NATIONAL AND KAPODISTRIAN
UNIVERSITY OF ATHENS
DEPARTMENT OF MATHEMATICS

Strategic behaviour in queueing systems in alternating environment

Thesis submitted to the National and Kapodistrian University of Athens, Department of Mathematics in partial fulfilment of the requirements for the MSc degree in Statistics and Operations Research

Dimitrios Galanis

Advisor: Antonis Economou

Athens, September 2018

The present MSc thesis was completed for the fulfilment of the requirements for the MSc degree in Statistics and Operations Research at the Department of Mathematics, National and Kapodistrian University of Athens, Greece.

Approved on .... by the Evaluation Committee composed of:

Name-Surname	Academic rank	Signature
Apostolos Burnetas	Professor	....
Antonis Economou (advisor)	Professor	....
Constantinos Melolidakis	Associate Professor	....

I would like to thank cordially professor Antonis Economou, advisor of my MSc thesis, for his help, encouraging guidance, valuable advice and the time that he dedicated to me during the composition of the thesis. I would also like to thank professor Apostolos Burnetas and associate professor Constantinos Melolidakis for their participation in the Evaluation Committee and for all the knowledge they transmitted to me during my studies. Last but not least, I would like to thank all of my relatives and friends that supported me in so many ways during the period of my studies.

Contents

Chapter 1

Introduction

The purpose of this essay is to examine the strategic behaviour of customers who arrive at a queue evolving in alternating environment. We begin by presenting the basic definitions and theorems that will be frequently used.

1.1 Queueing theory preliminaries

A **queue** or **queueing system** is a system that provides a **service** to arriving **customers**. After the customers are served they depart from the system immediately.

Generally, the **arrival times** and the **service times** of consecutive customers are random variables. Therefore, the system's progress in time (e.g. the number of customers in the system) cannot be calculated with certainty. However, we can describe and study the system by using appropriate stochastic processes.

1.1.1 Common performance measures of a queueing system

The main characteristics of a queue are the arrival process, the service process and the service discipline.We will briefly describe each of them.

Arrival Process

The arrival process describes the mechanism by which consecutive customers $C_1, C_2, \ldots, C_n, \ldots$ arrive at the system and is determined by the joint distributions of the arrival moments $t_0 = 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq \ldots$ or equivalently the **interarrival times**

$$T_n = t_{n+1} - t_n, \qquad n \in \mathbb{N}.$$

The most common cases are listed below.

1. ***Poisson arrival process.*** This is the most common arrival process. It is also called *completely random arrival process* and is the most suitable model for systems with great amount of potential customers that use the service rarely and independently from each other. The models that will be examined in this essay refer to this arrival process. It is denoted by the capital letter (M) which is derived by the Memoryless or Markovian property of the Poisson.

2. ***Deterministic arrival process.*** In this process customers arrive in equal time intervals of length a, which means $T_n = a$ with probability 1. This is an appropriate process to model systems that serve by making appointments. It is denoted by the capital letter (D).

3. ***General renewal arrival process.*** In this case,the interarrival times $T_1, T_2 ...$ are independent identically distributed random variables with a general distribution $A(x)$ $(x \geq 0)$ and finite mean value

$$a = \int_0^{\infty} x dA(x) = \int_0^{\infty} (1 - A(x)) dx = \int_0^{\infty} A^c(x) dx.$$

The parameter $\lambda = 1/a$,which is the mean number of arrivals per time unit, is referred to as the *arrival rate*. The deterministic arrival process (D) is a special case of the general renewal arrival process with $A(x) = 0$, if $x < a$, and $A(x) = 1$, if $x \geq a$. The Poisson process is also a special case of this process with $A(x) = 0$, $x < 0$ and $A(x) = 1 - e^{-\lambda x}$, $x \geq 0$ which means that $T_1, T_2, \ldots$ are independent identically distributed random variables that follow the exponential distribution. The general renewal arrival process is denoted by the capital letters (GI) which refer to the generally independent interarrival times.

Service process

The service process describes how the customers are served. It is defined by the number of servers in the system, k, $(k \in \{1, 2, ..., \infty\})$ and the distribution of the service times. By denoting X_n the *service time* of the n-th customer that joins the system, we assume that $X_1, X_2, ...$ are independent identically distributed random variables that follow an arbitrary probability distribution $B(x)$ $(x \geq 0)$ with finite mean value

$$b = \int_0^{\infty} x dB(x) = \int_0^{\infty} (1 - B(x)) dx = \int_0^{\infty} B^c(x) dx.$$

In a queue with one server who is constantly busy the parameter $\mu = 1/b$ represents the mean number of customers that depart from the system per time unit which is also referred to as service rate. The service process that does not assume a particular distribution for the time that is required for a customer to be served by a server is denoted by (G). The most common case is when $B(x) = 1 - e^{-\mu x} (x \geq 0)$, the exponential distribution, and is denoted by (M).

Service discipline

The service discipline determines in which order arriving customers will be served. The most commonly used discipline is the FCFS (First-Come-First-Served) or FIFO (First-In-First-Out), where the customers are served according to the order of their arrivals at the system. Other important disciplines are LCFS (Last-Come-First-Served) or LIFO (Last-In-First-Out), where, whenever a server becomes available, the customer that arrived most recently in the system is served and SIRO (Service-In-Random-Order), where the customer to be served is chosen randomly. The disciplines mentioned above do not take directly into consideration the service time of the queued customers. There are others, like SSTF (Shortest-Service-Time-First), where the customer with the least service time among those queued is chosen to be served.

Another queueing regime is the one that sets *priorities* for a certain type of customers. Those disciplines can be with *preemption* or *without preemption.* Regarding those without preemption, when a new arrival is placed at the head of the queue, the customer in service is allowed to complete it.When the discipline is with preemption, a customer that has priority interrupts the service of a low priority customer.There are two cases of priority service. The *conservative* case, where the service, when resumed, is continued from the point where it was interrupted and the *non-conservative* case, where the service begins anew when resumed.

A common example of a preemptive discipline is the *LCFS/P-R(Last-Come-First-Served/Preemptive-Resume).* Under this queueing regime, a new arrival preempts a customer who might be in service. The interrupted customer returns to the queue and is chosen to continue her service with the rule of the LCFS discipline. The service is resumed from the point it was interrupted and no work is lost.

Kendall's Notation

To classify queueing models we use the notation $A|B|k$

A refers to the arrival process

B refers to the service time distribution

k denotes the number of servers

Therefore, $GI|G|1$ refers to a general queue with one server and $M|D|3$ refers to a queue that has 3 servers, constant service times and a Poisson arrival process.

The *capacity* of the system is the maximum number of customers allowed in the system including those in service. Assume that the capacity of the system is a finite number s. Then the previous notation is expanded by adding s and we use $A|B|k|s$. After this notation the service discipline is declared. For example, $D|G|1|10(LCFS)$ refers to a queue with constant inter-arrival times, general service times, one server, capacity for 10 customers and the LCFS discipline.

Performance Measures

Let $j \in \mathbb{N}\backslash\{0\}$, then:

- S_j is the *sojourn time* of customer C_j,
- W_j is the *waiting time* of customer C_j,
- X_j is the *service time* of customer C_j.

Obviously

$$S_j = W_j + X_j \qquad (j \in \mathbb{N}).$$

For $t \geq 0$, we have the following notations

- $Q(t)$ is the *number of customers in the system (queue size)*
- $Q_q(t)$ is the *number of queued customers* and
- $Q_s(t)$ is the *number of customers currently served*

at some point in time t.We observe that

$$Q(t) = Q_q(t) + Q_q(s).$$

The most useful stochastic processes for studying a queue are $\{Q(t) : t \geq 0\}$, $\{Q_q(t) : t \geq 0\}$, $\{S_j : j \in \mathbb{N}\backslash\{0\}\}$ and $\{W_j : j \in \mathbb{N}\backslash\{0\}\}$.

Consider the following:

- $\overline{Q} = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t Q(x)dx$ is the (long-term) *average number of customers in the system (average queue length)*,
- $\overline{Q}_q = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t Q_q(x)dx$ is the (long-term) *average number of waiting customers*
- $\overline{Q}_s = \lim\limits_{t\to\infty} \frac{1}{t}\int_0^t Q_s(x)dx$ is the (long-term) *average number of customers currently being served.*

Obviously , $\overline{Q}$, $\overline{Q}_q$ and $\overline{Q}_s$ are averages with respect to time *(time averages).*

Consider also

- $\overline{S} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} S_j$ is the (long-term) *average sojourn time*,
- $\overline{W} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} W_j$ is the (long-term) *average waiting time*,
- $\overline{X} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j$ is the (long-term) *average service time.*

Obviously , $\overline{S}$, $\overline{W}$ and $\overline{X}$ are averages with respect to the customers *(customer averages).*

1.1.2 Queue length

The most useful stochastic process for describing a queueing system is the *queue length* $\{Q(t) \ : t \geq 0\}$. Therefore, we are interested in computing the probabilities

$$p_j(t) = P(Q(t) = j) \quad j \in \mathbb{N}$$

of j customers in the system at any point t. However, those are usually very difficult to compute. Moreover, after a short period of time, the system will achieve a *steady state*, which means that $\{Q(t)\}$ becomes stationary. Therefore, our attention focuses on the steady state probabilities

$$p_j = \lim_{n\to\infty} p_j(t) \quad j \in \mathbb{N}$$

which are much simpler to compute.

If the interarrival times or the service times of a queue are continuously distributed, which is usually the case, the steady state probabilities can always be defined.

An important parameter is the **utilization rate** $\rho = \frac{\lambda}{\mu}$ where λ is the mean arrival rate and μ is the mean service rate. Since λ customers enter the system per time unit and each one of them adds $\frac{1}{\mu}$ workload to the system,the mean workload that enters the system per time unit is ρ. If $\rho \geq k$ then the queue length keeps increasing in time and the steady state probabilities $p_j(t)$ are all zero. On the other hand, if $\rho < k$, the probabilities p_j are not equal to 0 and the system reaches a steady state.

Embedded processes

It is much easier to study $\{Q(t)\}$ when the Markov property holds. If the Markov property holds, given the value of $\{Q(t)\}$ (the present) the random variables $\{Q(u) : u > t\}$ (the future) are independent from $\{Q(s) : s < t\}$ (the past). In many applications this property does not hold. Therefore, we need to inspect the queue at specific time points $t \in [0, \infty]$ where the Markov property holds. Those time points are those of *consecutive arrivals* or *consecutive departures* of customers. We define the following random variables

- $Q_n^- = Q(t_n^-)$, $n \in \mathbb{N}$, the queue length before the n-th arrival
- $Q_n^+ = Q(\tau_n^+)$, $n \in \mathbb{N}$, the queue length after the n-th departure.

The stochastic processes $\{Q_n^- \, , \, n \in \mathbb{N}\}$, $\{Q_n^+, \, n \in \mathbb{N}\}$ which describe the system specifically at times of arrivals and departures respectively, are called **embedded processes** of $\{Q(t) : t \geq 0\}$. We denote by

- $r_j = \lim\limits_{n\to\infty} P(Q_n^- = j)$, $j \in \mathbb{N}$,
- $d_j = \lim\limits_{n\to\infty} P(Q_n^+ = j)$, $j \in \mathbb{N}$,

the limiting distributions of $\{Q_n^- = j\}$ and $\{Q_n^+ = j\}$ respectively. In a system where we do not have multiple arrivals or departures at the same moment it is proven that

$$r_j = d_j.$$

The PASTA property

The limiting distributions in continuous time $\{p_j\}$ and at arrival moments $\{r_j\}$ or departure moments $\{d_j\}$ do not always coincide. However, this is the case when we have a Poisson arrival process.Therefore,

$$p_j = r_j \quad j \in \mathbb{N},$$

when the arrival process is Poisson. Thus, an arrival from a Poisson process observes the system as if it happens at a random moment in time. Therefore, any performance measure of the queue at the instant of a Poisson arrival is simply the long term time average of that measure. This property is referred to as PASTA (Poisson Arrivals See Time Averages).

Little's Law

When long queues are formed in a system, we intuitively expect to have long sojourn times.This intuition is justified by Little's Law:

Under steady state conditions, the long-term average queue length in a queueing system equals the average rate at which customers arrive multiplied by the long-term average sojourn time of a customer,

$$\overline{Q} = \lambda \overline{S} \text{ (with probability 1).}$$

Assume that the limiting distributions of the stochastic processes $\{Q(t) : t \geq 0\}$ and $\{S_n : n \in \mathbb{N} \backslash \{0\}\}$ are defined and let Q, S be the respective limit random variables. The following holds :

$$E(Q) = \overline{Q} \quad \textit{and} \quad E(S) = \overline{S}.$$

Now , Little's Law takes the following alternative form

$$E(Q) = \lambda E(S).$$

An important aspect of the Little's Law is that it also holds in subsystems within systems. Therefore, if we consider the subsystem of waiting customers, without the customers currently served, Little's Law yields :

$$E(Q_q) = \lambda E(W),$$

where Q_q, W are the limit random variables that describe the number of waiting customers in the queue and the waiting time respectively.

If we consider the customers currently being served as our system, Little's Law yields

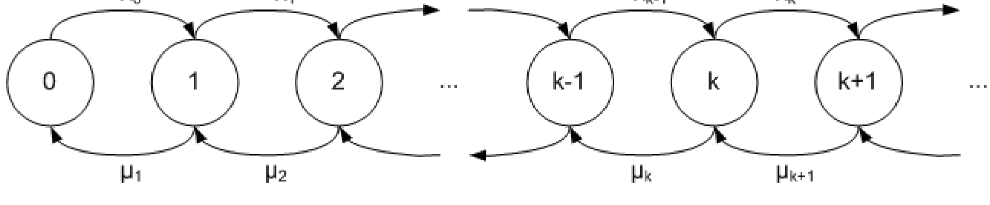
$$E(Q_s) = \lambda E(X) = \lambda b = \rho,$$

where $b = E(X)$ is the average service time and Q_s is the limit random variable that describes the number of customers currently served or equivalently, the number of busy servers.

1.1.3 The M/M/1 queue

Birth-death process

A birth-death process is a special case of a Markov process where the state space is $S = \mathbb{N}$ or $S = \{0, 1, \ldots, s\}$, and the state transitions are of only two types : "births" which increase the state variable by one, from state n to state $n+1$ and "deaths" which decrease the state variable by one, from state n to state $n-1$. The process is specified by birth rates $\{\lambda_i\}$ and death rates $\{\mu_i\}$. The state diagram is the following :



The stochastic process $\{X(t) : t \geq 0\}$ is called **birth-death** process. The unique *limiting (stationary) distribution*

$$p_n = \lim_{t \to \infty} p_n(t) \quad n \in S$$

exists if and only if

$$B^{-1} = \sum_{n \in S} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} < \infty,$$

and can be computed by

$$p_n = B \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} > 0, \, n \in S.$$

In this case $\{X(t)\}$ *is positive recurrent.* When $B^{-1} = \infty$, $p_n = 0$ $(n \in S)$ and $\{X(t\}$ is null recurrent or transient.

The M|M|1 queue

The M|M|1 queue is one of the simplest and most common service systems. In this system customers arrive according to a Poisson arrival process with parameter λ. There is only one server and the waiting capacity is infinite. Therefore a customer that finds the system empty begins her service immediately whereas a customer that finds the server busy (the system not empty) joins the queue. The consecutive service times are independent identically distributed random variables that follow the exponential distribution $exp(\mu)$ and are independent of the arrival times. When customers complete their service, they depart from the system and the server selects another customer to serve (if the queue is not empty).

Consider the stochastic process of the queue length $\{Q(t)\}$ at a specific time moment t. The process $\{Q(t), t \geq 0\}$ has a discrete state space $\{\mathcal{S} \subseteq \mathbb{N}\}$ and is a continuous time Markov chain since the interarrival times and the successive service times are independent and follow the exponential(λ) and the exponential(μ) distributions respectively.

If $Q(t) = n$ we can only move to state $n+1$ with rate λ or to $n-1$ with rate μ. Thus, we have a birth-death process like the one described in the previous section. Consequently, the unique limiting distribution exists if and only if

$$B^{-1} = \sum_{n=0}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} = \sum_{n=0}^{\infty} \frac{\lambda^n}{\mu^n} = \sum_{n=0}^{\infty} \rho^n < \infty,$$

where $\rho = \lambda/\mu$ is the utilization rate.

In order for $B^{-1} < \infty$, $\rho < 1$ is a necessary condition and then :

$$B^{-1} = \frac{1}{1-\rho}.$$

Therefore, in order for the limiting distribution of $\{Q(t), t \geq 0\}$ to exist, ρ must be less than 1 in which case the limiting distribution is $p_n = (1-\rho)\rho^n$, $n \in \mathbb{N}$, the Geometric(ρ) distribution. If $\rho \geq 1$, then $p_n = 0, \quad n \in \mathbb{N}$ or

$$\lim_{t \to \infty} P(Q(t) > n) = 1, \qquad n \in \mathbb{N},$$

and the queue length becomes infinite.

When $\rho < 1$, the system, after some time, will reach a steady state. Let Q be the number of customers in the system while steady state has been achieved. Then, the limiting random variable Q follows the geometric distribution with parameter ρ. Therefore, the mean value and the variance of the queue length are given respectively by

$$E(Q) = \frac{\rho}{1-\rho},$$

$$V(Q) = \frac{\rho}{1-\rho^2}.$$

Also, if S is the limiting random variable that represents the sojourn time of a customer in the system while the steady state has been achieved , Little's law yields

$$\begin{aligned} E(S) &= \frac{1}{\lambda} E(Q) \\ &= \frac{\frac{1}{\mu}}{1-\rho}. \end{aligned}$$

The following results are also true.

- $p_0 = P(Q = 0) = 1 - \rho$ *the probability that the system is empty*
- $P(Q \geq k) = \rho^k$, $k \in \mathbb{N}$.

1.2 Game theory preliminaries

In this section we will define and discuss elements of Game Theory that will be used throughout this essay.

1.2.1 Description of non-cooperative games

We begin by giving the definition of a game between two or more players.

Definition 1.2.1. *A game is specified by the following parameters*

a. A finite set of players $\mathcal{N} = \{1,2,3,\ldots,n\}$

b. A set of ***actions*** *available for player* $i \in \mathbb{N}$ *which is denoted by* A_i

c. *The **payoff function** which assigns to every set of a players' actions a real number.*

Remark 1. *In the applications of Game Theory to Operations Research, the set of players $\mathcal{N}$ is usually infinite.In this essay the players are the customers that arrive at a queue and are considered potentially infinite.*

1.2.2 Equilibrium strategies and payoff function

A **strategy** of a player represents a complete plan of actions that the player will follow under any circumstances that may arise during the game. A strategy profile is a set of strategies, one for each player of the game. Therefore, a strategy profile is a predetermined course of action for every possible situation throughout the game.

Assume a game that has r ($r \in \mathbb{N}$) different stages. A **pure strategy** for player i is a vector of actions from A_i, denoted by $s_i = (a_1, a_2, ..., a_r)$ which declares the action that player i will take when the game is at one of those r stages.

A **mixed strategy** is a vector that assigns a probability to all pure strategies a player has available and is the probability a player uses the corresponding strategy. For example, if a player has 4 different possible pure strategies, some mixed strategies are $m_1 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, $m_2 = \left(\frac{5}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{4}\right)$, $m_3 = \left(\frac{3}{10}, \frac{1}{10}, \frac{3}{5}, 0\right)$. Obviously, the coordinates of a mixed strategy add up to 1.

Let $\{s_{i1}, s_{i2}, ..., s_{ij}\ j \in \mathbb{N}\}$ be the available pure strategies of player i.Then the set of mixed strategies M_i is the set M_i=$\{(p_1, p_2, ..., p_j) : \sum_{k=1}^{j} p_k = 1,\ p_k \geq 0\}$.

Every pure strategy can be expressed as a mixed strategy. Consider a pure strategy of the i-th player s_{ik}. Then the mixed strategy $(0, ..., 0, 1, 0, ..0)$ with 1 at the k-th coordinate corresponds to s_{ik}

Definition 1.2.2. *The set $M = M_1 \times ... \times M_n = \{m_1, ..., m_n : m_1 \in M_1, ..., m_n \in M_n\}$ is the set of strategy profiles. $m \in M$ is a **strategy profile** that defines a strategy for all players.*

Payoff function

Each player is associated with a real payoff function $F_i(m)$. This function specifies the payoff received by player i given that the strategy profile m is adopted by the players. Denote by m_{-i} a strategy profile for the set of players $\mathcal{N}\backslash\{i\}$.Then, if $m = (m_1, ..., m_N)$, $m_{-i} = (m_1, ...m_{i-1}, m_{i+1}, ..., m_N)$ and we can also express the strategy profile m as $m = (m_{-i}, m_i)$.

We can now define the payoff function $F_i(m)$ corresponding to player i, as a function that assigns real numbers to the elements of the set of strategy profiles M :

$$F_i : (m_i, m_{-i}) \rightarrow r_i \in \mathbb{R}.$$

We assume that the function $F_i(s)$ is linear in m_i.
Therefore, by letting $m_i = p \cdot a + (1-p) \cdot b$, $p \in [0,1], a, b \in M_i$ we get:

$$F(m_i, m_{-i}) = p \cdot F_i(a, m_{-i}) + (1-p) \cdot F_i(b, m_{-i}).$$

Dominating strategies

Definition 1.2.3. *A strategy m_i^1 is said to **weakly dominate** strategy m_i^2 (for player i), if for any m_{-i}, $F_i(m_i^1, m_{-i}) \geq F_i(m_i^2, m_{-i})$ and for at least one m_{-i} the inequality is strict. A strategy m_i is said to be **weakly dominant** if it weakly dominates all other strategies in M_i.*

Definition 1.2.4. *A strategy m_i^* is said to be a **best response** for player i against the profile m_{-i} if*

$$m_i^* \in arg \max_{m_i \in M_i} F(m_i, m_{-i})$$

Therefore, m_i^* maximizes the utility of player i when the other players use the m_{-i} strategy.

We denote by $BR_i(\hat{m}_{-i})$ the set of best responses of the i-th player when all others follow the $\hat{m}_{-i}$ strategy:

$$\begin{aligned} BR_i(\hat{m}_{-i}) &= \{m_i^* : F_i(m_i^*, \hat{m}_{-i}) \geq F_i(m_i, \hat{m}_{-i}),\ \forall m_i \in M_i\} \\ &= \arg\max_{m_i \in M_i} F(m_i, \hat{m}_{-i}). \end{aligned}$$

Equilibrium

Definition 1.2.5. *A strategy profile m^e is an **equilibrium profile** if for every $i \in N, m_i^e$ is a best response for player i against m_{-i}^e i.e.,*

$$m_i^e \in arg \max_{m_i \in M_i} F_i(m_i, m_{-i}^e), \qquad i \in N$$

Remark 2. *If a best response m_i^* is a mixture of strategies then all those strategies are also best responses.This property does not hold when "best response" is replaced by "equilibrium".*

1.3 Game theory in queues

In this section we will discuss a queue from the perspective of game theory. The arriving customers are the players of the game, therefore the players are usually indistinguishable and infinitely many. When a customer arrives at the queue she needs to make a decision. The possible decisions usually are join or balk (not join). The options a customer has available are the elements of each player's action set. Depending on their choice, the customers will receive a payment at the completion of their service, which is given by their payoff function.

Let R be the reward customers gain when they complete their service and C be the cost per time unit a customer incurs while in the system. We consider that the different states of the game are the number of customers in the queueing system given by the variable $Q(t)$.

Considering that the players are indistinguishable, denote the common set of strategies and the payoff function by S and F respectively and let $F(a, b)$ be the payoff of a player who chooses strategy a while everyone else selects strategy b .We now have the following definition:

Definition 1.3.1. *A* ***symmetric equilibrium*** *is a strategy $s^e \in S$ such that*

$$s^e \in \underset{s \in S}{argmax} F(s, s^e)$$

which means that s^e is a best response against itself.

1.3.1 Information and strategies

The information a customer receives upon arriving at the system is usually very important when choosing a strategy. Consider the $M|M|1$ queue (FCFS). If a customer arrives and observes an empty queue, she will probably join since her sojourn time is expected to be short. On the other hand, if she observes a lot of customers in the system, her sojourn time increases along with the cost she will suffer and thus, she will be more reluctant to join. Therefore, we classify queues according to whether or not their length can be observed before making a decision.

The objective of this essay will be to examine models with *alternating environment*, i.e. the parameters of the system alternate randomly between two or more states. For example, the system may alternate between two service modes, a slow and a fast one. Naturally, a customer will prefer to join when the fast mode is active in order to lower her mean sojourn time. For this reason, we also need to classify queues according to whether or not we can observe the state of the system before taking an action.

Let $N(t)$ and $I(t)$ be the number of customers and the state of the system at a certain point in time $t \in [0, \infty]$. Then, a customer may have the following information levels available upon arrival:

1. **Fully observable queue**: In this case, the customers can observe both the number of customers in the system $N(t)$ as well as the state of the system $I(t)$ before making a decision.

2. **Almost observable**: In this case, arriving customers observe the number of customers in the system $N(t)$, but can't identify the state of the system $I(t)$ before deciding whether to join or balk.

3. **Almost unobservable**: In this model, customers can tell apart the different states of the system but cannot observe the number of customers in the queue and the set of information each customer has available is the set of possible states of the system $\{I(t) : t \in [0, \infty]\}$.

4. **Fully unobservable**: In this case, customers have no information regarding the number of customers in the system and they cannot tell apart the different states of the system.

Threshold strategies

Due to the structure of the payoff function, especially in the observable models, the equilibrium strategy usually is a *threshold strategy.* When a player follows a threshold strategy, she exhibits a certain behaviour until a specific threshold is reached. When the threshold is exceeded , the customer follows a different course of action. A simple example of a threshold strategy is when a customer joins a queue if she observes less than 10 customers in the system and balks when 10 or more customers are observed.

Definition 1.3.2.

- *A* ***pure threshold strategy*** *with threshold* $n \in \mathbb{N}$ *dictates that a customer will take an action A while the system is in states 0,1,2,...,n-1 and some other action B while the system is in any other state.*

- *A* ***mixed threshold strategy*** *with threshold n+p,* $n \in \mathbb{N}, p \in [0, 1)$ *dictates that*

  - *a player will take an action A while the system is in the states* $0, 1, 2, \ldots, n-1$*,*
  - *while in state n, a player will take action A with probability p and some other action B with probability* $1 - p$
  - *a player will take another action B for all other states of the system.*

1.3.2 Steady-state

A pure strategy prescribes an action to each state of the system. A strategy profile and an initial state induce a probability distribution over the states of the system. The player's payoff is determined by her strategy, the strategy of the other players and the state of the system, while every player is interested only in maximizing her expected payoff.

When we calculate a player's expected payoff while she follows a strategy x against all others using strategy y we assume that steady-state conditions have been reached. As stated above, the steady-state has the meaning that the probability distribution over the states is the limiting distribution, which makes calculations much simpler. Moreover, convergence in steady state is usually very fast, so the analysis remains valid if we neglect the transient effects on the customer equilibrium behaviour.

1.3.3 Avoid the crowd or Follow the crowd

We have already stated that the payoff function of a customer is a function of the strategy selected by other customers. In queueing models, the strategies are usually represented by a single number. For example, in the observable case we have a natural number that represents the threshold, while in the unobservable case customers usually enter the queue with probability $p \in [0, 1]$. In such cases it is meaningful to consider whether an individual's best response is a monotone function of the strategy selected by other customers.

Let $F(x, y)$ be the payoff function of a tagged customer who selects strategy x when all others select strategy y. Assume that for every y there is a unique best response $x(y) = \underset{x}{argmax} F(x, y)$. Let $x(y)$ be monotone in y.

- If $x(y)$ is increasing in y then, when y increases, $x(y)$ also increases, meaning that the tagged customer follows the behaviour of the other customers. This behaviour is called ***follow the crowd*** (FTC).
- If $x(y)$ is decreasing in y then, when y increases, $x(y)$ decreases, meaning that the tagged customer does not follow the behaviour of the other customers. This behaviour is called ***avoid the crowd*** (ATC).

An equilibrium strategy y satisfies that $x(y) = y$, which means that y is a fixed point of function x. An interesting property of the FTC behaviour is that multiple equilibria are possible, whereas in the ATC behaviour at most one equilibrium is possible.

1.3.4 Costs and objectives

In order to calculate the equilibrium strategies we need to compute the payoff function for a tagged customer who follows a strategy s_{tagged} when all others follow a strategy s_{others}. The welfare of the tagged customer consists of the benefit she gains upon completing her service minus the direct costs (e.g. the price of a ticket to enter the queue) and the indirect costs that are associated with waiting. Let R be the benefit a customer gains from completed service, C the cost per time unit[1] while the customer remains in the system, p the direct costs and t the time a customer remains in the system. Then, we may have the following optimization objectives:

[1]We assume that the waiting cost is linear in time. While true for most applications, this assumption does not always hold.

- *individual optimization* Each customer maximizes her net benefit while ignoring the costs she may inflict on other customers in the system.The net benefit is given by

$$F(s_{tagged}, s_{others}) = R - Ct - p.$$

- *social optimization* A social planner dictates the strategy all players (customers, servers, ...) must follow in order to maximize the *social welfare.* In this case, payments between players are considered transfer costs and do not affect the social welfare. The social welfare is computed by the sum of the benefits from completed services minus the costs incurred by the system's operation and the sum of the waiting costs of all players.

1.3.5 Joining, balking and reneging

The action set a customer has available when arriving in a queuing system usually consists of the following three options:

- *join*: A customer arrives at the system and decides to enter in order to be served.

- *balk*: A customer arrives at the system and decides not to enter. This is called *balking.* Usually, after a customer balks, they do not have the option of joining the system later on. This is the case for the models that will be examined throughout this essay.

- *renege* After a customer has entered the queue, she sometimes has the option to abandon the system while waiting. This is called *reneging.* Again, a customer that reneges is usually considered lost by the system and cannot rejoin at a later time. In the models of this essay, reneging is not allowed.

In section 1.1.1 we discussed the meaning of the arrival rate λ. After taking into account that an arriving customer does not always enter the queue we make the following adjustment:
We denote by Λ the rate customers arrive at the system -whether they join or not- and we denote by λ the rate of customers that arrive and decide to join the system. Λ is referred to as the *potential arrival rate*, while λ is referred to as the *effective arrival rate.*

1.4 Strategic behaviour in queueing systems in alternating environment

As stated in the previous sections, the economic analysis of customer behaviour in queueing systems is based on some reward-cost structure which is imposed on the system and reflects the customers' desire for service and their unwillingness to wait. Customers are allowed to make decisions about their actions in the system, for example they decide whether to join or balk, to abandon the queue, to buy priority or not etc. The customers want to maximize their benefit while taking into account that other customers have the same objective. This situation can be considered a game among the players. The basic problem is to find an equilibrium and socially optimal strategies. These ideas go back at least to the works of **Naor (1969)** and **Edelson and Hildebrand (1975)** who studied equilibrium and socially optimal strategies for whether to join or balk in an $M|M|1$ queue with a simple linear reward-cost structure. **Naor (1969)** assumed that an arriving customer observes the queue length before making her decision to join (observable case). His study was complemented by **Edelson and Hildebrand (1975)** who studied the unobservable case, where the customers had to make their decision without information about the state of the system. Since then, there is a growing number of papers that deal with the economic analysis of the balking behaviour of customers in variants of the $M|M|1$ queue, see e.g. **Hassin and Haviv (1997)** ($M|M|1$ queue with priorities), **Burnetas and Economou (2007)** ($M|M|1$ queue with setup times), **Guo and Zipkin (2007)** ($M|M|1$ queue with various levels of information and uncertainty in the system parameters), **Economou and Kanta (2008)** ($M|M|1$ queue with compartmented waiting space), **Sun et al. (2010)** ($M|M|1$ queue with setup/closedown times) and **Zhang and Wang (2010)** ($M|M|1$ queue with delayed repairs).

One of the models that will be discussed in this study is the $M|M|1$ queue with an unreliable server. The strategic behaviour in vacation queueing systems, where the server may become unavailable in between services, is a quite recent endeavor. **Burnetas and Economou (2007)** studied the $M|M|1$ queue with setup times under a strategic perspective. Subsequently, **Economou and Kanta (2008)** and **Jagannathan, Menache, Modiano and Zussman(2011)** studied the strategic joining/balking behaviour of the observable and unobservable models of the $M|M|1$ queue with unreliable server. **Guo and Hassin (2011;2012)** studied the strategic behaviour of customers in an $M|M|1$ vacation queue with an N-policy. The study of the strategic customer behaviour in vacation queues has been also extended in models with additional characteristics such as closedown times, general service or vacation time etc., see e.g. **Sun Guo and Tian (2010)**, **Economou,Gómez-Corral and Kanta (2011)**, **Li and Han (2011)**, **Do, Tran, Nquyen, Hong, and Lee (2012)**, **Liu, Ma, and Li (2012)**, **Zhang, Wang, and Liu (2013)** and **Yang, Wang, and Zhang (2014)**.

For the study of vacation queues it is usually convenient to consider *fluid queues.* Fluid queues are suitable for representing systems where the process of the customers is very fast in comparison with changes in the server status. A fluid queue is an input-output system, where the customers are modeled as a continuous fluid that enters and leaves a storage space, called the buffer, according to rates that depend on some underlying stochastic process that is related to the state of the system. Fluid queues have been used extensively as an approximation of the standard queues with discrete units in applications such as high-speed data networks, automated manufacturing systems, traffic/transportation networks etc.

