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The Kauffman Bracket and Genus of Alternating Links

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The Kauffman Bracket and Genus of Alternating Links

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Bryan Minh Nhut Nguyen

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Abstract

Giving a knot, there are three rules to help us finding the Kauffman bracket polynomial. Choosing knot's orientation, then applying the Seifert algorithm to find the Euler characteristic and genus of its surface. Finally finding the relationship of the Kauffman bracket polynomial and the genus of the alternating links is the main goal of this paper.

Acknowledgements

I would like to express my deepest appreciation to my immediate adviser, Dr. Rolland Trapp for his great interest and inspiration in the pursuit of my studies and in the preparation that led to the development of this thesis.

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Chapter 1

Introduction

1.1 History of Knot

Knot theory is the study of mathematical knots. In the 18^{th} century, Alexandre Theophile Vandermonde, a French mathematician, introduced the first reference to knots. He noted some history of ideas that led up to the development of modern knot theory. In the 19^{th} century, Carl Friedrich Gauss made his first study about knot theory and introduced Gauss Linking Number.

In this chapter, we will learn some basic concepts of Knot theory. Lets start with the question *what is knot?*

If we tie a knot in a piece of rope then glue the two ends of the rope together, we will have a knot. We cannot untangle this rope if we do not use the scissors to cut it.

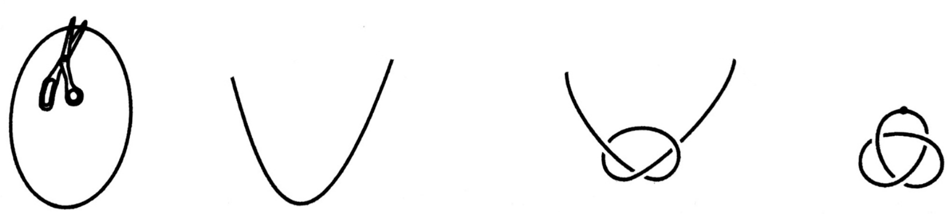


Figure 1.1: How to Make a Knot.

Definition 1.1. A closed loop of rope that does not intersect itself is called a **knot**.

There are so many type of knots. The simplest knot is called the **unknot**, or **trivial knot**, which is a closed loop of rope that has no knot in it. The next simplest knot is called the **trefoil knot**.

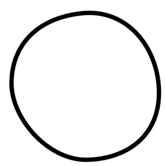


Figure 1.2: The Unknot.

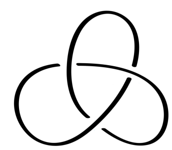
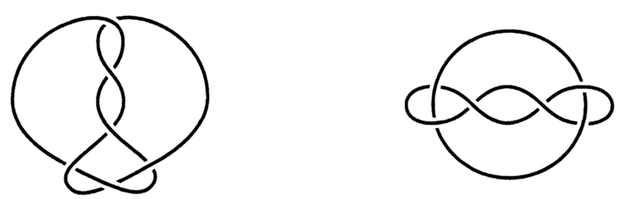


Figure 1.3: The Trefoil Knot.

Definition 1.2. A **link** is a collection of knots that interlaced together. Each knot in the link is called a component of the link.



Each Link Above is the Union of Two Knots.

Figure 1.4: Example of Links

If two knots have different picture, but if we can deform one knot to look like the other knot, then we say these two are the same knot. There are many different pictures of the same knot. We call such a picture of a knot a **projection** of the knot. For example, Figure 1.5 shows three projections of the trefoil knot. Even though they look completely different to each other, as we deform one knot, we will get to the other two knots.

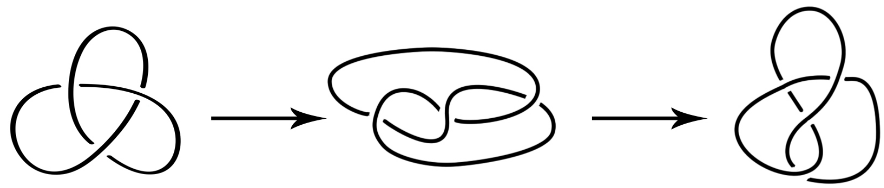


Figure 1.5: Three Projections of the Trefoil Knot.

The sections where the knot crosses itself are called the **crossings** of the projection. The Trefoil knot has three crossings as we can see in Figure 1.3. Knot and link diagrams are represented by a projection onto the plane where over crossings are denoted as a solid line and under crossings are denoted as a dash line.

An **alternating knot** is a knot with a projection that each crossing alternate under, over, under, over, as one travels around in a fixed direction. For instance, the trefoil knot in Figure 1.3 is alternating. In contrast, if there is at least one pair of consecutive crossings that are both under or both over crossings, then a projection is **non-alternating**.

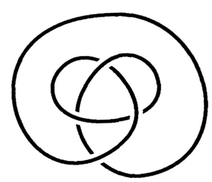
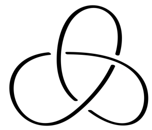
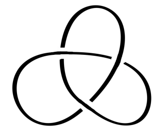


Figure 1.6: Non-alternating Projection.

If we reflect the projection of a knot onto a mirror, then we will have its mirror image. A **mirror image** of a projection is the knot whose has same projection but all crossing changed from under to over and vise versa. This can sometimes be obtained by a series of Reidemeister moves that we will discuss in next section.



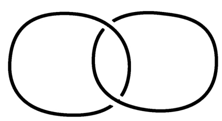
(a) Right Trefoil



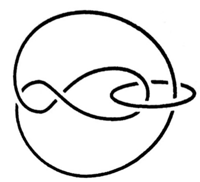
(b) Left Trefoil

Figure 1.7: Mirror Image of the Trefoil Knot.

An **alternating link** is a link that has an alternating projection. In contrast, if every projection of a link is non-alternating, we call it a **non-alternating link**.



(a) Hopf Link



(b) Whitehead Link

Figure 1.8: Alternating Link and Non-alternating Link.

A nontrivial knot has at least one crossing. If the knot has only one crossing, then we can deform it to be the trivial knot. The readers should convince themselves that any two-crossing projection of a knot corresponds to the unknot as well. Thus any projection of a nontrivial knot has at least three crossings.

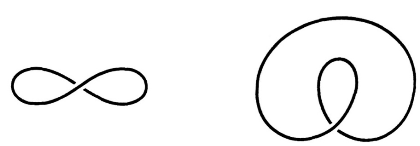


Figure 1.9: One-crossing Projections.

Note: Hopf link is the only link with a two-crossing projection, as Figure 1.8a.

1.2 Reidemesister Moves

How do we know if two knots are the same or different? If we look at Figure 1.3, there are three different projections of the same Trefoil knot. We can deform three of them to make one look like another. This deformation is a list of steps that we rearrange the string of the knot to resemble it to another projection. We must follow certain rules of rearrangement, called **ambient isotopy**. Rearrangement of the string under ambient isotopy is moving the string through the three dimensional space that will change the relation between the crossings. The changes in projections were shown by Reidemeister to always be a sequence of certain moves. There are three types of moves called the **Reidemeister Moves** [Ada04].

The first type of move is called **Reidemeister I**. It allows us to twist or untwist the knot in either direction, as in Figure 1.10. The second type, **Reidmeister II**, gives us an option to add or remove two crossings by moving one loop over another, as in Figure 1.11. The third type, **Reidmeister III**, allows us to slide a string completely past a crossing, as in Figure 1.12.

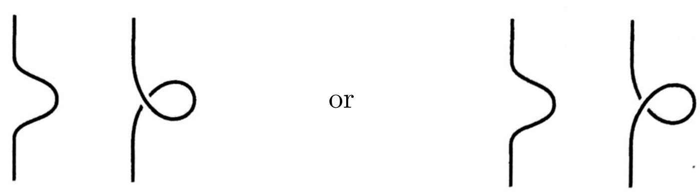


Figure 1.10: Type I Reidemeister Move.

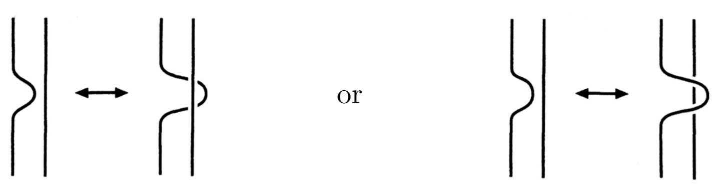


Figure 1.11: Type II Reidemeister Move.

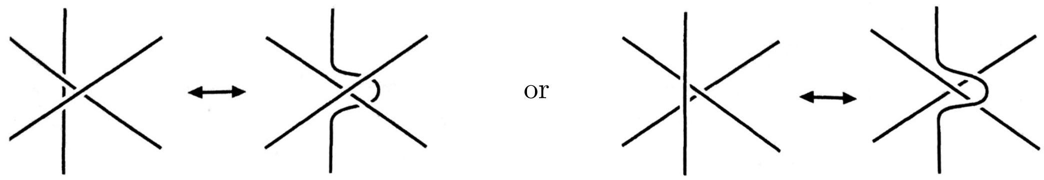
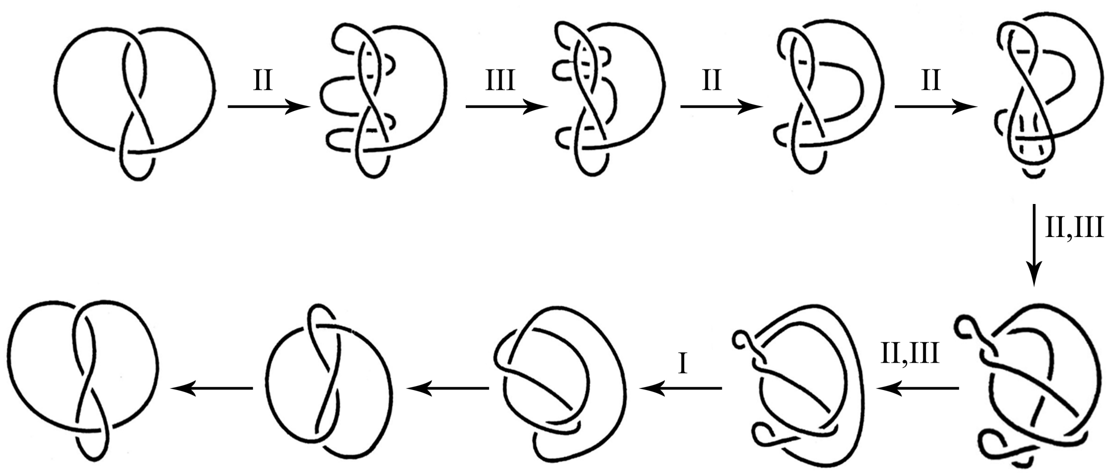


Figure 1.12: Type III Reidemeister Move.

Reidemeister's Theorem shows that any two projections of the same link are related by sequence of these three moves.

Example 1.1. Show the Figure 8 knot is equivalent to its mirror image.



The Figure 8 knot is equivalent to its mirror image.

Figure 1.13: The Figure 8 Knot Equivalency.

This is the series of moves showing that by doing Reidmeister move I, II, and III, we can show the Figure 8 knot is equivalent to its mirror image. It changes every crossing in the Figure 8 knot to the opposite crossing.

We will have use for another property of diagrams, call writhe. At every crossing of the projection, we have either a $+1$ or -1 sign. This can be used to introduce the writhe of an oriented diagram.

Definition 1.3. The **writhe** of an oriented diagram is the sum of the signs of its crossings.

(a) Positive Crossing (b) Negative Crossing

Figure 1.14: Positive Crossing and Negative Crossing.

Example 1.2. Find the writhe of the Trefoil knot.

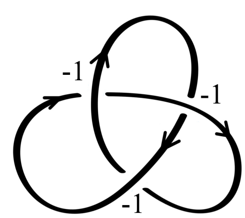


Figure 1.15: The Writhe of the Trefoil Knot.

