model M, $\Sigma^* \subseteq F^*$. Particularly, if Σ is true for some PE-model M, then F is also true in M.

It is expected that every axiom of PE is true in any PE-model. The next lemma states exactly this fact.

Lemma 4.78. For any axiom A of PE and any PE-model $M = \langle W, H, \mu, * \rangle$, it holds that

$$A^* = W.$$

As we have mentioned, in PE we are trying to estimate the probability of some unspecified event, by providing a proper evidence of the event, whose probability is computable and approaches the probability of the event.

The unspecified event meant to be proven can be interpreted by a purely propositional formula F, which will be interpreted by $*$ to some subset of W. This event logically follows by a set of (specified or unspecified) events $\Sigma = \{F_1, \ldots, F_n\}$, which is again a set of purely propositional formulae which are meant to describe the unspecified event F.

Let us assume that for each event F_i in Σ there is a corresponding evidence t_i; i.e., a specified event which is an instance of event F_i. By instance of F_i we mean that the outcomes described by t_i are among the possible outcomes of F_i; i.e., we want ${t_i}^* \subseteq {F_i}^*$, or equivalently $\overline{{t_i}^*} \cup {F_i}^* = W$. Note, that this is exactly the requirement for $M \Vdash t_i{:}F_i$ to hold. We usually fix for each such $\Sigma = \{F_1, \ldots, F_n\}$ a set of corresponding evidences $\boldsymbol{t} = \{t_1, \ldots, t_n\}$ and we will write

$$\boldsymbol{t}{:}\Sigma \coloneqq \{t_1{:}F_1, \ldots, t_n{:}F_n\}.$$

Our purpose is to construct via $\boldsymbol{t}$ a corresponding evidence $t \in \mathfrak{L}_n$ s.t.

$$\boldsymbol{t}{:}\Sigma \vdash_{\mathsf{PE}} t{:}F.$$

Such an evidence t is called *evidence for proposition F given Σ*.

Note, that for every $\Delta = \{F_{i_1}, \ldots, F_{i_k}\} \subseteq \Sigma$ s.t. $\Delta \vdash_{\mathsf{CL}} F$, we have from Lemma 4.74 that

$$t_{i_1}{:}F_{i_1}, \ldots, t_{i_k}{:}F_{i_k} \vdash_{\mathsf{PE}} t_{i_1} \cdot \ldots \cdot t_{i_k}{:}F.$$

Thus,

$$\boldsymbol{t}{:}\Sigma \vdash_{\mathsf{PE}} t_{i_1} \cdot \ldots \cdot t_{i_k}{:}F,$$

i.e., $t_{i_1} \cdot \ldots \cdot t_{i_k}$ is an evidence for F given Σ.

Definition 4.79 (Aggregated Evidence)**.** Let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$.
The *aggregated evidence $AE^{\Sigma}(F)$ for F given Σ* is the term

$$AE^{\Sigma}(F) \coloneqq \bigcup \{t \mid t \text{ an evidence for } F \text{ given } \Sigma\}.$$

Lemma 4.80. Let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$. The *aggregated evidence* $AE^{\Sigma}(F)$ *for* F *given* Σ is an evidence for F given Σ.

Corollary 4.81. Let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$.
A lattice term $t \in \mathfrak{L}_n$ is evidence for F given Σ iff

$$t \preceq AE^{\Sigma}(F).$$

The previous corollary explicitly states that $AE^{\Sigma}(F)$ is the largest term in the evidence lattice $\mathfrak{L}_n$, which is evidence for F in Σ. Hence, the probability seems to provide a tight lower bound of the probability of F, as described via Σ.

We are finally ready to state the corresponding theorem of (finite) strong soundness and strong completeness completeness for PE.

Theorem 4.82 (Strong Soundness and Completeness for PE)**.**
Let a set of formulae $\Sigma' \cup \{F'\} \subseteq \mathcal{L}_{\mathsf{PE}}$. Then,

$$\Sigma' \vdash_{\mathsf{PE}} F' \Leftrightarrow \Sigma' \Vdash F'.$$

Particularly, let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$ and $t \in \mathfrak{L}_n$. Then, it holds that

$$\boldsymbol{t}{:}\Sigma \Vvdash t{:}F \Rightarrow \boldsymbol{t}{:}\Sigma \vdash_{\mathsf{PE}} t{:}F.$$

CHAPTER 5

SUBSET MODELS FOR JUSTIFICATION LOGIC

Last three years a new semantics for justification logic has arisen, introduced by E. Lehmann and T. Studer [14, 15, 16]; that is the *subset model* semantics. In this formalism, we interpret each term t, in some world w, with a set of possible worlds. In that manner, we perceive the notion of justification as follows:

> In order to justify a proposition F, in some world w, we give an argument t that describes a set of possible worlds. Then, if in all of these worlds, the proposition F holds, we have a valid justification of F and we write $t{:}F$.

Therefore, the evidence term t is no longer connected with the proposition F, which is meant to be proven, but only with the perspective of reality after the announcement of it (, i.e., the set of possible worlds). This is a fundamental difference with the previous semantics, we have defined.

We remind the reader that in the basic model's formalism, any arbitrary basic model $*$ interprets each term $t \in \mathsf{Tm}$, to the set of formulas that this term justifies, i.e.,

$$*(t) = t^* = \{F \in \mathcal{L}_\mathsf{J} \mid * \models t{:}F\}\,.$$

In this sense, the basic model semantics are *proposition centered*, in contrast with subset model semantics. As a result the same holds for Mkrtychev models and modular models.

Furthermore, for the Fitting models the evidence function is defined as $\mathcal{E}\colon \mathsf{Tm} \times \mathcal{L}_\mathsf{J} \to W$, and associates to every pair of term t and formula F the set of worlds that t is an evidence of F[1].

5.1 Subset Model Semantics

Definition 5.1 (Subset Models)**.** A *subset model* $\mathcal{M} = \langle W, W_0, V, E\rangle$ is defined as follows:

- W is the *set of worlds*.

[1] But does not necessary justify it.

- W_0 is the *set of normal worlds*, where $W_0 \subseteq W$ and $W_0 \neq \emptyset$.
- $V\colon W \times \mathcal{L}_\mathsf{J} \to \{0,1\}$, called *valuation function*, s.t. for all $w \in W_0,\ t \in \mathsf{Tm},\ F, G \in \mathcal{L}_\mathsf{J}$:
  - $V(w, \bot) = 0$;
  - $V(w, F \to G) = 1 \quad \text{iff} \quad V(w, F) = 0$ or $V(w, G) = 1$;
  - $V(w, t{:}F) = 1 \quad \text{iff} \quad E(w, t) \subseteq [F]$.
- $E\colon W \times \mathsf{Tm} \to \mathcal{P}(W)$, called *evidence function*, s.t.[2] for all $w \in W_0,\ s, t \in \mathsf{Tm}$:
  - $E(w, s + t) \subseteq E(w, s) \cap E(w, t)$, called the *+-evidence condition*
  - $E(w, s \cdot t) \subseteq \mathfrak{W}_w(s, t)$, called the ***J**-evidence condition* where we have

$$\mathfrak{W}_w(s,t) \coloneqq \{v \in W \mid \forall F \in \mathsf{APP}_w(s,t) \quad v \in [F]\}$$

and

$$\mathsf{APP}_w(s,t) \coloneqq \{F \in \mathcal{L}_\mathsf{J} \mid \exists G \in \mathcal{L}_\mathsf{J} : E(w,s) \subseteq [G \to F] \ \mathit{and}\ E(w,t) \subseteq [G]\}\,.$$

Definition 5.2 (Truth in Subset Models)**.** Truth of justification formulae in subset models is interpreted on pairs $(\mathcal{M}, w)$, where $\mathcal{M} = \langle W, W_0, V, E\rangle$ is an arbitrary subset model and w is some wolrd in W. Specifically, we define that a formula $F \in \mathcal{L}_\mathsf{J}$ is true (or satisfied) in $(\mathcal{M}, w)$, denoted as $\mathcal{M}, w \models F$, as follows:

$$\mathcal{M}, w \models F \Leftrightarrow V(w, F) = 1.$$

This time there is no use of changing the symbol of truth $\models$ as the subset models, in contrast to all the other models for justification logic we have defined, are denoted by $\mathcal{M}$ instead of M. Let us give the following useful definition.

Definition 5.3. Let $\mathcal{M} = \langle W, W_0, V, E\rangle$ a subset model and some world $w \in W$. We define by $\mathtt{Th}(w)$, the set of formulae satisfied in w, i.e.,

$$\mathtt{Th}(w) \coloneqq \{F \in \mathcal{L}_\mathsf{J} \mid \mathcal{M}, w \models F\}$$

As usual, according to the axiom schemes that we meant to presume, we restrict the subset models to corresponding classes, by defining suitable requirements for their evidence functions.

Definition 5.4. For any normal world $w \in W_0$ and any justification term $t \in \mathsf{Tm}$, the axiom schemes with their corresponding evidence conditions are given in Table 5.1.

Axiom Schemes	Evidence Conditions
JT	$w \in E(w,t)$
JD	$E(w,t) \setminus [\bot] \neq \emptyset$
J4	$E(w, !t) \subseteq \{u \in W \mid (\forall F \in \mathcal{L}_\mathsf{J})\, [V(w, t{:}F) = 1 \Rightarrow V(v, t{:}F) = 1]\}$
J5	$E(w, ?t) \subseteq \{u \in W \mid (\forall F \in \mathcal{L}_\mathsf{J})\, [V(w, t{:}F) = 0 \Rightarrow V(v, t{:}F) = 0]\}$

Table 5.1: Axiom Schemes & Evidence Conditions for Subset Models

[2]In fact we could define the two following conditions as the *minimum evidence conditions for subset models*.

Just like the other semantics which we have defined, given a constant specification CS, we have to define some additional evidence conditions to the subset models, so that they respect CS.

Definition 5.5. Let CS a constant specification. We say that a subset model $\mathcal{M} = \langle W, V, E \rangle$ *meets constant specification* iff for each normal world $w \in W_0$ and each formula $c{:}F \in \mathsf{CS}$, it holds that

$$E(w, c) \subseteq [F] .$$

Theorem 5.6 (Soundness and Completeness for Subset Models)**.** Let CS an arbitrary constant specification for J_0.
$\mathsf{J}_0(\mathsf{CS})$ is sound and complete with respect to the class of subset models that meet CS.

For any of the defined justification logics, $\mathsf{JL}(\mathsf{CS})$, where CS is a constant specification for justification logic JL, $\mathsf{JL}(\mathsf{CS})$ is sound and complete with respect to the class of subset models that meet CS and fulfil the corresponding evidence conditions, as given in Table 5.1.

5.2 Impossible worlds & Hyperintensionality

It is easy to observe that in subset models, we distinguish two sets of worlds. Specifically, we distinguish the set of *normal / possible worlds*, W_0, and the set of *non-normal / impossible worlds*, $W \setminus W_0$ (which is probably empty). We perceive normal worlds as realities where the rules of logic apply. In contrast, non-normal worlds are realities that the rules of logic might fail. The existence of non-normal worlds should not sound bizarre. If we adopt the doxastic notion of justification logic, we can accept that someone can believe in some realities where contradictory propositions hold[3].

This perception of normal and non-normal worlds clarifies the fact that all the restrictions on the valuation function, as also all the evidence conditions were applied only for normal worlds in W_0. Particularly, given a specific justification logic JL and some corresponding constant specification CS, each normal world $w \in W_0$ is associated with a maximal $\mathsf{JL}(\mathsf{CS})$-consistent set of formulae, i.e., the set $\mathtt{Th}(w)$ of formulae that it satisfies. On the other hand, the corresponding set of a non-normal worlds could probably be $\mathsf{JL}(\mathsf{CS})$-inconsistent, or at least it could be $\mathsf{JL}(\mathsf{CS})$-consistent, but not maximal. In fact, a non-normal world could probably satisfy even a proposition F and its negation $\neg F$, simultaneously (thus $\mathsf{JL}(\mathsf{CS})$-inconsistent), or none of them (if $\mathsf{JL}(\mathsf{CS})$-consistent, for sure not maximal). Besides, it is trivial to observe that the truth in a non-normal world is completely determined by the valuation function, which is not assumed to have any restriction for non-normal worlds.

The existence of non-normal worlds is closely related with the notion of hyperintentionality. If we did not allow non-normal worlds, the hyperintentional concept of justification would fail. In order to see this, it suffices to understand the following remark.

Remark 5.7. *In every subset model $\mathcal{M} = \langle W, W_0, V, E \rangle$, s.t. the set of non-normal worlds is empty; i.e., $W \setminus W_0 = \emptyset$, the hyperintensional concept fails; i.e., for any* necessary equivalent *formulae $F, G \in \mathcal{L}_\mathsf{J}$ it holds that*

$$\mathcal{M} \models t{:}F \leftrightarrow t{:}G.$$

[3]For instance, I might thing that my thesis is descent, even though I read it.

Proof. Let $F, G \in \mathcal{L}_{\mathsf{J}}$, s.t. F and G are necessary equivalent. Then, for any subset model $\mathcal{M} = \langle W, W_0, V, E \rangle$, s.t. all their worlds are normal, F and G will be true in the same worlds, i.e., $[F] = [G]$. But then, for any term $t \in \mathsf{Tm}$ and any world $w \in W$, we have $E(w,t) \subseteq [F]$ iff $E(w,t) \subseteq [G]$. That is, $\mathcal{M}, w \models t{:}F$ iff $\mathcal{M}, w \models t{:}G$. Therefore, as w was an arbitrary world, we have $\mathcal{M} \models t{:}F \leftrightarrow t{:}G$. Thus, the hyperintansional concept fails. ☐

The concept of hyperintensionality and impossible (non-normal) worlds have been studied extensively in [50, 51], by Mark Jago, and Mark Jago and Franz Berto, respectively. Particularly, as stated in [51]

> Possible worlds are ways things might have been. They find applications in analysing possibility and necessity; propositions; knowledge and belief; information; and indicative and counterfactual conditionals. But possible worlds semantics faces the issue of hyperintensionality, generated by concepts that require distinctions between logical or necessary equivalents. The problems of distinguishing equivalent propositions, of logical omniscience, of information overload, of irrelevant conditionals, and of counterpossible conditionals, are all instances of the general issue. Adding impossible worlds promises to help with these puzzles. But can we genuinely think about the impossible? We argued that we can.

5.3 Ontology of Justification in Subset Models

5.3.1 Subset models vs Other Semantics

In section 3.6 we dealt with the ontological perception of justification in basic, Mkrtychev, Fitting, JYB-modular and modular models. In each one of them, we tacitly assumed that the term t, which was given as an evidence of the truth of the corresponding proposition F, that was meant to be proven (or believed true), was purely connected with F in the sense that t was per se, an explicit evidence for F.

Particularly, in basic and Mkrtychev models, the truth of a formula of the form $t{:}F$ depends entirely whether the proposition F belongs in the set of propositions that evinces from the evidence t, i.e. whether $F \in t^*$, where $*$ the corresponding model. As an extension, the same holds for JYB-modular and modular models, as the truth evaluation on formulae of the form $t{:}F$, in some world w, is determined by the truth evaluation in the corresponding basic model $*_w$ of this world. But even in the Fitting models the truth of formulae of the form $t{:}F$, in some world w is determined on the one hand from whether proposition F is believable, i.e., all the worlds that could be considered as the real worlds satisfy F, that is for all $u \in R[w]$ it holds that $M, u \Vdash F$; but at the same time from whether the world w that we are in, belongs to the set of worlds in which the evidence t evinces the truth of F, i.e., whether $w \in E(t, F)$.

On the contrary, subset models seems to deal with the notion of justification quite differently. Specifically, in any normal world of some subset model, the truth on a formula of the form $t{:}F$ is evaluated as follows:

> *Given an evidence t in some normal world w, we imagine of a set of different realities of how the facts could happened, i.e., the set of different worlds*[4] $E(w,t)$. *This consideration is at first level separated from the proposition F meant to be*

[4]viz. definition of *worlds* quoted by Wittgenstein in subsection 1.2.1.

proven. Afterwards, if in all of this worlds that we considered plausible due to evidence t, the proposition F holds, i.e, if $E(w, t) \subseteq [F]$, then we accept the formula $t{:}F$ as true.

This consideration of the truth of formulae of the form $t{:}F$ seems distant from the initial formulation of justification logic, i.e., the Logic of Proof LP, which was related to mathematical proving. Having said that, this perception of justification seems more relatable to the notion of justification in rhetoric. As an example:

In the trial, the prosecution presents the evidence t, which evince the guilt, F, of the defendant. At first, the association of the evidence t with F, exists only in the mind of the prosecution. By the evidence presented to the judge, she imagines of the different ways of how the facts could have happened, i.e., the set of worlds $E(w, t)$. Then, if in all the realities which she assumes plausible, she believes that the defendant is guilty, then she accepts the accusation $t{:}F$.

Once again, as in the other semantics, it is not clear whether the evidence t is convincing. One indirect way of defining this reliability is in correlation with the restriction of the set of worlds $E(w, t)$, that are assumed plausible. For instance, for an unreliable, in some world w, evidence t we can demand the evidence not restricting the set of worlds, i.e., we can demand $E(w, t) := W$.

This last observation is closely related with the notion of informativeness, e.g., an unreliable evidence is an uninformative evidence, that is it does not restrict the set of worlds. In this thesis we will not consider the notion of information in the justification logic setting, even though such a perception of justification logic seems promising.

5.3.2 Main Weakness of Subset Models

Subset semantics for justification logic seems promising for future research, as its definition is determined only by relations on sets. Nevertheless, they suffer from an important philosophical drawback, that is the interpretation of the **J**-evidence condition.

As mentioned before, the application axiom represents the application of modus ponens at a justification level. According to the formulation of subset semantics, the set of worlds $E(w, s \cdot t)$ that evince from the term $s \cdot t$, in some world w, is subset of the set $\mathfrak{W}_w(s, t)$. The set $\mathfrak{W}_w(s, t)$ is meant to represent the set of worlds in which any formula that the application axiom is *applicable* (, i.e., any formula in $\mathsf{APP}_w(s, t)$) is true. With the verb "applicable" for some formula F in the previous sentence, we mean that there is a formula G, s.t. the term s justifies the formula $G \rightarrow F$ and the term t justifies G, in w. But in order to assume that modus ponens is meaningful in the justification level, we would expect the evidences s and t to evince common worlds, i.e., we would expect $E(w, s) \cap E(w, t) \neq \emptyset$. Otherwise, the application of modus ponens between $G \rightarrow F$ and G could, probably, not be feasible, as the worlds in which $G \rightarrow F$ holds could be different from those that G holds. Let us give the following Christmas example:

Let us travelling with a time machine in which there is a small screen, which displays the date that we are on. Let t be the fact that our time machine is *on 20 April 343 CE and s be the fact that our time machine is on 25 December 2021. Then, the date t justifies the fact F which stands for "Santa Clause[5] exists.", as Santa Clause was alive on this date. On the other hand, the date s*

[5]A.k.a. Saint Nicholas

justifies the proposition $F \to G$, where G stands for "Santa Clause enters in my house through the chimney.", as it is generally believed, nowadays[6]*, that if Santa Clause exists, then on Christmas Eve he brings me presents by entering into my house through the chimney. Let our time machine is on both s and t dates synchronously. Then clearly $s \cdot t$ must justify the fact G, but this is impossible as my house has no fireplace and of course any human being would stuck in the chimney of my radiator. Clearly, we all believe that Santa Clause was capable of doing all his miracles only after his death; thus all the worlds that he exists, i.e., he is alive, are different from those that he passes through chimneys and flying with his sleigh. Particularly, $E(w, s) \cap E(w, t) = \emptyset$.*

In the previous example, we ended up justifying an impossible proposition G, as a result of the fact that the justification terms were evincing completely different worlds, i.e., $E(w, s) \cap E(w, t) = \emptyset$. Therefore, we applied modus ponens, but in fact there was not any world in which modus ponens on $F \to G$ and F was applicable. Probably, on such an observation Lehmann et al., originally formulated subset models on a different way. Specifically, she changed the usual justification language by defining in a different way the justification terms.

Definition 5.8. A $\star$-justification term is recursively defined through the following BNF-notation:

$$t \ ::= \ c \mid x \mid \mathsf{c}^\star \mid (t + t)$$

where c is a constant term, x is a variable term and $\mathsf{c}^\star$ is a distinct constant term.

A $\mathsf{c}^\star$-term is recursively defined by the following BNF-notation:

$$t \ ::= \ \mathsf{c}^\star \mid s + t \mid t + s,$$

where s an arbitrary $\star$-term.

We also define, for any $\star$-terms s, t the following abbreviation:

$$s \cdot t := (s + t) + \mathsf{c}^\star$$

The corresponding language $\mathcal{L}_{\mathsf{J}^\star}$ is the justification language $\mathcal{L}_{\mathsf{J}}$, where the terms are replaced by $\star$-terms.

She then defined a new axiomatic system $\mathsf{JL}^\star$, where the axiom scheme **J** was replaced by a new axiom scheme **Jc***.

Definition 5.9 (The Logic $\mathsf{JL}^\star$)**.** Let JL an arbitrary justification logic. We define as $\mathsf{JL}^\star$ the axiomatic system resulting by the replacement of the axiomatic scheme **J** from the axiomatic scheme **Jc***.

The axiomatic scheme **Jc*** is defined as follows:

For any $\mathsf{c}^\star$-term c, it holds that $c{:}F \land c{:}(F \to G) \to c{:}G$.

The corresponding definitions for constant specification and derivation is in the common way. She also defined the corresponding subset model semantics.

Definition 5.10. Given some logic $\mathsf{JL}^\star$, and a corresponding constant specification CS, then an $\mathsf{JL}^\star(\mathsf{CS})$-*subset model* $\mathcal{M} = \langle W, W_0, V, E\rangle$ is defined as an $\mathsf{JL}(\mathsf{CS})$-subset model, with the replacement of the **J**-evidence condition by the **Jc***-*evidence condition*

[6] And hopefully, until 25 December 2021.

- $E(w, \mathsf{c}^\star) \subseteq W_{MP}$, where

$W_{MP} := \{w \in W \mid (\forall F, G \in \mathcal{L}_{\mathsf{J}^\star})\, [V(w, F) = 1 \text{ and } V(w, F \to G) = 1 \Rightarrow V(w, G) = 1]\}$

Of course, the corresponding soundness and completeness axiom holds.

Theorem 5.11 (Soundness and Completeness for $\mathsf{JL}^\star(\mathsf{CS})$-Subset Models)**.** Let CS an arbitrary constant specification for $\mathsf{J_0}^\star$.
$\mathsf{J_0}^\star(\mathsf{CS})$ is sound and complete with respect to the class of $\mathsf{J_0}^\star(\mathsf{CS})$-subset models that meet CS.

For any $\mathsf{JL}^\star(\mathsf{CS})$, where CS is a constant specification for $\mathsf{JL}^\star$, $\mathsf{JL}^\star(\mathsf{CS})$ is sound and complete with respect to the class of subset models that meet CS and fulfil the corresponding evidence conditions.

If we take in consideration the **Jc***- and the $+$-evidence condition, as also the abbreviation $s \cdot t$, it is not hard to observe that this formalism determines that the modus ponuns on justification level, i.e. the application axiom is applied on the worlds that respects modus ponens. Indeed, if in a normal world w of some $\mathsf{JL}^\star(\mathsf{CS})$-subset model $\mathcal{M} = \langle W, W_0, V, E\rangle$, the formula $s \cdot t : F$ is true, then equivalently we have that $E(w, s \cdot t) \subseteq [F]$, but also we know that

$$\begin{aligned} E(w, s \cdot t) &= E(w, (s + t) + \mathsf{c}^\star) \\ &\subseteq E(w, s) \cap E(w, t) \cap E(w, \mathsf{c}^\star) \\ &\subseteq E(w, s) \cap E(w, t) \cap W_{MP}. \end{aligned}$$

That is, we consider in $E(w, s \cdot t)$ the worlds that are defined by both evidences s and t and on which the rule of modus ponens holds and we have that $s \cdot t : F$ is true w if all these worlds satisfy the proposition F. Perhaps, this is some kind of overkill. As we have mentioned before, due to the existence of non-normal worlds, we would probably like to assume that we were not restricted in worlds that modus ponens holds, i.e., worlds in W_{MP}, but we were restricted in worlds that satisfy any formula F that the application axiom is applicable due to the existence of a formula G s.t. $s{:}(G \to F)$ and $t{:}G$ both hold, i.e., worlds in $\mathfrak{W}_w(s, t)$. Nevertheless, both of this directions, even if intuitively more appealing, they have some serious logical drawbacks.

Remark 5.12. *In any* $\mathsf{JL}^\star(\mathsf{CS})$*, where* CS *a constant specification for* $\mathsf{JL}^\star$*, the application operator* $\cdot$ *is monotonic, i.e., for any formula* $F \in \mathcal{L}_{\mathsf{J}^\star}$ *and any* $\star$*-terms* s, t *it holds that*

$$\vdash_{\mathsf{JL}^\star(\mathsf{CS})} s{:}F \to s \cdot t{:}F.$$

Moreover, for any formulae $F, G \in \mathcal{L}_{\mathsf{J}^\star}$ *and any* $\star$*-terms* s, t *it holds that*

$$\vdash_{\mathsf{JL}^\star(\mathsf{CS})} s{:}(F \to G) \to t{:}F \to t \cdot s{:}G.$$

Proof. Let $\mathcal{M} = \langle W, W_0, V, E\rangle$ be an arbitrary $\mathsf{JL}^\star$-subset model that meets CS and some normal world $w \in W$ s.t. $\mathcal{M}, w \models s{:}F$. Then by the definition of truth in $\mathsf{JL}^\star$-subset models, we have that $E(w, s) \subseteq [F]$. Then, trivially we have that

$$E(w, s \cdot t) \subseteq E(w, s) \cap E(w, t) \subseteq [F].$$

Thus, $\mathcal{M}, w \models s \cdot t{:}F$. Therefore, $\mathcal{M}, w \models s{:}F \to s \cdot t{:}F$. Because $\mathcal{M}$ was an arbitrary $\mathsf{JL}^\star$-subset model, by completeness theorem 5.11 we have

$$\vdash_{\mathsf{JL}^\star(\mathsf{CS})} s{:}F \to s \cdot t{:}F,$$

as wanted. For a derivation of this same formula cf. [15, 16]. The second formula can be proved with similar procedures. □

From Remark 5.12 it is clear, that $\mathsf{JL}^\star(\mathsf{CS})$ is not capable of keeping track of the steps of the justification project, one of the main features of justification logic (viz. subsection 1.2.3). Thus, this logic seems insufficient. If we observe the previous proof, it is clear that logical drawbacks arise due to the requirement

$$E(w, s \cdot t) \subseteq E(w, s) \cap E(w, t).$$

But this requirement is closely related with our intuition. Therefore, by using subset semantics we will have to chose between these two weaknesses, i.e., either we would have an axiomatic system, missing some useful properties of justification logic, or we would have semantics which seems distant from our intuition.