There is an extensive literature devoted to the study of the fluid flow models. Several early computational approaches and results can be found in the papers of **Kosten (1974a)(1974b)** and **Kosten and Vrieze (1975)**. **Anick, Mitra, and Sondhi (1982)** introduced a benchmark model that is now known as AMS model. This is a fluid queue that represents a single buffer which receives data from several independent sources, each of which switches between on and off states according to a continuous time Markov chain. For a smooth introduction in the area and a literature review of classical references see **Kulkarni (1997)**, **Schwartz (1996)** or **Gautam (2012)**.

The fluid queues are in some sense semi-deterministic counterparts of vacation queueing models in random environment. Indeed, the on-off alterations of the independent sources that occur in fluid models can be seen as vacation/failures of the server of the system. Even though fluid queues treat situations similar to vacation queues, the literature has been devoted mainly on performance evaluation issues and control problems under a central planer (see e.g. **Rajagopal, Kulkarni, and Stidham (1995)** for the optimal flow control problem). There are also a number of studies with strategic considerations and fluid models that concern completely different situations from vacation queues, see e.g. **Maglaras (2006)**, **Jain, Juneja, and Shimkin (2013)**, and **Haviv (2013)**. The strategic behaviour regarding the joining/balking dilemma in an observable fluid queue with a Fist-Come-First-Served (FCFS) discipline, where the system alternates between exponentially distributed fast and slow service periods was studied by **Economou and Manou (2016)**.

In the two queuing models mentioned so far the studies in the changes of the environment mainly refer to the service process. When both the service and arrival processes alternate, the analysis becomes much more complicated. An interesting queueing model, where we can consider that the system alternates between states with different arrival and service rates, is the stochastic clearing system. In this model, the customers are accumulated in a waiting room and the server removes all customers at the completion of a service cycle. Stochastic clearing systems have been studied by **Stidham (1974)**, **Serfozo and Stidham (1978)**, **Artajelo and Gomez-Corral (1998)** and **Yang et al. (2002)**. They have been also studied in the framework of stochastic systems subject to (total) catastrophes or disasters, where catastrophic events are assumed to remove all customers/units of the system/population (see e.g. **Kyriakidis (1994)**, **Economou and Fakinos**

(2003, 2008), **Stirzaker (2006, 2007)** and **Gani and Swift (2007)**). In the majority of such studies the interest of investigators lies in the transient and/or the stationary distribution of the process of interest. However, optimization issues for this class of systems have also attracted the interest in the literature (see e.g. **Kyriakidis (1999a, 1999b)**, **Economou (2003)**, **Kyriakidis and Dimitrakos (2005)**.

The optimization questions that have been studied in the context of stochastic clearing systems concern the central planning of the systems and the objective is the determination of optimal strategies for the server, about when he should remove the customers from the system (see e.g. **Stidham (1977)**, **Kim and Seila (1993)**, **Economou (2003)**, **Kyriakidis and Dimitrakos (2005)**). The behaviour of customers in a clearing system in alternating environment when they are free to make decisions to maximize their benefit has been studied by **Economou and Manou (2013)**.

In this essay we will examine customer strategic behaviour in queuing systems in alternating environments. The essay is organized as follows. In chapter 2, we will describe a queue with an unreliable server. The customers arrive at the system, observe the number of customers in it and decide whether to join or balk. We will consider two information cases, the fully observable and the almost observable case and identify the equilibrium balking strategies.

In chapter 3, we will consider a stochastic clearing system in random environment. We will explore the customers' strategies under various information cases. In all cases, we will show that the expected net benefit of a customer depends only on her strategy and not on the strategies followed by other customers, a fact that implies the existence of dominant strategies. This is a special feature of the system and is related to the nature of the stochastic clearing mechanism. In the almost observable case, we notice that the number of waiting customers does not imply an additional cost on the individual, but their presence provides a signal about the state of the system.Then, we will characterize all the equilibrium strategies within the class of threshold and reverse threshold strategies and provide an algorithm to compute those equilibrium strategies.

In chapter 4, we will study the customers' join/balk dilemma in a fluid queue in alternating environment. Again, we will explore two informational cases and we will determine equilibrium customer strategies. We will also compute the expected social benefit per time unit under a given strategy and consider the related optimization problems. We then compare the expected social benefits per time unit under various combinations regarding the nature of the customers.

Chapter 2

Equilibrium balking strategies for an observable queue with breakdowns and repairs

2.1 The model

Consider a M|M|1 queue with infinite capacity. The customers arrive according to a Poisson process with rate λ and the service times are exponentially distributed with rate μ. In this model the server is not always active, but alternates between on and off periods. We assume that the *on* and *off* periods are also exponentially distributed with rates ζ and θ respectively.

Let $N(t)$ be the queue length at time t and $I(t)$ the state of the server at time t ($I(t) = 0$ corresponds to the state where the server is *off* and $I(t) = 1$ to the state where the server is *on*). The process $\{N(t), I(t) : t \geq 0\}$ is a continuous time Markov chain where the transition rates are given by

$$q_{(n,i)(n+1,i)} = \lambda, \quad n = 0, 1, 2, ... \; and \; i = 0, 1$$

$$q_{(n,i)(n-1,i)} = \mu, \quad n = 1, 2, 3, ...$$

$$q_{(n,0)(n,1)} = \theta, \quad n = 0, 1, 2, ...$$

$$q_{(n,1)(n,0)} = \zeta, \quad n = 0, 1, 2, ... \; .$$

We assume that customers arrive at the queueing system and receive some information. Then, they decide whether to join or not. When a customer joins the system, she receives a reward or R units upon completion of her service and she incurs a waiting cost of C units per time unit that is continuously accumulated while she remains in the system. When a customer does not join, her net benefit is considered to be 0. Each customer tries to

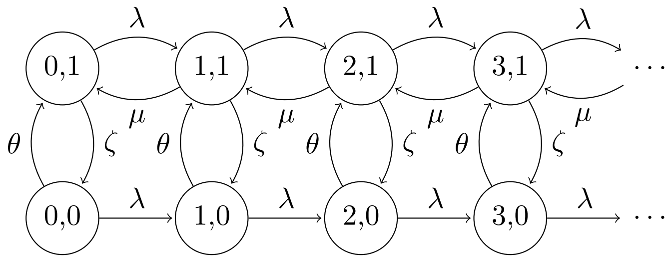


Figure 2.1: Transition diagram

maximize her expected net benefit. When a customer decides whether to join or not, her decision is irrevocable, meaning that reneging of entering customers or retrials of balking customers is not allowed.

We will consider two informational cases separately. We will study the *fully observable* case first, where the customers observe both the queue length $N(t)$ and the state of the system $I(t)$.

2.2 Equilibrium threshold strategies in the fully observable case

As stated above, in this information case, customers observe the queue size $N(t)$ and the the state of the system $I(t)$ before joining. The equilibrium threshold strategy has the form $(n_e(0), n_e(1))$, which declares that when customers observe the system in state i, they should join if $N(t) \leq n_e(i)$, and they should balk otherwise. In the *fully observable case* we will conclude that the strategy of a customer is independent of the strategy of the other customers which is a trait of dominant strategies.

We have the following result.

Theorem 2.2.1. *In the fully observable case of an $M|M|1$ queue with breakdowns and repairs there exist a pair of thresholds*

$$(n_e(0), n_e(1)) = \left(\left\lfloor \frac{R\mu\theta}{C(\theta+\zeta)} - \frac{\mu}{\theta+\zeta} \right\rfloor - 1, \left\lfloor \frac{R\mu\theta}{C(\theta+\zeta)} \right\rfloor - 1 \right) \tag{2.1}$$

such that the strategy "While arriving at time t, observe $(N(t), I(t))$; enter if $N(t) \leq n_e(I(t))$ and balk otherwise" is a weakly dominant strategy.

Proof. Consider a tagged customer that follows a strategy s_{tagged} when all other customers follow a strategy s_{others}. The tagged customers' net benefit is given by

$$F(n, i) = R - CT(n, i), \tag{2.2}$$

where $T(n,i) = E[S|N^- = n, I^- = i]$ is the average sojourn time of a customer that finds the system at state (n,i) upon arrival.
Let $m_{n,i}$ be the expected time the system remains in state (n,i) and $q_{(n_1,i),(n_2,i')}$ be the transition rate from state (n_1,i) to state (n_2,i').

When the system is in state $(n,1)$ $n = 1,2,...$ two possible transitions are possible (ignoring arrivals which do not affect the sojourn time of a customer already in the system) : $(n-1,1)$, which corresponds to the service of a customer, or $(n,0)$, which corresponds to a breakdown of the system. The time spent in state $(n,1)$ is the minimum of the time it takes for a customer to be served or for a breakdown to happen. Therefore, it follows $exp(\mu + \zeta)$ as the minimum of two independent variables that follow exponential distributions. Consequently, $m_{n,1} = \frac{1}{\mu+\zeta}$.
From state $(n,1)$ the service of a customer may be completed with probability $p((n,1) \to (n-1,1)) = \frac{q_{(n,1)(n-1,1)}}{m_{n,1}} = \frac{\mu}{\mu+\zeta}$ and the system moves to state $(n-1,1)$ or a breakdown may happen with probability $p((n,1) \to (n,0)) = \frac{q_{(n,1)(n,0)}}{m_{n,1}}$ where the system moves to state $(n,0)$.
By a similar analysis for states $(n,0)$ and $(0,1)$ and using a first step argument we have the following equations.

$$T(n,0) = \frac{1}{\theta} + T(n,1). \quad n = 0,1,2,... \tag{2.3}$$

$$T(0,1) = \frac{\mu}{\mu+\zeta} + \frac{\zeta}{\zeta+\mu}T(0,0). \tag{2.4}$$

$$T(n,1) = \frac{1}{\zeta+\mu} + \frac{\mu}{\zeta+\mu}T(n-1,1) + \frac{\zeta}{\zeta+\mu}T(n,0). \quad n = 1,2,3,... \tag{2.5}$$

By solving the system (2.3) for $n = 0$ and (2.4) we obtain $T(0,0)$ and $T(0,1)$. By plugging (2.3) in (2.5) we obtain a first order recursive relation for $T(n,1)$,

$$T(n,1) = \frac{1}{\mu}(1 + \frac{\zeta}{\theta}) + T(n-1,1),$$

which yields

$$T(n,i) = (n+1)(1 + \frac{\zeta}{\theta})\frac{1}{\mu} + (1-i)\frac{1}{\theta}. \tag{2.6}$$

A customer joins if

$$F(n,i) > 0 \Leftrightarrow R - CT(n,i) > 0 \Leftrightarrow R - C(n+1)(1+\frac{\zeta}{\theta})\frac{1}{\mu} + (1-i)\frac{C}{\theta} \Leftrightarrow$$

$$n < \frac{R\mu\theta}{C(\theta+\zeta)} - 1 - (i-1)\frac{\mu}{\theta+\zeta}.$$

Therefore, when $i = 1$ a customer joins if she observes less than $\left\lfloor \frac{R\mu\theta}{C(\theta+\zeta)} - 1 \right\rfloor$ customers in the system, is indifferent between entering or balking if she observes $\left\lfloor \frac{R\mu\theta}{C(\theta+\zeta)} - 1 \right\rfloor$ and balks otherwise. In other words, she follows a threshold strategy with threshold $n_e(1) = \left\lfloor \frac{R\mu\theta}{C(\theta+\zeta)} - 1 \right\rfloor$.

Similarly, when $i = 0$ customers follow a threshold strategy with threshold $n_e(0) = \left\lfloor \frac{R\mu\theta}{C(\theta+\zeta)} - \frac{\mu}{\theta+\zeta} \right\rfloor - 1$.

Thus, an arriving customer prefers to enter if $n \leq n_e(i)$, where $(n_e(0), n_e(1))$ is given by (2.1). This strategy is preferable, independently of what the other customers do i.e. it is a weakly dominant strategy.

□

Remark 3. One would expect that the relationship between $n_e(1)$ and $n_e(0)$ depends on the value of ζ and θ, since those parameters affect the time the system is on and off respectively. This may be the case in other models with similar breakdowns (ex. Burnetas and Economou (2007)). In our model, by comparing the two thresholds, we notice that $n_e(0) \leq n_e(1)$ is always true, regardless of the parameters of the system. In other words, customers that observe the server to be down are more reluctant to join the system. This happens because the only difference between the two states of the server is that if a customer enters while the system is down, she will have to wait for the server to be activated and thus incur an extra waiting cost.

Remark 4. We assume that $n_e(1) > 0$ or by (2.1) $R > C\frac{1}{\mu}(1 + \frac{\zeta}{\theta})$, which leads to $R > T(0, 1)$. If this inequality does not hold, then customers will not enter the queue even if they find it empty with an active server since they will incur a negative benefit. Therefore, the system will remain always empty.

2.3 Equilibrium threshold strategies in the almost observable case

We move on to the almost observable case, where arriving customers observe the number of customers in the system but not the state of the server. We will search for an equilibrium strategy within the class of pure threshold strategies. In order to do so, we must first compute the stationary distribution of the system assuming that all customers follow a given pure threshold strategy.

Proposition 2.3.1. *Consider the almost observable M|M|1 queue with breakdowns and repairs where the customers enter the system according to a threshold strategy "While arriving at time t, observe $N(t)$; enter if $N(t) \leq n_e$ and balk otherwise". Then, the stationary distribution $(p(n, i) : (n, i) \in \{0, 1, 2, ..., n_e + 1\} \times \{0, 1\})$ is given as follows:*

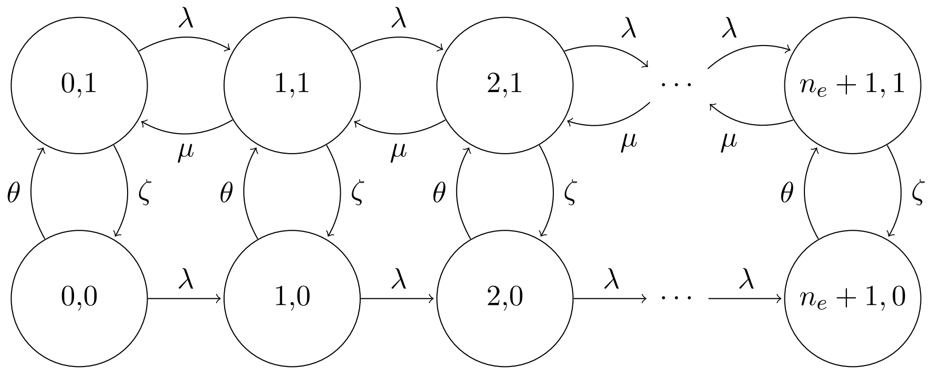


Figure 2.2: Transition diagram of the almost observable model in pure strategies

$$p(n,0) = A(\rho_1^{n+1} - \rho_2^{n+1}), \quad n = 0,1,2,...,n_e \tag{2.7}$$

$$p(n,1) = A(\nu_1\rho_1^{n+1} - \nu_2\rho_2^{n+1}), \quad n = 0,1,2,...,n_e \tag{2.8}$$

$$p(n_e+1,0) = \frac{\lambda A}{\theta}\left(\left(1+\frac{\zeta}{\mu}(1+\nu_1)\right)\rho_1^{n_e+1} - \left(1+\frac{\zeta}{\mu}(1+\nu_2)\right)\rho_2^{n_e+1}\right) \tag{2.9}$$

$$p(n_e+1,1) = \frac{\lambda A}{\mu}\left((1+\nu_1)\rho^{n_e+1} - (1+\nu_2)\rho_2^{n_e+1}\right) \tag{2.10}$$

where A is computed using the normalization equation and

$$\rho_{1,2} = \frac{\lambda}{2\mu(\lambda+\theta)}\left(\mu+\zeta+\lambda+\theta \pm \sqrt{(\mu+\zeta+\lambda+\theta)^2 - 4\mu(\lambda+\theta)}\right) \tag{2.11}$$

$$\nu_i = \frac{(\lambda+\theta)\rho_i - \lambda}{\zeta\rho_i}, \quad i = 1,2. \tag{2.12}$$

Proof. When all customers follow a threshold strategy of the form "While arriving at time t, observe $N(t)$; enter if $N(t) \leq n_e$ and balk otherwise", the transitions of the system $N(t), I(t) : t \geq 0$ are shown in figure 2.2.

The stationary distribution $(p(n,i))$ is obtained using the balance equations:

$$(\lambda+\theta)p(0,0) = \zeta p(0,1) \tag{2.13}$$

$$(\lambda+\theta)p(n,0) = \lambda p(n-1,0) + \zeta p(n,1), \quad n = 1,2,...n_e \tag{2.14}$$

$$\theta p(n_e+1,0) = \lambda p(n_e,0) + \zeta p(n_e+1,1) \tag{2.15}$$

$$\mu p(n+1,1) = \lambda p(n,1) + \lambda p(n,0), \quad n = 0,1,2...,n_e \tag{2.16}$$

Solving (2.14) with respect to $p(n,1)$ and substituting in (2.16) we obtain

$$\mu(\lambda+\theta)p(n+1,0)-\lambda(\mu+\zeta+\lambda+\theta)p(n,0)+\lambda^2 p(n-1,0)=0, \quad n=1,2,...,n_e-1$$

which is a homogeneous second-order difference equation with solution

$$p(n,0)=c_1\rho_1^n+c_2\rho_2^n, n=0,1,2,...,n_e \tag{2.17}$$

where ρ_1, ρ_2, as given by (2.11), are the roots of the corresponding characteristic equation

$$\mu(\lambda+\theta)x^2-\lambda(\mu+\zeta+\lambda+\theta)x+\lambda^2=0$$

and c_1, c_2 are to be determined. We can easily see that $\rho_1 \neq \rho_2$. Plugging (2.17) in (2.14) we obtain

$$p(n,1)=c_1\nu_1\rho_1^n+c_2\nu_2\rho_2^n, \quad n=1,2,...,n_e \tag{2.18}$$

where ν_i are given by (2.12). By (2.13) and (2.17) we obtain

$$p(0,1)=\frac{\lambda+\theta}{\zeta}(c_1+c_2). \tag{2.19}$$

Furthermore, $p(n_e+1,1)$ is computed by inserting $n=n_e$ in (2.16) and considering (2.17) and (2.18). Therefore,

$$p(n_e+1,1)=\frac{\lambda}{\mu}c_1(1+\nu_1)\rho_1^{n_e}+\frac{\lambda}{\mu}c_2(1+\nu_2)\rho_2^{n_e}.$$

Also, $p(n_e+1,0)$ is given by (2.15) and using the computations above we obtain

$$p(n_e+1,0)=\frac{\lambda}{\theta}c_1(1+\frac{\zeta}{\mu}(1+\nu_1)\rho_1^{n_e})+\frac{\lambda}{\theta}c_2(1+\frac{\zeta}{\mu}(1+\nu_2)\rho_2^{n_e})$$

We have expressed all the stationary probabilities in terms of the constants c_1 and c_2. By plugging $n=0$ in (2.16) we obtain $\mu p(1,1)=\lambda p(0,1)+\lambda p(0,0)$. We now plug in the probabilities yielded by (2.18) by setting $n=1$ and (2.17) by setting $n=0$ and using (2.19) we obtain after some algebra that

$$c_2=-\frac{\rho_2}{\rho_1}c_1.$$

Then, the unique unknown constant c_1 is computed using the normalization equation

$$\sum_{i=0}^{1}\sum_{n=0}^{n_e+1}p(n,i)=1$$

as an explicit but involved sum. Letting $A=\frac{c_1}{\rho_1}$ we have $c_1=A\rho_1$ and $c_2=-A\rho_2$ and the stationary probabilities are given from $(2.7-2.10)$. □

We are now in position to find the expected net reward of a customer that observes n customers ahead of her and decides to enter. We have the following.

Proposition 2.3.2. *Consider the almost observable $M|M|1$ queue with breakdowns and repairs where the customers enter to the system according to a threshold strategy "While arriving at time t, observe $N(t)$; enter if $N(t) \leq n_e$ and balk otherwise". The net benefit of a customer that observes n customers and decides to enter is given by*

$$F(n) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) - \frac{C}{\theta}\frac{\sigma^{n+1}-1}{(1+\nu_1)\sigma^{n+1}-(1+\nu_2)}, \quad n = 0, 1, ..., n_e \tag{2.20}$$

$$F(n_e+1) = R - C\frac{n_e+2}{\mu}\left(1+\frac{\zeta}{\theta}\right) - \frac{C}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^{n_e+1}-(\mu+\zeta(1+\nu_2))}{(\mu+(\zeta+\theta)(1+\nu_1))\sigma^{n_e+1}-(\mu+(\zeta+\theta)(1+\nu_2))}, \tag{2.21}$$

where $\sigma = \frac{\rho_1}{\rho_2}$.

Proof. The expected net benefit for a customer that joins the system when she observes n customers is

$$F(n) = R - CT(n), \tag{2.22}$$

where $T(n) = E[S|N^- = n]$ denotes her expected mean sojourn time given that she finds n customers in the system just before her arrival. Conditioning on the state of the server that she finds upon arrival we obtain

$$\begin{aligned} T(n) &= T(n,1)Pr[I^- = 1|N^- = n] + T(n,0)Pr[I^- = 0|N^- = n] \\ &\overset{(2.3)}{=} T(n,1) + \frac{1}{\theta}Pr[I^- = 0|N^- = n] \end{aligned} \tag{2.23}$$

where $T(n,1)$ is given by (2.6) for $i = 1$. The probability $Pr[I^- = 0|N^- = n]$ that a customer finds the server inactive upon her arrival given that she finds n customers in front of her is $\frac{\lambda p(n,0)}{\lambda p(n,1)+\lambda p(n,0)}, n = 0, 1, ..., n_e + 1$. Using the stationary probability obtained in Proposition 2.3.1 we obtain the probabilities

$$Pr[I^- = 0|N^- = n] = \frac{\rho_1^{n+1} - \rho_2^{n+1}}{(1+\nu_1)\rho_1^{n+1} - (1+\nu_2)\rho_2^{n+1}}, \quad n = 0, 1, ..., n_e$$

$$Pr[I^- = 0|N^- = n_e+1] = \frac{(\mu+\zeta(1+\nu_1))\rho_1^{n_e+1} - (\mu+\zeta(1+\nu_2))\rho_2^{n_e+1}}{(\mu+(\zeta+\theta)(1+\nu_1))\rho_1^{n_e+1} - (\mu+(\zeta+\theta)(1+\nu_2))\rho_2^{n_e+1}}$$

Setting $\sigma = \frac{\rho_1}{\rho_2}$ and using (2.6) and the probabilities we obtained above we conclude that

$$T(n) = \frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) + \frac{1}{\theta}\frac{\sigma^{n+1}-1}{(1+\nu_1)\sigma^{n+1}-(1+\nu_2)}, \quad n = 0, 1, ..., n_e \tag{2.24}$$

$$T(n_e+1) = \frac{n_e+2}{\mu}\left(1+\frac{\zeta}{\theta}\right) \tag{2.25}$$
$$\frac{1}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^{n_e+1}-(\mu+\zeta(1+\nu_2))}{(\mu+(\zeta+\theta)(1+\nu_1))\sigma^{n_e+1}-(\mu+(\zeta+\theta)(1+\nu_2))}.$$

Now the substitution of $T(n)$ in (2.22) yields (2.20) and (2.21) $\square$

For excluding the trivial case where a customer does not enter the system even if she finds no customers in front of her we assume that (2.20) is positive for $n = 0$ or $R > CT(0)$ and after some algebra we see that the equivalent to this condition is

$$R > \frac{C}{\mu}\left(1+\frac{\zeta}{\theta}\right) + \frac{C}{\theta}\frac{\zeta}{\lambda+\theta+\zeta} \tag{2.26}$$

which we assume from now on.

We can now describe the equilibrium balking threshold strategies in the almost observable case. We have the following result.

Theorem 2.3.1. *Define the sequences* $(f_1(n) : n = 0, 1, 2, ...)$ *and* $(f_2(n) : n = 0, 1, 2, ...)$ *by*

$$f_1(n) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) - \frac{C}{\theta}\frac{\sigma^{n+1}-1}{(1+\nu_1)\sigma^{n+1}-(1+\nu_2)} \quad n = 0, 1, ... \tag{2.27}$$

$$f_2(n) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) - \frac{C}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^{n}-(\mu+\zeta(1+\nu_2))}{(\mu+(\zeta+\theta)(1+\nu_1))\sigma^{n}-(\mu+(\zeta+\theta)(1+\nu_2))}, \quad n = 0, 1, ... \tag{2.28}$$

Then there exist finite non-negative integers $n_L \leq n_U$ *such that*

$$f_1(0), f_1(1), f_1(2), ..., f_1(n_U) > 0 \ \ \textit{and} \ \ f_1(n_U+1) \leq 0 \tag{2.29}$$

and

$$f_2(n_U+1), f_2(n_U), f_2(n_U-1), ..., f_2(n_L+1) \leq 0 \ \ \textit{and} \ \ f_2(n_L) > 0 \tag{2.30}$$

or

$$f_2(n_U+1), f_2(n_U), f_2(n_U-1), ..., f_2(0) \leq 0. \tag{2.31}$$

In the almost observable M|M|1 queue with breakdowns and repairs the pure threshold strategy of the form "While arriving at time t, observe N(t); enter if N(t)≤ n_e and balk otherwise", for $n_e \in \{n_L, n_{L+1}, ..., n_U\}$, are equilibrium strategies.

Proof. We have that $f_1(0) > 0$ because of (2.26). Moreover, $lim_{n\to\infty} = -\infty$, so if $n_U + 1$ is the subscript of the first non-positive term of the sequence $(f_1(n))$, we have that for the finite number n_U the condition (2.29) holds. Note also that $f_1(n)$ given by (2.27) can be written in the alternative form

$$\begin{aligned} f_1(n) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) \\ - \frac{C}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^n - (\mu+\zeta(1+\nu_2))}{(\mu+(\zeta+\lambda+\theta)(1+\nu_1))\sigma^n - (\mu+(\zeta+\lambda+\theta)(1+\nu_2))}. \end{aligned} \tag{2.32}$$

Let

$$\begin{aligned} f(n,x) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) \\ - \frac{C}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^n - (\mu+\zeta(1+\nu_2))}{(\mu+(\zeta+\lambda x+\theta)(1+\nu_1))\sigma^n - (\mu+(\zeta+\lambda x+\theta)(1+\nu_2))}. \end{aligned} \tag{2.33}$$

By comparing (2.33) with (2.32) and (2.28), we can see that $f_1(n) = f(n,1)$ and $f_2(n) = f(n,0)$. By writing $f(n,x)$ in the alternative form

$$\begin{aligned} f(n,x) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) \\ - \frac{C}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^n - (\mu+\zeta(1+\nu_2))}{\lambda x((1+\nu_1)\sigma^n - (1+\nu_2)) + (\mu+(\zeta+\theta)(1+\nu_1))\sigma^n - (\mu+(\zeta+\theta)(1+\nu_2))}, \end{aligned} \tag{2.34}$$

we can see that $f(n,x)$ is increasing in x. Therefore,

$$f_1(n) > f_2(n), \quad n = 0, 1, 2, ... \tag{2.35}$$

In particular we conclude that $f_2(n_U+1) < f_1(n_U+1) \leq 0$. We begin to go backwards, starting from the subscript $n_U + 1$ towards 0 and we let n_L be the subscript of the first positive term of the sequence $(f_2(n))$. Then we have (2.30). If all terms of $(f_2(n))$ going backwards from $n_u + 1$ towards 0 are non-positive we have (2.31).

Suppose, now, that we have the model where customers follow a pure threshold strategy of the form "While arriving at time t, observe $N(t)$; enter if $N(t) \leq n_e$ and balk otherwise" for some fixed threshold $n_e \in \{n_L, n_L+, ..., n_U\}$. We consider a tagged customer at her arrival instant.

Then, her net benefit if she observes n customers and decides to join is given by $(2.20) - (2.21)$.

If the tagged customer observes $n \leq n_e$ customers, her expected net benefit is $F(n)$ as given by (2.20) which, by (2.32), equals to $f_1(n)$. Therefore, $F(n) = f_1(n) > 0$ for $n = 0, 1, 2..., n_e$ when $n_e \in \{n_L, n_L+1, ..., n_U\}$ because of (2.29) and the customer prefers to join the system.

Likewise, if the tagged customer observes $n = n_e+1$ customers in the system her net benefit $F(n_e + 1) = f_2(n_e + 1) \leq 0$ when $n_e \in \{n_L, n_L + 1, ..., n_U\}$ because of $(2.21), (2.28)$ and (2.30) or (2.31) and the customer prefers to balk.

Therefore, we conclude that the strategy "While arriving at time t, observe N(t); enter if N(t)$\leq n_e$ and balk otherwise" for any $n_e \in \{n_L, n_L+1, ..., n_U\}$ is best response against itself, i.e. an equilibrium. $\square$

Theorem 2.3.1 provides an algorithm in order to identify the equilibrium strategies of the almost observable model. One has to start computing $(f_1(n))$ up to the first negative term. This "forward" procedure yields the highest equilibrium threshold n_U. Then, one has to start computing $f_2(n)$ starting from $f_2(n_U+1)$ and going towards 0 till the first positive term. The "backward" procedure yields the lowest equilibrium threshold n_L.

Remark 5. The equilibrium strategy "While arriving at time t, observe $N(t)$; enter if $N(t) \leq n_e$ and balk otherwise" is the pure threshold strategy with threshold $n_e + 1$. In other words, it can be represented by the vector $(1, 1, 1, ...1, 0, ...)$ which consists of $(n_e + 1)$ ones.

Follow the crowd

In the present model the customers adopt a **"follow the crowd"**(FTC) behaviour where customers tend to follow the behaviour of other customers. FTC behaviour is expressed when the best response of a customer against strategy x of other customers is increasing in x. Let $F_{n_e}(n)$ be the expected net reward of a tagged customer that observes a queue length of n customers and enters the system, while all other customers follow an n_e threshold strategy. The best response against this strategy, $BR(n_e)$, is defined by $BR(n_e) = max\{n : F_{n_e}(n) > 0, n = 0, 1, 2..., n_e + 1\}$. Note that $BR(n_e) \leq n_e + 1$. Suppose that all other customers adopt a threshold strategy with threshold n_e+1. Then, the best response of the tagged customer is defined by $BR(n_e+1) = max\{n : F_{n_e+1}(n) > 0, n = 0, 1, 2, ..., n_e+2\}$, where $F_{n_e+1}(n)$ is the expected net benefit of a customer that observes n customers and joins the system when all others follow an $n_e + 1$ threshold strategy which is (by $(2.20) - (2.30)$)

$$F_{n_e+1}(n) = R - C\frac{n+1}{\mu}\left(1+\frac{\zeta}{\theta}\right) - \frac{C}{\theta}\frac{\sigma^{n+1}-1}{(1+\nu_1)\sigma^{n+1}-(1+\nu_2)}, \quad n = 0, 1, ..., n_e+1,$$

$$F_{n_e+1}(n_e+2) = R - C\frac{n_e+2}{\mu}\left(1+\frac{\zeta}{\theta}\right)$$
$$- \frac{C}{\theta}\frac{(\mu+\zeta(1+\nu_1))\sigma^{n_e+1}-(\mu+\zeta(1+\nu_2))}{(\mu+(\zeta+\theta)(1+\nu_1))\sigma^{n_e+1}-(\mu+(\zeta+\theta)(1+\nu_2))}$$

We can see that $F_{n_e+1}(n)$ is equal to $F_{n_e}(n)$, for $n = 0, 1, ..., n_e$. Therefore, considering the definition of $BR(n_e)$, $F_{n_e+1}(n) = F_{n_e}(n) > 0, n = 0, 1, ...BR(n_e)$. If $BR(n_e) = n_e + 1$ then $F_{n_e+1}(n_e + 1) = f_1(n_e + 1) > f_2(n_e + 1) = F_{n_e}(n_e + 1) = F_{n_e}(BR(n_e)) > 0$. The two equalities are given by (2.27),(2.28) respectively and the inequality is true because $f_1(n) > f_2(n)$ by (2.35). From this analysis, we conclude that $BR(n_e+1) \geq BR(n_e)$ which means that the threshold that is a best response of a tagged customer when all others follow a threshold strategy $n_e + 1$ is greater than the threshold that is a best response when all others adopt the threshold n_e. This proves that the behaviour of the customers is FTC.

Remark 6. We can see that in this model multiple equilibria are possible since any threshold $n_e \in \{n_L, n_L+1, ..., n_U\}$ is an equilibrium strategy. This is a common property of FTC behaviour.

Chapter 3

Equilibrium balking strategies for a clearing queueing system in alternating environment

In this chapter, we consider a Markovian clearing queueing system, where the customers are accumulated according to a Poisson process and the server removes all present customers at the completion epochs of exponential service cycles. This system may represent the visits of a transportation facility with unlimited capacity at a certain station. The system evolves in an alternating environment that influences the arrival and service rates. We assume that the arriving customers decide whether to join or balk, based on a natural linear reward-cost structure. We will study the balking behaviour of the customers and derive the corresponding Nash equilibrium strategies under various information levels.

