



California State University, San Bernardino
CSUSB ScholarWorks

Electronic Theses, Projects, and Dissertations

Office of Graduate Studies

6-2015

SYMMETRIC PRESENTATIONS OF NON-ABELIAN SIMPLE GROUPS

Leonard B. Lamp

Cal State San Bernardino, leonarddotlamp@gmail.com

Follow this and additional works at: <http://scholarworks.lib.csusb.edu/etd>

Recommended Citation

Lamp, Leonard B., "SYMMETRIC PRESENTATIONS OF NON-ABELIAN SIMPLE GROUPS" (2015). *Electronic Theses, Projects, and Dissertations*. Paper 222.

This Thesis is brought to you for free and open access by the Office of Graduate Studies at CSUSB ScholarWorks. It has been accepted for inclusion in Electronic Theses, Projects, and Dissertations by an authorized administrator of CSUSB ScholarWorks. For more information, please contact scholarworks@csusb.edu.

SYMMETRIC PRESENTATIONS OF NON-ABELIAN SIMPLE GROUPS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Leonard Bo Lamp

June 2015

SYMMETRIC PRESENTATIONS OF NON-ABELIAN SIMPLE GROUPS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

by

Leonard Bo Lamp

June 2015

Approved by:

Dr. Zahid Hasan, Committee Chair

Date

Dr. Gary Griffing, Committee Member

Dr. Joseph Chavez, Committee Member

Dr. Charles Stanton, Chair,
Department of Mathematics

Dr. Corey Dunn
Graduate Coordinator,
Department of Mathematics

ABSTRACT

The goal of this thesis is to show constructions of some of the sporadic groups such as the Mathieu group, M_{12} , the Janko group J_1 , Projective Special Linear groups, $PSL(2, 8)$, and $PSL(2, 11)$, Unitary group $U(3, 3)$ and many other non-abelian simple groups. Our purpose is to find all simple non-abelian groups as homomorphic images of permutation or monomial progenitors, as well as grasping a deep understanding of group theory. Extension theory is used to determine groups up to isomorphisms. The progenitor, developed by Robert T. Curtis, is an infinite semi-direct product of the following form: $P \cong 2^{*n} : N = \{\pi w | \pi \in N, w \text{ is a reduced word in the } t'_i s\}$ where 2^{*n} denotes a free product of n copies of the cyclic group of order 2 generated by involutions t_i , for $1 \leq i \leq n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutory generators by conjugation. Thus we develop methods for factoring by a suitable number of relations in the hope of finding all finite non-abelian simple groups, and in particular one of the 26 sporadic simple groups. Then the algorithm for double coset enumeration together with the First Isomorphism Theorem aids us in proving the homomorphic image of the group we have constructed. After being presented with a group G , we then compute the composition series to solve extension problems. Given a composition such as $G = G_0 \geq G_1 \geq \dots \geq G_{n-1} \geq G_n = 1$ and the corresponding factor groups $G_0/G_1 = Q_1, \dots, G_{n-2}/G_{n-1} = Q_{n-1}, G_{n-1}/G_n = Q_n$. We note that $G_1 = 1$, implying $G_{n-1} = Q_n$. As we move to the next composition factor we see that $G_{n-2}/Q_n = Q_{n-1}$, so that G_{n-2} is an extension of Q_{n-1} by Q_n . Following this procedure we can recapture G from the products of Q_i and thus solve the extension problem. The Jordan-Holder theorem then allows us to develop a process to analyze all finite groups. If we knew all finite simple groups and could solve their extension problem, we would arrive at the isomorphism type. We will present how we solve extensions problems while our main focus will lie on extensions that will include the following: semi-direct products, direct products, central extensions and mixed extensions. Lastly, we will discuss Iwasawa's Lemma and how double coset enumeration aids us in showing the simplicity of some of our groups.

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to my immediate adviser, Dr. Hasan for his great interest and inspiration in the pursuit of my studies and in the preparation that led to the development of this thesis. I could not have done it without his commitment and 2 a.m. emails.

To Dr. Joseph Chavez and Dr. Gary Griffing I owe a deep sense of gratitude for prompt inspiration, positive attitude, and enthusiasm that enabled me to complete my thesis. I am also extremely thankful for my friends and family for all their understanding and helpful encouragement through my research period, without you I would of been done much earlier. I would like to express a special gratitude and thanks to Dr. Stanton and the entire CSUSB math department for imparting their knowledge and love for mathematics they graciously bestowed upon me.

I would also like to express my immeasurable appreciation to my colleague, Dustin Grindstaff. I could not have finished this thesis without your encouragement and competitive edge. I am deeply grateful for all our lunch breaks, especially when you paid. Best Duo Ever.

Lastly, I would like to thank the person that has been there for me throughout my entire education career, Chelsie Lynn Green. Her patience, encouragement, and unwavering love has been the foundation that I have built my life on for the past eight years. Words cannot explain how much of your love and devotion has encouraged me to finish this project, and for that I thank you.

Table of Contents

Abstract	iii
Acknowledgements	iv
List of Tables	viii
List of Figures	x
1 Introduction	1
1.1 The Origin of the Progenitor	1
1.2 S_4 as a Homomorphic Image of $2^{*3} : S_3$	2
1.2.1 Curtis M_{24} Example	2
1.2.2 Find a Relation that Produces $2^{*3} : S_3 \cong S_4$	3
2 Progenitors and Group Related Preliminaries	4
2.1 Preliminaries	4
2.2 Examples of Presentations	8
2.3 Blocks, Transitivity, Primitivity, and Iwasawa's Lemma	11
3 Writing Progenitors	14
3.1 Writing Relations	14
3.1.1 Factoring by the Famous Lemma	14
3.1.2 First Order Relations	16
3.2 Permutation Progenitors	17
3.3 Monomial Presentation Progenitors	18
3.3.1 Character Theory Preliminaries	18
3.3.2 Orthogonality Relations	19
3.4 Finding a Monomial Representation	24
3.5 Wreath Product Progenitors	29
3.5.1 Method to Writing Wreath Products	35
3.5.2 More Examples of Jesse Train's Method	35
3.6 Finding a Permutation Representation for Wreath Products	36
3.6.1 Using a Simple Loop	36
3.6.2 Using Classes to Find Permutations of Wreath Products	37

4	Finite Extensions	40
4.1	Extensions and Related Theorems	40
4.1.1	Simple Extension Examples	42
4.1.2	Extension Problems Related to the Progenitor $2^{*4} : S_4$	51
5	Progenitors with Isomorphism Types	67
5.1	$7^{*3} :_m S_3$	67
5.2	$2^{*9} : D_{18}$	68
5.3	$2^{*6} : ((C_3 \times C_3) : C_2)$	68
5.4	$2^{*7} : (C_7 : C_3)$	69
5.5	$2^{*5} : ((C_5 : C_2) : C_2)$	69
5.6	$2^{*12} : (2^2 : 3)$	70
5.7	$2^{*11} : D_{22}$	70
5.8	$2^{*10} : D_{20}$	71
5.9	$2^{*14} : D_{28}$	71
5.10	$2^{*11} : L_2(7)$	72
5.11	$3^{*3} :_m S_4$	72
5.12	$2^{*7} : (7 : 3)$	73
5.13	$2^{*7} : ((7 : 3) : 2)$	73
5.14	$7^{*2} :_m D_{18}$	73
5.15	$2^{*10} : (2^\bullet(5 : 2) : 2)$	74
5.16	$2^{*6} : (3^\bullet(3 : 2))$	74
6	Manual Double Coset Enumeration	75
6.1	Definition for Double Coset Enumeration	75
6.2	Double Coset Enumeration $2^{*5} : A_5$	76
6.3	Finding the Center of the Cayley Diagram	80
7	Double Coset Enumeration over Maximal Subgroups	88
7.1	Double Coset Enumeration of $S_5 \times 2$ over S_4	88
7.2	Double Coset Enumeration of H over N	92
7.3	Double Coset Enumeration of G over H	92
7.4	Computing Double Coset Enumeration of G over N	93
7.4.1	Mathematical Insight	94
7.5	Double Coset Enumeration of U(3,3) over a Maximal Subgroup	94
8	Use of Iwasawa's Theorem	99
8.1	Use of Iwasawa's	99
9	Double Coset Enumeration of $L_2(8)$ over D_{18}	104
9.1	Manual Double Coset Enumeration	104
9.2	Iwasawa's Lemma to Show $G \cong L_2(8)$	107
9.2.1	G Acts Faithfully X	107
9.2.2	G Acts Primitively on X	108
9.2.3	G is Perfect, $G' = G$	108

9.2.4	Conjugates of a Normal Abelian Subgroup K Generate G	110
9.3	J_1 is Simple	111
10	Conclusion	114
Appendix A	Double Coset Enumeration Codes	121
A.1	Double Coset Enumeration of $PSL(2, 8)$ over D_{18}	121
A.2	Double Coset Enumeration of $PSL(2, 23)$ to Find Relations	125
A.3	Double Coset Enumeration of $U(3, 3)$ over a Maximal Subgroup	130
Appendix B	J_1 is Simple Using Iwasawa's	138
Appendix C	Solved Composition Factor of the $6^\bullet : (PSL(2, 4) : 2)$	139
Appendix D	Unsuccessful Progenitors and their Relations	143
D.1	Wreath Product $Z_2 \text{ Wr } A_5$	143
D.2	Wreath Product $Z_2 \text{ Wr } S_4$	145
D.3	Wreath Product $Z_2 \text{ Wr } A_4$	147
D.4	Wreath Product $Z_3 \text{ Wr } Z_2$	149
Appendix E	Difficult Extension Problems	150
E.1	Mixed Extension Using Database	150
E.2	Extension Problem $(2 \times 11)^\bullet : (PGL(2, 11))$	153
E.3	Extension Problem $2^\bullet(U(3, 4) : 2)$	155
Bibliography		157

List of Tables

3.1	Classes of A_4	16
3.2	First Order Relations	17
3.3	Classes of S_3	21
3.4	Character Table of \mathbb{Z}_2	21
3.5	Character Table of S_3	22
3.6	Character Table of S_3	23
3.7	Character Table of H	24
3.8	Character Table of G	24
3.9	Labeling our t 's	27
3.10	Labeling our t 's	27
3.11	Labeling our t 's	28
3.12	Labeling Z	30
3.13	$\gamma(3)$	31
3.14	$\gamma(4)$	31
3.15	$\gamma(5)$	31
3.16	$\gamma(6)$	32
3.17	$\gamma(7)$	32
3.18	$(3, 4)(5, 6)^*$	33
3.19	$(3, 7, 6)^*$	33
5.1	$7^{*3} :_m S_3$	67
5.2	$2^{*9} : D_{18}$	68
5.3	$2^{*6} : ((C_3 \times C_3) : C_2)$	68
5.4	$2^{*7} : (C_7 : C_3)$	69
5.5	$2^{*5} : ((C_5 : C_2) : C_2)$	69
5.6	$2^{*12} : (2^2 : 3)$	70
5.7	$2^{*11} : D_{22}$	70
5.8	$2^{*10} : D_{20}$	71
5.9	$2^{*14} : D_{28}$	71
5.10	$2^{*11} : L_2(7)$	72
5.11	$3^{*3} :_m S_4$	72
5.12	$2^{*7} : (7 : 3)$	73
5.13	$2^{*7} : ((7 : 3) : 2)$	73

5.14	$7^{*2} :_m D_{18}$	73
5.15	$2^{*10} : (2^\bullet(5 : 2) : 2)$	74
5.16	$2^{*6} : (3^\bullet(3 : 2))$	74
10.1	$2^{*32} : (2^\bullet(2^4 : 5))$	117
10.2	$2^{*7} : (7 : 3)$	119
10.3	$2^{*7} : (7 : 3)$	119
10.4	$2^{*7} : (7 : 2)$	120
10.5	$2^{*7} : (7 : 2)$	120
10.6	$2^{*5} : (5 : 2)$	120

List of Figures

6.1	Cayley Diagram of $2^{*5} : A_5$	80
6.2	Cayley Diagram of $2^{*5} : A_5$ Factored by the Center	87
7.1	Cayley Diagram of S_5 over S_4	91
7.2	First Double Coset	95
7.3	Cayley Graph of $U(3, 3)$	97
8.1	Cayley Diagram of $L_2(23)$ over A_4	100
9.1	Cayley Diagram of $L_2(8)$	107

Chapter 1

Introduction

1.1 The Origin of the Progenitor

The aim of our research is to find representations for all simple groups through the use of a group construction, developed by Robert T. Curtis [Cur07], called a progenitor. The progenitor was developed when Curtis was asked if the Mathieu group M_{24} could contain two copies of the linear group $PSL(2, 3)$ which intersect in a subgroup isomorphic to the symmetric group S_4 . The use of this information was needed to construct a group with particular properties. Through Curtis' analysis, he found that M_{24} had seven generating involutions whose set normalizer was the maximal subgroup $PSL(2, 3)$. Given the simplicity of M_{24} together with the maximality of $PSL(2, 3)$, implied that these seven involutions must generate M_{24} . These ideas developed the creation of the progenitor which showed that M_{24} is a homomorphic image of $PSL(2, 3)$.

Through Curtis' efforts we further investigate progenitors of our own in the hope of finding finite homomorphic images as simple groups. In essence, if we wanted a representation for a group, G , such that $G = \langle t_1, t_2, \dots, t_n \rangle$ where $|t_i| = 2$, for $1 \leq i \leq n$, then we need an $N = Normalizer(G, \{\langle t_1 \rangle, \langle t_2 \rangle, \dots, \langle t_n \rangle\})$ where N acts transitively on $\{\langle t_1 \rangle, \langle t_2 \rangle, \dots, \langle t_n \rangle\}$. Given these conditions we then can say G is a homomorphic image of the progenitor $2^{*n} : N$. To further understand this concept we will show a smaller example using $G = S_4$, and also how the group is expressed as the progenitor $2^{*3} : S_3$.

1.2 S_4 as a Homomorphic Image of $2^{*3} : S_3$

First we note that our G has to be generated by involutions. Thus, $G = S_4 = \langle (1, 2), (1, 3), (1, 4) \rangle$. We let $(1, 2)$, $(1, 3)$ and $(1, 4)$ act as t_1, t_2 , and t_3 respectively in our discussion from above. Now, to compute the normalizer of $\{\langle (1, 2) \rangle, \langle (1, 3) \rangle, \langle (1, 4) \rangle\}$, we need to find the set of permutations in S_4 that upon conjugation of the members of the set $\{\langle (1, 2) \rangle, \langle (1, 3) \rangle, \langle (1, 4) \rangle\}$ send the set to itself. We find that such a set is $\{e, (2, 3), (2, 4), (3, 4), (2, 3, 4), (2, 4, 3)\}$. Notice that this set is generated by $\langle (2, 3, 4), (2, 3) \rangle$. Hence, the normalizer of $T = \{\langle (1, 2) \rangle, \langle (1, 3) \rangle, \langle (1, 4) \rangle\}$ in G is $S_3 = \langle (2, 3, 4), (2, 3) \rangle$ and the normalizer acts transitively on T , since there exists a permutation of our S_3 that will take us from one generator of S_4 to any other generator, namely $(2, 3)$, $(2, 4)$ and $(3, 4)$.

1.2.1 Curtis M_{24} Example

In M_{24} , [Cur07] found that $N = PSL(2, 3)$ with $N \cap N^t = S_4$ and the centralizer, $C_{M_{24}}(S_4) = \langle t \rangle$, then t commuted element wise and $[PSL(2, 3) : S_4] = 7$. This implied $M_{24} = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_7 \rangle$. Applying these concepts to $G = S_4$ above, we see that $S_4 = \langle (1, 2), (1, 3), (1, 4) \rangle$ and our N , is S_3 . If we let $t = (1, 4)$ then $|S_3 \cap S_3^t| = 2$ since

$$\begin{aligned} S_3 &= \{(2, 3, 4), (2, 4, 3), (2, 3), (2, 4), (3, 4), e\} \\ S_3^{(1,4)} &= \{(2, 3, 1), (2, 1, 3), (2, 3), (2, 1), (3, 1), e\} \end{aligned}$$

Thus, $S_3 \cap S_3^t = \{e, (2, 3)\}$. The centralizer in S_4 of the set $\{e, (2, 3)\}$ is equal to the number of conjugates in S_4 of $(1, 4)$. We find that the centralizer is given as follows: $C(S_4, \{e, (2, 3)\}) = \{e, (1, 4), (2, 3), (1, 4)(2, 3)\}$. Computing the transversals of $\{e, (2, 3)\}$ in $S_3 = \langle (2, 3, 4), (2, 3) \rangle$ and putting them in a set T , we get $T = \{e, (2, 4), (3, 4)\}$. Therefore $(1, 4)$ has 3 conjugates $(1, 4)^e = (1, 4)$, $(1, 4)^{(2,4)} = (1, 2)$ and $(1, 4)^{(3,4)} = (1, 3)$. Notice that these conjugates generate $G = S_4$. In summary, $S_4 = \langle (1, 2), (1, 3), (1, 4) \rangle$, where the normalizer $(S_4, \langle (1, 2), (1, 3), (1, 4) \rangle) = S_3 = \langle (2, 3, 4), (2, 3) \rangle$. In addition, $|S_3 \cap S_3^t| = 2$ implies $|\text{Stabilizer}(S_3, 3)| = 2$ and t_3 has 3 conjugates under conjugation by S_3 . Hence, S_4 is a homomorphic image of $2^{*3} : S_3$.