The writhe of the trefoil knot is $\omega(\boldsymbol{L}) = -1 - 1 - 1 = -3$. ■

1.3 Flypes

Definition 1.4. A **tangle** is a region of knot or link projection that is surrounded by a circle. There are two strands entering the circle and two strands exiting the circle. They across the circle exactly four times.

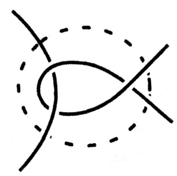
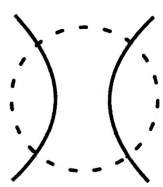
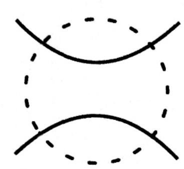


Figure 1.16: Tangle.

There are two basic tangles: ∞ tangle and $\bigcirc$ tangle. We denoted the tangle that has two vertical strands is the ∞ tangle, and the tangle that has two horizontal strands is the $\bigcirc$ tangle.



(a) ∞ Tangle.



(b) $\bigcirc$ Tangle.

Figure 1.17: ∞ Tangle and $\bigcirc$ Tangle.

Definition 1.5. A **twist** is a region of knot involving two strands that intertwine each other. An example of twist is shown in Figure 1.18

Figure 1.18: Twist.

Definition 1.6. A **flype** is a kind of manipulation of knot and link diagrams when we twist a region of the knot forcing nearby crossings to change. Note that flypes can diminish the number of twists in a link. With this, we mention that a diagram is twist reduced if we flype all crossings in a twist to one section as shown in Figure 1.20

Figure 1.19: Flype with Tangle R and T.

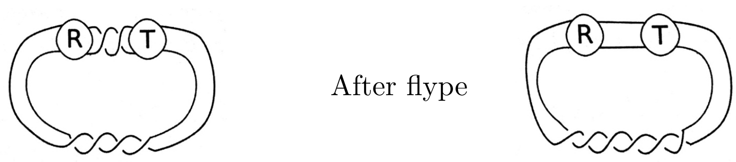


Figure 1.20: Flypes Reduced the Number of Twists.

There are two important facts we will need:

• Any two reduced alternating diagram of the same link are related by the sequence of flypes [MT91].

Note that non-alternating diagram of the same link are related by the sequence of Reidemeister moves.

• 3-variable bracket is invariant under flypes [Laf13].

Putting those two together we can say 3-variable bracket is an invariant of reduced alternating link. The main goal of this work is to uncover some of what the 3-variable bracket tells us about the topology of an alternating link.

Chapter 2

Link Polynomials

We will approach the Jones polynomial by going through the Kauffman Bracket polynomial first. Polynomials are a good source of link variants to distinguish different knots. If two knots have two different polynomials, then they are two different knots.

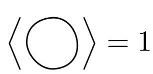
2.1 Bracket Polynomial

2.1.1 3-Variable Bracket

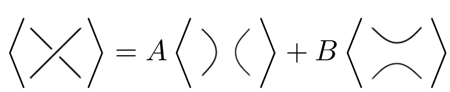
Louis Kauffman is approach to finding the Jones polynomial is called the Kauffman polynomial, also know as Bracket polynomial. He developed three rules to compute the Bracket polynomial of any link diagram $\boldsymbol{L}$, denoted $\langle \boldsymbol{L} \rangle$.

Let A, B, and d be variables, the rules are:

Rule 1:



Rule 2:



<u>*Rule 3:*</u> $$\langle \bigcirc \boldsymbol{L} \rangle = d \langle \boldsymbol{L} \rangle .$$

We have some remarks about these rules.

- The polynomial of any trivial diagram with one component is 1.
- We use A-smoothing and B-smoothing to split the link open vertically and horizontally.

In order to do A-smoothing and B-smoothing, we consider the link as the 4 quadrant graph created by one over strand and one under strand. We are going on the under strand. The quadrant II and IV are labeled as A, and the quadrant I and III are labeled as B as shown in the Figure 2.1

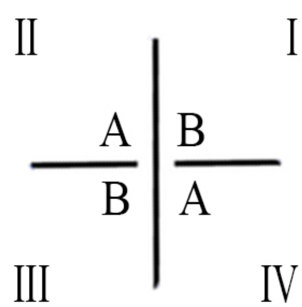


Figure 2.1: A-smoothing and B-smoothing.

- When the link diagram has two or more components, and one of them is trivial, we can erase the trivial component at the expense of multiplying the entire polynomial by d. Let us do some example calculations.

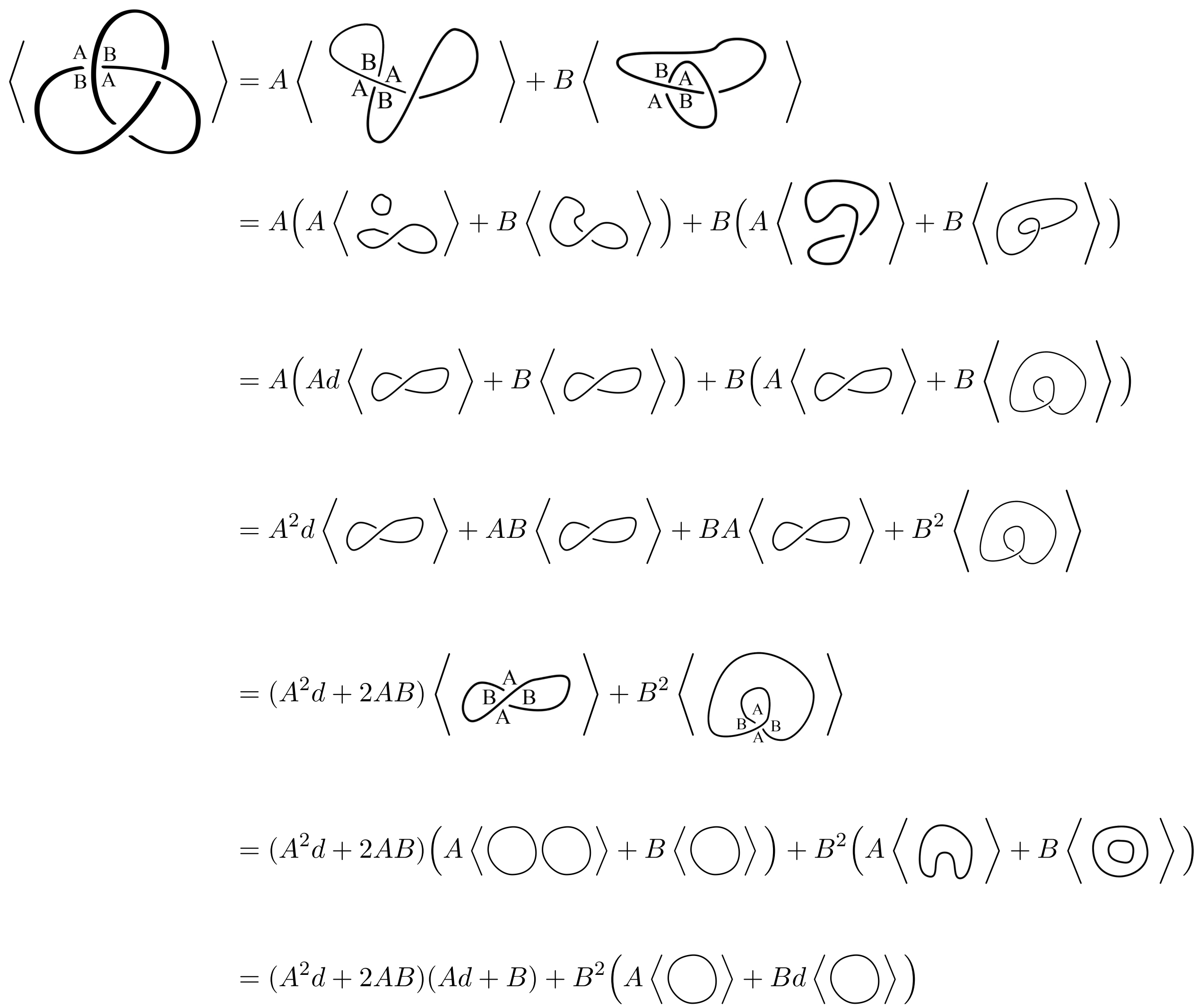
Example 2.1. Find the bracket polynomial of the 1-crossing diagram below.

$$= A \langle \bigcirc \bigcirc \rangle + B \langle \bigcirc \rangle$$

$$= Ad \langle \bigcirc \rangle + B \cdot 1$$

$$= Ad + B. \qquad \blacksquare$$

Example 2.2. Find the bracket polynomial of the projection of the Trefoil knot (Figure 1.3).



$$= A^3d^2 + A^2Bd + 2A^2Bd + 2AB^2 + B^2A + B^3d.$$

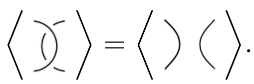
Hence, the 3-variable bracket polynomial of the 3-crossing diagram is

$$A^3d^2 + A^2Bd + 2A^2Bd + 2AB^2 + B^2A + B^3d. \qquad \blacksquare$$

2.1.2 1-Variable Bracket

In this section we show how to get the Jones polynomial from the Kauffman Bracket polynomial. To do so

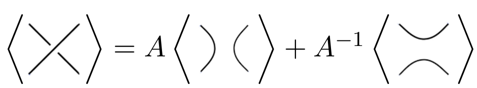
$$\text{we set } \begin{cases} AB = 1, & \text{then } B = A^{-1} \\ A^2 + ABd + B^2 = 0, & \text{then } d = -A^2 - A^{-2} \end{cases}$$

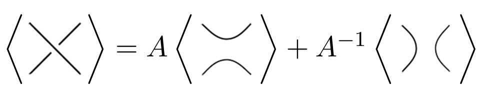
since we want 

Therefore, for the remainder of this section we have updated 3 new rules computing the bracket polynomial.

<u>*Rule 1:*</u> $\langle \bigcirc \rangle = 1$

<u>*Rule 2:*</u>





<u>*Rule 3:*</u> $\langle \bigcirc \ \boldsymbol{L} \rangle = (-A^2 - A^{-2}) \langle \boldsymbol{L} \rangle .$

This is what we called 1-variable bracket polynomial. It can be shown that the 1-variable bracket polynomial is invariant under Type II and III Reidemeister moves, but not Type I Reidemeister moves.

Definition 2.1. The Kauffman polynomial is defined on a link diagram $\boldsymbol{L}$ as follows:

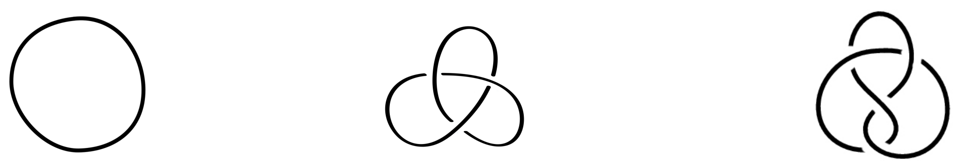
$$\chi(\boldsymbol{L}) = (-A^3)^{-\omega(\boldsymbol{L})} \langle \boldsymbol{L} \rangle . \tag{2.1}$$

where $\omega(\boldsymbol{L})$ is the writhe of the diagram, and $\langle \boldsymbol{L} \rangle$ is the Bracket polynomial.

2.2 Jones Polynomial

In 1984 Vaughan Jones, a New Zealand mathematician, introduced the knot polynomial that be as his name. Every knot has its polynomial which can be computed from a projection of the knot. Even though one knot has many different projections, it has only one polynomial. The polynomial is an invariant of the knot.

If two knots have two different polynomials, we can tell they are two different knots.



(a) Unknot $V_{\bigcirc}(t) = 1$

(b) Trefoil Knot $V(t) = -t^{-4} + t^{-3} + t^{-1}$

(c) Figure 8 Knot $V(t) = -t^{2} - t + 1 - t^{-1} + t^{-2}$

Figure 2.2: Examples of Jones Polynomial.