Queueing systems with batch services are often used to represent the visits of a transportation facility at a certain station. This allows for the quantification of the congestion of the station and can be used to take control measures (e.g. the changing the frequency of the visits), so that the quality of service is kept within acceptable limits. The capacity of the system is usually assumed unlimited. This is justified, because in most applications the capacity of the facility is chosen large enough, so that the probability that some waiting customers cannot be accommodated is negligibly small. Moreover, the waiting customers that cannot be served at a visit of the facility are not in general willing to wait for its next visit and abandon the system. Therefore, it is realistic to assume that all present customers are removed at the visit points of the facility. Such stochastic systems are referred to as **stochastic clearing systems**.

3.1 The model

We consider a transportation station with infinite waiting space that operates in an alternative environment. The environment is specified by a 2-state continuous-time Markov chain $\{I(t)\}$, with state space $S^I = \{1, 2\}$ and transition rates $q_{ii'}$, for $i \neq i'$. Whenever the environment is at state i, customers arrive according to a Poisson process at rate λ_i, whereas the transportation facility visits the station according to a Poisson process at rate μ_i. The two Poisson processes are assumed independent. At the visit epochs of the transportation facility all customers are served simultaneously and removed from the station. Therefore, we have a stochastic clearing system in an alternating environment.

We represent the state of the station by a pair $(N(t), I(t))$, where $N(t)$ records the number of customers at the station and $I(t)$ denotes the environmental state. The stochastic process $\{N(t), I(t)\}$ is a continuous time Markov chain with state space $S^{N,I} = \{(n, i) : n \geq 0, i = 1, 2\}$ and its non-zero transition rates given by

$$q_{(n,i)(n+1,i)} = \lambda_i, \quad n \geq 0,\ i = 1, 2, \tag{3.1}$$

$$q_{(n,i)(0,i)} = \mu_i, \quad n \geq 1,\ i = 1, 2, \tag{3.2}$$

$$q_{(n,1)(n,2)} = q_{12}, \quad n \geq 0, \tag{3.3}$$

$$q_{(n,2)(n,1)} = q_{21}, \quad n \geq 0. \tag{3.4}$$

We define $\rho_i = \frac{\lambda_i}{\mu_i}, i = 1, 2$. The value of ρ_i can be thought as the measure of congestion of the system under the environmental rate i, as it expresses the mean number of customers accumulated between two successive visits of the transportation facility (given that the environment remains continuously in state i).

We are interested in the behaviour of customers that arrive on the station and decide whether to join or balk. We assume that joining customers gain reward of R units upon completion of their service and accumulate waiting costs at the rate of C units per time unit they remain in the system. We also assume that customers are risk neutral and they try to maximize their net benefit. Finally, the decision of joining or balking is irrevocable in the sense that joining customers are not allowed to renege and balking customers are not allowed to re-enter the system.

Since all customers are assumed indistinguishable, we can consider the situation as a symmetric game among them. Denote the common set of available strategies and the payoff function as S and F respectively. Let $F(s_i, s_{-i})$ be the payoff of customer i who follows strategy s_i when all others follow strategy s_{-i}.

In these chapter we will obtain equilibrium strategies for joining/balking. We distinguish between four cases depending on the information available to customers upon their arrival instants, before the decision is made:

- **Fully unobservable case** : Customers do not observe $N(t)$ or $I(t)$ before joining.
- **Almost unobservable case** : Customers do not observe $N(t)$, but observe $I(t)$.
- **Fully observable case** : Customers observe $N(t)$ and $I(t)$.
- **Almost observable case** : Customers observe $N(t)$, but not $I(t)$.

From a methodological point of view the first three cases are similar so we will start by examining them first. The almost observable case which is the most interesting and methodologically demanding will be examined in section 3.3.

3.2 The unobservable and the fully observable case

Let T_i be the time until the next arrival of the transportation facility, given that the environment is at state i. A moment of reflection shows that T_i is independent from the number of customers in the system, because of the mechanism of the total removals of customers at the visits of the facility and the memoryless property of the exponential distribution.

By a first-step analysis argument, conditioning on the next transition of the Markov chain $(N(t), I(t))$, which is either a visit of the facility or a change in the environment, we obtain the following equations

$$\begin{aligned} E(T_1) &= [\text{time until next transition happens}] \\ &\quad + \Pr[\text{visit of the facility}] \cdot 0 + \Pr[\text{change in environment}] \cdot E(T_2) \\ &= \frac{1}{\mu_1 + q_{12}} + \frac{\mu_2}{\mu_2 + q_{12}} \cdot 0 + \frac{q_{12}}{\mu_1 + q_{12}}, \end{aligned} \tag{3.5}$$

$$E(T_2) = \frac{1}{\mu_2 + q_{21}} + \frac{\mu_2}{\mu_2 + q_{21}} \cdot 0 + \frac{q_{21}}{\mu_2 + q_{21}} \cdot E(T_1). \tag{3.6}$$

Solving (3.5) – (3.6) yields

$$E(T_1) = \frac{\mu_2 + q_{12} + q_{21}}{\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12}} \tag{3.7}$$

$$E(T_2) = \frac{\mu_1 + q_{12} + q_{21}}{\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12}} \tag{3.8}$$

Value of $\frac{R}{C}$	$\frac{R}{C} < V_{fu}$	$\frac{R}{C} = V_{fu}$	$\frac{R}{C} > V_{fu}$
Dominant strategy(ies)	0	$q \in [0, 1]$	1

Table 3.1: Dominant strategies in the fully unobservable case

3.2.1 The fully unobservable case

In this section we examine the fully unobservable case, where customers neither observe the number of customers in the system $N(t)$ nor the the environment of the facility. A balking strategy in the fully observable case is specified by a single joining probability q. The case $q = 0$ corresponds to the pure strategy "always balk" whereas the $q = 1$ strategy corresponds to the "always join" strategy. Any value of $q \in (0, 1)$ corresponds to the mixed strategy "to join with probability q or balk with probability $1-q$". We have the following theorem.

Theorem 3.2.1. *In the fully unobservable model of the stochastic clearing system in alternating environment, there always exists a dominant strategy. The dominant strategies depend on the relative value of the ratio $\frac{R}{C}$ with respect to the critical value*

$$V_{fu} = \frac{\lambda_1 q_{12}\mu_2 + \lambda_2 q_{12}\mu_1}{(\lambda_1 q_{21} + \lambda_2 q_{12})(\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12})} + \frac{q_{21} + q_{12}}{\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12}} \tag{3.9}$$

We have three cases that are summarized in Table 3.1.

Proof. Suppose that customers follow a certain strategy and consider a tagged customer upon arrival. The probability that she finds the system at environment i is

$$p_{I-arrival}(i) = \frac{\lambda_i p_I(i)}{\lambda_1 p_I(1) + \lambda_2 p_I(2)}, \tag{3.10}$$

where $(p_I(i), i = 1, 2)$ is the stationary distribution of the environment which is given by

$$p_I(1) = \frac{q_{21}}{q_{12} + q_{21}} \tag{3.11}$$

$$p_I(2) = \frac{q_{12}}{q_{12} + q_{21}}. \tag{3.12}$$

The expected net benefit of the tagged customer if she decides to join is given by

$$F_{fu} = \sum_{i=1}^{2} p_{I-arrival}(i)(R - CE(T_i)), \tag{3.13}$$

where $E(T_i)$ are given by (3.7), (3.8). Plugging (3.11) and (3.12) in (3.10) and substituting in (3.13) yields

$$\begin{aligned} F_{fu} &= R - C\frac{\lambda_1 q_{21} E[T_1] + \lambda_2 q_{12} E[T_2]}{\lambda_1 q_{21} + \lambda_2 q_{12}} \\ &= R - C\left(\frac{\lambda_1 q_{21}\mu_2 + \lambda_2 q_{12}\mu_1}{(\lambda_1 q_{21} + \lambda_2 q_{12})(\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12})} + \frac{q_{21} + q_{12}}{\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12}}\right). \end{aligned} \tag{3.14}$$

The tagged customer prefers to join when $F_{fu} > 0$, is indifferent when $F_{fu} = 0$ and prefers to balk otherwise. Solving with respect to $\frac{R}{C}$, we obtain the three cases in table 3.1.

□

3.2.2 The almost unobservable case

We now proceed to the almost unobservable case, where the customers observe the environment $I(t)$ but not the number of customers in the system $N(t)$. A general balking strategy in the almost unobservable case is specified by an ordered pair of joining probabilities (q_1, q_2), where q_i is the joining of a customer if the environment state upon arrival is $i, i = 1, 2$. We have the following Theorem.

Theorem 3.2.2. *In the almost unobservable model of the stochastic clearing system in alternating environment, there always exists a dominant strategy. The dominant strategies depend on the relative value of the ratio $\frac{R}{C}$ with respect to the critical values*

$$V_{au}^{min} = \frac{min(\mu_1, \mu_2) + q_{21} + q_{12}}{\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12}}, \quad V_{au}^{max} = \frac{max(\mu_1, \mu_2) + q_{21} + q_{12}}{\mu_1\mu_2 + \mu_1 q_{21} + \mu_2 q_{12}}. \tag{3.15}$$

If $\mu_1 \neq \mu_2$, then $V_{au}^{min} < V_{au}^{max}$ and we have the five cases that are summarized in Table 3.2.
If $\mu_1 = \mu_2$, then $V_{au}^{min} = V_{au}^{max}$. Let V_{au} denote the common value of V_{au}^{min} and V_{au}^{max}. We have the three cases summarized in Table 3.3

Proof. Consider a tagged customer that observes the state of the environment upon arrival. If she decides to join given that she finds the environment at state i, then her expected benefit will be

$$F_{au}(i) = R - CE(T_i), \tag{3.16}$$

where $E(T_i)$ are given by (3.5) − (3.6). The customer decides to join if $F_{au}(i) > 0$, which is written equivalently as $\frac{R}{C} > E(T_i)$. Similarly, she prefers

Value of $\frac{R}{C}$	$\frac{R}{C} < V_{au}^{min}$	$\frac{R}{C} = V_{au}^{min}$	$V_{au}^{min} < \frac{R}{C} < V_{au}^{max}$	$\frac{R}{C} = V_{au}^{max}$	$\frac{R}{C} > V_{au}^{max}$
Dominant strategy(ies) when $\mu_1 < \mu_2$	$(0,0)$	$(0,q_2)$, $q_2 \in [0,1]$	$(0,1)$	$(q_1,1)$, $q_1 \in [0,1]$	$(1,1)$
Dominant strategy(ies) when $\mu_1 > \mu_2$	$(0,0)$	$(q_1,0)$, $q_1 \in [0,1]$	$(1,0)$	$(1,q_2)$, $q_2 \in [0,1]$	$(1,1)$

Table 3.2: Dominant strategies in the almost unobservable case, when $\mu_1 \neq \mu_2$

Value of $\frac{R}{C}$	$\frac{R}{C} < V_{au}$	$\frac{R}{C} = V_{au}$	$\frac{R}{C} > V_{au}$
Dominant strategy(ies), when $\mu_1 = \mu_2$	$(0,0)$	(q_1,q_2), $q_1, q_2 \in [0,1]$	$(1,1)$

Table 3.3: Dominant strategies in the almost unobservable case, when $\mu_1 = \mu_2$

to balk when $\frac{R}{C} > E(T_i)$ and is indifferent between balking and joining when $\frac{R}{C} = E(T_i)$. By considering the various possible cases with regard to the order of the quantities $\frac{R}{C}$, $E(T_1)$ and $E(T_2)$, we obtain the corresponding cases in the statement of Theorem 3.2.2. Note that the strategies prescribed in Theorem 3.2.2 are dominant, since they do not depend on what other customers do, i.e. they are the best responses against any strategy of others. $\square$

3.2.3 The fully observable case

In this case the customers observe both the environment $I(t)$ and the number of waiting customers $N(t)$ upon arrival. We can see that the mean sojourn time of a customer depends on the state of the environment, but it does not depend on the number of customers that are in the system, since all customers are served simultaneously. Therefore, if a customer arrives and observes the environment at state i while n customers are in the system, the information about the number of customers is consider superfluous and is discarded which makes this case identical to the almost unobservable case. We conclude that the dominant balking strategies are the ones given in Theorem 3.2.2.

3.3 The almost observable case

In this chapter we consider the almost observable case where customers observe the number of customers $N(t)$ but not the state of the environment $I(t)$. Thus, a general balking strategy can be specified by a vector of probabilities $(\theta_0, \theta_1, \theta_2, ...)$, where θ_i is the probability a customer joins when she observes i customers in the system upon arrival (excluding herself).

Suppose that a tagged customer observes n customers upon arrival. Although the number of customers in the system does not influence her sojourn time, the information of the number of customers influences the probabilities that the environment is at state 1 or 2. We expect intuitively that there are two cases: Either the "slow service" environmental state with $\mu_i = min(\mu_1, \mu_2)$ coincides to the "more congested" environmental state i' with $\rho_{i'} = max(\rho_1, \rho_2)$, or it coincides with the "less congested" environmental state i'' with $\rho_{i''} = min(\rho_1, \rho_2)$. In the former case, a large number of customers signals that the environment is probably in the "slow service" state and an arriving customer is less inclined to join. Thus, we assume that the customer will benefit from joining the system if the number of customers n is below a certain threshold, i.e. she will adopt the a threshold strategy. On the contrary, in the latter case, the situation is reversed. Now, greater number of customers in the system means a higher probability that the environment is in the "fast service" state. Therefore, we expect that a tagged customer will benefit from joining the system, if the number of customers n exceeds a certain threshold i.e. she will adopt a so called reverse-threshold strategy. Following this reasoning, we will limit our search for equilibrium strategies within the class of threshold and reverse-threshold strategies. As we will see, this family is rich enough to ensure the existence of an equilibrium strategy for any values of the underlying parameters of the model.

Definition 3.3.1. A balking strategy $(\theta_0, \theta_1, \theta_2, ...)$, where θ_i is the joining probability of a customer that sees i customers in the system upon arrival (excluding herself) is said to be a mixed threshold strategy, if there exist $n_0 \in \{0, 1, ...\}$ and $\theta \in [0, 1]$ such that $\theta_i = 1$ for $i < n_0$, $\theta_{n_0} = \theta$ and $\theta_i = 0$ for $i > n_0$. Such strategy will be referred to as the (n_0, θ)-mixed threshold strategy (symbolically the $\lceil n_0, \theta \rceil$ strategy) and it prescribes to join if you see less than n_0 customers, to join with probability θ if you see exactly n_0 customers and to balk if you see more than n_0 customers.

An $(n_0, 0)$-mixed threshold strategy which prescribes to join if you see less than n_0 customers and and to balk otherwise will be referred to as the n_0-pure threshold strategy (symbolically the $\lceil n_0 \rceil$ strategy).

A balking strategy $(\theta_0, \theta_1, \theta_2, ...)$ is said to be a mixed reverse-threshold strategy, if there exist $n_0 \in \{0, 1, ...\}$ and $\theta \in [0, 1]$ such that $\theta_i = 0$ for $i < n_0$, $\theta_{n_0} = \theta$ and $\theta_i = 1$ for $i > n_0$. Such strategy will be referred to as the (n_0, θ)-mixed reverse-threshold strategy (symbolically the $\lfloor n_0, \theta \rfloor$ strategy) and it prescribes to balk if you see less than n_0 customers, to join with probability θ if you see n_0 customers and to balk if you see more than n_0 customers.

An $(n_0, 1)$-mixed reverse-threshold strategy which prescribes to join if you see at least n_0 customers and and to balk otherwise will be referred to as the n_0-pure reverse-threshold strategy (symbolically the $\lfloor n_0 \rfloor$ strategy).

The strategy which prescribes to join in any case is considered to be both a threshold and a reverse threshold strategy (symbolically the $\lceil \infty \rceil$or $\lfloor 0 \rfloor$ strategy). The same is true for the strategy which prescribes to balk in any case (symbolically $\lceil 0 \rceil$ or $\lfloor \infty \rfloor$ strategy).

3.3.1 Stationary distributions

In this subsection, we determine the stationary distributions of the system, when the customers follow any given strategy from the ones that have been described in Definition 3.3.1. We will first determine the stationary distribution of the original system when all customers join. The result is reported in the following Proposition 3.3.1.

Proposition 3.3.1. *Consider the stochastic clearing system in alternating environment, where all customers join. The stationary distribution* $(p(n,i))$ *is given by the formulas*

$$p(n,1) = A_1 \left(\frac{1}{1-z_1} \right)^n + B_1 \left(\frac{1}{1-z_2} \right), \quad n \geq 0. \tag{3.17}$$

$$p(n,2) = A_2 \left(\frac{1}{1-z_1} \right)^n + B_2 \left(\frac{1}{1-z_2} \right), \quad n \geq 0. \tag{3.18}$$

$$A_1 = \frac{(\mu_1 \lambda_2 z_1 + \mu_1 \mu_2 + \mu_2 q_{12} + \mu_1 q_{21}) p_I(1)}{\sqrt{\Delta}(1-z_1)}, \tag{3.19}$$

$$B_1 = -\frac{(\mu_1 \lambda_2 z_2 + \mu_1 \mu_2 + \mu_2 q_{12} + \mu_1 q_{21}) p_I(1)}{\sqrt{\Delta}(1-z_2)}, \tag{3.20}$$

$$A_2 = \frac{(\mu_2 \lambda_1 z_1 + \mu_1 \mu_2 + \mu_2 q_{12} + \mu_1 q_{21}) p_I(2)}{\sqrt{\Delta}(1-z_1)}, \tag{3.21}$$

$$B_2 = -\frac{(\mu_2 \lambda_1 z_2 + \mu_1 \mu_2 + \mu_2 q_{12} + \mu_1 q_{21}) p_I(2)}{\sqrt{\Delta}(1-z_2)}, \tag{3.22}$$

$$\Delta = [\lambda_2(\mu_1 + q_{12}) - \lambda_1(\mu_2 + q_{21})]^2 + 4\lambda_1 \lambda_2 q_{12} q_{21}, \tag{3.23}$$

$$z_{1,2} = \frac{-\lambda_1(\mu_2 + q_{21}) - \lambda_2(\mu_1 + q_{12}) \pm \sqrt{\Delta}}{2\lambda_1 \lambda_2} \tag{3.24}$$

and $p_I(1), p_I(2)$ *are the stationary probabilities of* $I(t)$ *given from* (3.11) – (3.12).

Proof. For the stationary analysis, note that the state of the system is described by a continuous Markov chain with state space $S^{N,I} = \{(n,i) : n \geq 0, i = 1,2\}$ with its non-zero transition rates given by (3.1)-(3.4). The corresponding stationary distribution $(p(n,i) : (n,i) \in S^{N,I})$ is obtained as the unique positive normalized solution of the following system of balance equations:

$$(\lambda_1 + \mu_1 + q_{12})p(0,1) = q_{21}p(0,2) + \sum_{n=0}^{\infty} \mu_1 p(n,1), \tag{3.25}$$

$$(\lambda_1 + \mu_1 + q_{12})p(n,1) = q_{21}p(n,2) + \lambda_1 p(n-1,1), n \geq 1, \tag{3.26}$$

$$(\lambda_2 + \mu_2 + q_{21})p(0,2) = q_{12}p(0,1) + \sum_{n=0}^{\infty} \mu_2 p(n,2), \tag{3.27}$$

$$(\lambda_2 + \mu_2 + q_{21})p(n,2) = q_{12}p(n,1) + \lambda_2 p(n-1,2), n \geq 1, \tag{3.28}$$

where we have included in (3.25)-(3.27) the pseudo-transitions from $(0,i)$ to $(0,i)$, $i = 1,2$, with rate μ_i, that correspond to visits of the facility at an empty system. Note also that the underlying Markov chain is always positive recurrent as the stochastic clearing mechanism ensures that starting from state $(0,1)$, the process will visit it again with probability 1 and the corresponding time is finite.

We define the partial stationary probability generating functions of the system as

$$G_i(z) = \sum_{n=0}^{\infty} p(n,i)z^n, |z| \leq 1, i = 1,2. \tag{3.29}$$

Then we have $G_1(1) = p_I(1), G_2(1) = p_I(2)$ with $p_I(1), p_I(2)$ given from (3.11)-(3.12). Summing equation (3.25) and equations (3.26) multiplied by $z^n, n \geq 1$, yields after some straightforward algebra

$$[\lambda_1(1-z) + \mu_1 + q_{12}]G_1(z) - q_{21}G_2(z) = \mu_1 p_I(1). \tag{3.30}$$

Similarly, equations (3.27)-(3.28), yield

$$-q_{12}G_1(z) + [\lambda_2(1-z) + \mu_2 + q_{21}]G_2(z) = \mu_2 p_I(2). \tag{3.31}$$

Solving the system of equations (3.30)-(3.31) with respect to $G_1(z)$ and $G_2(z)$ yields

$$G_1(z) = \frac{p_I(1)\{q_{12}\mu_2 + \mu_1[\lambda_2(1-z) + \mu_2 + q_{21}]\}}{[\lambda_1(1-z) + \mu_1 + q_{12}][\lambda_2(1-z) + \mu_2 + q_{21}] - q_{21}q_{12}} \tag{3.32}$$

$$G_2(z) = \frac{p_I(2)\{q_{21}\mu_1 + \mu_2[\lambda_1(1-z) + \mu_1 + q_{12}]\}}{[\lambda_1(1-z) + \mu_1 + q_{12}][\lambda_2(1-z) + \mu_2 + q_{21}] - q_{21}q_{12}} \tag{3.33}$$

Let $g(z)$ be the common denominator of $G_1(z)$ and $G_2(z)$ in (3.32)-(3.33) i.e. $g(z)$ is given as

$$
\begin{aligned}
g(z) &= [\lambda_1(1-z)+\mu_1+q_{12}][\lambda_2(1-z)+\mu_2+q_{21}]-q_{21}q_{12} \\
&= \lambda_1\lambda_2(1-z)^2+(\mu_1\lambda_2+q_{12}\lambda_2+\mu_2\lambda_1+q_{21}\lambda_1)(1-z) \\
&+\mu_1\mu_2+\mu_1q_{21}+\mu_2q_{12}.
\end{aligned} \tag{3.34}
$$

We can factorize $g(z)$ in the form

$$
g(z)=\lambda_1\lambda_2(1-z_1)\left[1-\frac{z}{1-z_1}\right](1-z_2)\left[1-\frac{z}{1-z_2}\right] \tag{3.35}
$$

where Δ and z_1, z_2 are given by (3.23)-(3.24), Note, now, that $G_1(z)$ is a rational function of z with a first degree numerator and a second degree denominator $g(z)$. By using partial fraction expansion we have that

$$
G_1(z)=\frac{A_1}{1-\frac{z}{1-z_1}}+\frac{B_1}{1-\frac{z}{1-z_2}}, \tag{3.36}
$$

with A_1 and B_1 given by (3.19)-(3.20). Expanding the powers of z yields

$$
G_1(z)=\sum_{n=0}^{\infty}\left[A_1\left(\frac{1}{1-z_1}\right)^n+B_1\left(\frac{1}{1-z_2}\right)^n\right]z^n \tag{3.37}
$$

and we deduce (3.17). Similarly, $G_2(z)$ is written as

$$
G_2(z)=\sum_{n=0}^{\infty}\left[A_2\left(\frac{1}{1-z_1}\right)^n+B_2\left(\frac{1}{1-z_2}\right)^n\right]z^n \tag{3.38}
$$

with A_2 and B_2 given by (3.21)-(3.22) and we deduce (3.18).

□

We will now deduce the stationary distribution of the system when customers follow a mixed threshold strategy. We have the following Proposition 3.3.2.

Proposition 3.3.2. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers join the system according to (n_0, θ)-mixed threshold strategy. The corresponding stationary distribution $(p_{ao}(n, i; \lceil n_0, \theta \rceil))$ is given by the formulas*

$$p_{ao}(n,i;\lceil n_0,\theta\rceil) = p(n,i), \quad 0 \le n \le n_0 - 1,\ i = 1,2, \tag{3.39}$$

$$p_{ao}(n_0,i;\lceil n_0,\theta\rceil) = \sum_{n=n_0}^{\infty} (1-\theta)^{n-n_0} p(n,i), \quad e = 1,2, \tag{3.40}$$

$$p_{ao}(n_0+1,i;\lceil n_0,\theta\rceil) = \sum_{n=n_0+1}^{\infty} [1-(1-\theta)^{n-n_0}] p(n,i), \quad e = 1,2, \tag{3.41}$$

$$p_{ao}(n,i;\lceil n_0,\theta\rceil) = 0, \quad n \ge n_0 + 2,\ i = 1,2, \tag{3.42}$$

where $p(n,i)$ are given by (3.17)-(3.18)

Proof. We assume that the customers follow the (n_0,θ)-mixed threshold strategy. Then the evolution of the system can be described by a Markov chain which is absorbed with probability 1 in the positive recurrent closed class of stated $S_{ao}^{N,I}(\lceil n_0,\theta\rceil) = \{(n,i) : (0 \le n \le n_0+1, i = 1,2)\}$. For the sake of brevity, we suppress the notation regarding $\lceil n_0,\theta\rceil$ in the rest of the proof. Thus, we refer to the corresponding stationary probabilities $p_{ao}(n,i;\lceil n_0,\theta\rceil)$ by $p_{ao}(n,i)$.
Since the Markov chain is finally absorbed in $S_{ao}^{N,I}(\lceil n_0,\theta\rceil)$ we obtain immediately (3.42). The vector of the stationary probabilities $(p_{ao}(n,i) : (n,i) \in S_{ao}^{N,I}(\lceil n_0,\theta\rceil))$ is obtained as the unique positive normalized solution of the system of balance equations

$$(\lambda_1 + \mu_1 + q_{12})p_{ao}(0,1) = q_{21}p_{ao}(0,2) + \sum_{n=0}^{n_0+1} \mu_1 p_{ao}(n,1), \tag{3.43}$$

$$(\lambda_1 + \mu_1 + q_{12})p_{ao}(n,1) = q_{21}p_{ao}(n,2) + \lambda_1 p_{ao}(n-1,1), 1 \le n \le n_0 - 1, \tag{3.44}$$

$$(\lambda_1\theta + \mu_1 + q_{12})p_{ao}(n_0,1) = q_{21}p_{ao}(n_0,2) + \lambda_1 p_{ao}(n_0-1,1), \tag{3.45}$$

$$(\mu_1 + q_{12})p_{ao}(n_0+1,1) = q_{21}p_{ao}(n_0+1,2) + \lambda_1\theta p_{ao}(n_0,1), \tag{3.46}$$

$$(\lambda_2 + \mu_2 + q_{21})p_{ao}(0,2) = q_{12}p_{ao}(0,1) + \sum_{n=0}^{n_0+1} \mu_2 p_{ao}(n,2), \tag{3.47}$$

$$(\lambda_2 + \mu_2 + q_{21})p_{ao}(n,2) = q_{12}p_{ao}(n,1) + \lambda_2 p_{ao}(n-1,2), 1 \le n \le n_0 - 1, \tag{3.48}$$

$$(\lambda_2\theta + \mu_2 + q_{21})p_{ao}(n_0,2) = q_{12}p_{ao}(n_0,1) + \lambda_2 p_{ao}(n_0-1,2), \tag{3.49}$$

$$(\mu_2 + q_{21})p_{ao}(n_0+1,2) = q_{12}p_{ao}(n_0+1,1) + \lambda_2\theta p_{ao}(n_0,2), \tag{3.50}$$

where we have included in (3.43) and (3.47) the pseudo transition from $(0,i)$ to $(0,i), i = 1,2$, with rate μ_i, that correspond to visits of the facility at an empty system.

For deducing the formulas (3.39)-(3.42) for the stationary probabilities, we may again follow the standard probability generating function approach as in the proof of Proposition 3.3.1.

□

We can now conclude the following corollaries.

Corollary 3.3.0.1. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers join according to the n_0-pure threshold strategy. The corresponding stationary distribution $(p_{ao}(n,i;\lceil n_0 \rceil))$ is given by the formulas*

$$p_{ao}(n,i;\lceil n_0 \rceil) = p(n,i), \quad 0 \leq n \leq n_0 - 1, \; i = 1,2, \tag{3.51}$$

$$p_{ao}(n,i;\lceil n_0 \rceil) = \sum_{n=n_0}^{\infty} p(n,i), \quad i = 1,2, \tag{3.52}$$

$$p_{ao}(n,i;\lceil n_0 \rceil) = 0, \quad n \geq n_0 + 1, \; i = 1,2, \tag{3.53}$$

where $p(n,i)$ are given by (3.17) − (3.18).

Corollary 3.3.0.2. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers always balk. The corresponding stationary distribution $(p_{ao}(n,i;\lceil 0 \rceil))$ is given by the formulas*

$$p_{ao}(0,i;\lceil 0 \rceil) = p_I(i), \quad i = 1,2, \tag{3.54}$$

$$p_{ao}(n,i;\lceil 0 \rceil) = 0, \quad n \geq 1, \; i = 1,2, \tag{3.55}$$

where $p_I(i), i = 1,2$ are given by (3.11) − (3.12).

We will now deduce the stationary distribution of the system when customers follow an (n_0, θ)-mixed reverse-threshold strategy.

Remark 7. Under an (n_0, θ)-mixed reverse-threshold strategy with $n_0 \geq 1$, we have that the customers balk when they arrive at an empty system. Thus, we have the stationary distribution of Corollary 3.3.0.2.

Next, we will examine the case where customers follow a $(0, \theta)$-mixed reverse-threshold strategy.

Proposition 3.3.3. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers join the system according to a $(0, \theta)$-mixed reverse-threshold strategy. For $\theta = 0$, the stationary distribution $(p_{ao}(n,i;\lfloor 0,0 \rfloor))$ is given by the formulas*

$$p_{ao}(0,i;\lfloor 0,0 \rfloor) = p_I(i), \quad i = 1,2, \tag{3.56}$$
$$p_{ao}(n,i;\lfloor 0,0 \rfloor) = 0, \quad n \geq 1, \; i = 1,2, \tag{3.57}$$

where $p_I(i), i = 1,2$ *are given by* $(3.11) - (3.12)$.

For $\theta \in (0,1)$*, the stationary distribution* $(p_{ao}(n,i;\lfloor 0,\theta \rfloor))$ *is given by the formulas*

$$p_{ao}(0,i;\lceil 0,\theta \rceil) = \sum_{n=0}^{\infty}(1-\theta)^n p(n,i), \quad i = 1,2, \tag{3.58}$$
$$p_{ao}(n,i;\lceil 0,\theta \rceil) = \theta \sum_{k=n}^{\infty}(1-\theta)^{k-n} p(k,i), \quad n \geq 1, \; i = 1,2, \tag{3.59}$$

where $p(n,i)$ *are given by* $(3.17) - (3.18)$

For $\theta = 1$*, the stationary distribution* $(p_{ao}(n,i;\lfloor 0,1 \rfloor))$ *is given by the formula*

$$p_{ao}(n,i;\lfloor 0,1 \rfloor) = p(n,i), \quad n \geq 0, i = 1,2, \tag{3.60}$$

where $p(n,i)$ *are given by* $(3.17) - (3.18)$.

The proof of Proposition 3.3.3 for $\theta = 0$ is immediate, as in this case the customers balk whenever they arrive at an empty system. Therefore, the continuous Markov chain is absorbed in the subset $\{(0,1),(0,2)\}$ of the state space and the stationary distribution is the one given in (3.54)-(3.55). In case $\theta = 1$, the customers always join and we apply Proposition 3.3.1. Thus, the only interesting case is for $\theta \in (0,1)$. Then, the proof of Proposition 3.3.3 follows the proofs of Proposition 3.3.1 and 3.3.2.

3.3.2 Expected net benefit functions

Based on the results of the previous section, we can now derive the expected net benefit of a tagged customers if she decides to join the system after observing n customers upon arrival. Thus, we have various cases, according to whether they follow a threshold or a reverse-threshold strategy. We have the following Propositions 3.3.4-3.3.6.

Proposition 3.3.4. *Consider the almost observable model of the stochastic clearing system in alternating environment, where all customers join the system. Then, the expected net benefit* $F_{ao}(n;\lceil \infty \rceil) \equiv F_{ao}(n;\lfloor 0 \rfloor)$ *of an arriving customer, if she decides to join, given that she finds* n *customers in the system, is given by*

$$F_{ao}(n; \lceil\infty\rceil) \equiv F_{ao}(n; \lfloor 0 \rfloor) = R - C\frac{A(\frac{1}{1-z_1})^n + B(\frac{1}{1-z_2})^n}{D(\frac{1}{1-z_1})^n + E(\frac{1}{1-z_2})^n} \quad n \geq 0, \quad (3.61)$$

where

$$A = \lambda_1 A_1 E(T_1) + \lambda_2 A_2 E(T_2), \quad (3.62)$$
$$B = \lambda_1 B_1 E(T_1) + \lambda_2 B_2 E(T_2), \quad (3.63)$$
$$D = \lambda_1 A_1 + \lambda_2, \quad (3.64)$$
$$E = \lambda_1 B_1 + \lambda_2 B_2 \quad (3.65)$$

and $E(T_1), E(T_2), A_1, A_2, B_1, B_2, z_1, z_2$ *are given by* (3.7) $-$ (3.8), (3.19) $-$ (3.22) *and* (3.24).