5.4 Aggregated Evidence & Subset Models

In section 4.6 we defined PE logic, which was introduced by Artemov in [33] as a logic suitable for constructing aggregated evidence for an unjustified event. In this section we will show how we can adapt subset models so that they respect PE. This work was done by Lehmann et al. in [14, 16] for $\mathsf{JL}^*(\mathsf{CS})$-models. Here we will work with $\mathsf{JL}(\mathsf{CS})$-models but all the parts are almost identical as in $\mathsf{JL}^*(\mathsf{CS})$-models.

First adaption must be done for justification terms, so that the operator $\cup$ belongs in our language.

Definition 5.13. The set $\mathsf{Tm}_\mathsf{s}^{\mathsf{PE}}$ of *terms* for PE is defined by the following BNF-notation:

$$t \ ::= \ 0 \mid 1 \mid c \mid x \mid t \cup t \mid t \cdot t,$$

where 0, 1 are distinguished constant terms, c is some constant term and x is a variable term.

We define as $\mathcal{L}_\mathsf{J}^{\mathsf{PE}}$ as the language $\mathcal{L}_\mathsf{J}$, where the justification terms belong in $\mathsf{Tm}_\mathsf{s}^{\mathsf{PE}}$.

Moreover, we have to adapt the evidence lattice $\mathfrak{L}$ so that is defined for terms in $\mathsf{Tm}_\mathsf{s}^{\mathsf{PE}}$. We define the corresponding lattice, $\mathfrak{L}_s$, as the free distributed lattice constructed by assuming each constant term c, each variable term x and 1 as its building blocks. The join of two terms s, t is the term $s \cup t$, while the meet of them is $s \cdot t$. Finally, 0 is added as a minimum element. The corresponding ordering is denoted as $\preceq_\mathsf{s}$.

Finally, we have to adapt subset models, so that they respect the axiomatic scheme PE, this time translated in language $\mathcal{L}_\mathsf{J}^{\mathsf{PE}}$.

Definition 5.14 (PE-Adapted Subset Models)**.** A PE-adapted subset model is a subset model $\mathcal{M} = \langle W, W_0, V, E\rangle$ s.t. for any $w \in W_0$ and for any $s, t \in \mathsf{Tm}_\mathsf{s}^{\mathsf{PE}}$ it holds that

- $E(w, 0) \coloneqq \emptyset$;
- $E(w, 1) \coloneqq W_0$;
- $E(w, s \cup t) = E(w, s \cup t) \cup E(w, s \cup t)$.

Definition 5.15 (Truth in PE-Adapted Subset Models)**.** Truth of $\mathcal{L}_\mathsf{J}^{\mathsf{PE}}$formulae in PE-adapted subset models is defined just like in subset models, in Definition 5.2.

We say that a formula $F \in \mathcal{L}_\mathsf{J}^{\mathsf{PE}}$ is PE-*valid* iff for any PE-adapted subset model $\mathcal{M} = \langle W, W_0, V, E\rangle$ and any world $w \in W_0$ it holds that $\mathcal{M}, w \models F$.

It is time to show that PE-adapted subset models indeed respect PE.

Theorem 5.16 (Soundness of PE in PE-Adapted Subset Models)**.** Any theorem F of PE in the language $\mathcal{L}_{\mathsf{J}}^{\mathsf{PE}}$ is PE-valid.

Finally, we have to assure that such models exist. The following theorem states exactly this fact.

Theorem 5.17. There exists a PE-adapted subset model.

5.5 Subset Models for Uncertain Justification

In Chapter 4, we described alternative logics and their associated semantics for perceiving the concept of uncertainty at the justification level. The most prominent, at least in our intuition, example of such logic is PPJ, which was introduced by Kokkinis et al. Yet again, this formalism fails to distinguish whether the uncertainty on the justification level arises from uncertainty about the suasiveness of the evidence (, i.e., case I) or uncertainty about the conclusiveness of the evidence over the statement (, i.e., case II).

Subset model semantics seems to provide a natural way of defining the suasiveness of an evidence t in some world w, as the evidence function $E\colon W \times \mathsf{Tm} \to \mathcal{P}(W)$ of subset models is completely independent from the statement F to be proven. That is, F does not belong neither in the domain, nor in the codomain of E. Therefore, it appears to be a promising idea to formalize a justification logic of uncertain reasoning, as in PPJ-models, by replacing the modular model setting by a subset model one.

Let us examine how the uncertainty can be perceived in such a formalism.

> *We are in some reality (world) w, but we do not know, for sure, which one. Nevertheless, the fact that we are in this reality determines our perception of the reality we believe we are in; i.e., it determines a probability for each reality. That is, it explicitly defines a probability space $\langle W_w, H_w, \mu_w\rangle$[7], so that for each set of realities $X \in H_w$, we give ourselves a probability $\mu_w(X)$ to be in some of these realities. Then, the probability that we give for some statement F to be true is the probability measure of the set of realities in which F holds; i.e., the probability*
>
> $$\mu_w\Big([F]_{\mathcal{M},w}\Big).$$
>
> *Moreover, given some evidence t, we can imagine of the set of plausible realities following the announcement of t; i.e., the set $E(w,t)$. Thus, a reasonable measure for the degree of the suasiveness of t in w is*
>
> $$\mu_w(E(w,t)).$$
>
> *We perceive a justification of statement F from evidence t as valid if the assertion F holds in all of the realities that we consider plausible after the announcement of t; i.e., if $E(w,t) \subseteq [F]_{\mathcal{M},w}$. In this manner, any reality that is considered plausible due to t and in which F holds (;i.e., any world in $E(w,t) \cap [F]_{\mathcal{M},w}$), constitutes an indication that t justifies F. Hence, the probability of the justification $t{:}F$ to be accepted can be calculated as*
>
> $$\mu_w\Big(E(w,t) \cap [F]_{\mathcal{M},w}\Big).$$

[7] W.l.o.g. we might assume that $W_w = W$ and setting $\mu_w(X)$, for any set of realities $X \in H_w$ that we consider impossible for us to be in.

This perception of the degree of conclusiveness of t over F is related to inductive reasoning. For t to be presumed as an evidence that justifies the truth of F, we count the worlds that t and F "coexist" and we establish a correlation between them.

Note, that this probability is different from

$$\mu_w\Big([t{:}F]_{\mathcal{M},w}\Big),$$

which was the probability for the acceptance of the justification $t{:}F$ in PPJ*. In subset models formalism, this probability represents the probability we give to ourselves to believe in such a justification $t:F$. Clearly, our reaction to some evidence differs depending on whether we are given some evidence or consider we are given some evidence. In the second case, we consider the possibility of accepting the justification even though it is false, or declining it even though it is true*[8]*. This might reflect the fact we are more gullible when we are given some evidence rather than when we are considering of someone to give us that evidence*

Based on the preceding, we can explicitly define the degree of conclusiveness of evidence t over assertion F as

$$\frac{\mu_w\Big(E(w,t)\cap[F]_{\mathcal{M},w}\Big)}{\mu_w(E(w,t))};$$

i.e., the conditional probability of accepting some justification of F from some evidence t given the fact that t is convincing (true). It is worth mentioning that if t justifies F in the usual notion in subset models; i.e., if $E(w,t)\subseteq[F]_{\mathcal{M},w}$, then we clearly have that

$$\frac{\mu_w\Big(E(w,t)\cap[F]_{\mathcal{M},w}\Big)}{\mu_w(E(w,t))}=\frac{\mu_w(E(w,t))}{\mu_w(E(w,t))}=1,$$

as wanted. Of course, in this case the probability for the acceptance of the justification $t{:}F$ will be the probability for the evidence to be true. Indeed,

$$\mu_w\Big(E(w,t)\cap[F]_{\mathcal{M},w}\Big)=\mu_w(E(w,t)).$$

Let us observe how the relevant language should be defined. On the surface, it appears that this language should remain the same as in PPJ; i.e., to assume language $\mathcal{L}_{\mathsf{PPJ}}$. However, because of the existence of non-normal worlds, we should be more mindful with the definition of the abbreviations $P_<$, $P_\leq$, $P_>$ and $P_=$ this time.

Let us start with the definition of $P_\leq$. In $\mathcal{L}_{\mathsf{PPJ}}$, it was defined as

$$P_{\leq\mathtt{p}}F\equiv P_{\geq 1-\mathtt{p}}\neg F,$$

with the intention

$$\mathcal{M},w\Vdash P_{\leq\mathtt{p}}F\Leftrightarrow\mu_w\Big([F]_{\mathcal{M},w}\Big)\leq\mathtt{p}.$$

In subset models, we have for any $w\in W_0$

$$\begin{aligned}\mathcal{M},w\Vdash P_{\geq 1-\mathtt{p}}\neg F&\Leftrightarrow\mu_w\Big([\neg F]_{\mathcal{M},w}\Big)\geq 1-\mathtt{p}\\&\not\Leftrightarrow\mu_w\Big([F]_{\mathcal{M},w}\Big)\leq\mathtt{p},\end{aligned}$$

[8]Every rational person should consider the possibility of being crazy.

where the second equivalence does not hold, as it might be the case that $[F]_{\mathcal{M},w} \cap [\neg F]_{\mathcal{M},w} \neq \emptyset$, due to the existence of non-normal worlds. Hence, the abbreviation for $P_{\leq}$ as defined for $\mathcal{L}_{\mathsf{PPJ}}$, is inappropriate for the formalism with subset models. As a result, we must enlarge $\mathcal{L}_{\mathsf{PPJ}}$ by adding a *new* probability operator $P_{\leq}$.

Similarly, $P_{<}$ was defined as

$$P_{<\mathtt{p}}F \equiv \neg P_{\geq \mathtt{p}}F,$$

so that

$$\mathcal{M}, w \Vdash P_{<\mathtt{p}}F \Leftrightarrow \mu_w\Big([F]_{\mathcal{M},w}\Big) < \mathtt{p}.$$

In subset models we have for any $w \in W_0$

$$\begin{aligned}\mathcal{M}, w \Vdash \neg P_{\geq \mathtt{p}}F &\Leftrightarrow \mathcal{M}, w \nVdash P_{\geq \mathtt{p}}F \\ &\Leftrightarrow \mu_w\Big([F]_{\mathcal{M},w}\Big) \geq \mathtt{p} \text{ is fasle} \\ &\Leftrightarrow \mu_w\Big([F]_{\mathcal{M},w}\Big) < \mathtt{p},\end{aligned}$$

as wanted. It should be noted that this would not necessarily hold if $w \in W \setminus W_0$, because $\neg P_{\geq \mathtt{p}}F$ and $P_{\geq \mathtt{p}}F$ could both be true in w. This should not be construed as a limitation because the soundness and completeness results are only considered for normal worlds. Based on the last observations, we conclude that

$$P_{<\mathtt{p}}F \equiv \neg P_{\geq \mathtt{p}}F$$

is an appropriate abbreviation for the subset model formalism of uncertainty.

Correspondingly, we can define the following abbreviations without the addition of some new probability operator

$$P_{>\mathtt{p}}F \equiv \neg P_{\leq \mathtt{p}}F \qquad\qquad P_{=\mathtt{p}}F \equiv P_{\geq \mathtt{p}}F \wedge P_{\leq \mathtt{p}}F.$$

As previously stated, under the subset model setting for uncertainty, there is a natural way of defining the notion of uncertainty on the suasiveness of an evidence t is some world w, via

$$\mu_w(E(w,t)).$$

It's probably a good idea to define an operator that expresses the suasiveness of any piece of evidence t. We can denote such an operator as $\mathbf{suas}_{\geq \mathtt{p}}(t)$ and defining the truth on it, in some normal world w as

$$\mathcal{M}, w \Vdash \mathbf{suas}_{\geq \mathtt{p}}(t) \Leftrightarrow \mu_w(E(w,t)) \geq \mathtt{p}.$$

Clearly, a corresponding operator $\mathbf{suas}_{<\mathtt{p}}(t)$ could be defined as an abbreviation

$$\mathbf{suas}_{<\mathtt{p}}(t) \equiv \neg \mathbf{suas}_{\geq \mathtt{p}}(t).$$

Yet again, we have define a *new* operator $\mathbf{suas}_{\leq \mathtt{p}}(t)$[9]. The abbreviations for $\mathbf{suas}_{>\mathtt{p}}(t)$ and $\mathbf{suas}_{=\mathtt{p}}(t)$ should be straight forward.

[9] Some idea of introducing for each term t some term $\bar{t}$ s.t. $E(w,\bar{t}) := W_w \setminus E(w,t)$, for each world $w \in W_0$ and denoting $\mathbf{suas}_{\leq \mathtt{p}}(t)$ as an abbreviation for $\mathbf{suas}_{\geq 1-\mathtt{p}}(\bar{t})$ might decline the unreliability concept of suasiveness.

Furthermore, we could define an operator $t{:}_{\geq \mathtt{p}}F$ which defines the conclusiveness of a justification of F from evidence t. Clearly, the truth on such operator, in some normal world w, would be defined as

$$\mathcal{M}, w \Vdash t{:}_{\geq \mathtt{p}}F \Leftrightarrow \frac{\mu_w\Big(E(w,t) \cap [F]_{\mathcal{M},w}\Big)}{\mu_w(E(w,t))} \geq \mathtt{p}.$$

The operators $t{:}_{\leq \mathtt{p}}F$, $t{:}_{< \mathtt{p}}F$, $t{:}_{> \mathtt{p}}F$ and $t{:}_{= \mathtt{p}}F$ should be defined as previously.

Additionally, we should define some operator for the measure of probability for a justification $t:F$ to be accepted. We could denote such an operator as $t:^{\geq \mathtt{p}}F$ and interpret the truth on it, in some normal world w as

$$\mathcal{M}, w \Vdash t{:}^{\geq \mathtt{p}}F \Leftrightarrow \mu_w\Big(E(w,t) \cap [F]_{\mathcal{M},w}\Big) \geq \mathtt{p}.$$

The operators $t{:}^{\leq \mathtt{p}}F$, $t{:}^{< \mathtt{p}}F$, $t{:}^{> \mathtt{p}}F$ and $t{:}^{= \mathtt{p}}F$ should be defined as previously.

Last but not least, it is worth mentioning that we could replace the usual justification operator $t{:}F$ for the operator $t{:}_{\geq 1}F$[10]. If $t{:}F$ holds then as shown above

$$\frac{\mu_w\Big(E(w,t) \cap [F]_{\mathcal{M},w}\Big)}{\mu_w(E(w,t))} = 1;$$

i.e., $t{:}_{\geq 1}F$ holds. The opposite direction does not necessary hold as it could be the case that

$$E(w,t) \setminus [F]_{\mathcal{M},w} \neq \emptyset$$

and

$$\mu_w\Big(E(w,t) \setminus [F]_{\mathcal{M},w}\Big) = 0;$$

i.e., the only worlds that follows from the announcement of t and F does not hold are considered impossible, but this should not be problematic because the certainty should be defined as probability equal to 1.

From the above a corresponding language $\mathscr{L}$ could be defined by the following BNF-notation:

$$\begin{aligned} F \ ::= \ & p \mid \bot \mid (F \to F) \mid \mathbf{suas}_{\geq \mathtt{p}}(t) \mid \mathbf{suas}_{\leq \mathtt{p}}(t) \mid \\ & t{:}_{\geq \mathtt{p}}F \mid t{:}_{\leq \mathtt{p}}F \mid t{:}^{\geq \mathtt{p}}F \mid t{:}^{\leq \mathtt{p}}F \mid P_{\geq \mathtt{p}}F \mid P_{\leq \mathtt{p}}F, \end{aligned}$$

where $p \in \mathsf{Prop}$, $t \in \mathsf{Tm}$ and $\mathtt{p} \in \mathtt{S}$.

Of course, if the non-normal worlds are excluded from the subset models, the abbreviation for $P_{\leq \mathtt{p}}F$, as defined for $\mathcal{L}_{\mathsf{PPJ}}$, remains valid. Therefore, the insertion of $P_{\leq \mathtt{p}}F$ as a new operator is superfluous. Similarly, for the operators $\mathbf{suas}_{\leq \mathtt{p}}(t)$, $t{:}_{\leq \mathtt{p}}F$ and $t{:}^{\leq \mathtt{p}}F$.

By the previous analysis, we could define the *probabilistic subset models* as the following definition indicates.

Definition 5.18 (Probabilistic Subset Models)**.** A *probabilistic subset model* $\mathcal{M} = \langle W, W_0, V, E, U, H, \mu\rangle$ is defined as follows:

- W is the *set of worlds*.

[10] Or the operator $t{:}_{=1}F$

- W_0 is the *set of normal worlds*, where $W_0 \subseteq W$ and $W_0 \neq \emptyset$.
- U, H and μ are functions over W, s.t. for each $w \in W$ $\langle U(w), H(w), \mu(w)\rangle$ is a finitely additive probability space,
- $V\colon W \times \mathscr{L} \to \{0,1\}$, called *valuation function*, s.t. for all $w \in W_0$, $t \in \mathsf{Tm}$, $F, G \in \mathcal{L}_\mathsf{J}$:

$$
\begin{aligned}
V(w,\bot) &= 0 \\
V(w, F\to G) = 1 \quad &\Leftrightarrow \quad V(w,F)=0 \text{ or } V(w,G)=1 \\
V(w, P_{\geq \mathtt{p}}F) = 1 \quad &\Leftrightarrow \quad \mu_w\Big([F]_{\mathcal{M},w}\Big) \geq \mathtt{p} \\
V(w, P_{\leq \mathtt{p}}F) = 1 \quad &\Leftrightarrow \quad \mu_w\Big([F]_{\mathcal{M},w}\Big) \leq \mathtt{p} \\
V(w, \mathbf{suas}_{\geq \mathtt{p}}(t)) = 1 \quad &\Leftrightarrow \quad \mu_w(E(w,t)) \geq \mathtt{p} \\
V(w, \mathbf{suas}_{\leq \mathtt{p}}(t)) = 1 \quad &\Leftrightarrow \quad \mu_w(E(w,t)) \leq \mathtt{p} \\
V(w, t{:}_{\geq \mathtt{p}}F) = 1 \quad &\Leftrightarrow \quad \frac{\mu_w\Big(E(w,t)\cap [F]_{\mathcal{M},w}\Big)}{\mu_w(E(w,t))} \geq \mathtt{p} \\
V(w, t{:}_{\leq \mathtt{p}}F) = 1 \quad &\Leftrightarrow \quad \frac{\mu_w\Big(E(w,t)\cap [F]_{\mathcal{M},w}\Big)}{\mu_w(E(w,t))} \leq \mathtt{p} \\
V\big(w, t{:}^{\geq \mathtt{p}}F\big) = 1 \quad &\Leftrightarrow \quad \mu_w\Big(E(w,t)\cap [F]_{\mathcal{M},w}\Big) \geq \mathtt{p} \\
V\big(w, t{:}^{\leq \mathtt{p}}F\big) = 1 \quad &\Leftrightarrow \quad \mu_w\Big(E(w,t)\cap [F]_{\mathcal{M},w}\Big) \leq \mathtt{p}
\end{aligned}
$$

- $E\colon W \times \mathsf{Tm} \to \mathcal{P}(W)$, called *evidence function*, s.t. for all $w \in W_0$, $s, t \in \mathsf{Tm}$:
  - $E(w, s+t) \subseteq E(w,s) \cap E(w,t)$, called the *$+$-evidence condition*
  - $E(w, s\cdot t) \subseteq \mathfrak{W}_w(s,t)$, called the ***J**-evidence condition* where we have

$$\mathfrak{W}_w(s,t) := \{v \in W \mid \forall F \in \mathsf{APP}_w(s,t) \quad v \in [F]\}$$

  and

$$\mathsf{APP}_w(s,t) := \{F \in \mathcal{L}_\mathsf{J} \mid \exists G \in \mathcal{L}_\mathsf{J} : E(w,s) \subseteq [G \to F] \text{ } \mathit{and} \text{ } E(w,t) \subseteq [G]\}\,.$$

The truth is defined as in subset models; i.e.,

$$\mathcal{M}, w \Vdash F \quad \Leftrightarrow \quad V(w,F) = 1.$$

Let us examine how the corresponding axiomatic system should be constructed. At first glance it seems reasonable to assume that PPJ should be a subsystem of the new logic. The existence of non-normal worlds should compel us to pay closer attention once more. It is not hard to observe that axiom schemes **PI**, **WE** and **LE**, as well as the rule of inference **ST**, are still valid in this logic. However, this is not the case with the axiom schemes **DIS** and **UN** as also the rule of inference **CE**.

Let us for instance consider the case of axiom scheme **DIS**. Clearly, this axiom scheme represents in PPJ how the probability measure treats the disjoint union of two sets. Let some normal world $w \in W_0$ s.t. the formulae $P_{\geq \mathtt{p}}F$, $P_{\geq \mathtt{q}}G$ and $P_{\geq 1}\neg(F \wedge G)$ are true in w. Equivalently, we have that

$$\mu_w\Big([F]_{\mathcal{M},w}\Big) \geq \mathtt{p} \qquad \mu_w\Big([G]_{\mathcal{M},w}\Big) \geq \mathtt{q} \qquad \mu_w\Big([\neg(F \wedge G)]_{\mathcal{M},w}\Big) \geq 1$$

Once again, the latter is not equivalent with

$$\mu_w\Big([F \wedge G]_{\mathcal{M},w}\Big) \leq 0$$

as it is possible

$$[F \wedge G]_{\mathcal{M},w} \cap [\neg(F \wedge G)]_{\mathcal{M},w} \neq \emptyset$$

and

$$\mu_w\Big([F \wedge G]_{\mathcal{M},w} \cap [\neg(F \wedge G)]_{\mathcal{M},w}\Big) > 0.$$

Of course, this can be replaced by defining the axiom scheme **DIS'**

$$P_{\geq \mathtt{p}}F \wedge P_{\geq \mathtt{q}}G \wedge P_{\leq 0}\neg(F \wedge G) \to P_{\geq \min\{1,\mathtt{p}+\mathtt{q}\}}(F \vee G).$$

However, once again due to the existence of non-normal worlds we might have that

$$[F \vee G]_{\mathcal{M},w} \neq [F]_{\mathcal{M},w} \cup [G]_{\mathcal{M},w}\,.$$

As a result, the disjoint union on probabilities cannot be expressed using the axiom scheme **DIS**, which is actually invalid in this formalism. Likewise, for **UN**.

In order to show that the rule of inference **CE** is not valid in these models, it is sufficient to observe that even the tautology $\top$ might not have probability equal to 1. Indeed, it might be the case that there are non-normal worlds in which $\top$ does not hold and which have probability greater than 0; i.e.,

$$\mu_w\Big(W \setminus [\top]_{\mathcal{M},w}\Big) > 0.$$

Once again, if we exclude the non-normal worlds from the subset models, the axiom schemes **DIS** and **UN**, as also the rule of inference **CE** remain valid.

Obviously, since all of the axioms in PPJ represent how the probability measure should behave, we expand those axioms for the new operators[11].

Finally, we should define some additional axiom schemes of how the different operators should co-behave. For instance an axiom scheme of the form

$$\mathbf{suas}_{\geq \mathtt{p}}(t) \wedge t{:}_{\geq \mathtt{q}}F \to t{:}^{\geq \mathtt{pq}}F,$$

seems reasonable.

If future work, we will try to construct a logic, which will be sound and complete with the corresponding probabilistic subset models, with or without the existence of non-normal worlds. Clearly, such a logic will be capable of distinguishing between the susiveness of the evidence and the conclusiveness of the evidence over the statement under justification. Therefore, such a logic is significant for the perception of uncertainty on justification.

[11] With this, we also mean the probability operator $P_{\leq \mathtt{p}}F$, where this time all the inequality symbols should be defines symmetrically.

APPENDIX A

PROOFS OF CHAPTER 2

MODAL LOGIC

Theorem 2.13 (Conservativity of Modal Logic)**.**
Any modal logic ML is a *conservative extension* of classical logic, CL.

Proof. The proof is similar to the proof of Theorem 2.29; thus, it may be omitted. □

Corollary 2.14 (Consistency of Modal Logic)**.** Any normal modal logic ML is consistent.

Proof. The proof is similar to the proof of Theorem 2.30; thus, it may be omitted. □

Theorem 2.15 (Deduction Theorem for Modal Logic)**.**
The deduction theorem holds for any normal modal logic.

Proof. The proof is similar to the proof of Theorem 2.31; thus, it may be omitted. □

Theorem 2.19 (Soundness & completeness for modal logic)**.** Axiom system K is sound and complete with respect to the semantic class $\mathcal{K}$, i.e., for every formula $F \in \mathcal{L}_\Box$, we have

$$\vdash_{\mathsf{K}} F \iff \mathcal{K} \models F.$$

$\vdash_{\mathsf{K}} F \Rightarrow \mathcal{K} \models F$	soundness
$\vdash_{\mathsf{K}} F \Leftarrow \mathcal{K} \models F$	completeness

The same holds for T w.r.t. $\mathcal{T}$, for KD w.r.t. $\mathcal{KD}$, for K4 w.r.t. $\mathcal{K}4$, for S4 w.r.t. $\mathcal{S}4$, for K5 w.r.t. $\mathcal{K}5$, for S5 w.r.t. $\mathcal{S}5$ and for KD45 w.r.t. $\mathcal{KD}45$.

Proof. The proof is similar to the proof of Theorem 3.25; thus, it may be omitted. □

Theorem 2.20 (Strong Soundness & completeness for modal logic)**.** Axiom system K is strongly sound and complete with respect to the semantic class $\mathcal{K}$, i.e., for every set of formulae $\Sigma \cup \{F\} \subseteq \mathcal{L}_\Box$, we have

$$\Sigma \vdash_{\mathsf{K}} F \iff \Sigma \models_{\mathcal{K}} F.$$

$$\Sigma \vdash_{\mathsf{K}} F \Rightarrow \Sigma \models_{\mathcal{K}} F \qquad \text{strong soundness}$$
$$\Sigma \vdash_{\mathsf{K}} F \Leftarrow \Sigma \models_{\mathcal{K}} F \qquad \text{strong completeness}$$

The same holds for T w.r.t. $\mathcal{T}$, for KD w.r.t. $\mathcal{KD}$, for $\mathsf{K4}$ w.r.t. $\mathcal{K}4$, for $\mathsf{S4}$ w.r.t. $\mathcal{S}4$, for $\mathsf{K5}$ w.r.t. $\mathcal{K}5$, for $\mathsf{S5}$ w.r.t. $\mathcal{S}5$ and for $\mathsf{KD45}$ w.r.t. $\mathcal{KD}45$.

JUSTIFICATION LOGIC

Theorem 2.29 (Conservativity of Justification Logic)**.**
Any justification logicJL is a *conservative extension* of classical logic, CL.

Proof. Let arbitrary justification logic $\mathsf{JL}(\mathsf{CS})$.

That any theorem of CL is also a theorem of $\mathsf{JL}(\mathsf{CS})$ is trivial, considering the axiom scheme **P** and Modus Ponens. For the other direction, we define the translation function $\mathbf{t}\colon \mathcal{L}_{\mathsf{J}} \to \mathcal{L}_{\mathsf{CL}}$, recursively defined as

$$\begin{aligned} \mathbf{t}(p) &:= p && \text{where } p \in \mathsf{Prop} \\ \mathbf{t}(\neg F) &:= \neg \mathbf{t}(F) && \text{where } F \in \mathcal{L}_{\mathsf{J}} \\ \mathbf{t}(F \to G) &:= \mathbf{t}(F) \to \mathbf{t}(G) && \text{where } F, G \in \mathcal{L}_{\mathsf{J}} \\ \mathbf{t}(t{:}F) &:= \mathbf{t}(F) && \text{where } t \in \mathsf{Tm} \text{ and } F \in \mathcal{L}_{\mathsf{J}} \end{aligned}$$

That is, **t** is the translation function that deletes all the terms and :'s of a justification formula.