Kauffman showed that his polynomial χ is equivalent to the Jones polynomial. To compute it, follow these steps:

Step 1: Compute the 3-variable bracket polynomial.

Step 2: Convert the 3-variable bracket polynomial into 1-variable bracket polynomial, $\langle \boldsymbol{L} \rangle$, by substitute $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

Step 3: Calculate the writhe $\omega(\boldsymbol{L})$.

Step 4: Compute the Kauffman polynomial $\chi(\boldsymbol{L}) = (-A^3)^{-\omega(\boldsymbol{L})} \langle \boldsymbol{L} \rangle$.

Step 5: Replacing A by $t^{-\frac{1}{4}}$ yields the Jones polynomial.

Example 2.3. Compute the Jones Polynomial of the Trefoil knot if already known the 3-variable bracket polynomial of the Trefoil knot is

$$\langle \boldsymbol{L} \rangle = A^3d^2 + A^2Bd + 2A^2Bd + 2AB^2 + B^2A + B^3d.$$

We already know the writhe of the trefoil knot is $\omega(\boldsymbol{L}) = -3$ from Figure 1.15. We compute the 1-variable bracket polynomial of the Trefoil knot by replacing $B = A^{-1}$ and $d = -A^2 - A^{-2}$. Therefore $\langle \boldsymbol{L} \rangle = A^7 - A^3 - A^{-5}$.

$$\begin{aligned} \text{The Kauffman polynomial } \chi(\boldsymbol{L}) &= (-A^3)^{-\omega(\boldsymbol{L})} \langle \boldsymbol{L} \rangle \\ &= (-A^3)^{-(-3)}(A^7 - A^3 - A^{-5}) \\ &= (-A^9)(A^7 - A^3 - A^{-5}) \\ &= -A^{16} + A^{12} + A^4. \end{aligned}$$

To obtain the Jones polynomial from the Kauffman polynomial, we replace A by $t^{-\frac{1}{4}}$.

$$\begin{aligned} \text{Hence, } V_{\boldsymbol{L}}(t) &= -(t^{-\frac{1}{4}})^{16} + (t^{-\frac{1}{4}})^{12} + (t^{-\frac{1}{4}})^4 \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned} \qquad \blacksquare$$

2.3 State Model

Back to 3-variables bracket, in this section we introduce a state model approach to the 3-variable bracket polynomial. This approach will be used in relating the 3-variable bracket to surface the link bounds.

Definition 2.2. A state S of a diagram is a choice of smoothing for each crossing. The number of components in a state is denoted $\mid \mathbf{S} \mid$.

$$\text{In general, if state S has: } \begin{cases} m : A\text{-smoothing} \\ n : B\text{-smoothing} \end{cases} \quad \mid \mathbf{S} \mid\text{: the number of components.}$$

Then its contribution to 3-variable bracket is $A^m B^n d^{|\mathbf{S}|-1}$.

$$\text{So, } \langle \boldsymbol{L} \rangle = \sum A^m B^n d^{|\mathbf{S}|-1}. \tag{2.2}$$

Example 2.4. Compute the bracket polynomial by state S contribution of the Trefoil knot as shown in Figure 2.3.

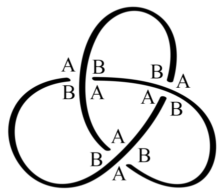


Figure 2.3: Trefoil Knot with A-smoothing and B-smoothing Labeling.

If we do all 3 A-smoothing, state $|\ \mathbf{S}\ | = 3$. The contribution is A^3d^2.

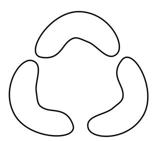


Figure 2.4: Trefoil Knot with 3 A-smoothing.

If we do 2 A-smoothing and 1 B-smoothing, state $|\ \mathbf{S}\ | = 2$. The contribution is $3A^2Bd$.

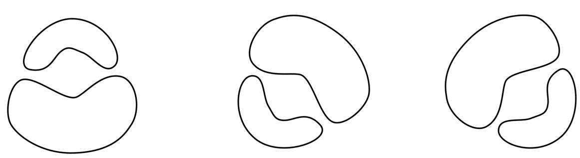


Figure 2.5: Trefoil Knot with 2 A-smoothing and 1 B-smoothing.

If we do 1 A-smoothing and 2 B-smoothing, state $\mid \mathbf{S} \mid = 1$. The contribution is $3AB^2$.

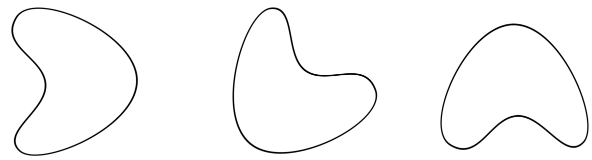


Figure 2.6: Trefoil Knot with 1 A-smoothing and 2 B-smoothing.

If we do all 3 B-smoothing, state $\mid \mathbf{S} \mid = 2$. The contribution is B^3d.

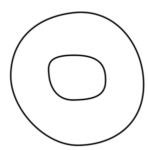
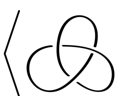


Figure 2.7: Trefoil Knot with 3 B-smoothing.

Therefore:

$\Big\langle$  $\Big\rangle$

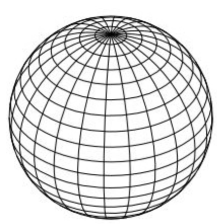
$$
\begin{aligned}
&= A^3d^2 + 3A^2Bd + 3AB^2 + B^3d \\
&= A^3d^2 + 3AB^2 + (3A^2B + B^3)d. \qquad \blacksquare
\end{aligned}
$$

Chapter 3

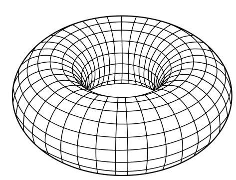
Surfaces

Definition 3.1. A closed **Surface** is the outside part or uppermost layer of a solid. A surface with boundary is one that has an edge. A disk is an example of a surface with boundary. A surface is orientable if it is not non-orientable. Surfaces are often used to describe the object's texture, form, or extent, and to distinguish knots. Every link is boundary of a surface, as we shall see.

3.1 Surfaces without Boundary



(a) Sphere

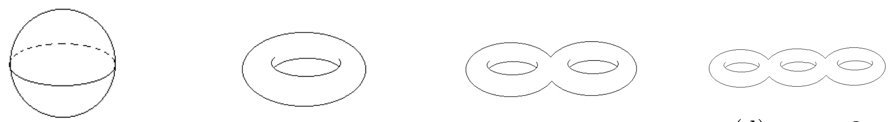


(b) Torus

Figure 3.1: Some Surfaces.

Figure 3.1 shows two basic examples of surfaces, the sphere and the doughnut. The surface of the doughnut is a **torus**. The number of holes in the doughnut is the

genus of the surface. Figure 3.1a shows the sphere has 0 holes, so it has genus 0. Figure 3.1b shows the torus has 1 hole, so it has genus 1.



(a) genus 0 (b) genus 1 (c) genus 2 (d) genus 3

Figure 3.2: Examples of Genus of the Surface.

We assume surfaces are made from rubber or elastic, and are easy to deform. Any two surfaces that can be deformed into each other are called **isotopic**. In order to work better with surfaces, we cut them into triangles. The triangles need to be flat with straight edges. This division is called a **triangulation** of the surface. An example of a triangulation of the sphere is given in Figure 3.3.

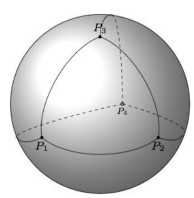


Figure 3.3: Triangulation of the Sphere.

Let's take a triangulation of the surface. Let V be the number of vertices in the triangulation. Let E be the number of edges. Let F be the number of triangles. Then the **Euler characteristic** of the triangulation be:

$$\chi = V - E + F. \tag{3.1}$$

Euler showed that any two triangulations of the same surfaces yield the same Euler characteristic.

Example 3.1. Compute the Euler characteristic of the triangulation of the disk in Figure 3.4.

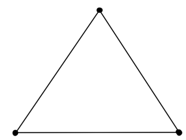
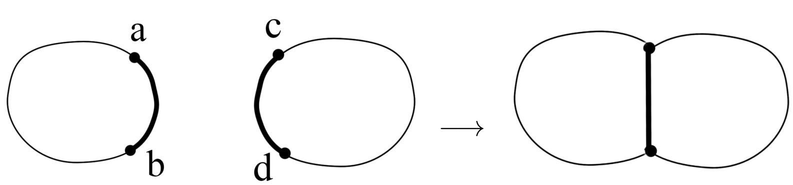


Figure 3.4: Triangulation of the Disk.

We can tell that there are 3 vertices, 3 edges, and 1 triangle in Figure 3.4. By the Euler characteristic formula, $\chi = V - E + F = 3 - 3 + 1 = 1$.

Hence the Euler characteristic of any disk is 1. ∎

Example 3.2. Compute the Euler characteristic of 2 disks glued together as shown in Figure 3.5c.



(a) $\chi = 1$ (b) $\chi = 1$ (c) χ_S

Figure 3.5: Glue Two Disks Together.

The Euler characteristic of each disk is $\chi = 1$. So adding the quantities $V - E + F$ of 2 disks would be $1 + 1 = 2$. However, when glueing 2 disks together, we loose 1 edge and 2 vertices. Therefore

$$\begin{aligned}\chi &= (V-2) - (E-1) + F \\ &= V - 2 - E + 1 + F \\ &= V - E + F - 1 \\ &= 2 - 1 \\ &= 1.\end{aligned}$$ ∎

This is not surprise because glueing two disks along an edge yields another disk.

Example 3.3. Compute the Euler characteristic of the triangulation of the sphere in Figure 3.3.

Looking at Figure 3.3, we see that there are 4 vertices, 6 edges, and 4 triangles. By the Euler characteristic formula, $\chi = V - E + F = 4 - 6 + 4 = 2$

Hence the Euler characteristic of the triangulation of this sphere is 2. ■

Example 3.4. Compute the Euler characteristic of the triangulation of the torus in Figure 3.6.

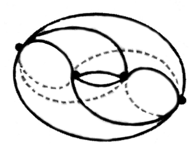


Figure 3.6: Triangulation of the Torus.

In the Figure 3.6 above, there are 4 vertices, 12 edges, and 8 triangles. By the Euler characteristic formula, $\chi = V - E + F = 4 - 12 + 8 = 0$

Hence the Euler characteristic of the triangulation of this torus is 0. ■

The Euler characteristic depends only on the type of surface, not on the particular triangulation of the surface. (reference - Adams). Therefore even though we have a different triangulation, we still obtain the Euler characteristic of the sphere is 2 and the torus is 0.

From above we already know the Euler characteristic of a torus is 0, so the Euler characteristic of two tori is still $\chi = 0$. If we connect two tori together by removing a disk from each of two tori, then connecting two tori together along the disks boundary. We will obtain the genus 2 of surface. This process is called the **connected sum**.

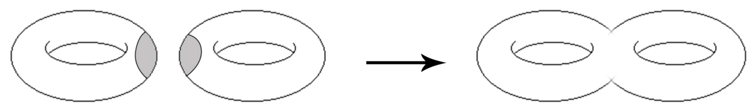


Figure 3.7: The Connected Sum of Two Tori.