Proof. The mean sojourn time of an arriving customer, if she decides to join, given that she finds n customers in the system is given by

$$p^-_{ao}(1|n; \lceil\infty\rceil)E(T_1) + p^-_{ao}(2|n; \lceil\infty\rceil)E(T_2), \quad (3.66)$$

where $p^-_{ao}(i|n; \lceil\infty\rceil), i = 1, 2$, is the probability that an arriving customer that observes n customers in the system finds the environment at state i, when the $\lceil\infty\rceil$ strategy is followed by other customers. The embedded probabilities $p^-_{ao}(i|n; \lceil\infty\rceil)$ are given by

$$p^-_{ao}(i|n; \lceil\infty\rceil) = \frac{\lambda_i p(n, i)}{\lambda_1 p(n, 1) + \lambda_2 p(n, 2)}, i = 1, 2, \quad (3.67)$$

where $p(n, i)$ are given by (3.17)-(3.18). Thus, the expected benefit of the tagged arriving customer, if she decides to join, is equal to

$$F_{ao}(n; \lceil\infty\rceil) = R - C[p^-_{ao}(1|n; \lceil\infty\rceil)E(T_1) + p^-_{ao}(2|n; \lceil\infty\rceil)E(T_2)]. \quad (3.68)$$

Plugging the formulas (3.17)-(3.18) into (3.67) and subsequently into (3.68) yields (3.61). □

Proposition 3.3.5. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers join the system according to the* (n_0, θ)*-mixed threshold strategy. Then, the expected net benefit* $F_{ao}(n; \lceil n_0, \theta\rceil)$ *of an arriving customer, if she decides to join, given that she finds n customers in the system, is given by*

$$F_{ao}(n;\lceil n_0,\theta\rceil) = R - C\frac{A(\frac{1}{1-z_1})^n + B(\frac{1}{1-z_2})^n}{D(\frac{1}{1-z_1})^n + E(\frac{1}{1-z_2})^n} \quad 0 \leq n \leq n_0 - 1, \tag{3.69}$$

$$F_{ao}(n_0;\lceil n_0,\theta\rceil) = R - C\frac{\sum_{k=n_0}^{\infty}(1-\theta)^{k-n_0}[A(\frac{1}{1-z_1})^k + B(\frac{1}{1-z_2})^k]}{\sum_{k=n_0}^{\infty}(1-\theta)^{k-n_0}[D(\frac{1}{1-z_1})^k + E(\frac{1}{1-z_2})^k]}, \tag{3.70}$$

$$F_{ao}(n_0+1;\lceil n_0,\theta\rceil) = R - C\frac{\sum_{k=n_0}^{\infty}[1-(1-\theta)^{k-n_0}][A(\frac{1}{1-z_1})^k + B(\frac{1}{1-z_2})^k]}{\sum_{k=n_0+1}^{\infty}[1-(1-\theta)^{k-n_0}][D(\frac{1}{1-z_1})^k + E(\frac{1}{1-z_2})^k]}, \tag{3.71}$$

where A,B,D,E,z_1, z_2 are given by (3.62)-(3.65) and (3.24).

Proof. Assume that customers join the system according to the (n_0, θ)-mixed threshold strategy. Then, the mean sojourn time of a tagged customer,if she decides to join, given that she finds n customers in the system before arrival is given by

$$p^-_{ao}(1|n;\lceil n_0,\theta\rceil)E(T_1) + p^-_{ao}(2|n;\lceil n_0.\theta\rceil)E(T_2), \tag{3.72}$$

where $p^-_{ao}(1|n;\lceil n_0,\theta\rceil), i = 1, 2$, is the probability that an arriving customer finds the environment at state i, given that there are n customers in the system and that the $\lceil n_0.\theta\rceil$-strategy is followed. The embedded probabilities are given by

$$p^-_{ao}(i|n;\lceil n_0,\theta\rceil) = \frac{\lambda_i p_{ao}(n,i;\lceil n_0,\theta\rceil)}{\lambda_1 p_{ao}(n,1;\lceil n_0,\theta\rceil) + \lambda_2 p_{ao}(n,2;\lceil n_0,\theta\rceil)}, i = 1, 2, \tag{3.73}$$

where $p_{ao}(n,i;\lceil n_0,\theta\rceil)$ are given by (3.39)-(3.41). Thus, the expected net benefit of the tagged customer, if she decides to join, is given equal to

$$F_{ao}(n;\lceil n_0,\theta\rceil) \equiv R - C[p^-_{ao}(1|n;\lceil n_0,\theta\rceil)E(T_1) + p^-_{ao}(2|n;;\lceil n_0,\theta\rceil)E(T_2)]. \tag{3.74}$$

Using the various forms of $p^-_{ao}(n,i;\lceil n_0,\theta\rceil)$ in (3.39)-(3.41) yields (3.69)-(3.71). □

In the case of the n_0-pure threshold, we obtain the following Corollary.

Corollary 3.3.0.3. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers join the system according to the (n_0)-pure threshold strategy. Then, the expected net benefit $F_{ao}(n;\lceil n_0\rceil)$ of an arriving customer, if she decides to join, given that she finds n customers in the system, is given by*

$$F_{ao}(n;\lceil n_0\rceil) = R - C\frac{A(\frac{1}{1-z_1})^n + B(\frac{1}{1-z_2})^n}{D(\frac{1}{1-z_1})^n + E(\frac{1}{1-z_2})^n} \quad 0 \le n \le n_0 - 1, \tag{3.75}$$

$$F_{ao}(n_0;\lceil n_0\rceil) = R - C\frac{\sum_{k=n_0}^{\infty}[A(\frac{1}{1-z_1})^k + B(\frac{1}{1-z_2})^k]}{\sum_{k=n_0}^{\infty}[D(\frac{1}{1-z_1})^k + E(\frac{1}{1-z_2})^k]}, \tag{3.76}$$

where A,B,D,E,z_1, z_2 are given by (3.62)-(3.65) and (3.24).

Remark 8. Applying Corollary 3.3.0.3 for $n_0 = 0$ yields the expected net benefit $F_{ao}(0;\lceil 0\rceil) \equiv F_{ao}(0;\lfloor\infty\rfloor)$ of an arriving customer, when the others follow the "always balk" strategy.

When the customers follow a $(0-\theta)$-mixed reverse threshold strategy, with $\theta \in (0,1)$, we can use the same line of argument with Propositions 3.3.4 and 3.3.5, using the stationary distribution given by (3.58)-(3.59). Then, we have the following proposition.

Proposition 3.3.6. *Consider the almost observable model of the stochastic clearing system in alternating environment, where the customers join the system according to the $(0,\theta)$-mixed reverse-threshold strategy for some $\theta \in (0,1)$. Then, the expected net benefit $F_{ao}(n;\lfloor 0,\theta\rfloor)$ of an arriving customer, if she decides to join, given that she finds n customers in the system, is given by*

$$F_{ao}(n;\lfloor 0,\theta\rfloor) = R - C\frac{\sum_{k=n}^{\infty}(1-\theta^{k-n})[A(\frac{1}{1-z_1})^k + B(\frac{1}{1-z_2})^k]}{\sum_{k=n}^{\infty}(1-\theta^{k-n})[D(\frac{1}{1-z_1})^k + E(\frac{1}{1-z_2})^k]}, \quad n \ge 0, \tag{3.77}$$

where A,B,D,E,z_1, z_2 are given by $(3.62)-(3.65)$ *and* (3.24).

To express the various formulas reported in Propositions 3.3.4, 3.3.5, 3.3.6 and in Corollary 3.3.0.3 for the expected net benefit function in a compact, unified way, we introduce the functions

$$S(n,\theta) = \sum_{k=n}^{\infty}(1-\theta)^{k-n}\left[(RD-CA)\left(\frac{1}{1-z_1}\right)^k + (RE-CB)\left(\frac{1}{1-z_2}\right)^k\right], \tag{3.78}$$

$$G(n,\theta) = \sum_{k=n}^{\infty}(1-\theta)^{k-n}\left[D\left(\frac{1}{1-z_1}\right)^k + E\left(\frac{1}{1-z_2}\right)^k\right], \tag{3.79}$$

$$H^U(n) = \frac{S(n,1)}{G(n,1)}, \qquad H^L(n) = \frac{F(n,0)}{G(n,0)}, n \ge 0. \tag{3.80}$$

Then we have

$$F_{ao}(n;\lceil\infty\rceil) \equiv F_{ao}(n;\lfloor 0\rfloor) = \frac{S(n,1)}{G(n,1)} = H^U(n), \quad n \geq 0 \tag{3.81}$$

$$F_{ao}(n;\lceil n_0,\theta\rceil) = \frac{S(n,1)}{G(n,1)} = H^U(n), \quad 0 \leq n \leq n_0 - 1, \tag{3.82}$$

$$F_{ao}(n_0;\lceil n_0,\theta\rceil) = \frac{S(n_0,\theta)}{G(n_0,\theta)}, \tag{3.83}$$

$$F_{ao}(n_0+1;\lceil n_0,\theta\rceil) = \frac{S(n_0,0) - S(n_0,\theta)}{G(n_0,0) - G(n_0,\theta)}, \tag{3.84}$$

$$F_{ao}(n;\lceil n_0\rceil) = \frac{S(n,1)}{G(n,1)} = H^U(n), \quad 0 \leq n \leq n_0 - 1, \tag{3.85}$$

$$F_{ao}(n_0;\lceil n_0\rceil) = \frac{S(n_0,0)}{G(n_0,0)} = H^L(n_0), \tag{3.86}$$

$$F_{ao}(0;\lceil 0\rceil) \equiv F_{ao}(0;\lfloor\infty\rfloor) = \frac{S(0,0)}{G(0,0)} = H^L(0), \tag{3.87}$$

$$F_{ao}(n;\lfloor 0,\theta\rfloor) = \frac{S(n,\theta)}{G(n,\theta)}, \quad n \geq 0. \tag{3.88}$$

3.3.3 Equilibrium strategies

As we have previously discussed, when the "fast service" coincides with the less congested environmental state, i.e. $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$, customers should adopt a threshold strategy. On the contrary, when the "fast service" coincides with the more congested environmental state, customers should adopt a reverse-threshold strategy. This intuitive finding is associated with the monotonicity of $H^U(n)$ which plays a key role in the following analysis. We have the following Proposition 3.3.7.

Proposition 3.3.7. *We have the following equivalences:*

$$H^U(n) \text{ is strictly decreasing} \Leftrightarrow AE - BD > 0 \Leftrightarrow (\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0 \tag{3.89}$$

$$H^U(n) \text{ is constant} \Leftrightarrow AE - BD = 0 \Leftrightarrow \mu_1 = \mu_2 \text{ or } \rho_1 = \rho_2 \tag{3.90}$$

$$H^U(n) \text{ is strictly increasing} \Leftrightarrow AE - BD < 0 \Leftrightarrow (\mu_1 - \mu_2)(\rho_1 - \rho_2) > 0 \tag{3.91}$$

The proof of this proposition is omitted, since the first case follows easily by simple algebraic manipulations that start from the relation $H^U(n+1) - H^U(n) < 0$ and lead to $AE - BD > 0$ and $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$, through successive equivalences. The other two cases are treated similarly. Moreover, the monotonicity if the function $\frac{S(n,\theta)}{G(n,\theta)}$ with respect to θ depends on the sign of $(\mu_1 - \mu_2)(\rho_1 - \rho_2)$. Specifically, we have the following Proposition 3.3.8.

Proposition 3.3.8.

$$\frac{S(n,\theta)}{G(n,\theta)} \textit{ is strictly increasing in } \theta \Leftrightarrow AE - BD > 0 \Leftrightarrow (\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0 \tag{3.92}$$

$$\frac{S(n,\theta)}{G(n,\theta)} \textit{ is constant in } \theta \Leftrightarrow AE - BD = 0 \Leftrightarrow \mu_1 = \mu_2 \textit{ or } \rho_1 = \rho_2 \tag{3.93}$$

$$\frac{S(n,\theta)}{G(n,\theta)} \textit{ is strictly decreasing in } \theta \Leftrightarrow AE - BD < 0 \Leftrightarrow (\mu_1 - \mu_2)(\rho_1 - \rho_2) > 0 \tag{3.94}$$

The proof of this proposition is also omitted, since the result is deduced easily after some algebra. We now state some properties of $S(n,\theta), G(n,\theta)$ and $H^U(n), H^L(n)$ that we will use in the sequel. Their proof is straightforward from their definition and is thus omitted.

Lemma 3.3.1. *The functions $S(n,\theta), G(n,\theta)$ satisfy the following properties:*

$$\begin{aligned} S(n,\theta) &= \sum_{k=n}^{\infty}(1-\theta)^{k-n}S(k,1) \\ &= S(n,1) + (1-\theta)S(n+1,\theta), \quad n \geq 0, \theta \in [0,1], \end{aligned} \tag{3.95}$$

$$\begin{aligned} G(n,\theta) &= \sum_{k=n}^{\infty}(1-\theta)^{k-n}G(k,1) \\ &= G(n,1) + (1-\theta)G(n+1,\theta), \quad n \geq 0, \theta \in [0,1], \end{aligned} \tag{3.96}$$

$$G(n,\theta) > 0, \quad n \geq 0, \theta \in [0,1], \tag{3.97}$$

$$G(n,\theta) \textit{ is strictly icreasing with respect to } \theta \textit{ for any fixed } n \geq 0. \tag{3.98}$$

Note that properties (3.97) and (3.98) of $G(n,\theta)$ assure that all denominators in (3.81)-(3.88) are positive.

The intuitive discussion at the beginning of section 3.3 in combination with propositions 3.3.7 and 3.3.8 suggests that we should methodologically proceed by considering three different cases corresponding to the sign (negative,positive or zero) of $(\mu_1 - \mu_2)(\rho_1 - \rho_2)$.

Case A: $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$

In case A, we will prove that an equilibrium strategy always exists. Moreover, we will present a systematic procedure for determining all equilibrium threshold strategies. We first introduce quantities that we will need in the sequel.

Definition 3.3.2. Suppose that

$$(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0. \tag{3.99}$$

We define

$$\begin{aligned}
n_U &= inf\{n \geq 0 : S(n,1) < 0\}, &(3.100)\\
n_L &= inf\{n \geq 0 : S(n,0) \leq 0\}, &(3.101)\\
n_U^- &= inf\{n \geq 0 : S(n,1) \leq 0\}, &(3.102)\\
n_L^+ &= inf\{n \geq 0 : S(n,0) < 0\}. &(3.103)
\end{aligned}$$

Then, we have several properties of n_U, n_L, n_U^-, n_L^+ that we summarize in the following Lemma 3.3.2.

Lemma 3.3.2. *Suppose that* (3.99) *holds. Then, there are three cases:*
Case I: $H^U(0) < 0$.
Then

$$\begin{aligned}
&n_U = n_L = n_U^- = n_L^+ = 0 &(3.104)\\
&S(n,\theta) < 0, \quad n \geq 0, \theta \in [0,1], &(3.105)\\
&S(n,0) - S(n,\theta) < 0, \quad n \geq 0, \theta \in (0,1]. &(3.106)
\end{aligned}$$

Case II: $H^U(0) \geq 0$. *and* $lim_{n\to\infty} H^U(n) < 0$.
Then

$$\begin{aligned}
&1 \leq n_U < \infty &(3.107)\\
&S(n,1) > 0, \quad 0 \leq n \leq n_U - 2, &(3.108)\\
&S(n_U - 1, 1) \geq 0, &(3.109)\\
&S(n,1) < 0, n \geq n_U. &(3.110)
\end{aligned}$$

and

$$\begin{aligned}
&0 \leq n_L \leq n_U, &(3.111)\\
&S(n,0) > 0, \quad 0 \leq n \leq n_L - 1, &(3.112)\\
&S(n_L, 0) \leq 0, &(3.113)\\
&S(n,0) < 0, \quad n \geq n_L + 1. &(3.114)
\end{aligned}$$

Moreover,

$$n_L^+ = \begin{cases} n_L, & \text{if } S(n_L, 0) < 0 \\ n_L + 1, & \text{if } S(n_L, 0) = 0 \end{cases} \tag{3.115}$$

$$n_U^- = \begin{cases} n_U, & \text{if } S(n_U - 1, 1) > 0 \\ n_U - 1, & \text{if } S(n_U - 1, 1) = 0 \end{cases} \tag{3.116}$$

For every $n_0 \in \{n_L^+, ..., n_U^-\}$, a unique solution $\theta(n_0) \in (0,1)$ of the equation $S(n_0, \theta) = 0$ exists with respect to θ, i.e.

$$S(n_0, \theta(n_0)) = 0, \quad n_L^+ \le n \le n_U^- - 1. \tag{3.117}$$

Case III: $lim_{n\to\infty} H^U(n) \ge 0$.
Then

$$n_U = n_L = n_U^- = n_L^+ = \infty \tag{3.118}$$
$$S(n, \theta) > 0, \quad n \ge 0, \theta \in [0,1], \tag{3.119}$$
$$S(n, 0) - S(n, \theta) > 0, \quad n \ge 0, \theta \in (0,1]. \tag{3.120}$$

Proof. In Case I, the condition $H^U(0) < 0$ in combination with monotonicity of $H^U(n)$ (due to (3.89)) implies that $H^U(n) < 0, n \ge 0$. By (3.80) and (3.97) we have that $S(n,1) < 0$ and therefore, $S(n,\theta) = \sum_{k=n}^{\infty}(1-\theta)^{k-n}S(k,1) < 0, n \ge 0, \theta \in [0,1]$ and $S(n,0) - S(n,\theta) = \sum_{k=n}^{\infty}[1-(1-\theta)^{k-n}] < 0, n \ge 0, \theta \in (0,1]$.

In Case II, the conditions $H^U(0) \ge 0.$ and $lim_{n\to\infty} H^U(n) < 0.$ combined with the conditions (3.89) for the monotonicity of $H^U(n)$ imply (3.107)-(3.110).

Equation (3.110) implies that $S(n_U, 0) = \sum_{k=n_U}^{\infty} S(k,1) < 0$ and we conclude (3.111). By the definition of n_L we have (3.112)-(3.113). Moreover, we have that $S(n,0) = \sum_{k=n}^{\infty} S(k,1) < 0$ for $n \ge n_U$.

For n with $n_L + 1 \le n \le n_U - 1$ we also have that $S(n,0) < 0$. Indeed, consider that there is an n such that $n_L+1 \le n \le n_U-1$ and $S(n,0) \ge 0$. By (3.95), we have that $S(n-1,0) = S(n-1,1)+S(n,0)$. Since $S(n-1,1) > 0$ by (3.108) and we assumed that $S(n,0) > 0$ we have that $S(n-1,0) > 0$ and inductively, we obtain $S(n_L, 0) > 0$ which contradicts (3.113). Thus $S(n,0) < 0$ for $n \ge n_L + 1$ and we obtain (3.114).

Equations (3.115)-(3.116) are immediate from (3.108)-(3.110) and (3.112)-(3.116). Consider now an $n_0 \in \{n_L^+, ...n_U^- - 1\}$. We have that $\frac{S(n_0,1)}{G(n_0,1)} > 0$. This is the case since $n_0 \le n_U^- - 1$ and there are two possibilities (see 3.116). Either $n_U^- = n_U$ and n_0 is strictly less than n_U and (by (3.108)-(3.109) and (3.115)) $\frac{S(n_0,1)}{G(n_0,1)} > 0$ is strictly positive or $n_U^- = n_U - 1$ which means that $n_0 \le n_U - 2$ and by (3.108) $\frac{S(n_0,1)}{G(n_0,1)} > 0$ is again positive. Also, since $n_0 \ge n_L^+$, considering the two possible values of n_L^+ given by (3.115) and the equations (3.113)-(3.114) we can easily conclude that $\frac{S(n_0,0)}{G(n_0,0)} < 0$. By condition (3.92) we have that $\frac{S(n_0,\theta)}{G(n_0,\theta)}$ is a strictly increasing and continuous function of θ, so by Bolzano's Theorem we conclude that there exists a unique solution $\theta(n_0) \in (0,1)$ of the equation $\frac{S(n_0,\theta)}{G(n_0,\theta)} = 0$. Thus, we obtain (3.117).

In Case III, the condition $lim_{n\to\infty} H^U(n) \ge 0$ in combination with monotonicity of $H^U(n)$ (due to (3.89)) implies that $H^U(n) > 0, n \ge 0$. By (3.80) and (3.97) we have that $S(n,1) > 0$ and therefore, $S(n,\theta) = \sum_{k=n}^{\infty}(1-$

$\theta)^{k-n}S(k,1) > 0, n \geq 0, \theta \in [0,1]$ and $S(n,0) - S(n,\theta) = \sum_{k=n}^{\infty}[1-(1-\theta)^{k-n}] < 0, n \geq 0, \theta \in (0,1]$. Thus, we conclude (3.118)-(3.120). $\square$

Using Lemma 3.3.2 we will now prove the existence of threshold equilibrium strategies, when (3.99) holds. We have the following Theorem 3.3.3.

Theorem 3.3.3. *In the almost observable model of the stochastic clearing system in alternating environment where (3.99) holds, equilibrium threshold strategies always exist. In particular, in the three cases of Lemma 3.3.2 we have:*

Case I *: $H^U(0) < 0$*
Then there exists a unique equilibrium threshold strategy,the $\lceil 0 \rceil$ strategy(always to balk).

Case II *: $H^U(0) \geq 0$. and $lim_{n\to\infty}H^U(n) < 0$.*
Then, an equilibrium pure threshold strategy always exists. Moreover, the equilibrium strategies within the class of all pure strategies are the strategies $\lceil n_0 \rceil$ with $n_0 = n_L, n_L+1, ..., n_U$. Also, the equilibrium strategies within the class of genuinely mixed threshold strategies are the strategies $\lceil n_0, \theta(n_0) \rceil$ with $n_0 \in \{n_L^+, ..., n_U^- - 1\}$ and $\theta(n_0)$ the unique solution of $S(n_0,\theta) = 0$ with respect to θ.

Case III *: $lim_{n\to\infty}H^U(n) \geq 0$.*
Then, there is a unique equilibrium threshold strategy, the $\lceil \infty \rceil$-strategy(always to join).

Proof. Case 1: Consider a tagged customer at his arrival instant and assume all other customers follow an $\lceil n_0 \rceil$ strategy for some $n_0 \geq 0$. Inequality (3.109) and relations (3.85)-(3.86) imply that the expected net benefit of the tagged customer, when she finds n customers and decides to join is $F_{ao}(n; \lceil n_0 \rceil) < 0$, for $0 \leq n \leq n_0$. Thus, she always prefers to balk and her best response against $\lceil n_0 \rceil$ is $\lceil 0 \rceil$.

We now assume that other customers follow an $\lceil n_0, \theta \rceil$ strategy, for some $n_0 \geq 0$ and $\theta \in (0,1)$. Then, if the tagged customer finds n customers at her arrival instant and decides to join, her expected net benefit will be $F_{ao}(n; \lceil n_0, \theta \rceil < 0$ for $0 \leq n \leq n_0 + 1$ from (3.105)-(3.106) and (3.83)-(3.84). Therefore, the tagged customer is always unwilling to join and her best response against $\lceil n_0, \theta \rceil$ is $\lceil 0 \rceil$.

If all customers follow the $\lceil \infty \rceil$ strategy, (3.109) and (3.81) yield $F_{ao}(n; \lceil \infty \rceil) < 0$ for $n \geq 0$. Again, due to negative expected benefit, it is preferable for the tagged customer to balk. So, her best response against $\lceil \infty \rceil$ is $\lceil 0 \rceil$. Thus, we conclude that the only best response against itself within the class of (pure and mixed) threshold strategies is $\lceil 0 \rceil$.

Case II: Consider a tagged arriving customer and suppose all other customers follow an $\lceil n_0 \rceil$ strategy for some $n_0 \leq n_L - 1$. If she finds n_0 customers and decides to join her expected net benefit will be $F_{ao}(n_0; \lceil n_0 \rceil) > 0$, from (3.112) and (3.86). This implies that when she finds n_0 customers, she is willing to join. Thus, $\lceil n_0 \rceil$ cannot be an equilibrium since it cannot be a best response against itself.

Consider a tagged arriving customer and suppose all other customers follow an $\lceil n_0 \rceil$ strategy for some $n_0 \geq n_U + 1$. Using (3.85) and (3.110) we have that $F_{ao}(n; \lceil n_0 \rceil) < 0$ for $n_U \leq n \leq n_0 - 1$. This implies that when she finds n customers, with $n_U \leq n \leq n_0 - 1$, she is unwilling to enter. Thus, the $\lceil n_0 \rceil$ cannot be an equilibrium strategy. We conclude that the search of equilibrium strategies should be restricted within the class of pure threshold strategies should be restricted to strategies $\lceil n_0 \rceil$ with $n_L \leq n_0 \leq n_U$.

We mark an arriving customer and we assume that all other customers follow an $\lceil n_0 \rceil$ strategy for some n_0 with $n_L \leq n_0 \leq n_U$. From (3.85), (3.86), (3.108), (3.109), (3.113) and (3.114) we have that the expected net benefit of a customer who finds n customers upon arrival and decides to join is $F_{ao}(n; \lceil n_0 \rceil) \geq 0$, for $0 \leq n \leq n_0 - 1$ and $F_{ao}(n_0; \lceil n_0 \rceil) \leq 0$. Thus, $\lceil n_0 \rceil$ is a best response against itself and we conclude that all such strategies are equilibrium strategies.

To finish our search of equilibrium strategies within the class of pure threshold strategies, we examine the $\lceil \infty \rceil$ strategy. This cannot be an equilibrium, since (3.110) and (3.81) imply that $F_{ao}(n; \lceil \infty \rceil) < 0$, for $n \geq n_U$, which means that it is not optimal for the tagged customer to join when she sees n customers for some $n \geq n_U$. Therefore, we conclude that the equilibrium strategies within the class of pure threshold strategies are exactly the strategies $\lceil n_0 \rceil$ for $n_L \leq n_0 \leq n_U$.

We will now search for equilibrium strategies in the class of genuinely mixed threshold strategies, i.e. among strategies $\lceil n_0, \theta_0 \rceil$ with $\theta_0 \in (0, 1)$. In order for a mixed threshold strategy $\lceil n_0, \theta_0 \rceil$ to be an equilibrium strategy, the following relations must be true : $S(n, 1) > 0$, for $0 \leq n \leq n_0 - 1$, $S(n_0, \theta_0) = 0$ and $S(n_0, 0) - S(n_0, \theta_0) \leq 0$. We derive those relations from the equations (3.82)-(3.84) which express a customers' expected benefit when she joins while observing less than n_0, exactly n_0 and $n_0 + 1$ customers in the system respectively. By comparing (3.82)-(3.84) with (3.85)-(3.86) we can easily see that $\lceil n_0, \theta_0 \rceil$ may be an equilibrium if and only if $\lceil n_0 \rceil$ is an equilibrium. Thus, we should restrict our search for equilibrium genuinely mixed threshold strategies to strategies $\lceil n_0, \theta_0 \rceil$ with $n_0 = n_L, ..., n_U$.

If $S(n_L, 0) = 0$, then there does not exist $\theta \in (0, 1)$ such that $S(n_L, \theta) = 0$, since $\frac{S(n,\theta)}{G(n,\theta)}$ is strictly decreasing. Therefore, $\lceil n_L, \theta \rceil$ cannot be an equilibrium strategy for any $\theta \in (0, 1)$. Similarly, if $S(n_U - 1, 1) = 0$, then the strategy $\lceil n_U - 1, \theta \rceil$ cannot be an equilibrium strategy for any $\theta \in (0, 1)$. Moreover, $\lceil n_U, \theta \rceil$ cannot be an equilibrium strategy for any $\theta \in (0, 1)$, since $S(n_u, \theta) < 0, \theta \in (0, 1)$. Therefore, a strategy $\lceil n_0, \theta \rceil$ with $\theta \in (0, 1)$ may be an equilibrium only if $n_L^+ \leq n_0 \leq n_U^- - 1$.

Now, for every $n_0 \in \{n_L^+, ...n_U^-\}$ the only $\lceil n_0, \theta_0 \rceil$ strategy that can be an equilibrium is the one that corresponds to $\theta_0 = \theta(n_0)$ since $S(n_0, \theta(n_0)) = 0$. Indeed, if all customers follow the $\lceil n_0, \theta(n_0) \rceil$ strategy, the expected benefit for a tagged customer, who finds n customers in the system and decides to join is $F_{ao}(n; \lceil n_0, \theta(n_0) \rceil) > 0$ for $0 \leq n \leq n_0 - 1$, $F_{ao}(n_0; \lceil n_0, \theta(n_0) \rceil) = 0$ and $F_{ao}(n_0 + 1; \lceil n_0, \theta(n_0) \rceil) < 0$, from (3.108), (3.114) and (3.117). Thus, $\lceil n_0, \theta(n_0) \rceil$ is an equilibrium strategy.

Case III : Following the same line of argument as in case I, we now find that when all customers follow a pure threshold strategy $\lceil n_0 \rceil$ or a mixed threshold strategy $\lceil n_0, \theta \rceil$ the expected net benefit function is always positive. Thus, the best response of a customer is always to join the system. Therefore, the only best response against itself in the class of threshold strategies is the $\lceil \infty \rceil$ strategy.

□

Note that although pure threshold strategies always exist, it is possible that genuinely mixed threshold strategies do not. This happens if $n_U^- - 1 < n_L^+$.

Case B:$(\mu_1 - \mu_2)(\rho_1 - \rho_2) > 0$

In case B, we seek equilibrium strategies in the class of reverse-threshold strategies. We will exclude strategies $\lfloor n_0 \rfloor$ and $\lfloor n_0, \theta \rfloor$ with $n_0 \geq 1$. Indeed, all these strategies prescribe to balk when the customer faces an empty system. Thus, under such strategy, the system continuously remains empty after the first service completion. Therefore, in steady state, these strategies are equivalent to the "always balk" strategy $\lfloor \infty \rfloor$. Thus, we seek equilibrium strategies only in the set $S_{r-t} = \{\lfloor 0 \rfloor, \lfloor \infty \rfloor\} \cup \{\lfloor 0, \theta_0 \rfloor : \theta_0 \in (0, 1)\}$. We first introduce several quantities that we will use in the sequel.

Definition 3.3.3. Suppose that

$$(\mu_1 - \mu_2)(\rho_1 - \rho_2) > 0. \tag{3.121}$$

We define

$$m_U = inf\{n \geq 0 : S(n, 1) > 0\}, \tag{3.122}$$
$$m_L = inf\{n \geq 0 : S(n, 0) \geq 0\}, \tag{3.123}$$
$$m_U^- = inf\{n \geq 0 : S(n, 1) \geq 0\}, \tag{3.124}$$
$$m_L^+ = inf\{n \geq 0 : S(n, 0) > 0\}. \tag{3.125}$$

Then, we have several properties of m_U, m_L, m_U^-, m_L^+ that we summarize in the following Lemma 3.3.4.

Lemma 3.3.4. *Suppose that* (3.121) *holds. Then, there are three cases:*
Case I: $H^U(0) > 0$.

Then

$$m_U = m_L = m_U^- = m_L^+ = 0 \tag{3.126}$$
$$S(n,\theta) > 0, \quad n \geq 0, \theta \in [0,1]. \tag{3.127}$$

Case II: $H^U(0) \leq 0$. *and* $lim_{n\to\infty} H^U(n) > 0$.
Then

$$1 \leq m_U < \infty \tag{3.128}$$
$$S(n,1) < 0, \quad 0 \leq n \leq m_U - 2, \tag{3.129}$$
$$S(m_U - 1, 1) \leq 0, \tag{3.130}$$
$$S(n,1) < 0, n \geq m_U. \tag{3.131}$$

and

$$0 \leq m_L \leq m_U, \tag{3.132}$$
$$S(n,0) < 0, \quad 0 \leq n \leq m_L - 1, \tag{3.133}$$
$$S(m_L, 0) \geq 0, \tag{3.134}$$
$$S(n,0) > 0, \quad n \geq m_L + 1. \tag{3.135}$$

Moreover,

$$m_L^+ = \begin{cases} m_L, & \text{if } S(m_L,0) > 0 \\ m_L + 1, & \text{if } S(m_L,0) = 0 \end{cases} \tag{3.136}$$

$$m_U^- = \begin{cases} m_U, & \text{if } S(m_U - 1,1) < 0 \\ m_U - 1, & \text{if } S(m_U - 1,1) = 0 \end{cases} \tag{3.137}$$

If $m_L^+ = 0$ *and* $m_U^- \geq 1$, *then there exists a unique* $\theta(0) \in (0,1)$ *such that*

$$S(0, \theta(0)) = 0, \tag{3.138}$$
$$S(n, \theta(0)) > 0, n \geq 1. \tag{3.139}$$

Case III: $lim_{n\to\infty} H^U(n) \leq 0$.
Then

$$m_U = m_L = m_U^- = m_L^+ = \infty \tag{3.140}$$
$$S(n,\theta) < 0, \quad n \geq 0, \theta \in [0,1]. \tag{3.141}$$

We omit the proof of Lemma 3.3.4 as it is completely analogous to the proof of Lemma 3.3.2. We are now in position to prove the existence and uniqueness of reverse-threshold strategies when (3.121) holds. We present the results in he following Theorem 3.3.5. The statements about the uniqueness of the reverse-threshold equilibrium strategies should be interpreted within the class $S_{r-t} = \{\lfloor 0 \rfloor, \lfloor \infty \rfloor\} \cup \{\lfloor 0, \theta_0 \rfloor : \theta_0 \in (0,1)\}$ of the reverse threshold strategies.