Let $F \in \mathcal{L}_{\mathsf{J}}$ is a theorem of $\mathsf{JL}(\mathsf{CS})$. We will prove by induction on the derivation of F that $\mathbf{t}(F)$ is a theorem of CL.

- Let F arrived in the derivation by **P**, i.e., it is an instance of propositional tautology. Then trivially it is a theorem in CL.

- Let $F := s{:}(G \to H) \to t{:}G \to s \cdot t{:}H$ arrived by axiom scheme **J**. We observe that

$$\begin{aligned} \mathbf{t}(s{:}(G \to H) \to t{:}G \to s \cdot t{:}H) &= \mathbf{t}(s{:}(G \to H)) \to \mathbf{t}(t{:}G) \to \mathbf{t}(s \cdot t{:}H) \\ &= (G \to H) \to G \to H, \end{aligned}$$

  which is a theorem of CL.

- The cases for the other axiomatic schemes are similar to **J**. Specifically, we have

$$\begin{aligned} \mathbf{t}(s{:}G \to s \cdot t{:}G) &= G \to G && + \\ \mathbf{t}(t{:}G \to G) &= G \to G && \mathbf{JT} \\ \mathbf{t}(\neg t{:}\bot) &= \neg\bot && \mathbf{JD} \\ \mathbf{t}(t{:}G \to !t{:}t{:}G) &= G \to G && \mathbf{J4} \\ \mathbf{t}(\neg t{:}G \to ?t{:}\neg t{:}G) &= \neg G \to \neg G && \mathbf{J5} \end{aligned}$$

  which are all theorems of CL.

- Let $F := c{:}A$, is a member of the constant specification CS. Then, by the definition of the translation function **t**, we have $\mathbf{t}(c{:}A) = \mathbf{t}(A)$, where A is an axiom of JL. But we have proven that the translation of any axiom of JL is a theorem of CL. Thus, $\mathbf{t}(F)$ is a theorem of CL.
- Let F arrived by Modus Ponens. Then, there is a justification formula G s.t. there are two steps of the derivation $G \to F$ and G arrived. By induction hypothesis $\mathbf{t}(G \to F) = \mathbf{t}(G) \to \mathbf{t}(F)$ and $\mathbf{t}(G)$ are theorems of CL. Therefore, there are some derivations that $\mathbf{t}(G) \to \mathbf{t}(F)$ and $\mathbf{t}(G)$ arrive. By combining those two derivations and by applying modus ponens between $\mathbf{t}(G) \to \mathbf{t}(F)$ and $\mathbf{t}(G)$, the formula $\mathbf{t}(F)$ arrives, as wanted.

By induction, we conclude that for any theorem F of $\mathsf{JL}\,(\mathsf{CS})$, $\mathbf{t}(F)$ is a theorem of CL. ☐

Corollary 2.30 (Consistency of Justification Logic)**.** Any justification logic $\mathsf{JL}(\mathsf{CS})$ is consistent.

Proof. Let us assume, in contradiction, that $\mathsf{JL}(\mathsf{CS})$ is inconsistent. Then, $\perp$ is a theorem of $\mathsf{JL}(\mathsf{CS})$. Thus, by Theorem 2.29, we have that $\mathbf{t}(\perp) = \perp$ is a theorem of CL, which leads in contradiction.

Of course, we could have the same result by the theorem of soundness, as any of the semantic models in this thesis does not satisfy $\perp$. ☐

Theorem 2.31 (Deduction Theorem for Justification Logic)**.**
The deduction theorem holds for any justification logic.

Proof. Let $\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} G$. We will prove that $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$, by induction on the complexity of the derivation.

- Let G arrived as a member of $\Sigma \cup \{F\}$. Then, if $G = F$, we trivially have that $F \to G$ is a propositional tautology, thus $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$. On the other hand, if $G \in \Sigma$, then $\Sigma \vdash_{\mathsf{JL(CS)}} G$. But it also holds that, $\Sigma \vdash_{\mathsf{JL(CS)}} G \to F \to G$, as a propositional tautology. Therefore, by applying modus ponens to $G \to F \to G$ and G, we have $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$.
- Let G arrived as an axiom of JL. Then, trivially $\Sigma \vdash_{\mathsf{JL(CS)}} G$. By the same trick as the $G \in \Sigma$ case, we have $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$.
- Let G arrived as a member of constant specification CS. Then again, $\Sigma \vdash_{\mathsf{JL(CS)}} G$ and we can apply the same trick, as previously.
- Let G arrived by modus ponens. Thus, there is some formula H, s.t. $\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} H \to G$ and $\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} H$. Therefore, by induction hypothesis we have that $\Sigma \vdash_{\mathsf{JL(CS)}} F \to H \to G$ and $\Sigma \vdash_{\mathsf{JL(CS)}} F \to H$. Then, as $(F \to H \to G) \to (F \to H) \to F \to G$ is a propositional tautology[1], by applying two times modus ponens, we have $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$.

By induction, we conclude that $\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} G$ implies $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$.

For the other direction, let $\Sigma \vdash_{\mathsf{JL(CS)}} F \to G$. Then, we also have $\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} F \to G$ and trivially that $\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} F$. Thus, by modus ponens we have

$$\Sigma \cup \{F\} \vdash_{\mathsf{JL(CS)}} G.$$

[1]Known also as *Frege's theorem in propositional logic*

□

Lemma 2.33 (Lifting Lemma)**.** Let JL be a justification logic that has the internalization property relative to some constant specification CS. Then it holds that

$$F_1, \ldots, F_n \vdash_{\mathsf{JL(CS)}} F \Rightarrow (\forall t_1, \ldots, t_n \in \mathsf{Tm})\,(\exists t \in \mathsf{Tm})\,\big[t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{JL(CS)}} t{:}F\big].$$

Proof. We will prove it by induction on the number of the premises.

- Let $n = 0$. Then the internalization property states exactly the Lifting lemma.
- Let it holds for some $n \in \mathbb{N}$. It suffices to show that it holds for $n + 1$. Let a set of formulae $\{F_1, F_2, \ldots, F_{n+1}, F\}$, s.t. $F_1, \ldots, F_{n+1} \vdash_{\mathsf{JL(CS)}} F$. Then by the deduction theorem for justification logic we have that

  $$F_1, \ldots, F_n \vdash_{\mathsf{JL(CS)}} F_{n+1} \to F.$$

  Thus, by induction hypothesis we have that for any $t_1, \ldots, t_n \in \mathsf{Tm}$, there is some $t \in \mathsf{Tm}$, s.t.

  $$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{JL(CS)}} t{:}(F_{n+1} \to F).$$

  Therefore, trivially we have that

  $$t_1{:}F_1, \ldots, t_n{:}F_n, t_{n+1}{:}F_{n+1} \vdash_{\mathsf{JL(CS)}} t{:}(F_{n+1} \to F)$$

  and

  $$t_1{:}F_1, \ldots, t_n{:}F_n, t_{n+1}{:}F_{n+1} \vdash_{\mathsf{JL(CS)}} t_{n+1}{:}F_{n+1},$$

  for any $t_{n+1} \in \mathsf{Tm}$. Hence, by axiom scheme **J** we have

  $$t_1{:}F_1, \ldots, t_n{:}F_n, t_{n+1}{:}F_{n+1} \vdash_{\mathsf{JL(CS)}} t \cdot t_{n+1}{:}F,$$

  i.e., there is such a some term $t \cdot t_{n+1} \in \mathsf{Tm}$.

□

Theorem 2.34. If CS an axiomatically appropriate constant specification for JL then JL has the strong internalization property relative to constant specification CS

Proof. We will prove by induction on the complexity of the derivation of F.

- Let F an instance of some axiom scheme in JL. Then, as CS is axiomatically appropriate, there is some $c \in \mathsf{Con}$, s.t. $\vdash_{\mathsf{JL(CS)}} c{:}F$.
- Let $F \in \mathsf{CS}$. That is $F \coloneqq c_n{:} \ldots {:}c_1{:}A$, where $c_1, \ldots, c_n \in \mathsf{Con}$ and A some axiom scheme in JL. Then, as CS is axiomatically appropriate, there is some $c_{n+1} \in \mathsf{Con}$, s.t.

  $$\vdash_{\mathsf{JL(CS)}} c_{n+1}{:}c_n{:} \ldots {:}c_1{:}A.$$

- Let F was derived by modus ponens. Then, there is a justification formula $G \in \mathcal{L}_{\mathsf{J}}$, s.t. G and $G \to F$ arrised as steps of the derivation. By induction hypothesis, there are some grounded terms $s, t \in \mathsf{Tm}$ s.t. $s{:}(G \to F)$ and $t{:}G$ are theorems of $\mathsf{JL(CS)}$. Therefore, by application axiom scheme **J** and modus ponens, we have that $s \cdot t{:}F$ is a theorem of $\mathsf{JL(CS)}$. Clearly, as s, t where grounded terms, so does $s \cdot t$.

□

Theorem 2.39. Let arbitrary modal language ML, and JL its correlated justification logic. Let also CS an axiomatically appropriate constant specification for JL. Then,

- for every theorem G of ML there is a normal realization F of G in JL.
- Therefore, JL is a counterpert of ML.

Proof. The first realization theorem was proved in 1995, in [1], by Artemov. The proof was between S4 and LP and was a constructive proof based on a cut-free sequent calculus for S4. The corresponding realization algorithm defining the realization could produce terms of exponential length in the size of the derivation. Vladimir Brezhnev and Roman Kuznets in [3] improved the realization algorithm so that it can produce terms of quadratic length. A non-constructive approach for the realization of S4 with respect to LP was first given in [7], by Fitting, in 2005. He made use of his semantics, which were also introduced in the same paper. Brezhnev, in [4] gave the realization theorem for K, KD and T, even though K and T could arise trivially, as sublogics of S4[2] from [1]. The realization theorem for S5 and JT45, in the standard axiomatization, was given by Natalia Rubtsova in [10]. Realization theorem for an infinite family of logics can be found in [8], where Fitting gave a general method for the proof of realization. A proof for all the logics defined in this thesis can be also found in [61], on chapters 6-8. □

[2]KD is also a sublogic of S4, but the corresponding justification logic JD and particularly the axiom scheme **JD** were firstly defined in [4].

APPENDIX B

PROOFS OF CHAPTER 3

BASIC MODELS

Theorem 3.5 (Soundness and Completeness for Basic Models)**.**

- Let CS a constant specification for J_0.

  $BM(\mathsf{J}_0(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J** and $+$ closure conditions, i.e., $* \in BM(\mathsf{J}_0(\mathsf{CS}))$ iff

$$\forall s, t \in \mathsf{Tm} \begin{cases} s^* \triangleright t^* \subseteq (s \cdot t)^* \\ s^* \cup t^* \subseteq (s+t)^* \end{cases} \quad \& \quad * \models \mathsf{CS}.$$

  The basic justification logic $\mathsf{J}_0(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{J}_0(\mathsf{CS}))$, i.e.,

$$\vdash_{\mathsf{J}_0(\mathsf{CS})} F \iff BM(\mathsf{J}_0(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JT.

  $BM(\mathsf{JT}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **JT** closure conditions, i.e., $* \in BM(\mathsf{JT}(\mathsf{CS}))$ iff

$$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,(\forall F \in \mathcal{L}_{\mathsf{J}})\,[F \in t^* \Rightarrow * \models F]\,.$$

  The justification logic $\mathsf{JT}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{JT}(\mathsf{CS}))$, i.e.,

$$\vdash_{\mathsf{JT}(\mathsf{CS})} F \iff BM(\mathsf{JT}(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JD.

  $BM(\mathsf{JD}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **JD** closure conditions, i.e., $* \in BM(\mathsf{JD}(\mathsf{CS}))$ iff

$$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,[\bot \notin t^*]\,.$$

  The justification logic $\mathsf{JD}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{JD}(\mathsf{CS}))$, i.e.,

$$\vdash_{\mathsf{JD}(\mathsf{CS})} F \iff BM(\mathsf{JD}(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JT.

  $BM(\mathsf{J4}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **J4** closure conditions, i.e. $* \in BM(\mathsf{J4}(\mathsf{CS}))$ iff

  $$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,(\forall F \in \mathcal{L}_\mathsf{J})\left[F \in t^* \Rightarrow t{:}F \in (!t)^*\right].$$

  The justification logic $\mathsf{J4}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{J4}(\mathsf{CS}))$, i.e.,

  $$\vdash_{\mathsf{J4}(\mathsf{CS})} F \iff BM(\mathsf{J4}(\mathsf{CS})) \models F.$$

- Let CS a constant specification for JT.

  $BM(\mathsf{J5}(\mathsf{CS}))$ is the class of basic CS-models, that respects the **J**, $+$ and **J5** closure conditions, i.e., $* \in BM(\mathsf{J5}(\mathsf{CS}))$ iff

  $$* \models \mathsf{CS} \quad \& \quad (\forall t \in \mathsf{Tm})\,(\forall F \in \mathcal{L}_\mathsf{J})\left[F \notin t^* \Rightarrow \neg t{:}F \in (?t)^*\right].$$

  The justification logic $\mathsf{J5}(\mathsf{CS})$ is sound and complete with respect to the $BM(\mathsf{J5}(\mathsf{CS}))$, i.e.,

  $$\vdash_{\mathsf{J5}(\mathsf{CS})} F \iff BM(\mathsf{J5}(\mathsf{CS})) \models F.$$

The $BM(\mathsf{LP}(\mathsf{CS}))$ classes and $BM(\mathsf{JD45}(\mathsf{CS}))$ are defined accordingly.

Proof. The original proof of this theorem, for the J_0 case, was given in [12]. The original proof of this theorem, for the other cases was given in [13]. A proof of this theorem can be also found in [61]. We will give only the counterpart for the axiom **JD**, which is not given in [61].

Let $* \in BM(\mathsf{JD}(\mathsf{CS}))$. Then for any term $t \in \mathsf{Tm}$, we have $* \models \neg t{:}\bot$, or equivalently $* \not\models t{:}\bot$. Equivalently, by the definition of basic models $\bot \notin t^*$.

Let the closure condition for **JD** holds for $*$. That is, for any term $t \in \mathsf{Tm}$ we have that $\bot \notin t^*$. Hence, $* \not\models t{:}\bot$, or equivalently $* \models \neg t{:}\bot$. □

MKRTYCHEV MODELS

Theorem 3.8. Every $BM(\mathsf{JT})$-model is a Mkrtychev model.

For every Mkrtychev model $*$, there is a basic model $*' \in BM(\mathsf{JT})$, s.t. for any formula $F \in \mathcal{L}_\mathsf{J}$

$$* \models F \iff *' \models F.$$

Proof. Let $* \in BM(\mathsf{JT})$. Let also $* \models t{:}F$, for some $t \in \mathsf{Tm}$ and $F \in \mathcal{L}_\mathsf{J}$. Then, by axiom scheme **JT** we have $* \models F$ and of course $F \in t^*$, i.e.,

$$* \models t{:}F \Rightarrow * \models F \;\&\; F \in t^*.$$

The other direction trivially holds by the definition of truth in $*$. Therefore,

$$* \models t{:}F \Leftrightarrow * \models F \;\&\; F \in t^*,$$

i.e., $*$ is a Mkrtychev model.

For the second proposition, let $*$ be a Mkrtychev model. We define the basic model $*'$, s.t.

$$*'(p) := *(p) \qquad p \in \mathsf{Prop}$$
$$*'(t) := \{F \in \mathcal{L}_{\mathsf{J}} \mid * \models t{:}F\} \qquad t \in \mathsf{Tm}$$

i.e., $t^{*'} := \{F \in \mathcal{L}_{\mathsf{J}} \mid * \models F \;\&\; F \in t^*\}$. Let us prove that $*'$ is in $BM(\mathsf{JT})$.

• **J**-closure condition

Let $F \to G \in s^{*'}$ and $f \in t^{*'}$. Equivalently, we have $* \models F \to G$ and $F \to G \in s^{*'}$, but also $* \models F$ and $F \in t^{*'}$. Then, by modus ponens we have $* \models G$, while by **J**-closure for $*$, we have $G \in (s \cdot t)^*$. Equivalently, we have $G \in (s \cdot t)^{*'}$.

• $+$-closure condition

Let w.l.o.g. that $F \in t^{*'}$. Equivalently, we have that $* \models F$ and $F \in t^*$. Thus, by $+$-closure we have $F \in (s+t)^*$ and of course, $* \models F$. Equivalently, we have that $F \in (s+t)^{*'}$.

• $+$-closure condition

Let $F \in t^{*'}$. Then, we have $* \models F$. Hence, it suffices to show that for any $F \in \mathcal{L}_{\mathsf{J}}$, it holds that

$$* \models F \Rightarrow *' \models F.$$

We will prove by induction on the complexity of F.

- Let $F := p \in \mathsf{Prop}$.
  Then, by the definition of $*'$ we have $*'(p) = *(p)$, thus, the requested proposition holds.

- Let $F := G \to H$, for some $G, H \in \mathcal{L}_{\mathsf{J}}$.
  Then we have

$$\begin{aligned} * \models G \to H &\Leftrightarrow * \not\models G \text{ or } * \models H \\ &\Rightarrow *' \not\models G \text{ or } *' \models H && \text{induction hypothesis} \\ &\Leftrightarrow *' \models G \to H \end{aligned}$$

- Let $F := t{:}G$, for some $t \in \mathsf{Tm}$ and $G \in \mathcal{L}_{\mathsf{J}}$.
  Then by the definition of $*'$, we have

$$\begin{aligned} * \models t{:}G &\Leftrightarrow G \in t^{*'} && \text{definition of } *' \\ &\Leftrightarrow *' \models t{:}G && \text{truth in basic m.} \end{aligned}$$

Therefore, by induction we have the requested property. □

Corollary 3.9 (Soundness and Completeness for Mkrtychev models)**.** Let CS a constant specification for JT.
$\mathsf{JT(CS)}$ is sound and complete with respect to Mkrtychev models.

Let CS a constant specification for LP.
$\mathsf{JT(CS)}$ is sound and complete with respect to Mkrtychev models that satisfies the **J4**-closure condition for basic models, i.e.,

$$F \in t^* \Rightarrow t{:}F \in (!t)^*.$$

Proof. The original proof of this corollary was given in [6]. In this thesis, it is given as a straight forward corollary of Theorems 3.5 and 3.8. □

FITTING MODELS

Theorem 3.15 (Soundness and Completeness for Fitting Models)**.** Let CS an arbitrary constant specification for the corresponding justification logic JL.

- $J_0(CS)$ is sound and complete with respect to the class of Fitting models that meet CS and respect the minimum evidence conditions, i.e., they respect the **J** and $+$ Evidence conditions.
- $JT(CS)$ is sound and complete with respect to the class of Fitting models that meet CS and respect the **JT** modal condition and the minimum evidence conditions.
- $JD(CS)$ is sound and complete with respect to the class of Fitting models that meet CS and respect the **JD** modal condition and the minimum evidence conditions.
- $J4(CS)$ is sound and complete with respect to the class of Fitting models that meet CS and respect the **J4** modal condition, the minimum evidence conditions, the monotonicity condition and the !-condition.
- $J5(CS)$ is sound and complete with respect to the class of Fitting models that meet CS and respect the **J5** modal condition, the minimum evidence conditions and the ?-condition and they have a strong evidence function.

We have the corresponding soundness and completeness theorems for the other justification logics. For instance, $LP(CS)$ is sound and complete with respect to the class of Fitting models based on a reflexive and transitive Kripke frame, which fulfill the monotonicity and ! evidence condition and which meet CS.

Proof. The original proof was given only for LP, by Fitting, in [7]. A proof for all the justification logics can be found in [61]. The proof is similar to the proof of Theorem 3.31; thus, it may be omitted. □

Theorem 3.17. Let CS arbitrary constant specification for JD. Then,

- $JD(CS)$ is sound and complete with respect to the class of Fitting models that meet CS and respect the minimum and **JD** evidence conditions.

We have the corresponding soundness and completeness theorems for the other justification logics, containing the axiom scheme **JD**.

Proof. The original proof can be found in [34], by Kuznets. □

MODULAR MODELS

Theorem 3.22 (Soundness and Completeness for Modular Models)**.** Let CS be an arbitrary constant specification for justification logic JL, where JL one of the defined justification logics.

The justification logic $JL(CS)$ is sound and complete with respect to the class of $JL(CS)$-models.

Proof. The original proof of this theorem, for the J_0 case, was given in [12]. The original proof of this theorem, for the other cases was given in [13]. A proof of this theorem can be also found in [61]. The proof is similar to the proof of Theorem 3.31; thus, it may be omitted. □

Theorem 3.25 (Soundness & Completeness of Modal-Justification Logics for Modular Models)**.** Each logic MLJL of the modal-justification logics is sound and complete with respect to the class of modular models that

- are based on Kripke frames in the class of Kripke frames which corresponds to the modal counterpart of the logic;
- and whose basic model function respects the requirements for the justification logic counterpart, i.e., they are JL-models.

If an arbitrary constant specification CS for JL is also given, then $\mathsf{MLJL(CS)}$ is sound and complete with respect to the class of $\mathsf{JL(CS)}$-models based on frames in the class of Kripke frames that corresponds to ML.

E.g., the modal-justification logic $\mathsf{K4J5(CS)}$ is sound and complete with respect to the class of $\mathsf{J5(CS)}$-models that are based on Kripke frames in $\mathcal{K}4$, where CS is some constant specification for $\mathsf{J5}$.

Proof. The proof is similar to the proof of Theorem 3.31; thus, it may be omitted. □

Theorem 3.28 (Conservativity of Modal-Justification Logic with Connection)**.** Let CS a constant specification for justification logic JL.
$\mathsf{MLJL(CS)} + \mathbf{C}$ is a conservative extension of CL.

Proof. We recursively define the translation function $\mathbf{t}\colon \mathcal{L}_{\Box\mathsf{J}} \to \mathcal{L}_{\mathsf{CL}}$, as follows

$$
\begin{aligned}
\mathbf{t}(p) &:= p && \text{where } p \in \mathsf{Prop}\\
\mathbf{t}(F \to G) &:= \mathbf{t}(F) \to \mathbf{t}(G) && \text{where } F, G \in \mathcal{L}_{\Box\mathsf{J}}\\
\mathbf{t}(t{:}F) &:= \mathbf{t}(F) && \text{where } t \in \mathsf{Tm} \text{ and } F \in \mathcal{L}_{\mathsf{J}}\\
\mathbf{t}(\Box F) &:= \mathbf{t}(F) && \text{where } F \in \mathcal{L}_{\Box\mathsf{J}}
\end{aligned}
$$

From Theorem 2.29, it suffices to show that the translation of any instance of axiom scheme $\mathbf{C}$ is a theorem of CL. But this is trivial as $\mathbf{t}(t{:}G \to \Box G) = G \to G$. □

Corollary 3.29 (Consistency of Modal-Justification Logic with Connection)**.** Any modal justification logic with connection is consistent.

Proof. The proof is similar to the proof of Theorem 2.30; thus, it may be omitted. □

Theorem 3.30 (Deduction Theorem for Modal-Justification Logic with Connection)**.** The deduction theorem holds for any modal-justification logic with connection.

Proof. This theorem follows easily from Theorems 2.15 and 2.31. □

Theorem 3.31 (Soundness & Completeness for Modal-Justification Logic with Connection)**.** Let MLJL be an arbitrary modal-justification logic.
$\mathsf{MLJL} + \mathbf{C}$ is sound and complete with respect to the class of JYB-modular models that

- are based on Kripke frames in the class of Kripke frames which corresponds to the modal counterpart of the logic;

- and whose basic model function respects the requirements for the justification logic counterpart, i.e., they are JYB-JL-models.

If an arbitrary constant specification CS for JL is also given, then $\mathsf{MLJL}(\mathsf{CS}) + \mathbf{C}$ is sound and complete with respect to the class of JYB-$\mathsf{JL}(\mathsf{CS})$-models based on frames in the class of Kripke frames that corresponds to ML.

Proof. We will prove the soundness and completeness of these semantics and corresponding logics. Ideas for the soundness and completeness of the other semantics and corresponding logics can be found here.

Soundness

Let as assume that we are given some modular model $M = \langle W, R, * \rangle$. Let us also assume that F, G are arbitrary justification formulae and $s, t \in \mathsf{Tm}$ are arbitrary terms. Clearly, modus ponens and instantiations of propositional tautologies are sound with respect to modular models, as those are also trivially sound with respect to basic models.

Justification Axioms

J Let for any world $w' \in W$, the basic model $*_{w'}$ has the **J**-closure condition. Let some $w \in W$, s.t. $M, w \Vdash s:(F \to G) \wedge t:F$. Equivalently, we have that $*_w \models s:(F \to G) \wedge t:F$, i.e., $F \to G \in s^{*_w}$ and $F \in t^{*_w}$. Therefore, $G \in s^{*_w} \rhd t^{*_w}$ and as $s^{*_w} \rhd t^{*_w} \subseteq (s \cdot t)^{*_w}$, we have $*_w \models t:G$. Equivalently, we have $M, w \Vdash s \cdot t:G$. Therefore, we indeed have

$$M \Vdash s:(F \to G) \to t:F \to s \cdot t:G,$$

as wanted.

+ Let for any world $w' \in W$, the basic model $*_{w'}$ has the $+$-closure condition. Let some $w \in W$, s.t. $M, w \Vdash s:F$. Equivalently, we have that $*_w \models s:F$, i.e., $F \in s^{*_w}$. Therefore, by $+$-closure condition $F \in s^{*_w} \cup t^{*_w} \subseteq (s+t)^{*_w}$ and thus $*_w \models s + t:F$. Equivalently, we have $M, w \Vdash s + t:F$. Therefore, we indeed have

$$M \Vdash s:F \to s + t:F,$$

as wanted. Simillarly, for $t:F \to s + t:F$.

JT Let for any world $w' \in W$, the basic model $*_{w'}$ has the **JT**-closure condition. Let some $w \in W$, s.t. $M, w \Vdash t:F$. Equivalently, we have that $F \in t^{*_w}$. Therefore, by **JT**-closure condition it holds that $*_w \models F$. Equivalently, we have $M, w \Vdash F$. Therefore, we indeed have

$$M \Vdash t:F \to F,$$

as wanted.

JD Let for any world $w \in W$, the basic model $*_w$ has the **JD**-closure condition,i.e., $\bot \notin t'^{*_w}$, for any $t' \in \mathsf{Tm}$. Equivalently, we have $*_w \not\models t':\bot$ or equivalently $M, w \not\Vdash t':\bot$. Therefore, we indeed have

$$M \Vdash \neg t:\bot,$$

as wanted.

J4 Let for any world $w' \in W$, the basic model $*_{w'}$ has the **J4**-closure condition. Let some $w \in W$, s.t. $M, w \Vdash t{:}F$. Equivalently, we have that $F \in t^{*_w}$. Therefore, by **J4**-closure condition we get $t : F \in (!)^{*_w}$, or equivalently $*_w \models !t : t : F$. Equivalently, we have $M, w \Vdash !t{:}t{:}F$. Therefore, we indeed have

$$M \Vdash t{:}F \to !t{:}t{:}F,$$

as wanted.