Since the Euler characteristic of each torus is 0, before glueing we have $V - E + F = 0$ when we have two tori. However, when we connecting them together by connected sum, we loose 3 vertices, 3 edges, and 2 triangles. Therefore the Euler characteristic of the genus 2 surface is:

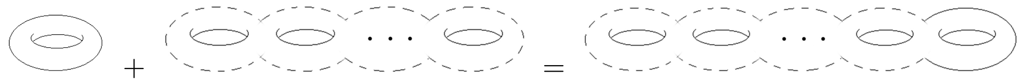
$$\begin{aligned}\chi &= (V-3) - (E-3) + (F-2) \\ &= V - 3 - E + 3 + F - 2 \\ &= V - E + F - 2 \\ &= 0 - 2 \\ &= -2.\end{aligned}$$

Proposition 3.1. *Use induction to show that the Euler characteristic of a surface of genus g is $2 - 2g$.*

Proof. We already know that the genus of the torus is $g = 1$, and its Euler characteristic is $\chi_1 = 2 - 2 \times 1 = 0$.

Suppose the genus of the surface is $g = n$, and its Euler characteristic is $\chi_n = 2 - 2 \times n = 2 - 2n$.

We need to show that the surface S with genus $g = n + 1$ has its Euler characteristic is $\chi_{n+1} = 2 - 2(n + 1)$.



(a) $\chi = 0$ (b) $\chi = 2 - 2n$ (c) χ_S

Figure 3.8: The Addition of Genus 1 Surface to Genus n Surface.

The Euler characteristic of surface with genus $n+1$ is

$$\chi_S = 0 + 2 - 2n + \Delta \tag{3.2}$$

where Δ is the net change of surface S. Figure 3.8c shows that we loose 3 vertices, 3 edges, and 2 triangles. Therefore $\Delta = -2$.

Hence, (3.2) is equal to

$$\begin{aligned} \chi_S &= 0 + 2 - 2n - 2 \\ &= 2 - 2(n+1). \end{aligned}$$

□

3.2 Surface with Boundary

Definition 3.2. A **boundary** is the edge of the surface or its knot curve.

The disk in Figure 3.9 and the annulus in Figure 3.10 are two basic examples of boundary of a surface.

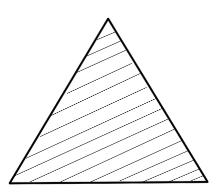


Figure 3.9: One Boundary Component.

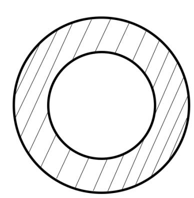


Figure 3.10: Two Boundary Component.

Some surfaces have boundary, and it can be shown that the genus of a connected surface S is given by:

$$g(S) = \frac{2 - \chi(S) - B}{2} \tag{3.3}$$

where B is the number of boundary components of the surface, and $\chi(S)$ is the Euler characteristic of S.

Example 3.5. Find the genus of surfaces of the disk and the annulus in Figure 3.9 and Figure 3.10.

We already know the Euler characteristic of the disk is 1, $\chi_{disk} = 1$. Using the equation (3.3) then the genus of the disk in Figure 3.9 is

$$g_{disk} = \frac{2 - \chi_{disk} - B}{2} = \frac{2 - 1 - 1}{2} = 0.$$

The Euler characteristic of the torus is 0, $\chi_{annulus} = 0$. Using the equation (3.3) then the genus of the torus in Figure 3.10 is

$$g_{annulus} = \frac{2 - \chi_{annulus} - B}{2} = \frac{2 - 0 - 2}{2} = 0. \qquad \blacksquare$$

From equation (3.3) we solve for $\chi(S)$ to derive the equation of Euler characteristic.

$$\chi(S) = 2 - 2g(S) - B. \tag{3.4}$$

Example 3.6. Find the Euler characteristic of the surface in Figure 3.11 without triangulating it.

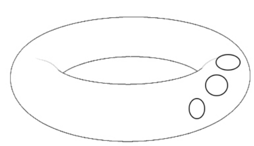


Figure 3.11: Torus Lost 3 Disks.

As we already did in previous examples, the Euler characteristic of a torus is $\chi = 0$. Now it is loosing 3 disks.

Since one disk is considered as a triangle, so loosing three disks means it will loose 9 vertices, 9 edges, and 3 triangles.

Hence a torus lost 3 disks has its Euler characteristic calculated as:

$$\begin{aligned}\chi &= (V-9)-(E-9)+(F-3)\\ &= V-E+F-3\\ &= 0-3\\ &= -3. \qquad \blacksquare\end{aligned}$$

Example 3.7. Given a disk and a band. Glue their ends together. Find the Euler characteristic of the surface in Figure 3.12d.

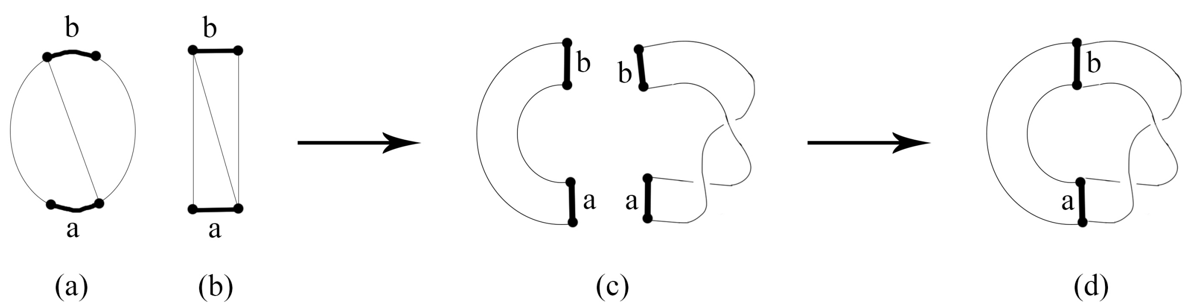


Figure 3.12: Glue a Disk and a Band.

The Euler characteristic of the surface in Figure 3.12a is calculated as

$$\chi_1 = V-E+F = 4-5+2 = 1.$$

The Euler characteristic of the surface in Figure 3.12b is calculated as

$$\chi_2 = V-E+F = 4-5+2 = 1.$$

So the $V-E+F$ of the two surfaces separately is

$$V-E+F = 1+1 = 2.$$

However, when glueing them together, we loose 4 vertices, 2 edges. Therefore the Euler characteristic of the surface in Figure 3.12d is

$$\begin{aligned}\chi_{total} &= (V-4)-(E-2)+F \\ &= V-E+F-2 \\ &= 2-2 \\ &= 0. \qquad \blacksquare\end{aligned}$$

Full twist is a 360 degree twist of a strip as shown in Figure 3.13 and half twist is a 180 degree twist of a strip as shown in Figure 3.14.

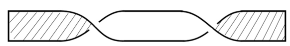


Figure 3.13: Full Twist.

Figure 3.14: Half Twist.

Example 3.8. Construct the surface out of paper. Cut out one large disk and two thin strips of paper. Full twist each strip and tape the two ends of the strip to the disk as in Figure 3.15. Compute the Euler characteristic and the genus of this surface.

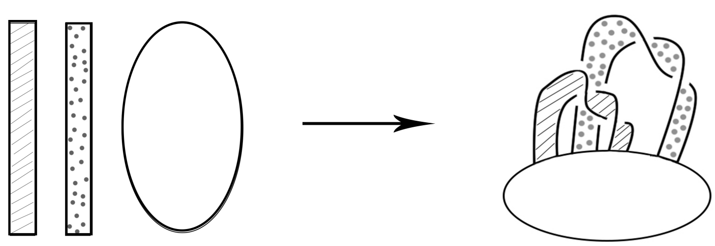


Figure 3.15: Glue 2 Strips to a Disk.

Before taping together, it is easy to tell the Euler characteristic of each strip is $\chi_{strip} = 1$ because each is topologically a disk, and the Euler characteristic of the disk is $\chi_{disk} = 1$ by cell decomposition. Therefore, the $V - E + F$ of 3 surfaces separately is

$$V - E + F = 1 + 1 + 1 = 3.$$

After taping together, each strip lost 4 vertices and 2 edges. So it lost the total of 8 vertices and 4 edges.

Hence, the Euler characteristic of the surface after taping together is

$$\begin{aligned}\chi &= (V-8)-(E-4)+F\\ &= V-E+F-4\\ &= 3-4\\ &= -1.\end{aligned}$$

Figure 3.16 is the deformation of the surface into the trefoil knot.

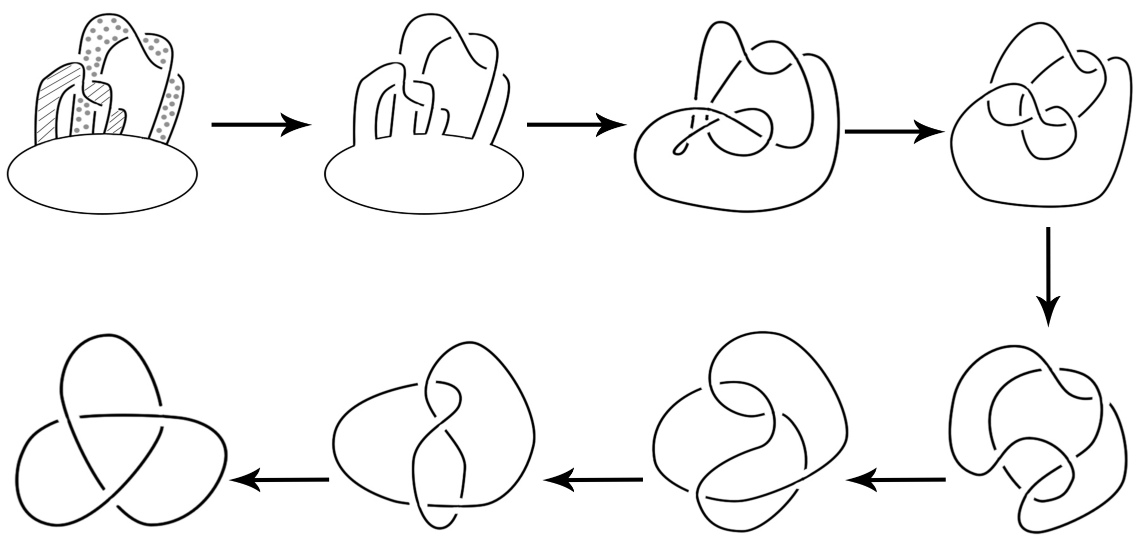


Figure 3.16: Surface Deformation.

In order to find the genus of this surface, we will use the formula (3.3)

$$g(S) = \frac{2-\chi(S)-B}{2} = \frac{2-(-1)-1}{2} = 1.$$

where B is 1 since the trefoil knot has only 1 boundary. ■

3.3 Orientable vs Non-orientable Surfaces

Definition 3.3. A surface in three dimensional space is orientable if it has two sides that can be painted different colors.

Let's take a look at some examples to have a better visualization for orientable surfaces. A trefoil knot in Figure 3.17 is an orientable surface since we can paint any connected surface in black and white such that two sections are next to each other of the surface do not have the same color.

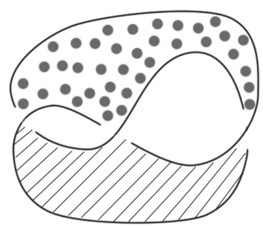


Figure 3.17: An Example of Orientable Surface.

Another example of an orientable surface is a disk, a torus, or a sphere. They are orientable since we can paint the exterior one color and the interior a different color.

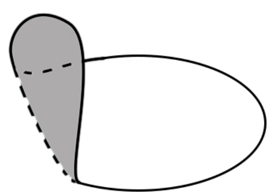


Figure 3.18: An Example of Orientable Surface in Three Dimensional.

Next, lets have a look at a surface that is non-orientable. The simplest example of the non-orientable surface is the Mobius band. A Mobius band is a surface that has only one side, so you can only paint the Mobius in one color because of the twist. It failed the definition of the orientable surface.

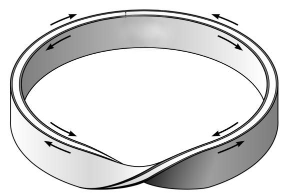


Figure 3.19: Mobius Band.

Classification of surfaces: Connected, orientable surfaces are homeomorphic if and only of they have the same Euler characteristic and number of boundary components.

