Change-Averse Nash Equilibria in Congestion Games

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Abstract

We introduce a new model in Congestion Games, where the players choose their strategy according to the new cost they incur, as well as the difference between their current state and the new state they are considering. The latter part of the decision-making process is based on the assumption that players who are considering a significant change are less prone to take it, than they do on a similar choice. This model has analogies with $\epsilon-$approximate equilibria. We can easily see that this new model provides a richer set of equilibria than approximate equilibria. Christodoulou et al. [7] prove that as far as Linear Congestion Games are concerned, we have good bounds on the Price of Anarchy. We prove that similar results are true in our case. We also prove that players do actually converge on such an equilibrium and relatively quickly.

Acknowledgments

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Contents

Chapter 1

Introduction

In the classical models of network congestion games [43], we have a number of players who pick their path based on the latencies of its edges, which in turn depend upon the traffic therein. A basic assumption made in those kinds of models is that latency is deterministically defined. However, this may significantly fail in cases where latencies can vary and thus introduce a factor of uncertainty regarding the actual latency as this is produced by its traffic, which players realistically take into account.

In this thesis, we will examine a case where players having this uncertainty are reluctant to change their path. The assumption we made is that the players' unwillingness to change their path is proportional to the estimated cost of the new path, as well as the *difference between the new path and the old*. More precisely, herein, this is given by the percentage of the old path that we don't find in the new path, which means the more different the path, the greater this percentage is. If, for example, we move to a completely new path, which doesn't have even one common edge with the old one, then we get one hundred percent on this percentage. On the other extreme case, when we examine zero change, that is between the same paths, the percentage obviously is zero percent. Other metrics on this are possible.

1.1 Results

Herein, we study the properties congestion games have under this equilibrium notion. The notion being that under the above assumptions given a path choice by each player, no player has an incentive to deviate. This we call *Change-Averse equilibrium.*

The first property we study is how much a change-averse nash equilibrium deteriorates in the worst case; this deterioration is called *Price of Anarchy.* More precisely, we find that in the multiplicative version of this equilibrium given a constant number α which models the

sensitivity players have to "difference between paths," the price of anarchy is bounded above the price of anarchy given by an $\alpha-$approximate nash equilibrium. This immediately gives us upper bounds to our price of anarchy based on the work done by Christodoulou et al. [7]. We also prove that each one of these bounds is tight (or partially tight) in our case as well. Therefore, in the atomic case, the price of anarchy for $\alpha \in [0, 1/3]$ is $\frac{5(1+\alpha)}{2-a}$; for $a \in [1/3, 5/4]$ is $\frac{11(1+a)}{4-a}$. Roughly it grows as $(1+a)(3+a)$. In the non-atomic case for $0 < a \leq 1$ we have $\frac{4(1+a)}{3-a}$ and for $a \geq 1$ we have $(1+a)^2$. There is also the case of the additive version of the change-averse equilibrium regarding which we prove a tight bound for the price of anarchy which is $(5d+3)/2d$, where d is the average cost of the optimal solution divided by a.
The second property we study is based on the following question: We know that equilibria exist, but what do we know about players reaching those equilibria in time? We use the following dynamics: among players who have a change-averse improvement at any point in the game, we pick the player with the largest relative cost gain and he makes the move. If the network congestion game is played like this, and under some reasonable assumptions, the players will reach the equilibrium in $\frac{n \cdot a}{h_{min}} \log \frac{\Phi(s^0)}{\Phi_{min}}$ iterations, where Φ is the potential function of the game, and $h_{min} \geq \frac{a}{a+|E|}$ where E is the set of edges in the network.

1.2 Related Work

The modeling and studying of uncertainties in routing games has received a lot of attention in recent years. An extensive survey on this topic is given by Cominetti [8].
Pieter Kleer et al. [22] studied worst-case deviations in network routing games. They included in their model a deviation for each path that has to be below a certain value. This models the uncertainty that is placed on each path. This work was inspired by the study of the Price of Risk Aversion by Nikolova and Stier-Moses [32]. There are several papers that study the problem of imposing tolls (which can be viewed as latency deviations) on the edges of a network to reduce the cost of the resulting nash flow. One closely related example is the *restricted network toll problem* by Bonifaci et al. [3]. The authors study the problem of computing non-negative tolls that have to obey some upper bound restrictions such that the cost of the resulting nash flow is minimized.
Beckmann et al. [1] generalized the selfish routing model to include uncertain delays by associating every edge to a random variable whose distribution depends on the edge flow. So, for one to find the shortest path, he must calculate the mean and the variability of the delay, and thus players end up solving stochastic shortest path problems. There is also Markowitz' mean-risk framework [28] where there is an optimization of the weighted linear combination of mean and risk. Other more classic measures of risk in finance etc have been criticized for

being unrealistic. Therefore, other work was done on this. [32] [42] [35] [25] [9]
The study of Price of Anarchy was first started by Koutsoupias and C. Papadimitriou [24], where congestion games of m parallel links were examined. In that case the price of anarchy measured by the social welfare function of maximum cost, was stated as function of m and is found to be $\Theta(\log m / \log\log m)$. The lower bound was proved in that paper and the upper bound in [23] [12]. Later, this was expanded for m parallel paths and the price of anarchy was found to be $\Theta(\log m / \log\log\log m)$. Czumaj et al. [11] studied more general latency functions, especially the ones relating to queueing theory. Gairing et al. [17] as well as Lucking et al. [26] considered other kinds of social costs and studied those price of anarchy problems.
Suri et al. [41] the special case of congestion games where each strategy is formed by only one resource (singleton strategy). They proved bounds in the case of the average social cost. As far as maximum social cost is concerned, Gairing et al. [18] found the price of anarchy to be $\Theta(\log N / \log\log N)$. On the other end of the spectrum, Christodoulou et al. [6] show a $\Theta(\sqrt{N})$ upper bound.
The case of congestion games where we do not have separate and distinct players but instead we work with a specific flow and consider the players infinitesimal is called *non-atomic case*. This type of congestion games, was studied by T. Roughgarden and E. Tardos in [40] and [39] where it was shown that as far as linear latencies are concerned, the average price of anarchy is 4/3. The also extended this result for polynomial latencies. Also, J.R Correa et al [10] and T. Roughgarden et al. [37] separately examined the social cost of maximum latency.
Christodoulou et al. established tight bounds on the Price of Anarchy and the Price of Stability for ϵ-Nash Equilibria as far as Linear Congestion Games are concerned. We can use their work to our advantage because of the similarity between ϵ-Nash Equilibria and Change-Aversion Equilibria.
Fabrikant, Papadimitriou and Talwar [15] examined systematically the complexity of finding Nash equilibria in congestion games. They showed that the computational complexity of finding a Nash equilibrium in symmetric congestion games is PLS-complete. They also created a polynomial algorithm for the case of symmetric network congestion games. However, this does not infer any local dynamics fast convergence.
Convergence questions similar to this thesis, meaning players with local dynamics and simple local algorithms, have been investigated by many authors. There exist numerous results on "load-balancing" games, which are restricted congestion games in which each strategy consists of just a single edge (or "machine"), but which may be generalized to allow either player-specific cost fucntions [29] or weights on the players [13] [14] [20]. Miltaich, Even-Dar et al. and Goldberg establish polynomial time convergence for versions of the Nash dynamics to (exact or approximate) Nash equilibria in these games, while Even-Dar and Mansour

consider a more complex dynamics in which all players move concurrently according to a certain rerouting mechanism. Kearns and Mansour [21] give polynomial time global and local algorithms that find additive ϵ-approximate equilibria for "large-population" games under a "bounded influence" assumption; however, this assumption appears not to hold for the general multiple-resource congestion games we consider here.

Recent papers by Fischer et al. [16] and Blum et al. [2] consider congestion games at a similar level of generality to ours, each with some version of a "bounded (relative) slope" assumption that is analogous to bounded jumps. However, these papers analyze the *non-atomic* setting where the number of players is taken to infinity and are infinitesimally small. Despite its apparent similarity, the non-atomic case is actually quite different from our discrete setting; for example, in the non-atomic case Nash equilibria can be computed in polynomial time [15] [1]. Fischer et al. establish polynomial bounds on the rate of convergence to approximate Nash equilibria (under a different notion of approximation) of a concurrent dynamics with moves based on "adaptive sampling," while Blum et al. give polynomial bounds when players use no-regret online learning algorithms.