J5 Let for any world $w' \in W$, the basic model $*_{w'}$ has the **J5**-closure condition. Let some $w \in W$, s.t. $M, w \Vdash \neg t : F$. Equivalently, we have that $F \notin t^{*_w}$. Therefore, by **J5**-closure condition we get $\neg t{:}F \in (?)^{*_w}$, or equivalently $*_w \models ?t{:}\neg t{:}F$. Equivalently, we have $M, w \Vdash ?t{:}\neg t{:}F$. Therefore, we indeed have

$$M \Vdash \neg t{:}F \to ?t{:}\neg t{:}F,$$

as wanted.

Modal Axioms

From now on we assume that $F, G \in \mathcal{L}_{\Box J}$ are arbitrary modal-justification formulae.

N Let F is true in M, i.e., for any world $w' \in W$ it holds that $M, w' \Vdash F$. Let arbitrary $w \in W$. Then for any $u \in R[w]$ we have that $M, u \Vdash F$ and thus $M, w \Vdash \Box F$. Therefore, $M \Vdash \Box F$ and thus we indeed have that

$$M \Vdash F \Rightarrow M \Vdash \Box F,$$

i.e., M respects the necessitation rule.

K Let some $w \in W$, s.t. $M, w \Vdash \Box(F \to G) \wedge \Box F$. That is, for any $u \in R[w]$ we have that $M, u \Vdash (F \to G) \wedge F$ and thus $M, u \Vdash G$. Therefore, $M, w \Vdash \Box G$ and thus we indeed have that

$$M \Vdash \Box(F \to G) \to \Box F \to \Box G,$$

i.e., M respects axiom scheme **K**.

T Let M is based on some Kripke frame in $\mathcal{T}$ and let $w \in W$, s.t. $M, w \Vdash \Box F$. That is, for any $u \in R[w]$ we have that $M, u \Vdash F$. As the accessibility relation R is reflexive we have $w \in R[w]$ and thus $M, w \Vdash F$. Therefore, we indeed have that

$$M \Vdash \Box F \to F,$$

i.e., M respects axiom scheme **T**.

D • Let M is based on some Kripke frame in $\mathcal{KD}$, i.e., for any world $w \in W$ it holds that $R[w] \neq \emptyset$. Therefore, for any world $w \in W$ it holds that $M, w \nVdash \Box\bot$, as otherwise there would be some $u \in R[w]$, s.t. $M, u \Vdash \bot$, which leads to contradiction. Therefore, we indeed have that

$$M \Vdash \neg\Box\bot,$$

i.e., M respects axiom scheme **D**.

4 Let M is based on some Kripke frame in $\mathcal{K}4$ and let $w \in W$, s.t. $M, w \Vdash \Box F$. That is, for any $u \in R[w]$ we have that $M, u \Vdash F$. As the accessibility relation R is transitive we have that for any $u \in R[w]$ and any $v \in R[u]$, it holds that $v \in R[w]$. Thus $M, v \Vdash F$ and as a result $M, u \Vdash \Box F$. Equivalently, we have that $M, w \Vdash \Box\Box F$. Therefore, we indeed have that

$$M \Vdash \Box F \to \Box\Box F,$$

i.e., M respects axiom scheme **4**.

5 Let M is based on some Kripke frame in $\mathcal{K}5$ and let $w \in W$, s.t. $M, w \Vdash \neg\Box F$. That is, there is some $u \in R[w]$ s.t. $M, u \nVdash F$. As the accessibility relation R is euclidean we have that for any $v \in R[w]$ it holds that $u \in R[v]$. Thus, for any $v \in R[w]$ we have that $M, v \Vdash \neg\Box F$ and as a result $M, w \Vdash \Box\neg\Box F$.Therefore, we indeed have that

$$M \Vdash \neg\Box F \to \Box\neg\Box F,$$

i.e., M respects axiom scheme **5**.

Axiom **C**

C Let M is an JYB-modular model. Let also some world $w \in W$, s.t. $M, w \Vdash t{:}F$, where $F \in \mathcal{L}_{\mathsf{J}}$ is a justification formula. Equivalently, we have that $F \in t^{*_w}$. As $t^{*_w} \subseteq \Box^w$ we get $M, w \Vdash \Box F$. Therefore, we indeed have that

$$M \Vdash t{:}F \to \Box F,$$

i.e., M respects axiom scheme **C**.

Completeness

Before starting with the proof of the completeness part, we have to give some definitions and lemmas.

Definition B.1. Let $\mathsf{MLJL(CS)}$ a modal-justification logic, where CS a constant specification for JL. Let also $\Sigma \subseteq \mathcal{L}_{\Box\mathsf{J}}$ a set of modal-justification formulae.

- We say that Σ is $\mathsf{MLJL(CS)} + \mathbf{C}$-inconsistent iff $\Sigma \vdash_{\mathsf{MLJL(CS)}+\mathbf{C}} \bot$.
  Otherwise, Σ is called $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent
- Σ is called maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent iff it is $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent and any proper superset of modal-justification formulae of it is $\mathsf{MLJL(CS)} + \mathbf{C}$-inconsistent.

Note that this definition is congruent with Definition 2.1. Clearly, the maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent sets, are exactly the maximal sets (according to Definition 2.1) that are also $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent.

Lemma B.2 (Properties of Maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-Consistent Sets)**.** Let Σ is a maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent set of modal-justification formulae. Then, for any modal-justification formula $F \in \mathcal{L}_{\Box\mathsf{J}}$, the following two propositions hold:

- $F \in \Sigma \Leftrightarrow \Sigma \vdash_{\mathsf{MLJL(CS)}+\mathbf{C}} F$
- $F \in \Sigma$ or $\neg F \in \Sigma$

Proof. Let Σ is maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent.

The right direction for the first proposition is trivial. For the left direction, let $\Sigma \vdash_{\mathsf{MLJL(CS)}+\mathbf{C}} F$. Clearly, $\Sigma \cup \{F\}$ is $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent. Thus, as Σ is also maximal, it must be the case that $F \in \Sigma$.

For the second proposition, let $F \notin \Sigma$. Then, $\Sigma \cup \{F\}$ is $\mathsf{MLJL(CS)}+\mathbf{C}$-inconsistent, i.e., $\Sigma \cup \{F\} \vdash_{\mathsf{MLJL(CS)}+\mathbf{C}} \bot$. Then, by the corrsponding deduction theorem, we have $\Sigma \vdash_{\mathsf{MLJL(CS)}+\mathbf{C}} \neg F$. Therefore, by the first of the two propositions, we have $\neg F \in \Sigma$. ☐

Lemma B.3 (Lindenbaum's Lemma for Modal-Justification Logic with Connection)**.** Let $\mathsf{MLJL(CS)}$ a modal-justification logic, where CS a constant specification for JL. Every $\mathsf{MLJL(CS)}+\mathbf{C}$-consistent set of formulae can be extended to a maximal $\mathsf{MLJL(CS)}+\mathbf{C}$-consistent set of formulae.

Proof. The proof of Lemma B.3 is similar to the classical proof of Lindenbaum's Lemma for classical logic and thus omitted. ☐

We will prove the completeness by defining for each modal-justification logic a corresponding modular model, called *canonical model*.

Definition B.4 (Canonical Model for Modal-Justification Logic with Connection)**.** Let $\mathsf{MLJL(CS)}$ an arbitrary modal-justification logic, where CS a constant specification for JL.

We define the *canonical model* $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, * \rangle$ for $\mathsf{MLJL(CS)} + \mathbf{C}$, as follows:

- $\mathscr{W}$ is the set of maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent sets.
- $\mathscr{R}$ is an accesibility relation on $\mathscr{W}$ defined as

$$\Gamma \mathscr{R} \Delta \Leftrightarrow \Box^{\Gamma} \subseteq \Delta.$$

- $*$ is a basic model function, defined as follows

$$*_{\Gamma}(p) = 1 \Leftrightarrow p \in \Gamma,$$

  for any $p \in \mathsf{Prop}$ and

$$*_{\Gamma}(t) := \{F \in \mathcal{L}_{\Box \mathsf{J}} \mid t{:}F \in \Gamma\},$$

  for any term $t \in \mathsf{Tm}$.

Let $\mathsf{MLJL(CS)} + \mathbf{C}$ an arbitrary modal-justification logic and $\mathscr{M}$ the corresponding canonical model. By Lemma B.3 we have that the set of worlds $\mathscr{W}$, of the canonical model $\mathscr{M}$ is well defined and as a result, $\mathscr{M}$ is a modular model. In order to prove that $\mathscr{M}$ is a JYB-$\mathsf{MLJL(CS)}$-modular model, we will need the next important lemma.

Lemma B.5 (Truth Lemma for Modal-Justification Logic with Connection)**.** Let $\mathsf{MLJL(CS)} + \mathbf{C}$ a modal-justification logic, where CS a constant specification for JL. Let also $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, * \rangle$ the corresponding canonical model.
For any modal-justification formula $F \in \mathcal{L}_{\Box \mathsf{J}}$ and any world $\Gamma \in \mathscr{W}$ it holds that

$$F \in \Gamma \Leftrightarrow \mathscr{M}, \Gamma \Vdash F.$$

Proof. We will give the proof for the $\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}$ case. The other cases can be treated similarly. We will prove that Truth lemma holds by induction on the complexity of F.

- Let $F := p \in \mathsf{Prop}$. Then for any world $\Gamma \in \mathscr{W}$ it is easy to observe that

$$p \in \Gamma \Leftrightarrow *_\Gamma(p) = 1 \Leftrightarrow \mathscr{M}, \Gamma \Vdash p,$$

  as wanted.

- The cases of the boolean connectives are trivial.

- Let $F := t{:}G$, where $G \in \mathcal{L}_\mathsf{J}$ and $t \in \mathsf{Tm}$. Let also some arbitrary world $\Gamma \in \mathscr{W}$. It is easy to observe that

$$\begin{aligned} t{:}G \in \Gamma &\Leftrightarrow \Gamma \in *_\Gamma(t) \\ &\Leftrightarrow *_\Gamma \models t{:}G \\ &\Leftrightarrow \mathscr{M}, \Gamma \Vdash t{:}G, \end{aligned}$$

  where the first equivalence holds from the definition of the basic model function, the second one from the definition of truth in basic models and the last one from the definition of truth in modular models.

- Let $F := \Box G$, where $G \in \mathcal{L}_{\Box\mathsf{J}}$. Let also some arbitrary world $\Gamma \in \mathscr{W}$, s.t. $\Box G \in \Gamma$. Then for any $\Delta \in \mathscr{R}[\Gamma]$ we have that $G \in \Delta$, as by the definition of the accessibility relation $\mathscr{R}$ we know that $\Box^\Gamma \subseteq \Delta$. Therefore, by induction hypothesis we have that for any $\Delta \in \mathscr{R}[\Gamma]$, it holds that $\mathscr{M}, \Delta \Vdash G$. Equivalently, we have that $\mathscr{M}, \Gamma \Vdash \Box G$, as wanted.

For the other direction, let $\Gamma \in \mathscr{W}$, s.t. $\mathscr{M}, \Gamma \Vdash \Box G$. Let us assume in contradiction that $\Box G \notin \Gamma$. Then, as Γ is a maximal $\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}$-consistent set of formulae, we have that $\neg\Box G \in \Gamma$. Firstly, we want to show that $\Box^\Gamma \cup \{\neg F\}$ is $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent. Let us assume, in contradiction that $\Box^\Gamma \cup \{\neg G\}$ is $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$-inconsistent. That is, there is some finite subset of formulae $\Sigma \subset \Box^\Gamma$, s.t.

$$\Sigma \cup \{\neg G\} \vdash_{\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}} \bot.$$

Then by the deduction theorem for justification logic, i.e. Theorem 2.31 and propositional reasoning, we have

$$\vdash_{\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}} \bigwedge \Sigma \wedge \neg G \to \bot,$$

or equivalently by propositional reasoning it holds that

$$\vdash_{\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}} \bigwedge \Sigma \to G.$$

By Example 3.27 we get that

$$\vdash_{\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}} \bigwedge \Box\Sigma \to \Box G,$$

Clearly, $\Box\Sigma \subseteq \Gamma$ and as Γ is a maximal $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent set of formulae, we have that $\bigwedge \Box\Sigma \in \Gamma$ and thus $\Box G \in \Gamma$. Hence, both $\neg\Box G$ and $\Box G$ belong in Γ, which leads to contradiction, as Γ is $\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}$-consistent. Therefore, $\Box^\Gamma \cup \{\neg F\}$ is indeed $\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}$-consistent. On that account, we can extend it to a maximal $\mathsf{KJ}_0(\mathsf{CS})+\mathbf{C}$-consistent set Δ. But then, because $\Box^\Gamma \subseteq \Delta$, we have that $\Delta \in \mathscr{R}[\Gamma]$ and $\neg G \in \Delta$. Then, by induction hypothesis we have that $\mathscr{M}, \Delta \Vdash \neg G$, thus $\mathscr{M}, \Gamma \not\Vdash \Box\Gamma$, which leads to contradiction. By that we conclude that $\Box G \in \Gamma$, as wanted. From all the above, by induction we have that the Truth Lemma for $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$ is holds. $\Box$

From now on we will prove for any modal-justification logic $\mathsf{MLJL}(\mathsf{CS}) + \mathbf{C}$ the corresponding canonical model is a JYB-$\mathsf{MLJL}(\mathsf{CS})$-modular model. Specifically, we will show for any of the basic cases of modal-justification logic that its canonical model respects the requested properties of accessibility relation and basic model function closure conditions. For any case, we will denote by $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, * \rangle$ the corresponding canonical model.

$\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$ We want to show that $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model.

J Let $\Gamma \in \mathscr{W}$, $s, t \in \mathsf{Tm}$ and $G \in \mathcal{L}_\mathsf{J}$, s.t. $G \in s^{*_\Gamma} \triangleright t^{*_\Gamma}$. Equivalently, there is some $F \in \mathcal{L}_\mathsf{J}$, s.t. $F \to G \in s^{*_\Gamma}$ and $F \in t^{*_\Gamma}$, i.e., $\mathscr{M}, \Gamma \Vdash s{:}(F \to G) \wedge t{:}F$. Then, as Γ is a maximal $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent set, by axiom scheme **J**, we have that $\mathscr{M}, \Gamma \Vdash s \cdot t{:}G$. Therefore, by Lemma B.5 we have that $s \cdot t{:}G \in \Gamma$. By the definition of the basic model function $*$, we have that $G \in (s \cdot t)^{*_\Gamma}$. As G was arbitrary, we have $s^{*_\Gamma} \triangleright t^{*_\Gamma} \subseteq (s \cdot t)^{*_\Gamma}$, i.e., the closure condition for **J** holds for Γ, which was an arbitrary world in $\mathscr{W}$.

From now on we will, many times, write our steps in the form of equivalences and implications, for brevity.

+ Let $\Gamma \in \mathscr{W}$, $s, t \in \mathsf{Tm}$ and $F \in \mathcal{L}_\mathsf{J}$, s.t. $F \in s^{*_\Gamma} \cup t^{*_\Gamma}$. W.l.o.g. let $F \in t^{*_\Gamma}$. We observe that

$$
\begin{aligned}
F \in t^{*_\Gamma} &\Leftrightarrow t{:}F \in \Gamma && \text{Def. of } * \\
&\Rightarrow s + t{:}F \in \Gamma && \text{max. cons., } + \\
&\Leftrightarrow F \in (s+t)^{*_\Gamma} && \text{Def. of } *
\end{aligned}
$$

Therefore, we conclude that $s^{*_\Gamma} \cup t^{*_\Gamma} \subseteq (s+t)^{*_\Gamma}$,i.e., the $+$-closure condition holds.

CS Let $\Gamma \in \mathscr{W}$ and $c{:}A \in \mathsf{CS}$. We want to show that $*_\Gamma$ is a CS-basic model. As Γ is a maximal $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent set of formulae, we have that $\mathsf{CS} \subset \Gamma$. Thus $c{:}A \in \Gamma$. Then by definition of the basic model function $*$ we have that $A \in c^{*_\Gamma}$, i.e., $*_\Gamma \models c{:}A$. Therefore, $*_\Gamma$ is indeed a CS-basic model.

JYB Let $\Gamma \in \mathscr{W}$, $t \in \mathsf{Tm}$ and $F \in \mathcal{L}_\mathsf{J}$, s.t. $F \in t^{*_\Gamma}$. Then we have

$$
\begin{aligned}
F \in t^{*_\Gamma} &\Leftrightarrow t{:}F \in \Gamma && \text{Def. of } * \\
&\Rightarrow \Box F \in \Gamma && \text{max. cons., } \mathbf{C} \\
&\Leftrightarrow \mathscr{M}, \Gamma \Vdash \Box F && \text{Lem.} B.5 \\
&\Leftrightarrow F \in \Box^\Gamma && \text{Def. of } \Box^\Gamma
\end{aligned}
$$

Therefore, we conclude that $t^{*_\Gamma} \subseteq \Box^\Gamma$,i.e., $\mathscr{M}$ is a JYB-modular model.

From all the above we have that the canonical model $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, * \rangle$ for $\mathsf{KJ}_0(\mathsf{CS}) + \mathbf{C}$ is indeed a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model.

$\mathsf{TJ}_0(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that the accessibility relation $\mathscr{R}$ is also reflexive, i.e., $\mathscr{M}$ is a JYB-$\mathsf{TJ}_0(\mathsf{CS})$-modular model.

Let arbitrary $\Gamma \in \mathscr{W}$ and some modal-justification formula F, s.t. $F \in \Box^{\Gamma}$. Equivalently, $\mathscr{M}, \Gamma \Vdash \Box F$. Then by Lemma B.5 we have $\Box F \in \Gamma$. As Γ is a maximal $\mathsf{TJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent set, by axiom scheme **T** we get $F \in \Gamma$. As F was an arbitrary modal-justification formula that belongs in $\Box^{\Gamma}$, we have that $\Box^{\Gamma} \subseteq \Gamma$. Therefore, by the definition of $\mathscr{R}$ we have that $\Gamma \mathscr{R} \Gamma$, i.e., $\mathscr{R}$ is reflexive.

$\mathsf{KDJ}_0(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that the accessibility relation $\mathscr{R}$ is also serial, i.e., $\mathscr{M}$ is a JYB-$\mathsf{KDJ}_0(\mathsf{CS})$-modular model.

Let arbitrary $\Gamma \in \mathscr{W}$. We have to show that $\mathscr{R}[\Gamma] \neq \emptyset$. It suffices to show that $\Box^{\Gamma}$ is $\mathsf{KDJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent, as then by Lemma B.3 we can extend to some maximal $\mathsf{KDJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent set Δ which belongs in $\mathscr{R}[\Gamma]$. Let us assume, in contradiction, that $\Box^{\Gamma}$ is $\mathsf{KDJ}_0(\mathsf{CS}) + \mathbf{C}$-inconsistent. Equivalently, there is some finite $\Sigma \subset \Box^{\Gamma}$, s.t. $\Sigma \vdash_{\mathsf{KDJ}_0(\mathsf{CS})+\mathbf{C}} \bot$. Then, by Exapmle 3.27 we have $\Box\Sigma \vdash_{\mathsf{KDJ}_0(\mathsf{CS})+\mathbf{C}} \Box\bot$. Therefore, $\Gamma \vdash_{\mathsf{KDJ}_0(\mathsf{CS})+\mathbf{C}} \Box\bot$ and by Lemma B.5 we have $\Box\bot \in \Gamma$, which leads to contradiction. Thus, $\Box^{\Gamma}$ is $\mathsf{KDJ}_0(\mathsf{CS}) + \mathbf{C}$-consistent, as wanted.

$\mathsf{K4J}_0(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that the accessibility relation $\mathscr{R}$ is also transitive, i.e., $\mathscr{M}$ is a JYB-$\mathsf{K4J}_0(\mathsf{CS})$-modular model.

Let arbitrary $\Gamma, \Delta, \Theta \in \mathscr{W}$, s.t. $\Gamma \mathscr{R} \Delta$ and $\Delta \mathscr{R} \Theta$, i.e., $\Box^{\Gamma} \subseteq \Delta$ and $\Box^{\Delta} \subseteq \Theta$. It suffices to show that $\Box^{\Gamma} \subseteq \Theta$. Let arbitrary $F \in \Box^{\Gamma}$. We observe that

$$\begin{aligned} F \in \Box^{\Gamma} &\Leftrightarrow \mathscr{M}, \Gamma \Box F && \\ &\Leftrightarrow \Box F \in \Gamma && \text{Lem.} B.5 \\ &\Rightarrow \Box\Box F \in \Gamma && \text{max. cons., } \mathbf{4} \\ &\Rightarrow \Box F \in \Delta && \Box^{\Gamma} \subseteq \Delta \\ &\Rightarrow F \in \Theta && \Box^{\Delta} \subseteq \Theta \end{aligned}$$

Thus, $\mathscr{R}$ is indeed transitive.

$\mathsf{K5J}_0(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that the accessibility relation $\mathscr{R}$ is also euclidean, i.e., $\mathscr{M}$ is a JYB-$\mathsf{K5J}_0(\mathsf{CS})$-modular model.

Let arbitrary $\Gamma, \Delta, \Theta \in \mathscr{W}$, s.t. $\Gamma \mathscr{R} \Delta$ and $\Gamma \mathscr{R} \Theta$, i.e., $\Box^{\Gamma} \subseteq \Delta$ and $\Box^{\Gamma} \subseteq \Theta$. It suffices to show that $\Box^{\Delta} \subseteq \Theta$. Let us assume, in contradiction that there is some $F \in \Box^{\Delta}$, s.t. $F \notin \Theta$. As $F \notin \Theta$, by $\Box^{\Gamma} \subseteq \Theta$ we get $\Box F \notin \Gamma$. Therefore, as Γ is maximal $\mathsf{K5J}_0(\mathsf{CS}) + \mathbf{C}$-consistent, we have $\neg\Box F \in \Gamma$ and by axiom scheme **5** $\Box\neg\Box F \in \Gamma$. Then, as $\Box^{\Gamma} \subseteq \Delta$, we have $\neg\Box F \in \Delta$, which leads to contradiction, as Δ is $\mathsf{K5J}_0(\mathsf{CS}) + \mathbf{C}$-consistent. Thus, $\Box^{\Delta} \subseteq \Theta$.

$\mathsf{KJT}(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that for any world $\Gamma \in \mathscr{W}$ the basic model $*_{\Gamma}$ respects the **JT**-closure condition, i.e., $\mathscr{M}$ is a JYB-$\mathsf{KJT}(\mathsf{CS})$-modular model.

Let some world $\Gamma \in \mathscr{W}$ and some modal-justification formula $F \int^{*_{\Gamma}}$. We observe that

$$\begin{aligned} F \in t^{*_\Gamma} &\Leftrightarrow t{:}F \in \Gamma \\ &\Rightarrow F \in \Gamma && \text{max. cons., } \mathbf{JT} \\ &\Leftrightarrow \mathscr{M}, \Gamma \Vdash F && \text{Lem. } B.5 \\ &\Leftrightarrow *_\Gamma \models F && \text{Truth in modular m.} \end{aligned}$$

Therefore, for the arbitrary $\Gamma \in \mathscr{W}$, the basic model $*_\Gamma$ respects the **JT**-closure condition, as wanted.

$\mathsf{KJD}(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that for any world $\Gamma \in \mathscr{W}$ the basic model $*_\Gamma$ respects the **JD**-closure condition, i.e., $\mathscr{M}$ is a JYB-$\mathsf{KJD}(\mathsf{CS})$-modular model.

Let some world $\Gamma \in \mathscr{W}$ and let us assume, in contradiction, that $\bot \in t^{*_\Gamma}$. Equivalently, by the definition of $*$, we have that $t{:}\bot \in \Gamma$. But as Γ is a maximal $\mathsf{KJD}(\mathsf{CS}) + \mathbf{C}$-consistent set, by axiom **JD**, we have that $\neg t{:}\bot \in \Gamma$, which leads to contradiction. Therefore, for the arbitrary $\Gamma \in \mathscr{W}$, $\bot \notin t^{*_\Gamma}$, i.e., the basic model $*_\Gamma$ respects the **JD**-closure condition, as wanted.

$\mathsf{KJ4}(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that for any world $\Gamma \in \mathscr{W}$ the basic model $*_\Gamma$ respects the **J4**-closure condition, i.e., $\mathscr{M}$ is a JYB-$\mathsf{KJ4}(\mathsf{CS})$-modular model.

Let some world $\Gamma \in \mathscr{W}$ and some modal-justification formula $F \in t^{*_\Gamma}$. We observe that

$$\begin{aligned} F \in t^{*_\Gamma} &\Leftrightarrow t{:}F \in \Gamma \\ &\Rightarrow !t{:}t{:}F \in \Gamma && \text{max. cons., } \mathbf{J4} \\ &\Leftrightarrow t{:}F \in (!t)^{*_\Gamma} && \text{Def. of } * \end{aligned}$$

Therefore, for the arbitrary $\Gamma \in \mathscr{W}$, the basic model $*_\Gamma$ respects the **J4**-closure condition, as wanted.

$\mathsf{KJ5}(\mathsf{CS}) + \mathbf{C}$ Clearly, $\mathscr{M}$ is a JYB-$\mathsf{KJ}_0(\mathsf{CS})$-modular model, similarly with previously. We have to show that for any world $\Gamma \in \mathscr{W}$ the basic model $*_\Gamma$ respects the **J4**-closure condition, i.e., $\mathscr{M}$ is a JYB-$\mathsf{KJ4}(\mathsf{CS})$-modular model.

Let some world $\Gamma \in \mathscr{W}$ and some modal-justification formula $F \notin t^{*_\Gamma}$. We observe that

$$\begin{aligned} F \notin t^{*_\Gamma} &\Leftrightarrow t{:}F \notin \Gamma \\ &\Leftrightarrow \neg t{:}F \in \Gamma && \text{max. cons.} \\ &\Rightarrow ?t{:}\neg t{:}F \in \Gamma && \text{max. cons., } \mathbf{J5} \\ &\Leftrightarrow \neg t{:}F \in (?t)^{*_\Gamma} && \text{Def. of } * \end{aligned}$$

Therefore, for the arbitrary $\Gamma \in \mathscr{W}$, the basic model $*_\Gamma$ respects the **J5**-closure condition, as wanted.

The proof that for any modal-justification logic $\mathsf{MLJL}(\mathsf{CS}) + \mathbf{C}$ the corresponding canonical model is a JYB-$\mathsf{MLJL}(\mathsf{CS})$-modular model follows from the cases we have already shown. We finish our proof with following lemma.