1.2.2 Find a Relation that Produces $2^{*3} : S_3 \cong S_4$

Recall that a progenitor is an infinite group, thus we may be able to show that S_4 is a finite homomorphic image of $2^{*3} : S_3$ from the criterion above. However, in order to get a finite image we need to find a relation that will create a finite presentation isomorphic to S_4 . Fortunately, this has been investigated by [Cur07] and we will explain his results. Consider the relation $x = t_1 t_2 t_1$. The presentation of the progenitor $2^{*3} : S_3$ is given as: $\langle x, y, t \rangle = \langle x^3, y^2, (xy)^2, t^2, (t, y) \rangle$. Inserting the relation we achieve $\langle x, y, t \rangle = \langle x^3, y^2, (xy)^2, t^2, (t, y), x = tt^x t \rangle$. Upon completion of the double coset enumeration we find that $f(x) = (2, 3, 4)$, $f(y) = (3, 4)$, and $f(t) = (1, 2)$. We find that $H = \langle f(x), f(y) \rangle$ is isomorphic to S_3 , which is maximal in S_4 . Also, $f(t) \in S_4$ but $f(t) \notin S_3$ implies that $G = \langle f(x), f(y), f(t) \rangle = S_4$. The details of this discovery and the double coset enumeration of this group can be found in the appendix. In the chapters that follow, we will provide charts of progenitors and the groups they produce with particular relations.

Chapter 2

Progenitors and Group Related Preliminaries

In this chapter, we will provide the foundation in which we have found all the homomorphic images of non-abelian simple groups presented in this paper. The progenitor is a construction that was developed by Robert T. Curtis in his search to find a presentation for all finite non-abelian simple groups. The progenitor is a semi-direct product of the following form: $P \cong 2^{*n} : N = \{\pi w | \pi \in N, w \text{ is a reduced word in the } t'_i s\}$ where 2^{*n} denotes a free product of n copies of the cyclic group of order 2 generated by involutions t_i for $1 \leq i \leq n$; and N is a transitive permutation group of degree n which acts on the free product by permuting the involuntary generators by conjugation. [Cur07] A progenitor is essentially a presentation for an infinite group, when factored by relations generate images for finite groups. Thus we develop methods for factoring by any number of relations in the hope of finding all non-abelian simple groups.

To get a full understanding of what is meant by the development of a progenitor we start introducing some elementary definitions.

2.1 Preliminaries

Definition 2.1. [Rot95] Let G be a set. A **binary operation** on G is a function that assigns each ordered pair of elements of G an element of G .

Definition 2.2. [Rot95] A **semigroup** $(G, *)$ is a nonempty set G equipped with an associative operation $*$.

Definition 2.3. [Rot95] A **group** is a semigroup G containing an element e such that

$$(i) \ e * a = a = a * e \text{ for all } a \in G$$

$$(ii) \text{ for every } a \in G, \text{ there is an element } b \in G \text{ with } a * b = e = b * a$$

Theorem 2.4. [Rot95] **First Isomorphism Theorem**

Let $f : G \rightarrow H$ be a homomorphism with kernel K . Then K is a normal subgroup of G and $G/K \cong \text{im} f$

Proof. Let $K \triangleleft G$. Define $\varphi : G/K \rightarrow H$ by the mapping $\varphi(Ka) = f(a)$. We will show that φ is a well defined one to one and onto homomorphism. To see that φ is well-defined, we assume that $Ka = Kb$, hence $ab^{-1} \in K$. Then

$$1 = f(ab^{-1}) = f(a)f(b)^{-1}, \text{ and } f(a) = f(b).$$

Thus $\varphi(Ka) = \varphi(Kb)$ as desired. Now we show that φ is a homomorphism:

$$\varphi(KaKb) = \varphi(Kab) = f(ab) = f(a)f(b) = \varphi(Ka)\varphi(Kb)$$

Clearly, $\text{im} \varphi = \text{im} f$. Now show that φ is an injection. Assume $\varphi(Ka) = \varphi(Kb)$, then $f(a) = f(b)$; hence $f(ab^{-1}) = 1$, $ab^{-1} \in K$ and $Ka = Kb$. We have shown that φ is an isomorphism. \square

Theorem 2.5. [Rot95] **Second Isomorphism Theorem**

Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$

Proof. We want to apply the First Isomorphism Theorem. Thus we will find a homomorphism from N onto NT/T with kernel $N \cap T$. Define a map $\theta : T \rightarrow TN/N$ by $\theta(a) = aN$. Suppose that $a, b \in T$. Then $\theta(ab) = abN = (aN)(bN) = \theta(a)\theta(b)$. Thus, θ is a homomorphism. Now, θ is onto since if $aN \in TN/N$ with $a \in TN$. We note that $a = tn$ where $t \in T$ and $n \in N$. Now $n^{-1}N = N$, thus $tN = an^{-1}N = aN$. We conclude that $\theta(t) = aN$. Finally, if $a \in T$, then $\theta(a) = N \iff a \in T \cap N$. So $\ker \theta = T \cap N$. Using the First Isomorphism Theorem we can now conclude that $T/(N \cap T) \cong \theta(N) = NT/N$ \square

Theorem 2.6. [Rot95] **Third Isomorphism Theorem**

Let $K \leq H \leq G$, where both K and H are normal subgroups of G . Then H/K is a normal subgroup of G/K and

$$(G/K)(H/K) \cong G/H.$$

Proof. This theorem is easily proved using the First Isomorphism Theorem. First we consider the natural map $G \rightarrow G/H$. The kernel of π , contains H . We define the mapping $f : G/K \rightarrow G/H$ by $f(Ka) = Ha$. We note that this mapping is well-defined since $K \leq H$. Notice that the mapping sends the left coset gK to the left coset gH . Now assume that gK is in the kernel. Then the left coset gH is the identity coset, such that $gH = H$, implying $g \in H$. Thus the kernel consists of those left cosets of the form gK for $g \in H$, that is H/K . Thus the result now follows by the First Isomorphism Theorem, and this completes the proof. \square

Theorem 2.7. [Rot95] If $K \leq H$ and $[H : K] = n$, then there is a homomorphism $\phi : H \rightarrow S_n$ with $\ker \phi \leq K$.

Proof. If $a \in H$ and X is the set of all the right cosets of H in K , we define a function $\phi_a : X \rightarrow X$ by $Kh \mapsto Kha$ for all $h \in H$. We know that $\phi : X \rightarrow X$ is 1-1 if and only if there exists a function $f : X \rightarrow X$ such that $\phi(f) = 1_X$. Therefore we need to show $(\phi_a)^{-1} = \phi_{a^{-1}}$. So, given that $\phi_a : Kh \mapsto Kha$, then $(\phi_a)^{-1} : Kha \mapsto Kh$. Now,

$$\phi_{a^{-1}} : Kh \rightarrow Kha^{-1}$$

Taking the composition $\phi_a \phi_{a^{-1}}$ we have,

$$\phi_a \phi_{a^{-1}} : Kh \rightarrow Kha^{-1} \rightarrow Kh$$

Therefore $(\phi_a)^{-1} = \phi_{a^{-1}}$ and $\phi_a \in S_X$ for all $a \in H$. We now note that, $a \mapsto \phi_a : X \rightarrow S_X$ is a homomorphism given by the mapping $\phi : H \rightarrow S_X \cong S_n$. To show this mapping is a homomorphism we let $Kh \in X$ then if $a, b \in H$ we have $(ab)\phi = (Kh)(ab)\phi = (Kha)(b\phi) = Khab = a\phi b\phi$. If $a \in \ker \phi$ this implies $Kha = Kh$ for all $h \in H$. Letting $h = Id$ then we have $Ka = K$ and by properties of cosets we know $a \in K$ and $\ker \phi \leq K$. \square

Theorem 2.8. [Rot95] *If $K \leq H$, then H acts transitively on the set of all right cosets of H over K .*

Proof. Let X be the set of all right cosets of H over K and assume $Kh \in X$. To show that H acts transitively on the set of right cosets we must find $h \in H$ such that $K \rightarrow Kh$, but $h \in H$ so there exists $h \in H$ such that $K \rightarrow Kh$, thus H acts transitively on the set of all right cosets of H in K . \square

Definition 2.9. [Rot95] *If X is a nonempty subset of a group F , then F is a **free group** with basis X if, for every group G and every function $f : X \rightarrow G$, there exists a unique homomorphism $\phi : F \rightarrow G$ extending f . Moreover, X generates F .*

Definition 2.10. [Rot95] *Given a set X , there exists a free group F with basis X .*

Theorem 2.11. [Rot95] *Every group G is a quotient of a free group.*

Proof. Let G be a group. Define a set $X = \{x_g | g \in G\}$ so that $f : X \rightarrow G$ is a bijection defined given by $x_g \rightarrow g$. By Theorem 2.10, we let F be a free group with basis X . Then by definition of a free group there exist a unique homomorphism $\phi : F \rightarrow G$ that extends f . Moreover, ϕ is onto since f is onto. Now $\frac{F}{\ker \phi} = G$, so G is a quotient of a free group. \square

Definition 2.12. [Rot95] *Let X be a set and let Δ be a family of words on X . A group G has **generators** X and **relations** Δ if $G \cong F/R$, where F is the free group with basis X and R is the normal subgroup of F generated by Δ . The ordered pair $(X|\Delta)$ is called a **presentation** of G .*

2.2 Examples of Presentations

Show that a presentation for D_8 is given as $\langle x, y \mid x^4 = 1, y^2 = 1, (x * y)^2 = 1 \rangle$

Proof. Let F be a free group with basis $X = \{x, y\}$ and define a homomorphism $\phi : F \rightarrow D_8$ by $\phi(x) = (1, 2, 3, 4)$ and $\phi(y) = (1, 3)$. From definition 2.7, we have ϕ is an onto homomorphism. Let $G = \frac{F}{R}$, where $R = \langle x^4, y^2, (x * y)^2 \rangle$. Now $\phi : F \rightarrow D_8$ and we have $\frac{F}{\ker\phi} \cong D_8$. Now we must show that $R = \ker\phi$. So,

$$\phi(x^4) = (\phi(x))^4 = (1, 2, 3, 4)^4 = 1 \implies x^4 \in \ker\phi$$

$$\phi(y^2) = (\phi(y))^2 = (1, 3)^2 = 1 \implies y^2 \in \ker\phi$$

$$\phi((x * y)^2) = (\phi(x)\phi(y))^2 = ((1, 2, 3, 4)(1, 3))^2 = ((12)(34))^2 = 1 \implies (x * y)^2 \in \ker\phi$$

Thus $x^4, y^2, (x * y)^2 \in \ker\phi$. Hence, $\langle x^4, y^2, (x * y)^2 \rangle = R \subseteq \ker\phi$.

Now, $|\frac{F}{R}| \geq |\frac{F}{\ker\phi}| \implies |\frac{F}{R}| \geq |D_8| = 8$. So, $|G| \geq 8$. Now we will show $|G| \leq 8$.

$$G = \frac{F}{R} \leq \{R, Rx, Rx^2, Rx^3, Ry, Rxy, Rx^2y, Rx^3y\}$$

The above set is closed under right multiplication by x and by y , since

$$(Rxy)y = xRy^2 = XR = RX \text{ and}$$

$$\begin{aligned} R(xy)^2 &= R \\ \implies Rxyxy &= R \\ \implies xRyx &= Ry^{-1} \\ \implies Ryx &= x^{-1}Ry^{-1} \\ \implies Ryx &= Rx^{-1}y^{-1} \\ \implies Ryx &= Rx^3Ry \\ \implies Ryx &= Rx^3y \end{aligned}$$

So, $Ryx = Rx^3y$ and $Rxyx = xRyx = xRx^3y = Rx^4y = Ry$ belongs to the set above. Thus, $|G| \leq 8$. Hence $|G| = 8$. By the Third Isomorphism Theorem, we have an onto homomorphism $\psi : \frac{F}{R} \rightarrow \frac{F}{\ker\phi}$ ($R < \ker\phi \leq F$). Thus, $\frac{F/R}{\ker\psi} \cong \frac{F}{\ker\phi}$. Hence $\ker\psi = 1$ and $F/R \cong F/\ker\phi \cong D_8$. \square

Definition 2.13. [Cur07] Let G be a group and $T = \{t_1, t_2, \dots, t_n\}$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the **progenitor** to be the semi direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_m .

Definition 2.14. [Rot95] If $H \leq G$ and $g \in G$, then the **conjugate** gHg^{-1} is $\{ghg^{-1} : h \in H\}$. The conjugate gHg^{-1} is often denoted by H^g

Definition 2.15. [Rot95] If $H \leq G$, then the **normalizer** of H in G , denoted by $N_G(H)$, is

$$N_G(H) = \{a \in G : aHa^{-1} = H\}$$

Definition 2.16. [Rot95] If $a \in G$, then the **centralizer** of a in G , denoted by $C_G(a)$, is the set of all $x \in G$ which commute with a .

Definition 2.17. [Rot95] If $a \in G$, the number of conjugates of a is equal to the index of its centralizer:

$|a^G| = [G : C_G(a)]$, and this number is a divisor of $|G|$ when G is finite.

Theorem 2.18. [Rot95] If G is a finite group and $H \leq G$, then the number of conjugates of H in G is $[G : N_G(H)]$.

Proof. Let $[H]$ denote the set of all the conjugates of H , and let G/N denote the set of all left cosets of $N = N_G(H)$ in G . We define a mapping $f : [H] \rightarrow G/N$ by $f(aHa^{-1}) = aN$. If $aHa^{-1} = bHb^{-1}$ for some $b \in G$ then $b^{-1}aHa^{-1}b = H$ and $b^{-1}a$ normalizes H . So, $b^{-1}a \in N$, and so by properties of cosets $bN = aN$, thus f is well defined. Now show f is one to one. If $aN = f(aHa^{-1}) = f(cHc^{-1}) = cN$ for some $c \in G$, then $c^{-1}a \in N$, $c^{-1}a$ normalizes H , $c^{-1}aHa^{-1}c = H$, and $aHa^{-1} = cHc^{-1}$. Clearly the function is onto, since if $a \in G$ then $aN = f(aHa^{-1})$. Therefore, we have f as a one to one and onto mapping so $|[H]| = |G/N| = [G : N_G(H)]$. \square

From the two previous theorems mentioned above, we can now discuss a way to naturally write a progenitor. As defined above, the presentation of a progenitor is of the form $2^{*n} : N$, where N acts transitively on the t 's. Since N is said to be transitive on the t 's we can determine the number of conjugates of our t by taking the index of the point stabilizer N^1 in N . In comparison to definition 2.10, we see that $N = \langle X | R \rangle$, implying that our progenitor will take the following form:

$$m^{*n} : N \cong \langle X, t | R, t^m, (t, N^1) \rangle$$

Notice that we allow our t to commute with the point stabilizer that corresponds with our associated t . Hence, $|t^n| = |N : N^1|$, as desired. We will illustrate the above description in the following example.

Example 2.19. Show that the presentation for $2^{*3} : S_3$ can be written in a simpler form.

Let $S_3 = \langle (1, 2, 3), (1, 2) \rangle$ and note 2^{*3} means that we have three elements of order two, and allow these to be represented as t 's. Then the presentation for $2^{*3} : S_3$ is given as

$$\{x, y, t_1, t_2, t_3 | x^3, y^2, (x * y)^2, t_1^2, t_2^2, t_3^2, t_1^x = t_2, t_1^y = t_2, t_2^x = t_3, t_2^y = t_1, t_3^x = t_1, t_3^y = t_3\}$$

Now the same presentation can be written as:

$$\{x, y, t_1 | x^3, y^2, (x * y)^2, t_1^2, (t, N^1 = (x * y))\}$$

When we say t_1^2 , this implies $(t^x)^2 = 1, (t^{x^2})^2 = 1$ and $(t^y)^2 = 1$. In addition, (t, N^1) is equivalent to writing $t_1^x = t_2, t_1^y = t_2, t_2^x = t_3, t_2^y = t_1, t_3^x = t_1, t_3^y = t_3$.

We then define $\{x, y, t_1 | x^3, y^2, (x * y)^2, t_1^2, (t, N^1 = x * y)\}$ as a symmetric presentation.

2.3 Blocks, Transitivity, Primitivity, and Iwasawa's Lemma

Definition 2.20. [Rot95] If X is a set and G is a group, then X is a G -set if there is a function $\alpha : G \times X \rightarrow X$ (called an action), denoted by $\alpha : (g, x) \mapsto gx$, such that:

(i) $1 * x = x$ for all $x \in X$

(ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. We say that G acts on X . If

$|X| = n$, then n is called the **degree** of the G -set X .

Definition 2.21. [Rot95] If X is a G -set, then a **block** is a subset B of X such that, for each $g \in G$, either $gB = B$ or $gB \cap B = \emptyset$. Note $gB = \{gx : x \in B\}$. **Nontrivial** blocks are \emptyset, X , and one-point subsets.

Definition 2.22. [Rot95] A G -set X is **transitive** if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$.