Let's look, lastly, at two recent developments for more general games. Goemans et al. [19] [30] study convergence of Nash dynamics not to an (approximate) Nash equilibrium but instead to a "sink equilibrium", for wider classes of games for which pure equilibria need not exist. They quantify the rate of convergence in various cases, and also the quality of the resulting solution as measured by a global utility function, rather than player-specific costs. And a subexponential time non-local algorithm for finding an approximate Nash equilibrium in general games with a fixed number of players and explicitely presented strategies was given by Lipton, Markakis and Mehta. [27]

Chapter 2

Preliminaries

In this chapter we will go through some basics regarding our subject so that the reader is familiar with the context of this thesis, as well as some fundamental concepts we will use throughout this thesis.

Our study is done in the context of Algorithmic Game Theory. Game Theory is the mathematical study of situations (called *games*) where we have different entities (called *players*) who have several options to take and try to optimize their behavior. The players have a payoff (or a cost) whose value depends on their choice and the other players' choices. The assumption is that players choose rationally based on their self-interest. Had we had to deal with one player, the game would be just an optimization problem, i.e a problem of maximizing payoff or minimizing cost just by manipulating a single player's strategy. When we get a second player in the game (or more), the structure of the problem significantly changes and becomes richer. Let's look at some examples.

Assume we want to travel by car to our destination through morning traffic. We have several choices to take and each choice we take gives us our total traveling time to destination. However, our total traveling time also depends on what choices the other drivers take. For example, I can find out somehow about a shortcut that gets me to the center of the city relatively quickly, but if enough people are convinced that this specific route is faster, we would have more drivers in that route, and therefore the traveling time of each one could significantly increase to the point where this option is no longer optimal. So, one needs to guess what others are doing based on the information one has in order to choose their best strategy.

2.1 General Definitions and Basic Solution Concepts

Formally, a *game* consists of n players. Each player has his own set of possible strategies, say S_i. To play this game, each player selects a strategy $s_i \in S_i$. We use $\boldsymbol{s}$ to denote the vector of strategies selected by players, otherwise called a strategy profile. We also define $\boldsymbol{S}$ as the set of all possible strategy profiles. The strategy profile selected by the players actually determines the outcome of each player; in general, the outcome will be different for different players. Another important factor is that there has to be a preference ordering for each player on his strategies. In this thesis, we will use cost functions, meaning we will assign a value to each player for each outcome $C_i(\boldsymbol{s}) : \boldsymbol{S} \to \mathbb{R}$. The larger this is, the less desirable it is for the player. To introduce our basic solution concept, let's see an example of a classic game.

The game is called **Tragedy of the commons**: [33] This is one in the context of sharing bandwidth. Suppose that n players each would like to have part of a share resource. For example, each player wants to send information along a shared channel of known maximum capacity, say 1. In this game each player will have an infinite set of strategies, player's i strategy is to send x_i units of flow along the channel for some value $x_i \in [0, 1]$.

Assume that each player would like to have a large fraction of the bandwidth , but assume also that the quality of the channel deteriorates with the total bandwidth used. We will describe this game by a simple model, using a benefit or payoff function for each set of strategies. If the total bandwidth $\sum_j x_j$ exceeds the channel capacity, no player get any benefit. if $\sum_j x_j < 1$ then the value for player i is $x_i(1 - \sum_j x_j)$. This models exactly the kind of trade-off we had in mind: the benefit for a player deteriorates as the total assigned bandwidth increases, but it increases with his own share (up to a point).

Now, let's see which are the stable strategies are for a player by concentrating on player i. Assume that $t = \sum_{j \neq i} x_j < 1$ flow is sent by all other players. Now player i faces a simple optimization problem for selecting his flow amount: sending x flow results in a benefit of $x(1 - t - x)$. Using elementary calculus, we get that the optimal solution for player i is $x = (1 - t)/2$. A set of strategies is stable if all players are playing their optimal selfish strategy, given the strategies of all other players. For this case, this means that $x_i = (1 - \sum_{j \neq i})/2$ for all i, which has a unique solution in $x_i = 1/(n + 1)$ for all i.

Why is this solution a tragedy? The total value of the solution is extremely low. The value for player i is $x_i(1 - \sum_{j \neq i} x_j) = 1/(n + 1)^2$, and the sum of the values over all players is then $n/(n + 1)^2 \approx 1/n$. In contrast, if the total bandwidth is $\sum_i x_i = 1/2$ then the total value is $1/4$, approximately $n/4$ times bigger. In this game the n users sharing the common resource overuse it so that the total value of the shared resource decreases quite dramatically.

Now, let's define more formally the stable solution concept above. It's not a dominant strategy, but instead is something less stringent, which gives us more possibilities. It is called a *Nash Equilibrium* and it is the concept that is used as a central solution concept in Game Theory. The idea is the one above: we consider a solution stable if no single player can individually improve his or her welfare by deviating. More formally,

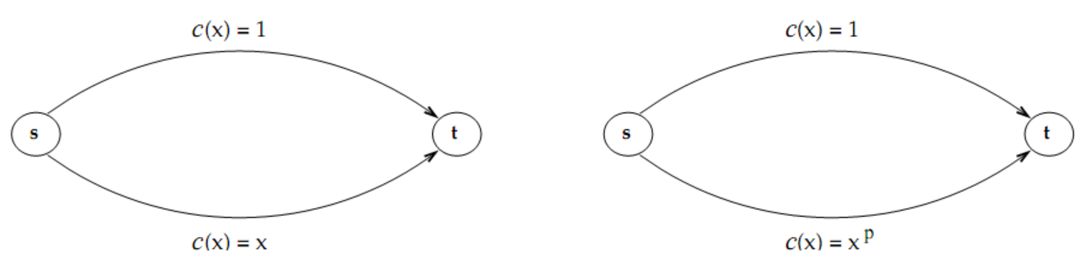
Definition (Nash Equilibrium). *A strategy profile $\boldsymbol{s} \in \boldsymbol{S}$ is said to be a Nash equilibrium if for all players i and each alternate strategy $s'_i \in S_i$, we have that*

$$C_i(s_i, s_{-i}) \leq C_i(s'_i, s_{-i}).$$

In other words, no player i can change his chosen strategy from s_i to s'_i and thereby improve his payoff, assuming that all other players stick to the strategies they have chosen in s. Observe that such a solution is self-enforcing in the sense that once the players are playing such a solution, it is in every player's interest to stick to his or her strategy.

2.2 Price of Anarchy

Internet users and service providers act selfishly and spontaneously, without an authority that monitors and regulates network operation in order to achieve some "social optimum" such as minimum total delay [4]. How much performance we lose because of this factor? [24] We already saw on the example of Prisoners dilemma that there is a socially optimal solution that is not the Nash equilibrium. So, how far can a nash equilibrium be from the optimal solution. This question introduces the concept of Price of Anarchy. Let's see two very representative examples of this given in [34].



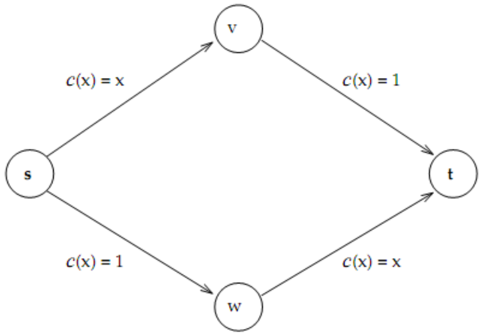
(a) Pigou's example (b) A nonlinear variant

Example([38]) Consider the simple network shown in figure (a) above, also called Pigou's network. Two disjoint edges connect a source vertex s to a sink vertex t. Each edge is labeled with a cost function $C(.)$, which describes the cost (e.g travel time) incurred by users of the edge, as a function of the amount of traffic routed on the edge. The upper edge has the constant cost function $C(x) = 1$, and thus represents a route that is relatively long but immune to congestion. The cost of the lower edge, by contrast, is governed by the function $C(x) = x$ and thus increases as the edge gets more congested. In particular, the lower edge is cheaper than the upper edge if and only if less than one unit of traffic uses it.