Lemma B.6. Let $\mathsf{MLJL(CS)} + \mathbf{C}$ a modal-justification logic, where CS a constant specification for JL. Let also $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, * \rangle$ the corresponding canonical model. For any modal-justification formula $F \in \mathcal{L}_{\Box \mathsf{J}}$ we have that

$$\mathsf{MLJL(CS)} + \mathbf{C} \nvdash F \Rightarrow \mathscr{M} \Vvdash\!\!\!\!/\ F.$$

Proof. Let some modal-justification formula $F \in \mathcal{L}_{\Box \mathsf{J}}$, s.t. $\mathsf{MLJL(CS)} + \mathbf{C} \nvdash F$. Then, by deduction theorem for justification logic we have $\{\neg F\}$ is $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent. Therefore, by Lemma B.3 it can be extended to a maximal $\mathsf{MLJL(CS)} + \mathbf{C}$-consistent set $\Gamma \in \mathscr{W}$. Clearly, as $\neg F \in \Gamma$, by Lemma B.5 we have that $\mathscr{M}, \Gamma \Vvdash \neg F$. Thus, $\mathscr{M}, \Gamma \Vvdash\!\!\!\!/\ F$ and as a result, $\mathscr{M} \Vvdash\!\!\!\!/\ F$. □

From the previous lemma we have that for any modal-justification formula F not derivable in some modal-justification logic $\mathsf{MLJL(CS)} + \mathbf{C}$, F is not true in the class of JYB-$\mathsf{MLJL(CS)}$-modular models, as it is not true in its canonical model, which is proved to belong in it. This last sentence is the contrapositive statement of completeness. □

Lemma 3.32. The following propositions hold:

- Axiom scheme **JT** is a theorem of $\mathsf{TJ}_0 + \mathbf{C}$, or equivalently,

$$\mathsf{TJ}_0 + \mathbf{C} = \mathsf{TJT}$$

.

- Axiom scheme **JD** is a theorem of $\mathsf{KDJ}_0 + \mathbf{C}$, or equivalently,

$$\mathsf{KDJ}_0 + \mathbf{C} = \mathsf{KDJD}$$

.

Proof.
- $\vdash_{\mathsf{TJ}_0 + \mathbf{C}} t{:}F \to F$

$$\begin{array}{ll} F_1 : t{:}F \to \Box F & \mathbf{C} \\ F_2 : \Box F \to F & \mathbf{T} \\ F_3 : t{:}F \to F & 1, 2\ \mathbf{P} \end{array}$$

- $\vdash_{\mathsf{KDJ}_0 + \mathbf{C}} \neg t{:}\bot$

$$\begin{array}{ll} F_1 : t{:}F \to \Box F & \mathbf{C} \\ F_2 : \Box \bot \to \bot & \mathbf{D} \\ F_3 : t{:}\bot \to \bot & 1, 2\ \mathbf{P} \end{array}$$

□

Corollary 3.33. The following propositions hold:

- TJT is sound and complete with respect to the class of JYB-modular models based on reflexive Kripke frames.

- KDJD is sound and complete with respect to the class of JYB-modular models based on serial Kripke frames.

Proof. This is a straight forward corollary of Thoerem 3.31 and Lemma 3.32. □

HIEARCHY OF JUSTIFICATION LOGIC SEMANTICS

Theorem 3.34 (Hiearchy of Justification Logic Semantics)**.** For each model in some level of the pyramid, there is a model of the next level of the pyramid that expresses that model, i.e., satisfies exactly the same justification formulae.

Proof. Mkrtychev vs basic models

By Theorem 3.8 for every Mkrtychev model $*$, there is a basic model $*' \in BM(\mathsf{JT})$.

Basic vs Fitting models

Let an arbitrary basic model $*$. We define the Fitting model $M_* = \langle W, R, V, E \rangle$, where $W := \{w\}$ is a singleton, $R := \emptyset$ is the empty accessibility relation,

$$V(p) := \begin{cases} \{w\} & , p^* = 1 \\ \emptyset & , \text{ else} \end{cases}$$

and

$$E(t, F) := \begin{cases} \{w\} & , F \in t^* \\ \emptyset & , \text{ else} \end{cases}.$$

It is easy to prove by induction on the complexity of the formula, that for any justification formula $F \in \mathcal{L}_\mathsf{J}$

$$* \models F \Leftrightarrow M_* \Vdash F.$$

Indeed, let $F := p \in \mathsf{Prop}$. Then we trivially have

$$\begin{aligned} * \models p &\Leftrightarrow p^* = 1 \\ &\Leftrightarrow w \in V(p) = \{w\} \\ &\Leftrightarrow M_*, w \Vdash p \\ &\Leftrightarrow M_* \Vdash p. \end{aligned}$$

The cases of boolean connectives are trivial.
Let $F := t{:}G$. Then we have

$$\begin{aligned} * \models t{:}G &\Leftrightarrow G \in t^* \\ &\Leftrightarrow w \in E(t, G) = \{w\} \\ &\Leftrightarrow \begin{array}{ll} w \in E(t, G) = \{w\} & \& \\ \forall u \in R[w] = \emptyset & M_*, u \Vdash G \end{array} \\ &\Leftrightarrow M_*, w \Vdash t{:}G \\ &\Leftrightarrow M_* \Vdash t{:}G. \end{aligned}$$

Fitting vs JYB-modular models

• Let $M = \langle W, R, V, E \rangle$ an arbitrary Fitting model. We define the JYB-modular model $M' := \langle W, R, * \rangle$, where for any $w \in W$, for any term $t \in \mathsf{Tm}$ and any atomic proposition $p \in \mathsf{Prop}$, we define

$$*_w(t) := \{F \mid w \in E(t,F)\} \cap \Box^w$$

and

$$*_w(p) := \begin{cases} 1 & , w \in V(p) \\ 0 & , \text{ else} \end{cases}$$

Clearly M' is a modular model. It is also trivially a JYB-modular model, as for any world $w \in W$ and any term $t \in \mathsf{Tm}$, it holds that

$$*_w(t) := \{F \mid w \in E(t,F)\} \cap \Box^w \subseteq \Box^w.$$

We can prove by induction on the complexity of the formula that for any justification formula $F \in \mathcal{L}_\mathsf{J}$, it holds that

$$M, w \Vdash F \Leftrightarrow M', w \Vvdash F.$$

Let $F := p \in \mathsf{Prop}$. Then we easily observe that for any world $w \in W$, we have

$$\begin{aligned} M, w \Vdash p &\Leftrightarrow w \in V(p) \\ &\Leftrightarrow *_w(p) = 1 \\ &\Leftrightarrow *_w \models p \\ &\Leftrightarrow M', w \Vvdash p. \end{aligned}$$

The boolean cases are straight forward.

Let $F := t{:}G$. Then we can easily observe that

$$\begin{aligned} M, w \Vdash t{:}G \Leftrightarrow & \begin{array}{l} w \in E(t,G) = \{w\} \quad \& \\ \forall u \in R[w] \;\; M, u \Vdash G \end{array} \\ \Leftrightarrow & \begin{array}{l} w \in E(t,G) = \{w\} \quad \& \\ \forall u \in R[w] \;\; M', u \Vvdash G \end{array} \qquad \text{induction hypothesis} \\ \Leftrightarrow & \; G \in *_w(t) \\ \Leftrightarrow & \; M', w \Vvdash t{:}G. \end{aligned}$$

• Let $M = \langle W, R, * \rangle$ an arbitrary JYB-modular model. We define the Fitting model $M' := \langle W, R, V, E \rangle$, where for any atomic propostion $p \in \mathsf{Prop}$

$$V(p) := \{w \in W \mid *_w(p) = 1\}$$

and for any term $t \in \mathsf{Tm}$ and any justification formula $F \in \mathcal{L}_\mathsf{J}$

$$E(t,F) := \{w \in W \mid F \in *_w(t)\}.$$

We once again, prove by induction on the complexity of the formula, that for any formula $F \in \mathcal{L}_\mathsf{J}$ and any world $w \in W$

$$M, w \Vvdash F \Leftrightarrow M', w \Vdash F.$$

The atomic and boolean cases are similar, with previously.

Let $F \coloneqq t{:}G$. Then it is not hard to see that for any world $w \in W$

$$
\begin{aligned}
M, w \Vdash\!\vdash t{:}G &\Leftrightarrow G \in *_w(t) \subseteq \Box^w \\
&\Leftrightarrow \begin{array}{l} w \in E(t,G) = \{w\} \quad \& \\ \forall u \in R[w]\;\; M, u \Vdash\!\vdash G \end{array} \\
&\Leftrightarrow \begin{array}{l} w \in E(t,G) = \{w\} \quad \& \\ \forall u \in R[w]\;\; M', u \Vdash G \end{array} && \text{induction hypothesis} \\
&\Leftrightarrow M, w \Vdash t{:}G.
\end{aligned}
$$

Fitting / JYB-modular models vs modular models

Clearly, every JYB-modular model is a modular model, while the converse does not hold. □

APPENDIX C

PROOFS OF CHAPTER 4

LOGIC OF UNCERTAIN JUSTIFICATION

Theorem 4.6 (Conservativity of Logic of Uncertain Justifications)**.**
UJ is a *conservative extension* of the basic justification logic, J_0.

Proof. It follows by the translation function $\mathbf{t}\colon \mathcal{L}_{UJ} \to \mathcal{L}_J$, which is recursively defined as

$$\begin{aligned} \mathbf{t}(p) &:= p & &\text{where } p \in \mathsf{Prop} \\ \mathbf{t}(\neg F) &:= \neg\mathbf{t}(F) & &\text{where } F \in \mathcal{L}_{UJ} \\ \mathbf{t}(F \to G) &:= \mathbf{t}(F) \to \mathbf{t}(G) & &\text{where } F, G \in \mathcal{L}_{UJ} \\ \mathbf{t}(t{:}_{\mathsf{p}}F) &:= t{:}\mathbf{t}(F) & &\text{where } t \in \mathsf{Tm},\ \mathsf{p} \in \mathtt{S}^* \text{ and } F \in \mathcal{L}_{UJ} \end{aligned}$$

that any theorem of UJ translated in the language $\mathcal{L}_J$ is a theorem of J_0, as any axiom in UJ is translated to some axiom in J_0.

That, any theorem of J_0 is also a theorem of UJ, is trivial if we interpret the justification operator :, as $:_1$. Note, that this is in a sense different from the translation function **t**, as it is actually a difference only on the symbolism, i.e., : and $:_1$ are two different ways to interpret the same thing. □

Corollary 4.7 (Consistency of Logic of Uncertain Justifications)**.** Let CS an arbitrary constant specification for UJ. Then, UJ(CS) is consistent.

Proof. By Theorem 4.6, UJ is a conservative extension of J_0, which is consistent according to Corollary 2.30. The introduction of constant specification does not affect the consistency. □

Theorem 4.8 (Deduction Theorem for Logic of Uncertain Justifications)**.** Let CS an arbitrary constant specification for UJ. Then, the deduction theorem holds for UJ(CS).

Proof. The standard strategy of the proof of deduction theorems (e.g. the one for Theorem 2.31) can be applied here. □

Lemma 4.9 (UJ-Internalization Property & Lifting Lemma for UJ)**.**
Let CS an axiomatically appropriate constant specification for UJ. Then, UJ has the UJ-*internalization property*, i.e., for any formula $F \in \mathcal{L}_{\mathsf{UJ}}$ it holds that

$$\vdash_{\mathsf{UJ(CS)}} F \Rightarrow (\exists t \in \mathsf{Tm})\big[\vdash_{\mathsf{UJ(CS)}} t{:}_1 F\big].$$

Let UJ has the UJ-internalization property relative to some constant specification CS (e.g. axiomatically appropriate). Then, if

$$F_1, \ldots, F_n \vdash_{\mathsf{UJ(CS)}} F$$

it holds that for every $t_1, \ldots, t_n \in \mathsf{Tm}$, there exists some $t \in \mathsf{Tm}$ s.t.

$$t_1{:}_1 F_1, \ldots, t_n{:}_1 F_n \vdash_{\mathsf{UJ(CS)}} t{:}_1 F.$$

Proof. The proof of the first proposition is similar to the proof of Theorem 2.34. It can also be found, for a *total* constant specification in [21].

The proof of the second proposition, i.e., the Lifting Lemma for UJ, is similar to the proof of Lemma 2.33, thus, it may be omitted. ☐

Theorem 4.14 (Soundness and Completeness for UJ)**.** Let CS an arbitrary constant specification for UJ.
UJ(CS) is sound and complete in respect with the class of UJ-Fitting models that meat constant specification CS and respect the minimum evidence conditions.

Proof. The soundness part is straight forward. A proof of it can be found in [21]. The completeness is again treated by the construction of a corresponding canonical model defined as follows:

Definition C.1 (Canonical Model for UJ)**.**
Let CS an arbitrary constant specification for UJ.

We define the *canonical model* $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, \mathscr{V}, \mathscr{E} \rangle$ for UJ(CS), as follows:

- $\mathscr{W}$ is the set of maximal UJ(CS)-consistent sets.
- $\mathscr{R}$ is an accesibility relation on $\mathscr{W}$, s.t. for any $\Gamma, \Delta \in \mathscr{W}$

$$\Gamma \mathscr{R} \Delta \Leftrightarrow \{F \in \mathcal{L}_{\mathsf{UJ}} \mid (\exists t \in \mathsf{Tm})\,(\exists \mathtt{p} \in \mathtt{S}_{>0})\,[t{:}_{\mathtt{p}} F \in \Gamma]\} \subseteq \Delta.$$

- $\mathscr{V} \colon \mathsf{Prop} \to \mathcal{P}(\mathscr{W})$ s.t. for any $p \in \mathsf{Prop}$

$$\mathscr{V}(p) := \{\Gamma \in \mathscr{W} \mid p \in \Gamma\}$$

- $\mathscr{E} \colon \mathscr{W} \times \mathsf{Tm} \times \mathcal{L}_{\mathsf{UJ}} \to \mathcal{P}([0,1])$ s.t. for any $\Gamma \in \mathscr{W}$, any $t \in \mathsf{Tm}$ and any $F \in \mathcal{L}_{\mathsf{PJL}}$

$$\mathscr{E}(\Gamma, t, F) := \{0\} \cup \{\mathtt{p} \in \mathtt{S}_{>0} \mid t{:}_{\mathtt{p}} F \in \Gamma\}.$$

The rest of the proof is treated as usually, thus it may be omitted. The complete proof can be found in [21]. ☐

PROBABILISTIC JUSTIFICATION LOGIC

Theorem 4.20 (Conservativity of PPJ)**.**
PPJ is a *conservative extension* of classical logic, CL.

Proof. We recursively define the translation function $\mathbf{t}\colon \mathcal{L}_{\Box\mathsf{J}} \to \mathcal{L}_{\mathsf{CL}}$, as follows

$$\begin{aligned} \mathbf{t}(p) &:= p && \text{where } p \in \mathsf{Prop} \\ \mathbf{t}(F \to G) &:= \mathbf{t}(F) \to \mathbf{t}(G) && \text{where } F, G \in \mathcal{L}_{\mathsf{PPJ}} \\ \mathbf{t}(t{:}F) &:= \mathbf{t}(F) && \text{where } t \in \mathsf{Tm} \text{ and } F \in \mathcal{L}_{\mathsf{PPJ}} \\ \mathbf{t}(P_{\geq \mathrm{p}}F) &:= \top && \text{where } \mathrm{p} \in \mathsf{S} \text{ and } F \in \mathcal{L}_{\mathsf{PPJ}} \end{aligned}$$

We will prove by transfinite induction[1] on the complexity of the derivation of PPJ-theorem F, that $\mathbf{t}(F)$ is a theorem of CL. From Theorem 2.29, it suffices to show that the translation of any instance of axiom schemes in Table 4.3 is a theorem of CL and also that the result of the application of any rule of inference in Table 4.3 is also a theorem of CL.

For the axiomatic schemes we have

PI $\mathbf{t}(P_{\geq 0}F) := \top$

WE $\mathbf{t}(P_{\leq \mathrm{p}}F \to P_{<\mathrm{q}}F) := \top \to \top$

LE $\mathbf{t}(P_{<\mathrm{p}}F \to P_{\leq \mathrm{p}}F) := \top \to \top$

DIS $\mathbf{t}\big(P_{\geq \mathrm{p}}F \land P_{\geq \mathrm{q}}G \land P_{\geq 1}\neg F \land G \to P_{\geq \min\{1,\mathrm{p}+\mathrm{q}\}}F \lor G\big) := \top \land \top \land \top \to \top$

UN $\mathbf{t}(P_{\leq \mathrm{p}}F \land P_{<\mathrm{q}}G \to P_{<\mathrm{p}+\mathrm{q}}F \lor G) := \top \land \top \to \top$

which are all theorems of CL.

For the rules of inference, we have

CE Let through the application of **CE** we conclude that $P_{\geq 1}G$ is a theorem of PPJ. Clearly, $\mathbf{t}(P_{\geq 1}G) = \top$ is a theorem of CL.

ST Let through the application of **ST** we conclude that $H \to P_{\geq \mathrm{p}}G$ is a theorem of PPJ. Clearly, $\mathbf{t}(H \to P_{\geq \mathrm{p}}G) = \mathbf{t}(F) \to \top$ is a theorem of CL.

□

Corollary 4.21 (Consistency of PPJ)**.** Let CS an arbitrary constant specification for PPJ. Then, $\mathsf{PPJ}(\mathsf{CS})$ is consistent.

Proof. The proof is similar to the proof of Theorem 2.30; thus, it may be omitted. □

Theorem 4.22 (Deduction Theorem for PPJ)**.** Let CS an arbitrary constant specification for PPJ. The deduction theorem holds for $\mathsf{PPJ}(\mathsf{CS})$.

Proof. We state the parts of the proof that are in a sense different from the already given deduction theorems 2.15, 2.31 and 3.30. We use **transfinite** induction on the complexity of the derivation of $\Sigma \cup \{F\} \vdash_{\mathsf{PPJ}} G$, to show that

$$\Sigma \cup \{F\} \vdash_{\mathsf{PPJ}} G \Rightarrow \Sigma \vdash_{\mathsf{PPJ}} F \to G.$$

[1] Simple induction is not sufficient for this proof, as the steps of the derivation might be denumerable, due to rule of inference **ST**.

CE Let $G = P_{\geq 1}H$ is the result of the application of **CE**. By the definition of derivation from premises we have that $\vdash_{\mathsf{PPJ}} H$[2] and thus by **CE** $\vdash_{\mathsf{PPJ}} P_{\geq 1}H$. Thus, trivially $\Sigma \vdash_{\mathsf{PPJ}} P_{\geq 1}H$ and as a result of propositional reasoning $\Sigma \vdash_{\mathsf{PPJ}} F \to P_{\geq 1}H$.

ST Let $G = F' \to P_{\geq \mathtt{p}}H$ is the result of the application of axiom scheme **ST**. Therefore, for any $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ we have that $\Sigma \cup \{F\} \vdash_{\mathsf{PPJ}} F' \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}}H$. By induction hypothesis, we have $\Sigma \vdash_{\mathsf{PPJ}} F \to F' \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}}H$. Equivalently, by propositional reasoning we have $\Sigma \vdash_{\mathsf{PPJ}} F \wedge F' \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}}H$, for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$. Hence, by axiom scheme **ST** we have $\Sigma \vdash_{\mathsf{PPJ}} F \wedge F' \to P_{\geq \mathtt{p}}H$ and propositional reasoning $\Sigma \vdash_{\mathsf{PPJ}} F \to F' \to P_{\geq \mathtt{p}}H$, as wanted.

□

Theorem 4.30 (Soundness and Completeness for PPJ)**.** Let CS an arbitrary constant specification for PPJ.
PPJ(CS) is sound and complete in respect with the class $\mathsf{PPJ(CS)}_{\mathsf{Meas}}$.

Proof. The proof of this theorem is a corollary of Theorem 4.31. □

Theorem 4.31 (Strong Soundness and Completeness for PPJ)**.** Let CS an arbitrary constant specification for PPJ.
PPJ(CS) is strongly sound and strongly complete in respect with the class $\mathsf{PPJ(CS)}_{\mathsf{Meas}}$, i.e., for any set of formulae $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PPJ}}$ it holds that

$$\Sigma \vdash_{\mathsf{PPJ(CS)}} F \Rightarrow \Sigma \Vdash_{\mathsf{CS}} F \qquad \text{strong soundness}$$
$$\Sigma \vdash_{\mathsf{PPJ(CS)}} F \Leftarrow \Sigma \Vdash_{\mathsf{CS}} F \qquad \text{strong completeness}$$

Proof. Before beginning with the proof of strong soundness and strong completeness we will give some auxiliary lemmata.

Lemma C.2 (Properties of Finitely Additive Measure)**.** Let $\langle W, H, \mu\rangle$ a finitely additive probability space and arbitrary $U, V \in H$. Then the following properties hold:

- $\mu(U \cup V) = \mu(U) + \mu(V) - \mu(U \cap V)$
- $\mu(U) + \mu(W \setminus U) = 1$
- $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$ □

Lemma C.3. Let CS some constant specification for PPJ and $\mathcal{M} = \langle U, W, H, \mu, *\rangle$ a PPJ(CS)-model. Then, for any world $w \in W$ and any formulae $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that

- $[F \wedge G]_{\mathcal{M},w} = [F]_{\mathcal{M},w} \cap [G]_{\mathcal{M},w}$,
- $[F \vee G]_{\mathcal{M},w} = [F]_{\mathcal{M},w} \cup [G]_{\mathcal{M},w}$,
- $[\neg F]_{\mathcal{M},w} = W \setminus [F]_{\mathcal{M},w}$.

[2]Note that this is stronger than $\Sigma \cup \{F\} \vdash_{\mathsf{PPJ}} H$

Proof. We will show it only for the $[F \wedge G]_{\mathcal{M},w}$. The other cases can be treated similarly.

$$\begin{aligned}[F \wedge G]_{\mathcal{M},w} &:= \{u \in W_w \mid \mathcal{M}, u \Vdash F \wedge G\} \\ &= \{u \in W_w \mid \mathcal{M}, u \Vdash F \text{ and } \mathcal{M}, u \Vdash G\} \\ &= \{u \in W_w \mid \mathcal{M}, u \Vdash F\} \cap \{u \in W_w \mid \mathcal{M}, u \Vdash G\} \\ &= [F]_{\mathcal{M},w} \cap [G]_{\mathcal{M},w}\end{aligned}$$

□

Lemma C.4. Let CS some constant specification for PPJ and $\mathcal{M} = \langle U, W, H, \mu, * \rangle \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$. Then, for any world $w \in W$, any formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ and any $\mathtt{p} \in \mathtt{S}$, it holds that

- $\mathcal{M}, w \Vdash P_{\leq \mathtt{p}} F \Leftrightarrow \mu_w\Big([F]_{\mathcal{M},w}\Big) \leq \mathtt{p}$
- $\mathcal{M}, w \Vdash P_{< \mathtt{p}} F \Leftrightarrow \mu_w\Big([F]_{\mathcal{M},w}\Big) < \mathtt{p}$

Proof. We will prove it for the $P_{\leq \mathtt{p}} F$. The other case can be treated similarly.

$$\begin{aligned} \mathcal{M}, w \Vdash P_{\leq \mathtt{p}} F & \quad \Leftrightarrow \\ \mathcal{M}, w \Vdash P_{\geq 1-\mathtt{p}} \neg F & \quad \Leftrightarrow \\ \mu_w\Big([\neg F]_{\mathcal{M},w}\Big) \geq 1 - \mathtt{p} & \quad \Leftrightarrow \text{Lemma C.3} \\ \mu_w\Big(W \setminus [F]_{\mathcal{M},w}\Big) \geq 1 - \mathtt{p} & \quad \Leftrightarrow \text{Lemma C.2} \\ 1 - \mu_w\Big([F]_{\mathcal{M},w}\Big) \geq 1 - \mathtt{p} & \quad \Leftrightarrow \\ \mu_w\Big([F]_{\mathcal{M},w}\Big) \leq \mathtt{p} & \end{aligned}$$

□

Strong Soundness

We prove that for any set of formulae $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PPJ}}$ $\Sigma \vdash_{\mathsf{PPJ(CS)}} F$ implies that $\Sigma \Vdash_{\mathsf{CS}} F$, by induction on the complexity of the derivation $\Sigma \vdash_{\mathsf{PPJ(CS)}} F$. We will only give the cases that are not covered as similar to the previous soundness theorems. Let $\mathcal{M} \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$ s.t. $\mathcal{M} \Vdash \Sigma$.

PI Let $F = P_{\geq 0} G$ be an instance of axiom scheme **PI**. As $\mathcal{M} \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$, we have that for any world $w \in U$, it holds that $[G]_{\mathcal{M},w} \in H_w$. Clearly, as $\mu_w\Big([G]_{\mathcal{M},w}\Big) \geq 0$, thus $\mathcal{M} \Vdash P_{\geq 0} G$.

WE Let $F = P_{\leq \mathtt{p}} G \to P_{< \mathtt{q}} G$ be an instance of axiom sceme **WE** and $\mathcal{M} \Vdash P_{\leq \mathtt{p}} G$. Then by the first bullet of Lemma C.4 we have $\mu_w\Big([G]_{\mathcal{M},w}\Big) \leq \mathtt{p} < \mathtt{q}$, for any world $w \in U$. Thus, by the second bullet of Lemma C.4 we have that $\mathcal{M} \Vdash P_{< \mathtt{q}} G$. Therefore, $\mathcal{M} \Vdash F$.

LE Similarly with **WE** case.

DIS Let $F = P_{\geq \mathtt{p}} G \wedge P_{\geq \mathtt{q}} H \wedge P_{\geq 1} \neg G \wedge H \rightarrow P_{\geq \min\{1, \mathtt{p}+\mathtt{q}\}} G \vee H$ be an instance of axiom sceme **DIS** and $\mathcal{M} \Vdash P_{\geq \mathtt{p}} G \wedge P_{\geq \mathtt{q}} H \wedge P_{\geq 1} \neg G \wedge H$, or equivalently as an abbreviation $\mathcal{M} \Vdash P_{\geq \mathtt{p}} G \wedge P_{\geq \mathtt{q}} H \wedge P_{\leq 0} G \wedge H$. Equivalently, we have that for any world $w \in U$, it holds that $\mu_w\Big([G]_{\mathcal{M},w}\Big) \geq \mathtt{p}$, $\mu_w\Big([H]_{\mathcal{M},w}\Big) \geq \mathtt{q}$ and by the first bullet of Lemmata C.3 and C.4

$$\mu_w\Big([G \wedge H]_{\mathcal{M},w}\Big) = \mu_w\Big([G]_{\mathcal{M},w} \cap [H]_{\mathcal{M},w}\Big) \leq 0.$$

But then, by Lemmata C.2 and C.3 we have that

$$\begin{aligned}\mu_w\Big([G \vee H]_{\mathcal{M},w}\Big) &= \mu_w\Big([G]_{\mathcal{M},w} \cup [H]_{\mathcal{M},w}\Big) \\ &= \mu_w\Big([G]_{\mathcal{M},w}\Big) + \mu_w\Big([H]_{\mathcal{M},w}\Big) - \mu_w\Big([G]_{\mathcal{M},w} \cap [H]_{\mathcal{M},w}\Big) \\ &\geq \mathtt{p} + \mathtt{q}.\end{aligned}$$

Thus, $\mathcal{M} \Vdash P_{\geq \min\{1, \mathtt{p}+\mathtt{q}\}} G \vee H$. Hence, $\mathcal{M} \Vdash F$.

UN Similarly with **DIS** case.