Theorem 3.3.5. *In the almost observable model of the stochastic clearing system in alternating environment where (3.121) holds, equilibrium reverse-threshold strategies always exist. In particular, in the three cases of Lemma 3.3.4 we have:*

Case I *:* $H^U(0) > 0$
Then there exists a unique equilibrium reverse-threshold strategy, the $\lfloor 0 \rfloor$ *strategy ("always to join").*

Case II *:* $H^U(0) \leq 0$. *and* $lim_{n\to\infty} H^U(n) > 0$.
If $m_U^- = 0$*, the* $\lfloor 0 \rfloor$ *strategy ("always to join") is the unique equilibrium reverse-threshold strategy. If* $m_L^+ \geq 1$ *then the* $\lfloor \infty \rfloor$ *strategy "always to balk" is the unique equilibrium reverse-threshold strategy. Otherwise, the* $\lfloor 0, \theta(0) \rfloor$ *strategy is the unique equilibrium reverse-threshold strategy.*

Case III *:* $lim_{n\to\infty} H^U(n) \leq 0$.
Then, there is a unique equilibrium reverse-threshold strategy, the $\lfloor \infty \rfloor$*-strategy ("always to balk").*

Proof. Case I : Consider a tagged customer at his arrival instant that observes n customers in the system, when all other customers follow the $\lfloor 0 \rfloor$ strategy. Then, the expected net benefit of the tagged customer is given by (3.81) and since $S(n, \theta) > 0$ for $n \geq 0, \theta \in [0, 1]$ by (3.127) and $G(n, \theta) > 0$, for $n \geq 0, \theta \in [0, 1]$ by (3.97), we have that $F_{ao}(n; [0]) > 0$ for $n \geq 0$. Thus, the tagged customer always prefers to join and her best response against $\lfloor 0 \rfloor$ is $\lfloor 0 \rfloor$.

Similarly, consider a tagged customer at his arrival instant that observes n customers in the system, when all other customers follow a $\lfloor 0, \theta_0 \rfloor$ strategy, for some $\theta_0 \in (0, 1)$. Then, the expected net benefit of the tagged customer is given by (3.88) and by (3.127) we have that $F_{ao}(n; [0, \theta_0]) > 0$ for $n \geq 0$. Thus, the tagged customer always prefers to join and her best response against $\lfloor 0, \theta_0 \rfloor$ is $\lfloor 0 \rfloor$.

If all other customers follow the $\lfloor \infty \rfloor$ strategy, the net benefit of a tagged customer that enters the system and observes n customers in the system $F_{ao}(n; [\infty])$ is positive for $n \geq 0$ by (3.87) and (3.127). Thus, the tagged customer always prefers to join and her best response against $\lfloor \infty \rfloor$ is $\lfloor 0 \rfloor$. So the only reverse-threshold strategy which is best response against itself is the $\lfloor 0 \rfloor$ strategy.

Case II: Assume that $m_U^- = 0$. Then $S(0, 1) = 0$ and $m_U = 1$. Consider now a tagged customer at her arrival instant and suppose all other customers follow the $\lfloor 0 \rfloor$ strategy. Inequality (3.131) and relation (3.81) imply that her expected net benefit, when she finds n customers in the system and decides to join is $F_{ao}(n; \lfloor 0 \rfloor) \geq 0$ for $n \geq 0$. Thus, $\lfloor 0 \rfloor$ is a best response against itself.

Assume, now, that $m_L^+ \geq 1$. By definition of m_L^+ is equal to m_L or $m_L + 1$. In the former case, we have that $0 \leq m_L - 1$ and (3.133) yields that $S(0,0) < 0$. In the latter case, we have that $m_L = 0$ or $m_L > 0$. If $m_L > 0$ again by (3.133) we have that $S(0,0) < 0$. If $m_L = 0$ then by (3.136) $F(m_L, 0) = F(0,0) = 0$. Therefore, if $m_L^+ \geq 1$ we have $F(0,0) \leq 0$. If we consider a tagged arriving customer and suppose all other customers follow the $\lfloor \infty \rfloor$ strategy, then the tagged customer, if she observes 0 customers and decides to join, has expected net benefit $F_{ao}(0; \lfloor \infty \rfloor) \leq 0$ due to (3.87). Thus, $\lfloor \infty \rfloor$ is a best response against itself i.e. it is an equilibrium strategy. Otherwise, we will have that $m_L^+ = 0$. Consider again a tagged customer and suppose that all other customers follow the $\lfloor 0, \theta(0) \rfloor$ strategy. If the tagged customer at her arrival instant finds n customers in the system and decides to join, her expected net benefit will be either $F_{ao}(0; \lfloor 0, \theta(0) \rfloor) = 0$ when $n = 0$ or $F_{ao}(n; \lfloor 0, \theta(0) \rfloor) \geq 0$ if $n \geq 1$ due to (3.88), (3.138), (3.139). Therefore, the $\lfloor 0, \theta(0) \rfloor$ strategy is an equilibrium strategy.

Case III : Following the same line of argument as in case I, we now conclude that the expected net benefit function is negative. Thus, the best response to every reverse-threshold strategy is $\lfloor \infty \rfloor$. Thus the only equilibrium reverse-threshold strategy is $\lfloor \infty \rfloor$ □

Case C:$(\mu_1 - \mu_2)(\rho_1 - \rho_2) = 0$

Case C occurs when $\mu_1 = \mu_2$ or $\frac{\lambda_1}{\rho_1} = \frac{\lambda_2}{\rho_2}$. In this case, the distinction "fast environmental state" and "slow environmental state" has no sense or the distinction "more congested environmental state" and "less congested environmental state" has no sense. Therefore, we conclude that the information on the number of customers in the system, does not affect the decision of a tagged arriving customer. A similar analysis is possible as in the other two cases and we have the following Theorem 3.3.6.

Theorem 3.3.6. *In the almost observable model of the stochastic clearing system in alternative environment, where*

$$\mu_1 = \mu_2 \text{ or } \rho_1 = \rho_2, \tag{3.142}$$

an equilibrium strategy exists within the class of threshold and reverse-threshold strategies. In particular, we have the following three cases:

Case I *:* $H^U(0) < 0$

Then the unique equilibrium strategy in the class of threshold and reverse-threshold strategies is the $\lceil 0 \rceil \equiv \lfloor \infty \rfloor$ *strategy("always to balk").*

Case II *:* $H^U(0) = 0$

Then every strategy in the class of threshold and reverse-threshold strategies is equilibrium strategy.

Case III *: $H^U(0) > 0$.*

Then, the unique equilibrium strategy in the class of threshold and reverse-threshold strategies is the $\lceil\infty\rceil \equiv \lfloor 0\rfloor$-strategy ("always to join").

Proof. By (3.142) and (3.90) we can see that $H^U(n)$ is constant. Therefore, in case I, $H^U(n)$ is always negative. We consider the cases where other customers follow $\lceil\infty\rceil \equiv \lfloor 0\rfloor$, $\lceil n_0\rceil$, $\lfloor 0,\theta\rfloor$ and $\lceil 0\rceil \equiv \lfloor\infty\rfloor$. By (3.81),(3.85),(3.87) and (3.88) we can see that the expected net benefit in each of those cases, when a tagged customer observes n customers in the system and joins, is negative (in the $\lceil n_0\rceil$ case we assume $n \leq n_0$) and therefore prefers to balk. Thus, the only equilibrium strategy is $\lceil 0\rceil \equiv \lfloor\infty\rfloor$("always to balk"). We do not need to check $\lceil n_0,\theta\rceil$ strategies because if $\lceil n_0\rceil$ is not an equilibrium, $\lceil n_0,\theta\rceil$ cannot be an equilibrium.

The other two cases are similar. In case II the expected net benefit is always 0, therefore any strategy is equilibrium, whereas in case III, the expected benefit is always positive, and customers prefer to "always join". □

This concludes the analysis of customer strategic behaviour in a clearing systems in an alternating environment. We identified four cases with respect to the information level of information provided to arriving customers and derived the equilibrium strategy for each case. It is important to notice that in each case, we searched for equilibrium strategies within the appropriate class of strategies. Moreover, in the almost observable case, which is the most interesting, Theorems 3.3.3, 3.3.5 and 3.3.6 suggest that the equilibrium strategies in the class of threshold and reverse-threshold strategies are completely characterized by the signs of the quantities $(\mu_1 - \mu_2)(\rho_1 - \rho_2), H^U(n), lim_{n\to\infty H^U(n)}$ and $H^L(n)$. Thus, we can easily combine these theorems and develop an algorithm for determining equilibrium strategies.

ATC or FTC?

We must also notice that, in the almost observable case, the two cases A and B concerning the sign of the quantity $(\mu_1-\mu_2)(\rho_1-\rho_2)$ are quite different. In case A, there is a general interval of thresholds which constitute equilibrium strategies. On the other hand, in case B, we have a unique reverse-threshold equilibrium strategy. As stated in the introduction, multiple equilibria are a property of the *follow the crowd* (FTC) behaviour, whereas in the *avoid the crowd* (ATC) behaviour, at most one equilibrium is possible. This is indeed the case here :

Consider case A, where $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$ and the "fast service" coincides with the "less congested" environmental state. We compare two threshold strategies n and $n+1$ as follows. Consider a tagged customer who arrives at the system and observes n customers, while all other customers follow an n threshold strategy. Then, she can deduce that *at least* n customers have arrived at the station since the last clearing epoch, since some customers may have come and balked when they faced n customers. If the customers follow an $n+1$ threshold strategy instead, the tagged customer knows that *exactly* n customers arrived at the system since the last clearing epoch. This gives the customer the sense that the system is less congested which is a signal that the environmental state is probably the “fast service” one. Therefore, the customer is more willing to join the system and adopts a higher threshold. Thus, if the customers adopt a higher threshold, an arriving customer tends to follow them in adopting a higher threshold and we have an FTC situation.

On the other hand, consider case B where $(\mu_1 - \mu_2)(\rho_1 - \rho_2) > 0$ and the “slow service” coincides with the “less congested” environmental state. In this case, we deal with reverse-threshold strategies. Also, if the customers follow a strategy $\lfloor n_0, \theta \rfloor$ with $n_0 \geq 1$ and $\theta \in [0, 1]$ the system remains empty. Therefore, we will limit our intuitive discussion in the case where customers follow an $\lfloor 0, \theta \rfloor$ with $\theta \in [0, 1]$. Now, consider a tagged customer who arrives at the system and observes 0 customers, while all other customers initially follow an $\lfloor 0, \theta \rfloor$ reverse-threshold strategy and then move to a $\lfloor 0, \theta' \rfloor$ with $\theta' > \theta$. When the customers follow the $\lfloor 0, \theta' \rfloor$ the information of an empty system means that the system is probably in the the “less congested” environmental state, which now coincides with the slow service. Therefore, the tagged customer becomes less willing to enter the system. Thus, when other customers enter the system with a higher probability, the tagged customer tends to decrease the probability of joining the system i.e. we have an ATC situation.

Social optimization

Due to the clearing mechanism all customers are removed simultaneously at the end of a clearing epoch. Therefore, customers do not impose any externalities on other customers, and by maximizing their expected net benefit, they also maximize the social benefit. Thus, in the fully observable, fully unobservable and the almost unobservable case, equilibrium strategies are also socially optimal. In the almost observable case, equilibrium strategies are also socially optimal, except from the case where $H^U(n)$ is strictly decreasing, $H^U(n) \geq 0$ and $lim_{n \to \infty} H^U(n) < 0$. In this case, when customers adopt an $\lceil n_0 \rceil$ with $n_0 \in \{n_L, ..., n_U\}$, joining customers have positive expected net benefit. Naturally, a social planner would want to have the highest threshold possible, while the customers' expected net benefit remains positive. This means that the unique socially optimal strategy is the $\lceil n_U \rceil$ strategy.

Chapter 4

Strategic behaviour in an observable fluid queue with an alternating environment

4.1 The model

We consider a fluid queue that represents a production facility, that alternates between fast and slow periods, which are independent and exponentially distributed with rates q_1 and q_0 respectively. The state of the machine is recorded by the 2-state continuous time Markov chain $\{Z_{(t)}\}$, where the states 1 and 0 correspond respectively to the fast and slow modes. The input rate of the fluid that represents the arrivals of new customers is λ. The output rate is μ_1, when the machine is in the fast service rate, and μ_0 otherwise. We assume that $0 < \mu_0 < \mu_1$. The case $\mu_0 = 0$ is qualitatively similar, but it is omitted from this presentation, as the various formulas do change and they should be evaluated by taking appropriate limits (see e.g. (4.2)). The waiting (buffer) capacity of the facility is infinite. The dynamics of the process $\{X_{(t)}\}$ that records the level of the fluid is given by

$$\frac{dX_{(t)}}{dt} = \begin{cases} \lambda - \mu_i & if\, X_{(t)} > 0\, and\, Z_{(t)} = i, \\ (\lambda - \mu_i)^+ & if\, X_{(t)} = 0\, and\, Z_{(t)} = i, \end{cases} \tag{4.1}$$

where $(x)^+ = max(x, 0)$. It is easy to see that the bivariate process $\{(X_{(t)}, Z_{(t)})\}$ is Markovian. In what follows, we will use the notational convention $i' = 1 - i$, $i = 0, 1$. Thus, if i refers to the current state of the machine, i' is the opposite state.

The customers are strategic and decide whether to join or balk upon arrival with the objective of maximizing their expected utility. Every customer receives a reward of R units for completing service. On the other hand, he accumulates waiting cost at rate C per time unit in the system. The decisions of the customers are assumed irrevocable. In particular no retrials of balking customers nor reneging of entering customers are allowed. Since all customers are indistinguishable and each one tries to maximize his own benefit by taking into account that the other customers do the same, we can consider this situation as a symmetric game among them. We are interested in the computation of the symmetric equilibrium strategies for this game. These strategies are best responses against themselves, so no customer has an incentive to deviate from such a strategy unilaterally.

The strategic behaviour of the customers is influenced by the level of information that they receive upon arrival, before making their decisions. We will consider the following two cases:

- The fully observable (**fo**) case: Customers observe $(X_{(t)}, Z_{(t)})$
- The almost observable (**ao**) case: Customers observe only $X_{(t)}$.

There are two other interesting informational cases:

- The almost unobservable (**au**) case: Customers observe only $Z_{(t)}$.
- The fully unobservable (**fu**) case: Customers observe neither $X_{(t)}$ nor $Z_{(t)}$.

4.2 The fully observable case

Suppose that a customer arrives at the production facility and observes the state (x, i). Then, to assess its expected utility if he joins, he needs to compute his conditional expected sojourn time in the system, $S_i(x)$. Note that this conditional expected sojourn time does not depend on the strategies of the other customers, given (x, i). Indeed, future customers do not influence $S_i(x)$ because of the FCFS discipline and past customers do not influence it either, since reneging is not allowed. We can easily obtain closed formulas for $S_i(x)$. Indeed, we have the following Lemma 4.2.1.

Lemma 4.2.1. *The conditional expected sojourn time of a customer in the system, given that the state of the server is i and the fluid level in front of him is x is given by the formula*

$$\begin{aligned} S_i(x) = & \frac{q_0 + q_1}{q_0\mu_1 + q_1\mu_0} x + \frac{q_i \mu_{i'}(\mu_{i'} - \mu_i)}{(q_0\mu_1 + q_1\mu_0)^2} \\ & \times \left(1 - e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x}\right), \ x \geq 0, \ i = 0, 1. \end{aligned} \tag{4.2}$$

Proof. To compute $S_i(x)$, we condition on the length T_i of the remaining sojourn time of the machine at state i, which is exponentially distributed with rate q_i, because of the memoryless property of the exponential distribution. Then, we obtain

$$S_i(x) = \frac{x}{\mu_i} e^{-q_i \frac{x}{\mu_i}} + \int_0^{\frac{x}{\mu_i}} (t + S_{i'}(x - \mu_i t)) q_i e^{-q_i t} dt. \tag{4.3}$$

Changing the variable of the integration to $u = x - \mu_i t$ and using integration by parts yields

$$S_i(x) = \frac{1}{q_i} - \frac{1}{q_i} e^{-\frac{q_i}{\mu_i} x} + \frac{q_i}{\mu_i} e^{-\frac{q_i}{\mu_i} x} \int_0^x S_{i'}(u) e^{\frac{q_i}{\mu_i} u} du.$$

Multiplying by $\mu_i e^{\frac{q_i}{\mu_i} x}$ and differentiating with respect to x, we obtain the linear system of ODEs

$$\frac{dS_i(x)}{dx} = -\frac{q_i}{\mu_i} S_i(x) + \frac{q_i}{\mu_i} S_{i'}(x) + \frac{1}{\mu_i},\ i = 0,\, 1,$$

with initial conditions $S_i(0) = 0, i = 0,\, 1$. Using the standard theory for first-order linear systems of ODEs with constant coefficients (see e.g. Braun, 1983, Chapter 3), we obtain the unique solution stated in (4.2). □

We can now derive the customer equilibrium strategies in the **fo** case. We have the following Theorem 4.2.2.

Theorem 4.2.2. *In the **fo** case, the equilibrium strategies are specified by two thresholds, that is they prescribe "While arriving at time t, observe $(X_{(t)}, Z_{(t)})$, join if $X_{(t)} < x_e(Z_{(t)})$ and balk if $X_{(t)} > x_e(Z_{(t)})$". The thresholds $x_e(i), i = 0,\, 1$ are given as the unique roots of the equations*

$$\frac{q_0 + q_1}{q_0\mu_1 + q_1\mu_0} x + \frac{q_i \mu_{i'}(\mu_{i'} - \mu_i)}{(q_0\mu_1 + q_1\mu_0)^2} \times \left(1 - e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x}\right) = \frac{R}{C},\ i = 0,\, 1, \tag{4.4}$$

with respect to x.

Proof. The expected utility of a customer that observes upon arrival the system at state $(X_{(t)}, Z_{(t)}) = (x, i)$ and decides to join is $U^{(fo)}(x, i) = R - CS_i(x)$. Note that this expected utility does not depend on the strategies followed by the other customers. The customer prefers strictly to join if $U^{(fo)}(x, i) > 0$, prefers strictly to balk if $U^{(fo)}(x, i) < 0$ and he is indifferent between joining and balking if $U^{(fo)}(x, i) = 0$. These conditions are equivalent respectively to $S_i(x) < \frac{R}{C}$, $S_i(x) > \frac{R}{C}$ and $S_i(x) = \frac{R}{C}$. Differentiation of (4.2) yields

$$\frac{dS_i(x)}{dx} = \frac{q_0 + q_1}{q_0\mu_1 + q_1\mu_0} + \frac{q_i \mu_{i'}(\mu_{i'} - \mu_i)}{(q_0\mu_1 + q_1\mu_0)^2} \times \left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right) e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x},$$
$$x \geq 0,\ i = 0,\, 1, \tag{4.5}$$

$$\frac{d^2 S_i(x)}{dx^2} = -\frac{q_i \mu_{i'}(\mu_{i'} - \mu_i)}{(q_0\mu_1 + q_1\mu_0)^2} \left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)^2 e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x},\ x \geq 0,\ i = 0,\, 1. \tag{4.6}$$

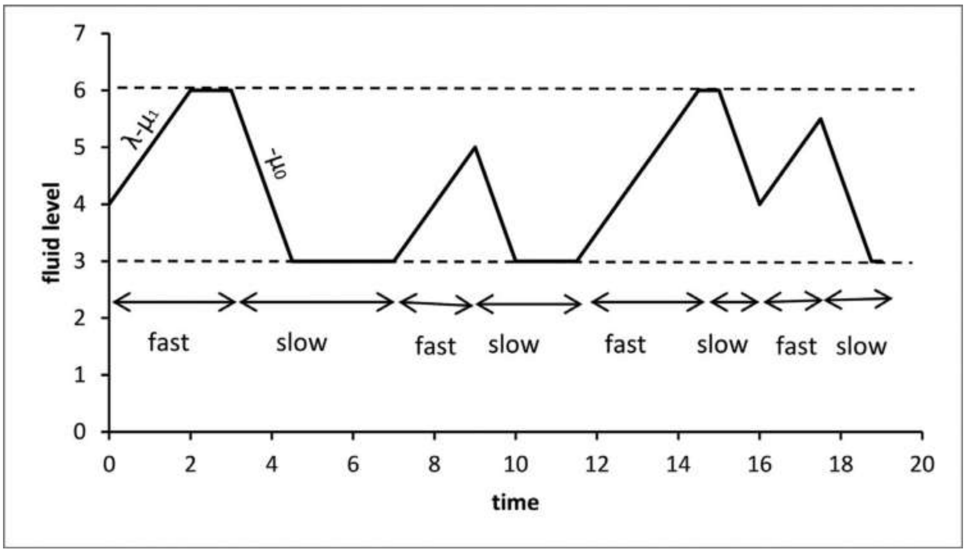


Figure 4.1: Typical sample path of fluid level for $x_*(0) = 3$, $x_*(1) = 6$, $\lambda = 5$, $\mu_0 = 2$ and $\mu_1 = 4$ (corresponding to case I of Lemma 4.2.3).

Therefore $\frac{dS_0(x)}{dx} > 0$, $\frac{d^2S_1(x)}{dx^2} > 0$ and $\frac{dS_1(0)}{dx} = \frac{1}{\mu_1} > 0$ and we conclude that $S_i(x)$ is strictly increasing in $[0, \infty)$ with image the interval $[0, \infty)$. This shows that there exists a unique root $x_e(i)$ of $S_i(x) = \frac{R}{C}$ with respect to x which is exactly Eq. (4.4). Then, a customer that finds the system at state (x, i) prefers to join for $x < x_e(i)$. He is indifferent between joining and balking if $x = x_e(i)$. Finally, he prefers to balk for $x > x_e(i)$. $\square$

We can see that $x_e(0) < x_e(1)$, that is a customer finding the system at the fast service rate affords higher fluid levels in front of him under the equilibrium strategy than a customer that finds the system at the slow service state. This is intuitively clear and can be formally shown since $S_1(x) < S_0(x)$, $x > 0$. Indeed

$$S_1(x) - S_0(x) = \frac{\mu_0 - \mu_1}{q_0\mu_1 + q_1\mu_0}\left(1 - e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x}\right),\ x \geq 0.$$

A subtle point is that an equilibrium strategy should specify necessarily joining or balking, according to whether $X_{(t)} < x_e(Z_{(t)})$ or not, only for states $(X_{(t)}, Z_{(t)})$ that are reachable with positive probability, given the initial state. Indeed, an equilibrium strategy can prescribe anything in states that are not reachable. For example, suppose that the system is initially empty and consider a strategy that prescribes joining, whenever ($Z_{(t)} = 0$ and $X_{(t)} \leq x_e(0)$) or ($Z_{(t)} = 1$ and $X_{(t)} \neq x_e(1)$). This is an equilibrium strategy, because what it does for states $(X_{(t)}, Z_{(t)}) = (x, 1)$ with $x > x_e(1)$

does not matter, since such states are never observed. However we can use the notion of subgame perfect equilibrium strategy to refine the equilibrium strategies and to eliminate such "pathological" cases (that occur also in the standard analysis of the $M/M/1$ queue - see e.g. Hassin and Haviv (2003) Remark 2.2 in p.24, or Hassin and Haviv (2002)). Then, the subgame perfect equilibrium strategies prescribe joining when $X_{(t)} < x_e(Z_{(t)})$, balking when $X_{(t)} > x_e(Z_{(t)})$ and whatever for $X_{(t)} = x_e(Z_{(t)})$.

Moreover, we note that in a game-theoretical terminology, the equilibrium strategies in the **fo** case are dominant strategies. Indeed, such a strategy is a best response of a tagged customer against any strategy of the other customers. Thus, we have a very strong equilibrium concept in the **fo** case. This does not happen in the other informational case as we will see in the following section.

We now move to the computation of the expected social benefit function per time unit, when the customers follow a threshold strategy. In the statement of Theorem 4.2.2, we have not explicitly stated what the customers do when they observe $(X_{(t)}, Z_{(t)})$ with $X_{(t)} = x_e(Z_{(t)})$. Clearly, an equilibrium strategy can prescribe anything at such a state. However, for the computation of the expected social benefit per time unit under a threshold strategy with threshold-vector $(x_*(0), x_*(1))$, it is necessary to take into account the fraction of entering customers, when the fluid level $x_*(i)$ is reached and the environment is in state i, $i = 0, 1$. Therefore, for a certain threshold strategy, we note that when the threshold has been reached, this fraction stabilizes the fluid level till the next change of the environmental state. Now, we compute the steady-state distribution of the fluid level, when the customers follow such a strategy, that is we compute the functions $F_i(x)$ with

$$F_i(x) = Pr[X_{(t)} \leq x,\, Z_{(t)} = i],\ x \geq 0,\ i = 0, 1 \tag{4.7}$$

where t is an arbitrary time point, assuming that the process $\{(X_{(t)}, Z_{(t)}\}$ has achieved stationarity. This is done in Lemma 4.2.3. To facilitate the reader to follow the proof, we provide some graphs that show the typical sample paths of $X_{(t)}$ under a strategy specified by two thresholds $x_*(0)$ and $x_*(1)$. These are given in Figs. 4.1 and 4.2.

Lemma 4.2.3. *Suppose that the customers follow a strategy specified by two thresholds $x_*(0)$ and $x_*(1)$, such that $x_*(0) \leq x_*(1)$. Then, we have the following cases.*

Case I. $\lambda > \mu_1$.

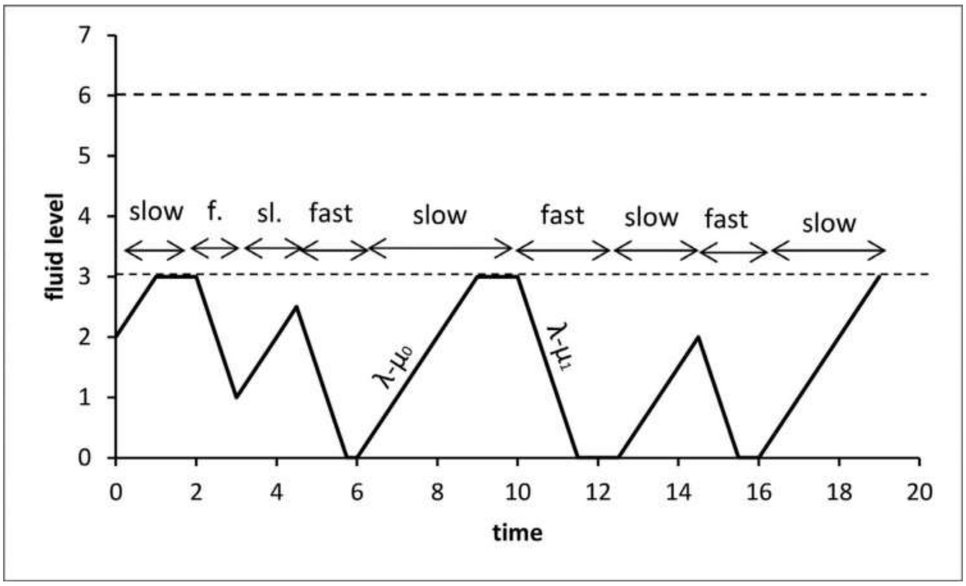


Figure 4.2: Typical sample path of fluid level for $x_*(0) = 3$, $x_*(1) = 6$, $\lambda = 3$, $\mu_0 = 2\, and\, \mu_1 = 5$ (corresponding to case III of Lemma 4.2.3).

The fluid level oscillates in $[x_*(0),\, x_*(1)]$. *In particular*

$$F_0(x) = \begin{cases} 0 \\ \quad if\, x < x_*(0) \\ \frac{q_1}{q_0+q_1} \cdot \frac{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x - x_*(0)]\right\}}{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1) - x_*(0)]\right\}} \\ \quad if\, x_*(0) \le x \le x_*(1) \\ \frac{q_1}{q_0+q_1} \\ \quad if\, x \ge x_*(1) \end{cases} \tag{4.8}$$

and

$$F_1(x) = \begin{cases} 0 \\ \quad if\, x \le x_*(0) \\ \frac{q_0}{q_0+q_1} \cdot \frac{-q_1\mu_0\left(1 - \exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x - x_*(0)]\right\}\right)}{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1) - x_*(0)]\right\}} \\ \quad if\, x_*(0) \le x \le x_*(1) \\ \frac{q_0}{q_0+q_1} \\ \quad if\, x \ge x_*(1) \end{cases} \tag{4.9}$$

In other words, the distribution $F_0(x)$ *is mixed with a point mass*

$p_0(x_*(0))$ at $x_*(0)$ given by

$$p_0(x_*(0)) = \frac{q_1}{q_0+q_1} \cdot \frac{-q_1\mu_0 + q_0(\lambda-\mu_1)}{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1)-x_*(0)]\right\}} \tag{4.10}$$

and probability density $f_0(x)$ in $(x_*(0), x_*(1))$ given by

$$f_0(x) = \frac{q_1}{q_0+q_1} \cdot \frac{q_0(\lambda-\mu_1)\left(\frac{q_0}{\mu_0} - \frac{q_1}{\lambda-\mu_1}\right)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x-x_*(0)]\right\}}{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1)-x_*(0)]\right\}}, \quad x_*(0) < x < x_*(1), \tag{4.11}$$

while $F_1(x)$ is mixed with a point mass $p_1(x_*(1))$ at $x_*(1)$ given by

$$p_1(x_*(1)) = \frac{q_0}{q_0+q_1} \cdot \frac{[q_0(\lambda-\mu_1) - q_1\mu_0]\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1)-x_*(0)]\right\}}{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1)-x_*(0)]\right\}} \tag{4.12}$$

and probability density $f_1(x)$ in $(x_*(0), x_*(1))$ given by

$$f_1(x) = \frac{q_0}{q_0+q_1} \cdot \frac{q_1\mu_0\left(\frac{q_0}{\mu_0} - \frac{q_1}{\lambda-\mu_1}\right)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x-x_*(0)]\right\}}{-q_1\mu_0 + q_0(\lambda-\mu_1)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\mu_0}\right)[x_*(1)-x_*(0)]\right\}}, \quad x_*(0) < x < x_*(1). \tag{4.13}$$

Case II. $\lambda = \mu_1$.

The fluid level stabilizes at $x_*(0)$.

Case III. $\mu_0 < \lambda < \mu_1$.

The fluid level oscillates in $[0, x_*(0)]$*. In particular*

$$F_0(x) = \begin{cases} 0 \\ \quad if\, x \le 0 \\ \frac{q_1}{q_0+q_1} \cdot \frac{q_0(\lambda-\mu_1)\left(1-\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x\right\}\right)}{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}} \\ \quad if\, 0 \le x < x_*(0) \\ \frac{q_1}{q_0+q_1} \\ \quad if\, x \ge x_*(0) \end{cases} \tag{4.14}$$

and

$$F_1(x) = \begin{cases} 0 \\ \quad if\, x < 0 \\ \frac{q_0}{q_0+q_1} \cdot \frac{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x\right\}}{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}} \\ \quad if 0 \le x \le x_*(0) \\ \frac{q_0}{q_0+q_1} \\ \quad if\, x \ge x_*(0) \end{cases} \tag{4.15}$$

In other words, both $F_0(x)$ *and* $F_1(x)$ *are mixed. The distribution* $F_0(x)$ *has a point mass* $p_0(x_*(0))$ *at* $x_*(0)$ *given by*

$$p_0(x_*(0)) = \frac{q_1}{q_0+q_1} \cdot \frac{[q_1(\lambda-\mu_0)+q_0(\lambda-\mu_1)]\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}}{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}} \tag{4.16}$$

and probability density $f_0(x)$ *in* $(0,\, x_*(0))$ *given by*

$$f_0(x) = \frac{q_0}{q_0+q_1} \cdot \frac{q_0(\lambda-\mu_1)\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x\right\}}{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}}, \quad 0 < x < x_*(0) \tag{4.17}$$

Regarding $F_1(x)$*, we can easily see that it has a point mass* $p_1(0)$ *at* 0 *given by*

$$p_1(0) = \frac{q_0}{q_0+q_1} \cdot \frac{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)}{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}} \tag{4.18}$$

and probability density $f_1(x)$ in $(0,\, x_*(0))$ given by

$$f_1(x) = \frac{q_0}{q_0+q_1} \cdot \frac{q_1(\lambda-\mu_0)\left(-\frac{q_1}{\lambda-\mu_1} - \frac{q_0}{\lambda-\mu_0}\right)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x\right\}}{q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}},$$
$$0 < x < x_*(0). \tag{4.19}$$

Case IV. The fluid level stabilizes at 0.

Proof. First, note that in all cases, the fluid level enters in levels smaller than or equal to $x_*(1)$, because of the threshold strategy and the fact that $x_*(0) \leq x_*(1)$.

Cases II and IV are immediate. Indeed, for case II, if the fluid starts from a level below $x_*(0)$, then during sojourn times in the fast mode it stays at the same level (since $\lambda - \mu_1 = 0$), while during sojourn times in the slow mode it grows linearly at rate $\lambda - \mu_0$, until it reaches the level $x_*(0)$. Thereafter, it stays at this level. If it starts from a level above $x_*(0)$, a similar situation occurs and the fluid reaches again eventually $x_*(0)$. Case IV is also clear, as there is a non-positive drift under the slow mode and a strictly negative drift under the fast mode, so the fluid level decreases (with some possible constant intervals) till it reaches zero.