3.4 Seifert Surfaces

Definition 3.4. A **Seifert surface** for any knot K is connected, orientable surface embedded in three dimensional with boundary K

An example of the Seifert surface is the trefoil knot as Figure 3.17 since it is orientable and connected.

Theorem 3.1. *Every link bounds a Seifert surface.*

Seifert's algorithm shows us steps how to construct a Seifert surface from a diagram.

Step 1: Choose an orientation of a provided link diagram.

Step 2: Smooth every crossing according to orientation. At each crossing, two strands come in and two strands come out. Smooth the crossing by connecting a coming in strand to its adjacent coming out strand as Figure 3.20 below.

The result of this step obtain closed curves that are Seifert circles. Each circle is considered as a disk in the plane.

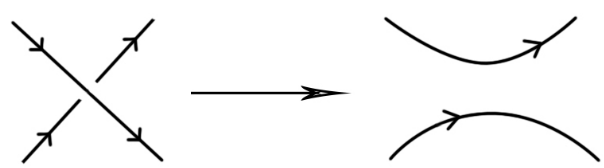


Figure 3.20: Smoothing the Srossings.

Step 3: Connect the disks by adding the twisted strips at each former crossing.

Proposition 3.2. *If a knot K has n crossings, and the Seifert algorithm produces s Seifert circles, then the genus of K is calculated by*

$$g = \frac{c - s + 1}{2}.$$

Example 3.9. Find the genus of the surface of the figure eight knot by Seifert's algorithm.

The figure eight knot is oriented as Figure 3.21. Next we are going to smooth four crossings. The result shows that we obtain 3 Seifert circles as Figure 3.22.

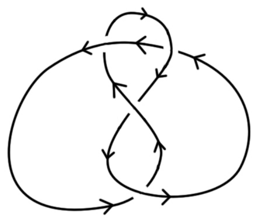


Figure 3.21: Figure Eight Knot.

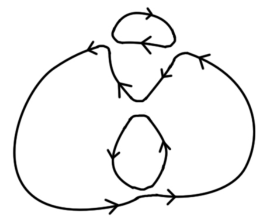
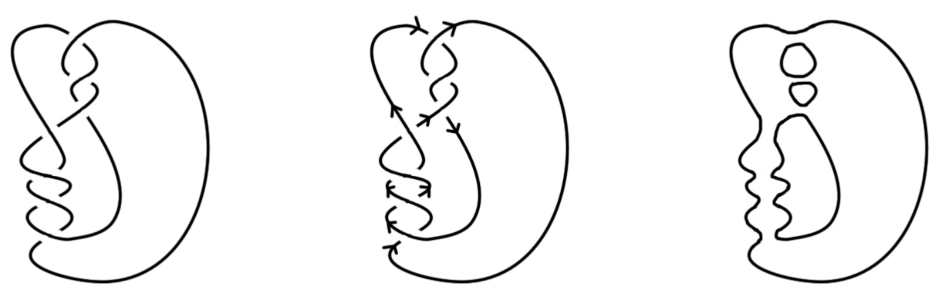


Figure 3.22: Smoothing the Crossings.

The genus of the surface is calculated by

$$g = \frac{c-s+1}{2} = \frac{4-3+1}{2} = 1. \qquad \blacksquare$$

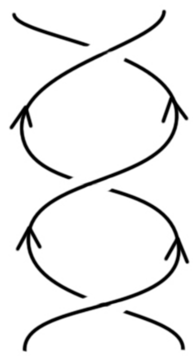
Definition 3.5. If an alternating diagram is unoriented, we can choose the direction to make it oriented. Then smoothing every crossing. This process obtains a **Seifert state S** of a diagram.



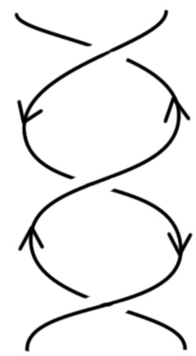
(a) Unoriented Link. (b) Choose Orientation. (c) Seifert State.

Figure 3.23: Examples of Seifert State.

If both strands of the twist follow the same orientation then they are **compatibly oriented**. In contrast, if they follow the opposite directions then they are **incompatibly orientated**. (See Figure 3.24)



(a) Compatible Orientation.



(b) Incompatible Orientation.

Figure 3.24: Compatible Orientation and Incompatible Orientation.

The unoriented two-bridge link diagram as Figure 3.23a has two twists. The first twist has 3 crossings, and the second twist has 4 crossings. We are going to box each twist in a rectangle as in Figure 3.25. Note that if a rectangle contains an even number of crossings, then the strand that enter the rectangle at the top left corner will leave the rectangle at the bottom left. Thus the strands enter and leave on the same side of the twist if the twist has an even number of crossings. On the other hand, if a rectangle contains an odd number of crossings, then the strand that enters the rectangle at the top left corner will leave the rectangle at the opposite corner. This explains that strands enter and leave on the opposite side of the twist if the twist has an odd number of crossings.

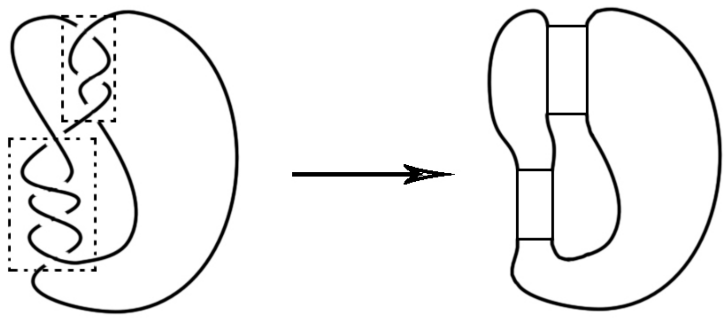


Figure 3.25: Box the Twist by a Rectangle.

We now apply that idea to two-bridge links. Lets say we choose clockwise as the orientation of the link. The arrow will enter the first rectangle that has 3 crossings. The arrow will leave the rectangle at the opposite corner as Figure 3.26. That arrow will continue moving along the strand and enter the next rectangle that has 4 crossings. The arrow will leave the rectangle at the same side. The arrow will keep moving along by that rule, and it will go through the entire strand of the link. This tells us the two-bridge link has only one component, so it is a knot. Choosing the orientation on one place of the link will determine the orientation of the entire link.

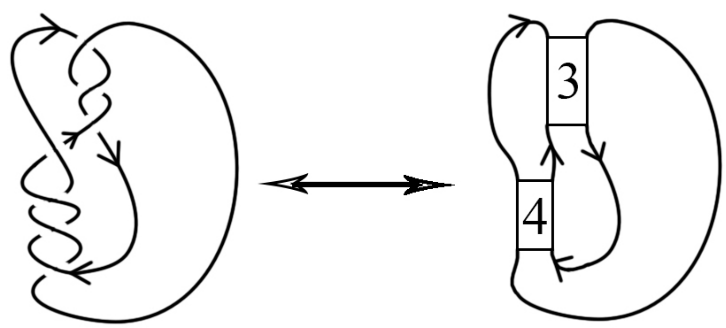


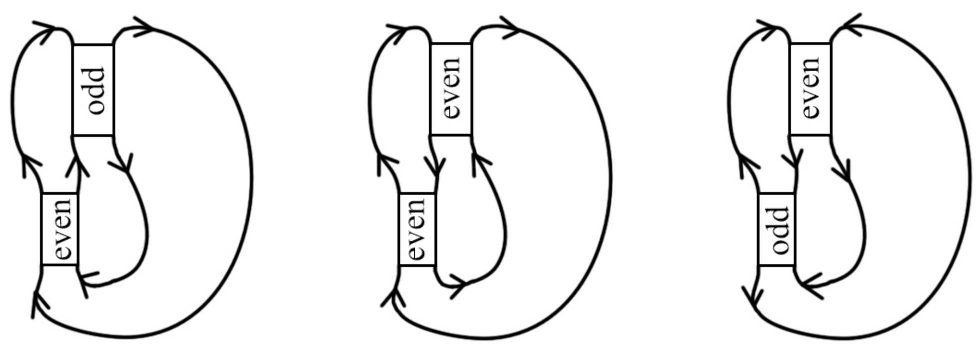
Figure 3.26: Link Orientation.

Lets go a little bit further to examine the orientation of the two-bridge diagram when we have different number of crossing in these two twists.

Case 1: The first twist has odd number of crossings and the second twist has even number of crossings. This case has been examined above.

Case 2: The first twist has even number of crossings and the second twist also has even number of crossings.

Case 3: The first twist has even number of crossings and the second twist has odd number of crossings.

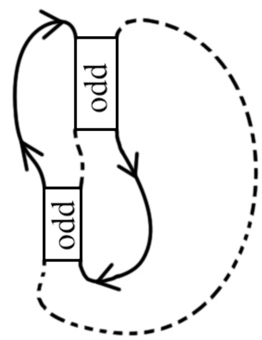


(a) Case 1. (b) Case 2. (c) Case 3.

Figure 3.27: Two-bridge Knot.

In each of these three cases, the two-bridge diagram has only one boundary since the arrow passes through every strand in the diagram. Choosing the orientation on one place of the knot will determine the orientation of the entire knot as shown in Figure 3.27.

Case 4: The first twist has odd number of crossings and the second twist also has odd number of crossings. This case is different than the other three cases. Follow the path of the arrow as we enter at the to left corner of the first rectangle which has odd number of crossings. It will leave the first rectangle at the opposite side corner. Next it will enter the bottom right corner of the second rectangle which also has odd number of crossings. Then it will leave the second rectangle at the opposite, the top left corner. The path that the arrow follows does not go through the dash strand as shown in Figure 3.28. Therefore this two-bridge diagram contains two components since choosing the orientation on one place of the link will not determine the orientation of the entire link. In order to go through the entire link, we need to have the second arrow that go through the dash strand path.



(d) Case 4.

Figure 3.28: Two-bridge Knot.

3.5 State Model and State Surface

Recall from the previous chapter, we already know the State of a diagram is a choice of A or B smoothing at each crossing of a diagram. In this section, we are going to find the relationship between the State model and the State surface. First we will associate with each State a corresponding State surface. Next we will calculate the Euler characteristic of each State surface. Then we will find an alternate formula for the bracket polynomial using the Euler characteristic of State surface.

Definition 3.6. Given any State, the **State surface** is constructed by glueing a half twisted band over each smoothed crossing.

Figure 3.29c shows an example of the State surface of the Figure eight knot.

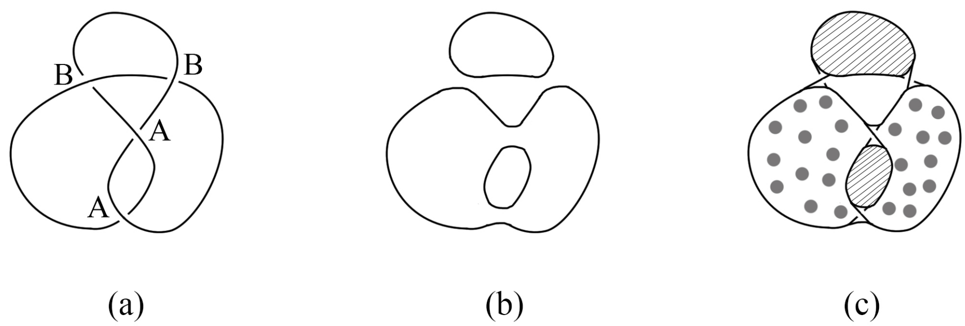


Figure 3.29: Maximal State Surface of Figure Eight Knot.

The surface associated with a Seifert State is the Seifert surface defined in section 3.4. Now we would like to calculate the Euler characteristic of State surface. We begin with an example.