CE Let $F = P_{\geq 1} G$ resulted by the application of rule of inference **CE**. Then, we know that $\vdash_{\mathsf{PPJ(CS)}} G$ and thus by induction hypothesis $\mathcal{M} \Vdash G$, i.e., for any $w \in U$ it holds that $\mathcal{M}, w \Vdash G$. Clearly, as $\mathcal{M} \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$ we have that for any world $w \in U$ $[G]_{\mathcal{M},w} \in H_w$. Furthermore, we have that

$$[G]_{\mathcal{M},w} = \{u \in W_w \mid \mathcal{M}, u \Vdash G\} = W_w,$$

as $\mathcal{M} \Vdash G$. Thus, $\mu_w\Big([G]_{\mathcal{M},w}\Big) = 1$, or equivalently $\mathcal{M}, w \Vdash P_{\geq 1} G$. Hence, $\mathcal{M} \Vdash F$.

ST Let $F = G \rightarrow P_{\geq \mathtt{p}} H$ resulted by the application of rule of inference **ST** and $\mathcal{M} \Vdash G$. Clearly, for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ we have that $\Sigma \vdash_{\mathsf{PPJ(CS)}} G \rightarrow P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} H$. Thus, by induction hypothesis we have that for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds $\mathcal{M} \Vdash G \rightarrow P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} H$ and as a result $\mathcal{M} \Vdash P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} H$. That is, for every $w \in U$ we have $\mu_w\Big([H]_{\mathcal{M},w}\Big) \geq \mathtt{p} - \frac{1}{\mathtt{k}}$. It suffices to show that for every $w \in U$ it holds that $\mu_w\Big([H]_{\mathcal{M},w}\Big) \geq \mathtt{p}$. Let us assume, in contradiction, that there is some $u \in U$ s.t. $\mu_w\Big([H]_{\mathcal{M},w}\Big) < \mathtt{p}$. Then by Archimedean Property there is some $\mathtt{n} \in \mathbb{N}$ s.t.

$$\mathtt{p} > \mathtt{p} - \mu_w\Big([H]_{\mathcal{M},w}\Big) > \frac{1}{\mathtt{n}}.$$

Thus, there is some $\mathtt{n} \in \mathbb{N}_{\geq \frac{1}{\mathtt{n}}}$ s.t. $\mu_w\Big([H]_{\mathcal{M},w}\Big) < \mathtt{p} - \frac{1}{\mathtt{n}}$, which leads to contradiction. ☐

Before continuing with the proof of strong completeness, we will give three auxiliary lemmata.

Lemma C.5 (Properties of $\mathsf{PPJ(CS)}$-Consistent Sets)**.** Let CS be an arbitrary constant specification for PPJ and $\Sigma \subseteq \mathcal{L}_{\mathsf{PPJ}}$ a $\mathsf{PPJ(CS)}$-consistent set of formulae. The following propositions hold:

- For any formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ at least one of the sets $\Sigma \cup \{F\}$ or $\Sigma \cup \{\neg F\}$ is $\mathsf{PPJ(CS)}$-consistent.
- If $\neg(F \rightarrow P_{\geq \mathtt{p}} G) \in \Sigma$, for some $\mathtt{p} \in \mathtt{S}_{>0}$, then there is some $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ s.t. $\Sigma \cup \left\{F \rightarrow P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}} G\right\}$ is $\mathsf{PPJ(CS)}$-consistent.

Proof. The first bullet is a simple corollary of deduction theorem 4.22.
For the second bullet let us assume, in contradiction that for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds that $\Sigma \cup \left\{\neg\Big(F \rightarrow P_{\geq \mathtt{p}'-\frac{1}{\mathtt{k}}} G\Big)\right\}$ is $\mathsf{PPJ(CS)}$-inconsistent. That is, for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds that

$$\Sigma \cup \left\{\neg\Big(F \rightarrow P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}} G\Big)\right\} \vdash_{\mathsf{PPJ(CS)}} \bot,$$

or equivalently by the deduction theorem 4.22

$$\Sigma \vdash_{\mathsf{PPJ(CS)}} \neg\Big(F \rightarrow P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}} G\Big) \rightarrow \bot.$$

Thus, by propositional reasoning we have that for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds that

$$\Sigma \vdash_{\mathsf{PPJ(CS)}} F \rightarrow P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}} G.$$

Hence, by axiom scheme **ST** we have that

$$\Sigma \vdash_{\mathsf{PPJ(CS)}} F \rightarrow P_{\geq \mathtt{p}} G.$$

But this leads to contradiction as clearly

$$\Sigma \vdash_{\mathsf{PPJ(CS)}} \neg F \rightarrow P_{\geq \mathtt{p}} G$$

and Σ was assumed $\mathsf{PPJ(CS)}$-consistent. ☐

Lemma C.6 (Properties of maximal $\mathsf{PPJ(CS)}$-Consistent Sets)**.** Let CS an arbitrary constant specification for PPJ and Σ a maximal $\mathsf{PPJ(CS)}$-consistent set of formulae in $\mathcal{L}_{\mathsf{PPJ}}$. Then, for every $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ the following propositions hold:

1. Exactly one of F, $\neg F$ belongs in Σ.
2. $\Sigma \vdash_{\mathsf{PPJ(CS)}} F \Leftrightarrow F \in \Sigma$
3. $F \vee G \in \Sigma \Leftrightarrow F \in \Sigma$ or $G \in \Sigma$
4. $F \wedge G \in \Sigma \Leftrightarrow F \in \Sigma$ and $G \in \Sigma$
5. $F, F \rightarrow G \in \Sigma \Rightarrow G \in \Sigma$
6. Let $\mathtt{p} := \sup\{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}} F \in \Sigma\}$. Then
   (a) for all $\mathtt{q} \in \mathtt{S}_{<\mathtt{p}}$ it holds $P_{>\mathtt{q}} F \in \Sigma$,
   (b) for all $\mathtt{q} \in \mathtt{S}_{<\mathtt{p}}$ it holds $P_{\geq \mathtt{q}} F \in \Sigma$,
   (c) if $\mathtt{p} \in \mathtt{S}$ then $P_{\geq \mathtt{p}} F \in \Sigma$

(d) for any $\mathtt{q} \in \mathtt{S}$ it holds

$$\mathtt{q} \leq \mathtt{p} \Leftrightarrow P_{\geq \mathtt{q}} F \in \Sigma.$$

Proof. The cases 1-5 are standard and may be omitted.

(6a) Let $q \in \mathtt{S}_{<\mathtt{p}}$. Clearly, by the definition of $\mathtt{p} := \sup \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}} F \in \Sigma\}$ there must be some $\mathtt{k} \in \mathtt{S} \cap (\mathtt{q}, \mathtt{p}]$ s.t. $P_{\geq \mathtt{k}} \in \Sigma$. It easy to observe that $P_{\geq \mathtt{k}} \rightarrow P_{>\mathtt{q}}$[3] is a theorem of $\mathsf{PPJ(CS)}$. Thus, by the case 2 we know that $P_{\geq \mathtt{k}} \rightarrow P_{>\mathtt{q}} \in \Sigma$ and by case 5 we have $P_{>\mathtt{q}} \in \Sigma$, as wanted.

(6b) The case for $P_{\geq \mathtt{q}} \in \Sigma$ is a corollary of the previous case.

(6c) For $\mathtt{p} = 0$, by axiom scheme **PI** we have that $P_{\geq 0} F \in \Sigma$. Let $\mathtt{p} \in \mathtt{S}_{>0}$. Then from the previous case we have that for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds that $P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} G \in \Sigma$. Thus, for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds that $\top \rightarrow P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} G \in \Sigma$ and by axiom scheme **ST** that $\top \rightarrow P_{\geq \mathtt{p}} G \in \Sigma$. Hence, by case 5 we get $P_{\geq \mathtt{p}} F \in \Sigma$.

(6d) The last case is a straight forward corollary of the previous.

□

Lemma C.7 (Lindenbaum's Lemma for PPJ)**.** Let CS an arbitrary constant specification for PPJ and $\Sigma \subseteq \mathcal{L}_{\mathsf{PPJ}}$ a $\mathsf{PPJ(CS)}$-consistent set. Then there exists a maximal $\mathsf{PPJ(CS)}$-consistent superset of Σ.

Proof. Let $\Sigma \subseteq \mathcal{L}_{\mathsf{PPJ}}$ an arbitrary $\mathsf{PPJ(CS)}$-consistent set. As $|\mathtt{S}| = \aleph_0$ and the countable union of countable sets is countable, we know that $|\mathcal{L}_{\mathsf{PPJ}}| = \aleph_0$. Therefore, there is an enumeration $\{F_i\}_{i \in \mathbb{N}}$ of the formulae in $\mathcal{L}_{\mathsf{PPJ}}$. We will define a countable family $\{\Delta_i\}_{i \in \mathbb{N}}$ of set of formulae in $\mathcal{L}_{\mathsf{PPJ}}$, as follows:

- $\Delta_0 := \Sigma$.
- for every $i \in \mathbb{N}$
  1. if $\Delta_i \cup \{F_i\}$ is $\mathsf{PPJ(CS)}$-consistent, then $\Delta i + 1 := \Delta_i \cup \{F_i\}$,
  2. else if F_i is of the form $G \rightarrow P_{\geq \mathtt{p}} H$, where $\mathtt{p} \in \mathtt{S}_{>0}$, then we choose some $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ s.t. $\Delta_i \cup \left\{\neg\Big(G \rightarrow P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} H\Big)\right\}$ is $\mathsf{PPJ(CS)}$-consistent[4] and we set $\Delta_{i+1} := \Delta_i \cup \left\{\neg F_i, \neg\Big(G \rightarrow P_{\geq \mathtt{p} - \frac{1}{\mathtt{k}}} H\Big)\right\}$,
  3. else we set $\Delta_{i+1} := \Delta_i \cup \{\neg F_i\}$.

We finally set $\Delta := \bigcup_{i \in \mathbb{N}} \Delta_i$. We will first show that for every $i \in \mathbb{N}$ the set Δ_i is $\mathsf{PPJ(CS)}$-consistent.

- Trivially, it holds for $\Delta_0 = \Sigma$.
- Trivially, it holds if Δ_i was constructed by step 1.

[3]Note that $P_{>\mathtt{q}}$ is an abbreviation.
[4]As we will prove later such a $\mathtt{k}$ exists.

- Let Δ_i was constructed by step 2. Therefore, $\Delta_i \cup \{F_i\}$ is $\mathsf{PPJ}(\mathsf{CS})$-inconsistent and by the first bullet of Lemma C.5 we have that $\Delta_i \cup \{\neg F_i\}$ is $\mathsf{PPJ}(\mathsf{CS})$-consistent. Thus, by the second bullet of Lemma C.5 we have that there exists some $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ s.t. $\Delta_i \cup \left\{\neg F_i, \neg\Big(G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}} H\Big)\right\}$ is $\mathsf{PPJ}(\mathsf{CS})$-consistent.
- Let Δ_i was constructed by step 3. As before, because $\Delta_i \cup \{F_i\}$ is $\mathsf{PPJ}(\mathsf{CS})$-inconsistent, by the first bullet of Lemma C.5 we have that $\Delta_i \cup \{\neg F_i\}$ is $\mathsf{PPJ}(\mathsf{CS})$-consistent.

We want to show that Δ is a maximal $\mathsf{PPJ}(\mathsf{CS})$-consistent set of formulae. It is not hard to see that it suffices to prove that Δ does not contain all the formulae and that it is deductively closed, i.e., for any $F \in \mathcal{L}_{\mathsf{PPJ}}$ s.t. $\Delta \vdash_{\mathsf{PPJ}(\mathsf{CS})} F$ it holds that $F \in \Delta$.

Clearly, $\bot \notin \Delta$, as elsewhere there would be some $i \in \mathbb{N}$ s.t. $\bot \in \Delta_i$ and thus Δ_i would be $\mathsf{PPJ}(\mathsf{CS})$-inconsistent.

Let some $F \in \mathcal{L}_{\mathsf{PPJ}}$ s.t. $\Delta \vdash_{\mathsf{PPJ}(\mathsf{CS})} F$. We will prove by transfinite induction on the complexity of the derivation of F that $F \in \Delta$.

- If $F \in \Delta$, or F is some axiom scheme of PPJ, or $F \in \mathsf{CS}$, or F results from the application of modus ponens, this can be treated similar with Lindenbaum's Lemma for the other logis (e.g. for justification logic) and it is assumed known.

CE Let $F = P_{\geq 1} G$ results by the application of rule of inference **CE**. Then, by the Definition 4.18 of the derivation in $\mathsf{PPJ}(\mathsf{CS})$ we know that $\vdash_{\mathsf{PPJ}(\mathsf{CS})} P_{\geq 1} G$, which is stronger than $\Delta \vdash_{\mathsf{PPJ}(\mathsf{CS})} P_{\geq 1} G$. As $\{F_i\}_{i \in \mathbb{N}}$ is an enumeration of $\mathcal{L}_{\mathsf{PPJ}}$, there exists some $i \in \mathbb{N}$ s.t. $P_{\geq 1} G = F_i$. Then, as Δ_i is $\mathsf{PPJ}(\mathsf{CS})$-consistent and $P_{\geq 1} G$ is a theorem of $\mathsf{PPJ}(\mathsf{CS})$, we have that $\Delta_i \cup \{P_{\geq 1} G\}$ is $\mathsf{PPJ}(\mathsf{CS})$-consistent and by construction of the Δ's family we have $\Delta_{i+1} = \Delta_i \cup \{P_{\geq 1} G\}$. Thus, $P_{\geq 1} G \in \Delta$.

ST Let $F = P_{\geq 1} G \to P_{\geq \mathtt{p}} H$, where $\mathtt{p} \in \mathtt{S}_{>0}$, results by the application of rule of inference **ST**. Then, for every $\mathtt{k} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ it holds that

$$\Delta \vdash_{\mathsf{PPJ}(\mathsf{CS})} G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{k}}} H.$$

Let us, in contradiction, assume that $F \notin \Delta$. Clearly, by the construction of Δ, we have that $\neg F \in \Delta$. Yet again, there is some $i \in \mathbb{N}$ s.t. $F = F_i$. By the construction of Δ, there is some $\mathtt{n} \in \mathbb{N}_{\geq \frac{1}{\mathtt{p}}}$ s.t. $F, \neg\Big(G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{n}}} H\Big) \in \Delta_{i+1}$. But $G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{n}}} H$ is a premise of **ST**, thus by induction hypothesis we know that $G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{n}}} H \in \Delta$. Hence, there is some $j \in \mathbb{N}$ s.t. $G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{n}}} H \in \Delta_j$. But then,

$$\left\{G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{n}}} H, \neg\Big(G \to P_{\geq \mathtt{p}-\frac{1}{\mathtt{n}}} H\Big)\right\} \subseteq \Delta_{\max\{i+1, j\}}.$$

From that we conclude that $\Delta_{\max\{i+1,j\}}$ is $\mathsf{PPJ}(\mathsf{CS})$-inconsistent, which leads to contradiction. Thus, $F \in \Delta$.

□

Strong Completeness

We once again have to define a canonical model for PPJ logic.

Definition C.8 (Canonical Model for PPJ)**.**
Let CS an arbitrary constant specification for PPJ.
We define the *canonical model* $\mathscr{M} := \langle \mathscr{U}, \mathscr{W}, \mathscr{H}, \mu, * \rangle$ for PPJ(CS), as follows:

- $\mathscr{U}$ is the set of maximal PPJ(CS)-consistent sets.
- $\mathscr{W}_\Gamma := \mathscr{U}$, for any $\Gamma \in \mathscr{U}$.
- $\mathscr{H}_\Gamma := \{(F)_\mathscr{M} \mid F \in \mathcal{L}_{\mathsf{PPJ}}\}$, for any $\Gamma \in \mathscr{U}$, where
$$(F)_\mathscr{M} := \{\Gamma' \in \mathscr{U} \mid F \in \Gamma'\}.$$
- $\mu_\Gamma((F)_\mathscr{M}) := \sup\{\mathsf{p} \in \mathsf{S} \mid P_{\geq \mathsf{p}} F \in \Gamma\}$, for any $\Gamma \in \mathscr{U}$ and any $F \in \mathcal{L}_{\mathsf{PPJ}}$.
- $*_\Gamma \colon \mathsf{Prop} \cup \mathsf{Tm} \to \{0,1\} \cup \mathcal{P}(\mathcal{L}_{\mathsf{PPJ}})$ s.t.
  - for any $p \in \mathsf{Prop}$,
$$*_\Gamma(p) = 1 \Leftrightarrow p \in \Gamma,$$
  - for any term $t \in \mathsf{Tm}$,
$$*_\Gamma(t) := \{F \in \mathcal{L}_{\mathsf{PPJ}} \mid t{:}F \in \Gamma\},$$

  where $\Gamma \in \mathscr{U}$ and $*_\Gamma := *(\Gamma)$.

We will sometimes write for brevity $\mu_\Gamma(F) := \mu_\Gamma((F)_\mathscr{M})$.

It is now time to prove that $\mathscr{M} \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$.

Theorem C.9. Let CS an arbitrary constant specification for PPJ and $\mathscr{M} := \langle \mathscr{U}, \mathscr{W}, \mathscr{H}, \mu, * \rangle$ the corresponding canonical model. Then, $\mathscr{M} \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$.

Proof.

- It holds trivially, by Lindenbaum's lemma C.7 that $\mathscr{U} \neq \emptyset$.
- We want to show that $\langle \mathscr{W}, \mathscr{H}, \mu \rangle$ is a finitely additive probability space. Let arbitrary $\Gamma \in \mathscr{U}$.

  Clearly, $\mathscr{W}_\Gamma = \mathscr{U} \subseteq \mathscr{U} \neq \emptyset$.

  We want to show that $\mathscr{H}_\Gamma$ is an algebra over $\mathscr{W}_\Gamma$.
  - $(\top)_\mathscr{M} = \mathscr{U} = \mathscr{W}_\Gamma \in \mathscr{H}_\Gamma$ and trivially $\mathscr{H} \subseteq \mathcal{P}(\mathscr{W}_\Gamma)$.
  - For every $X = (F)_\mathscr{M} \in \mathscr{H}_\Gamma$, for some $F \in \mathcal{L}_{\mathsf{PPJ}}$, we have
$$\begin{aligned} \mathscr{W}_\Gamma \setminus (F)_\mathscr{M} &= \{\Delta \in \mathscr{U} \mid F \notin \Delta\} && \text{L. C.6.1} \\ &= \{\Delta \in \mathscr{U} \mid F \notin \Delta\} \\ &= (\neg F)_\mathscr{M} \in \mathscr{H}_\Gamma \end{aligned}$$
  - For every $X, Y \in \mathscr{H}_\Gamma$, i.e., $X = (F)_\mathscr{M} \in \mathscr{H}_\Gamma$ and $Y = (F)_\mathscr{M} \in \mathscr{H}_\Gamma$ for some $F, G \in \mathcal{L}_{\mathsf{PPJ}}$, we have
$$\begin{aligned} (F)_\mathscr{M} \cup (G)_\mathscr{M} &= \{\Delta \in \mathscr{U} \mid F \in \Delta \text{ or } G \in \Delta\} && \text{L. C.6.3} \\ &= \{\Delta \in \mathscr{U} \mid F \vee G \in \Delta\} \\ &= (F \vee G)_\mathscr{M} \in \mathscr{H}_\Gamma \end{aligned}$$

Finally, we want to show that μ_Γ is a finitely additive measure on $\mathscr{H}_\Gamma$.

- We will show that $\mu_\Gamma(F)_{\mathscr{M}}$ is well defined, for any $F \in \mathcal{L}_{\mathsf{PPJ}}$. Firstly, by axiom scheme **PI** we know that $P_{\geq 0}F \in \Gamma$. Hence the set $\{\mathtt{p} \in \mathtt{S} \mid P_{\geq \mathtt{p}}F \in \Gamma\}$ is a non-empty and upper bounded subset of $\mathbb{R}$. Thus, by the *least-upper-bound property* of reals, we have that there exists the

$$\mu_\Gamma(F)_{\mathscr{M}} := \sup\{\mathtt{p} \in \mathtt{S} \mid P_{\geq \mathtt{p}}F \in \Gamma\}.$$

  Moreover, We want to show that for any $X \in \mathscr{H}_\Gamma$, there exists a unique value for $\mu_\Gamma(X)$, i.e., for any $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ s.t. $X = (F)_{\mathscr{M}} = (G)_{\mathscr{M}}$, we want to show that $\mu_\Gamma((F)_{\mathscr{M}}) = \mu_\Gamma((F)_{\mathscr{M}})$. It suffices to show that, if $(F)_{\mathscr{M}} \subseteq (G)_{\mathscr{M}}$ then $\mu_\Gamma((F)_{\mathscr{M}}) \leq \mu_\Gamma((F)_{\mathscr{M}})$. Let $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ s.t. $(F)_{\mathscr{M}} \subseteq (G)_{\mathscr{M}}$. By the definition of $(\ )_{\mathscr{M}}$ we know that for any $\Delta \in \mathscr{U}$ it holds that

$$F \in \Delta \Rightarrow G \in \Delta,$$

  or equivalently

$$F \notin \Delta \text{ or } G \in \Delta.$$

  But then, by Lemma C.6.1 we have

$$\neg F \in \Delta \text{ or } G \in \Delta,$$

  while by Lemma C.6.3 we have $\neg F \vee G \in \Delta$. But then, by propositional reasoning and Lemma C.6 we have $F \rightarrow G \in \Delta$.

  Let us assume in contradiction that $F \rightarrow G$ is not a theorem of $\mathsf{PPJ}(\mathsf{CS})$. Then, by deduction Theorem 4.22 we can easily take that $\{\neg(F \rightarrow G)\}$ is $\mathsf{PPJ}(\mathsf{CS})$-consistent. Therefore, by Lindenbaum's Lemma C.7, there must be some maximal $\mathsf{PPJ}(\mathsf{CS})$-consistent extension Δ' of $\{\neg(F \rightarrow G)\}$. But then, by the previous paragraph we have that $F \rightarrow G$ is also in Δ'. Hence, Δ' is $\mathsf{PPJ}(\mathsf{CS})$-inconsistent which leads to contradiction. Hence, $F \rightarrow G$ is a theorem of $\mathsf{PPJ}(\mathsf{CS})$.

  Therefore, by the second bullet of Example 4.19 we have that for any $\mathtt{p} \in \mathtt{S}$ it holds that $P_{\geq \mathtt{p}}F \rightarrow P_{\geq \mathtt{p}}G$ is a theorem of $\mathsf{PPJ}(\mathsf{CS})$ and as a result it belongs to any maximal $\mathsf{PPJ}(\mathsf{CS})$-consistent $\Delta \in \mathscr{U}$.

  Let some $\mathtt{p} \in \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}}F \in \Gamma\}$. That is, $P_{\geq \mathtt{p}}F \in \Gamma$. Then, by the previous paragraph we have $P_{\geq \mathtt{p}}F \rightarrow P_{\geq \mathtt{p}}G \in \Gamma$. Hence, by Lemma C.6.5 we have that $P_{\geq \mathtt{p}}G \in \Gamma$. Thus, $\mathtt{p} \in \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}}G \in \Gamma\}$, i.e., $\{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}}F \in \Gamma\} \subseteq \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}}G \in \Gamma\}$. Therefore, clearly

$$\mu_\Gamma((F)_{\mathscr{M}}) \leq \mu_\Gamma((G)_{\mathscr{M}}), \tag{$*$}$$

  as wanted.

- Trivially, we have that for any formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that

$$\mu_\Gamma((F)_{\mathscr{M}}) \in [0, 1].$$

- Trivially, we have that

$$\mu_\Gamma(\mathscr{W}_\Gamma) = \mu_\Gamma((\top)_{\mathscr{M}}) = 1,$$

  as by axiom scheme **CE** and maximal $\mathsf{PPJ}(\mathsf{CS})$-consistency of Γ we have $1 \in \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}}F \in \Gamma\}$.

– We want to show that μ_Γ respects the finite addition property, i.e., for every $U, V \in \mathscr{H}_\Gamma$ it holds that

$$U \cap V = \emptyset \Rightarrow \mu_\Gamma(U \cup V) = \mu_\Gamma(U) + \mu(V).$$

We will first prove the auxiliary proposition that for any formula $F \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that

$$\mu_\Gamma(F) + \mu_\Gamma(\neg F) \leq 1.$$

Let $\mathtt{p} \in \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}} \neg F \in \Gamma\}$[5]. If $1 - \mathtt{p} < \mu_\Gamma(F)$ then by Lemma C.6.6a we would have

$$P_{>1-\mathtt{p}} F = \neg P_{\geq \mathtt{p}} \neg F \in \Gamma$$

which leads to contradiction as Γ is $\mathsf{PPJ}(\mathsf{CS})$-consistent. Thus, $1-\mu_\Gamma(F) \geq \mathtt{p}$ and as $\mathtt{p}$ was arbitrary we have that $1 - \mu_\Gamma(F)$ is an upper-bound of $\{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}} \neg F \in \Gamma\}$. Hence, $\mu_\Gamma(\neg F) \leq 1 - \mu_\Gamma(F)$, i.e.,

$$\mu_\Gamma(F) + \mu_\Gamma(\neg F) \leq 1,$$

as wanted.

At last, let arbitrary $U, V \in \mathscr{H}_\Gamma$ s.t. $U \cap V = \emptyset$, i.e., let $F, G \in \mathcal{L}_{\mathsf{PPJ}}$ s.t. $(F)_{\mathscr{M}} \cap (G)_{\mathscr{M}} = \emptyset$. That is, $(G)_{\mathscr{M}} \subseteq (\neg F)_{\mathscr{M}}$ and by $(*)$ we have

$$\mu_\Gamma(G) \leq \mu_\Gamma(\neg F).$$

Thus, by the previous paragraph we get

$$\mu_\Gamma(G) + \mu_\Gamma(F) \leq 1.$$

Moreover, we know that

$$\mu_\Gamma(\neg(F \wedge G)) = \mu_\Gamma(\mathscr{W}_\Gamma \setminus (F \wedge G)_{\mathscr{M}}) = 1,$$

which by Lemma C.6.6c implies that $P_{\geq 1} \neg(F \wedge G) \in \Gamma$. We set for clarity

$$\mu_\Gamma(F) \coloneqq \mathtt{p} \qquad \mu_\Gamma(G) \coloneqq \mathtt{q}.$$

* Let $\mathtt{p}, \mathtt{q} > 0$. Then by Lemma C.6.6c, we have that for any $\mathtt{p}' \in \mathtt{S} \cap [0, \mathtt{p})$ and for any $\mathtt{q}' \in \mathtt{S} \cap [0, \mathtt{q})$ it holds that $P_{\geq \mathtt{p}'} F, P_{\geq \mathtt{q}'} G \in \Gamma$ and trivially $\mathtt{p}' + \mathtt{q}' < \mathtt{p} + \mathtt{q} \leq 1$. Thus, by axiom scheme **DIS** and Lemma C.6.4 and 5 we get $P_{\geq \mathtt{p}'+\mathtt{q}'}(F \vee G) \in \Gamma$. Hence,

  $$\mathtt{k} \coloneqq \mu_\Gamma(F \vee G) \geq \mathtt{p} + \mathtt{q}.$$

  Clearly, if $\mathtt{p} + \mathtt{q} = 1$, then $\mathtt{k} = \mathtt{p} + \mathtt{q}$. So let us assume that $\mathtt{p} + \mathtt{q} < 1$. Thus, $\mathtt{p}, \mathtt{q} \in (0, 1)$. Let us also, in contradiction, assume that $\mathtt{k} > \mathtt{p} + \mathtt{q}$. Then, by Lemma C.6.6b for every $\mathtt{k}' \in \mathtt{S} \cap (\mathtt{p} + \mathtt{q}, \mathtt{k})$ we have

  $$P_{\geq \mathtt{k}'}(F \vee G) \in \Gamma \qquad (**)$$

  Let $\mathtt{p}'', \mathtt{q}'' \in \mathtt{S}$ s.t. $\mathtt{p}'' + \mathtt{q}'' = \mathtt{k}'$, $\mathtt{p}'' > \mathtt{p}$ and $\mathtt{q}'' > \mathtt{q}$[6]. Clearly, $P_{\geq \mathtt{p}''} F, P_{\geq \mathtt{q}''} G \notin \Gamma$, elsewhere we would have $\mathtt{p}'' \leq \mathtt{p}$ and $\mathtt{q}'' \leq$

[5]We have already shown that all set of this form are non-empty due to axiom scheme **PI**.