Definition 2.23. [Rot95] Let G be a transitive group on X and B be a nontrivial block then

(1) B is a block $\forall g \in G$

(2) $\exists g_1, g_2, g_3, \dots, g_m$ such that $X = Bg_1, Bg_2, \dots, Bg_m$ and $Bg_i \cap Bg_j = \emptyset \forall i \neq j$

(3) $|Bg| = |X| \forall g \in G$

Theorem 2.24. [Rot95] If X is a transitive G -set of degree n , and if $x \in X$, then

$$|G| = n|G_x|$$

Proof. From definition 2.17 we have $|G_x| = [G : G_x]$. Since we assume that X is transitive, $Gx = X$, and so $n = |G|/|G_x|$ □

Definition 2.25. [Rot95] A G -set X is transitive if it has only one orbit; that is, for every $x, y \in X$, there exists $\sigma \in G$ with $y = \sigma x$.

Definition 2.26. [Rot95] A transitive G -set X is **primitive** if it contains no nontrivial block; otherwise, it is **imprimitive**.

Theorem 2.27. [Rot95] Let X be a finite G -set, and let $H \leq G$ act transitively on X . Then $G = HG_x$ for each $x \in X$.

Proof. Let $g \in G$ and $x \in X$ and $g = hg'$. G is transitive on X then $gx \in X$ for $g \in G$ but H is transitive on X . Thus $Hx = \{h_1x, h_2x, \dots, h_nx\} = X$. Now there $\exists h \in H$ such that $hx = gx$ implying $x = h^{-1}gx$. Note that $h^{-1}g$ stabilizes x . Therefore $h^{-1}g \in G_x$. So, $g = h(h^{-1}g)$. If $g \in G$ then $g \in HG_x$, and $G \subseteq HG_x$. However, $HG_x \subseteq G$. Thus since, $G_x \subseteq G$ and $H \subseteq G$ this implies $HG_x = G$ \square

Theorem 2.28. [Rot95] Every doubly-transitive G – set is primitive.

Proof. Let B be a nontrivial block that contains x, y . So $B = \{x, y, \dots\}$. Let $z \in X$ be such that $x \neq z, y \neq z$ and $z \notin B$. Since x is doubly transitive $\exists g \in G$ such that $gx = x$ and $gy = z$. Now, $gB = \{x, z\}$ and $x \in B \cap gB$ but $B \neq gB$. B is not a non trivial block, contradiction. Thus there are no nontrivial blocks, so X is primitive. \square

Theorem 2.29. [Rot95] Let X be a G – set and $x, y \in X$.

- (i) If $H \leq G$, then $H_x \cap H_y \neq \emptyset \implies H_x = H_y$
- (ii) If H is normal in G , then the subsets Hx are blocks of X .

Proof. For the proof of (i), let X be a G – set and $x, y \in X$ and $z \in H_x \cap H_y$. Then $z \in H_x$ and $z \in H_y \implies \exists h_1, h_2 \in H$ such that $z = h_1x$ and $z = h_2y$. So, $h_1x = h_2y$ and $Hh_1x = Hh_2y \implies Hx = Hy$.

For the proof of (ii) let $g \in G$ and assume $gH_x \cap Hx = \emptyset$. Now $Hg = gH$ since H is normal in G . So, $gH_x = (gH)x = (Hg)x = Hgx$. Assume that $gB \cap G \neq \emptyset$ and show that $gB = B$. Thus $GHx \cap Hx \neq \emptyset \implies Hgx \cap Hx \neq \emptyset$. Then $\exists h_1, h_2 \in H$ such that $h_1gx = h_2x$. Then by taking H of both sides we see that $Hgx = Hx \implies gHx = Hx$. \square

Theorem 2.30. [Rot95] If X is a faithful primitive G – set of degree $n \geq 2$. If H is normal in G and if $H \neq 1$, then X is a transitive H – set.

Proof. We need to show that $Hx = x$. Let $x \in X$, we know from definition 2.28 that Hx is block, but x is primitive implies Hx must be a trivial block. So $Hx \neq \emptyset$ since $Hx = \{hx|h \in H\} \geq \{x\}$. Assume $Hx = \{x\}$ then $hx = x$ for all $h \in H$, but we have $Gx = \{g \in G|gx = x\}$. Thus $H \leq Gx$ but H is normal in G . Then $gHg^{-1} = H$ for all $g \in G$. Let $g \in G$. Thus $gHg^{-1} \subseteq gG_xg^{-1} \implies H \subseteq gG_xg^{-1}$, but $gG_xg^{-1} = G_{gx}$. From this we obtain $H \subseteq G_{gx}$ for all $g \in G, x \in X$. So, $H \subseteq G_y \exists y \in X$. So $H \subseteq \bigcap_{y \in X} G_y$. Let $a \neq e \in \bigcap_{y \in X} G_y$. Then $a \in G_y$ for all $y \in X \implies ax = x$ for all $x \in X$. Then $a \in \ker f$ (taking the mapping $f : G \rightarrow S_x$ into account). Therefore $\ker f \neq 1$, and a is a

non-identity element of G that fixes all elements of X implying that X is not faithful, a contradiction. \square

Finally, with all the theorems and lemma's mentioned above we arrive at our intended destination.

Theorem 2.31. *Iwasawa's Lemma*

Let $G' = G$ (such a group is called **perfect**) and let X be a faithful primitive G -set. If there is $x \in X$ and an abelian normal subgroup K of G_x whose conjugates $\{ghg^{-1}\}$ generate G , then G is simple.

Proof. Let $H \neq 1$ be a normal subgroup of G . Show that $H = G$ and $K^G = \{gk^{-1}g|k \in K, g \in G\}$ generates G . Let $g \in G$ then $g = \prod g_i k_i g_i^{-1}$, since H is normal G and X is faithful and primitive by definition 2.25. Hence, $G = Hx$ implies $g_i = h_i s_i$ where $h_i \in H$ and $s_i \in Gx$. So $g = \prod g_i k_i g_i^{-1} = \prod h_i s_i k_i (h_i s_i)^{-1} = \prod h_i s_i k_i s_i^{-1} h_i^{-1} \subseteq HKH$. Since H is normal $K \leq G \implies HK \leq G$. Thus for all $g \in G$, $g \in HK$ implies $G \subseteq HK$. Therefore we have $G = HK$. Now by the use of the Second Isomorphism Theorem we have $HK/H \cong K/(H \cap K)$. Then $G/H \cong K/(H \cap K)$, since $G = HK$. Now $K/(H \cap K)$ is abelian since K is abelian. Thus G/H is abelian. Now from theorem 2.30 we have $G' \subseteq H$, but $G = G'$, so $G \subseteq H$. Recall that H is normal in G , so $G = H$. Therefore G is simple and this completes the proof. \square

Chapter 3

Writing Progenitors

In this chapter we will discuss how to write symmetric presentations for several progenitors. For clarity, we will include examples. Our aim is to use these progenitors to find finite homomorphic images of finite non-abelian simple groups. To find such homomorphic images one must factor the progenitor by relations. We wish to discuss two famous methods for writing relations first.

3.1 Writing Relations

3.1.1 Factoring by the Famous Lemma

Frequently, we take a progenitor of the form $m^{*n} : N$ factored by a single relator to produce a simple group. However, finding such a relation raises much difficulty. Naturally, one may ask what forms these relations should take in order to produce groups of interest. [Why06] Recall that a progenitor is an infinite group. In order to achieve a finite homomorphic image we must factor our progenitor by elements of both N and the free product group m^{*n} together. Robert Curtis explored a way to write elements of our control group N in terms of symmetric generators. His efforts helped develop the following lemma. The lemma, which is used extensively through this thesis, gives us a way in which to factor the progenitor so that we can find such images.

Theorem 3.1. *Famous Lemma*

$N \cap \langle t_i, t_j \rangle \leq C_N(N_{ij})$ where N_{ij} denotes the stabilizer in N of the two points i and j .

Proof. Let $w \in N \cap \langle t_i, t_j \rangle$. We need to show $w \in C = \text{Centralizer}(N, N^{t_i t_j})$. Recall the definition of centralizer:

$$\text{Centralizer}(N, H) = \{n \in N \mid nh = hn \forall h \in H\}$$

Hence $w \in C$ if w commutes with every elements of N^{ij} . Let $\pi \in N^{ij}$ then

$$\begin{aligned} w^\pi &= w \\ \implies \pi^{-1} w \pi &= w \\ \implies w \pi &= \pi w \end{aligned}$$

Thus π commutes with every element of N_{ij} , completing the proof. \square

The above lemma is powerful and we will consider the relations it produces as we begin to factor our progenitors. This lemma provides a method for constructing groups given that our progenitor is of the form $2^{*n} : N$ where N is transitive on n letters. Note $\langle t_i, t_j \rangle = \{t_i^2, t_j^2, (t_i t_j)^k = 1\} = D_{2k}$ the dihedral group of order $2k$. Also, its well known that

$$\text{Center}(D_{2k}) = \begin{cases} 1, & \text{if } k \text{ is odd} \\ \langle (t_i t_j)^{\frac{k}{2}} \rangle, & \text{if } k \text{ is even} \end{cases}$$

Thus for each two point stabilizer we compute the centralizer of the two point stabilizer in N and then write elements of N in terms of $\langle t_i, t_j \rangle$ in the following way.

$$\begin{cases} (x t_i)^m = 1, & \text{where } m \text{ is odd and } x \text{ sends } 1 \text{ to } 2 \\ (t_i t_j)^m = x, & \text{where } m \text{ is even and } x \text{ fixes both } 1 \text{ and } 2 \end{cases}$$

An example of using this famous lemma is provided below.

Example 3.2. Take $N = S_4 = \langle (1, 2, 3, 4), (1, 2) \rangle$. Now we compute $\text{Centralizer}(S_4, \text{Stabilizer}(S_4, [1, 2]))$. Note that we are stabilizing the points 1 and 2. Therefore $\text{Stabilizer}(N, [1, 2]) = \langle (3, 4) \rangle$. Next we compute the centralizer by finding which elements of N commute with $\{e, (3, 4)\}$.

Thus $Centralizer(N, Stabiliser(S_4, [1, 2])) = \{e, (1, 2), (3, 4), (1, 2)(3, 4)\}$. Using the above definition to write these as relations we have the following:

$$\begin{aligned}(t_1 t_2)^m &= (3, 4), m \text{ is even} \\ ((1, 2)t_1)^m &= 1, m \text{ is odd} \\ ((12)(34)t_1)^m &= 1, m \text{ is odd}\end{aligned}$$

3.1.2 First Order Relations

We wish to find a method to exhaust all possible relations of a particular progenitor that would allow us to find all possible finite homomorphic images. J.N. Bray, A.N.A Hammus, and R.T. Curtis developed a way to exhaust all relations of the form $(\pi t_i^a)^b = 1$, where $\pi \in N$ and w is a word in the t_i 's, which they called the first order relations. In order to find these relations, we begin by finding the $Centralizer(N, \text{elements in conjugacy class})$. Next we compute the orbits of the centralizer and then write relations by taking the class representative and right multiplying by a single t_i from each orbit respectively. We will illustrate this through the following example.[Why06]

Example 3.3. Let $N = A_4 = \langle (1, 2)(3, 4), (1, 2, 3) \rangle$. To find all first order relations we first compute the classes of A_4 . The classes of A_4 are as follows:

Table 3.1: Classes of A_4

Class Number	Order	Class Representative	Length
[1]	1	e	1
[2]	2	(1,2)(3,4)	3
[3]	3	(1,2,3)	4
[4]	3	(1,3,2)	4

We need to compute the centralizer of each class representative in $N = A_4$ and then find the orbits of the corresponding centralizer. The following table is constructed to show how to write each relation from each class.

Table 3.2: First Order Relations

Cl. Num	Cl. Rep	Cent(N,Class Rep)	Orb	Rels
[2]	(1,2)(3,4)	$\langle (1,2)(3,4), (1,3)(2,4) \rangle$	{1,2,3,4}	(1,2)(3,4) t_1
[3]	(1,2,3)	$\langle (1,2,3) \rangle$	{1,2,3} {4}	(1,2,3) $t_1, (1,2,3)t_4$
[4]	(1,3,2)	$\langle (1,3,2) \rangle$	{1,2,3} {4}	(1,3,2) $t_1, (1,3,2)t_4$

Keeping these techniques in mind we reach the main concept of this chapter, writing progenitors.

3.2 Permutation Progenitors

From chapter 1 we know a progenitor is an infinite semi-direct product, $m^{*n} : N$, where m represents the order of the t 's, n represents the number of t 's, and N is our control group. When we write a permutation progenitor we take N to be transitive on n letters, hence $m^{*n} : N$. The general form of a permutation progenitor is of the form:

$$\langle x, y, t | \langle x, y \rangle \cong N, t^m, (t, N^i) \rangle, \text{ where } N^i \text{ stands for the stabilizer of } i \text{ in } N$$

Definition 3.4. [Cur07] *A symmetric presentation of a group G is a definition of G of the form*

$$G \cong \frac{2^{*n}:N}{\pi_1 w_1, \pi_2 w_2, \dots}$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators $\pi_1 w_1, \pi_2 w_2, \dots$ have been factored out.

From definition 2.16 we note (t, N^i) gives the number of conjugates of t . Definition 2.16 states that the number of conjugates of H in G is equal to $[G : C_g(a)]$. Applying this concept to our case we have the index of the $Centralizer(N, t)$ equal to the number of conjugates of t , which is equal to the stabilizer of a single point in N .

Example 3.5. We will illustrate how to write a permutation progenitor by using the following example. Let $N = S_3$ and note S_3 is transitive on 3 letters. Let $S_3 = \langle (1, 2, 3), (2, 3) \rangle$ and take $x \sim (1, 2, 3)$ and $y \sim (2, 3)$. Then a presentation for S_3 is $\langle x, y | x^2, y^2, (xy)^2 \rangle$. Now we must introduce a symmetric generator which we routinely

use as t . Allowing our $t \sim t_1$ and also of order 2, we will begin to write the progenitor $2^{*3} : S_3$. Since we want to label our t as t_1 , we must compute N^1 . Now $N^1 = \langle (2, 3) \rangle$ and we notice $y \sim (2, 3)$. We have performed all the needed computations for writing our progenitor. Thus

$$2^3 : S_3 = \langle x, y, t | x^3, y^2, (xy)^2, t^2, (t, y) \rangle$$

We have wrote progenitors for many groups and the presentations of each group will be presented in the chapters that follow.

3.3 Monomial Presentation Progenitors

Definition 3.6. [Cur07] A **monomial representation** of a group G is a homomorphism from G into $GL(n, F)$, the group of nonsingular $n \times n$ matrices over the field F , in which the image of every element of G is a monomial matrix over F .

Definition 3.7. [Led87] A matrix in which there is precisely one non-zero term in each row and in each column is said to be monomial.

3.3.1 Character Theory Preliminaries

Definition 3.8. [Led87] Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m . We consider the characteristic polynomial of $A(x)$, namely

$$\det(\lambda I - A(x)) = \begin{pmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & \lambda - a_{mm}(x) \end{pmatrix}$$

This is a polynomial of degree m in λ , and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi(x) = a_{11}(x) + a_{22}(x) + \dots + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated as $\text{tr}A(x)$, so that

$$\phi(x) = \text{tr}A(x)$$

We regard $\phi(x)$ as a function on G with values in K , and we call it the **character** of $A(x)$.

Definition 3.9. Equivalent Representation

[Led87] Let $\rho : G \rightarrow GL(n, \mathbb{F})$ and $T \in GL(n, \mathbb{F})$. Then $T^{-1}\rho T$ is also a representation of G and $T^{-1}\rho T$ and ρ are called equivalent.

Definition 3.10. Trivial Character

[Led87] The **Trivial Character** is the character χ of the trivial representation, where $\chi : G \rightarrow \mathbb{F}$ given by $\chi(g) = 1 \forall g \in G$.

Definition 3.11. Character Table of a Cyclic Group

[Led87] Let G be a cyclic group of order n . The $G = \langle z \rangle$, and $|z| = n$. Let $\epsilon_r = e^{\frac{2\pi ir}{n}}$, where $r = 0, 1, 2, \dots, n$, be the n^{th} roots of unity. For $z^s \in G$, $s = 0, 1, 2, \dots, n$ the values of the n irreducible characters χ^r are given by $\chi^r(z^s) = e^{\frac{2\pi irs}{n}}$, where $r = 0, 1, 2, \dots, n$.

Theorem 3.12. [Led87] The number of irreducible character of G is equal to the number of conjugacy classes of G

3.3.2 Orthogonality Relations

Definition 3.13. [Led87] Let G be a finite group having the distinct irreducible characters $\chi^{(1)}, \chi^{(2)}, \dots, \chi^{(k)}$. Let $1 \leq i, j \leq k$. Then $\langle \chi^i, \chi^j \rangle = \delta_{ij}$. Note that δ_{ij} denotes the Kronecker delta defined as $\delta_{ij} = 0$, if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$

If we treat the character table as a matrix the following additional relations hold:

(a) In a character table, let X be the row vector $(h_\alpha \chi_\alpha^{(i)})$ and \bar{Y} be the conjugate of the row vector $(\chi_\alpha^{(j)})$. If $i \neq j$, then the ordinary dot product $X \cdot Y = 0$

(b) In a character table, let X be the row vector $(h_\alpha \chi_\alpha^{(i)})$. Then the ordinary dot product $X \cdot \overline{\chi_\alpha^{(j)}} = |G|$. Note that $\chi_\alpha^{(i)} = \chi_\alpha(g)$, where g is an element the conjugacy class C_α .