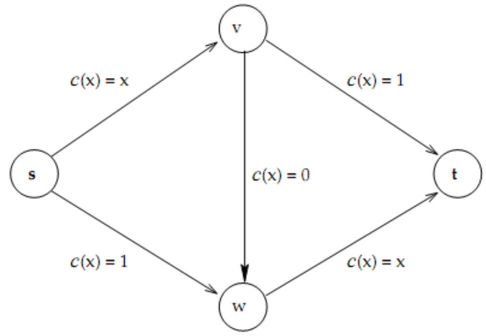
Now, let's assume that we have one unit of traffic, that represents a major part of the network users, and that each of them chooses independently among the two routes from s to t. Each user wants to minimize his cost, so it is expected that all traffic should follow the lower edge. So, each user would think as follows: the lower route is never worse than the upper one, even when it is fully congested, and it is superior whenever some of the other users are foolish enough to take the upper route. So, we end up with all users incurring one unit of cost each.

Now if we have the authority to choose how the traffic is routed how can we improve this outcome for all players? Imagine that we assign half of the traffic on the upper route and half of the traffic on the lower route. The users using the upper route, each experiences one unit of cost and therefore are not worse than previously. However, the users on the lower route, experience 1/2 unit of cost. So, the average cost incurred by traffic has decreased from 1 to 3/4.

Pigou's example shows that selfish routing does not produce an optimal outcome, exactly like the Prisoner's dilemma. If we modify the model, this phenomenon is even worse. Let's replace the cost function $C(x) = x$ with a $C(x) = x^P$ for P large (look at the figure b above). As before, when we let players to behave according to their own self-interest, they would all choose the lower route, thus incurring a cost of 1. However, if we had the authority to control the traffic and directed a small ϵ fraction of traffic along the upper route, then the average cost would drop to $\epsilon + (1 - \epsilon)^{p+1}$, which approaches 0 as ϵ tends to 0 and p tends to infinity. That means that we can make this divergence as large as possible.



(a) Initial network



(b) Augmented network

Example [5]: Let's consider the network shown in the figure (a) above. We have two separate paths from s to t, with a total cost of $1 + x$ each, where x is the amount of traffic using that specific path. The routes are equivalent and so selfish traffic should split between them. If we assume there is 1 unit of total traffic, all users experience 3/2 units of cost on the outcome.

Suppose we want to improve this outcome and the solution we come up with is to build a short, high-capacity edge connecting the vertices in the middle of each path. We have the new network in the figure (b) just above. There, there is a new edge (v, w) with a cost 0 regardless of the traffic on it.

The previous traffic pattern will now change. Let's examine the path $s \to v \to w \to t$. This path is not worse than each of the original paths and sometimes it is strictly better. So, it stands to reason that all users will follow this new path. However, we now have heavy traffic on (s, v) and (w, t), namely a cost of 1 for each player there. Therefore, all traffic experiences a total cost of 2, instead of 3/2 they had before the "help." This shows us an at first counter-intuitive result; that adding a new edge can increase the cost that the traffic experiences instead of helping them.
Now, let's look at this concept more formally. *The Price of Anarchy* of a game is defined as the ratio between the worst objective value of an equilibrium of the game and that of an optimal outcome [AGT Book]. Note that the price of anarchy depends on which is your objective function of choice, as well as the equilibrium concept at hand. In any case, the essence of this is that the closer the Price of Anarchy is to 1, the better off we are. In this thesis, we study the Price of Anarchy that relates to our notion of equilibrium, namely the Change-Averse Equilibrium, which we prove it's not a bad one.

However, this concept has a drawback. In the case of multiple equilibria, even though it gives us a good look into the worst-case scenario, it identifies games with significant inefficiency differences. Consider two different games which have 3 equilibria. Game A has three "bad" equilibria (we don't usually consider constant values bad, but let's use it in favor of making an example), namely the ratio is 10, 9, 11 respectively. Game B has one "bad" and two "good" equilibria, namely 1.2, 1.1, 11 respectively. The price of anarchy will not differentiate between those two games, although the second one is obviously better in relation to its equilibria inefficiency. Therefore we introduce the concept of *Price of Stability,* which is defined as the ratio between the *best* objective value of an equilibrium of the game and that of an optimal outcome. This, although it is a weaker measure of the equilibria inefficiency than the Price of Anarchy, it has its advantages. One of the benefits of the *Price of Stability* concept is that the players are not always completely independent and they do indeed have some kind of cooperation, which the price of stability will better incorporate in the model of the game's equilibria inefficiency [33].

2.3 Convergence

Another important issue is this: In the analysis above, we were examining the games and the demand was that there is some kind of stable solution, namely a Nash Equilibrium. However, do players ever reach this equilibrium? We know it's there, but if we make reasonable assumptions on how each games begins and how players behave on the information they have at each point in time, will they reach this stable state, so that they can stay there?
Even if they can reach it, how quickly will they do that? There is the case of players taking so long to reach the equilibrium, that is no longer realistic to consider that they will practically get there, because in real life the games are played for a finite amount of time.
Another thing to consider is what are the behaviors of players that lead them to converge to an equilibrium? There could be many variations of behaviors and not each one necessarily leads to a stable state.
The best and most natural behavior one can probably think of, is the one of the "best response." Let us have a strategy profile $\boldsymbol{s}$ and player i. This player has a cost $C_i(\boldsymbol{s})$. Let's assume that s_i' is another strategy available to player i and if he changes to that while everyone else sticks to their chosen strategy, i will incur a cost of $C_i(s_i', s_{-i})$. Now, a change from strategy s_i to strategy s_i' is called an *improving response* for player i if $C_i(s_i', s_{-i}) < C_i(\boldsymbol{s})$. Also, this change is called *best response* if s_i' minimizes cost over all available options of i, i.e $min_{s_i' \in S_i} C_i(s_i', s_{-i})$. Players choosing those kinds of moves in turn gives us a natural way for players to behave during a play of the game.

We will use this concept in this thesis in our model and will prove that the players making best-response moves according to our kind of equilibrium do actually converge and relatively fast as well, which gives more validity to our option of notion of equilibrium.

2.4 Congestion Games

Congestion games [36] are a special class of games that are in a category called potential games [31]. A congestion game is a game where each player wants some combination of resources. Each resource (which also can be called facility) can be used by many players. The more players use the same resource, the more costly is for each player to have the combination that includes it.

The traffic example is most probably the most representative one in this case of games. Players want to get to a destination. Depending on the starting point and the destination each player might need a different combination (set) of roads (which correspond to edges in the modeled network). Now, the more traffic there is on one edge, the more delay each player will have on that edge. So, each player gets to decide which set of edges to pick, in order to arrive at his chosen destination, given what the other players are choosing.

Another example would be n firms engaged in production. Each firm has several alternative production processes available, each of which uses a subset of the primary factors. The use of any primary factor involves a set-up cost but no variable cost. The set-up costs are functions of the number of other firms demanding the same factor. The cost of each process is the sum of the factor costs.

More formally, a congestion game is a tuple $(N, E, (S_i)_{i\in N}, (f_e)_{e\in E})$, where $N = \{1, ..., n\}$ is a set of n players, E is a set of facilities, $\Sigma_i \subseteq 2^E$ is a collection of pure strategies of player i: a pure strategy $A_i \in \Sigma_i$ is a set of facilities, and finally f_e is a cost(or latency) function associated with facility j.

This model usually uses linear cost functions of the form $f_e(k) = \alpha^e \cdot k + b_e$ for nonnegative constants α_e and b_e. Especially in the chapter where we examine the price of anarchy in congestion games satisfying the Change-averse equilibrium, we will use the identity latency function, that is $f_e(k) = k$. Why is this not extremely restrictive? Well, when α_e is an integer, one can find a similar game with α_e facilities. Moreover, even α_e is not an integer, the identity function can be used in the place of a linear function without the loss of generality.

A pure strategy profile $A = (A_1, ..., A_n)$ is a vector of strategies, as we saw before. The cost function is given by the following formula: $C_i(A) = \sum_{e\in A_i} f_e(n_e(A))$, where $n_e(A)$ is the number of players using e in A. A pure strategy profile A is a Nash equilibrium if no player has any reason to unilaterally deviate to another pure strategy, that is more formally:

$\forall i \in N, \forall S \in (\Sigma_i)\ C_i(A) \leq C_i(A_{-i}, S)$, where (A_{-i}, S) is the strategy profile produced if we just have player i deviating from A_i to S.
Why are Congestion games a special class of Potential games? First, let's define potential games. A *potential game* is a game where there exists a *potential function* $\Phi : A \to \mathbb{R}$, such that

$$C_i(a_i, a_{-i}) - C_i(a_i\, a_{-i}) = \Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i}),$$

for every $i \in N$, $a_i, a_i' \in A_i$ and $a_{-i} \in A_{-i}$.