[6]Trivially, there exist such $\mathtt{p}''$ and $\mathtt{q}''$.

q. Hence, by Lemma C.6.1 we have $\neg P_{\geq \mathtt{p}''} F, \neg P_{\geq \mathtt{q}''} G \in \Gamma$, i.e., $P_{<\mathtt{p}''} F, P_{<\mathtt{q}''} G \in \Gamma$. Moreover, by axiom scheme **LE** and Lemma C.6.5 we have that $P_{\leq \mathtt{p}''} F \in \Gamma$. Hence, by axiom scheme **UN** and Lemma C.6.4 and C.6.5 we get

$$P_{<\mathtt{p}''+\mathtt{q}''}(F \vee G) = \neg P_{\geq \mathtt{p}''+\mathtt{q}''}(F \vee G) = \neg P_{\geq \mathtt{k}'}(F \vee G).$$

But this contradicts equation $(**)$, as Γ is $\mathsf{PPJ}(\mathsf{CS})$-consistent. Thus, $\mathtt{k} = \mathtt{p} + \mathtt{q}$, i.e.,

$$\mu_\Gamma(F \vee G) = \mu_\Gamma(F) + \mu_\Gamma(G),$$

as wanted.

* The cases where at least one of $\mathtt{p}$ or $\mathtt{q}$ is equal to 0 is similar.

Therefore, μ_Γ is a finitely additive measure on $\mathscr{H}_\Gamma$. Hence, $\langle \mathscr{W}, \mathscr{H}, \mu \rangle$ is a finitely additive probability space.

- We want to show that $*$ is a $\mathsf{PPJ}(\mathsf{CS})$-modular model over $\mathscr{U}$. This can be treated similarly as in Theorem 3.31.

  Thus $\mathscr{M}$ is a $\mathsf{PPJ}(\mathsf{CS})$-*model*. Thus we can define the truth on this model, as in Definition 4.27.

- We want to show that $\mathscr{M}$ is also measurable, i.e., we want to show that for every $\Gamma \in \mathscr{U}$ and for every $F \in \mathcal{L}_{\mathsf{PPJ}}$ $[F]_{\mathscr{M},\Gamma} \in \mathscr{H}_\Gamma$. It suffices to show that $[F]_{\mathscr{M},\Gamma} = (F)_{\mathscr{M}}$. We will prove it by induction on the complexity of F.

  – Let $F := p \in \mathsf{Prop}$. Then,

$$\begin{aligned}
[p]_{\mathscr{M},\Gamma} &= \{\Delta \in \mathscr{W}_\Gamma \mid \mathscr{M}, \Delta \Vdash p\} \\
&= \{\Delta \in \mathscr{W}_\Gamma \mid p^{*\Delta} = 1\} \\
&= \{\Delta \in \mathscr{W}_\Gamma \mid p \in \Delta\} \\
&= (p)_{\mathscr{M}}.
\end{aligned}$$

  – Let $F := \neg G$. Then,

$$\begin{aligned}
[\neg G]_{\mathscr{M},\Gamma} &= \mathscr{W}_\Gamma \setminus [\neg G]_{\mathscr{M},\Gamma} && \text{Lem. C.3} \\
&= \mathscr{W}_\Gamma \setminus (\neg G)_{\mathscr{M}} && \text{Ind. Hyp.} \\
&= \{\Delta \in \mathscr{W}_\Gamma \mid G \notin \Delta\} \\
&= \{\Delta \in \mathscr{W}_\Gamma \mid \neg G \in \Delta\} && \text{Lem. C.6.1} \\
&= (\neg G)_{\mathscr{M}}
\end{aligned}$$

  – Let $F := G \to H$. Then,

$$\begin{aligned}
[G \to H]_{\mathscr{M},\Gamma} &= [\neg G \vee H]_{\mathscr{M},\Gamma} && \text{Abbr. for } \vee \\
&= [\neg G]_{\mathscr{M},\Gamma} \cup [H]_{\mathscr{M},\Gamma} && \text{Lem. C.3} \\
&= (\neg G)_{\mathscr{M}} \cup (H)_{\mathscr{M}} && \text{Ind. Hyp.} \\
&= \{\Delta \in \mathscr{W}_\Gamma \mid \neg G \in \Delta\} \cup \{\Delta \in \mathscr{W}_\Gamma \mid H \in \Delta\} \\
&= \{\Delta \in \mathscr{W}_\Gamma \mid \neg G \vee H \in \Delta\} && \text{Lem. C.6.3} \\
&= (G \to H)_{\mathscr{M}}.
\end{aligned}$$

– Let $F \coloneqq t{:}G$, where $t \in \mathsf{Tm}$. Then,

$$\begin{aligned}[t{:}G]_{\mathscr{M},\Gamma} &= \{\Delta \in \mathscr{W}_\Gamma \mid \mathscr{M}, \Delta \Vdash t{:}G\} \\ &= \{\Delta \in \mathscr{W}_\Gamma \mid G \in t^{*\Delta}\} \\ &= \{\Delta \in \mathscr{W}_\Gamma \mid t{:}G \in \Delta\} \\ &= (t{:}G)_{\mathscr{M}} .\end{aligned}$$

– Let $F \coloneqq P_{\geq \mathtt{p}} G$, where $\mathtt{p} \in \mathtt{S}$. Then,

$$\begin{aligned}[P_{\geq \mathtt{p}} G]_{\mathscr{M},\Gamma} &= \{\Delta \in \mathscr{W}_\Gamma \mid \mathscr{M}, \Delta \Vdash P_{\geq \mathtt{p}} G\} \\ &= \Big\{\Delta \in \mathscr{W}_\Gamma \mid \mu_\Delta\Big([G]_{\mathscr{M},\Gamma}\Big) \geq \mathtt{p}\Big\} \\ &= \{\Delta \in \mathscr{W}_\Gamma \mid \mu_\Delta((G)_{\mathscr{M}}) \geq \mathtt{p}\} && \text{Ind. Hyp.} \\ &= \{\Delta \in \mathscr{W}_\Gamma \mid P_{\geq \mathtt{p}} G \in \Delta\} && \text{Lem. C.6.6d} \\ &= (P_{\geq \mathtt{p}} G)_{\mathscr{M}}\end{aligned}$$

Therefore, by induction hypothesis we have that for any $\Gamma \in \mathscr{U}$ $F \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that $[F]_{\mathscr{M},\Gamma} = (F)_{\mathscr{M}}$, as wanted.

From all the above we conclude that $\mathscr{M} \in \mathsf{PPJ(CS)}_{\mathsf{Meas}}$, as wanted. □

It is time to give the corresponding *truth lemma* for PPJ.

Lemma C.10 (Truth Lemma for PPJ)**.** Let CS an arbitrary constant specification for PPJ and $\mathscr{M} \coloneqq \langle \mathscr{U}, \mathscr{W}, \mathscr{H}, \mu, * \rangle$ the corresponding canonical model. Then, for every $\Gamma \in \mathscr{U}$ and $F \in \mathcal{L}_{\mathsf{PPJ}}$ it holds that

$$F \in \Gamma \Leftrightarrow \mathscr{M}, \Gamma \Vdash F.$$

Proof. The proof is similar to the proof of Theorem B.5. We will only give the case which was not covered in that proof.

$$\begin{aligned} \mathscr{M}, \Gamma \Vdash P_{\geq \mathtt{p}} F & \quad \Leftrightarrow \\ \mu_\Gamma [F]_{\mathscr{M},\Gamma} \geq \mathtt{p} & \quad \Leftrightarrow \text{as shown} \\ \mu_\Gamma (F)_{\mathscr{M}} \geq \mathtt{p} & \quad \Leftrightarrow \\ \sup \{\mathtt{q} \in \mathtt{S} \mid P_{\geq \mathtt{q}} F \in \Gamma\} \geq \mathtt{p} & \quad \Leftrightarrow \text{Lem. C.6.6d} \\ P_{\geq \mathtt{p}} F \in \Gamma & \end{aligned}$$

□

It is time to conclude the strong completeness theorem. Let CS an arbitrary constant specification for PPJ and $\mathscr{M} \coloneqq \langle \mathscr{U}, \mathscr{W}, \mathscr{H}, \mu, * \rangle$ the corresponding canonical model. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PPJ}}$ s.t. $\Sigma \nvdash_{\mathsf{PPJ(CS)}} F$. Clearly, by Deduction Theorem 4.22 we conclude that $\Sigma \cup \{\neg F\}$ is PPJ(CS)-consistent. Thus, by Lindenbaum's Lemma C.7 we can expand it to a maximal PPJ(CS)-consistent set Γ. Clearly, by Truth Lemma C.10, we have $\mathscr{M}, \Gamma \Vdash \Sigma$ and $\mathscr{M}, \Gamma \nVdash F$. Therefore, we have $\Sigma \nVdash_{\mathsf{CS}} F$. Hence, by contraposition we have the strong soundness theorem for PPJ(CS). □

PAVELKA STYLE FUZZY JUSTIFICATION LOGIC

Lemma 4.35. Łukasiewicz t-norm is a *continuous t-norm*, i.e., for any $\mathtt{p}, \mathtt{q}, \mathtt{r} \in [0, 1]$

- $\mathtt{p} *_{Ł} \mathtt{q} = \mathtt{q} *_{Ł} \mathtt{p}$;
- $(\mathtt{p} *_{Ł} \mathtt{q}) *_{Ł} \mathtt{r} = \mathtt{p} *_{Ł} (\mathtt{q} *_{Ł} \mathtt{r})$;
- if $\mathtt{p} \leq \mathtt{q}$, then $\mathtt{p} *_{Ł} \mathtt{r} \leq \mathtt{q} *_{Ł} \mathtt{r}$;
- $1 *_{Ł} \mathtt{p} = \mathtt{p}$;
- $*_{Ł}$ is continuous function.

Moreover, Łukasiewicz implication is the corresponding *residuum*, i.e., for any $\mathtt{p}, \mathtt{q} \in [0, 1]$

$$\mathtt{p} \Rightarrow_{Ł} \mathtt{q} = \max \{\mathtt{r} \in [0, 1] \mid \mathtt{p} *_{Ł} \mathtt{r} \leq \mathtt{q}\}.$$

Proof. The proof is trivial, thus omitted. □

Theorem 4.39 (Deduction Theorem for Fuzzy Justification Logic)**.** Let JL an arbitrary justification logic and CS a constant specification for RPL(JL). Let also arbitrary set of formulae $\Sigma \cup \{F, G\} \subseteq \mathcal{L}_{\mathsf{RPL(JL)}}$ s.t. $\Sigma \cup \{F\} \vdash_{\mathsf{RPL(JL(CS))}} G$. Then, there is some $n \in \mathbb{N}_{>0}$ s.t.

$$\Sigma \vdash_{\mathsf{RPL(JL(CS))}} F^n \to G,$$

where

$$F^n := \underbrace{F\& \cdots \&F}_{n}.$$

Proof. This proof is given in the standard way. The reason why $n \in \mathbb{N}_{>0}$ appears is that

$$(F \to G)\&(F \to H) \to F \to G\&H$$

is not a theorem of RPL(JL), while

$$(F \to G)\&(F \to H) \to F\&F \to G\&H$$

is. A proof of this Theorem can be found in [27] (cf. Theorem 2.2.18, Remark 3.3.3.). □

Lemma 4.40 (RPL(JL)-Internalization Property & Lifting Lemma for RPL(JL))**.** Let JL an arbitrary justification logic and CS a constant specification for RPL(JL). Then, RPL(JL(CS)) has the RPL(JL)-*internalization property*, i.e., for any formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$ it holds that

$$\vdash_{\mathsf{RPL(JL(CS))}} F \Rightarrow (\exists t \in \mathsf{Tm})\big[\vdash_{\mathsf{RPL(JL(CS))}} t{:}F\big].$$

Let RPL(JL) has the RPL(JL)-internalization property relative to some constant specification CS (e.g. axiomatically appropriate). Then if

$$F_1, \ldots, F_n \vdash_{\mathsf{RPL(JL(CS))}} F,$$

then it holds that for every $t_1, \ldots, t_n \in \mathsf{Tm}$, there exists some $t \in \mathsf{Tm}$ s.t.

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{RPL(JL(CS))}} t{:}F.$$

Proof. The proofs of the two propositions are similar to the proof of Lemma 4.9 (viz. Theorem 2.34 and Lemma refFitLem). □

Lemma 4.42. Let $M = \langle W, R, V, E\rangle$ an arbitrary fuzzy Fitting model. Then, for any world $w \in W$, any formulae $F, G \in \mathcal{L}_{\mathsf{RPL(JL)}}$, any term $t \in \mathsf{Tm}$ and any $\mathtt{p} \in \mathtt{S}$, it holds:

- $V_w(\neg F) := 1 - V_w(F)$,
- $V_w(F \&, G) := V_w(F) *_{Ł} V_w(G)$
- $V_w(F \wedge G) := \min\{V_w(F), V_w(G)\}$,
- $V_w(F \veebar G) := \min\{1, V_w(F) + V_w(G)\}$,
- $V_w(F \vee G) := \max\{V_w(F), V_w(G)\}$,
- $V_w(F \rightleftharpoons G) := 1 - |V_w(F) - V_w(G)|$,
- $V_w(F \leftrightarrow G) := 1 - |V_w(F) - V_w(G)|$,
- $V_w(t{:}_{\mathtt{p}}F) := \mathtt{p} \Rightarrow_{Ł} V_w(t{:}F)$,
- $V_w(t{:}^{\mathtt{p}}F) := V_w(t{:}F) \Rightarrow_{Ł} \mathtt{p}$,
- $V_w\left(t \overset{\mathtt{p}}{:} F\right) := \min\{V_w(t{:}_{\mathtt{p}}F), V_w(t{:}^{\mathtt{p}}F)\}$.

Proof. The proof is trivial, thus omitted. □

Lemma 4.44. Let $M = \langle W, R, V, E\rangle$ an arbitrary fuzzy Fitting model. Then, for any world $w \in W$, any formula $F \in \mathcal{L}_{\mathsf{RPL(JL)}}$, any term $t \in \mathsf{Tm}$ and any $\mathtt{p} \in \mathtt{S}$, it holds that

- $M, w \Vdash t{:}_{\mathtt{p}}F \Leftrightarrow V_w(t{:}F) \geq \mathtt{p}$,
- $M, w \Vdash t{:}^{\mathtt{p}}F \Leftrightarrow V_w(t{:}F) \leq \mathtt{p}$,
- $M, w \Vdash t \overset{\mathtt{p}}{:} F \Leftrightarrow V_w(t{:}F) = \mathtt{p}$.

Proof. The proof is trivial, thus omitted. A proof can be found in [28]. □

Theorem 4.48 (Strong Soundness & Pavelka-Style Completeness for $\mathsf{RPL(JL)}$)**.**
Let JL an arbitrary justification logic and CS a constant specification for $\mathsf{RPL(JL)}$. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{RPL(JL)}}$.

$$\Sigma \vdash_{\mathsf{RPL(JL(CS))}} F \Rightarrow \Sigma \Vdash_{\mathsf{RPL(JL(CS))}} F \qquad \text{strong soundness}$$

$$\|F\|_{\Sigma}^{\mathsf{RPL(JL(CS))}} = |F|_{\Sigma}^{\mathsf{RPL(JL(CS))}} \qquad \text{Pavelka-style completeness}$$

Proof. The proof of this theorem can be found in [28]. □

POSSIBILISTIC JUSTIFICATION LOGIC

Theorem 4.54 (Conservativity of Possibilistic Justification Logic).
PJL is a *conservative extension* of the basic justification logic, J_0.

Proof. The proof is similar to the proof of Thoerem 4.6, where

$$\mathbf{t}\big(t{:}^{+}_{\mathrm{p}}F\big) := t{:}\mathbf{t}(F),$$

in the recursive definition.

A different proof can be found in [32]. □

Corollary 4.55 (Consistency of Possibilistic Justification Logic). Let CS an arbitrary constant specification for PJL. Then, $\mathsf{PJL}(\mathsf{CS})$ is consistent.

Proof. The proof is similar to the proof of Corollary 4.7, thus it may be omitted. □

Theorem 4.56 (Deduction Theorem for Possibilistic Justification Logic). Let CS an arbitrary constant specification for PJL. Then, the deduction theorem holds for PJL.

Proof. The proof of the deduction theorem for PJL is achieved in the standard way, thus it may be omitted. □

Lemma 4.57 (PJL-Internalization Property & Lifting Lemma for PJL).
Let CS an axiomatically appropriate constant specification for PJL. Then, PJL has the PJL-*internalization property*, i.e., for any formula $F \in \mathcal{L}_{\mathsf{PJL}}$ it holds that

$$\vdash_{\mathsf{PJL}(\mathsf{CS})} F \Rightarrow (\exists t \in \mathsf{Tm})\big[\vdash_{\mathsf{PJL}(\mathsf{CS})} t{:}_1 F\big].$$

Let PJL has the PJL-internalization property relative to some constant specification CS (e.g. axiomatically appropriate). Then it holds that, if

$$F_1, \ldots, F_n \vdash_{\mathsf{PJL}(\mathsf{CS})} F$$

then for any $t_1, \ldots, t_n \in \mathsf{Tm}$, for any $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathsf{S}$ and any $I \subseteq \{1, \ldots, n\}$ there exist some $t \in \mathsf{Tm}$ s.t.

$$\{t_i{:}_{\mathrm{p}_i} F_i \mid i \in I\} \cup \big\{t_i{:}^{+}_{\mathrm{p}_i} F_i \mid i \notin I\big\} \vdash_{\mathsf{PJL}(\mathsf{CS})} t{:}_{\mathrm{p}} F,$$

where $\mathrm{p} := \min_{i \in \{1,\ldots,n\}} \{\mathrm{p}_i\}$.

Proof. The proof is similar with the proofs of Lemmata 4.9, 4.57 etc. (viz. Theorem 2.34 and Lemma refFitLem). This lemma can be also found in [32]. □

Theorem 4.63 (Soundness and Completeness for PJL). Let CS an arbitrary constant specification for PJL.
$\mathsf{PJL}(\mathsf{CS})$ is sound and complete in respect with the class of PJL-models that meat constant specification CS and respect the PJL-minimum evidence conditions.

Proof. This theorem arises as a corollary of Theorem 4.64. □

Theorem 4.64 (Finitely-Strong Soundness and Completeness for PJL). Let CS an arbitrary constant specification for PJL. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PJL}}$ s.t. $\mathcal{G}(\Sigma \cup \{F\})$ is finite. Then, it holds that

$$\Sigma \vdash_{\mathsf{PJL}(\mathsf{CS})} F \Leftrightarrow \Sigma \Vdash_{\mathcal{C}} F,$$

where $\mathcal{C}$ the class of PJL-models that meat constant specification CS and respect the PJL-minimum evidence conditions.

Proof. The finitely-strong soundness proof is treated as usual. We will only cover the finitely-strong completeness part. Let CS some constant specification for PJL. Let also $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PJL}}$ s.t. $\mathcal{G}(\Sigma \cup \{F\})$ is finite. We define the set

$$\mathscr{X} := \{\mathrm{p}, 1-\mathrm{p} \mid \mathrm{p} \in \mathcal{G}(\Sigma \cup \{F\}) \cup \{0\}\}.$$

As always, we will define a corresponding canonical model, but this time explicitly for $\Sigma \cup \{F\}$. Before that, we have to give a corresponding Lindenbaum's lemma, which is proven in the standard way.

Lemma C.11 (Lindenbaum's Lemma for PJL)**.** Let CS an arbitrary constant specification for PJL and $X \subseteq \mathtt{S}$, s.t. $0, 1 \in X$. Let also $\Sigma \subseteq \mathcal{L}_{\mathsf{PJL}}(X)$ a $\mathsf{PJL}(\mathsf{CS})$-consistent set. Then there exists a maximal $\mathsf{PJL}(\mathsf{CS})$-consistent superset of Σ in the language of $\mathcal{L}_{\mathsf{PJL}}(X)$, i.e., it is maximal consistent in the logic which results from the restriction of the axiom schemes in Table 4.6 to those instances that $\mathrm{p}, \mathrm{q} \in X$.

We are now capable to define the $\mathsf{PJL}(\mathsf{CS})$-canonical model for $\Sigma \cup \{F\}$.

Definition C.12 (Canonical Model for PJL)**.**
Let CS an arbitrary constant specification for PJL. Let also $X \subseteq \mathtt{S}$ s.t. $0, 1 \in X$.

We define the *canonical model* $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, \mathscr{V}, \mathscr{E} \rangle$ for $\mathsf{PJL}(\mathsf{CS})$ in the language of $\mathcal{L}_{\mathsf{PJL}}(X)$, as follows:

- $\mathscr{W}$ is the set of maximal $\mathsf{PJL}(\mathsf{CS})$-consistent sets in the language of $\mathcal{L}_{\mathsf{PJL}}(X)$.
- $\mathscr{R} \colon \mathscr{W} \times \mathscr{W} \to [0, 1]$, s.t. for any $\Gamma, \Delta \in \mathscr{W}$
  - for any $\mathrm{p} \in X_{>0}$, it holds that

    $$\mathscr{R}(\Gamma, \Delta) > 1 - \mathrm{p} \Leftrightarrow AL(\Gamma, \mathrm{p}) \subseteq \Delta;$$

  - for any $\mathrm{p} \in X$, it holds that

    $$\mathscr{R}(\Gamma, \Delta) \geq 1 - \mathrm{p} \Leftrightarrow AL^{+}(\Gamma, \mathrm{p}) \subseteq \Delta,$$

  where $AL, AL^{+} \colon \mathcal{P}(\mathcal{L}_{\mathsf{PJL}}(X)) \times X \to \mathcal{P}(\mathcal{L}_{\mathsf{PJL}}(X))$ s.t.

  $AL(\Gamma, \mathrm{p}) := \left\{G \in \mathcal{L}_{\mathsf{PJL}}(X) \mid (\exists \mathrm{q} \in \mathtt{S}_{\geq \mathrm{p}})\, (\exists t \in \mathsf{Tm})\, \left[t{:}_{\mathrm{q}} G \in \Gamma \text{ or } t{:}^{+}_{\mathrm{q}} G \in \Gamma\right]\right\}$

  and

  $$\begin{aligned} AL^{+}(\Gamma, \mathrm{p}) := \{&G \in \mathcal{L}_{\mathsf{PJL}}(X) \mid (\exists \mathrm{q} \in \mathtt{S}_{>\mathrm{p}})\, (\exists t \in \mathsf{Tm})\, [t{:}_{\mathrm{q}} G \in \Gamma]\} \cup \\ &\left\{G \in \mathcal{L}_{\mathsf{PJL}}(X) \mid (\exists \mathrm{q} \in \mathtt{S}_{\geq \mathrm{p}})\, (\exists t \in \mathsf{Tm})\, \left[t{:}^{+}_{\mathrm{q}} G \in \Gamma\right]\right\}. \end{aligned}$$

- $\mathscr{V} \colon \mathsf{Prop} \to \mathcal{P}(\mathscr{W})$ s.t. for any $p \in \mathsf{Prop}$

  $$\mathscr{V}(p) := \{\Gamma \in \mathscr{W} \mid p \in \Gamma\}$$

- $\mathscr{E} \colon \mathsf{Tm} \times \mathcal{L}_{\mathsf{PJL}} \to \mathcal{P}(\mathscr{W})$ s.t. for any $t \in \mathsf{Tm}$ and any $F \in \mathcal{L}_{\mathsf{PJL}}$

  $$\mathscr{E}(t, F) := \left\{\Gamma \in \mathscr{W} \mid (\exists \mathrm{p} \in \mathtt{S})\, \left[t{:}_{\mathrm{p}} F \in \Gamma \text{ or } t{:}^{+}_{\mathrm{p}} F \in \Gamma\right]\right\}$$

Let us now show that the canonical model for $\mathsf{PPJ}(\mathsf{CS})$ in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ belongs in the class of PJL-models that meats CS and respects the PJL-minimum evidence conditions.

Theorem C.13. Let CS an arbitrary constant specification for PJL, $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PJL}}$ s.t. $\mathcal{G}(\Sigma \cup \{F\})$ is finite and

$$\mathscr{X} := \{\mathtt{p}, 1 - \mathtt{p} \mid \mathtt{p} \in \mathcal{G}(\Sigma \cup \{F\}) \cup \{0\}\}.$$

Then, the canonical model, $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, \mathscr{V}, \mathscr{E} \rangle$, for $\mathsf{PPJ(CS)}$ in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ belongs in the class of PJL-models that meats CS and respects the PJL-minimum evidence conditions.

Proof. We have to show that all the requirements for a $\mathsf{PJL(CS)}$-model are fulfilled by $\mathscr{M}$, but also that $\mathscr{M}$ is well defined.

($\mathscr{W}$) $\mathscr{W} \neq \emptyset$ due to Lindenbaum's Lemma C.11.