Definition 3.14. [Led87] Let G be a finite group having the distinct irreducible characters $\chi^{(1)}, \chi^{(2)}, \dots, \chi^{(k)}$. Let $1 \leq i, j \leq k$. Then $\sum_{\alpha=1}^k \chi_\alpha^{(i)} \overline{\chi_\alpha^{(j)}} = \frac{|G|}{h_\alpha} \delta_{\alpha\beta}$

(a) In a character table, the dot product of any column with the conjugate of any other column is 0.

(b) In a character table, the dot product of the column α with its own conjugate is $\frac{|G|}{h_\alpha}$.

Definition 3.15. [Led87] The sum of squares of the degrees of the distinct irreducible characters of G is equal to $|G|$. The **degree of a character** χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character.

Definition 3.16. Lifting Process

[Isa76] Let N be a normal subgroup of G and suppose that $A_0(Nx)$ is a representation of degree m of the group G/N . Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N . If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m$, $\phi(u) = m = \phi(1)$. The lifting process preserves irreducibility.

Definition 3.17. Induced Character

[Isa76] Let $H \leq G$ and $\phi(u)$ be a character of H and define $\phi(x) = 0$ if $x \notin H$, then

$$\phi^G(x) = \begin{cases} \phi(x), & x \in H \\ 0 & x \notin H \end{cases}$$

is an induced character of G .

Definition 3.18. Formula for Induced Character

[Isa76] Let G be a finite group and H be a subgroup such that $[G : H] = \frac{|G|}{|H|} = n$. Let C_α , $\alpha = 1, 2, \dots, m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha$, $\alpha = 1, 2, \dots, m$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G . The values of ϕ^G on the m classes of G are given by:

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w), \alpha = 1, 2, 3, \dots, m.$$

Example 3.19. We wish to obtain the character table for $S_3 = \langle (1, 2, 3), (1, 2) \rangle$. Now we know by theorem 3.12 the number of conjugacy classes of S_3 is equal to the number of irreducible characters. Note to find the conjugacy classes of a group G , we take an element $g \in G$ and conjugate it by every element of G . The elements that this set produces lie in the same conjugacy class. For instance, take $(1, 2) \in S_3$. Then

$$(1, 2)^e = (1, 2)$$

$$(1, 2)^{(1,2)} = (2, 1)$$

$$(1, 2)^{(1,3)} = (3, 2)$$

$$(1, 2)^{(2,3)} = (1, 3)$$

$$(1, 2)^{(1,2,3)} = (2, 3)$$

$$(1, 2)^{(1,3,2)} = (3, 1).$$

Thus the elements $\{(1, 2), (1, 3), (2, 3)\}$ are in one conjugacy class of S_3 . The list of classes of S_3 are given below.

Table 3.3: Classes of S_3

Class Number	Order	Class Representative	Length
[1]	1	e	1
[2]	2	(1,2)	3
[3]	3	(1,2,3)	2

Now for us to complete the character table of S_3 , we must first apply the lifting process to induce a character from a subgroup of S_3 . We let $H \leq G$ where $H = \langle (1, 2) \rangle$ such that $[G : H] = \frac{|G|}{|H|} = \frac{6}{2} = 3$. Using theorem 3.11, the character table for $\mathbb{Z}_2 \cong H$ follows:

Table 3.4: Character Table of \mathbb{Z}_2

Conjugacy Classes	C_1	C_2
Order	1	2
ϕ_1^0	1	1
ϕ_1^1	1	$e^{\pi i} = -1$

Next we lift ϕ_1^1 from H up to G by using the formula, $\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w)$, where C_α are the classes of G , h_α is the size of the class α , and α in the number of classes of G . We will begin by lifting character ϕ_1 from H . Therefore,

$$\phi_1^2 = \frac{3}{3} \sum_{w \in C_1 \cap H} \phi(w) = 1$$

$$\phi_2^2 = \frac{3}{3} \sum_{w \in C_2 \cap H = \{(1,2)\}} \phi(w) = -1$$

This gives us the value for the second character of S_3 . Now, up to this point our character table for S_3 is

Table 3.5: Character Table of S_3

Conjugacy Classes	C_1	C_2	C_3
Order	1	2	3
χ_1	1	1	1
χ_2	1	-1	
χ_3			

To find the remaining values in the character table we will use our orthogonality relations. For instance, by definition 3.15, the sum of squares of the degree of distinct irreducible characters of G is equal to $|G|$. Let the unknown character of χ_3 be labeled as x , then by using definition 3.15 we have,

$$\begin{aligned} 1^2 + 1^2 + x^2 &= 6 \\ \implies x^2 &= 4 \\ \implies x &= 2 \end{aligned}$$

Thus the degree of χ_3 of S_3 is 2. Then using definition 3.13 the value of χ_3 for $(1, 2)$ is given as

$$\begin{aligned} 1 * 1 + 1 * -1 * 2 * x &= 0 \\ \implies x &= 0 \end{aligned}$$

By definition 3.14 the missing value for character 2 is found by :

$$\begin{aligned} 1 * 1 * 1 + 3 * 1 * -1 + 2 * 1 * x &= 0 \\ \implies 1 - 3 + 2x &= 0 \\ \implies 2x &= 2 \\ \implies x &= 1 \end{aligned}$$

Finally we can find the last values of our character table by using definition 3.13 once again,

$$\begin{aligned}
1 * 1 + 1 * 1 * -1 + 2 * x &= 0 \\
\implies 2 + 2x &= 0 \\
\implies 2x &= -2 \\
\implies x &= -1
\end{aligned}$$

Thus our character table for S_3 is given below.

Table 3.6: Character Table of S_3

Conjugacy Classes	C_1	C_2	C_3
Order	1	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Definition 3.20. Formula for Monomial Representation

Let ϕ be a linear character of the subgroup H of index n in G and let $G = H \cup Ht_1 \cup Ht_2 \cup \dots \cup Ht_n$. Let $x \in G$. Then the monomial representation of G has the formula:

$$A(x) = \begin{pmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \cdots & \phi(t_1xt_n^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \cdots & \phi(t_2xt_n^{-1}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi(t_nxt_1^{-1}) & \phi(t_nxt_2^{-1}) & \cdots & \phi(t_nxt_n^{-1}) \end{pmatrix}$$

Theorem 3.21. Monomial Character

[Isa76] Let χ be a character of G . Then χ is monomial if $\chi = \lambda^G$, where λ is a linear character of some subgroup (not necessarily proper) of G . If χ is monomial then χ is afforded by a monomial representation X of G ; that is, each row and column of $X(g)$ has exactly one nonzero entry for each $g \in G$. Moreover, the nonzero entries of $X(g)$, for any $g \in G$, are n^{th} roots of unity for some, since G is finite.

3.4 Finding a Monomial Representation

Now that we have some background of character theory we are now ready to find a monomial representation. After finding said representation we can use this to write our monomial progenitor. To find a monomial presentation we find a subgroup H of a group G whose index is equal to the degree of an irreducible character of G . We then induce a character from H up to G . Finally we use the formula for finite monomial representation to find our desired representations of G . We will illustrate the process described above for $G = A_4$.

Example 3.22. Write a Presentation for $3^3 :_m A_4$

Given that A_4 has a monomial irreducible representation in dimension 3 write a progenitor for $3^3 :_m A_4$. Knowing that A_4 has a monomial irreducible in dimension 3 implies the order of our subgroup H must be of order 4, since $\frac{|A_4|}{|H|} = 3 \implies \frac{12}{|H|} = 3$. So, $|H| = 4$. Let $H = \langle x, y \rangle \leq G$, where $x \sim (1, 2)(3, 4)$, and $y \sim (1, 3)(2, 4)$. The character table of H and G are given below:

Table 3.7: Character Table of H

Conjugacy Classes	C_1	C_2	C_3	C_4
Order	1	2	2	2
ϕ_1	1	1	1	1
ϕ_2	1	-1	1	-1
ϕ_3	1	1	-1	-1
ϕ_4	1	-1	-1	1

Table 3.8: Character Table of G

Conjugacy Classes	C_1	C_2	C_3	C_4
Order	1	2	3	3
χ_1	1	1	1	1
χ_2	1	1	J	-1-J
χ_3	1	1	-1-J	J
χ_4	3	-1	0	0

Next we induce a linear character from H up to G . Each character of H has degree 1, thus every character of H is linear. We will induce ϕ_2 . Recall the formula for inducing a character is

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in C_\alpha \cap H} \phi(w)$$

Taking a representative from each class and using the formula above we have:

For the first class of G

$$\begin{aligned} \phi_2(e) &= \frac{3}{1} \sum_{w \in C_1 \cap H} \phi_2(w) \\ &= \frac{3}{1} \sum_{w=\{e\}} \phi_2(e) \\ &= 3 * 1 \\ &= 3 \end{aligned}$$

Now we compute for the second class of G .

$$\phi_2((1, 2)(3, 4)) = \frac{3}{3} \sum_{w \in C_2 \cap H} \phi_2(w)$$

We note that $w = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$.

$$\begin{aligned} &= \frac{3}{3} \sum_{w \in C_2 \cap H} \phi_2(w) \\ &= 1 * [\phi_2((1, 2)(3, 4)) + \phi_2((1, 3)(2, 4)) + \phi_2((1, 4)(2, 3))] \\ &= 1 * [-1 + 1 + -1] \\ &= -1 \end{aligned}$$

Now $\phi_2((1, 2, 3)) = 0$, since $C_3 \cap H = \emptyset$ and $\phi_2((1, 3, 2)) = 0$, since $C_4 \cap H = \emptyset$. In addition $\phi_2^G = 3 \ -1 \ 0 \ 0 = \chi_4$. Finally we can find a monomial representation of G using definition 3.20. Computing the transversals of H in G , we find the set is given as $T = \{e, (1, 2, 3), (1, 3, 2)\}$. Let $T_1 = e$, $T_2 = (1, 2, 3)$, $T_3 = (1, 3, 2)$ and $B(e) = 1$, $B(x) = -1$, $B(y) = 1$, $B(xy) = -1$, and $B(g) = 0$ if $g \notin H$.

Then,

$$A(x) = \begin{bmatrix} B(T_1 x T_1^{-1}) & B(T_1 x T_2^{-1}) & B(T_1 x T_3^{-1}) \\ B(T_2 x T_1^{-1}) & B(T_2 x T_2^{-1}) & B(T_2 x T_3^{-1}) \\ B(T_3 x T_1^{-1}) & B(T_3 x T_2^{-1}) & B(T_3 x T_3^{-1}) \end{bmatrix}$$

To compute the above matrix we substitute in the values we have defined above. Doing so will produce the following matrix:

$$A(x) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now doing the same process for matrix $A(y)$, we get the following:

$$A(y) = \begin{bmatrix} B(T_1 y T_1^{-1}) & B(T_1 y T_2^{-1}) & B(T_1 y T_3^{-1}) \\ B(T_2 y T_1^{-1}) & B(T_2 y T_2^{-1}) & B(T_2 y T_3^{-1}) \\ B(T_3 y T_1^{-1}) & B(T_3 y T_2^{-1}) & B(T_3 y T_3^{-1}) \end{bmatrix}$$

Substituting our values for y and using how we defined our transversals we have:

$$A(y) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The entries of both matrices are ± 1 , and the smallest field that contains these entries is \mathbb{Z}_3 . This implies that our t 's are of order 3. The following is a monomial representation for A_4 , since $A_4 = \langle x, y | x^2, y^3, (xy)^3 \rangle$ and it is easily checked that $|A(x)| = 2$, $|A(y)| = 3$, and $|A(x)A(y)| = 3$. Now we will convert the above representation into permutations using the following definition.

Definition 3.23. Convert Matrix into Permutations

Let A be a matrix where a_{ij} stands for the i^{th} row and j^{th} column of the matrix.

Then if

$$\begin{cases} a_{ij} = -1 \implies t_i \rightarrow t_j^{-1} \\ a_{ij} = 1 \implies t_i \rightarrow t_j^1 \end{cases}$$

Applying the definition to the Matrix $A(x)$ we get

$$\begin{aligned}
 a_{11} = -1 &\implies t_1 \rightarrow t_1^{-1} \\
 a_{22} = 1 &\implies t_2 \rightarrow t_2 \\
 a_{33} = -1 &\implies t_3 \rightarrow t_3^{-1}
 \end{aligned}$$

Now converting $A(y)$ we get

$$\begin{aligned}
 a_{12} = 1 &\implies t_1 \rightarrow t_2 \\
 a_{23} = 1 &\implies t_2 \rightarrow t_3 \\
 a_{31} = 1 &\implies t_3 \rightarrow t_1
 \end{aligned}$$

Recall that the number of t 's are equal to $[G : H] = 3$, but now our t_i 's have order 3. Therefore we will label them as shown in the table below:

Table 3.9: Labeling our t 's

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}

Using the labeling above and the outcomes from $A(x)$, we will find a permutation representation of $A(x)$.

Table 3.10: Labeling our t 's

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}
↓	↓	↓	↓	↓	↓
t_1^{-1}	t_2	t_3^{-1}	t_1	t_2^{-1}	t_3
4	2	6	1	5	3

Following the above labeling we get $xx = (1, 4)(3, 6)$.

Now for $A(y)$

Table 3.11: Labeling our t 's

1	2	3	4	5	6
t_1	t_2	t_3	t_1^{-1}	t_2^{-1}	t_3^{-1}
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
t_2	t_3	t_1	t_2^{-1}	t_3^{-1}	t_1^{-1}
2	3	1	5	6	4

Following the above labeling we get $yy = (1, 2, 3)(4, 5, 6)$. Again we note that $(xx)^2$, $(yy)^3$ and $(xx * yy)^3$, so $\langle xx, yy \rangle \cong A_4$

Now, we are at the point where we can write a monomial progenitor. Similar to the way we wrote permutation progenitors, monomial progenitors are of the form

$$\langle x, y, t \mid \langle x, y \rangle \cong N, t^m, \text{Normalizer}(N, \langle t \rangle) \rangle$$

where N is the permutation group that came from converting the matrices above. The only notable difference between permutation progenitors and monomial progenitors is that we compute the normalizer of our t , instead of the stabilizer of one point in N . To write our progenitor we need to choose a symmetric generator t . Let us choose $t \sim t_1$. Then the $\text{Normalizer}(N, \langle t_1 \rangle) = \langle (2, 5)(3, 6) \rangle$. If $x \sim (1, 4)(3, 6)$ and $y \sim (1, 2, 3)(4, 5, 6)$ then $xyx^{-1} \sim (2, 5)(3, 6)$. We note that t_1 commutes with this permutation. Thus our presentation of our progenitor $3^*3 :_m A_4$ is given as:

$$\langle x, y, t \mid x^2, y^3, (xy)^3 t^3, (t, yxy^{-1}) \rangle$$

Our progenitor is infinite so we need to add the first order relations mentioned at the beginning of this chapter, to find finite homomorphic images. Lists of these images are presented in charts in the chapters that follow.

3.5 Wreath Product Progenitors

The wreath product of the group H by K , denoted $H \wr K$ is a semi-direct product $H^n : K$, where n is the number of letters on which K acts and H^n is the direct product of n isomorphic copies of H . Since the extension of a wreath product, H^n by K , is a semi-direct product, K also permutes the n copies of H . The official definition of a wreath product is given below.

Definition 3.24. Wreath Product

The wreath product is a semi-direct product of two groups, X and Y , such that $X \cap Y = \emptyset$. Define $H \leq S_X$ and $K \leq S_Y$. Let $Z = X \times Y$, such that $X \cap Y = \emptyset$. Define a permutation group Z , and let $\gamma \in H$, $y \in Y$ and $k \in K$, where

$$\gamma(y) = \begin{cases} (x, y) \mapsto (x\gamma, y) \\ (x, y_1) \mapsto (x, y_1), y \neq y_1 \end{cases}$$

Note $\gamma \in S_Z$ since $(\gamma(y))^{-1} = \gamma^{-1}(y)$. Then

$$\phi : H \longrightarrow S_Z \implies H = \{\gamma(y) | \gamma \in H\} = H(y)$$

$\gamma \mapsto \gamma(y)$, such that $\langle H(y) | y \in Y \rangle = D\gamma_{y \in Y} H(y)$ Note: $\gamma(y)$ and $\gamma(y_1)$ does not move the same element of Z and

$$\prod H(y_i) \langle H(y) | y \in Y, y \neq y_i \rangle = 1$$

$$H(y_i) \triangleleft \langle H(y) | y \in Y \rangle.$$

Then define $k \in K$ as $k^*(x, y) \mapsto (x, yk)$. Given,

$\psi : K \longrightarrow S_Z$ and $k \mapsto k^*$ is one to one then $K \cong \{k^* | k \in K\} = K^*$. Therefore, the functions $\gamma \mapsto \gamma(y)$, where y is a fixed element of Y with image $H(y)$, and $k \mapsto k^*$ with image K^* are monomorphism from H and K to $Sym(Z)$. This is written as $H \wr K = \langle H(y), k^*y \in Y \rangle$, this is referred to the **Base, B**. [May14]

For more details see text book [Rot95]. For a better understanding of how wreath products are written, consider the following example of $\mathbb{Z}_2 \wr A_5$.