So, Congestion games do have a function with this property which is

$$\Phi(A) = \sum_{e \in E} \sum_{k=1}^{n_e(A)} f_e(k).$$

Then, by the property of potential games, we have that Congestion Games have indeed a pure nash equilibrium. Let's say we find a minimal value for Φ. That means that on this strategy profile, no player can reduce his cost, because in the case he could, by the definition of potential function, the potential would decrease. This is a contradiction. This function we chose for Congestion Games does indeed have this property.

2.5 Change-Averse Equilibria

A key question in making decisions is how to make a good decision under uncertainty, especially when people making those decisions are risk-averse. Applications relating to this include alleviating congestion in transportation networks, as well as improving telecommunications, robotics, security and others, all having uncertainty in their core and require reliable solutions.
Peter Kleer et al. introduce a model where each path has a bounded deviation and players weigh their options based on their expected delay plus the path's bounded deviation. The bounded deviations on this model could be any number less than a specific value which can be different for each path. In our case, we attempt to specify the nature of this deviation. Although our model is different in other minor ways, our players don't just have a deviation on each path and compare them to reach a decision, but instead they have a function that computes the difference between the paths and we make the assumption that the player that perceives the most difference, will also perceive the most risk in taking that path.

More formally,

Definition (Additive Change-Averse equilibrium). *Let α be a constant nonnegative number. A change-averse equilibrium is a strategy profile $\boldsymbol{x}^*$ such that for all $i \in \{1, ..., n\}$ and for all x_i that belong to the set of strategies of player i,*

$$C_i(\boldsymbol{x}^*) \leq C_i(x_i, x^*_{-i}) + \alpha f(x^*_i, x_i),$$

where f is a measure of the change perceived by the player i, that is

$$f(x, y) = \frac{|x \backslash y|}{|x|}$$

Also, another version of this is the multiplicative one, which goes like this:

Definition (Multiplicative Change-Averse equilibrium). *Let α be a constant nonnegative number. A change-averse equilibrium is a strategy profile $\boldsymbol{x}^*$ such that for all $i \in \{1, ..., n\}$ and for all x_i that belong to the set of strategies of player i,*

$$C_i(\boldsymbol{x}^*) \leq (1 + \alpha f(x^*_i, x_i)) \cdot C_i(x_i, x^*_{-i}),$$

where f is a measure of the change perceived by the player i, that is

$$f(x, y) = \frac{|x \backslash y|}{|x|}$$

Chapter 3

Change-Averse Equilibria in Congestion Games and Price of Anarchy

Now, in this chapter we will present our results regarding the Price of Anarchy in the Change-Averse Equilibrium setting with linear latencies in a Congestion game as mentioned in the introduction. First, we will prove tight bounds for the price of anarchy for the multiplicative version of the Change-Averse equilibrium. For that we will consider separately the atomic case where we have n number of players, and the non-atomic case (also called *selfish routing*) where we have an infinite amount of players infinitesimally small. Then, we will finish off by proving a tight bound for the additive change-averse equilibrium. The way this is done is by finding an upper bound for the price of anarchy for each case and then finding an example of a game where that specific price of anarchy actually manifests.

3.1 Multiplicative Change-Averse Equilibrium

Just as a reminder, we give the multiplicative definition of change-averse equilibria, and after that we prove the aforementioned proposition.

Definition (Multiplicative Change-Averse equilibrium). *Let α be a constant nonnegative number. A change-averse equilibrium is a strategy profile $\boldsymbol{x}^*$ such that for all $i \in \{1, ..., n\}$ and for all x_i that belong to the set of strategies of player i,*

$$C_i(\boldsymbol{x}^*) \leq (1 + \alpha f(x_i^*, x_i)) \cdot C_i(x_i, x_{-i}^*),$$

where f is a measure of the change perceived by the player i, that is

$$f(x,y) = \frac{|x \backslash y|}{|x|}$$

The following proposition describes an obvious relationship between $\epsilon-$Nash Equilibria and Change-Aversion Equilibria concerning the concept of Price of Anarchy.

Proposition 1. *Let G be a Congestion Game. Let $\alpha \geq 0$ be the uniform sensitivity for change that each player has in G. The Price of Anarchy in a Change-Aversion Equilibrium is at most the Price of Anarchy in a α-approximate Nash Equilibrium.*

Proof. First, let's deal with the non-atomic case. Let P be a flow-currying path under f which is a Change-Averse equilibrium flow, and P' any other path. By Change-Aversion Equilibrium we have

$$l_P(f) \leq (1 + \alpha \cdot g(P, P')) \cdot l_{P'}(f) \leq (1 + \alpha) \cdot l_{P'}(f)$$

The above holds since $g \leq 1$.
Next, the atomic case. Let x be the strategy profile that is a Change-Averse Equilibrium. For each player i and each alternative strategy $x_i' \in S_i$, by definition of the equilibrium we have

$$C_i(\boldsymbol{x}) \leq (1 + \alpha \cdot g(P, P')) \cdot C_i(x_i', x_{-i}) \leq (1 + \alpha) \cdot C_i(x_i', x_{-i})$$

So, *every* Change-Aversion equilibrium is also an α-approximate Equilibrium.

Let f_{eq} be the Change-Aversion equilibrium flow with the worst total cost; let f_{eq}' be the α-approximate equilibrium flow with the worst total cost; let PoA be the Price of Anarchy for the Change-Aversion Equilibrium and let PoA' be the Price of Anarchy for the α-approximate equilibrium. Therefore,

$$PoA = \frac{C(f_{eq})}{OPT} \leq \frac{C(f_{eq}')}{OPT} = PoA'$$

Similarly, for the atomic case.

□

3.1.1 Atomic Case

Now, for the atomic case the following proofs are from Christodoulou et al. [7], which (by the previous proposition) hold for our case as well. What follows is a lemma used in the proof thereafter regarding a tight Price of Anarchy upper bound for ϵ-Nash Equilibria.

Lemma 1 [Christodoulou et al.] . *For every α,β,$z \in \mathbb{N}$:*

$$\beta(\alpha+1) \leq \frac{1}{2z+1}\alpha^2 + \frac{z^2+3z+1}{2z+1}\beta^2$$

Proof. Consider the function f(α, β) which we obtain when we subtract the left part of the statements inequality from the right part and multiply the result by $2z+1$.

$$f(\alpha,\beta) = \alpha^2 + (z^2+3z+1)\beta^2 - (2z+1)\beta(\alpha+1)$$

$$= \left(\alpha - \frac{2z+1}{2}\beta\right)^2 + \frac{(8z+3)\beta^2 - (8z+4)\beta}{4}.$$

For $\beta = 0$ and for any $\beta \geq 2$, $f(\alpha,\beta)$ is clearly positive. For $\beta = 1$ it takes the form of $f(\alpha, 1) = (\alpha - z)(\alpha - z - 1) \geq 0$, and the lemma follows.

□

Theorem [Christodoulou et al.]. *For any positive real ϵ, the approximate Price of Anarchy of general congestion games with linear latencies is at most,*

$$(1+\epsilon)\frac{z^2+3z+1}{2z-\epsilon},$$

where $z \in \mathbb{N}$ is the maximum integer with $\frac{z^2}{z+1} \leq 1+\epsilon$ (or equivalently for $z = \lfloor \frac{1+\epsilon+\sqrt{5+6\epsilon+\epsilon^2}}{2} \rfloor$).