For cases I and III, let us consider a fluid queue that alternates between two environmental states $-$ and $+$, with exponential sojourn times with rates q_- and q_+ respectively, where the fluid drift is $\eta_- < 0$ for the environmental state $-$ and $\eta_+ > 0$ for the environmental state $+$. If the fluid is constrained to oscillate in $[0, T]$ (i.e., the fluid rates become zero when the fluid hits the boundary states 0 and T) and $F_-(x)$ (respectively $F_+(x)$) denotes the limiting probability that the fluid level is smaller than or equal to x and the environment is at state $-$ (respectively $+$), then we have that $F_-(x)$ and $F_+(x)$ are differentiable in $(0, T)$ and satisfy the linear system of ODEs

$$\eta_- \frac{dF_-(x)}{dx} = -q_-F_-(x) + q_+F_+(x), \tag{4.20}$$
$$\eta_+ \frac{dF_+(x)}{dx} = q_-F_-(x) - q_+F_+(x), \tag{4.21}$$

with boundary conditions

$$F_-(T) = \frac{q_+}{q_- + q_+},\ F_+(0) = 0 \tag{4.22}$$

(see e.g. Kulkarni (1997)). Using the standard theory for first-order linear systems of ODEs with constant coefficients and computing the constants by the boundary conditions (see e.g. Braun (1983), Chapter 3), we have that this system has a unique solution which is given by the relations

$$F_{-}(x)=\begin{cases}0 & if\, x<0\\ \frac{q_{+}}{q_{-}+q_{+}}\cdot\frac{q_{+}\eta_{-}+q_{-}\eta_{+}\exp\left\{-\left(\frac{q_{-}}{\eta_{-}}+\frac{q_{+}}{\eta_{+}}\right)x\right\}}{q_{+}\eta_{-}+q_{-}\eta_{+}\exp\left\{-\left(\frac{q_{-}}{\eta_{-}}+\frac{q_{+}}{\eta_{+}}\right)T\right\}} & if\, 0\leq x\leq T\\ \frac{q_{+}}{q_{-}+q_{+}} & if\, x\geq T\end{cases} \tag{4.23}$$

and

$$F_{+}(x)=\begin{cases}0 & if\, x\leq 0\\ \frac{q_{-}}{q_{-}+q_{+}}\cdot\frac{q_{+}\eta_{-}\left(1-\exp\left\{-\left(\frac{q_{-}}{\eta_{-}}+\frac{q_{+}}{\eta_{+}}\right)x\right\}\right)}{q_{+}\eta_{-}+q_{-}\eta_{+}\exp\left\{-\left(\frac{q_{-}}{\eta_{-}}+\frac{q_{+}}{\eta_{+}}\right)T\right\}} & if\, 0\leq x\leq T\\ \frac{q_{+}}{q_{-}+q_{+}} & if\, x\geq T\end{cases} \tag{4.24}$$

For case I, a moment of reflection shows that the fluid oscillates in $[x_*(0), x_*(1)]$ with negative drift $-\mu_0$ when the machine is in the slow mode and positive drift $\lambda-\mu_1$ when then machine is in the fast mode. We can then use the formulas (4.23) and (4.24), substituting $x-x_*(0)$ for x and setting $q_-=q_0$, $q_+=q_1$, $\eta_-=-\mu_0$ and $\eta_+=\lambda-\mu_1$. This yields (4.8) and (4.9).

Similarly, in case III, the fluid is easily seen to oscillate in $[0, x_*(0)]$ and we use formulas (4.23) and (4.24) with $q_-=q_1$, $q_+=q_0$, $\eta_-=\lambda-\mu_1$ and $\eta_+=\lambda-\mu_0$ to obtain (4.14) and (4.15). □

We are now ready to compute the function of the expected social benefit per time unit, $B^{(fo)}(x_*(0),x_*(1))$, when the customers follow a threshold strategy with thresholds $x_*(0)$ and $x_*(1)$. We have Theorem 4.2.4.

Theorem 4.2.4. *Suppose that the customers follow a strategy specified by two thresholds $x_*(0)$ and $x_*(1)$, such that $x_*(0)\leq x_*(1)$. Then, we have the following cases.*

Case I. $\lambda>\mu_1$

The expected social benefit per time unit is given by

$$B^{(fo)}(x_*(0),x_*(1))=\left(\frac{q_1}{q_0+q_1}\mu_0+\frac{q_0}{q_0+q_1}\mu_1\right)R-CE[X], \tag{4.25}$$

where

$$E[X]=x_*(0)p_0(x_*(0))+\int_{x_*(0)}^{x_*(1)}x(f_0(x)+f_1(x))dx+x_*(1)p_1(x_*(1)) \tag{4.26}$$

with $p_0(x_*(0))$, $f_0(x)$, $p_1(x_*(1))$ and $f_1(x)$ given by (4.10)-(4.13).

Case II. $\lambda = \mu_1$.

The expected social benefit per time unit is given by

$$B^{(fo)}(x_*(0), x_*(1)) = \lambda \left(\frac{q_1}{q_0 + q_1} \cdot \frac{\mu_0}{\lambda} + \frac{q_0}{q_0 + q_1} \right) R - Cx_*(0). \quad (4.27)$$

Case III. $\mu_0 < \lambda < \mu_1$.

The expected social benefit per time unit is given by

$$\begin{aligned} B^{(fo)}(x_*(0), x_*(1)) = \lambda & \left(\int_0^{x_*(0)} f_0(x)dx + p_0(x_*(0))\frac{\mu_0}{\lambda} + \frac{q_0}{q_0 + q_1} \right) R \\ & - CE[X], \end{aligned} \quad (4.28)$$

where

$$E[X] = x_*(0)p_0(x_*(0)) + \int_0^{x_*(0)} x(f_0(x) + f_1(x))dx \quad (4.29)$$

with $p_0(x_*(0))$, $f_0(x)$ *and* $f_1(x)$ *given by (4.16), (4.17) and (4.19), respectively.*

Case IV. $\lambda \leq \mu_0$.

The expected social benefit per time unit is given by

$$B^{(fo)}(x_*(0), x_*(1)) = \lambda R. \quad (4.30)$$

Proof. In all cases, the expected social benefit per time unit is given as

$$B^{(fo)}(x_*(0), x_*(1)) = \lambda_{eff} R - CE[X], \quad (4.31)$$

where λ_{eff} is the effective fluid arrival rate, which counts the arrivals of customers that do enter in the system and $E[X]$ is the expected stationary fluid level. In case I, the fluid fluctuates in $[x_*(0), x_*(1)]$. Moreover, the fluid once at level $x_*(0)$ during an environmental sojourn time at the slow mode, it stays there till the next environmental change. Therefore, a fraction $\frac{\mu_0}{\lambda}$ of the customers that find the system at the slow mode and the fluid at level $x_*(0)$ do enter in the system, in order to ensure that the net fluid change rate is 0. On the other hand, customers that find the system at the slow mode but the fluid level strictly above $x_*(0)$ do not enter. Similarly, all customers that find the system at the fast mode and the fluid level strictly below $x_*(1)$ do enter, whereas only a fraction $\frac{\mu_1}{\lambda}$ of the customers that find the system at the fast mode and the fluid at level $x_*(1)$ enter in the system. Thus, the effective arrival rate in case I is

$$\lambda_{eff} = \lambda \left(p_0(x_*(0))\frac{\mu_0}{\lambda} + \int_{x_*(0)}^{x_*(1)} f_1(x)dx + p_1(x_*(1))\frac{\mu_1}{\lambda} \right). \quad (4.32)$$

The expected stationary fluid level $E[X]$ is computed by (4.26), as the distribution of the fluid level has point masses $p_0(x_*(0))$ and $p_1(x_*(1))$ at $x_*(0)$ and $x_*(1)$, given by (4.10) and (4.12), and probably density function $f_0(x) + f_1(x)$ with $f_0(x)$ and $f_1(x)$ given by (4.11) and (4.13). Substitution of (4.32) and (4.26) in (4.31) yields after some straightforward algebra (4.25). Another, more intuitive, way to justify (4.25) is to observe that in case I the server is continuously busy, so the effective arrival rate is equal to the mean service rate, which is $\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1$.

In the case II, the fluid level stabilizes at $x_*(0)$ so $E[X] = x_*(0)$ and all customers see upon arrival this state. Therefore, all customers that find the machine at the fast mode enter, while only a fraction $\frac{\mu_0}{\lambda}$ of those that find the machine at the slow mode do enter. Hence the effective arrival rate is now

$$\lambda_{eff} = \lambda \left(\frac{q_1}{q_0 + q_1} \cdot \frac{\mu_0}{\lambda} + \frac{q_0}{q_0 + q_1} \right) \tag{4.33}$$

and we obtain readily (4.27).

Case III is proved similarly to case I, so we omit the details. Finally, in case IV, all customers enter and are served immediately without delay. Hence $\lambda_{eff} = \lambda$ and $E[X] = 0$, so we obtain (4.30). □

The formulas for the expected social benefit per time unit in Theorem 4.2.4 can be further reduced in more explicit expressions, as the relevant integrals are computable in closed form (indeed, a moment of reflection shows that only exponential functions are involved). Nevertheless, the formulas are too complicate and there is no need to be reported. Due to this complexity, the socially optimal strategies cannot be computed in closed form in all cases. More concretely, in cases I, II and IV, it is easy to determine the socially optimal strategies. However, in case III, an explicit solution is not possible, so we have performed several numerical experiments and we present the main results.

In cases I and II, the effective arrival rate is independent of the strategy $(x_*(0), x_*(1))$ and equals the mean service rate $\overline{\mu} = \frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1$, i.e.,

$$\lambda_{eff} = \overline{\mu}. \tag{4.34}$$

Indeed, in these two cases, we have that $\lambda \geq \max(\mu_0, \mu_1)$ and consequently the server uses his maximum service capacity at both environmental states. So, a strategy is socially optimal, if it minimizes the expected fluid level. Thus, in case I, the unique socially optimal strategy is the threshold strategy with threshold vector $(0, 0)$, whereas in case II, every threshold strategy with threshold vector $(0, x), x \geq 0$, is socially optimal.

In case IV, the arrival rate λ is so small that the buffer remains empty under any strategy. Therefore, the strategy is not relevant for the fluid level. So, the socially optimal behaviour is always to join as a joining customer receives the reward from service and has no waiting cost.

In case III, the expected social benefit per time unit under a threshold strategy $(x_*(0), x_*(1))$, with $x_*(0) \leq x_*(1)$, depends only on the threshold $x_*(0)$. So, we can write $B^{(fo)}(x_*(0))$ instead of $B^{(fo)}(x_*(0), x_*(1))$ for this case. As the threshold $x_*(0)$ increases, we have a socially favourable effect, the increase of the effective arrival rate, and a socially unfavourable one, the increase of the expected fluid level. The effective arrival rate is given as

$$\begin{aligned}\lambda_{eff} &= \lambda \left(\int_0^{x_*(0)} f_0(x)dx + p_0(x_*(0))\frac{\mu_0}{\lambda} + \frac{q_0}{q_0+q_1} \right) \\ &= \frac{q_0(\lambda-\mu_1) + q_1(\lambda-\mu_0)\left(\frac{q_1\mu_0}{q_0+q_1}\right)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}}{q_0(\lambda-\mu_1) + q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}}\end{aligned} \tag{4.35}$$

which can be seen to be an increasing and concave function in $x_*(0)$. Moreover,

$$\lim_{x_*(0)\to\infty} \lambda_{eff} = \lambda \tag{4.36}$$

The expected fluid level can be computed by evaluating the integrals in (4.29). After some straightforward algebraic manipulations, it yields

$$\begin{aligned}E[X] &= \frac{q_1(q_0+q_1)(\lambda-\mu_0)x_*(0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}}{q_0+q_1)\left[q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}\right]} \\ &+ \frac{q_0q_1(\mu_1-\mu_0)\left[\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}-1\right]}{(q_0+q_1)\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)\left[q_0(\lambda-\mu_1)+q_1(\lambda-\mu_0)\exp\left\{-\left(\frac{q_1}{\lambda-\mu_1}+\frac{q_0}{\lambda-\mu_0}\right)x_*(0)\right\}\right]}.\end{aligned} \tag{4.37}$$

The expected fluid is also an increasing function in $x_*(0)$ but the limit as $x_*(0) \to \infty$ depends on the relative order of the arrival rate λ and the mean service rate $\overline{\mu}$ when all customers join. In particular, taking $x_*(0) \to \infty$ yields

$$\lim_{x_*(0)\to\infty} E[X] = \begin{cases} \frac{q_1(\mu_1-\mu_0)(\lambda-\mu_0)}{(q_0+q_1)[q_1(\mu_0-\lambda)+q_0(\mu_1-\lambda)]} & if\ \frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1 > \lambda, \\ \infty & if\ \frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1 \leq \lambda. \end{cases} \tag{4.38}$$

Indeed, if the mean service rate $\overline{\mu}$ exceeds the arrival rate λ, we see that the mean drift $\lambda - \overline{\mu}$ of the fluid process is negative and therefore the expected fluid level should tend to a finite number, as $x_*(0) \to \infty$. In the other hand, if the mean service rate is smaller than or equal to the arrival rate λ, the fluid increases beyond any finite number in the long run. From the above discussion, we realize that the behavior of the function $B^{(fo)}(x_*(0))$ as

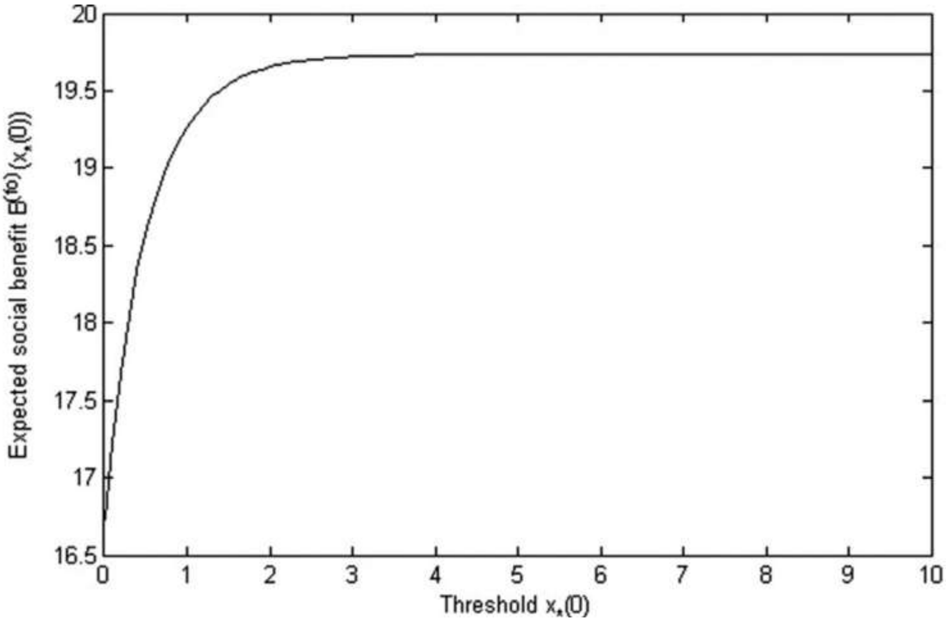


Figure 4.3: Expected social benefit $B^{(fo)}(x_*(0))$ with respect to the threshold $x_*(0)$ for $q_0 = 2$, $q_1 = 1$, $\lambda = 2$, $\mu_0 = 1\,\mu_1 = 5$, $R = 10$ and $C = 1$.

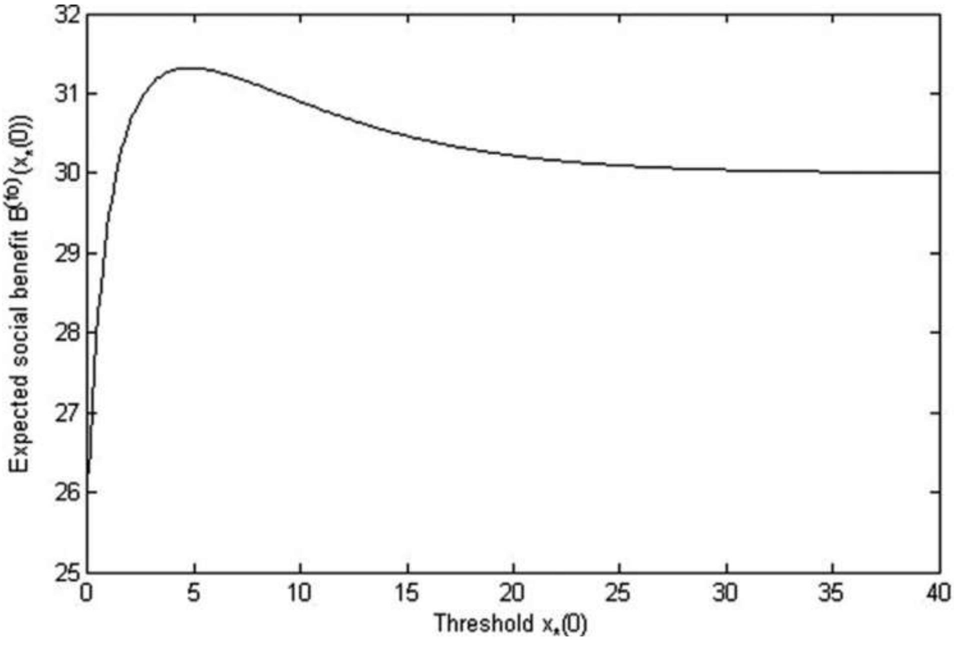


Figure 4.4: Expected social benefit $B^{(fo)}(x_*(0))$ with respect to the threshold $x_*(0)$ for $q_0 = 2$, $q_1 = 1$, $\lambda = 3.4$, $\mu_0 = 1\,\mu_1 = 5$, $R = 10$ and $C = 1$.

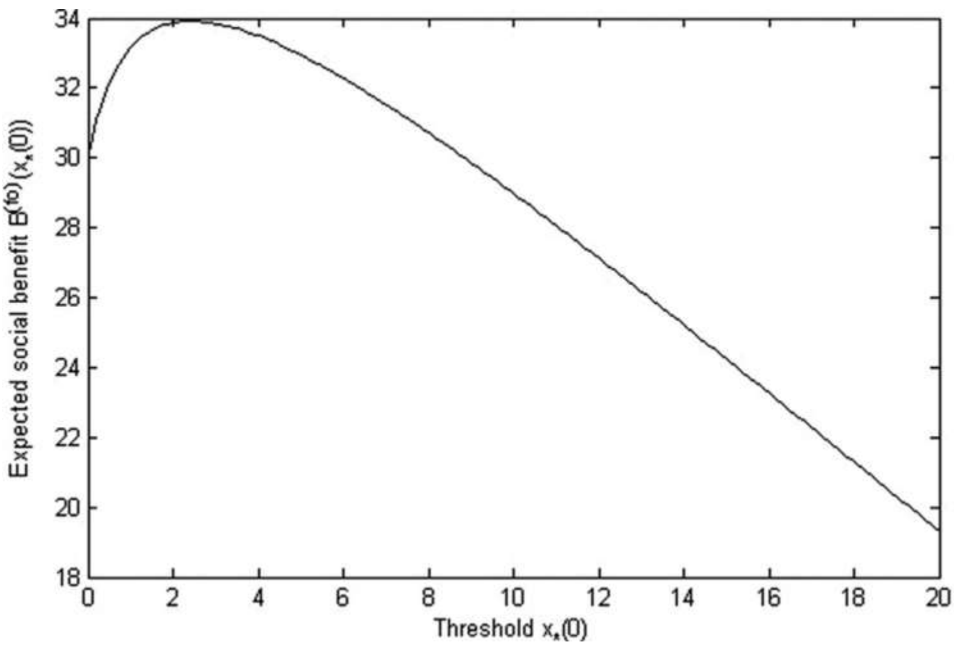


Figure 4.5: Expected social benefit $B^{(fo)}(x_*(0))$ with respect to the threshold $x_*(0)$ for $q_0 = 2$, $q_1 = 1$, $\lambda = 4$, $\mu_0 = 1\,\mu_1 = 5$, $R = 10$ and $C = 1$.

$x_*(0)$ increases depends on the trade-off between the increase of the effective arrival rate and the increase of the expected fluid level.

In Figs. 4.3-4.5, we present the three typical cases for the behavior of the expected social benefit function $B^{(fo)}$ with respect to $x_*(0)$. We have considered a basic scenario with rates $q_0 = 2$, $q_1 = 1$, $\mu_0 = 1$, $\mu_1 = 5$, $R = 10$ and $C = 1$. The arrival rate λ varies and takes the values 2, 3.4 and 4 for Figs. 4.3-4.5, respectively.

In Fig.4.3, the arrival rate is small enough so that the mean drift is negative. The expected social benefit is an increasing function of the threshold $x_*(0)$ and tends to 19.7333 (taking into account (4.36) and (4.38)). The socially optimal strategy in this case corresponds to $x_*(0) = \infty$, i.e., it is the 'always join' strategy. This case (increasing expected social benefit) occurs generally when the arrival rate dominates the negative effect on the expected fluid level.

In Fig. 4.4, the arrival rate is a bit larger, but still the mean drift is negative. In this case the expected social benefit is a unimodal function of the threshold $x_*(0)$ that stabilizes as $x_*(0) \to \infty$. In particular, we see that the expected social benefit is increasing in $x_*(0)$, for $x_*(0) \in [0, 4.8]$, and decreasing in $x_*(0)$, for $x_*(0) \in [4.8, \infty)$. As $x_*(0) \to \infty$, we have that $B^{(fo)}(x_*(0)) \to 30$. So, every threshold strategy with threshold-vector $(x_*(0), x_*(1))$, where $x_*(0) = 4.8$ and $x_*(1) \geq x_*(0)$ is socially optimal. In general, this is the typical case (unimodal expected social benefit with finite limit) for intermediate values of λ, i.e., when the drift is negative, but λ is not very close to μ_0.

Finally, in Fig. 4.5, the arrival rate is higher and the mean drift becomes positive so that the expected fluid level tends to infinity as $x_*(0)$ tends to infinity. The expected social benefit is increasing in $x_*(0)$, for $x_*(0) \in [0, 24]$, and decreasing in $x_*(0)$, for $x_*(0) \geq 2.4$. So, every threshold strategy with threshold vector $(x_*(0), x_*(1))$, where $x_*(0) = 2.4$ and $x_*(1) \geq x_*(0)$ is socially optimal. This typical case (unimodal expected social benefit with infinite limit) occurs when the arrival rate is so large that the drift is non-negative.

4.3 The almost observable case

In this section, we identify the equilibrium customer strategies regarding the joining/balking dilemma, when the customers observe only the level of the fluid before making their decisions. We limit ourselves to the class of threshold strategies, where customers decide to join if they find the fluid at levels below some threshold x_*, while they decide to balk if the fluid level exceeds x_*. As in the **fo** case (see the relevant discussion just before Lemma 4.2.3), we need to specify the fraction of entering customers when the fluid level reaches the threshold x_*. Again, when the threshold has been reached, then the maximum fraction of customers is entered so that the fluid level does not exceed the threshold. To assess the best response of a tagged customer, given that the others follow a given threshold strategy with threshold x_*, we need to compute the steady-state distribution of the fluid level, that is the distribution functions $F_i(x)$ defined in (4.7). This can be done using Lemma 4.2.3 for $x_*(0) = x_*(1) = x_*$. Then, we can obtain the conditional expected sojourn time of a customer in the system, given that the fluid level he finds upon arrival is x and the others follow a threshold strategy x_*. This is done in Lemma 4.3.1.

Lemma 4.3.1. *Let $S^{(ao)}(x; x_*)$ denote the conditional expected sojourn time of a customer in the system , given that the fluid level is x and the other customers follow a threshold strategy x_*.*

Case I. $\lambda \geq \mu_1$.

$$S^{(ao)}(x_*; x_*) = \frac{q_0 + q_1}{q_0\mu_1 + q_1\mu_0} x_* + \frac{q_0 q_1 (\mu_1 - \mu_0)^2}{(q_0 + q_1)(q_0\mu_1 + q_1\mu_0)^2} \left(1 - e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x_*}\right). \quad (4.39)$$

Case II. $\mu_0 < \lambda < \mu_1$.

$$S^{(ao)}(x;x_*) = \begin{cases} 0 \\ \quad if\, x = 0 \\ \frac{q_0+q_1}{q_0\mu_1+q_1\mu_0}x - \frac{q_1\mu_0(\lambda-\mu_0)+q_0\mu_1(\lambda-\mu_1)}{(q_0\mu_1+q_1\mu_0)^2}\left(1-e^{-\left(\frac{q_0}{\mu_0}+\frac{q_1}{\mu_1}\right)x_*}\right) \\ \quad if\, 0 < x < x_* \\ \frac{q_0+q_1}{q_0\mu_1+q_1\mu_0}x_* - \frac{q_0\mu_1(\mu_0-\mu_1)}{(q_0\mu_1+q_1\mu_0)^2}\left(1-e^{-\left(\frac{q_0}{\mu_0}+\frac{q_1}{\mu_1}\right)x_*}\right) \\ \quad if\, x = x_*. \end{cases} \tag{4.40}$$

Case III. $\lambda \le \mu_0$.

$$S^{(ao)}(0;x_*) = 0. \tag{4.41}$$

Proof. By conditioning on the (unobservable) state of the machine at the arrival time of a tagged customer, we have that

$$S^{(ao)}(x;x_*) = \pi_{Z|X}(0|x;x_*)S_0(x) + \pi_{Z|X}(1|x;x_*)S_1(x),\ x \ge 0, \tag{4.42}$$

where $\pi_{Z|X}(i|x;x_*)$ is the probability that the tagged customer finds the machine at mode i, given that he observes fluid level x and the other customers follow the threshold strategy x_*.

In case I (which corresponds to cases I and II of Lemma 4.2.3), the fluid level stabilizes at x_*. Therefore, all customers observe this level upon arrival, so the observation is uninformative. So, in case I, we have $\pi_{Z|X}(0|x;x_*) = \frac{q_1}{q_0+q_1}$ and $\pi_{Z|X}(1|x;x_*) = \frac{q_0}{q_0+q_1}$. Using (4.42) and (4.2) we obtain easily (4.39).

In case II (which corresponds to case III of Lemma 4.2.3), the fluid level oscillates in $[0, x_*]$. For $x_* > 0$, recall that the distributions $F_0(x)$ and $F_1(x)$ are given by (4.14) and (4.15), with $x_*(0) = x_*$. Since $F_1(x)$ has a point mass at 0, while $F_0(x)$ has not, we conclude that $\pi_{Z|X}(0|0;x_*) = 0$. Similarly, since $F_0(x)$ has a point mass at x_*, while $F_1(x)$ has not, we conclude that $\pi_{Z|X}(0|x_*;x_*) = 1$. Finally, for $x \in (0, x_*)$, we have that $\pi_{Z|X}(0|x;x_*) = \frac{f_0(x)}{f_0(x)+f_1(x)}$, with $f_0(x)$ and $f_1(x)$ given by (4.17) and (4.19) respectively, with $x_*(0) = x_*$. This yields, after a few simplifications, $\pi_{Z|X}(0|x;x_*) = \frac{\mu_1-\lambda}{\mu_1-\mu_0}$, $0 < x < x_*$. Thus in a nutshell,

$$\pi_{Z|X}(0|x;x_*) = \begin{cases} 0 & x = 0 \\ \frac{\mu_1-\lambda}{\mu_1-\mu_0} & 0 < x < x_*, \\ 1 & x = x_* \end{cases}$$

$$\pi_{Z|X}(1|x;x_*) = \begin{cases} 1 & x = 0 \\ \frac{\lambda-\mu_0}{\mu_1-\mu_0} & 0 < x < x_*, \\ 0 & x = x_*. \end{cases} \tag{4.43}$$

Using (4.43), (4.42) and (4.2), we obtain easily (4.40), whenever $x_* > 0$. For $x_* = 0$, all customers observe fluid level equal to 0 upon arrival, so $S_{(ao)}(0; x_*) = 0$ and we can see that (4.40) remains valid. The same argument yields also (4.41), in case III. $\square$

We can now identify all equilibrium threshold strategies in the **ao** case. We have Theorem 4.3.2.

Theorem 4.3.2. *In the* ***ao*** *case, we have the following cases for the existence of equilibrium threshold strategies that prescribe "While arriving at time t, observe $X_{(t)}$, join if $X_{(t)} < x_e$ and balk if $X_{(t)} > x_e$".*

Case I. $\lambda \geq \mu_1$.

There exists a unique equilibrium threshold strategy with threshold x_e which is the unique root of the equation

$$\frac{q_0 + q_1}{q_0\mu_1 + q_1\mu_0} x + \frac{q_0 q_1 (\mu_1 - \mu_0)^2}{(q_0 + q_1)(q_0\mu_1 + q_1\mu_0)^2} \cdot \left(1 - e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x}\right) = \frac{R}{C} \tag{4.44}$$

with respect to x.

Case II. $\mu_0 < \lambda < \mu_1$.

There exists a unique equilibrium threshold strategy with threshold x'_e which is the unique root of the equation

$$\frac{q_0 + q_1}{q_0\mu_1 + q_1\mu_0} x + \frac{q_0 \mu_1 (\mu_1 - \mu_0)}{(q_0\mu_1 + q_1\mu_0)^2} \cdot \left(1 - e^{-\left(\frac{q_0}{\mu_0} + \frac{q_1}{\mu_1}\right)x_*}\right) = \frac{R}{C} \tag{4.45}$$

with respect to x.

Case III. $\lambda \leq \mu_0$

All threshold strategies are equilibrium strategies.

Proof. Suppose that the customers follow a threshold strategy x_*. The expected utility of a tagged customer that observes upon arrival the fluid at level $X_{(t)} = x$ and decides to join is $U^{(ao)}(x; x_*) = R - CS^{(ao)}(x; x_*)$.

In case I, because the fluid level stabilizes at x_*, the tagged customer will see necessarily this level and so his conditional expected sojourn time in the system is given by (4.39). The threshold strategy x_* is a best response against itself, when $U^{(ao)}(x_*; x_*) = 0$. Therefore, x_* is a root of (4.44). Note that the left side of (4.44) is monotone in x, since it is equal to $\frac{q_1}{q_0+q_1} S_0(x) + \frac{q_0}{q_0+q_1} S_1(x)$ and both $S_0(x)$ and $S_1(x)$ are monotone, as we have established in the proof of Theorem 4.2.2. Therefore, (4.44) has a unique root that gives the unique equilibrium threshold strategy in this case.

In case II, a threshold strategy x_*, is a best response against itself, if and only if $U^{(ao)}(x;x_*) \geq 0$ for $0 \leq x < x_*$, and $U^{(ao)}(x_*;x_*) = 0$. These conditions are seen to be equivalent to

$$\frac{q_0+q_1}{q_0\mu_1+q_1\mu_0}x - \frac{q_1\mu_0(\lambda-\mu_0)+q_0\mu_1(\lambda-\mu_1)}{(q_0\mu_1+q_1\mu_0)^2} \times \left(1-e^{-\left(\frac{q_0}{\mu_0}+\frac{q_1}{\mu_1}\right)x_*}\right) \leq \frac{R}{C},\ 0 \leq x < x_* \tag{4.46}$$

$$\frac{q_0+q_1}{q_0\mu_1+q_1\mu_0}x_* - \frac{q_0\mu_1(\mu_0-\mu_1)}{(q_0\mu_1+q_1\mu_0)^2}\left(1-e^{-\left(\frac{q_0}{\mu_0}+\frac{q_1}{\mu_1}\right)x_*}\right) = \frac{R}{C}. \tag{4.47}$$

Due to the monotonicity of the left side of (4.46) with respect to x (which is equal to $\frac{\mu_1-\lambda}{\mu_1-\mu_0}S_0(x) + \frac{\lambda-\mu_0}{\mu_1-\mu_0}S_1(x)$) and the fact that the left side of (4.47) (which is equal to $S_0(x_*)$) exceeds the left side of (4.46) for $x = x_*$, we can easily conclude that the relations are valid if and only if x_* is the unique solution x'_e of Eq. (4.45).

In case III, any threshold strategy is a best response against itself. Indeed, whatever threshold the other customers may follow, a tagged customer may use the same threshold, since he always observes the fluid at state 0 and he is willing to enter since $U^{(ao)}(0;x_*) = R - CS^{(ao)}(0;x_*) = R > 0$. □

It is interesting to notice that the equilibrium threshold does not depend on the exact value of the arrival rate λ (since the corresponding equations do not involve λ), but only on its relative order with respect to the service rates μ_0 and μ_1. We can now compute the function of the expected social benefit per time unit, $B^{(ao)}(x_*)$, when the customers follow a threshold strategy x_*, using Theorem 4.2.4 and setting $B^{(ao)}(x_*) = B^{(fo)}(x_*, x_*)$. We have Theorem 4.3.3.

Theorem 4.3.3. *In the **ao** case, suppose that the customers follow a threshold strategy x_*. Then, we have the following cases.*

Case I. $\lambda \geq \mu_1$.

The expected social benefit per time unit is given by

$$B^{(ao)}(x_*) = \left(\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1\right)R - Cx_*. \tag{4.48}$$

Case II. $\mu_0 < \lambda < \mu_1$. *The expected social benefit per time unit is given by*

$$B^{(ao)}(x_*) = \lambda\left(\int_0^{x_*} f_0(x)dx + p_0(x_*)\frac{\mu_0}{\lambda} + \frac{q_0}{q_0+q_1}\right) \times R - CE[X], \tag{4.49}$$

where

$$E[X] = x_*p_0(x_*) + \int_0^{x_*} x(f_0(x)+f_1(x))dx \tag{4.50}$$

with $p_0(x_*)$, $f_0(x)$ and $f_1(x)$ given by (4.16), (4.17) and (4.19), respectively.