Example 3.25. Writing the Progenitor on $\mathbb{Z}_2 \wr A_5$

Let H and K be permutation groups on $X = \{1, 2\}$ and $Y = \{3, 4, 5, 6, 7\}$ respectively, where $H = \mathbb{Z}_2$ and $K = A_5$. Let $Z = X \times Y$. We will then define a permutation group on Z , called the wreath product of H by K .

Let,

$$H = \mathbb{Z}_2 = \langle (1, 2) \rangle$$

$$K = A_5 = \langle (3, 4)(5, 6), (3, 7, 6) \rangle$$

Then we define

$Z = X \times Y = \{(1, 3), (1, 4), (1, 6), (1, 7), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)\}$. Consider the following labeling of the elements of Z :

Table 3.12: Labeling Z

Labeling	Element
8	(1,3)
9	(1,4)
10	(1,5)
11	(1,6)
12	(1,7)
13	(2,3)
14	(2,4)
15	(2,5)
16	(2,6)
17	(2,7)

Let $\gamma \in H$ and y be a element of Y and compute $\gamma(3), \gamma(4), \gamma(5), \gamma(6)$, and $\gamma(7)$ using the relation defined below:

$$\gamma(y) = \begin{cases} (x, y) \mapsto (x\gamma, y) \\ (x, y_1) \mapsto (x, y_1), y \neq y_1 \end{cases}$$

Letting $\gamma = (1, 2)$ and computing the $\gamma(3), \gamma(4), \gamma(5), \gamma(6)$, and $\gamma(7)$ we get:

Table 3.13: $\gamma(3)$

Labeling	Element	Element	Labeling	
8	(1,3)	\mapsto	(2,3)	13
9	(1,4)	\mapsto	(1,4)	9
10	(1,5)	\mapsto	(1,5)	10
11	(1,6)	\mapsto	(1,6)	11
12	(1,7)	\mapsto	(1,7)	12
13	(2,3)	\mapsto	(1,3)	8
14	(2,4)	\mapsto	(2,4)	14
15	(2,5)	\mapsto	(2,5)	15
16	(2,6)	\mapsto	(2,6)	16
17	(2,7)	\mapsto	(2,7)	17

Table 3.14: $\gamma(4)$

Labeling	Element	Element	Labeling	
8	(1,3)	\mapsto	(1,3)	8
9	(1,4)	\mapsto	(2,4)	14
10	(1,5)	\mapsto	(1,5)	10
11	(1,6)	\mapsto	(1,6)	11
12	(1,7)	\mapsto	(1,7)	12
13	(2,3)	\mapsto	(2,3)	13
14	(2,4)	\mapsto	(1,4)	9
15	(2,5)	\mapsto	(2,5)	15
16	(2,6)	\mapsto	(2,6)	16
17	(2,7)	\mapsto	(2,7)	17

Table 3.15: $\gamma(5)$

Labeling	Element	Element	Labeling	
8	(1,3)	\mapsto	(1,3)	8
9	(1,4)	\mapsto	(1,4)	9
10	(1,5)	\mapsto	(2,5)	15
11	(1,6)	\mapsto	(1,6)	11
12	(1,7)	\mapsto	(1,7)	12
13	(2,3)	\mapsto	(2,3)	13
14	(2,4)	\mapsto	(2,4)	14
15	(2,5)	\mapsto	(1,5)	10
16	(2,6)	\mapsto	(2,6)	16
17	(2,7)	\mapsto	(2,7)	17

Table 3.16: $\gamma(6)$

Labeling	Element	Element	Labeling	
8	(1,3)	\mapsto	(1,3)	8
9	(1,4)	\mapsto	(1,4)	9
10	(1,5)	\mapsto	(1,5)	10
11	(1,6)	\mapsto	(2,6)	16
12	(1,7)	\mapsto	(1,7)	12
13	(2,3)	\mapsto	(2,3)	13
14	(2,4)	\mapsto	(2,4)	14
15	(2,5)	\mapsto	(2,5)	15
16	(2,6)	\mapsto	(1,6)	11
17	(2,7)	\mapsto	(2,7)	17

Table 3.17: $\gamma(7)$

Labeling	Element	Element	Labeling	
8	(1,3)	\mapsto	(1,3)	8
9	(1,4)	\mapsto	(1,4)	9
10	(1,5)	\mapsto	(1,5)	10
11	(1,6)	\mapsto	(1,6)	11
12	(1,7)	\mapsto	(2,7)	17
13	(2,3)	\mapsto	(2,3)	13
14	(2,4)	\mapsto	(2,4)	14
15	(2,5)	\mapsto	(2,5)	15
16	(2,6)	\mapsto	(2,6)	16
17	(2,7)	\mapsto	(1,7)	12

Thus from our computations above we have

$$\gamma(3) = ((1, 3), (2, 3)) \sim (8, 13)$$

$$\gamma(4) = ((1, 4), (2, 4)) \sim (9, 14)$$

$$\gamma(5) = ((1, 5), (2, 5)) \sim (10, 15)$$

$$\gamma(6) = ((1, 6), (2, 6)) \sim (11, 16)$$

$$\gamma(7) = ((1, 7), (2, 7)) \sim (12, 17)$$

Now we will label $a = (8, 13)$, $b = (9, 14)$, $c = (10, 15)$, $d = (11, 16)$, and $e = (12, 17)$, which we will use later when writing our progenitor.

Let $k \in K$ and define $K^* : (x, y) \mapsto (x, (y)k)$, as in the definition above, then $k = (3, 4)(5, 6)^*$

Table 3.18: $(3, 4)(5, 6)^*$

Labeling	Element	Element	Labeling
8	(1,3)	\mapsto	(1,4) 9
9	(1,4)	\mapsto	(1,3) 8
10	(1,5)	\mapsto	(1,6) 11
11	(1,6)	\mapsto	(1,5) 10
12	(1,7)	\mapsto	(1,7) 12
13	(2,3)	\mapsto	(2,4) 14
14	(2,4)	\mapsto	(2,3) 13
15	(2,5)	\mapsto	(2,6) 16
16	(2,6)	\mapsto	(2,5) 15
17	(2,7)	\mapsto	(2,7) 17

Thus $(3, 4)(5, 6)^* : ((1, 3), (1, 4))((1, 5), (1, 6)), ((2, 3), (2, 4)), ((2, 5)(2, 6)) \sim (8, 9)(10, 11)(13, 14)(15, 16)$. Label $(8, 9)(10, 11)(13, 14)(15, 16) = f$.

Table 3.19: $(3, 7, 6)^*$

Labeling	Element	Element	Labeling
8	(1,3)	\mapsto	(1,7) 12
9	(1,4)	\mapsto	(1,4) 9
10	(1,5)	\mapsto	(1,5) 10
11	(1,6)	\mapsto	(1,3) 8
12	(1,7)	\mapsto	(1,6) 11
13	(2,3)	\mapsto	(2,7) 17
14	(2,4)	\mapsto	(2,4) 14
15	(2,5)	\mapsto	(2,5) 15
16	(2,6)	\mapsto	(2,3) 13
17	(2,7)	\mapsto	(2,6) 16

Thus $(3, 7, 6)^* : ((1, 3), (1, 7), (1, 6))((2, 3), (2, 7), (2, 6)) \sim (8, 12, 11)(13, 17, 16)$.
 Label $(8, 12, 11)(13, 17, 16) = g$.

Therefore by definition, $\mathbb{Z}_2 \wr A_5 = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) : A_5$. Before we write the full presentation for the above group we need to compute the action of $A_5 = \langle f, g \rangle$ on $\mathbb{Z}_2^5 = \langle a, b, c, d, e \rangle$. Then,

$$\begin{aligned}
 (8, 13)^{(8,9)(10,11)(13,14)(15,16)} &= (9, 14) \implies a^f = b \\
 (8, 13)^{(8,12,11)(13,17,16)} &= (12, 17) \implies a^g = e \\
 (9, 14)^{(8,9)(10,11)(13,14)(15,16)} &= (8, 13) \implies b^f = a \\
 (9, 14)^{(8,12,11)(13,17,16)} &= (9, 14) \implies b^g = b \\
 (10, 15)^{(8,9)(10,11)(13,14)(15,16)} &= (11, 16) \implies c^f = d \\
 (10, 15)^{(8,12,11)(13,17,16)} &= (10, 15) \implies c^g = c \\
 (11, 16)^{(8,9)(10,11)(13,14)(15,16)} &= (10, 15) \implies d^f = c \\
 (11, 16)^{(8,12,11)(13,17,16)} &= (8, 13) \implies d^g = a \\
 (12, 17)^{(8,9)(10,11)(13,14)(15,16)} &= (12, 17) \implies e^f = e \\
 (12, 17)^{(8,12,11)(13,17,16)} &= (11, 16) \implies e^g = d
 \end{aligned}$$

So a presentation for $\mathbb{Z}_2 \wr A_5$ is given as:

$$\langle a, b, c, d, e, f, g \mid a^2, b^2, c^2, d^2, e^2, (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), f^2, g^3, (fg)^5, a^f = b, a^g = e, b^f = a, b^g = b, c^f = d, c^g = c, d^f = c, d^g = a, e^f = e, e^g = d \rangle$$

Now since we have a permutation representation for $N = \mathbb{Z}_2 \wr A_5$ we can write a progenitor using the same method we took for writing permutation progenitors. Finding the wreath products can be tedious and time consuming, however there is a shortcut to finding a presentation for the wreath product that was developed by Jesse Train.

3.5.1 Method to Writing Wreath Products

Jesse Train [Tra13], a masters student and Alumni of CSUSB developed a method in writing wreath products. We will illustrate his method on the same example presented above to show that his method does produce the same outcome.

Example 3.26. We have $\mathbb{Z}_2^5 : A_5$, where $\mathbb{Z}_2 \cong \langle a|a^2 \rangle$ and we name the five copies of \mathbb{Z}_2 as

$\mathbb{Z}_2^5 \cong \langle a|a^2 \rangle \times \langle b|b^2 \rangle \times \langle c|c^2 \rangle \times \langle d|d^2 \rangle \times \langle e|e^2 \rangle$. Since \mathbb{Z}_2^5 is a direct product then the presentation for \mathbb{Z}_2^5 is as follows:

$$\langle a, b, c, d, e | a^2, b^2, c^2, d^2, e^2, (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e) \rangle.$$

We also note that a well known presentation for A_5 is $A_5 \cong \langle x, y | x^2, y^3, (xy)^5 \rangle$. Next we label the five copies of \mathbb{Z}_2 as $a = 1, b = 2, c = 3$, and $d = 4$. We know that a permutation representation for A_5 is given as $\langle (1, 2)(3, 4), (1, 5, 4) \rangle$. By letting $x \sim (1, 2)(3, 4)$ and $y \sim (1, 5, 4)$, we have

$$a^x = b, a^y = e, b^x = a, b^y = b, c^x = d, c^y = c, d^x = c, d^y = a, e^x = e, e^y = d.$$

Using all the above information we can write the wreath product of $\mathbb{Z}_2 \wr A_5$. Note, this presentation is identical to the previous example.

$$\langle a, b, c, d, e, f, g | a^2, b^2, c^2, d^2, e^2, (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e), f^2, g^3, (fg)^5, a^f = b, a^g = e, b^f = a, b^g = b, c^f = d, c^g = c, d^f = c, d^g = a, e^f = e, e^g = d \rangle$$

3.5.2 More Examples of Jesse Train's Method

Example 3.27. Construct the Wreath Product of $\mathbb{Z}_2 \wr S_4$ [Tra13] .

By definition, $\mathbb{Z}_2 \wr S_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 : S_4$. Given $S_4 \cong \langle (1, 2, 3, 4), (1, 2) \rangle$ and labeling the generators as follows we have $\langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle : \langle (1, 2, 3, 4), (1, 2) \rangle$. The presentation of the wreath product is given as:

$$\langle a, b, c, d, e, f | a^2, b^2, c^2, d^2, (a, b), (a, c), (a, d), (b, c), (b, d), (c, d), e^4, f^2, (ef)^3, a^e = b, a^f = b, b^e = c, b^f = a, c^e = d, c^f = c, d^e = a, d^f = d \rangle.$$

Example 3.28. Construct the Wreath Product of $\mathbb{Z}_3 \wr \mathbb{Z}_2$. By definition, $\mathbb{Z}_3 \wr \mathbb{Z}_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 : \mathbb{Z}_2$. Given $\mathbb{Z}_2 \cong \langle (1, 2) \rangle$ and labeling the generators as follows we have $\langle x_1 \rangle \times \langle x_2 \rangle : \langle (1, 2) \rangle$. Finding the action of $\mathbb{Z}_2 = \langle (1, 2) \rangle$ on the two copies of \mathbb{Z}_3 we have the presentation of the wreath product given below:

$$\langle a, b, c | a^3, b^3, (a, b), c^2, a^c = b, b^c = a \rangle$$

3.6 Finding a Permutation Representation for Wreath Products

In the first example, when we wrote the presentation for the wreath product, we also found its permutation representation. However, using this method, we notice that we have a presentation for the wreath product, but we lack the permutations that correspond to each generator. Without these permutations we cannot define a symmetric generator t and thus cannot form our progenitor. Fortunately, we can use *MAGMA* to help in solving this problem. Depending on the size of the group there are two ways we have to finding the required permutation representation.

3.6.1 Using a Simple Loop

MAGMA has a command for wreath products. Using the command for finding the wreath product of $\mathbb{Z}_2 \wr S_4$ is given below:

$$W := \text{WreathProduct}(\text{CyclicGroup}(2), S_4)$$

In the example above, we have found a presentation using the method for $\mathbb{Z}_2 \wr S_4$, which was

$$\langle a, b, c, d, e, f | a^2, b^2, c^2, d^2, (a, b), (a, c), (a, d), (b, c), (b, d), (c, d), e^4, f^2, (ef)^3, a^e = b, a^f = b, b^e = c, b^f = a, c^e = d, c^f = c, d^e = a, d^f = d \rangle$$

To find a permutation that corresponds to the generators a, b, c, d, e and f , we run the loop given below.

```

for A,B,C,D,E,F in W do if Order(A) eq 2 and Order(B) eq 2
and Order(C) eq 2 and Order(D) eq 2 and (A,B) eq Id(W)
and (A,C) eq Id(W) and (A,D) eq Id(W) and (B,C) eq Id(W)
and (B,D) eq Id(W) and (C,D) eq Id(W) and Order(E) eq 4
and Order(F) eq 2 and Order((e*f)) eq 3 and A^E eq B and
B^E eq C and C^E eq D and D^E eq A and A^F eq B and
B^F eq A and C^F eq C and D^F eq D and W eq
sub<W|A,B,C,D,E,F> then A,B,C,D,E,F; break; end if;
end for;

```

The loop essentially looks for permutations in W that correspond to our given presentation. This method works, however as the wreath products get larger, *MAGMA* takes a much longer time to compute the permutations. If this is the case, we then use the classes of W to find our needed permutations.

3.6.2 Using Classes to Find Permutations of Wreath Products

MAGMA can compute classes of groups using the command *Classes*. We note *MAGMA* stores the classes in the following form $\langle \text{Order of Element, Number of Elements, Class Representative} \rangle$. Hence if $C := \text{Class}(G) \implies C[1] = \langle 1, 1, \text{Id}(G) \rangle$ and $C[1][3]$ gives the permutation representative for the first class of G .

Using *MAGMA* the classes of W are shown below.