Example 3.10. Compute the Euler characteristic of a Seifert state for the (3,3,3)-pretzel knot as Figure 3.30

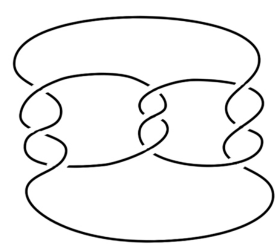
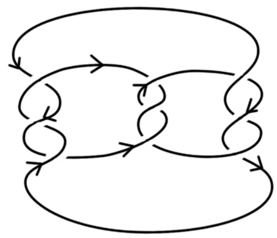


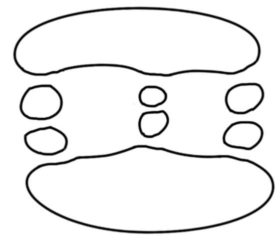
Figure 3.30: (3,3,3)-Pretzel Knot.

We can compute the Euler characteristic of a Seifert state for the (3,3,3)-pretzel knot by a couple different methods.

<u>*Method 1*:</u>



(a) Choose Orientation for the (3,3,3)-Pretzel.



(b) Seifert Circles of the (3,3,3)-Pretzel Knot.

Figure 3.31: (3,3,3)-Pretzel Knot.

The (3,3,3)-pretzel knot has 9 crossings. Suppose we choose the orientation of the pretzel as shown in Figure 3.31a. Since this is a knot, its boundary is 1. Smoothing the crossing then we obtain 8 Seifert circles as shown in Figure 3.31b.

The genus of the (3,3,3)-pretzel knot is calculated as:

$$g = \frac{c - s + 1}{2} = \frac{9 - 8 + 1}{2} = 1$$

where c is the number of crossing and s is the number of Seifert circle. Then

$$\chi = 2 - 2g - B = 2 - 2 \times 1 - 1 = -1.$$

The Euler characteristic of the (3,3,3)-pretzel knot is -1.

Method 2:

It is easy to notice that the (3,3,3)-pretzel knot in Figure 3.30 is created from two large disks and three twisted bands.

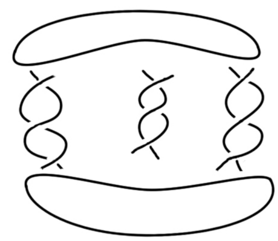


Figure 3.32: (3,3,3)-Pretzel Knot is Created from 2 Disks and 3 Twisted Bands.

Since the Euler characteristic of each disk or twisted band is $\chi = 1$, the $V - E + F$ of five separate disks and twisted bands in Figure 3.32 is

$$1 + 1 + 1 + 1 + 1 = 5.$$

When we glue one twisted band to two disks, we loose 4 vertices and 2 edges. There are 3 twisted bands so we loose 12 vertices and 6 edges in total.

Hence, the Euler characteristic of the (3,3,3)-pretzel knot is:

$$\begin{aligned} \chi &= (V - 12) - (E - 6) + F \\ &= V - E + F - 6 \\ &= 5 - 6 \\ &= -1. \qquad \blacksquare \end{aligned}$$

Proposition 3.3. *Let S be a state surface, then*

$$\chi(S) = |\, \boldsymbol{S} \,| - c. \tag{3.5}$$

Proof. Let S be a state of link diagram L, with $|\, \mathbf{S} \,|$ be state circles. To build the state surface, we will attach c bands to the state circles. Before glueing, then we have $|\, \mathbf{S} \,| + c$ disks, where c is the number of crossings, and $|\, \mathbf{S} \,|$ is the number of Seifert circles after smoothing all the crossings.

The Euler characteristic of the surface before glueing is

$$\chi(S) = V - E + F = c + |\, \mathbf{S} \,|$$

since the Euler characteristic of each disk is 1.

For every crossing we glue a band to two disks, and we loose two edges and four vertices. Therefore, the change in Euler characteristic is

$$\begin{aligned} \chi(S) &= (V-4) - (E-2) + F \\ &= V - E + F - 2 \\ &= c + |\, \mathbf{S} \,| -2, \end{aligned}$$

for each crossing. Hence for c crossings, we have

$$\chi(S) = c + |\, \mathbf{S} \,| -2c = |\, \mathbf{S} \,| - c.$$

$\square$

Now lets go back and solve example 3.10 by using equation (3.5). We can call this is the third method to compute the Euler characteristic of a Seifert state for the (3,3,3)-pretzel knot.

Method 3: (of example 3.10)

The (3,3,3)-pretzel (see Figure 3.30) has 9 crossings and 8 Seifert circles (see Figure 3.31b). So its Euler characteristic is calculated by using equation (3.5):

$$\chi = |\, \mathbf{S} \,| - c = 8 - 9 = -1.$$

Proposition 3.4. *The State sum Model of an alternating diagram L is given by*

$$\langle \boldsymbol{L} \rangle = \sum_S A^m B^n d^{\ \chi(S)+c-1}$$

where $\chi(S)$ is the Euler characteristic of the corresponding State.

Proof. Solving for $\mid \mathbf{S} \mid$ from equation (3.5), we have

$$\mid \mathbf{S} \mid = \chi(S) + \ c. \tag{3.6}$$

The bracket polynomial of L can be calculated as shown in equation (2.2)

$$\langle \boldsymbol{L} \rangle = \sum_S A^m B^n d^{|\mathbf{S}|-1}$$

$$= \sum_S A^m B^n d^{\ \chi(S)+c-1}.$$

$\square$

Lets call M is the highest degree of d which is the maximal state of surface S.

$$\text{Then} \qquad M = \chi(S) + c - 1.$$

Solving for $\chi(S)$, we have

$$\chi(S) = M - c + 1.$$

Example 3.11. Calculate the Euler characteristic of the Figure eight knot.

The bracket polynomial shows that the Figure eight knot has the highest degree of d is 2. Also, Figure eight knot is a four crossing knot. Therefore

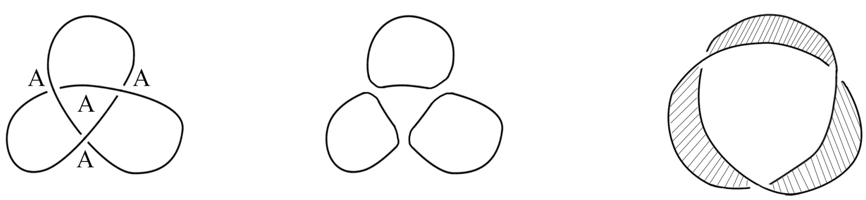
$$\chi(S) = 2 - 4 + 1 = -1.$$

Definition 3.7. A **maximal state** is one that contributes to the highest degree of d in the bracket polynomial of L. A **maximal state surface** is a state surface corresponding to a maximal state.

Figure 3.29 shows an example of the maximal state surface of the Figure eight knot.

Theorem 3.2. *[Nov] Let L be a reduced twisted diagram with at least three crossings in each twist, then only the smoothing across the twists will give us the unique maximal state of L.*

Figure 3.33 shows an example of the Trefoil knot. Recall the three variable bracket of the Trefoil knot is $\langle\ \rangle = A^3d^2+3A^2Bd+3AB^2+B^3d$. The highest degree of d that shows the maximal state of the Trefoil knot is the 3 A-smoothing state as shown in Figure 3.33a. Also notice that this state is obtained by smoothing across the twist, as in Theorem 3.2. Use three half twisted bands to glue those three circles together, we will have the maximal state surface of the Trefoil knot as shown in Figure 3.33c.



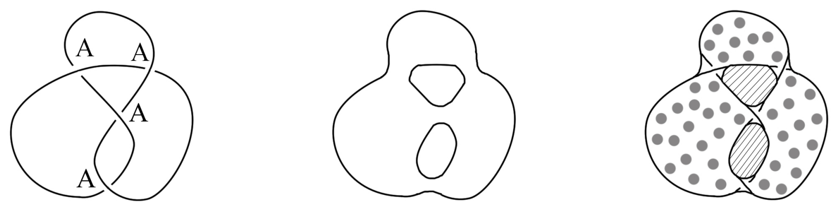
(a) (b) (c)

Figure 3.33: Maximal State Surface of Trefoil Knot.

Theorem 3.2 applies to the example of the Trefoil knot since it has three crossings in its twist. However, we cannot apply Theorem 3.2 to the Figure eight knot since it has only two crossings in each twist. When a twist has fewer than three crossings, then it might give us more than one maximal state. From [Laf13], she found that the three variable bracket of the Figure eight knot is $\langle\ \rangle = A^4d^2+4A^3Bd+5A^2B^2+A^2B^2d^2+4AB^3d+B^4d^2$. The highest degree of d that shows the maximal state of the Figure eight knot is 2. As expected, the Figure eight knot has three different maximal states, which are $A^4d^2, A^2B^2d^2$, and B^4d^2.

<u>*Case 1:*</u>

If we do A-smoothings to all four crossings of the Figure eight as shown in Figure 3.34a, then we obtain the maximal state A^4d^2 of the surface. It is clear that we cannot paint two sides in different colors, see Figure 3.34c. Therefore, this state gives us a non-orientable surface.

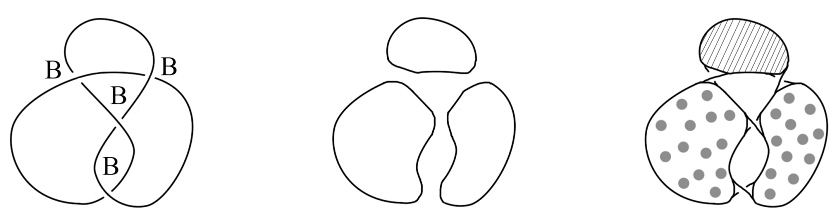


(a) (b) (c)

Figure 3.34: Maximal State Surface of Figure Eight Knot in State A^4d^2.

<u>*Case 2:*</u>

If we do B-smoothings to all four crossings of the Figure eight as shown in Figure 3.35a, then we obtain the maximal state B^4d^2 of the surface. It is clear that we cannot paint two sides in different colors, as in Figure 3.35c. Therefore, this state gives us a non-orientable surface.



(a) (b) (c)

Figure 3.35: Maximal State Surface of Figure Eight Knot in State B^4d^2.

<u>*Case 3:*</u>

If we do 2 A-smoothings and 2 B-smoothings to four crossings of the Figure eight as shown in Figure 3.29a, then we obtain the maximal state $A^2B^2d^2$ of the surface. For this case, we can paint two sides in different colors, as in Figure 3.29c. In this case we obtain an orientable surface.

We want to know when a unique maximal state gives information about the orientable genus of link L. To facilitate this, we rephrase Proposition 3.4 to write the bracket polynomial in term of the genus of a link.

Proposition 3.5. *The State sum Model of an alternating diagram L is given by*

$$\langle \boldsymbol{L} \rangle = \sum_{S} A^m B^n d^{\ 1-2g(S)-B+c}$$

where the highest degree of d is the maximal state of a surface, and g(S) is the genus of the state surface, and B is the number of components of L.

Proof. The Euler characteristic for orientable surfaces with boundary is

$$\chi(S) = 2 - 2g(S) - B,$$

and from equation (3.6) $\mid \mathbf{S} \mid = \chi(S) + c = 2 - 2g(S) - B + c$.

The bracket polynomial of L can be calculated as shown in equation (2.2)

$$\langle \boldsymbol{L} \rangle = \sum_{S} A^m B^n d^{|\mathbf{S}|-1}$$

$$= \sum_{S} A^m B^n d^{\ 2-2g(S)-B+c-1}$$

$$= \sum_{S} A^m B^n d^{\ 1-2g(S)-B+c}.$$

$\square$

We just showed that $\langle \boldsymbol{L} \rangle = \sum A^m B^n d^{\,1-2g(S)-B+c}$. The exponent of d is $(1 - 2g(S) - B + c)$ and contains g, which tells us the genus of state surface. Lets call M is the highest degree of d which corresponds to the maximal state of surface S. Then solving

$$M = 1 - 2g(S) - B + c$$

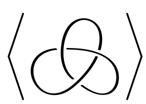
for g gives

$$g = \frac{1 - B + c - M}{2}.$$

This equation can also tells us if the surface S is non-orientable. If the genus g is not a non-negative integer then the surface S is non-orientable. If the genus g is an integer, the surface S could be either orientable or non-orientable.