($\mathscr{R}$) We have to show that there exists such a fuzzy accessibility relation.
Clearly, as $\mathcal{G}(\Sigma \cup \{F\})$ is finite, we have that $\mathscr{X}$ is also finite. Let the decreasing enumeration $\{\mathtt{p}_i \mid i \in \{1, \ldots, n\}\}$ of $\mathscr{X}$, i.e.

$$1 = \mathtt{p}_1 > \mathtt{p}_2 > \cdots > \mathtt{p}_n = 0.$$

Trivially, for every $\Gamma \in \mathscr{W}$

$$\begin{aligned} AL^+(\Gamma, \mathtt{p}_1) = \{G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X}) \mid (\exists \mathtt{q} \in \mathtt{S}_{>1})\, (\exists t \in \mathsf{Tm})\, [t{:}_{\mathtt{q}} G \in \Gamma]\} \cup \\ \left\{G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X}) \mid (\exists \mathtt{q} \in \mathtt{S}_{\geq \mathtt{p}})\, (\exists t \in \mathsf{Tm})\, \left[t{:}^+_{\mathtt{q}} G \in \Gamma\right]\right\} \\ = \emptyset \cup \emptyset. \end{aligned}$$

Moreover, trivially we have that for any $\mathtt{p} \in \mathscr{X}$ it holds that

$$AL^+(\Gamma, \mathtt{p}) \subseteq AL(\Gamma, \mathtt{p}).$$

Furthermore, let some $\mathtt{p}, \mathtt{p}' \in \mathscr{X}$ s.t. $\mathtt{p} > \mathtt{p}'$ and $G \in AL(\Gamma, \mathtt{p})$. Then, there is some $\mathtt{q} \in \mathscr{X}_{\geq \mathtt{p}'}$ s.t. $G = t{:}_{\mathtt{q}} F \in \Gamma$ or $G = t{:}^+_{\mathtt{q}} F \in \Gamma$. But then, clearly as $\mathtt{p} > \mathtt{p}'$ we have that $\mathtt{q} \in \mathscr{X}_{\geq \mathtt{p}}$ and thus in either case $G \in AL^+(\Gamma, \mathtt{p}')$, i.e,

$$AL(\Gamma, \mathtt{p}) \subseteq AL^+(\Gamma, \mathtt{p}').$$

Hence, for any $i \in \{1, \ldots, n-1\}$ we have

$$AL^+(\Gamma, \mathtt{p}_i) \subseteq AL(\Gamma, \mathtt{p}_i) \subseteq AL^+(\Gamma, \mathtt{p}_{i+1}).$$

Therefore, we can define the fuzzy accessibility relation $\mathscr{R}$ as follows:
For any $\Gamma, \Delta \in \mathscr{W}$

– if $AL^+(\Gamma, \mathtt{p}_i) \subseteq \Delta \not\supseteq AL(\Gamma, \mathtt{p}_i)$ for some $i \in \{1, \ldots, n-1\}$, then

$$\mathscr{R}(\Gamma, \Delta) := 1 - \mathtt{p}_i;$$

– else, if $AL(\Gamma, \mathtt{p}_i) \subseteq \Delta \not\supseteq AL^+(\Gamma, \mathtt{p}_{i+1})$ for some $i \in \{1, \ldots, n-1\}$, then

$$\mathscr{R}(\Gamma, \Delta) := 1 - \mathtt{p},$$

for some $\mathtt{p} \in (\mathtt{p}_{i+1}, \mathtt{p}_i)$;

– else, if $AL^+(\Gamma, \mathtt{p}_n) \subseteq \Delta$, then

$$\mathscr{R}(\Gamma, \Delta) \coloneqq 1.$$

Clearly, $\mathscr{R}$ is a fuzzy accessibility relation. We have to show that it also respects the requested properties. Let arbitrary $j \in \{1, \ldots, n-1\}$ and arbitrary $\Gamma, \Delta \in \mathscr{W}$. We distinguish the following cases

– Let there exist some $i \in \{1, \ldots, n-1\}$ s.t. $AL^+(\Gamma, \mathtt{p}_i) \subseteq \Delta \not\supseteq AL(\Gamma, \mathtt{p}_i)$. Then, by the definition of AL and AL^+ we have that there must exist some $G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and some term $t \in \mathsf{Tm}$ s.t. $t{:}_{\mathtt{p}_i} G \in \Gamma$ and $G \notin \Delta$.

$j < i$ Then, as shown before, we have

$$AL(\Gamma, \mathtt{p}_j) \subseteq AL^+(\Gamma, \mathtt{p}_i) \subseteq \Delta$$

and

$$\mathscr{R}(\Gamma, \Delta) \coloneqq 1 - \mathtt{p}_i > 1 - \mathtt{p}_j.$$

$j \geq i$ Then $AL(\Gamma, \mathtt{p}_i) \subseteq AL(\Gamma, \mathtt{p}_j)$, as $\mathtt{p}_i \in \mathscr{X}_{\geq \mathtt{p}_j}$. Thus,

$$AL(\Gamma, \mathtt{p}_j) \not\subseteq \Delta$$

and

$$\mathscr{R}(\Gamma, \Delta) \leq 1 - \mathtt{p}_j.$$

– Let there exist some $i \in \{1, \ldots, n-1\}$ s.t. $AL(\Gamma, \mathtt{p}_i) \subseteq \Delta \not\supseteq AL^+(\Gamma, \mathtt{p}_{i+1})$. Then, by the definition of AL and AL^+ we have that there must exist some $G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and some term $t \in \mathsf{Tm}$ s.t. $t{:}^+_{\mathtt{p}_{i+1}} G \in \Gamma$ and $G \notin \Delta$.

$j \leq i$ Then, we have

$$AL(\Gamma, \mathtt{p}_j) \subseteq AL^+(\Gamma, \mathtt{p}_i) \subseteq \Delta$$

and

$$\mathscr{R}(\Gamma, \Delta) \coloneqq 1 - \mathtt{p} > 1 - \mathtt{p}_i \geq 1 - \mathtt{p}_j.$$

$j > i$ Then $AL^+(\Gamma, \mathtt{p}_i) \subseteq AL(\Gamma, \mathtt{p}_j)$. Thus,

$$AL(\Gamma, \mathtt{p}_j) \not\subseteq \Delta$$

and

$$\mathscr{R}(\Gamma, \Delta) < 1 - \mathtt{p}_j.$$

– Let $AL^+(\Gamma, \mathtt{p}_n) \subseteq \Delta$. Then, $j < i$, hence

$$AL(\Gamma, \mathtt{p}_j) \subseteq AL^+(\Gamma, \mathtt{p}_n) \subseteq \Delta$$

and

$$\mathscr{R}(\Gamma, \Delta) \coloneqq 1 > 1 - \mathtt{p}_j.$$

Finally, let $AL^+(\Gamma, \mathtt{p}_n) \subseteq \Delta$. Clearly,

$$\mathscr{R}(\Gamma, \Delta) \coloneqq 1 \geq 1 - \mathtt{p}_n = 1.$$

Otherwise, let $AL^+(\Gamma, \mathtt{p}_n) \not\subseteq \Delta$. As

$$AL^+(\Gamma, \mathtt{p}_1) = \emptyset \subseteq \Delta,$$

there must be some $i \in \{1, \ldots, n-1\}$ s.t. either $AL^+(\Gamma, \mathsf{p}_i) \subseteq \Delta \not\supseteq AL(\Gamma, \mathsf{p}_i)$, or $AL(\Gamma, \mathsf{p}_i) \subseteq \Delta \not\supseteq AL^+(\Gamma, \mathsf{p}_{i+1})$. In either case,

$$\mathscr{R}(\Gamma, \Delta) < 1 - \mathsf{p}_n = 1.$$

Therefore, $\mathscr{R}$ is as wanted.

($\mathscr{V}$) $\mathscr{V}$ is trivially well defined.

($\mathscr{E}$) Note that for any $F \in \mathcal{L}_{\mathsf{PJL}} \setminus \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and for any $t \in \mathsf{Tm}$, it holds that $\mathscr{E}(t, F) = \emptyset$. Thus, it is well defined for every $t \in \mathsf{Tm}$ and $F \in \mathcal{L}_{\mathsf{PJL}}$. That, $\mathscr{E}$ meats CS and respects the PJL-minimum evidence conditions can be shown in the standard way. We will only give the **PJ**, **PJ**$^+$ condition.

The cases where F or G belong in $\mathcal{L}_{\mathsf{PJL}} \setminus \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ are trivial. Let $F, G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$. Trivially, $F \to G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$. Let some

$$\Gamma \in \mathscr{E}(s, F \to G) \cap \mathscr{E}(t, F).$$

Then, by the definition of $\mathscr{E}$ we have that there exist $\mathsf{p}, \mathsf{q} \in \mathsf{S}$ s.t.

$$s{:}_{\mathsf{p}}(F \to G) \in \Gamma \text{ or } s{:}^+_{\mathsf{p}}(F \to G) \in \Gamma$$

and

$$t{:}_{\mathsf{q}}F \in \Gamma \text{ or } t{:}^+_{\mathsf{q}}F \in \Gamma.$$

Particularly, as $F, G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$, we clearly have $\mathsf{p}, \mathsf{q} \in \mathscr{X}$. Let w.l.o.g. that $\mathsf{q} > \mathsf{p}$. Then, as Γ is maximal $\mathsf{PJL}(\mathsf{CS})$-consistent in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and

$$t{:}_{\mathsf{q}}F \to t{:}^+_{\mathsf{p}}F$$

is an axiom of $\mathsf{PJL}(\mathsf{CS})$ in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$, we get that $t{:}^+_{\mathsf{p}}F \in \Gamma$. Moreover, as

$$s{:}^+_{\mathsf{p}}(F \to G) \to s{:}_{\mathsf{p}}(F \to G)$$

and

$$t{:}^+_{\mathsf{p}}F \to t{:}_{\mathsf{p}}F$$

are axioms of $\mathsf{PJL}(\mathsf{CS})$ in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$, we have $s{:}_{\mathsf{p}}(F \to G), t{:}_{\mathsf{p}}F \in \Gamma$. Therefore, as

$$s{:}_{\mathsf{p}}(F \to G) \to t{:}_{\mathsf{p}}F \to s \cdot t{:}_{\mathsf{p}}G$$

is an axiom of $\mathsf{PJL}(\mathsf{CS})$ in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$, we get that $s \cdot t{:}_{\mathsf{p}}G \in \Gamma$. Hence, $\Gamma \in \mathscr{E}(s \cdot t, G)$. As Γ was arbitrary we get

$$\mathscr{E}(s, F \to G) \cap \mathscr{E}(t, F) \subseteq \mathscr{E}(s \cdot t, G),$$

as wanted.

□

Let us give the corresponding *truth lemma* for PJL.

Lemma C.14. Let CS an arbitrary constant specification for PJL, $\Sigma \cup \{F\} \subseteq \mathcal{L}_{\mathsf{PJL}}$ s.t. $\mathcal{G}(\Sigma \cup \{F\})$ is finite,

$$\mathscr{X} := \{\mathtt{p}, 1 - \mathtt{p} \mid \mathtt{p} \in \mathcal{G}(\Sigma \cup \{F\}) \cup \{0\}\}.$$

and $\mathscr{M} := \langle \mathscr{W}, \mathscr{R}, \mathscr{V}, \mathscr{E} \rangle$ the canonical model for $\mathsf{PPJ}(\mathsf{CS})$ in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$. Then, for any $F \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and any $\Gamma \in \mathscr{W}$ it holds that

$$\mathscr{M}, \Gamma \Vdash F \Leftrightarrow F \in \Gamma.$$

Proof. The proof is given by induction in the complexity of F. We will only give the cases that are not similar with the previous truth lemmata, i.e., the cases where $F := t{:}_{\mathtt{p}} G$ or $F := t{:}^{+}_{\mathtt{p}} G$. Particularly, we will only give the former case.

Let $t{:}_{\mathtt{p}} G \in \Gamma$, where $t \in \mathsf{Tm}$, $G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and $\mathtt{p} \in \mathscr{X}_{>} 0$[7]. Then, by the definition of $\mathscr{E}$ and AL we have that $\Gamma \in \mathscr{E}(t, G)$ and that $G \in AL(\Gamma, \mathtt{p})$. We want to show that $\mathscr{M}, \Gamma \Vdash F$. As $\Gamma \in \mathscr{E}(t, G)$, it suffices to show that $N_\Gamma([G]_{\mathscr{M}}) \geq \mathtt{p}$, i.e.,

$$N_\Gamma([G]_{\mathscr{M}}) = 1 - \Pi_\Gamma(W \setminus [G]) = 1 - \Pi_\Gamma([\neg G]_{\mathscr{M}}) \geq \mathtt{p},$$

or equivalently

$$\Pi_\Gamma([\neg G]_{\mathscr{M}}) = \sup_{\Delta \in [\neg G]_{\mathscr{M}}} \{\pi_\Gamma(\Delta)\} = \sup_{\Delta \in [\neg G]_{\mathscr{M}}} \{\mathscr{R}(\Gamma, \Delta)\} \leq 1 - \mathtt{p}.$$

Let arbitrary $\Delta \in [\neg G]_{\mathscr{M}}$. Then we have

$$\begin{aligned}
\Delta \in [\neg G]_{\mathscr{M}} &\Leftrightarrow \mathscr{M}, \Delta \Vdash \neg G && \\
&\Leftrightarrow \mathscr{M}, \Delta \nVdash G && \\
&\Rightarrow G \notin \Delta && \text{Ind. Hyp.} \\
&\Rightarrow AL(\Gamma, \mathtt{p}) \nsubseteq \Delta && G \in AL(\Gamma, \mathtt{p}) \\
&\Leftrightarrow \mathscr{R}(\Gamma, \Delta) \leq 1 - \mathtt{p} && \text{1st requirement for } \mathscr{R}.
\end{aligned}$$

Thus, as Δ was arbitrary we have that

$$\sup_{\Delta \in [\neg G]_{\mathscr{M}}} \{\mathscr{R}(\Gamma, \Delta)\} \leq 1 - \mathtt{p},$$

as wanted.

For the other direction we assume that $t{:}_{\mathtt{p}} G \notin \Gamma$, where $t \in \mathsf{Tm}$, $\mathtt{p} \in \mathscr{X}$ and $G \in \mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$ and we continue by contraposition. Let also $\mathtt{p}$, the smallest $\mathtt{q} \in \mathscr{X}$ s.t. $t{:}_{\mathtt{q}} G \notin \Gamma$. If $\mathtt{p} = 0$ then by **G1**, **G2** and maximal consistency of Γ, we have that $\Gamma \notin \mathscr{E}(t, F)$. Thus, $\mathscr{M}, \Gamma \nVdash F$ and the requested proposition holds. Hence, we may assume that $\mathtt{p} \in \mathscr{X}_{>0}$. Let us assume that $AL(\Gamma, \mathtt{p}) \cup \neg G$ is $\mathsf{PJL}(\mathsf{CS})$-inconsistent in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$. Then, we have

$$\begin{aligned}
AL(\Gamma, \mathtt{p}) \cup \neg G \vdash \bot & && \Leftrightarrow \text{Th. 4.56} \\
AL(\Gamma, \mathtt{p}) \vdash \neg G \to \bot & && \Leftrightarrow \\
AL(\Gamma, \mathtt{p}) \vdash G & && \Rightarrow \text{Lem. 4.57} \\
\exists s \in \mathsf{Tm}\ \Gamma \vdash s{:}_{\mathtt{p}} G & && \Leftrightarrow \text{L. B.2 for } \mathsf{PJL} \\
s{:}_{\mathtt{p}} G \in \Gamma & && \Rightarrow \textbf{SJ}, t{:}_0 G \in \Gamma \\
t{:}_{\mathtt{p}} G \in \Gamma & &&
\end{aligned}$$

[7]The case for which $\mathtt{p} = 0$ is trivial.

which leads to contradiction. Thus, $AL(\Gamma, \mathtt{p}) \cup \neg G$ is $\mathsf{PJL}(\mathsf{CS})$-consistent in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$. Hence, by Lindenbaum's Lemma C.11 it can be expanded to some maximal $\mathsf{PJL}(\mathsf{CS})$-consistent set, Δ, in the language of $\mathcal{L}_{\mathsf{PJL}}(\mathscr{X})$. Clearly, by the first requirement for $\mathscr{R}$ we have that $\mathscr{R}(\Gamma, \Delta) > 1 - \mathtt{p}$ and also $\neg G \in \Delta$. But then, as previously $\Pi_\Gamma([\neg G]_{\mathscr{M}}) > 1 - \mathtt{p}$. Thus, as previously $N_\Gamma([G]_{\mathscr{M}}) < \mathtt{p}$. But then, $\mathscr{M}, \Gamma \not\models G$, as wanted □

The rest of the proof is, as usual, by contraposition and may be omitted. □

AGGREGATED PROBABILISTIC EVIDENCE

Lemma 4.74 (Lifting Lemma for PE)**.** If $\{F_1, \ldots, F_n, F\}$ is a set of purely propositional formulae s.t.

$$F_1, \ldots, F_n \vdash_{\mathsf{CL}} F,$$

then for every $t_1, \ldots, t_n \in \mathsf{Tm}$ it holds that

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \ldots \cdot t_n{:}F.$$

Proof. The proof will be given by induction on the complexity of the derivation of F. We will assume Łukasiewicz's axiom system for classical logic, as defined in Table 2.2.

- If F is some axiom scheme of CL, then by axiom scheme **SS** of PE we have $\vdash_{\mathsf{PE}} 1{:}F$ and by axiom scheme **G** we have $\vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \ldots \cdot t_n{:}F$. Hence, trivially

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \ldots \cdot t_n{:}F.$$

- If $F = F_i$ is some member of $\{F_1, \ldots, F_n\}$, then trivially

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_i{:}F.$$

  But then, by the definition of $\mathfrak{L}$ and axiom scheme **G** we have

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \ldots \cdot t_n{:}F.$$

- If F arrived by some application of modus ponens then there must be some purely propositional G s.t.

$$F_1, \ldots, F_n \vdash_{\mathsf{CL}} G \to F \quad \& \quad F_1, \ldots, F_n \vdash_{\mathsf{CL}} G.$$

  Then, by induction hypothesis we have that

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1{\cdot}t_2{\cdot}\ldots{\cdot}t_n{:}(G \to F) \quad \& \quad t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1{\cdot}t_2{\cdot}\ldots{\cdot}t_n{:}G.$$

  Thus, by axiom scheme **J** of PE and modus ponens we have that

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \ldots \cdot t_n \cdot t_1 \cdot t_2 \cdot \ldots \cdot t_n{:}F.$$

  By the definition of $\mathfrak{L}$ we trivially have that

$$t_1 \cdot t_2 \cdot \ldots \cdot t_n \cdot t_1 \cdot t_2 \cdot \ldots \cdot t_n = t_1 \cdot t_2 \cdot \ldots \cdot t_n.$$

  Therefore, by axiom scheme **G** we have

$$t_1{:}F_1, \ldots, t_n{:}F_n \vdash_{\mathsf{PE}} t_1 \cdot t_2 \cdot \ldots \cdot t_n{:}F,$$

  as wanted.

□

Lemma 4.76. For any PE-model $M = \langle W, H, \mu, *\rangle$ and any formulae $F, G \in \mathcal{L}_{\mathsf{PE}}$ it holds

- $(\neg F)^* = \overline{F^*}$,
- $(F \wedge G)^* = F^* \cap G^*$,
- $(F \vee G)^* = F^* \cup G^*$.

Proof. The proof is trivial. We will only give the second bullet.

$$
\begin{aligned}
(F \wedge G)^* &= (\neg(F \to \neg G))^* && \\
&= \overline{(F \to \neg G)^*} && \text{1st bullet} \\
&= \overline{\overline{F^*} \cup (\neg G)^*} && \\
&= \overline{\overline{F^*} \cup \overline{G^*}} && \text{1st bullet} \\
&= F^* \cap G^* && \text{De Morgan's Law}
\end{aligned}
$$

□

Lemma 4.78. For any axiom A of PE and any PE-model $M = \langle W, H, \mu, *\rangle$, it holds that

$$A^* = W.$$

Proof. The proof is trivial, thus it may be omitted. One proof of this lemma can be found in [33]. □

Lemma 4.80. Let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$. The *aggregated evidence* $AE^{\Sigma}(F)$ *for* F *given* Σ is an evidence for F given Σ.

Proof. The proof is trivial, thus it may be omitted. One proof of this lemma can be found in [33]. □

Corollary 4.81. Let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$.
A lattice term $t \in \mathfrak{L}_n$ is evidence for F given Σ iff

$$t \preceq AE^{\Sigma}(F).$$

Proof. This is an immediate corollary of Lemma 4.80. □

Theorem 4.82 (Strong Soundness and Completeness for PE)**.**
Let a set of formulae $\Sigma' \cup \{F'\} \subseteq \mathcal{L}_{\mathsf{PE}}$. Then,

$$\Sigma' \vdash_{\mathsf{PE}} F' \Leftrightarrow \Sigma' \Vdash F'.$$

Particularly, let a set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \ldots, F_n\}$ and $t \in \mathfrak{L}_n$. Then, it holds that

$$\boldsymbol{t}{:}\Sigma \Vvdash t{:}F \Rightarrow \boldsymbol{t}{:}\Sigma \vdash_{\mathsf{PE}} t{:}F.$$

Proof. The proof of the soundness part is straight forward and can be found in [33]. The completeness part can be proven in the usual fashion. The only part which seems different is the one explicitly mentioned, i.e. that for any set of purely propositional formulae $\Sigma \cup \{F\}$, where $\Sigma = \{F_1, \dots, F_n\}$ and $t \in \mathfrak{L}_n$, it holds that

$$\boldsymbol{t}{:}\Sigma \Vdash t{:}F \Rightarrow \boldsymbol{t}{:}\Sigma \vdash_{\mathsf{PE}} t{:}F.$$

We will prove this part by contraposition. Let us assume that $\boldsymbol{t}{:}\Sigma \nvdash_{\mathsf{PE}} t{:}F$. By Corollary 4.81 we know that $t \not\preceq_n AE^{\Sigma}(F)$. Clearly, as $\mathfrak{L}_n$ is a free distributive lattice, $\cup, \cdot$ are Boolean. Thus, any term in $\mathfrak{L}_n$ can be written in DNF form. That is,

$$t = \bigcup_{(t_{i_1},\dots,t_{i_k}) \in I} t_{i_1} \cdot \ldots \cdot t_{i_k},$$

where I a family of subsequences on $\boldsymbol{t}$. Since, $t \not\preceq_n AE^{\Sigma}(F)$ it must be the case that there is some $t_{i_1} \cdot \ldots \cdot t_{i_k}$ is t which does not belong in $AE^{\Sigma}(F)$. Hence, we have

$$\boldsymbol{t}{:}\Sigma \nvdash_{\mathsf{PE}} t_{i_1} \cdot \ldots \cdot t_{i_k}{:}F$$

and thus,

$$\{t_{i_1}{:}F_{i_1}, \dots, t_{i_k}{:}F_{i_k}\} \nvdash_{\mathsf{PE}} t_{i_1} \cdot \ldots \cdot t_{i_k}{:}F.$$

Then, by Lifting Lemma 4.74 we have

$$\{F_{i_1}, \dots, F_{i_k}\} \nvdash_{\mathsf{CL}} F.$$

Thus, by completeness of CL, we have that there is some *truth assignment* v s.t.

$$v \models \{F_{i_1}, \dots, F_{i_k}\}$$

and

$$\not\models F.$$

Let $\langle W, H, \mu\rangle$ an arbitrary probability space and $*$ an interpretation s.t. $M = \langle W, H, \mu, *\rangle$ is a PE-model and for any $p \in \mathsf{Prop}$, any $t \in \{t_1, \dots, t_n\}$ and any $i_j \in \{i_1, \dots, i_k\}$

$$\begin{array}{lcl} p^* = \emptyset & \Leftrightarrow & v(p) = 0 \\ p^* = W & \Leftrightarrow & v(p) = 1 \\ t^* = \emptyset & \Leftrightarrow & t \neq t_{i_j} \\ t^* = W & \Leftrightarrow & t = t_{i_j}. \end{array}$$

It is not hard to show by induction that for any $i \in \{1, \dots, n\}$ it holds that $(t_i{:}F_i)^* = W$. But clearly, $(t{:}F)^* = \emptyset$. Thus,

$$\boldsymbol{t}{:}\Sigma \nVdash t{:}F,$$

as wanted. ∎

APPENDIX D

PROOFS OF CHAPTER 5

Theorem 5.6 (Soundness and Completeness for Subset Models)**.** Let CS an arbitrary constant specification for J_0.
$\mathsf{J}_0(\mathsf{CS})$ is sound and complete with respect to the class of subset models that meet CS.

For any of the defined justification logics CS, $\mathsf{JL}(\mathsf{CS})$, where CS is a constant specification for justification logic JL, $\mathsf{JL}(\mathsf{CS})$ is sound and complete with respect to the class of subset models that meet CS and fulfil the corresponding evidence conditions, as given in Table 5.1.

Proof. A proof of this theorem can be found in [14, 15, 16]. The proof is similar to the proof of Theorem 3.31; thus, it may be omitted. □

Theorem 5.11 (Soundness and Completeness for $\mathsf{JL}^\star(\mathsf{CS})$-Subset Models)**.** Let CS an arbitrary constant specification for ${\mathsf{J}_0}^\star$.
${\mathsf{J}_0}^\star(\mathsf{CS})$ is sound and complete with respect to the class of ${\mathsf{J}_0}^\star(\mathsf{CS})$-subset models that meet CS.

For any $\mathsf{JL}^\star(\mathsf{CS})$, where CS is a constant specification for $\mathsf{JL}^\star$, $\mathsf{JL}^\star(\mathsf{CS})$ is sound and complete with respect to the class of subset models that meet CS and fulfil the corresponding evidence conditions.

Proof. A proof of this theorem can be found in [14, 15, 16]. The proof is similar to the proof of Theorem 3.31; thus, it may be omitted. □

Theorem 5.16 (Soundness of PE in PE-Adapted Subset Models)**.** Any theorem F of PE in the language $\mathcal{L}_{\mathsf{J}}^{\mathsf{PE}}$ is PE-valid.

Proof. The proof can be given in the standard way. Let arbitrary PE-adapted subset model $\mathcal{M} = \langle W, W_0, V, E \rangle$.

P,**J**,**MP** The proof follows from the corresponding proof for subset models (viz. Theorem 5.1).

U Let $\mathcal{M} \models s{:}F \wedge t{:}F$. Then, $E(w, s), E(w, t) \subseteq [F]$. Thus, clearly

$$E(w, s \cup t) = E(w, s) \cup E(w, t) \subseteq [F]$$

and as a result $\mathcal{M}, w \models s \cup t{:}F$.

SS Clearly, for any $w' \in W_0$ and any propositional tautology A, it holds that $\mathcal{M}, w' \models A$; i.e., $E(w, 1) = W_0 \subseteq [A]$. Hence, $\mathcal{M}, w \models 1{:}F$.

ES It is trivial.

G Let $\mathcal{M}, w \models s{:}F$. Let also $t \preceq_{\mathsf{s}} s$, thus, $s = s \cup t$. Then, we have that

$$
\begin{aligned}
E(w,t) &\subseteq E(w,s) \cup E(w,t) && \\
&\subseteq E(w, s \cup t) && E(w, s\cup t) = E(w,s) \cup E(w,t) \\
&\subseteq E(w,s) && s = s \cup t \\
&\subseteq [F] && \mathcal{M}, w \models s{:}F.
\end{aligned}
$$

Hence, $\mathcal{M}, w \models t{:}F$. □

Theorem 5.17. There exists a PE-adapted subset model.

Proof. Such a model $\mathcal{M} \models s{:}F \wedge t{:}F$ can be constructed, as follows:

- $W = W_0 = w$.
- $V \colon W \times \mathcal{L}_{\mathsf{J}}^{\mathsf{PE}} \to \{0, 1\}$ recursively defined as
  - $V(w, \bot) \coloneqq 0$;
  - $V(w, p) \coloneqq 1$, where $p \in \mathsf{Prop}$;
  - $V(w, F \to G) \coloneqq 1$ iff $V(w, F) = 0$ or $V(w, G) = 1$;
  - $V(w, t{:}F) \coloneqq 1$ iff $1 \preceq_{\mathsf{s}} t \Rightarrow V(w, F) = 1$.
- $E(w,t) \coloneqq \begin{cases} W & 1 \preceq_{\mathsf{s}} t \\ \emptyset & \text{otherwise} \end{cases}$.

It is straightforward to show that $\mathcal{M}$ is indeed a PE-adapted subset model. The complete proof can be found in [14, 16]. □

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