Conjugacy Classes of group W

```

-----
[1]   Order 1      Length 1
      Rep Id(W)

[2]   Order 2      Length 1
      Rep (1, 2)(3, 4)(5, 6)(7, 8)

[3]   Order 2      Length 4
      Rep (5, 6)

[4]   Order 2      Length 4
      Rep (3, 4)(5, 6)(7, 8)

[5]   Order 2      Length 6
      Rep (1, 2)(5, 6)

```


- [6] Order 2 Length 12
Rep (1, 5)(2, 6)(3, 7)(4, 8)
- [7] Order 2 Length 12
Rep (1, 3)(2, 4)
- [8] Order 2 Length 12
Rep (1, 7)(2, 8)(3, 4)(5, 6)
- [9] Order 2 Length 24
Rep (1, 3)(2, 4)(5, 6)
- [10] Order 3 Length 32
Rep (1, 5, 7)(2, 6, 8)
- [11] Order 4 Length 12
Rep (1, 5, 2, 6)(3, 7, 4, 8)
- [12] Order 4 Length 12
Rep (1, 3, 2, 4)
- [13] Order 4 Length 12
Rep (1, 2)(3, 4)(5, 8, 6, 7)
- [14] Order 4 Length 24
Rep (1, 5, 2, 6)(3, 7)(4, 8)
- [15] Order 4 Length 24
Rep (1, 3, 2, 4)(5, 6)
- [16] Order 4 Length 48
Rep (1, 3, 5, 7)(2, 4, 6, 8)
- [17] Order 6 Length 32
Rep (3, 7, 5, 4, 8, 6)
- [18] Order 6 Length 32
Rep (1, 5, 7)(2, 6, 8)(3, 4)
- [19] Order 6 Length 32
Rep (1, 5, 8, 2, 6, 7)(3, 4)
- [20] Order 8 Length 48
Rep (1, 7, 5, 4, 2, 8, 6, 3)

Now there are 20 classes of W and we notice that classes 2 – 9 contain elements of order 2, and our generators a, b, c, d are of this order. Therefore, our permutation representation for each of those generators lie in one of these classes. Now we union the classes together and ask *MAGMA* to find our permutations for these generators. We take a similar approach in finding the permutations for generators e and f . The generator e is an element of order 4 which will lie in the union of classes 11 and 12, while f is of order 2 which will lie in classes 2 – 9 mentioned above. Following the presentation as we did before in the loop above, we generate the following loop:

```
for A,B,C,D,F in Class(W,CC[2][3]) join Class(W,CC[3][3])
join Class(W,CC[4][3]) join Class(W,CC[5][3]) join
Class(W,CC[6][3]) join Class(W,CC[7][3]) join
Class(W,CC[8][3]) do for E in Class(W,CC[11][3]) join
Class(W,CC[12][3]) join Class(W,CC[13][3]) join
Class(W,CC[14][3]) join Class(W,CC[15][3]) join
Class(W,CC[16][3]) do if Order(E*F) eq 3 and (A,B) eq
Id(W) and (A,C) eq Id(W) and (A,D) eq Id(W) and (B,C)
eq Id(W) and (B,D) eq Id(W) and (C,D) eq Id(W) and A^E
eq B and B^E eq C and C^E eq D and D^E eq A and A^F
eq B and B^F eq A and C^F eq C and D^F eq D and W eq
sub<W|A,B,C,D,E,F> then A,B,C,D,E,F; break; end if;
end for; end for;
```

Throughout the thesis we have wrote many progenitors involving the three types mentioned in this chapter. These presentations along with the homomorphic images they produced are presented in the next chapters.

Chapter 4

Finite Extensions

4.1 Extensions and Related Theorems

Definition 4.1. [Rot95] If K and Q are groups, then an **extension** of K by Q is a group G having a normal subgroup $K_1 \cong K$ with $G/K_1 \cong Q$.

Definition 4.2. [Rot95] If H and K are groups, then their **direct product**, denoted by $H \times K$, is the group with elements all ordered pairs (h, k) , where $h \in H$ and $k \in K$ and with operation

$$(h, k)(h', k') = (hh', kk')$$

Definition 4.3. [Rot95] A group G is a **semi-direct product** of the subgroups K by the subgroups Q , denoted by $G = K : Q$, if K is normal in G and K has a complement $Q_1 \cong Q$.

Definition 4.4. [Rot95] A **central extension** of K by Q is an extension G of K by Q with $K \leq Z(G)$.

Definition 4.5. A **mixed extension** combines the properties of both a semi-direct product and central extension, where $G = NK$ and N is a normal subgroup of a group G but is not central.

Definition 4.6. [Rot95] A **normal series**

$$G = H_0 \geq H_1 \geq \cdots \geq H_m = 1$$

is a refinement of a normal series

$$G = H_0 \geq H_1 \geq \cdots \geq H_m = 1$$

if G_0, G_1, \dots, G_n is a subsequence of H_0, H_1, \dots, H_m .

Definition 4.7. [Rot95] A **composition series** is a normal series

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

in which, for all i either G_{i+1} is a maximal normal subgroup of G_i or $G_{i+1} = G_i$.

Theorem 4.8. Jordan Hölder

[Rot95] Every two composition series of a group G are equivalent.

Proof. Since every composition series are normal series, then every two composition series of G have equivalent refinements. Now every composition series is a normal series with maximal length. A refinement repeats several of the factors so its new factor group have order 1. Thus, two composition series of G are equivalent. \square

Definition 4.9. [Rot95] If G has a composition series, then the factor groups of this series are called the **composition factors** of G

Definition 4.10. [Rot95] If $K \leq G$, then a **(right) transversal** of K in G is a subset T of G consisting of one element from each right coset of K in G .

We now wish to express how groups can be represented as many different extensions. We will first show this in simpler examples and expand this idea to the composition factors of homomorphic images of some progenitors.

4.1.1 Simple Extension Examples

Example 4.11. Prove the following and write a presentation in each case.

- (a) $C_8 \cong C_2^\bullet C_4$ and $C_8 \cong C_4^\bullet C_2$
 (b) $(C_4 \times C_2) \cong C_2^\bullet C_2^2$ and $C_2^3 \cong (C_2 \times C_2^2)$

Proof. (Proof of part a) Now we know

$C_8 = \langle (1, 2, 3, 4, 5, 6, 7, 8) \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$. We say that C_8 is an extension of C_2 by C_4 where C_2 is a normal subgroup of C_8 . We let

$N = C_2 = \langle a^4 \rangle = \{e, (1, 5)(2, 6)(3, 7)(4, 8)\}$, and a presentation for $N = \langle a | a^2 = 1 \rangle$.

Using the idea of factor sets we then compute the right cosets of N in G . Thus,

$G/N = \{N, N(1, 2, 3, 4, 5, 6, 7, 8), N(1, 3, 5, 7)(2, 4, 6, 8), N(1, 8, 7, 6, 5, 4, 3, 2, 1)\}$. We then

let the element $(1, 2, 3, 4, 5, 6, 7, 8) = b$, which will act as the second generator of our final presentation. Now we note that $b^4 = a$. We have now written C_4 as products of elements

from the center which corresponds to the definition of a central extension written above.

Therefore, our final presentation is $C_2^\bullet C_4 = \langle a, b | a^2, b^4 = a \rangle$.

Now to show that $C_8 \cong C_4^\bullet C_2$, we again recognize that

$C_8 = \langle (1, 2, 3, 4, 5, 6, 7, 8) \rangle = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$. However, in this case we find that the central element is of order 4. Let the central element be represented as

$a = (1, 3, 5, 7)(2, 4, 6, 8)$ and a presentation for the central element be:

$N = \langle a | a^4 = 1 \rangle = \{e, (1, 3, 5, 7)(2, 4, 6, 8), (1, 5)(3, 7)(2, 6)(3, 7)(4, 8),$

$(1, 7, 5, 3)(2, 8, 6, 4)\}$. Computing the right cosets of G over N gives the following: $G/N =$

$\{N, N(1, 2, 3, 4, 5, 6, 7, 8)\}$. We let $b = (1, 2, 3, 4, 5, 6, 7, 8)$ and notice that $b^2 = a$. There-

fore, our final presentation is $C_4^\bullet C_2 = \langle a, b | a^4, b^3 = a \rangle$.

(Proof of part b) Let $G = C_4 \times C_2$ be an extension of $N = C_2$ by $H = C_2^2$. Now a

presentation for $G = \langle a, b | a^4, b^2, (a, b) \rangle$ and the permutation representation of G is

given by the following set:

$$\{e, (1,3)(2,6)(4,7)(5,8), (1,4,5,2)(3,7,8,6), (1,7,5,6)(2,3,4,8), (1,8)(2,7)(3,5)(4,6), \\ (1,2,5,4)(3,6,8,7), (1,5)(2,4)(3,8)(6,7), (1,6,5,7)(2,8,4,3)\}.$$

Since we are writing a central extension, where our center is $N = C_2$, we let

$N \cong C_2 = \langle a | a^2 = 1 \rangle = \langle (1, 5)(2, 4)(3, 8)(6, 7) \rangle$.

Computing the right transversals of G/N we get

$G/N = \{N, N(1, 2, 5, 4)(3, 6, 8, 7), N(1, 3)(2, 6)(4, 7)(5, 8), N(1, 6, 5, 7)(2, 8, 4, 3)\}$. Now our H is generated by two elements, say b and c . Let $b = (1, 2, 5, 4)(3, 6, 8, 7)$ and $c = (1, 3)(2, 6)(4, 7)(5, 8)$. Notice that $b^2 = a$ and $c^2 = e$. Now the element in question is $(1, 6, 5, 7)(2, 8, 4, 3)$. This element can be expressed in terms of a and b . Thus, $(1, 6, 5, 7)(2, 8, 4, 3) = (b * c)$. Then squaring this element we get $(b * c)^2 = a$. Thus, our final presentation of $C_2^3 C_2^2$ is

$$\langle a, b, c | a^2, b^2 = a, c^2, (b * c)^2 = a \rangle.$$

Now show that $C_2^3 \cong (C_2^2 \times C_2)$.

Let $G = C_2^3 = \langle a, b, c | a^2, b^2, c^2, (a, b), (a, c), (b, c) \rangle$. The elements of G are given as:

$$\{e, (1, 7)(2, 8)(3, 4)(5, 6), (1, 6)(2, 4)(3, 8)(5, 7), (1, 5)(2, 3)(4, 8)(6, 7), (1, 4)(2, 6)(3, 7)(5, 8), \\ (1, 3)(2, 5)(4, 7)(6, 8), (1, 8)(2, 7)(3, 6)(4, 5), (1, 2)(3, 5)(4, 6)(7, 8)\}.$$

Now $N = C_2$ and its extended by $H = C_2^2$. A presentation for $N = \langle a | a^2 = 1 \rangle = \langle (1, 7)(2, 8)(3, 4)(5, 6) \rangle$. Then, $G/N = \{N, N(1, 2)(3, 5)(4, 6)(7, 8), N(1, 3)(2, 5)(4, 7)(6, 8), N(1, 5)(2, 3)(4, 8)(6, 7)\}$. We let $(1, 2)(3, 5)(4, 6)(7, 8)$, $(1, 3)(2, 5)(4, 7)(6, 8)$, $(1, 5)(2, 3)(4, 8)(6, 7)$ be representatives for the elements b, c, d , respectively. We see that, $(1, 2)(3, 5)(4, 6)(7, 8)^{(1, 3)(2, 5)(4, 7)(6, 8)} = (1, 2)(3, 5)(4, 6)(7, 8) \implies b^c = b$ or (b, c) . Also, $(1, 7)(2, 8)(3, 4)(5, 6)^{(1, 2)(3, 5)(4, 6)(7, 8)} = (1, 7)(2, 8)(3, 4)(5, 6) \implies (a, b)$ and $(1, 7)(2, 8)(3, 4)(5, 6)^{(1, 3)(2, 5)(4, 7)(6, 8)} = (1, 7)(2, 8)(3, 4)(5, 6) \implies (a, c)$. Note $(b * c) = d$, thus for a more efficient presentation of G , we can omit d from our final presentation. Thus our final presentation is given as $(C_2 \times C_2^2) = \langle a, b, c | a^2, b^2, c^2, (b, c), (a, b), (a, c) \rangle$. \square

Lets do one more example with a much larger group.

Example 4.12. We will prove that the subgroup G of S_{24} , given below, is a central extension of \mathbb{Z}_2 by A_5 and we will find a presentation for G .

$$\begin{aligned} S &:= \text{Sym}(24); \\ a &:= S!(1, 2, 5, 4)(3, 6, 8, 7)(9, 13, 11, 14)(10, 15, 12, 16)(17, 19, 18, 20)(21, 24, 23, 22); \\ b &:= S!(1, 3, 2)(4, 5, 8)(6, 9, 10)(7, 11, 12)(13, 16, 17)(14, 15, 18)(19, 21, 22)(20, 23, 24); \\ G &:= \text{sub} \langle S|a, b \rangle; \end{aligned}$$

Proof. Since the group is so large, we use MAGMA for many of the calculations. However, the process in which we show the isomorphism is the same as above. First we let

$G := \text{sub} \langle S|a, b \rangle$ and then we compute the center by using the command, $C := \text{Center}(G)$. MAGMA then returns

$(1, 5)(2, 4)(3, 8)(6, 7)(9, 11)(10, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 23)(22, 24)$, as our cen-

ter. We then create a group with this being the generator, i.e $N := \text{sub} \langle G|C \rangle$; We then compute the quotient group of G/N by doing the following:

$q, ff := \text{quo} \langle G|N \rangle$; We see that the quotient group, labeled as q , is generated by the elements $(1, 2)(3, 4)$, and $(1, 3, 2)(4, 5, 6)$, also referred to as $q.1$ and $q.2$ respectively. Now the generators of G , in this case a and b , are given as $G.1$ and $G.2$. Since we are showing we have a central extension of \mathbb{Z}_2 by A_5 , we essentially want to express elements of q in terms of our a and b . So we compute the following:

```
> q;
Permutation group q acting on a set of cardinality 6
Order = 60 = 2^2 * 3 * 5
  (1, 2)(3, 4)
  (1, 3, 2)(4, 5, 6)
> ff(G.1^2);
Id(q)
> ff(G.1^3);
(1, 2)(3, 4)
> ff(G.2^3);
Id(q)
> ff((G.1*G.2)^5);
Id(q)
```

□

Allowing $G.1 = a, G.2 = b$ and the central element to be represented as c , we can write our presentation for a central extension of \mathbb{Z}_2 by A_5 . Let

$H \langle a, b, c \rangle := \text{Group} \langle a, b, c \mid a^2 = c, b^3 = c, (a * b)^5 = c, c^2 \rangle$. We then check this isomorphism with MAGMA.

```
> H<a,b,c>:=Group<a,b,c|a^2=c,b^3=c,(a*b)^5=c,c^2>;
> #H;
120
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(G,H1);
> s;
true
```

Now we would like to apply these concepts to some more difficult groups. We will consider the composition series from the progenitor $2^{*4} : S_4$. Consider the group $G \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^4, y^2, (x * y)^3, t^2, (t, y), (t, y^x)(x^3 * t * t^x * t^y)^3 \rangle$; We have MAGMA compute the composition factors of G .

```
> CompositionFactors(G1);
G
|  Cyclic(2)
*
|  Alternating(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
*
|  Cyclic(5)
1
>
```

Through the use of the Jordan-Holder Theorem, G can be decomposed in the following fashion:

$$G \supset G_1 \supset G_2 \cdots \supset G_5, \text{ where } G = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5) = (G/C_2)(C_2/A_5)(A_5/C_5)(C_5/C_5)(C_5/1)$$

We wish to solve this extension problem. We continue by finding the normal subgroup lattice of G . Using *MAGMA* to compute the normal lattice of G we have,


```

> NL:=NormalLattice(G1);
> NL;
Normal subgroup lattice
-----
[1] Order 1      Length 1 Maximal Subgroups:
---
[2] Order 125    Length 1 Maximal Subgroups: 1
---
[3] Order 7500   Length 1 Maximal Subgroups: 2
---
[4] Order 15000  Length 1 Maximal Subgroups: 3

```

We first note that G does not have a center and thus cannot be written as a central extension. Through inspection it would seem as though $NL[2]$ could be generated by a direct product of three cyclic groups of order 5. This happened to be the case. During this part of our research we assumed that our extension problem would be $C_5^3 : S_5$, after writing a presentation for this group and asking MAGMA if both $C_5^3 : S_5$ and our G were isomorphic we found that this was not the case. After some time we then noticed there was not a subgroup of order 60 inside of $NL[3]$ that could extend us to from $NL[2]$ to $NL[3]$. This gave us the inclination that we have a mixed extension. Now MAGMA has a database for perfect groups so if we have a perfect group we can use MAGMA'S presentation for that group.

Definition 4.13. [Rot95] If $a, b \in G$, the *commutator* of a and b , denoted by $[a, b]$, is

$$[a, b] = aba^{-1}b^{-1}$$

The *commutator subgroup* (or *derived subgroup*) of G , denoted by G' , is the subgroup of G generated by all the commutators.

Definition 4.14. [Rot95] If $G = G'$, where G' denotes the derived group, then G is said to be *perfect*.

So, we ask MAGMA for the derived group of $G1$ and we get,

```

> D:=DerivedGroup(G1);
> #D;
7500
> IsPerfect(D);
true
> DerivedGroup(D) eq D;
true

```

This shows us that $NL[3]$ is a perfect group. We need to find a presentation for this, by doing the following:

```

> DB := PerfectGroupDatabase();
> "A5" in TopQuotients(DB);
> ExtensionPrimes(DB, "A5");
{ 2, 3, 5, 7, 11, 19 }
Above we wanted to see a 5 since we have  $5^3$ .
> ExtensionExponents(DB, "A5", 5);
{ 3, 4, 5, 6 }
Above we wanted to see a 3, since we have  $5^3$ 
> ExtensionNumbers(DB, "A5", 5, 3);
{ 1, 2 }
Above tells us that we have 2 presentations for the perfect group.
Let H1 is the first one.
> H1:=Group(DB, "A5", 5, 3,1);
> H1;
Finitely presented group H1 on 5 generators Relations
a^2 = Id(H1)
b^3=Id(H1)
(a * b)^5 = Id(H1)
x^5 = Id(H1)
y^5 = Id(H1)
z^5 = Id(H1)
(x, y) = Id(H1)
(x, z) = Id(H1)
(y, z) = Id(H1)
a^-1 * x * a * z^-1 = Id(H1)
a^-1 * y * a * y = Id(H1)
a^-1 * z * a * x^-1 = Id(H1)
b^-1 * x * b * z^-1 = Id(H1)
b^-1 * y * b * z^-1 * y = Id(H1)
b^-1 * z * b * z^-1 * y^2 * x^-1 = Id(H1)
> P1:=PermutationGroup(DB, "A5", 5, 3,1);
> s:=IsIsomorphic(NL[3],P1);
> s;
false
This shows that H1 was not the presentation that we needed.
So we try H2.
> P2:=PermuationGroup(DB,"A5",5,3,2);
> s:=IsIsomorphic(NL[3],P2);
> s;
true
> H2:=Group(DB, "A5", 5, 3,2);

```

```

> H2;
Finitely presented group H1 on 5 generators Relations
a^2 = Id(H2)
b^3=Id(H2)
a * b * a * b * a * b * a * b * a * b * z^-1 = Id(H2)
x^5 = Id(H2)
y^5 = Id(H2)
z^5 = Id(H2)
(x, y) = Id(H2)
(x, z) = Id(H2)
(y, z) = Id(H2)
a^-1 * x * a * z^-1 = Id(H2)
a^-1 * y * a * y = Id(H2)
a^-1 * z * a * x^-1 = Id(H2)
b^-1 * x * b * z^-1 = Id(H2)
b^-1 * y * b * z^-1 * y = Id(H2)
b^-1 * z * b * z^-1 * y^2 * x^-1 = Id(H2)

```

Notice the difference between $H1$ and $H2$. In the presentation of $H2$, we have the generator of A_5 , in this case a and b , as a product of an element of the abelian group. Then, at the bottom of the presentation we have, $a^{-1} * x * a * z^{-1} = Id(H2)$, which is equivalent to $x^a = z$. This expresses how a acts on our abelian group, a property of semi direct products.