Proof. Let $A = (A_1, ..., A_n)$ be an ϵ-approximate pure Nash, and $P = (P_1, ..., P_n)$ be the optimum allocation. From the definition of ϵ-equilibria we get

$$\sum_{e \in A_i} n_e(A) \leq (1+\epsilon)\sum_{e \in P_i}(n_e(A)+1).$$

If we sum up for every player i and use the previous lemma we get

$$SUM(A) = \sum_{i \in N} c_i(A)$$

$$
\begin{aligned}
&= \sum_{i \in N} \sum_{e \in A_i} n_e(A) \\
&= \sum_{e \in E} n_e^2(A) \le (1+\epsilon) \sum_{e \in E} n_e(P)(n_e(A)+1) \\
&\le \frac{1+\epsilon}{2z+1} \sum_{e \in E} n_e^2(A) + \frac{(1+\epsilon)(z^2+3z+1)}{2z+1} \sum_{e \in E} n_e^2(P) \\
&= \frac{1+\epsilon}{2z+1} SUM(A) + \frac{(1+\epsilon)(z^2+3z+1)}{2z+1} OPT.
\end{aligned}
$$

From this we obtain the theorem

$$SUM(A) \le (1+\epsilon) \frac{z^2+3z+1}{2z-\epsilon} OPT.$$

□

Therefore, combining the previous results with the first proposition, we trivially get

Theorem 1. *For any positive real uniform change-aversion sensitivity α, the Change-Averse Price of Anarchy of general congestion games with linear latencies is at most,*

$$(1+\alpha)\frac{z^2+3z+1}{2z-\alpha},$$

where $z \in \mathbb{N}$ is the maximum integer with $\frac{z^2}{z+1} \le 1+\alpha$ (or equivalently for $z = \lfloor \frac{1+\alpha+\sqrt{5+6\alpha+\alpha^2}}{2} \rfloor$).

This bound is proved tight by Christodoulou et al. for ϵ-Nash Equilibria. Their instance is proved tight in our case as well.

Theorem 2. *For any positive real uniform change-aversion sensitivity α , there are instances of congestion games with linear latencies, for which the Change-Averse Price of Anarchy of general congestion games with linear latencies, is at least*

$$(1+\alpha)\frac{z^2+3z+1}{2z-\alpha},$$

where $z \in \mathbb{N}$ is the maximum integer with $\frac{z^2}{z+1} \le 1+\alpha$ (or equivalently for $z = \lfloor \frac{1+\alpha+\sqrt{5+6\alpha+\alpha^2}}{2} \rfloor$).

Proof. Let $z \in \mathbb{N}$ be the maximum integer with $\frac{z^2}{z+1} \le 1+\alpha$. We will construct an instance with $z+2$ players and $2z+4$ facilities. There are two types of facilities:

- $z+2$ facilities of type a with latency $l_e(x) = x$ and
- $z+2$ facilities of type b with latency $l_e(x) = \gamma x = \frac{(z+1)^2-(1+\alpha)(z+2)}{(1+\alpha)(z+1)-z^2}x$.

Player i has two alternative pure strategies, S_i^1 and S_i^2.

- The first strategy is to play the two facilities a_i and b_i, i.e. $S_i^1 = \{a_i, b_i\}$.
- The second strategy is to play every facility of type a except for a_i and $z+1$ facilities of type b starting at facility b_{i+1}. more precisely, the second strategy has the facilities

$$S_i^2 = \{a_1, ..., a_{i-1}, a_{i+1}, ..., a_{z+2}, b_{i+1}, ..., b_{i+1+z}\},$$

where the indices may require computations ($\mod z+2$).

First we prove that playing the second strategy $S^2 = (S_1^2, ..., S_n^2)$ is a Change-Aversion Equilibrium. The cost of player i is

$$c_i(S^2) = (z+1)^2 + \gamma z^2,$$

as there exactly $z+1$ players using facilities of type a and exactly z players using facilities of type b.

If player i unilaterally switches to the other available strategy S_i^1 he has cost

$$c_i(S_i^2, S_{-i}^2) = (z+2) + \gamma(z+1) = \frac{c_i(S^2)}{1+\alpha},$$

which shows that S^2 is a Change-Aversion Equilibrium because of the fact that the alternatives are disjoint sets, and so we conclude that our function $g = 1$.

The optimum allocation is the strategy profile S^1, where every player has cost $c_i(S^1) = 1+\gamma$ and so the price of anarchy is

$$\frac{c_i(S^2)}{c_i(S^1)} = \frac{(z+1)^2 + \gamma z^2}{1+\gamma} = (1+\alpha)\frac{z^2+3z+1}{2z-\alpha}.$$

Notice that the parameter z is an integer because it expresses a number of facilities.

□

To sum up, the price of anarchy in relation to ϵ-Nash Equilibria and Change-Aversion Equilibria with α uniform sensitivity, have identical structure. The intuition behind this is that in the bad instances of Change-Aversion Equilibria, a player has to choose between disjoint strategies and so we fall into the approximate equilibria case.

So, in agreement with Christodoulou et al., we have for any positive real α, an integer $z(\alpha)$, which is the maximum integer that satisfies $\frac{z^2}{z+1} \leq 1+\alpha$. So for $\alpha \in [0, 1/3]$, $z(\alpha) = 1$ and the price of anarchy is $\frac{5(1+\alpha)}{2-\alpha}$, for $\alpha \in [1/3, 5/4]$, $z(\alpha) = 2$ and the price of anarchy is $\frac{11(1+\alpha)}{4-\alpha}$ and so on. Roughly the price of anarchy grows as $(1+\alpha)(3+\alpha)$.

3.1.2 Non-Atomic Case

In the same manner, let's examine the following proofs for upper bounds by Christodoulou et al. regarding the non-atomic case.

Lemma 2 [Christodoulou et al.] . *For every real numbers α, β, λ we have,*

$$\beta\alpha \leq \frac{1}{4\lambda}\alpha^2 + \lambda\beta^2,$$

Proof. Simply because $\alpha^2 + 4\lambda^2\beta^2 - 4\lambda\alpha\beta = (\alpha - 2\lambda\beta)^2 \geq 0$. □

Theorem [Christodoulou et al.]. *For any positive real ϵ, and for every $\lambda \geq 1$, the approximate price of anarchy of non-atomic congestion games with linear latencies are at most*

$$\frac{4\lambda^2(1+\epsilon)}{4\lambda - 1 - \epsilon}$$

Proof. Let f be an $\epsilon-$approximate Nash flow, and $f*$ be the optimum flow (or any other feasible flow). From the definition of approximate Nash equilibria, we get that for every path p with non-zero flow in f and any other path p':

$$\sum_{e \in p} l_e(f_e) \leq (1+\epsilon) \sum_{e \in p'} l_e(f_e*).$$

If we sum up these inequalities for all pairs of paths p and p' weighted with the amount of flow of f and $f*$ on these paths.

$$\sum_{p,p'} f_p f^*_{p'} \sum_{e \in p} l_e(f_e) \leq (1+\epsilon) \sum_{p,p'} f_p f^*_{p'} \sum_{e \in p} l_e(f^*_e)$$

$$\sum_{p'} f^*_{p'} \sum_{e\in E} l_e(f_e)f_e \leq (1+\epsilon)\sum_{p} f_p \sum_{e\in E} l_e(f^*_e)f^*_e$$

$$(\sum_{p'} f^*_{p'}) \sum_{e\in E} l_e(f_e)f_e \leq (1+\epsilon)(\sum_{p} f_p) \sum_{e\in E} l_e(f^*_e)f^*_e$$

But $\sum_p f_p = \sum_{p'} f^*_{p'}$ is equal to the total rate for the feasible flows f and f^*. Simplifying, we get

$$\sum_{e\in E} l_e(f_e)f_e \leq (1+\epsilon)\sum_{e\in E} l_e(f_e)f^*_e.$$

This is the generalization to approximate equilibria of the inequality established by Beckmann, McGuire and Winston [] for the exact Wardrop equilibria.
Since we consider linear functions of the form $l_e(f_e) = a_e f_e + b_e$, we get

$$\sum_{e\in E}(a_e f_e^2 + b_e f_e) \leq (1+\epsilon)\sum_{e\in E} a_e f_e f^*_e + (1+\epsilon)\sum_{e\in E} b_e f^*_e$$

By applying lemma 2 of Chirstodoulou et al, we get

$$\sum_{e\in E}(a_e f_e^2 + b_e f_e) \leq (1+\epsilon)\sum_{e\in E} \alpha_e\Big(\frac{1}{4\lambda}f_e^2 + \lambda f_e^{*2}\Big) + (1+\epsilon)\sum_{e\in E} b_e f^*_e.$$

From that we get

$$\sum_{e\in E}\Big(a_e\Big(1-(1+\epsilon)\frac{1}{4\lambda}\Big)f_e^2 + b_e f_e\Big) \leq \lambda(1+\epsilon)\sum_{e\in E} a_e f_e^{*2} + (1+\epsilon)\sum_{e\in E} b_e f^*_e.$$

and for $\lambda geq 1$

$$\frac{4\lambda-1-\epsilon}{4\lambda}SC(f) \leq (1+\epsilon)\lambda SC(f^*).$$

This gives us a price of anarchy at most of

$$\frac{4\lambda^2(1+\epsilon)}{4\lambda-1-\epsilon},$$

for every $\lambda \geq 1$.