Example 3.12. Determine if the Trefoil knot is orientable or non-orientable by its maximal state.

From previous, we found that the bracket polynomial of the Trefoil knot is



$$= A^3d^2 + 3A^2Bd + 3AB^2 + B^3d.$$

Notice that the Trefoil knot has one boundary and three crossings. The highest degree of d of the Trefoil knot is $M = 2$. Therefore

$$\begin{aligned} g &= \frac{1 - B + c - M}{2} \\ &= \frac{1 - B + c - 2}{2} \\ &= \frac{1 - 1 + 3 - 2}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Since $g = \frac{1}{2}$ is not a non-negative integer, the Trefoil knot is a non-orientable surface. It shows clearly that we only can paint the maximal state surface of the Trefoil knot in one color as show in Figure 3.33c. ■

Example 3.13. Determine if the maximal state surface of the Figure eight knot is orientable or non-orientable by its maximal state.

From [Laf13] , she found that the bracket polynomial of the Figure eight knot is

$\Big\langle$  $\Big\rangle = A^4d^2 + 4A^3Bd + 5A^2B^2 + A^2B^2d^2 + 4AB^3d + B^4d^2.$

Notice that the Figure eight knot has one boundary and four crossings. The highest degree of d is $M = 2$. Therefore similar as we solved for the genus g in example 3.12, the genus g of the Figure eight knot is $g = 1$, which is a non-negative integer. However, it appears that some maximal states (A^4d^2 state and B^4d^2 state) are non-orientable as shown in Figure 3.34c and Figure 3.35c are non-orientable, and the maximal state $A^2B^2d^2$ is orientable. Therefore the genus of the surface does not give us enough information to tell if the surface is orientable or non-orientable. ■

Definition 3.8. The 3-variable bracket of an alternating link L determines the orientable genus g of link L if

$$g = \frac{1 - B + c - M}{2} \tag{3.7}$$

where B, c, M are defined above.

Theorem 3.3. *Given L is an alternating twist reduced diagram in which each twist has at least three crossings, and also each twist his incompatibly oriented. Then the three-variable bracket $\langle L \rangle$ determines the genus of L.*

Proof. Since L is an alternating twist reduced diagram in which each twist has at least three crossings, Theorem 3.2 [Nov] gives us a unique maximal state of L if we smooth across the twists. Also, because each twist is incompatibly oriented, Seifert's algorithm smooths across the twists. That implies the Seifert surface is the unique maximal state surface. □

Figure 3.31b shows us an example of the maximal state of the Seifert surface of the (3,3,3)-pretzel knot.

As mentioned from previous section, the Euler characteristic of the orientable surface is $\chi = 2 - 2g_o - B$ (the subscript "o" stands for orientable). From [CK12], Adams and Kindred recalled that the Euler characteristic of the non-orientable surface is $\chi = 2 - k - B$ where k is a cross-cap number, the number of Mobius strips that glue together to get the surface. They define the non-orientable genus g_n by $k = 2g_n$ (the subscript "n" stands for non-orientable). Therefore $\chi = 2 - 2g_n - B$. Now we get the same formula whether we are dealing with orientable or non-orientable surfaces. This gives the same relationship between χ and g_n as χ and g_o have. The calculations in example 3.13 highlight the ambiguity if the genus turns out to be an integer. In that case the genus is 1, and it is realized by both orientable and non-orientable surfaces.

To understand this better, Adams and Kindred [CK12] prove in their article that for alternating links there is a **minimal genus state surface**. This is important because it is not true in general. If we make the state surface from the non-alternating diagram, we are no longer guaranteed to obtain the minimal genus surface. In the alternating case they prove there is always a state that gives you a minimal genus surface.

Equation (3.7) shows $g = \frac{1-B+c-M}{2}$ is true whether the surface is orientable or non-orientable. The number of boundary components B and the number of crossings c are fixed, so in order to maximize M, we need to minimize g. The largest M always gives smallest g.

Theorem 3.4. *The maximal degree of the bracket polynomial always gives the minimal genus surface.*

Proof. Adams and Kindred give algorithm that shows there is always a state surface which is minimal genus (it could be orientable or not). We know if we have the highest degree of d then we can tell what is the minimal genus. $\square$

Adams and Kindred [CK12] show in their article that $g_n \leq g_o + \frac{1}{2}$. If we add a half twist by making a twisted band to an orientable surface, then we will obtain a non-orientable surface. We are going to use the (3,3,3)-pretzel knot in Figure 3.30 as an example to demonstrate for their proof.

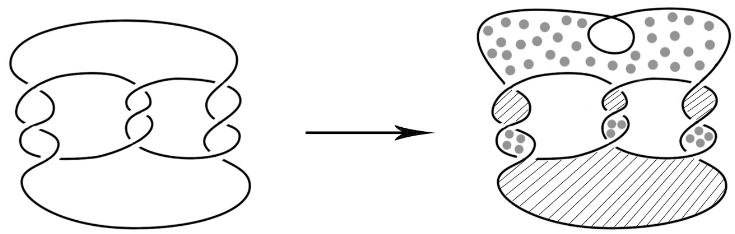


Figure 3.36: Adding a Half Twist to a (3,3,3)-Pretzel Knot.

The (3,3,3)-pretzel knot as shown in Figure 3.30 has three twists. Each twist has three crossings. Let's generalize this pretzel by calling a to be the number of crossings in the twist on the left, b to be the number of crossings in the twist in the middle, and c to be the number of crossings in the twist on the right as shown in the Figure 3.37 below.

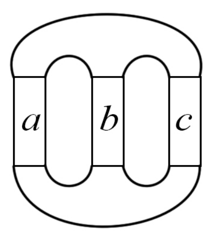


Figure 3.37: (a,b,c)-Pretzel.

We are trying to figure out how accurately the 3-variable bracket tells us the orientable genus for an alternating links. In order to do that, we are going to examine the (a,b,c)-pretzel in 4 different cases.

<u>*Case 1:*</u> (odd, odd, odd)-pretzel

Let's start this case by finding an orientable genus of the (3,3,3)-pretzel knot.

Example 3.14. The orientable genus of the (3,3,3)-pretzel knot in Figure 3.30.

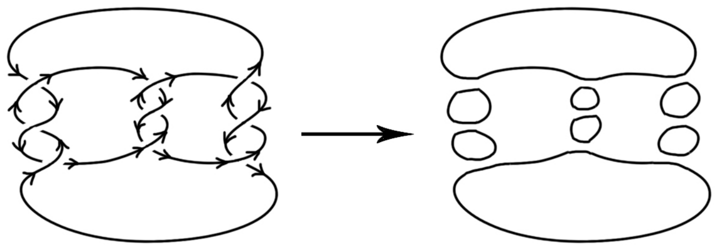


Figure 3.38: (3,3,3)-Pretzel.

In order to find the maximal state of the (3,3,3)-pretzel, we need to have each twist in the diagram incompatibly oriented. Smoothing all crossings in each twist by Seifert's algorithm we have:

Twist a has 3 crossings. So we obtain 2 circles.

Twist b has 3 crossings. So we obtain 2 circles.

Twist c has 3 crossings. So we obtain 2 circles.

The total number of circles in this stage is the sum of all circles we obtain above plus 2 more circles from the base. Therefore $\mid \mathbf{S} \mid = 2+2+2+2 = 8$.

The total number of crossing in this knot is $c = 3+3+3 = 9$.

Therefore the Euler characteristic of the (3,3,3)-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 8 - 9 = -1.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2-\chi-B}{2} = \frac{2-(-1)-1}{2} = 1. \qquad \blacksquare$$

In the next example, we will make more crossings in each twist to see if we obtain the similar result as we did in previous example.

Example 3.15. The orientable genus of the (5,3,5)-pretzel knot in Figure 3.39.

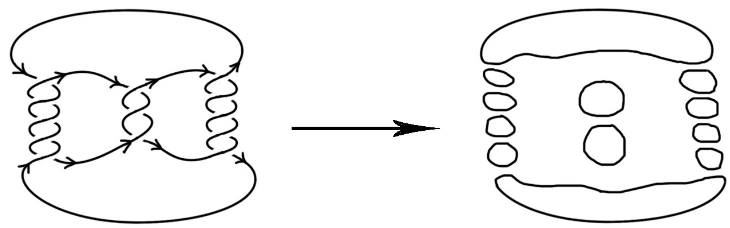


Figure 3.39: (5,3,5)-Pretzel.

Similar as what we just did in example 3.14, the number of circles we obtain in each twist after smoothing each crossing by Seifert's algorithm is $4, 2, 4$ with respect to the order of the twist a, b, c.

The total number of circles in this stage is the sum of all circles we obtain above plus 2 more circles from the base. Therefore $\mid \mathbf{S} \mid = 4 + 2 + 4 + 2 = 12$.

The total number of crossing in this knot is $c = 5 + 3 + 5 = 13$.

Therefore the Euler characteristic of the (5,3,5)-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 12 - 13 = -1.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2 - \chi - B}{2} = \frac{2 - (-1) - 1}{2} = 1. \qquad \blacksquare$$

Now let's generalize this case by making the (odd, odd, odd)-pretzel where $2n+1, 2m+1$, and $2k+1$ represent for the odd number of crossings in twist a, b, and c respectively.

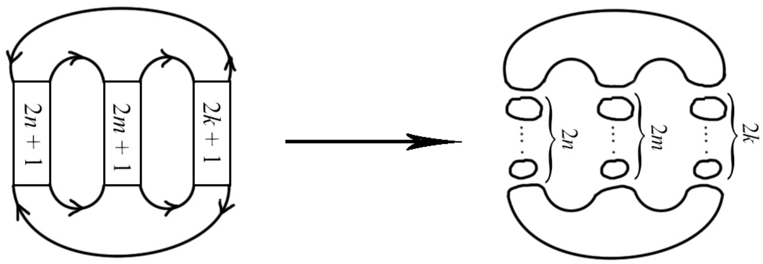


Figure 3.40: $(2n+1, 2m+1, 2k+1)$-Pretzel.

From previous two examples, we can tell the number of circle that we obtain after smoothing all the crossings in each twist by Seifert's algorithm is the number of crossing in that twist reduces by 1. So we will obtain $2n, 2m$, and $2k$ circles from each twist respectively.

The total number of circles in this stage is the sum of all circles we obtain above plus 2 more circles from the base. Therefore $\mid \mathbf{S} \mid = 2n + 2m + 2k + 2$.

The total number of crossing in this knot is

$$c = 2n + 1 + 2m + 1 + 2k + 1 = 2n + 2m + 2k + 3.$$

Therefore the Euler characteristic of the $(2n+1, 2m+1, 2k+1)$-pretzel is

$$\chi = \mid \mathbf{S} \mid - \, c = 2n + 2m + 2k + 2 - (2n + 2m + 2k + 3) = -1.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2 - \chi - B}{2} = \frac{2 - (-1) - 1}{2} = 1.$$

The generality of case 1 tells us the orientable genus of (odd, odd, odd)-pretzel is always equal 1.

<u>*Case 2:*</u> (even, even, even)-pretzel

Let's start this case by finding an orientable genus of the (2,2,2)-pretzel link.

Example 3.16. The orientable genus of the (2,2,2)-pretzel link in Figure 3.41.