Recall that mixed extension combines the properties of both semi-direct products and central extensions. This is clear in the presentation of $H2$. Now we can use this presentation above for $NL[3]$ and then extend it by the *CyclicGroup*(2) and we would be done. However, we will try and find this presentation ourselves. There are two things we need to answer. The first being how to find the representation of the two elements that generate A_5 in $NL[2]$ and second how to express those generators as products of elements of the generators of $NL[2]$. We start by factoring $NL[3]$ by $NL[2]$ and generating the quotient group q .

```

> q,ff:=quo<NL[3]|NL[2]>;
q is the name of this quotient group while ff is the corresponding
mapping.
> A:=q.1;
> B:=q.2;
> ff(NL[3].1) eq A;
true
> ff(NL[3].2) eq B;

```

```

true
> T:=Transversal(NL[3],NL[2]);
Note this decomposing NL[3] into right coset of NL[2]
> ff(T[2]) eq A;
true
> ff(T[3]) eq B;
true
> (A*B)^5;
Id(q)
(T[2]*T[3])^5 in NL[2];
true
> (T[2]*T[3])^5;

```

We have now found the two elements that generate A_5 and that their product lies in $NL[2]$. Using the Schreier System for $NL[2]$, we can convert the above elements of $NL[2]$ and then we can also see how those elements act on the generators of $NL[2]$. Consider the following:

```

> X:=NL[2].2;
> Y:=NL[2].3;
> Z:=NL[2].4;
> N:=sub<G1|X,Y,Z>;
> NN<k,1,m>:=Group<k,1,m|k^5,1^5,m^5,(k,1),(k,m),(1,m)>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..125]];
> for i in [2..125] do
for> P:=[Id(N):1 in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=X; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=X^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=Y; end if;
for|for> if Eltseq(Sch[i])[j] eq -2 then P[j]:=Y^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=Z; end if;
for|for> if Eltseq(Sch[i])[j] eq -3 then P[j]:=Z^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;

> for i in [1..125] do if ArrayP[i] eq (T[2]*T[3])^5 then
Sch[i]; end if; end for;

```

```

k * m^-2
> A:=[Id(NN) : i in [1..6]];
> for i in [1..125] do if X^T[2] eq ArrayP[i] then A[1]:=Sch[i];
Sch[i]; end if; end for;
1
> for i in [1..125] do if X^T[3] eq ArrayP[i] then A[2]:=Sch[i];
Sch[i]; end if; end for;
m * k^-2 * l^-2
> for i in [1..125] do if Y^T[2] eq ArrayP[i] then A[3]:=Sch[i];
Sch[i]; end if; end for;
k
> for i in [1..125] do if Y^T[3] eq ArrayP[i] then A[4]:=Sch[i];
Sch[i]; end if; end for;
l * k^-1
> for i in [1..125] do if Z^T[2] eq ArrayP[i] then A[5]:=Sch[i];
Sch[i]; end if; end for;
k^2 * l^2 * m^-1
> for i in [1..125] do if Z^T[3] eq ArrayP[i] then A[6]:=Sch[i];
Sch[i]; end if; end for;
m
> A;
[ 1, m * k^-2 * l^-2, k, l * k^-1, k^2 * l^2 * m^-1, m ]
> NN<a,b,k,l,m>:=Group<a,b,k,l,m|k^5,l^5,m^5,(k,l),(k,m),(1,m),
a^2,b^3,(a*b)^5=k * m^-2,k^a=1, k^b=m*k^-2*l^-2,l^a=k,
l^b=l*k^-1,m^a=k^2*l^2*m^-1,m^b=m>;
> #NN;
7500
> N1:=CosetAction(NN,sub<NN|Id(NN)>);
> f1,N1,k1:=CosetAction(NN,sub<NN|Id(NN)>);
> #N1;
7500
> s:=IsIsomorphic(N1,NL[3]);
> s;
true

```

This completes the isomorphism type of this group. We now consider some more interesting homomorphic images that resulted from the progenitor $2^{*4} : S_4$.

4.1.2 Extension Problems Related to the Progenitor $2^{*4} : S_4$

The table presented below is a table of groups that we found from the progenitor $2^{*4} : S_4$. We will solve some of the extension problems of these groups.

a	b	c	d	e	Order(G1)	G
0	0	0	0	4	240	$2^{\bullet}S_6$
0	0	0	3	4	120	S_6
0	0	2	0	8	672	$2^{\bullet}PGL(2, 7)$
0	5	10	0	10	241920	$6^{\bullet} : (PSL(3, 4)) : C_2$
0	0	4	0	10	28800	$2^{\bullet}((A_5 \times A_5) : C_2)$
7	10	0	0	0	24360	$PGL(2, 29)$
9	7	0	0	0	178920	$PSL(2, 71)$

Consider the first group listed in the table whose symmetric presentation is given as:

$$G \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^4, y^2, (x * y)^3, t^2, (t, y), (t, y^x), (x * y * t)^4 \rangle;$$

The composition factors of the group are given as:

```
> CompositionFactors(G1);
G
| Cyclic(2)
*
| Alternating(5)
*
| Cyclic(2)
1
```

We can decompose G as follows:

$G = (G/G1)(G1/G2)(G2/G3)(G4/G5) = (G/C_2)(C_2/A_5)(A_5/C_2)(C_2/1)$. Now we wish to solve this extension problem by first considering the normal lattice of G . The normal lattice of G is given as,

Normal subgroup lattice

```

[7] Order 240 Length 1 Maximal Subgroups: 4 5 6
---
[6] Order 120 Length 1 Maximal Subgroups: 3
[5] Order 120 Length 1 Maximal Subgroups: 2 3
[4] Order 120 Length 1 Maximal Subgroups: 3
---
[3] Order 60 Length 1 Maximal Subgroups: 1
---
[2] Order 2 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:

```

We then use MAGMA to give us a computer based proof of the construction of this group. We continue by finding that our group contains a central element of order two. Since a center of a group is always normal and within the normal lattice we see that the second group in our normal lattice is of order 2, we proceed to ask if that subgroup is equal to the center, and this is indeed true. At this point we are continuing in such a way that this group is a central extension. Thus, we then factor G by the center and name this group q . Now our job is to solve the extension problem of q . The composition factors of q are given as:

```

> NL[2] eq Center(G1);
true
> q,ff:=quo<G1|NL[2]>;
> CompositionFactors(q);
  G
  | Cyclic(2)
  *
  | Alternating(5)
  1

```

From previous knowledge of extension, when we extend an alternating group by a order 2 element, we often get a symmetric group. Now q would seem to be isomorphic to S_5 . We check our assumption by first writing a presentation for S_5 and then asking MAGMA if our group S_5 is isomorphic to our group q .

```

> H<a,b>:=Group<a,b|a^2,b^4,(a*b)^5,(a,b)^3>;

```

```

> f2,H1,k2:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,q);
> s;
true

```

Now that we have our presentation for q , the only thing left to accomplish is to see which, elements of q can be written in terms of the central element. Note that we say central element, since our center consists of a single element of order two. As before, we compute the right transversals of the center in our group G . Now we run the following loops in MAGMA to see if we can write any elements our current presentation in terms of the center. Let A, B be the first and second generators of our S_5 respectively, and let C represent the central element. (For completeness, one can check the appendix for the entire code).

```

> for i in [0..2] do if A^2 eq C^i then i; break;
end if;end for;
0
> for i in [0..2] do if B^4 eq C^i then i; break;
end if;end for;
0
> for i in [0..2] do if (A*B)^5 eq C^i then i; break;
end if;end for;
0
> for i in [0..2] do if (A,B)^3 eq C^i then i; break;
end if;end for;
0
> for i in [0..2] do if B^8 eq C^i then i; break; end if;
end for;
0
> for i in [0..2] do for j in [0..4] do if A^C eq A^i*B^j then i;
break; end if;end for;
1 0
> for i in [0..2] do for j in [0..4] do if B^C eq A^i*B^j then i;
break; end if;end for;
0 1

```

The last two lines of our code imply that the central element commutes with both of the generators of S_5 , a property that we already know from the definition of the center. At last we can write our final presentation for G and see if our efforts give us the isomorphism type of G .


```

> H<a,b,c>:=Group<a,b,c|a^2,b^4,(a*b)^5,(a,b)^3,c^2,(c,a),(c,b)>;
> #H;
240
> f2,H1,k2:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,G1);
> s;
true

```

Now consider the group whose isomorphism type is presented as $2 \bullet PGL(2, 7)$. The symmetric presentation of this group is given as $G \langle x, y, t \rangle := Group \langle x, y, t | x^4, y^2, (x * y)^3, t^2, (t, y), (t, y^x), (x * y * t)^8 \rangle$. The composition factors of the group are given as,

```

CompositionFactors(G1);
G
| Cyclic(2)
*
| A(1, 7)           = L(2, 7)
*
| Cyclic(2)
1

```

As in the group above, this group also has a center of order two. Thus, we factor our group by this central element, and solve the extension problem of the new group which we routinely label as q . The composition factors of q can be decomposed as $q = (G/G1)(G1/G2)(G2/G3) = (G/C_2)(C_2/PSL(2, 7))(PSL(2, 7)/1)$. Our assumption is that our q is isomorphic to $PGL(2, 7)$ and we could proceed by using the well known presentation of $PGL(2, 7)$. However, we tried this approach and it was not the case. Thus we use the presentation that is stored in MAGMA for $PSL(2, 7)$ and we will then extend this by an element of order two, and thus expressing q as a semi-direct product.

The normal lattice for q is given below:

```

Normal subgroup lattice(q)
-----

```

```

[3] Order 336 Length 1 Maximal Subgroups: 2
---
[2] Order 168 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:
>FPGGroup(nl[2]);

```

Finitely presented group on 2 generators

Relations

```
$.1^2 = Id($)  
$.2^4 = Id($)  
$.2^-2 * $.1 * $.2^2 * $.1 * $.2^2 * $.1 = Id($)  
($.1 * $.2^-1)^7 = Id($)
```

$\$.1$, $\$.2$ represent the first and generators of $nl[2]$ respectively. We then create a presentation for $nl[2]$ using a and b as the generators.

```
> H<a,b>:=Group<a,b|a^2,b^4,(a*b)^7,b^-2*a*b^2*a*b^2*a>;  
> #H;  
168  
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);  
> s,t:=IsIsomorphic(H1,NL2[2]);  
> s;  
true
```

We are currently at $nl[2]$ of the normal subgroup lattice, now we need an element of order 2 that is not in $nl[2]$ that sends us to $nl[3]$ which will give us a presentation for our q . We do so by using the subgroups of q in the following way.

```
> for i in [1..#S] do if #S[i]‘subgroup eq 2 and S[i]‘subgroup.1  
notin nl[2] then i; end if; end for;  
3  
> S[3]‘subgroup.1;  
(1, 8)(2, 5)(3, 4)(6, 24)(9, 27)(10, 19)(11, 17)(12, 18)(13, 16)  
(15, 21)(22, 25)(23, 28)  
> Z1:=q!(1, 8)(2, 5)(3, 4)(6, 24)(9, 27)(10, 19)(11, 17)(12, 18)  
(13, 16)(15, 21)(22, 25)(23, 28);
```

Now that we have our element that will extend us to $nl[3]$, we need to see how this element acts on the elements that generate $PSL(2, 7)$, since we are expressing this extension as a semi-direct product. To do this effectively, we use the Schreier System of our presentation of $PSL(2, 7)$ where A, B represent the generators of this group.

```
> N:=sub<q|A,B>;  
> NN<i,j>:=Group<i,j|i^2,j^4,(i*j)^7,j^-2*i*j^2*i*j^2*i>;  
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);  
> ArrayP:=[Id(N): i in [1..#N]];  
> for i in [2..#N] do  
for> P:=[Id(N): l in [1..#Sch[i]]];  
for> for j in [1..#Sch[i]] do
```

```

for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
for|for> if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..#N] do if ArrayP[i] eq A^Z1 then print Sch[i];
for|if> end if; end for;
i^j
This shows that a^z=a^b
> for i in [1..#N] do if ArrayP[i] eq B^Z1 then print Sch[i];
for|if> end if; end for;
j^-1
This shows that b^z=b^-1

```

This completes the extension problem for q . However we still need to see if we can express the products of elements of q in terms of the central element. We find that this is not the case, but we do know that the central element commutes with all the generators of q and this is enough to show the isomorphism type of G .

```

> H<a,b,c,d>:=Group<a,b,c,d|a^2,b^4,(a*b)^7,b^-2*a*b^2*a*b^2*a,
c^2,a^c=a^b,b^c=b^-1,d^2,(d,a),(d,b),(d,c)>;
> #H;
672
> f3,H3,k3:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H3,G1);
> s;
true

```

Now that we are comfortable with semi-direct products, central extensions, and direct products, we will see how they all come together to find the isomorphism type of the group presented above given as $2^\bullet((A_5 \times A_5) : C_2)$. This will be the last composition factor that we show for this group.

The symmetric presentation of G is given as:

$$G \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^4, y^2, (x*y)^3, t^2, (t, y), (t, y^x), (x*y*t*t^x*t^y)^4, (x*y*t)^{10} \rangle$$

As before, we compute the normal lattice of G . We also find that G has a center of order two which again happens to be equal to the second group in the normal lattice. However, instead of factoring out by the center, we factor out by the largest abelian subgroup which happens to be $NL[5]$. In this case we are continuing by assuming that we have a mixed extension. It will not be until we compute the action of the generators of $NL[5]$ on the elements of q to determine which type of extension we ultimately have. We then factor this subgroup and find the isomorphism type of the remaining group q . Now the composition factors of q and normal subgroup lattice of q is given as:

```
> CompositionFactors(q);
G
| Cyclic(2)
*
| Alternating(5)
*
| Alternating(5)
1
Normal subgroup lattice
-----
[3] Order 7200 Length 1 Maximal Subgroups: 2
---
[2] Order 3600 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:
```

Observing the composition factors and the normal lattice together, we assume that the second normal subgroup listed might be the direct product of both of the Alternating(5) groups listed. We check this in the next step.

```
> H<a,b,c,d>:=Group<a,b,c,d|a^2,b^3,(a*b)^5,c^2,d^3,(c*d)^5,
(a,c),(a,d),(b,c),(b,d)>;
> #H;
3600
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,NL[2]);
> s;
true
```

Note that we used $\text{Alt}(5) = \langle x^2, y^3, (xy)^5 \rangle$ and we used two copies

of then commuted all the elements since we have a direct product.

Lastly, we need to find an element of order 2 that's not in our current normal subgroup that will send us to the last normal subgroup and thus completing this extension problem of q .