□

So, by the proposition before, we have:

Theorem 3. *For any positive real α, and for every $\lambda \geq 1$, the Change-Averse price of anarchy of non-atomic congestion games with linear latencies are at most*

$$\frac{4\lambda^2(1+\alpha)}{4\lambda - 1 - \alpha}$$

Also, let's present their corollaries to this modified for our model:

Corollary 1. *For any nonnegative real $\alpha \leq 1$, the Change-Averse price of anarchy of non-atomic congestion games with linear latencies is at most*

$$\frac{4(1+\alpha)}{3-\alpha}.$$

Corollary 2. *For any nonnegative real $\alpha \geq 1$, the Change-Averse price of anarchy of non-atomic congestion games with linear latencies is at most*

$$(1+\alpha)^2.$$

The question is now whether these are tight. In agreement with their results, Corollary 1 is tight and Corollary 2 is tight only for integral values of α.

Theorem 4. *For any nonnegative real $\alpha \geq 1$, there are instances of congestion games with linear latencies, for which the Chage-Averse price of anarchy of general congestion games with linear latencies, is at least*

$$\frac{4(1+\alpha)}{3-\alpha}.$$

Proof. We will construct an instance with 3 commodities, each of them with unit flow, and 6 facilities. There are two types of facilities:

- 3 facilities of type a, with latency $\ell(x) = x$ and
- 3 facilities of type b with constant latency $\ell(x) = \gamma = \frac{2(1-\alpha)}{1+\alpha}$.

Commodity i has two alternative pure strategies, S_i^1 and S_i^2.

- The first strategy is to choose both the facilities a_i and b_i, i.e. $S_i^1 = \{a_i, b_i\}$.

- As a second alternative, players of commodity i may choose every facility of type a except a_i; we denote this set by $S_i^2 = \{a_{-i}\}$.

First we prove that playing the second strategy $S^2 = (S_i^2, S_2^2, S_3^2)$ is an α-Change-Averse Equilibrium. The cost of every player in commodity i is $c_i(S^2) = 4$, as each one of the 2 facilities in a_{-i} is used by one other commodity.

If a player in commodity i switches to the other available strategy S_i^1 he gets

$$c_i(S_i^1, S_{-i}^2) = 2 + \gamma = \frac{c_i(S^2)}{1+\alpha}$$

and so S^2 is an Change-Averse equilibrium since the two alternatives are disjoint and $g = 1$.

In the optimum case, the players use strategy profile S^1, where commodity i has cost $c_i(S^1) = 1 + \gamma$ and so the price of anarchy is

$$\frac{c_i(S^2)}{c_i(S^1)} = \frac{4}{1+\gamma} = \frac{4(1+\alpha)}{3-\alpha}$$

□

Theorem 5. *For any real positive α, there are instances of congestion games with linear latencies, for which the Change-Averse price of anarchy of general congestion games with linear latencies, is at least*

$$(1+\alpha)\frac{z(z+1)}{2z-\alpha} = (1+\alpha)\frac{z^2+z}{2z-\alpha},$$

where $z = \lfloor 1+\alpha \rfloor$.

Proof. Let $z = \lfloor 1+\alpha \rfloor$. We will construct an instance with $z+2$ commodities and $2z+4$ facilities. There are two types of facilities:

- $z+2$ facilities of type a, with latency 1 and
- $z+2$ facilities of type b, with latency $\gamma = \frac{(z+1)^2-(1+\alpha)(z+1)}{(1+\alpha)z-z^2}$.

Commodity i has two alternative pure strategies, S_i^1 and S_i^2.

- The first strategy is to play the two facilities a_i and b_i, i.e. $S_i^1 = \{a_i, b_i\}$.

- The second strategy is to play every facility of type a except for a_i and $z+1$ facilities of type b starting at facility b_{i+1}. more precisely, the second strategy has the facilities

$$S_i^2 = \{a_1, ..., a_{i-1}, a_{i+1}, ..., a_{z+2}, b_{i+1}, ..., b_{i+1+z}\},$$

where the indices may require computations ($\mod z+2$).

First we prove that playing the second strategy $S^2 = (S_1^2, ..., S_n^2)$ is a Change-Aversion Equilibrium. The cost of commodity i is

$$c_i(S^2) = (z+1)^2 + \gamma z^2,$$

as there exactly $z+1$ commodities using facilities of type a and exactly z commodities using facilities of type b.

If commodity i unilaterally switches to the other available strategy S_i^1 he has cost

$$c_i(S_i^2, S_{-i}^2) = (z+1) + \gamma z = \frac{c_i(S^2)}{1+\alpha},$$

which shows that S^2 is a Change-Aversion Equilibrium because of the fact that the alternatives are disjoint sets, and so we conclude that our function $g = 1$.

The optimum allocation is the strategy profile S^1, where every player has cost $c_i(S^1) = 1+\gamma$ and so the price of anarchy is

$$\frac{c_i(S^2)}{c_i(S^1)} = \frac{(z+1)^2 + \gamma z^2}{1+\gamma} = (1+\alpha)\frac{z(z+1)}{2z-\alpha}.$$

Notice that the parameter z is an integer because it expresses a number of facilities.

□

3.2 Additive Change-Averse Equilibrium

So far we saw the results relating to multiplicative Change-Averse Equilibria. Now, we will examine the results relating to the additive definition of a change-averse equilibrium. As a reminder, this definition follows:

Definition (Additive Change-Averse equilibrium). *Let α be a constant nonnegative number. A change-averse equilibrium is a strategy profile $\boldsymbol{x}^*$ such that for all $i \in \{1, ..., n\}$ and for all x_i that belong to the set of strategies of player i,*

$$C_i(\boldsymbol{x}^*) \leq C_i(x_i, x^*_{-i}) + \alpha f(x^*_i, x_i),$$

where f is a measure of the change perceived by the player i, that is

$$f(x, y) = \frac{|x \backslash y|}{|x|}$$

So, for this case the following lemma and theorem hold true:

Lemma 3. *For every pair of non-negative integers a, b it holds that:*

$$b(a+1) \leq a^2/3 + 5b^2/3$$

.

Proof. Special case of previous lemma. □

Theorem 6. *Let G a linear Congestion Game. Let d be the positive real $d = avg(P)/\alpha$, where P is the optimal solution and α is the uniform change-sensitivity of the players. Then, the Price of Anarchy is at most $\frac{5d+3}{2d}$.*

Proof. Let A be a Change-Aversion Nash Equilibrium and P be an optimal (or any other) allocation. The cost of player i at the equilibrium is $c_i(A) = \sum_{e \in A_i} n_e(A)$, where $n_e(A)$ denotes the number of players that use facility e in A. We want to bound the social cost, the sum of the cost of the players: $SUM(A) = \sum_i c_i(A) = \sum_{e \in E} n_e^2(A)$, with respect to the optimal cost $SUM(P) = \sum_i c_i(P) = \sum_{e \in E} n_e^2(P)$.

At the Change-Aversion Equilibrium, the cost of the player i should not increase more than the factor of change, the player experiences by changing:

$$c_i(A) = \sum_{e \in A_i} n_e(A) \leq \sum_{e \in P_i} n_e(A_{-i}, P_i) + \alpha f(A_i, P_i) \leq \sum_{e \in P_i} (n_e(A) + 1) + \alpha f(A_i, P_i)$$

If we sum over all players i, we can see that:

$$SUM(A) = \sum_i c_i(A) \leq \sum_{i \in N} \sum_{e \in P_i} (n_e(A)+1) + \sum_{i \in N} \alpha f(A_i, P_i) = \sum_{e \in E} n_e(P)(n_e(A)+1) + \sum_{i \in N} \alpha f(A_i, P_i)$$

Because of the inequality of nonnegative integers: $b(a+1) \leq a^2/3+5b^2/3$ (Lemma above) we have that:

$$\sum_{e \in E} n_e(P)(n_e(A)+1) \leq \frac{1}{3}SUM(A) + \frac{5}{3}SUM(P)$$

Also, we can see that:

$$\sum_i \alpha \frac{|A_i \backslash P_i|}{|A_i|} \leq n \cdot \alpha$$

By definition of d, then

$$\sum_i \alpha \frac{|A_i \backslash P_i|}{|A_i|} \leq n \cdot a \leq SUM(P)/d$$

Therefore if we combine all of them:

$$SUM(A) \leq \frac{1}{3}SUM(A + \frac{5}{3}SUM(P) + \frac{1}{d}SUM(P)$$

which means that the Price of Anarchy is at most $\frac{5d+3}{2d}$

□

Now, is this bound tight? We prove below that it is.