Case III. $\lambda \le \mu_0$. *The expected social benefit per time unit is given by*

$$B^{(ao)}(x_*) = \lambda R. \tag{4.51}$$

A moment of reflection shows that, in case I, the unique socially optimal strategy is the threshold strategy 0. In case II, a threshold strategy x_* is socially optimal in the **ao** case if and only if the threshold strategies (x_*, x'_*), with $x'_* \ge x_*$, are socially optimal in the **fo** case. Finally, in case III, any threshold strategy x_* is socially optimal as the buffer is always empty. Thus, in all cases, we have that $\max_{(x_*(0),x_*(1))} B^{(fo)}(x_*(0), x_*(1)) = \max_{x_*} B^{(ao)}(x_*)$, i.e., the optimal expected social benefit in the **fo** case coincides with the optimal expected social benefit in the **ao** case.

4.4 Numerical results - qualitative insight - discussion

In this chapter we studied the **fo** and the **ao** cases of the fluid queue with alternating service process. We aimed at determining the equilibrium and the socially optimal strategies within appropriate sets of strategies for each level of information. We have the following results in the three cases regarding the relative order of the arrival rate λ with respect to the service rates μ_0 and μ_1.

Case I. $\lambda \ge \mu_1$

In this case that corresponds to case I and II - **fo** and case I - **ao**, we proved that the unique equilibrium threshold strategy, in the **fo** case, is the $(x_e(0), x_e(1))$ strategy, where $x_e(i)$ is the unique root of Eq. (4.4), $i = 0, 1$. In the **ao** case, the unique equilibrium threshold strategy is the x_e strategy, where x_e is the unique root of Eq. (4.44). We have that $x_e(0) < x_e < x_e(1)$.

The socially optimal strategy, in the **fo** case, is the threshold strategy (0, 0) and, in the **ao** case, is the threshold strategy 0. Moreover, the effective arrival rate under any threshold strategy is equal to the mean service rate, a the server is never idle. Thus, the effective arrival rate under any threshold strategy in the **fo** and **ao** cases equals $\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1$. So, the fraction of customers that join under the equilibrium strategies is equal to the fraction of customers that join under the socially optimal strategies. Nevertheless, under equilibrium strategies the customers keep the fluid at high levels, whereas, under socially optimal strategies the buffer is always empty. This happens because, in the decentralized version, the arriving customers ignore the waiting cost they impose to future customers by joining and tend to join even if the fluid is at high levels. On the other hand, in the centralized version,

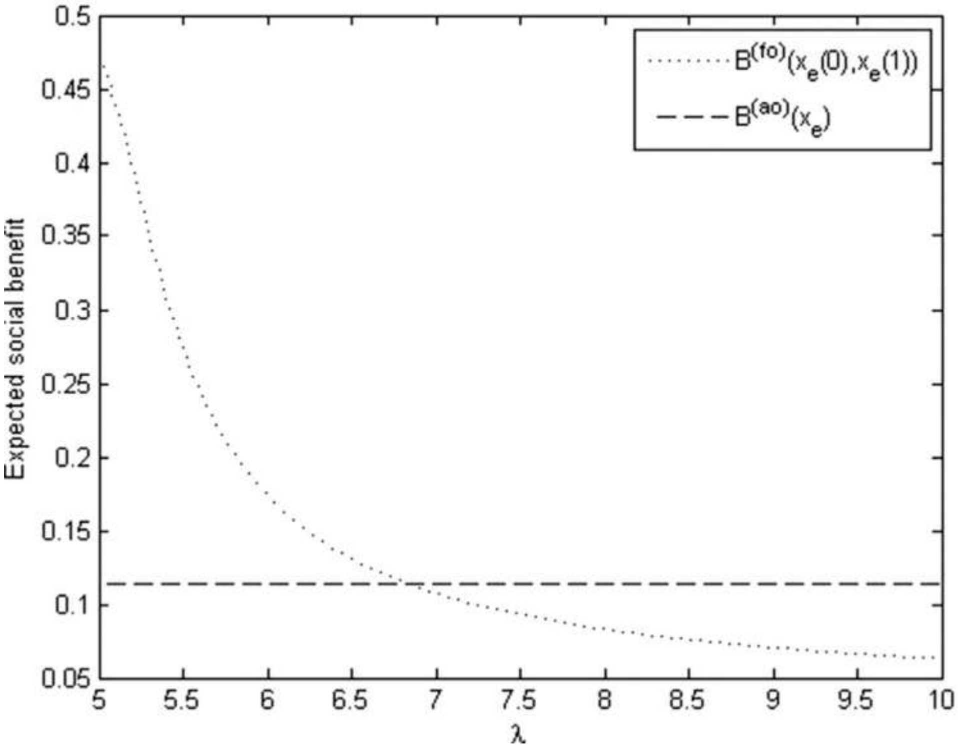


Figure 4.6: Expected social benefits $B^{(fo)}(x_e(0), x_e(1))$ and $B^{(ao)}(x_e)$ with respect to the arrival rate λ for $q_0 = 3$, $q_1 = 2$, $\mu_0 = 2$, $\mu_1 = 5$, $R = 5$ and $C = 1$.

the central planner takes into account these negative externalities and keep the fluid level at 0.

Case II. $\mu_0 < \lambda < \mu_1$.

In this case that corresponds to case III - **fo** and case II - **ao**, formulas (4.14) and (4.15) show that under any threshold strategy $(x_*(0), x_*(1))$, with $x_*(1) \geq x_*(0)$, the fluid oscillates in $[0, x_*(0)]$ and the steady-state distribution of the fluid level is the same as under the threshold strategy $x_*(0)$. In the **fo** case, the unique subgame perfect equilibrium is the threshold strategy $(x_e(0), x_e(1))$, where $x_e(i)$ is the unique root of Eq. (4.4), $i = 0$, 1. However, any threshold strategy $(x_e(0), x)$ with $x \geq x_e(0)$, is equilibrium as the customers will never find the fluid level strictly above $x_e(0)$. In the **ao** case, the unique equilibrium threshold strategy is the x'_e strategy, where the x'_e is the unique root of Eq. (4.45). So, $x'_e = x_e(0)$. Thus, the equilibrium strategy x'_e in the **ao** case yields the equilibrium strategies (x'_e, x) for every $x \geq x'_e$ in the **fo** case and the expected social benefit under the equilibrium strategies in both cases is the same. So, $B^{(fo)}(x'_e, , x) = B^{(ao)}(x'_e)$, for $x \geq x'_e$. Thus, in the decentralized version of the model, given that the fluid level is known, the customers do not benefit not lose by knowing also the state of the server.

The situation is similar in the centralized version, where, if the threshold strategy x_* is socially optimal in the **ao** case, any threshold strategy (x_*, x) with $x \geq x_*$, is socially optimal in the **fo** case and the expected social benefit under these socially optimal strategies is the same.

Case III. $\lambda \leq \mu_0$.

In this case that corresponds to case IV - **fo** and case III - **ao**, the fluid level stabilizes at 0. Thus, any threshold strategy is equilibrium and socially optimal strategy in the **fo** case. Also, any threshold strategy is equilibrium and socially optimal strategy in the **ao** case. The expected social benefit under all these strategies is λR. So, as in the previous case, given that the fluid level is known, the knowledge of the state of the server does not change the benefit in the decentralized and the centralized versions.

We would also like to compare the effect of information on the expected social benefit when the customers decide selfishly and when they make their decisions so that they maximize the expected social benefit. From above, it is obvious that the optimal expected social benefits in the **fo** and **ao** cases do coincide. So, if the customers are altruistic/cooperative and want to maximize the expected social benefit, they are indifferent between the two levels of information. Also, the expected social benefit under any equilibrium threshold strategy in the **fo** case is equal to the expected social

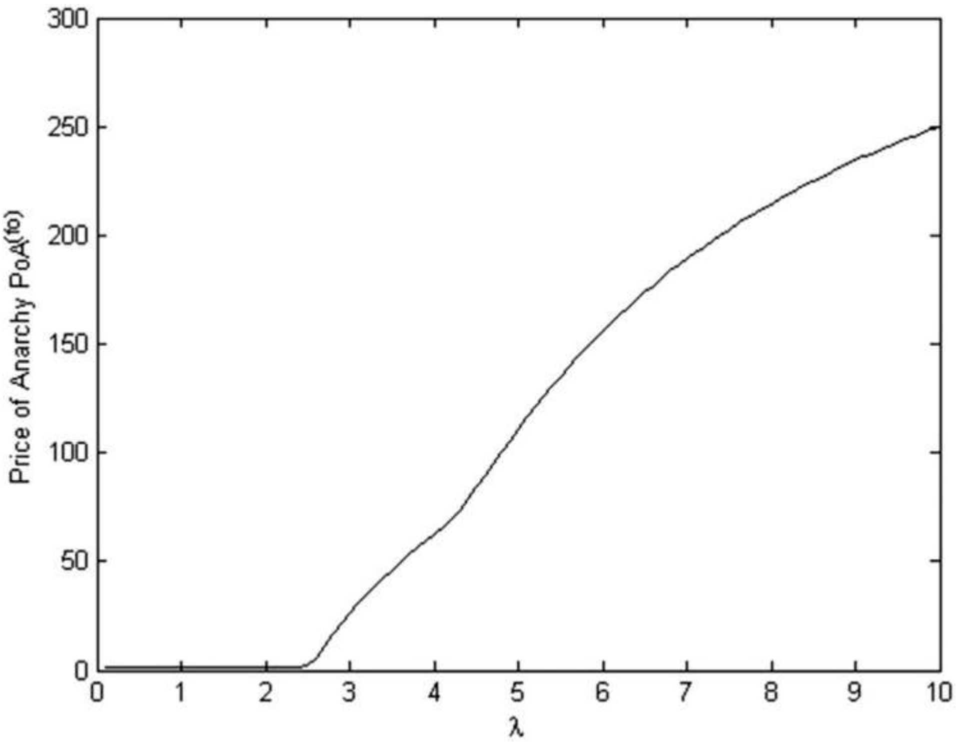


Figure 4.7: PoA in the **fo** case with respect to the arrival rate λ for $q_0 = 1$, $q_1 = 3$, $\mu_0 = 2\,\mu_1 = 4$, $R = 5$ and $C = 1$.

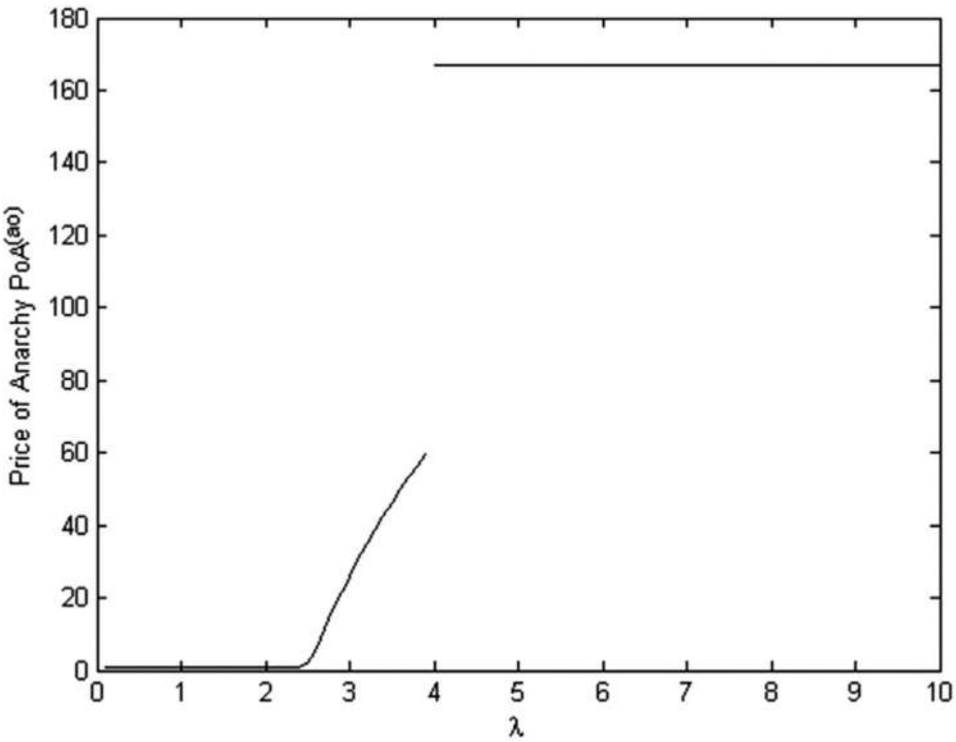


Figure 4.8: PoA in the **ao** case with respect to the arrival rate λ for $q_0 = 1$, $q_1 = 3$, $\mu_0 = 2\,\mu_1 = 4$, $R = 5$ and $C = 1$.

benefit under any equilibrium threshold strategy in the **ao** case except for the case where $\lambda \geq \mu_1$. In Fig. 4.6, we present the expected social benefit in equilibrium in the **fo** case, $B^{(fo)}(x_e(0), x_e(1))$, and the expected social benefit in equilibrium in the **ao** case, $B^{(ao)}(x_e)$, in an example with $q_0 = 3$, $q_1 = 2$, $\mu_0 = 2$, $\mu_1 = 5$, $R = 5$, $C = 1$, and $\lambda \in [5, 10]$. We observe that $B^{(ao)}(x_e)$ is a constant function of λ. This is clear because of (4.48) and the fact that x_e does not depend on λ (see relevant comment just after the proof of Theorem 4.3.2). On the other hand, $B^{(fo)}(x_e(0), x_e(1))$ is a decreasing function of λ, so there exists a critical value $\lambda*$, which is equal to 6.8 in the present example, such that $B^{(fo)}(x_e(0), x_e(1)) \geq B^{(ao)}(x_e)$, for $\lambda \leq \lambda*$, while $B^{(fo)}(x_e(0), x_e(1)) \leq B^{(ao)}(x_e)$, for $\lambda \geq \lambda*$. Thus, it is better to reveal the service mode to selfish customers for values of $\lambda \in (\mu_1, \lambda*)$, while it is better to conceal it for $\lambda \in (\lambda*, \infty)$.

Finally, we study the Price of Anarchy (PoA). In the current context, for the **fo** case, the PoA is defined as the ratio of the optimal expected social benefit per time unit over the corresponding equilibrium expected social benefit per time unit, i.e,

$$PoA^{(fo)} = \frac{B^{(fo)}(x_{SOC}(0), x_{SOC}(1))}{B^{(fo)}(x_e(0), x_e(1))}, \tag{4.52}$$

where $x_e(i)$ and $x_{SOC}(i)$ denote the equilibrium and socially optimal thresholds respectively, Similarly, in the **ao** case, we define

$$PoA^{(ao)} = \frac{B^{(ao)}(x_{SOC})}{B^{(ao)}(x_e)}, \tag{4.53}$$

where x_e and x_{SOC} denote the equilibrium and socially optimal thresholds, respectively.

In the **fo** case, when the arrival rate is small (up to a bit higher than μ_0), the PoA is equal to 1, as the optimal expected social benefit is equal to the equilibrium expected social benefit. As the arrival rate increases, the PoA increases and as λ tends to infinity, the limit of the PoA is

$$\lim_{\lambda \to \infty} PoA^{(fo)} =$$
$$\frac{R\left(\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1\right)}{R\left(\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1\right) - C\frac{[(q_0+q_1)q_0 x_e(1) - q_1\mu_0]\exp\left\{\frac{q_0}{\mu_0}(x_e(1)-x_e(0))\right\} + q_1\mu_0}{(q_0+q_1)q_0\exp\left\{\frac{q_0}{\mu_0}(x_e(1)-x_e(0))\right\}}} \tag{4.54}$$

where $x_e(i)$ is the unique root of Eq. (4.4), $i = 0, 1$. So, the PoA in the **fo** case is bounded. In Fig. 4.7, we consider the PoA with respect to λ in an example with $q_0 = 1$, $q_1 = 3$, $\mu_0 = 2$, $\mu_1 = 4$, $R = 5$, and $C = 1$. In this example, we have that $\lim_{\lambda\to\infty} PoA^{(fo)} = 393.0964$.

In the **ao** case, the PoA is equal to 1, for small arrival rates, and, when the arrival rate is greater than μ_1, the PoA becomes constant. We have that

$$\lim_{\lambda\to\infty} PoA^{(ao)} = \frac{R\left(\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1\right)}{R\left(\frac{q_1}{q_0+q_1}\mu_0 + \frac{q_0}{q_0+q_1}\mu_1\right) - Cx_e} \tag{4.55}$$

where x_e is the unique root of Eq. (4.42). In Fig. 4.8, we have the PoA with respect to λ in an example with $q_0 = 1$, $q_1 = 3$, $\mu_0 = 2$, $\mu_1 = 4$, $R = 5$, and $C = 1$.

As a conclusion, the graphs of the PoA as a function of the arrival rate λ coincide for the two informational cases, for values of λ smaller than the fast service rate μ_1. Both graphs start from the value 1 and keep this value till a bit higher than the low service rate (which is $\mu_0 = 2$ in the considered scenario). Then, they continue almost linearly till the high service rate (which is $\mu_1 = 4$ in the considered scenario). But after this point, in the **fo** case the PoA increases in a continuous way and approaches its limit given by (4.54), while in the **ao** case, it has a discontinuity at μ_1 and becomes constant after that point, assuming the value given by (4.55). The limiting value of the PoA for the **fo** case exceeds considerably the corresponding value for the **ao** case. Therefore, we conclude that for low arrival rates the socially optimal and the equilibrium thresholds do coincide, while this is not the case for higher arrival rates.

We close our discussion by referring to two directions that may lead to interesting generalizations of the results. Both of them were suggested by an anonymous referee. The first idea is to generalize the analysis in the case of more complex but yet natural utility functions for the customers. In the present essay, the utility function of the customers has the form $U(x,i) = R - CS_i(x) = R - CE_{(x,i)}[S]$, where S denotes a sojourn time of a customer in the system and the subscript (x,i) refers to the state seen upon arrival. An interesting option would be to consider a utility function that involves the variance, i.e., $U(x,i) = R - C(E_{(x,i)}[S] + \alpha Var_{(x,i)}[S])$. With such a utility function, a customer may be tempted to balk, if there is a lot of variability in the sojourn time. However, the computations become much more involved. For example, we have that $Var_{(x,i)}[S] = E_{(x,i)}[S^2] - (E_(x,i)[S])^2$ and to compute $E_{(x,i)}[S^2]$, we can condition similarity to (4.3) and we will have that

$$E_{x,i}[S^2] = \left(\frac{x}{\mu_i}\right)^2 e^{-q_i\frac{x}{\mu_i}} + \int_0^{\frac{x}{\mu_i}} (t^2 + 2tE_{(x-\mu_i t,i')}[S] + E_{(x-\mu_i t,i')}[S^2])q_i e^{-q_i t} dt. \tag{4.56}$$

These equations can be solved similarly with the methodology of Lemma 4.2.1. However, the corresponding formulas are too involved and do not allow to continue analytically. Considering an exponential utility function of the form $U(x,i) = R - CE_{(x,i)}[e^{\alpha S}]$ (as it is common in finance/insurance models) leads also to computationally intractable expressions. Yet, a numerical investigation of the present model with such utility functions seems interesting from an economic viewpoint.

Another interesting direction of generalizing the model is to consider the case where the machine mode process $\left\{Z_{(t)}\right\}$ has more than two states. The generalization is not easy since the sample paths of the fluid level assume very different forms because of the various machine modes and the associated thresholds. Finally it would be interesting to study a diffusion counterpart of the process $\left\{X_{(t)}\right\}$, i.e., a Brownian motion with drift, also modulated by a machine mode process $\left\{Z_{(t)}\right\}$. This could have an influence in computing the corresponding expected social benefit in both the **fo** and **ao** cases and their numerical optimization.

Chapter 5

Summary in Greek - Περίληψη

Στρατηγική συμπεριφορά σε συστήματα αναμονής σε εναλλασσόμενο περιβάλλον

Ένα σύστημα αναμονής ή αλλιώς ουρά, είναι ένα σύστημα στο οποίο πελάτες εισέρχονται για να δεχθούν κάποια υπηρεσία η οποία έχει κάποια διάρκεια. Όταν ένας πελάτης εισέρχεται στο σύστημα, αν δεν υπάρχει κάποιος υπηρέτης διαθέσιμος για να ξεκινήσει η εξυπηρέτηση του, ο πελάτης εισέρχεται στον χώρο αναμονής του συστήματος όπου περιμένει τη σειρά του να εξυπηρετηθεί και έτσι δημιουργείται μια ουρά αναμονης.

Η οικονομική ανάλυση της συμπεριφοράς των πελατών βασίζεται σε μία δομή αμοιβής-κόστους η οποία εισάγεται στο σύστημα ώστε να αντανακλά την επιθυμία των πελατών για την υπηρεσία που προσφέρεται, καθώς και την απροθυμία τους να περιμένουν. Στους πελάτες επιτρέπεται να αποφασίσουν για τις ενέργειές τους μέσα στο σύστημα, π.χ. αποφασίζουν αν θα εισέλθουν ή θα αποχωρήσουν από το σύστημα, αν θα εγκαταλείψουν το σύστημα ενώ βρίσκονται στο χώρο αναμονής, αν θα αγοράσουν προτεραιότητα κ.α. Οι πελάτες θέλουν να μεγιστοποιήσουν το καθαρό κέρδος τους γνωρίζοντας ότι και οι άλλοι πελάτες προσπαθούν να κάνουν το ίδιο. Θεωρούμε λοιπόν ότι ένα τέτοιο σύστημα αποτελεί ένα συμμετρικό παίγνιο ανάμεσα στους πελάτες και για την ανάλυσή του χρησιμοποιούμε εργαλεία από τη θεωρία ουρών και τη θεωρία παιγνίων. Στόχος μιας τέτοιας ανάλυσης είναι η εύρεση στρατηγικών ισορροπίας, όπου ο κάθε πελάτης βελτιστοποιεί την προσωπική του ωφέλεια και η εύρεσ κοινωνικά βέλτιστων στρατηγικών, που μεγιστοποιούν τη συνολική κοινωνική ωφέλεια, όταν υιοθετηθούν από τους πελάτες.

Σε αυτή την εργασία μελετάμε τη στρατηγική συμπεριφορά πελατών που φθάνουν σε ένα σύστημα αναμονης το οποίο εξελίσσεται σε εναλλασσόμενο περιβάλλον, δηλαδή σε ένα σύστημα που ο ρυθμός άφιξης των πελατών, αλλά και ο ρυθμός εξυπηρέτησης τους μεταβάλλεται. Η στρατηγική που υιοθετούν οι πελάτες εξαρτάται και από την πληροφόρηση που δέχονται φτάνοντας στο σύστημα. Εξετάζουμε λοιπόν τη συμπεριφορά των πελατών στα παρακάτω πιθανά επίπεδα πληροφόρησης:

- Παρατηρήσιμη περίπτωση, όπου ο πελάτης παρατηρεί το πλήθος των πελατών που βρίσκονται στο σύστημα, αλλά και την κατάσταση του περιβάλλοντος του συστήματος.
- Μερικώς παρατηρήσιμη περίπτωση, όπου ο πελάτης παρατηρεί το πλήθος των πελατών που βρίσκονται στο σύστημα, αλλά δεν γνωρίζει την κατάσταση του περιβάλλοντος του συστήματος.
- Μερικώς μη-παρατηρήσιμη περίπτωση, όπου ο πελάτης δεν παρατηρεί το πλήθος των πελατών που βρίσκονται στο σύστημα, αλλά παρατηρεί την κατάσταση του περιβάλλοντος στο οποίο βρίσκεται το σύστημα.
- Μη-παρατηρήσιμη περίπτωση, όπου ο πελάτης δεν παρατηρεί το πλήθος των πελατών που βρίσκονται στο σύστημα, ούτε την κατάσταση του περιβάλλοντος του συστήματος.

Συνήθως, αν οι πελάτες εξυπηρετούνται με τη σειρά που φτάσανε στο σύστημα ($FCFS$ πειθαρχία ουράς), όσο αυξάνεται το πλήθος των πελατών στο σύστημα, αυξάνεται και ο μέσος χρόνος αναμονής των πελατών, και συνεπώς το κόστος τους. Αυτό σημαίνει ότι όταν ένας πελάτης παρατηρεί μεγάλο αριθμό πελατών μπροστά του, είναι λιγότερο πρόθυμος να εισέλθει στο σύστημα. Έτσι λοιπόν, στα συστήματα που ένας αφικνούμενος πελάτης μπορεί να παρατηρήσει το πλήθος των πελατών μπροστά του, καταλήγουμε συνήθως σε στρατηγικές κατωφλίου, όπου ένας πελάτης που ενεργεί σύμφωνα με τη στρατηγική κατωφλίου (n_0), αν παρατηρήσει μέχρι n_0 άτομα εισέρχεται στην ουρά, αλλιώς αποχωρεί.
Στην παρούσα εργασία εξετάζουμε 3 συστήματα αναμονής σε εναλλασσόμενο περιβάλλον και εντοπίζουμε βέλτιστες στρατηγικές των πελατών για το δίλημμα της εισόδου-αποχώρησης στο σύστημα. Τα συστήματα που εξετάζουμε και τα άρθρα στα οποία μελετήθηκαν είναι τα παρακάτω :

- $M/M/1$ ουρά με αναξιόπιστο υπηρέτη (Οικονόμου και Καντά (2008))
- Σύστημα αναμονής με πλήρη εκκαθάριση σε εναλλασσόμενο περιβάλλον (Οικονόμου και Μάνου (2013))
- Σύστημα εξυπηρέτησης ρευστού σε εναλλασσόμενο περιβάλλον (Οικονόμου και Μάνου (2016))

M/M/1 ουρά με αναξιόπιστο υπηρέτη

Θεωρούμε πρώτα μια $M/M/1$ ουρά με αναξιόπιστο υπηρέτη, ο οποίος εναλλάσσεται ανάμεσα σε ενεργή και ανενεργή κατάσταση, με τη διάρκεια της κάθε κατάστασης να ακολουθεί εκθετική κατανομή με ρυθμό ζ και θ ανίστοιχα. Οι πελάτες φτάνουν στο σύστημα σύμφωνα με μια διαδικασία $Poisson$ με ρυθμό λ. Όταν ο υπηρέτης βρίσκεται στην ενεργή κατάσταση, ο χρόνος εξυπηρέτησης ακολουθεί την εκθετική κατανομή με ρυθμό μ, ενώ στην ανενεργή κατάσταση δεν εξυπηρετεί. Οι πελάτες λαμβάνουν R μονάδες ωφέλειας όταν ολοκληρώνεται η εξυπηρέτησή τους και υπόκεινται σε κόστος C ανά μονάδα χρόνου. Οι πελάτες αποφασίζουν αν θα εισέλθουν ή αν θα αποχωρήσουν τη στιγμη που φτάνουν στο σύστημα και η απόφασή τους είναι αμετάκλητη. Εξετάζουμε τη συμπεριφορά των πελατών, όταν βελτιστοποιούν το καθαρό κέρδος τους, σε δύο διαφορετικά επίπεδα πληροφόρησης, στην παρατηρήσιμη ουρά και στη μερικώς παρατηρήσιμη ουρά.

Στην παρατηρήσιμη περίπτωση, καταλήγουμε ότι η βέλτιστη στρατηγική είναι της μορφής $(n_e(0), n_e(1))$ που είναι ένα ζεύγος στρατηγικών κατωφλίου. Όταν ένας πελάτης παρατηρεί τον υπηρέτη στην ανενεργή κατάσταση υιοθετεί $n_e(0)$-στρατηγική κατωφλίου, ενώ όταν παρατηρεί τον υπηρέτη στην ενεργή κατάσταση υιοθετεί $n_e(1)$-στρατηγική κατωφλίου. Η στρατηγική αυτή είναι ασθενώς κυριαρχούσα στρατηγική καθώς είναι βέλτιστη απάντηση έναντι οποιασδήποτε στρατηγικής των άλλων πελατών. Επίσης, ισχύει ότι $n_e(0) < n_e(1)$, καθώς όταν ένας αφικνούμενος πελάτης παρατηρεί τον υπηρέτη στην ανενεργή του κατάσταση, θα υποστεί ένα επιπλέον κόστος αναμονής μέχρι την ενεργοποίησή του.

Για τη μελέτη της μερικώς παρατηρήσιμης περίπτωσης, χρειαζόμαστε τη στάσιμη κατανομή του συστήματος. Υποθέτουμε ότι οι πελάτες ακολουθούν μία στρατηγική κατωφλίου n_e και υπολογίζουμε τη στάσιμη κατανομή του συστήματος μέσω των εξισώσεων ισορροπίας. Έπειτα, υπολογίζοντας το αναμενόμενο καθαρό κέρδος ενός επιλεγμένου πελάτη, ενώ οι υπόλοιποι ακολουθούν μια n_e-στρατηγική κατωφλίου, μελετώντας τις ιδιότητες των συναρτήσεων πληρωμής του επιλεγμένου πελάτη, καταλήγουμε ότι η στρατηγική ισορροπίας είναι οποιαδήποτε στρατηγική κατωφλίου με κατώφλι $n_e \in \{n_L, n_L + 1, ..., n_U\}$, με τα n_L και n_U να προκύπτουν από τις ιδιότητες της συνάρτησης πληρωμής. Η ύπαρξη πολλαπλών στρατηγικών ισορροπίας είναι ένδειξη πως οι πελάτες συμπεριφέρονται σύμφωνα με το πλήθος (ΣΤΠ), δηλαδή όταν οι υπόλοιποι πελάτες υιοθετούν ένα υψηλότερο ή αντίστοιχα χαμηλότερο κατώφλι, το ίδιο κάνει και ο επιλεγμένος πελάτης.

Σύστημα αναμονής με πλήρη εκκαθάριση σε εναλλασσόμενο περιβάλλον

Το επόμενο σύστημα που μελετάμε είναι ένα σύστημα αναμονής με εκκαθάριση σε μεταβαλλόμενο περιβάλλον. Μπορούμε να σκεφτόμαστε ένα τέτοιο σύστημα σαν έναν σταθμό ενός μεταφορικού μέσου, όπου οι πελάτες εισέρχονται στο σταθμό και εξυπηρετούνται όλοι μαζί, ταυτόχρονα και ακαριαία, όταν το μέσο επισκέπτεται το σταθμό. Το περιβάλλον του μέσου μεταφοράς έχει δύο καταστάσεις : **α)** στην κατάσταση 1 οι πελάτες φθάνουν στο σταθμό σύμφωνα με μία διαδικασία $Poisson$ ρυθμού λ_1 και ο χρόνος εξυπηρέτησης ακολουθεί εκθετική κατανομή με ρυθμο μ_1 **β)** στην κατάσταση 2 οι πελάτες φθανουν με μία διαδικασία $Poisson$ ρυθμού λ_2 και ο χρόνος εξυπηρέτησης ακολουθεί εκθετική κατανομή με ρυθμό μ_2. Κατά την ολοκλήρωση της εξυπηρέτησης τους οι πελάτες λαμβάνουν R μονάδες ωφέλειας και κατά την αναμονή τους υφίστανται ένα κόστος C μονάδων ανά χρονική μονάδα. Οι πελάτες αποφασίζουν αν θα εισέλθουν ή θα αποχωρήσουν από το σύστημα κατα την άφιξή τους και αυτές οι αποφάσεις είναι αμετάκλητες. Αναζητούμε τις στρατηγικές ισορροπίας των πελατών για το δίλημμα εισόδου-αποχώρησης από το σύστημα στα παρακάτω επίπεδα πληροφόρησης :**α)** παρατηρήσιμη περίπτωση, **β)** μερικώς παρατηρήσιμη περίπτωση **γ)** μερικώς μη παρατηρήσιμη περίπτωση **δ)** μη παρατηρήσιμη περίπτωση. Η μεθοδολογία για την εύρεση στρατηγικών ισορροπίας των περιπτώσεων α), γ), δ) είναι παρόμοια και αυτές τις περιπτώσεις τις εξετάζουμε μαζί, ενώ η περίπτωση β) που είναι η πιο ενδιαφέρουσα εξετάζεται μόνη της.

Μη παρατηρήσιμη περίπτωση

Στη μη παρατηρήσιμη περίπτωση, η κυριαρχούσα στρατηγική εξαρτάται από τη σχέση της τιμής του κλάσματος $\frac{R}{C}$ με μία κρίσιμη τιμή, η οποία εξαρτάται από το μέσο χρόνο παραμονής του πελάτη στο σύστημα. Αν η τιμή του $\frac{R}{C}$ είναι μικρότερη από τη κρίσιμη τιμή, οι πελάτες πάντα αποχωρούν από το σύστημα, αν η τιμή του $\frac{R}{C}$ είναι ίση με τη κρισιμη τιμή είναι αδιάφοροι ως προς το αν θα εισέλθουν ή όχι και αν η τιμή του $\frac{R}{C}$ είναι μεγαλύτερη από την κρίσιμη τιμή, οι πελάτες πάντα εισέρχονται στο σύστημα.