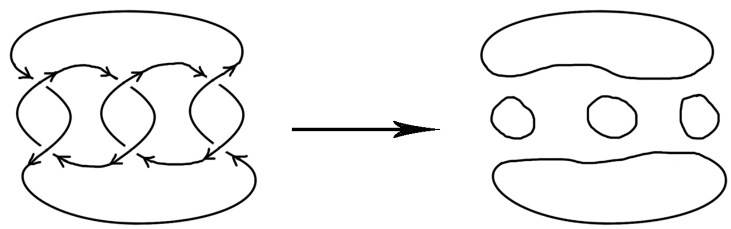


Figure 3.41: (2,2,2)-Pretzel.

This is the pretzel link with 3 boundaries. We need to choose the orientation for each boundary so that each twist is incompatibly oriented. Hence, we have the maximal state of the surface when applying Seifert's algorithm.

We know that the number of circle that we obtain after smoothing all the crossings in each twist by Seifert's algorithm is the number of crossing in that twist reduces by 1. So we will obtain $1, 1,$ and 1 circles from each twist respectively.

The total number of circles in this stage is the sum of all circles we obtain above plus 2 more circles from the base. Therefore $\mid \mathbf{S} \mid = 1 + 1 + 1 + 2 = 5$.

The total number of crossing in this knot is $c = 2 + 2 + 2 = 6$.

Therefore the Euler characteristic of the (2,2,2)-pretzel is

$$\chi = \mid \mathbf{S} \mid - \, c = 5 - 6 = -1.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2 - \chi - B}{2} = \frac{2 - (-1) - 3}{2} = 0. \qquad \blacksquare$$

Now let's generalize case 2 by making the (even, even, even)-pretzel. We denote $2n, 2m$, and $2k$ represent for the even number of crossings in twist a, b, and c respectively.

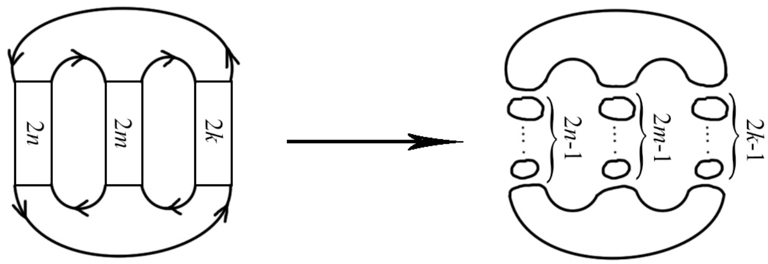


Figure 3.42: $(2n, 2m, 2k)$-Pretzel.

There is no different than what we have been discussing in case 1, the number of circle that we obtain after smoothing all the crossings in each twist by Seifert's algorithm is the number of crossing in that twist reduces by 1. So we will obtain $2n-1, 2m-1$, and $2k-1$ circles from each twist respectively.

The total number of circles in this stage is the sum of all circles we obtain above plus 2 more circles from the base. Therefore $\mid \mathbf{S} \mid = 2n-1+2m-1+2k-1+2 = 2n+2m+2k-1$.

The total number of crossing in this knot is $c = 2n+2m+2k$.

Therefore the Euler characteristic of the $(2n, 2m, 2k)$-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 2n+2m+2k-1-(2n+2m+2k) = -1.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2-\chi-B}{2} = \frac{2-(-1)-3}{2} = 0.$$

The generality of case 2 tells us the orientable genus of (even, even, even)-pretzel is equal 0. This is homeomorphic to a sphere with 3 holes taken out. That explains why the genus of this case is always 0.

<u>*Case 3:*</u> (odd, even, even)-pretzel

We are going to start this case by finding an orientable genus of the (3,2,4)-pretzel link.

Example 3.17. The orientable genus of the (3,2,4)-pretzel link in Figure 3.43.

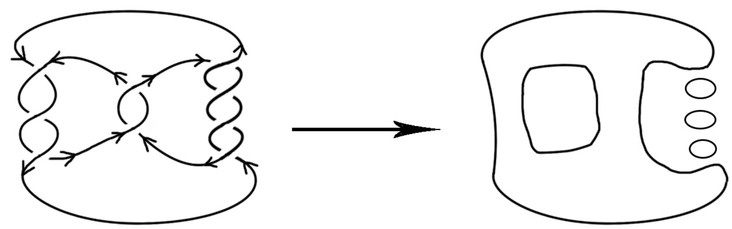


Figure 3.43: (3,2,4)-Pretzel.

This pretzel is a link with 2 boundaries. Twist a carries the odd number of crossings which already be a compatibly oriented. In order to get the maximal state, we need to choose the orientation to the other boundary so that we can get at least one twist incompatibly oriented.

This case is a little bit different than the previous 2 cases. After smoothing all the crossings by Seifert's algorithm, we obtain the state in sideway as shown in Figure 3.43. Twist a and b are compatibly oriented, so we do not get any circle after smoothing. Twist c has 4 crossings which gives us 3 circles after smoothing. The number of circles we get is the sum of all circles we obtain above plus 2 more circles.

Therefore, $\mid \mathbf{S} \mid = 0 + 0 + 3 + 2 = 5$.

The total number of crossing in this knot is $c = 3 + 2 + 4 = 9$.

Therefore the Euler characteristic of the (3,2,4)-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 5 - 9 = -4.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2 - \chi - B}{2} = \frac{2 - (-4) - 2}{2} = 2. \qquad \blacksquare$$

Next we are going to generalize this case by making the (odd, even, even)-pretzel. We denote $2n+1$ be the odd number of crossings in twist a. We let $2m$ and $2k$ represent for the even number of crossings in twist b and c respectively.

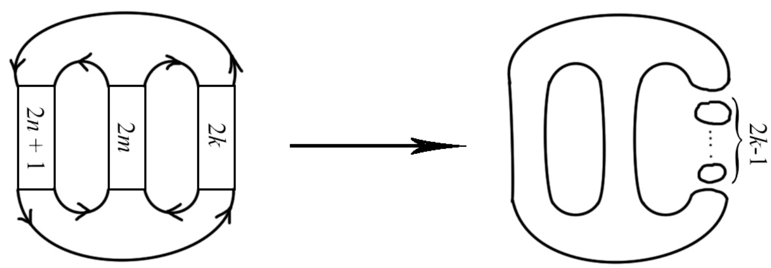


Figure 3.44: $(2n+1, 2m, 2k)$-Pretzel.

After smoothing all the crossings by Seifert's algorithm, we obtain 0 circle from twist a, b and $2k-1$ circles from twist c. The number of circles we get is the sum of all circles we obtain above plus 2 more circles.

Therefore, $\mid \mathbf{S} \mid= 0+0+2k-1+2 = 2k+1$.

The total number of crossing in this knot is $c = 2n{+}1{+}2m{+}2k = 2n{+}2m{+}2k{+}1$.

Therefore the Euler characteristic of the $(2n+1, 2m, 2k)$-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 2k-1-(2n+2+2k+1) = -2n-2m.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2-\chi-B}{2} = \frac{2-(-2n-2m)-2}{2} = n+m.$$

We can see as the number of crossing goes to infinity the orientable genus goes to infinity as well.

Case 4: (even, odd, odd)-pretzel

Let's start this case by finding an orientable genus of the (2,1,3)-pretzel knot.

Example 3.18. Find the orientable genus of the (2,1,3)-pretzel knot in Figure 3.45.

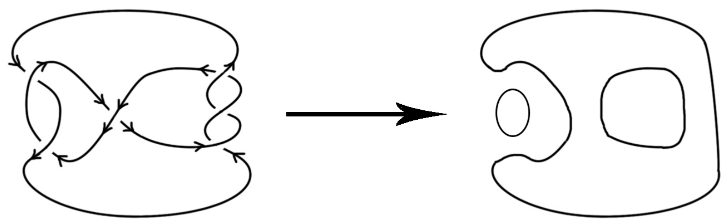


Figure 3.45: (2,1,3)-Pretzel.

After smoothing all the crossings by Seifert's algorithm, we obtain the sideway state surface as shown in Figure 3.45. Twist b and c are compatibly oriented, so we do not get any circle after smoothing. Twist a has 2 crossings which gives us 1 circles in its maximal state. The number of circles we get is the sum of all circles we obtain above plus 2 more circles.

Therefore, $\mid \mathbf{S} \mid = 1 + 0 + 0 + 2 = 3$.

The total number of crossing in this knot is $c = 2 + 1 + 3 = 6$.

Therefore the Euler characteristic of the (2,1,3)-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 3 - 6 = -3.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2 - \chi - B}{2} = \frac{2 - (-3) - 1}{2} = 2. \qquad \blacksquare$$

Next we are going to generalize this case by making the (even,odd, odd)-pretzel where $2n$ represents the even number of crossing in twist a while $2m+1$ and $2k+1$ represent for the odd number of crossings in twist b and c respectively.

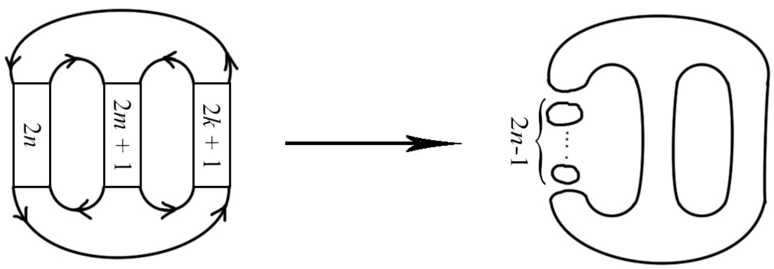


Figure 3.46: $(2n, 2m+1, 2k+1)$-Pretzel.

Twist b and c are compatibly oriented, so we do not get any circle after smoothing after smoothing all the crossings by Seifert's algorithm. Twist a has $2n$ crossings which gives us $2n-1$ circles in its maximal state. The number of circles we get is the sum of all circles we obtain above plus 2 more circles.

Therefore, $\mid \mathbf{S} \mid = 0+0+2n-1+2 = 2n+1$.

The total number of crossing in this knot is $c = 2n+2m+1+2k+1 = 2n+2m+2k+1$.

Therefore the Euler characteristic of the $(2n, 2m+1, 2k+1)$-pretzel is

$$\chi = \mid \mathbf{S} \mid - c = 2n+1-(2n+2m+2k+1) = -2m-2k-1.$$

Hence, the orientable genus of this surface is

$$g_o = \frac{2-\chi-B}{2} = \frac{2-(-2m-2k-1)-1}{2} = m+k+1.$$

The orientable genus is approaching infinity as we increase an even number of crossings in twist a.

The last 2 cases tells us the non-orientable genus is minimal which is equal to 1, but the orientable genus is increasing to infinity as we increase the number of crossing in each twist.

Adams and Kindred [CK12] show in their article that $g_n \leq g_o + \frac{1}{2}$. Even though the non-orientable genus is bounded above the orientable genus adding a half, there is no corresponding upper bound on g_o as a function of g_n. From what we examine those cases above, g_n is 1 while g_o is increasing without bound.

Chapter 4

Conclusion

Summary

In the first chapter, we went over a history of knot and discussed some basic definitions in Knot Theory. We learned how to make some simple knots and how to deform the knot by three types of Reidemeister moves and flypes. In the second chapter, we learn Jones polynomial and how to compute the Kauffman bracket, or they usually refer it as a 3-variable bracket polynomial.

The third chapter is very important which is about Surfaces. In this chapter, we discussed some common definitions about sufaces and boundaries, then learned how to calculate for the Euler characteristic and genus of the surface. Next we talked about Seifert's circle. Then we look at the differences between the orientable and non-orientable surfaces. Finally, we work on our main goal of this paper, finding the relationship between the Kauffman bracket and the genus of the alternating links.

While Adams and Kindred show that 3-variable bracket always gives us a genus, this paper shows that we cannot expect to find the orientable genus from the exponent of d in 3-variable bracket.

All we were looking at is the exponent of d, but there might be some other thing or other coefficient in the Kauffman bracket polynomial that may help us to discover more about this topic. Let's Knot Theory continues challenging us to discover it in near future.

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