Theorem 7. *Let d be the positive real $d = avg(P)/\alpha$, where P is the optimal solution of a Congestion Game and α is the uniform change-sensitivity of the players. We have games with 3 or more players where the Price of Anarchy is $\frac{5d+3}{2d}$.*

Proof. We will construct a congestion game where for N players and $|E| = 2N$ facilities, we have a Price of Anarchy of $\frac{5d+3}{2d}$. We divide the set E into 2 subsets $E_1 = \{h_1, ..., h_N\}$, $E_2 = \{g_1, ..., g_N\}$ with N facilities each. Player i has two pure strategies $\{h_i, g_i\}$ and $\{g_{i+1}, h_{i-1}, h_{i+1}\}$. The Optimal Solution is when each player i chooses the first strategy which incurs a cost of 2. So the cost of the optimal solution is $2N$. We claim that there exist a Change-Averse Equilibrium for arbitrarily small α where each player chooses the second strategy.

Let's say each player picks the second strategy. So, let's examine the traffic on the three resources player i picked. He is the only one choosing g_{i+1} and he incurs a unit cost from

that. Resource h_{i-1} is chosen by one other player, as well as h_{i+1}. That means that he has a total cost of 5.
Now let's examine his cost if he were to change strategy. Resource h_i is chosen by two other players and g_i by one other player. So, again, a total cost of 5.

Now, the function that measures change perception, i.e $f(x, y) = \frac{|x \setminus y|}{|x|}$, is 1 because of the two alternative strategies are disjoint. So, there exist arbitrarily small α such that the strategy profile described, *is* in fact a Change-Averse Equilibrium with a cost of $5N$. That means that the Price of Anarchy is 5/2.

So, let's check if this is the actual result of the formula above.

$$\frac{5d + 3}{2d} = \frac{5 \cdot (avg(P)/\alpha) + 3}{2 \cdot (avg(P)/\alpha)} = \frac{10 + 3\alpha}{4} \xrightarrow{\alpha \to 0} \frac{5}{2}.$$

$\square$

Chapter 4

Convergence and Change-Averse Equilibria

In the previous chapter we computed some bounds on the price of anarchy regarding the change-averse equilibrium in the linear latency case. However, how valid of an inefficiency result would a ratio of the optimal solution cost to the equilibrium cost be, if the equilibrium itself is not realistic? Following this line of reasoning, one major way an equilibrium concept would become unworkable would be if we prove that players will never arrive to such an equilibrium, as we said in the introduction. On the other hand, we could prove that players, having reasonable dynamics, will indeed arrive at this equilibrium and relatively quickly as well. The latter is the case for the results that follow. In general, therefore, we are examining the existence of convergence and the speed of that convergence. In this chapter we will at first see best response dynamics and based on that, we'll see that players playing with "change-averse moves" do actually converge without having put attention to the speed of convergence. Next, we will examine $\epsilon-$Nash dynamics and its convergence speed proof in order to use it later for our case. Finally, we will adapt their results to our model, proving our own model's speed of convergence for the case of network congestion games.

4.1 Basic Convergence results

To understand the results that immediately follow we need to understand first the notion of *best-response dynamics. Best-Response dynamics* is a procedure by which players search for a pure nash equilibrium. More exactly: When the current strategy profile $\boldsymbol{s}$ is not a pure nash equilibrium, we pick an arbitrary player i and an arbitrary beneficial deviation s_i' for player i, and move to the profile (s_i', s_{-i}). Now, the thing is that it doesn't necessarily matter if a nash equilibrium exist; these dynamics might go into cycles; so a separate approach is needed than the mere proof of existence. Now, each player may have several deviating moves; later, we will specialize it as needed. However, this best-response dynamics fits well with potential games, because we track these movements by our potential function, remember:

$$\Phi(s_i', s_{-i}) - \Phi(\boldsymbol{s}) = C_i(s_i', s_{-i}) - C_i(\boldsymbol{s}).$$

With this information it can easily be proved that in a finite potential game, these dynamics do converge. More formally

Proposition [Monderer et al.]. *In a finite potential game, from an arbitrary initial outcome, best-response dynamics converges to a PNE*

Proof. In every iteration of best-response dynamics, the deviator's cost strictly decreases. By the definition of the potential function, the potential function strictly decreases. Thus, no cycles are possible. Since the game is finite, best-response dynamics eventually hals, necessarily at a PNE. □

So, we see that there is a natural procedure by which the players actually arrive at a PNE. Is this the case however for the Change-Averse equilibrium? We next see that indeed this is the case.

Proposition 2. *For every congestion game, every sequence of change-averse improvement steps is finite.*

Proof. We know by Rosenthal's proposition that this is true for improvement steps that reduce the player's cost at any given strategy profile. Now, by change-averse improvement steps we mean:

$$C_i(\mathbf{x}^*) > C_i(x_i, x_{-i}^*) + \alpha_i f(x_i^*, x_i),$$

or even

$$C_i(\mathbf{x}^*) > (1 + \alpha_i f(x_i^*, x_i)) C_i(x_i, x_{-i}^*).$$

We don't mind whether we have the multiplicative definition, or the additive one. In any case, if you have a change-averse improvement move, you definitely have a regular improvement move, but the converse is not true.

So, we can visualize Rosenthal's proposition with a directed tree with strategy profiles as vertices. The root of the tree is an arbitrary strategy profile. Every edge from a vertex shows that a specific player can and will improve the cost and thus creating a new strategy profile (arrival vertex). By the proposition, the height of this tree is finite. Now, based on the change-averse improvement move, that means that at each vertex, we just might cut some subtree from there on down (the change-averse moves are a subset of the cost improving moves). That means that at the end of this procedure, we also create a finite tree with change-averse improvement steps.

□

4.2 ϵ-Nash Dynamics

Because of the similarity of Change-Averse equilibrium and $\epsilon-$nash equilibrium, we will examine this kind of dynamics along with its corresponding convergence result and then adapt this to our case even though there are some important differences. First, the definition of an $\epsilon-$Nash equilibrium:

Definition (ϵ-Nash equilibrium). *For $\epsilon \in [0, 1)$, a state $s = (s_1, ..., s_n) \in S_1 \times ... \times S_n$ is an ϵ-Nash equilibrium if for all players p_i, $C_i(s_1, ..., s'_i, ..., s_n) \geq (1-\epsilon)C_i(s_1, ..., s_i, ..., s_n)$ for all $s'_i \in S_i$.*

So, based on this definition the analogous to a best-response move would be an ϵ-move which is a move where a player can increase his cost by a factor at least ϵ, that is if the player is playing s_i and wants to change to s'_i, we have $C_i(s_1, ..., s'_i, ..., s_n) < (1-\epsilon)C_i(s_1, ..., s_i, ..., s_n)$. If no one has this kind of move, we reached an $\epsilon-$Nash equilibrium.

One assumption that we must make is the **bounded jump** assumption. If an edge *satisfies* the $a-$bounded jump condition that means that its latency function has the following property:

$$\forall t \geq 1 \; \exists a \geq 1 \; : d_e(t+1) \leq a d_e(t).$$

This is not a restrictive quality as we can easily see. Most natural latency functions pass this test. The following convergence theorem deals with the non-atomic case.

Another very important factor is how we break ties in the case of more than one players have an $\epsilon-$move available. The technique we use is to pick the player who maximizes the relative cost gain. In other words, the move is made by a player p_i who maximizes $\frac{C_i(\boldsymbol{s})-C_i(s_1,...,s'_i,...,s_n)}{C_i(\boldsymbol{s})}$.