Μερικώς μη παρατηρήσιμη περίπτωση

Στο μερικώς μη παρατηρήσιμο μοντέλο, αν οι χρόνοι εξυπηρέτησης έχουν διαφορετικό ρυθμό, δηλαδή $\mu_1 \neq \mu_2$, τότε έχουμε μία γρήγορη λειτουργία του υπηρέτη και μια αργή. Καθώς οι πελάτες παρατηρούν την κατάσταση του υπηρέτη κατά την άφιξή τους, η κυριαρχούσα στρατηγική εξαρτάται απο τη σχετική τιμή του $\frac{R}{C}$ ως προς δύο κρίσιμες τιμές που αντιστοιχούν στη γρήγορη και στην αργή λειτουργία αντίστοιχα. Έχουμε τις παρακάτω περιπτώσεις.

- Αν το $\frac{R}{C}$ είναι μικρότερο από τη μικρότερη κρίσιμη τιμή, τότε ο πελάτης πάντα αποχωρεί από το σύστημα.
- Αν το $\frac{R}{C}$ είναι ίσο με τη μικρότερη κρίσιμη τιμή, τότε ο πελάτης πάντα αποχωρεί, αν το σύστημα βρίσκεται στην αργή λειτουργία και είναι αδιάφορος ως προς το να εισέλθει ή όχι όταν παρατηρεί την γρήγορη λειτουργία.
- Αν το $\frac{R}{C}$ είναι ανάμεσα στις δύο κρίσιμες τιμές, τότε ο πελάτης πάντα αποχωρεί, αν το σύστημα βρίσκεται στην αργή λειτουργία και πάντα εισέρχεται όταν παρατηρεί την γρήγορη λειτουργία.

- Αν το $\frac{R}{C}$ είναι ίσο με τη μεγαλύτερη κρίσιμη τιμή, τότε ο πελάτης είναι αδιάφορος ως προς το αν θα εισέλθει στο σύστημα όταν παρατηρεί τον υπηρέτη στην αργή λειτουργία και πάντα εισέρχεται στο συστημα όταν παρατηρεί τον υπηρέτη στη γρήγορη λειτουργία.
- Αν το $\frac{R}{C}$ είναι μεγαλύτερο από τη μεγαλύτερη κρίσιμη τιμή, τότε ο πελάτης πάντα εισέρχεται στο σύστημα.

Αν οι χρόνοι εξυπηρέτησης έχουν ίδιο ρυθμό, τότε οι δυο κρίσιμες τιμές συμπίπτουν και οι στρατηγικές ισορροπίας των πελατών όταν παρατηρούν την κατάσταση του υπηρέτη είναι παρόμοιες με την μη παρατηρήσιμη περίπτωση.

Παρατηρήσιμη περίπτωση

Στην παρατηρήσιμη περίπτωση, ο χρόνος παραμονής των πελατών εξαρτάται μόνο από την κατάσταση του υπηρέτη και όχι από το πλήθος των πελατών στο σύστημα και έτσι το πλήθος των πελατών θεωρείται περιττή πληροφορία και αγνοείται. Οπότε οι ανάλυση είναι ακριβώς η ίδια με τη μερικώς μη παρατηρήσιμη περίπτωση.

Μερικώς παρατηρήσιμη περίπτωση

Στην μερικώς παρατηρήσιμη περίπτωση, οι αφικνούμενοι πελάτες παρατηρούν το πλήθος των πελατών πριν εισέλθουν στο σύστημα. Αν και το πλήθος των πελατών δεν επηρεάζει τον μέσο χρόνο παραμονής ενός πελάτη στο σύστημα άμεσα, δίνει κάποια πληροφορία για την κατάσταση του συστήματος. Ορίζουμε ως $\rho_i = \frac{\lambda_i}{\mu_i}$ το μέτρο συνωστισμού του συστήματος στην κατάσταση $i = 1, 2$. Έχουμε τις παρακάτω περιπτώσεις:

- $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$, όπου όταν ένας αφικνούμενος πελάτης παρατηρεί μια συνωστισμένη κατάσταση του συστήματος, αυξάνεται η πιθανότητα ο υπηρέτης να βρίσκεται στην αργή λειτουργία. Συνεπώς, όταν βλέπει υψηλό αριθμό πελατών στο σύστημα, ένας επιλεγμένος πελάτης είναι λιγότερο διαθετιμένος να εισέλθει, άρα περιμένουμε να ακολουθήσει μια στρατηγική κατωφλίου.
- $(\mu_1 - \mu_2)(\rho_1 - \rho_2) > 0$, όπου όταν ένας αφικνούμενος πελάτης παρατηρεί μια συνωστισμένη κατάσταση του συστήματος, αυξάνεται η πιθανότητα ο υπηρέτης να βρίσκεται στην γρήγορη λειτουργία. Σε αυτή την περίπτωση, όταν βλέπει υψηλό αριθμό πελατών στο σύστημα, ένας επιλεγμένος πελάτης είναι περισσότερο διαθετιμένος να εισέλθει, άρα περιμένουμε να ακολουθήσει μια στρατηγική αντιστρόφου κατωφλίου, υπό την οποία ένας παίκτης εισέρχεται στο σύστημα όταν παρατηρεί πάνω από ένα συγκεκριμένο αριθμό πελατών.
- $(\mu_1 - \mu_2)(\rho_1 - \rho_2) = 0$, όπου δεν έχει νόημα ο διαχωρισμός ανάμεσα σε λιγότερο ή περισσότερο συνωστισμένη κατάσταση.

Περίπτωση 1: $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$

Από τη διαισθητική συζήτηση που έγινε παραπάνω, σε αυτή την περίπτωση αναζητούμε στρατηγικές ισορροπίας στην κλάση των στρατηγικών κατωφλίου. Υπολογίζοντας τη στάσιμη κατανομή του συστήματος υπό τις πιθανές στρατηγικές κατωφλίου που υποθέτουμε πως ακολουθούν οι υπόλοιποι πελάτες, βρίσκουμε τη συνάρτηση πληρωμής ενός επιλεγμένου πελάτη ενάντια σε αυτές τις στρατηγικές. Ορίζοντας ως $H^U(n)$ τη συνάρτηση πληρωμής ενός επιλεγμένου πελάτη όταν οι υπόλοιποι πελάτες ακολουθούν μία μεικτή ή καθαρή στρατηγική n_0-κατωφλίου και ο επιλεγμένος πελάτης παρατηρεί $n < n_0$ πελάτες, αποδεικνύουμε μια σειρά από ιδιότητες για τις συναρτήσεις πληρωμής υπό οποιαδήποτε στρατηγική κατωφλίου και έχουμε τις παρακάτω περιπτώσεις :

Περίπτωση A : $H^U(0) < 0$
: Υπάρχει μοναδική στρατηγική ισορροπίας, η $\lceil 0 \rceil$ στρατηγική(πάντα να αποχωρεί).

Περίπτωση B : $H^U(0) \geq 0$. *και* $lim_{n\to\infty} H^U(n) < 0$.
: Τότε, υπάρχουν πάντα καθαρές στρατηγικές ισορροπίας. Επιπλέον, οι στρατηγικές ισορροπίας μέσα στην κλάση όλων των καθαρών στρατηγικών είναι οι στρατηγικές $\lceil n_0 \rceil$ με n_0 ανάμεσα σε δυο ακραίες τιμές n_L και n_U. Επίσης, οι στρατηγικές ισορροπίας μέσα στην κλάση των αμιγώς μικτών στρατηγικών είναι οι στρατηγικές $\lceil n_0, \theta(n_0) \rceil$ με n_0 ανάμεσα σε δύο ακραίες τιμές n_L^+ και $n_U^- - 1$ και $\theta(n_0)$ η μοναδική τιμή του θ που μηδενίζει την αναμενόμενη πληρωμή του πελάτη, όταν παρατηρεί n_0 πελάτες στο συστημα και οι υπόλοιποι πελάτες ακολουθούν μεικτή στρατηγική με κατώφλι n_0.

Περίπτωση Γ : $lim_{n\to\infty} H^U(n) \geq 0$.
: Τότε, υπάρχει πάντα στρατηγική ισορροπίας, η $\lceil \infty \rceil$-στρατηγική (πάντα να εισέρχεται).

Περίπτωση 2: $(\mu_1 - \mu_2)(\rho_1 - \rho_2) < 0$

Σε αυτήν την περίπτωση αναζητούμε στρατηγικές ισορροπίας στην κλάση των στρατηγικών αντιστρόφου κατωφλίου. Μια καθαρή στρατηγική n_0-κατωφλίου $\lfloor n_0 \rfloor$ υπαγορεύει σε έναν πελάτη να εισέρχεται όταν παρατηρεί $n \geq n_0$ άτομα στο σύστημα και να αποχωρεί αλλιώς. Μια μικτή στρατηγική κατωφλίου $\lfloor n_0, \theta \rfloor$ υπαγορεύει στους πελάτες να εισέρχονται όταν παρατηρούν $n > n_0$, να εισέρχονται με πιθανότητα θ αν παρατηρούν n_0 πελάτες και να αποχωρούν αλλιώς. Ακολουθώντας παρόμοια ανάλυση με την περίπτωση 1, έχουμε τα παρακάτω αποτελέσματα :
Περίπτωση A: $H^U(0) > 0$
Τότε, υπάρχει μοναδική στρατηγική ισορροπίας αντιστρόφου κατωφλίου, η

στρατηγική $\lfloor 0 \rfloor$.
Περίπτωση B: $H^U(0) \leq 0$ και $lim_{n\to\infty} H^U(n) > 0$.
Εδώ, ανάλογα με τις παραμέτρους του συστήματος, υπάρχει μοναδική στρατηγική ισορροπίας αντιστρόφου κατωφλίου και μπορεί να είναι της μορφής $\lfloor 0 \rfloor$, $\lfloor \infty \rfloor$ ή $\lfloor 0, \theta(0) \rfloor$.
Περίπτωση Γ: $\lim_{n\to\infty} H^U(n) \leq 0$. Τότε, υπάρχει μοναδική στρατηγική ισορροπίας αντιστρόφου κατωφλίου, η $\lfloor \infty \rfloor$ στρατηγική.

Περίπτωση 3: $(\mu_1 - \mu_2)(\rho_1 - \rho_2) = 0$

Στην τελευταία περίπτωση δεν υπάρχει γρηγορότερη ή λιγότερο συνωστισμένη κατάσταση του συστήματος και η πληροφορία του αριθμού των πελατών που παρατηρεί ένας αφικνούμενος πελάτης δεν επηρεάζει την απόφαση του. Έχουμε τις παρακάτω περιπτώσεις:
Περίπτωση A: $H^U(0) < 0$
Τότε, η μοναδική στρατηγική ισορροπίας μέσα στην κλάση των στρατηγικών κατωφλίου και αντιστρόφου κατωφλίου είναι η $\lceil 0 \rceil \equiv \lfloor \infty \rfloor$ (πάντα αποχωρεί)
Περίπτωση B: $H^U(0) = 0$
Τότε, κάθε στρατηγική στην κλάση των στρατηγικών κατωφλίου και αντίστροφου κατωφλίου είναι στρατηγική ισορροπίας.
Περίπτωση Γ: $H^U(0) > 0$.
Τότε, η μοναδική στρατηγική ισορροπίας μέσα στη κλάση των στρατηγικών κατωφλίου και αντίστροφου κατωφλίου είναι η $\lceil \infty \rceil \equiv \lfloor 0 \rfloor$ στρατηγική (πάντα να εισέρχεται).

Σύστημα εξυπηρέτησης ρευστού σε εναλλασσόμενο περιβάλλον

Το τελευταίο σύστημα που μελετάμε είναι ένα σύστημα εξυπηρέτησης ρευστού σε εναλλασσόμενο περιβάλλον. Ένα σύστημα εξυπηρέτησης ρευστού (ή ουρά ρευστού) είναι ένα σύστημα εισόδου-εξόδου, όπου οι πελάτες μοντελοποιούνται ως ένα ρευστό που εισέρχεται και εξέρχεται από το χώρο εξυπηρέτησης. Θεωρούμε ότι ο υπηρέτης εναλλάσσεται μεταξύ δυο καταστάσεων i, μιας γρήγορης ($i = 1$) και μιας αργής ($i = 0$), με ρυθμό q_1 και q_0 αντίστοιχα. Ο ρυθμός εισόδου του ρευστού είναι λ και ο ρυθμός εξόδου είναι μ_1 στη γρήγορη κατάσταση και μ_0 αλλιώς. Θεωρούμε ότι $0 < \mu_0 < \mu_1$. Η ολοκλήρωση της εξυπηρέτησης ενός πελάτη αξίζει R μονάδες ωφέλειας και το κόστος αναμονής είναι C ανά μονάδα χρόνου. Εξετάζουμε δυο επίπεδα πληροφόρησης, την παρατηρήσιμη περίπτωση και τη μερικώς παρατηρήσιμη περίπτωση.

Παρατηρήσιμη περίπτωση

Στρατηγικές ισορροπίας

Σε αυτή την περίπτωση, ένας αφικνούμενος πελάτης παρατηρεί και το επίπεδο του ρευστού x και την κατάσταση του υπηρέτη i. Θέλουμε να υπολογίσουμε το αναμενόμενο καθαρό κέρδος ενός επιλεγμένου πελάτη $U = R - CS_i(x)$, όπου με $S_i(x)$ ορίζουμε το μέσο χρόνο παραμονής ενός πελάτη στο σύστημα όταν αυτός παρατηρεί τη στάθμη του ρευστού στο επίπεδο x και τον υπηρέτη στην κατάσταση i, κατά την άφιξή του. Για να υπολογίσουμε λοιπόν το αναμενόμενο κέρδος ενός επιλεγμένου πελάτη, πρέπει να υπολογίσουμε το μέσο χρόνο παραμονής του στο σύστημα, το οποίο το κάνουμε δεσμεύοντας ως προς τον χρόνο παραμονής του υπηρέτη στην κατάσταση i. Ένας επιλεγμένος πελάτης εισέρχεται στο σύστημα όταν $S_i(x) > \frac{R}{C}$, αποχωρεί όταν $S_i(x) < \frac{R}{C}$ και είναι αδιάφορος ως προς την είσοδο ή αποχώρηση από το σύστημα αν $S_i(x) = \frac{R}{C}$. Παρατηρώντας ότι η $S_i(x)$ είναι αύξουσα συνάρτηση στο $[0, \infty)$ και έχει εικόνα το διάστημα $[0, \infty)$, η εξίσωση $S_i(x) = \frac{R}{C}$ έχει μοναδική λύση $x_e(i)$. Άρα η στρατηγική ισορροπίας ορίζεται απο τα δύο κατώφλια $(x_e(0), x_e(1))$.

Κοινωνική βελτιστοποίηση

Συνεχίζουμε στον υπολογισμό των κοινωνικά βέλτιστων στρατηγικών. Αρχικά, παρατηρούμε ότι αν η στάθμη του ρευστού $x(t)$ είναι $x_e(i)$ όταν ο υπηρέτης βρίσκεται στην κατάσταση i, οποιαδήποτε στρατηγική είναι βέλτιστη. Για τον υπολογισμό της μέσης κοινωνικής ωφέλειας θα υποθέσουμε ότι στην κατάσταση $(x_e(i), i)$ ένα ποσοστό $min\{1, \frac{\mu_i}{\lambda}\}$ των αφικνούμενων πελατών εισέρχεται στο σύστημα, δηλαδή εισέρχεται τέτοιο ποσοστό ώστε η στάθμη του ρευστού να παραμένει στο επίπεδο $x_e(i)$. Ξεκινάμε υπολογίζοντας τη στάσιμη κατανομή του συστήματος όταν οι πελάτες ακολουθούν μια στρατηγική κατωφλίου $(x_*(0), x_*(1))$, όπου $x_*(0) < x_*(1)$. Καταλήγουμε στις παρακάτω περιπτώσεις:

1. $\boldsymbol{\lambda > \mu_1}$: Το ρευστό ταλαντεύεται στο διάστημα $[x_*(0), x_*(1)]$.
2. $\boldsymbol{\lambda = \mu_1}$: Το ρευστό σταθεροποιείται στο $x_*(0)$.
3. $\boldsymbol{\mu_2 < \lambda < \mu_1}$: Το ρευστό ταλαντεύεται στο διάστημα $[0, x_*(0)]$.
4. $\boldsymbol{\lambda \le \mu_2}$: Το ρευστό σταθεροποιείται στο 0.

Έχοντας υπολογίσει τη στάσιμη κατανομή, υπολογίζουμε το μέσο κοινωνικό όφελος ανα χρονική μονάδα το οποίο δίνεται για κάθε μια από τις παραπάνω περιπτώσεις από τη σχέση $\lambda_{eff}R - CE(x)$, όπου το λ_{eff} είναι ο πραγματικός ρυθμός εισόδου των πελατών που εισέρχεται στο σύστημα υπό τη στρατηγική $(x_*(0), x_*(1))$ και $E(x)$ είναι η μέση στάθμη του ρευστού υπό τη στρατηγική $(x_*(0), x_*(1))$.

Στις περιπτώσεις (1) και (2), ο πραγματικός ρυθμός άφιξης πελατών είναι ανεξάρτητος από τη στρατηγική $(x_*(0), x_*(1))$ και ισούται με το μέσο ρυθμό εξυπηρέτησης $\overline{\mu}$, καθώς ο υπηρέτης εξυπηρετεί στη μέγιστη δυνατότητά του και στις δυο καταστάσεις, αφού ο ρυθμός άφιξης του ρευστού είναι μεγαλύτερος και από τους δυο ρυθμούς εξυπηρέτησης. Άρα, στην περίπτωση (1), η κοινωνικά βέλτιστη στρατηγική είναι η $(0,0)$, ενώ στην περίπτωση (2), η κοινωνικά βέλτιστη στρατηγική είναι οποιαδήποτε στρατηγική $(0,x)$ με $x \geq 0$.

Στην περίπτωση (4), ο ρυθμός άφιξης των πελατών είναι μικρός σε σχέση με τους ρυθμούς εξυπηρέτησης και οι πελάτες παρατηρούν το ρευστό πάντα στο επίπεδο 0. Συνεπώς, η κοινωνικά βέλτιστη στρατηγική είναι να εισέρχεται πάντα ένας αφικνούμενος πελάτης, αφού λαμβάνει το κέρδος R και έχει μηδενικό κόστος αναμονής.

Στην περίπτωση (3), παρατηρούμε ότι η μέση κοινωνική ωφέλεια είναι ανεξάρτητη του $x_*(1)$. Καθώς το $x_*(0)$ αυξάνεται έχουμε ένα κοινωνικά θετικό αποτέλεσμα, την αύξηση του πραγματικού ρυθμού εισόδου του ρευστού στο σύστημα, και ένα κοινωνικά αρνητικό αποτέλεσμα που είναι η αύξηση της μέσης στάθμης του ρευστού. Συνεπώς η συμπεριφορά της συνάρτησης της μέσης κοινωνικής ωφέλειας εξαρτάται από το βαθμό που επηρεάζονται αυτές οι ποσότητες.

Η μερικώς παρατηρήσιμη περίπτωση

Στρατηγικές ισορροπίας

Σε αυτό το κομμάτι εντοπίζουμε τις στρατηγικές ισορροπίας των πελατών που φτάνουν στο σύστημα και παρατηρούν τη στάθμη του ρευστού χωρίς να παρατηρούν την κατάσταση του υπηρέτη. Η στρατηγική ισορροπίας εδώ είναι ένα κατώφλι x_e , δηλαδή οι πελάτες εισέρχονται όταν παρατηρούν τη στάθμη του ρευστού κάτω από το x_e, αποχωρούν όταν η στάθμη του ρευστού είναι πάνω από το x_e και όταν παρατηρούν τη στάθμη του ρευστού στο επίπεδο x_e εισέρχεται ένα ποσοστό πελατών έτσι ώστε η στάθμη να παραμένη σταθερή. Για να υπολογίσουμε το κατώφλι x_e πρέπει να υπολογίσουμε πρώτα το μέσο χρόνο παραμονής ενός πελάτη στο σύστημα, όταν παρατηρεί το ρευστό στη στάθμη x και οι υπόλοιποι πελάτες ακολουθούν στρατηγική κατωφλίου x_*. Υπολογίζουμε τη στάσιμη κατανομή του ρευστού και δεσμεύοντας ως προς την κατάσταση του ρευστού όταν παρατηρούμε στάθμη x έχουμε ότι ο μέσος χρόνος παραμονής ενός πελάτη είναι

$$S(x;x_*) = \sum_{i=0}^{i=1} \pi(i|x;x_*)S_i(x),$$

όπου $\pi(i|x;x_*)$ είναι η πιθανότητα ο υπηρέτης να είναι στην κατάσταση i ενώ ένας αφικνούμενος πελάτης παρατηρεί στάθμη x υπό τη στρατηγική κατωφλίου x_* και $S_i(x)$ ο μέσος χρόνος παραμονής του πελάτη στο σύστημα ενώ ο υπηρέτης βρίσκεται στην κατάσταση i και η στάθμη του ρευστού είναι στο επίπεδο x. Η στάσιμη κατανομή της στάθμης του ρευστού στη μερικώς παρατηρήσιμη περίπτωση ταυτίζεται με τη στάσιμη κατανομή της παρατηρήσιμης περίπτωσης όταν θέσουμε $x_*(0) = x_*(1) = x_*$.

Η συνάρτηση κέρδους ενός πελάτη δίνεται από τη σχέση $R - CS(x; x_*)$. Όταν ο ρυθμός λ είναι μεγαλύτερος από το ρυθμό γρήγορης εξυπηρέτησης μ_1, το ρευστό ισορροπεί στη στάθμη x_*. Συνεπώς, κάθε αφικνούμενος πελάτης παρατηρεί το ρευστό στη στάθμη x_* και έτσι δεν του δίνεται κάποια πληροφορία για την κατάσταση του υπηρέτη. Το κατώφλι δίνεται από τη λύση της εξίσωσης $R - CS(x; x_*) = 0$. Στην περίπτωση όπου ο ρυθμός άφιξης των πελατών βρίσκεται ανάμεσα στους ρυθμούς εξυπηρέτησης μ_0, μ_1, αποδεικνύουμε ότι υπάρχει μοναδική λύση στην εξίσωση $R - CS(x; x_*) = 0$ η οποία αποτελεί το κατώφλι x_*. Τέλος, αν ο ρυθμός λ είναι μικρότερος του ρυθμού μ_0, το σύστημα παραμένει πάντα στη στάθμη 0 και οι πελάτες πάντα εισέρχονται.

Κοινωνική βελτιστοποίηση

Για να υπολογίσουμε τη μέση κοινωνική ωφέλεια ανά χρονική μονάδα, χρησιμοποιούμε τη συνάρτηση κοινωνικής ωφέλειας της παρατηρήσιμης περίπτωσης θέτοντας $x_*(0) = x_*(1) = x_*$. Καταλήγουμε λοιπόν ότι στην περίπτωση που ο ρυθμός λ είναι μεγαλύτερος από το ρυθμό γρήγορης εξυπηρέτησης μ_1, η κοινωνικά βέλτιστη στρατηγική είναι το κατώφλι 0. Όταν ο ρυθμός άφιξης των πελατών βρίσκεται ανάμεσα στους ρυθμούς εξυπηρέτησης μ_0, μ_1 η κάθε στρατηγική x_* είναι κοινωνικά βέλτιστη στη μερικώς παρατηρήσιμη περίπτωση αν και μόνο αν η στρατηγική (x_*, x'_*) με $x'_* > x_*$ είναι κοινωνικά βέλτιστη στην παρατηρήσιμη περίπτωση. Τέλος, αν ο ρυθμός λ είναι μικρότερος του ρυθμού μ_0 κάθε στρατηγική κατωφλίου x_* είναι κοινωνικά βέλτιστη. Παρατηρούμε ότι οι κοινωνικά βέλτιστες στρατηγικές συμπίπτουν και στα δυο επίπεδα πληροφόρησης.

Τιμή της αναρχίας

Η τιμή της αναρχίας ποσοτικοποιεί την απόκλιση των στρατηγικών ισορροπίας από τις κοινωνικά βέλτιστες στρατηγικές. Στην παρατηρήσιμη περίπτωση ορίζεται ως το κλάσμα

$$PoA = \frac{B(x_*(0), x_*(1))}{B(x_e(0), x_e(1))},$$

δηλαδή το κλάσμα με αριθμητή την μέση κοινωνική ωφέλεια ανα χρονική μονάδα υπό κοινωνικά βέλτιστες στρατηγικές προς την μέση κοινωνική ωφέλεια ανά χρονική μονάδα υπό στρατηγικές ισορροπίας. Αντίστοιχα, στη μερικώς παρατηρήσιμη περίπτωση η τιμή της αναρχίας δίνεται από το κλάσμα $PoA = \frac{B(x_*)}{B(x_e)}$. Εξετάζοντας αυτά τα κλάσματα σε σχέση με το ρυθμό άφιξης λ καταλήγουμε ότι η τιμή της αναρχίας είναι 1 για μικρές τιμές του λ και πέρνοντας το όριο καθώς το λ τείνει στο άπειρο τη τιμή της αναρχίας φράσεται από κάποιον αριθμό.

Bibliography

[1] Anick, D. , Mitra, D. , & Sondhi, M. M. (1982). *Stochastic theory of a data-handling system with multiple sources.* The Bell System Technical Journal, 61 , 1871–1894.

[2] Artalejo, J. R., & Gomez-Corral, A. (1998). *Analysis of a stochastic clearing system with repeated attempts.* Communications in Statistics. Stochastic Models, 14, 623–645.

[3] Braun, M. (1983). *Differential equations and their applications (3rd ed.).* New York: Springer-Verlag .

[4] Burnetas, A. , & Economou, A. (2007). *Equilibrium customer strategies in a single server Markovian queue with setup times.* Queueing Systems, 56 , 213–228 .

[5] Do, C. T. , Tran, N. H. , Nguyen, M. V. , Hong, C. S. , & Lee, S. (2012). *Social optimization strategy in unobserved queueing systems in cognitive radio networks.* IEEE Communication Letters, 16 , 1944–1947.

[6] Economou, A. (2003). *On the control of a compound immigration process through total catastrophes.* European Journal of Operational Research, 147, 522–529.

[7] Economou, A., & Fakinos, D. (2003). *A continuous-time Markov chain under the influence of a regulating point process and applications in stochastic models with catastrophes.* European Journal of Operational Research, 149, 625–640.

[8] Economou, A., & Fakinos, D. (2008). *Alternative approaches for the transient analysis ofMarkov chains with catastrophes.* Journal of Statistical Theory and Practice, 2, 183–197.

[9] Economou, A. , Gómez-Corral, A. , & Kanta, S. (2011). *Optimal balking strategies in single-server queues with general service and vacation times.* Performance Evaluation, 68 , 967–982

[10] Economou, A., & Kanta, S. (2008a). *Optimal balking strategies and pricing for the single server Markovian queue with compartmented waiting space.* Queueing Systems, 59, 237–269.

[11] Economou, Antonis and Kanta, Spyridoula (2008b) *Equilibrium threshold strategies in the observable single-server queue with breakdowns and repairs.* Operations Research Letters 36 (2008) 696-699.

[12] Economou, Antonis and Manou, Athanasia (2011) *Equilibrium threshold strategies for a clearing queuing system in alternating environment.* Ann Oper Res (2013) 208 489-514.

[13] Economou, Antonis and Manou, Athanasia (2016) *Strategic behavior in an observable fluid queue with an alternating service process.* European Journal of Operational Research 254 (2016) 148-160.

[14] Edelson, N. M., & Hildebrand, K. (1975). *Congestion tolls for Poisson queueing processes.* Econometrica, 43, 81–92.

[15] Gani, J., & Swift, R. J. (2007). *Death and birth-death and immigration processes with catastrophes.* Journal of Statistical Theory and Practice, 1, 39–48.

[16] Gautam, N. (2012). *Analysis of queues: methods and applications .* Boca Raton: CRC Press .

[17] Guo, P., & Zipkin, P. (2007). *Analysis and comparison of queues with different levels of delay information.* Management Science, 53, 962–970.

[18] Guo, P. , & Hassin, R. (2011). *Strategic behavior and social optimization in Markovian vacation queues.* Operations Research, 59 , 986–997.

[19] Guo, P. , & Hassin, R. (2012). *Strategic behavior and social optimization in Markovian vacation queues: The case of heterogeneous customers.* European Journal of Operational Research, 222 , 278–286 .

[20] Hassin, R., & Haviv, M. (1997). *Equilibrium threshold strategies: the case of queues with priorities.* Operations Research, 45, 966–973.

[21] Haviv, M. (2013). *When to arrive at a queue with tardiness costs?* Performance Evaluation, 70 , 387–399 .

[22] Jagannathan, K. , Menache, I. , Modiano, E. , & Zussman, G. (2011). *Non-cooperative spectrum access –The dedicated vs. free spectrum choice.* In Proceedings of the international symposium on mobile ad hoc networking and computing (MobiHoc'11) . Article No. 10

[23] Jain, R. , Juneja, S. , & Shimkin, N. (2011). *The concert queueing game: To wait or to be late?* Queueing Systems, 74 , 369–402 .

[24] Kim, K., & Seila, A. F. (1993). *A generalized cost model for stochastic clearing systems.* Computers and Operations Research, 20, 67–82.

[25] Kosten, L. (1974a). *Stochastic theory of multi-entry buffer, part 1.* Delft Progress Report, Series F, 1 , 10–18 .

[26] Kosten, L. (1974b). *Stochastic theory of multi-entry buffer, part 2.* Delft Progress Report, Series F, 1 , 44–55 .

[27] Kosten, L. , & Vrieze, O. J. (1975). *Stochastic theory of multi-entry buffer, part 3.* Delft Progress Report, Series F, 1 , 103–115 .

[28] Kulkarni, V. G. (1997). *Fluid models for single buffer systems.* In J. H. Dshalalow (Ed.), Frontiers in queueing. models and applications in science and engineering (pp. 321–338). CRC Press . (Ed.)

[29] Kyriakidis, E. G. (1994). *Stationary probabilities for a simple immigration-birth-death process under the influence of total catastrophes.* Statistics and Probability Letters, 20, 239–240.

[30] Kyriakidis, E. G. (1999a). *Optimal control of a truncated general immigration process through total catastrophes.* Journal of Applied Probability, 36, 461–472.

[31] Kyriakidis, E. G. (1999b). *Characterization of the optimal policy for the control of a simple immigration process through total catastrophes.* Operations Research Letters, 24, 245–248.

[32] Kyriakidis, E. G., & Dimitrakos, T. D. (2005). *Computation of the optimal policy for the control of a compound immigration process through total catastrophes.* Methodology and Computing in Applied Probability, 7, 97–118.

[33] Li, H. , & Han, Z. (2011). *Socially optimal queueing control in cognitive radio networks subject to service interruptions: To queue or not to queue?* IEEE Transactions on Wireless Communications, 10 , 1656–1666 .

[34] Liu, W. , Ma, Y. , & Li, J. (2012). *Equilibrium threshold strategies in observable queueing systems under single vacation policy.* Applied Mathematical Modelling, 12 , 6186–6202 .

[35] Maglaras, C. (2006). *Revenue management for a multiclass single-server queue via a fluid model analysis.* Operations Research, 54 , 914–932 .

[36] Naor, P. (1969). *The regulation of queue size by levying tolls.* Econometrica, 37 , 15–24 .

[37] Rajagopal, S. , Kulkarni, V. G. , & Stidham, S., Jr. (1995). *Optimal flow control of a stochastic fluid-flow system.* IEEE Journal on Selected Areas in Communications, 13 , 1219–1228 .

[38] Schwartz, M. (1996). *Broadband integrated networks .* Prentice-Hall .

[39] Serfozo, R., & Stidham, S. (1978). *Semi-stationary clearing processes.* Stochastic Processes and their Applications, 6, 165–178.

[40] Stidham, S. Jr. (1977). *Cost models for stochastic clearing systems.* Operations Research, 25, 100–127.

[41] Stidham, S., Jr. (2009). *Optimal design of queueing systems .* Boca Raton: CRC Press, Taylor and Francis Group .

[42] Stirzaker, D. (2006). *Processes with catastrophes.* The Mathematical Scientist, 31, 107–118.

[43] Stirzaker, D. (2007). *Processes with random regulation.* Probability in the Engineering and Informational Sciences, 21, 1–17.

[44] Sun, W., Guo, P., & Tian, N. (2010). *Equilibrium threshold strategies in observable queueing systems with setup/closedown times.* Central European Journal of Operations Research, 18, 241–268.

[45] Yang, W. S., Kim, J. D., & Chae, K. C. (2002). *Analysis of M/G/1 stochastic clearing systems.* Stochastic Analysis and Applications, 20, 1083–1100.

[46] Yang, T. , Wang, J. , & Zhang, F. (2014). *Equilibrium balking strategies in the Geo / Geo /1 queues with server breakdowns and repairs.* Quality Technology & Quantitative Management, 11 , 231–243 .

[47] Zhang, F., & Wang, J. (2010). *Equilibrium analysis of the observable queue with balking and delayed repairs.* In 3rd international joint conference on computational sciences and optimization, CSO 2010: theoretical development and engineering practice 2, art. no. 5533079 (pp. 125–129).

[48] Zhang, F. , Wang, J. , & Liu, B. (2013). *Equilibrium balking strategies in Markovian queues with working vacations.* Applied Mathematical Modelling, 37 , 8264–8282 .

[49] Hassin, R. and Haviv M. (2003) *To Queue or Not To Queue: Equilibrium Behavior in Queueing Systems.* Kluwer Academic Publishers, Boston.

[50] **Φακίνος, Δ. (2008) *Ουρές Αναμονής: Θεωρία και Ασκήσεις, 2η έκδοση.* Εκδόσεις Συμμετρία, Αθήνα.**