Theorem [Chien, Sinclair]. *Consider an atomic selfish routing game where:*

1. *All players have a common source vertex and a common sink vertex.*
2. *Cost functions satisfy the "a-bounded jump condition," meaning* $c_e(x+1) \in [c_e(x), a \cdot c_e(x)]$ *for every edge* e *and positive integer* x.
3. *Traffic follows the* $\epsilon-$*Nash dynamics from any initial state*

Then an ϵ*-Nash equilibrium is reached in* $(\frac{n \cdot a}{\epsilon} \log \frac{\Phi(s^0)}{\Phi_{min}})$ *iterations.* $\Phi(s^0)$ *is the initial value of the potential function and* Φ_{min} *its minimum value at the end.*

It is useful to know that the main tool Chien and Sinclair used for this proof because we will make a use of it in the next section.

Lemma [Chien, Sinclair]. *In a symmetric congestion game in which every edge has a-bounded jumps, if in the* $\epsilon-$*Nash dynamics with state* s *the next move is made by player* p_i*, then* $C_j(\boldsymbol{s}) \leq aC_i(\boldsymbol{s})$ *for all* j.

Proof. Suppose player p_i moves from s_i to s_i', taking the game from state $s = (s_1, ..., s_n)$ to $s' = (s_1, ..., s_i', ..., s_n)$. Consider an arbitrary player p_j, and resulting state if p_j, rather than p_i, had adopted s_i', denote this state $s'' = (s_1, ..., s_i, ..., s_j'' = s_i', ..., s_n)$. Since p_i moved and not p_j, we can conclude that p_j's relative gain for this move is at most p_i's relative gain, regardless of whether this in an ϵ-move for p_j. (If it is an ϵ-move, then p_i's relative gain must be at least as large by the definition of the dynamics; if it is not an ϵ-move, then p_j's relative gain is at most ϵ while p_i's relative gain is more than ϵ.) Thus we have

$$\frac{C_j(s) - C_j(s'')}{C_j(s)} \leq \frac{C_i(s) - C_i(s')}{C_i(s)}.$$

Now let us compare the cost p_i pays for adopting s_i', namely $C_i(s')$, with how much p_j would have paid for the same strategy, namely $C_j(s'')$. For each edge $e \in s_i'$, either p_i is already occupying it before the move ($e \in s_i$), or not. In the former case, p_j may have to pay as much as $d_e(f_s(e) + 1)$ to use e, while p_i only pays $d_e(f_s(e))$; by the bounded jump assumption, these differ by at most a factor of a. In the latter case, p_i pays $d_e(f_s(s)+1)$ and p_j pays at most the same amount. Summing over all edges $e \in s_i'$, we obtain $C_j(s'') \leq ac_i(s')$.

Combining this with the previous inequality, we obtain $\frac{C_j(s) - aC_i(s')}{C_j(s)} \leq \frac{C_i(s) - C_i(s')}{C_i(s)}$, from which we can see that $C_j(s) \leq aC_i(s)$, as required. □

4.3 Change-Averse Dynamics and Speed of Convergence

Now, we can use a result of Chien et al.[] for approximate equilibria and use it and modify it for Change-Averse Equilibria.

Theorem 8. *Consider an atomic selfish routing game where:*

1. *All players have a common source vertex and a common sink vertex.*
2. *Cost functions satisfy the "a-bounded jump condition," meaning $c_e(x+1) \in [c_e(x), a \cdot c_e(x)]$ for every edge e and positive integer x.*
3. *The REL-MAXGAIN variant of Change-Averse-best-response dynamics is used: in every iteration, among players with a change-averse-move available, the player who can obtain the biggest relative cost decrease moves to its minimum-cost deviation.*

Then an Change-Averse-PNE is reached in $(\frac{n \cdot a}{h_{min}} \log \frac{\Phi(s^0)}{\Phi_{min}})$ iterations.

Proof. Let's define a Change-Averse move. A player makes this move from s_i to s_i' when:

$$C_i(s_i', s_{-i}) < (1 - h(s_i, s_i'))C_i(s),$$

where $h(s_i, s_i') = \frac{\alpha g(s_i, s_i')}{\alpha g(s_i, s_i') + 1}$.

Let's consider the player with the maximum cost, j. So, for j it holds that $C_j(s) \geq \sum_{i=1}^{n} C_i(s)/n$. We know that the Potential function $\Phi(s) \leq \sum_{i=1}^{n} C_i(s)/n$, so we have

$$C_j(s) \geq \Phi(s)/n.$$

Next, we will bound the cost of our choice to the player with the maximum cost.

First off, for two unrelated change-averse functions, namely $g(x)$, $g(y)$, we have the following:

$$g(x) \leq (N-1)g(y).$$

The idea is that the minimum possible value of g for two different strategies is $1/(N-1)$ and the maximum 1, where N the number of vertices in the graph. Since, as g grows, so does h, and by some calculations it's relatively easy to see for two different h, namely $h(x)$, $h(y)$, we have that

$$h(x) \leq (N-1)h(y)$$

Assume that player i moves from s_i to s'_i. So, we have $s' = (s_1, ..., s'_i, ..., s_n)$. Also, from symmetry, let's define this strategy profile: $s'' = (s_1, ..., s_i, ..., s''_j = s'_i, ..., s_n)$.

Let's make the assumption that j does not have a Change-Averse move to s'_i.

So, from the above it holds:

$$\frac{C_j(s) - C_j(s'')}{C_j(s)} \leq h(s_j, s''_j) \leq (N-1)h(s_i, s'_i) \leq (N-1)\frac{C_i(s) - C_i(s')}{C_i(s)}.$$

From Chien et al.[] and the a-jump condition, we know that $C_j(s'') \leq a \cdot C_i(s')$.

Now, we have that:

$$\frac{C_j(s) - aC_i(s')}{C_j(s)} \leq (N-1) \cdot \frac{C_i(s) - C_i(s')}{C_i(s)}.$$

By some calculations, we have that

$$C_j(s) \leq \frac{aC_i(s')C_i(s)}{(N-1)C_i(s') - (N-2)C_i(s)} \leq \frac{aC_i(s)}{(N-2)\frac{C_i(s')-C_i(s)}{C_i(s')}} < 0$$

which cannot be the case. So, j does have a Change-Averse move to $s'_i = s''_j$. That means, that since we choose the largest relative gain from all Change-Averse moves, we have that

$$\frac{C_j(s) - C_j(s'')}{C_j(s)} \leq \frac{C_i(s) - C_i(s')}{C_i(s)}.$$

This leads us to the result from Chien et al. [] that $C_j(s) \leq aC_i(s)$.

So, let's put everything together.
From the above we have that $C_i(s) \geq \frac{1}{an}\Phi(s)$. Therefore,

$$\Phi(s) - \Phi(s') = C_i(s) - C_i(s') \geq h_{min}C_i(s) \geq \frac{h_{min}}{an}\Phi(s).$$

Therefore at each move Φ will decrease by a factor at least $\frac{h_{min}}{an}$. So, we have the result.

□

Chapter 5

Conclusions - Future Work

So, in this thesis we looked over some basic results regarding the change-averse equilibrium. We've seen some good bounds on the price of anarchy, all of which, are at least partially tight. Therein, we used one basic assumption: that the latency functions are linear. One extension of those results could go in the direction of examining polynomial latency functions or even general latency functions.

On the last chapter we've shown that players converge at an equilibrium and at a speed of $(\frac{n \cdot a}{h_{min}} \log \frac{\Phi(s^0)}{\Phi_{min}})$. Here, we could go in many directions; for example, we could see what happens if we drop combinations of the assumptions made: $a-$bounded jumps, or symmetry; or if we used general congestion games and not just network congestion games. Also, do we have convergence under different dynamics which lead to a change-averse equilibrium?

Another factor is the price of stability. As was said at the second chapter, the price of anarchy has some drawbacks relaying with full accuracy the complete structure of the inefficiency of the equilibrium; namely, in a game with multiple equilibria, taking into account only the worst might not lead you to understand the problem adequately. So, an examination of the price of stability would be a very good way to complete the inefficiency results.

And finally in the model itself. We used a very specific function to measure the change a player experiences, which is $f(x, y) = \frac{|x \setminus y|}{|x|}$. Although this is a very straightforward measure, other measures are possible and if we study price of anarchy, the convergence and other such properties under that measure, we can see how much of a factor this choice plays in the outcome of the game and in the case that these properties are not wildly different, it gives credit to the model in general.

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