Now we look for the element of order 2 in the third normal subgroup of q .

```
> for z1 in NL[3] do if Order(z1) eq 2 and z1 notin NL[2]
and NL[3] eq sub<q|NL[2],z1> then Z1:=z1;break;
end if; end for;
> Z1;
(1, 29)(2, 14)(3, 5)(4, 34)(6, 35)(7, 17)(8, 30)(9, 27)
(10, 20)(12, 22)(13, 16)(18, 31)(19, 23)(24, 26)(32, 33)
> N:=sub<q|A,B,C,D>;
We then need to see how this element acts on elements
(Alt(5) X Alt(5)), since we are using the properties of a
semi-direct product.
> NN<a,b,c,d>:=Group<a,b,c,d|a^2,b^3,(a*b)^5,c^2,d^3,(c*d)^5,
(a,c),(a,d),(b,c),(b,d)>;
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..#N]];
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..#N]];
> for i in [2..#N] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
for|for> if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
for|for> if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
for|for> if Eltseq(Sch[i])[j] eq -4 then P[j]:=D^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..#N] do if ArrayP[i] eq A^Z1 then print Sch[i];
for|if> end if; end for;
c * d * c * d^-1 * c * d * c * d^-1 * c
>
> for i in [1..#N] do if ArrayP[i] eq B^Z1 then print Sch[i];
```

```

for|if> end if; end for;
d * c * d^-1 * c * d * c
> for i in [1..#N] do if ArrayP[i] eq C^Z1 then print Sch[i];
for|if> end if; end for;
a^b
> for i in [1..#N] do if ArrayP[i] eq D^Z1 then print Sch[i];
for|if> end if; end for;
b^-1 * a * b * a * b^-1 * a * b^-1

```

Using this above information we write our final presentation

```

for $q$.
> H<a,b,c,d,e>:=Group<a,b,c,d,e|a^2,b^3,(a*b)^5,c^2,d^3,
(c*d)^5,
(a,c),(a,d),(b,c),(b,d),e^2,a^e=c*d*c*d^-1*c*d*c*d^-1*c,
b^e=d*c*d^-1*c*d*c,c^e=a^b,d^e=b^-1*a*b*a*b^-1*a*b^-1>;
> f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H2,NL[3]);
> s;
true

```

To complete the isomorphism type of G , we wish to find how the elements of q can be written in term of the central element. Without loss of generality we wish to use the presentation for q that MAGMA has stored. So, the presentation for q is

$$H \langle a, b, c, d, e \rangle := \text{Group} \langle a, b, c, d, e \mid a^2, b^3, (a * b)^5, c^2, d^3, (c * d)^5, (a, c), (a, d), (b, c), (b, d), e^2, a^e = c * d * c * d^{-1} * c * d * c * d^{-1} * c, b^e = d * c * d^{-1} * c * d * c, c^e = a^b, d^e = b^{-1} * a * b * a * b^{-1} * a * b^{-1} \rangle;$$

Now we compute the right transversals of the the subgroup $NL[5]$ and then see if we can write any of the elements of q in terms of the generators of $NL[5]$. Note that if we have a mixed extension it will contain the properties of both a semi-direct product and a central extension. We let E, F be the generators of $NL[5]$ and compute the following:

Now we have checked that the second transversal is equal to the generator a in our presentation for q . We have done similar checked for b and d as well. So, we label A, B, D accordingly.

```

> A:=T[2];
> B:=T[3];
> D:=T[4];
> for i,j in [0..2] do if (T[3]*T[4])^2 eq C.1^i*C.2^j then i,j;
end if;end for;
0 0
> for i,j in [0..2] do if (A)^2 eq C.1^i*C.2^j then i,j;

```

```

end if;end for;
> for i,j in [0..2] do if (A)^4 eq C.1^i*C.2^j then i,j;
end if;end for;
0 0
> for i,j in [0..2] do if (B)^2 eq C.1^i*C.2^j then i,j;
end if;end for;
0 0
> for i,j in [0..2] do if (D)^2 eq C.1^i*C.2^j then i,j;
end if;end for;
0 0
> for i,j in [0..2] do if (B * A^-1)^3 eq C.1^i*C.2^j then i,j;
end if;end for;
0 0
>
> for i,j in [0..2] do if
A^-1 * D * A * D * A * B * A^-1 * D * A^-1 * D * A * D
eq C.1^i*C.2^j then i,j; end if;end for;
1 0
> for i,j in [0..2] do if (D * A^-1)^15 eq C.1^i*C.2^j then i,j;
end if;end for;
0 1
Now using the above information we find a possible
presentation for G
> H<i,j,k,l,m>:=Group<i,j,k,l,m|i^4,j^2,k^2,(j * k)^2,
(j * i^-1)^3,(i * k * i^-1 * j)^2,i^-1 * k * i * k * i * j
* i^-1 * k * i^-1 * k * i * k=1,(k * i^-1)^15=m,l^2,m^2,
(m,l),(l,i),(l,j),(l,k),(m,i),(m,j),(m,k)>;
Notice that we made the generators of NL[5] commute with all
the generators. If this is the presentation for G then we have
shown that this is a central extension rather than mixed, since
H does not possess properties of a semi-direct product.
> #H;
28800
> #G1;
28800
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,G1);
> s;
true

```

This solves the extension problem for G , and so G is $2^{2^{\bullet}}((A_5 \times A_5) : C_2)$.

We have an understanding of semi-direct product, central, and direct extension problems. Lets take a look at the mixed extension $6^{\bullet} : (PSL(3,4) : C_2)$. The symmetric

presentation of this group is:

$G \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^4, y^2, (x * y)^3, t^2, (t, y), (t, y^x), (x * t * t^x)^5, (x * y * t * t^x * t^y)^{10}, (x * y * t)^{10} \rangle$; Computing the composition factors of the group we get

```
> CompositionFactors(G1);
G
| Cyclic(2)
*
| A(2, 4)                = L(3, 4)
*
| Cyclic(2)
*
| Cyclic(3)
1
```

We can decompose G such that $G = (G/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)$
 $= (G/C_2)(C_2/PSL(3, 4))(PSL(3, 4)/C_2)(C_2/C_3)(C_3/1)$. Now we wish to solve this extension problem by first considering the normal lattice of G . The normal lattice of G is given as

Normal subgroup lattice

```
-----
[6] Order 241920 Length 1 Maximal Subgroups: 5
---
[5] Order 120960 Length 1 Maximal Subgroups: 4
---
[4] Order 6 Length 1 Maximal Subgroups: 2 3
---
[3] Order 3 Length 1 Maximal Subgroups: 1
[2] Order 2 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:
```

We continue in our normal fashion by using MAGMA to tell us whether or not G has a center. In fact, G does have a center and it is equivalent to the second normal subgroup listed in the lattice. However, we wish to take a more efficient approach in solving this extension. As we have seen in the beginning of the chapter involving extensions, many groups can be expressed as several different extension types. Therefore we set out to express this group as a mixed extension. Recall, that mixed extensions possess properties of both semi-direct products and central extensions, and it must also

be noted that mixed extensions are formed upon extension of a normal abelian group. We find that the subgroup of order 6 listed in the normal lattice is the largest abelian subgroup of G . Now we also find that subgroup is generated by the second and third normal subgroup. We find the isomorphism type of this normal abelian subgroup of order 6 is a direct product of the normal subgroup of order 3 and the normal subgroup of order 2. A presentation for this group is written as $H \langle d, e \rangle := \langle d, e \mid d^3, e^2(d, e) \rangle$. We will return to the importance of this presentation towards the end of the extension problem. Now we factor the subgroup of order 6, $NL[6]$, from G , and again label it q . As before, our concern now is the isomorphism type of q . Thus we look at the composition factors and the normal lattice of q . So,

```
> CompositionFactors(q);
  G
  | Cyclic(2)
  *
  | A(2, 4)                = L(3, 4)
```

Normal subgroup lattice

```
[3] Order 40320 Length 1 Maximal Subgroups: 2
---
[2] Order 20160 Length 1 Maximal Subgroups: 1
---
[1] Order 1      Length 1 Maximal Subgroups:
```

Since q does not have a center, then it clearly cannot be written as a central extension. From the normal lattice we see that we don't have a normal subgroup of order 2 which cancels out the possibility for expressing q as a direct product. Thus we continue by writing q as a semi-direct product. From experience solving extension problems, we assume that q is possibly isomorphic to $PGL(3, 4)$. However this is not the case. Note that $PSL(3, 4)$ is a group of order 20160. We use [WB99] as reference for a presentation for $PSL(3, 4)$ and then use MAGMA to tell us if our presentation is correct. Hence,

```

> H<a,b>:=Group<a,b|a^2,b^4,(a*b)^7,(a*b^2)^5,(a*b*a*b^2)^7,
(a*b*a*b*a*b^2*a*b^-1)^5>;
> f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H1,NL[2]);
> s;
true

```

Following our assumption that q is a semi-direct product of $PSL(3,4)$ by C_2 we need to find an element of order 2 that lies in q but not in $NL[2]$ but also produces q . We do so by using the following loop.

```

for z1 in NL[3] do if Order(z1) eq 2 and z1 notin NL[2] and NL[3]
eq sub<q|NL[2],z1> then Z1:=z1;break; end if; end for;

```

Once we have stored this element, we use the Schreier System to see how this element acts on the generators of $NL[2]$, hence a and b . The results are shown below:

```

> for i in [1..#N] do if ArrayP[i] eq A^Z1 then print Sch[i];
end if; end for;
a*b*a*b^-1*a*b*a*b*a*b^-1*a*b^-1*a*b*a*b^-1*a*
b^2 * a * b^-1 * a * b^2
> for i in [1..#N] do if ArrayP[i] eq B^Z1 then print Sch[i];
end if; end for;
a * b * a * b^-1 * a * b * a * b^-1 * a * b^-1 * a * b * a *
b * a * b^-1

```

We enter this information into a new presentation, which we name as H , and use MAGMA to verify that this presentation is isomorphic to q .

```

H<a,b,c>:=Group<a,b,c|a^2,b^4,(a*b)^7,(a*b^2)^5,
(a*b*a*b^2)^7,(a*b*a*b*a*b^2*a*b^-1)^5,c^2,a^c=a*b*a*b^-1
*a*b*a*b*a*b^-1*a*b^-1*a*b*a*b^-1*a*b^2*a*b^-1*a*b^2,b^c
=a*b*a*b^-1*a*b*a*b^-1*a*b^-1*a*b*a*b*a*b^-1>;
> f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H2,q);
> s;
true

```

Now that we have the isomorphism type of q , we must find how the elements of the normal abelian group act on q . To do so, we find the transversals of $NL[4]$ in $G1$. At this point we need to find a permutation representation of the elements a, b , and c that we have used in our presentation. Note that $f2$, in the above code is a mapping that

sends a to a permutation that is written on the number of letters that is equal to the order of H . In addition, in the code $s, t := IsIsomorphic(H2, q)$; s represents a boolean value while t is a mapping that sends elements of $H2$ to elements of $q \subset G$. We run the following loops to make sure we have the exact permutation elements that correspond to our presentation.

```
T:=Transversal(G1,NL[4]);
> A:=t(f2(a));
> B:=t(f2(b));
> C:=t(f2(c));
> for i in [1..#T] do if ff(T[i]) eq A then i; end if;end for;
16890
> for i in [1..#T] do if ff(T[i]) eq B then i; end if;end for;
6838
> for i in [1..#T] do if ff(T[i]) eq C then i; end if;end for;
14720
> A:=T[16890];
> B:=T[6838];
> C:=T[14720];
```

We now have the permutations that we desire. As mentioned above, we know that $NL[4]$ is a direct product of $NL[2]$ by $NL[3]$, where $NL[2]$ is generated by an element of order 2 and $NL[3]$ is generated by an element of order 3. Thus, we need to find such elements that correspond to the presentation, $H < d, e > := < d, e | d^3, e^2(d, e) >$. Writing a mixed extension requires us to know whether we can express any of the elements in our presentation for q in terms of our normal abelian subgroup and also how the generators of that subgroup act on the generators of $NL[4]$. Therefore we run the following loops.

```
> for d,e in NL[4] do if Order(e) eq 2 and Order(d) eq 3 and e^d
eq e then D:=d; E:=e; end if; end for;
> sub<G1|E,D> eq NL[4];
true
> for i in [0..3] do for j in [0..2] do if A^2 eq D^i*E^j then
i,j; break; end if;end for; end for;
0 0
This tells us that a^2 is just equal to the identity of G.
> for i in [0..3] do for j in [0..2] do if B^4 eq D^i*E^j then
i,j; break; end if;end for; end for;
0 0
This tells us that b^4 is just equal to the identity of G.
> for i in [0..3] do for j in [0..2] do if (A*B)^7 eq D^i*E^j
```

```

    then i,j; break; end if;end for; end for;
1 1
This tells us that  $(a*b)^7=d*e$ 
> for i in [0..3] do for j in [0..2] do if  $(A*B^2)^5$  eq  $D^iE^j$ 
  then i,j; break; end if;end for; end for;
0 0
This tells us that  $(a*b^2)^5$  is just equal to the identity of G.
> for i in [0..3] do for j in [0..2] do if  $(A*B*A*B^2)^7$  eq
 $D^iE^j$  then i,j; break; end if;end for; end for;
1 0
This tells us that  $(a*b*a*b^2)^7=d$ 
> for i in [0..3] do for j in [0..2] do if
 $(A * B * A * B * A * B^2 * A * B^{-1})^5$  eq  $D^iE^j$  then i,j;
break; end if;end for
for|for> ; end for;
2 0
This tells us that  $(a*b*a*b*a*b^2*a*b^{-1})^5=d$ 
> for i in [0..3] do for j in [0..2] do if  $C^2$  eq  $D^iE^j$  then
i,j; break; end if;end for; end for;
0 0
This tells us that  $c^2$  is just equal to the identity of G.
> for i in [0..3] do for j in [0..2] do if  $A^C$  eq  $D^iE^j$  then
i,j; break; end if;end for; end for;
> for i in [0..3] do for j in [0..2] do if  $B^C$  eq  $D^iE^j$  then
i,j; break; end if;end for; end for;
> for i,k in [0..2] do for j in [0..4] do if  $A^D$  eq  $A^iB^jC^k$ 
then i,j,k; break; end if; end for;end for;
1 0 0
This implies that a commutes with d.
> for i,k in [0..2] do for j in [0..4] do if  $A^E$  eq  $A^iB^jC^k$ 
then i,j,k; break; end if; end for; end for;
1 0 0
This implies that a commutes with e.
> for i,k in [0..2] do for j in [0..4] do if  $B^E$  eq  $A^iB^jC^k$ 
  then i,j,k; break; end if; end for; end for;
0 1 0
This implies that b commutes with e.
> for i,k in [0..2] do for j in [0..4] do if  $B^D$  eq  $A^iB^jC^k$ 
then i,j,k; break; end if; end for; end for;
0 1 0
This implies that b commutes with d.
> for l in [0..3] do for i,k,m in [0..2] do for j in [0..4] do
if  $C^D$  eq  $A^iB^jC^kD^lE^m$  then i,j,k,l,m; break; end if;
end for; end for;

```

```

for> end for;
0 0 1 2 0
This implies that  $c^d = c*d^2$ 
> for i,k in [0..2] do for j in [0..4] do if C^E eq A^i*B^j*C^k
then i,j,k; break; end if; end for;
for> end for;
0 0 1
This implies that  $c$  commutes with  $e$ .

```

Note, the second to the last loop reassures us that we do indeed have a mixed extension. If a, b , and c had commuted with both d and e then we would have had a central extension. Finally, we check to see if we have accomplished our goal by entering in the above information into our presentation and using MAGMA to see if the two groups are isomorphic. Thus we have,

```

> H<a,b,c,d,e>:=Group<a,b,c,d,e|d^3,e^2,(d,e),a^2,b^4,(a*b)^7=
d*e,(a*b^2)^5,(a*b*a*b^2)^7=d,(a*b*a*b*a*b^2*a*b^-1)^5=d^2,
c^2,a^c=a*b*a*b^-1*a*b*a*b*a*b^-1*a*b^-1*a*b*a*b^-1*a*
b^2*a*b^-1*a*b^2,b^c=a*b*a*b^-1*a*b*a*b^-1*a*b^-1*a*b*a
*b*a*b^-1,a^d=a,a^e=a,b^d=b,b^e=b,c^d=c*d^2,c^e=c>;
> #H;
241920
> #G1;
241920
> f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
> s,t:=IsIsomorphic(H2,G1);
> s;
true

```

This completes the isomorphism type of G .

Chapter 5

Progenitors with Isomorphism Types

5.1 $7^{*3} :_m S_3$

$G\langle x, y, t \rangle := \text{Group}\langle x, y, t \mid x^3, y^2, (x*y)^2, t^7, t^x = t^2, (y*t)^i, (x*t*t^x*t^{x^2})^j, (y*t^2)^k, (x*y*t^3)^l \rangle;$

Table 5.1: $7^{*3} :_m S_3$

i	j	k	l	G
0	0	0	4	$6 \times PSL(2, 7)$
0	0	0	3	$PSL(2, 7)$
5	6	7	14	A_7
15	0	15	5	J_1

5.2 $2^{*9} : D_{18}$

$G\langle x, y, t \rangle := \text{Group}\langle x, y, t \mid x^{-9}, y^2, (x^{-1}y)^2, t^2, (t, y*x), (x^3t)^i, (x^4t^x)^j, (y*t)^k, (x*t*t^{x^2}*t)^l \rangle;$

Table 5.2: $2^{*9} : D_{18}$

i	j	k	l	G
0	0	3	5	$PSL(2, 19)$
0	16	4	2	$2 \times PSL(2, 17)$
0	3	6	0	$((2 \times PSL(2, 19)) : 2)$
0	9	7	2	$PSL(2, 8)$
2	0	7	9	$PSL(2, 71)$
2	0	7	10	$PGL(2, 29)$
2	0	7	11	$PSL(2, 43)$
2	0	8	7	$PGL(2, 41)$
2	0	13	4	$PGL(2, 27)$
2	9	18	4	$PGL(2, 17)$
2	10	9	0	$6 \times PSL(2, 19)$
2	10	10	10	$2^\bullet(U(3, 4) : 2)$
2	10	15	5	J_1
2	10	13	6	$(U(3, 4) : 2)$
2	11	12	6	$(M_{12} : 2)$
2	12	15	5	J_2
2	13	7	0	$PSL(2, 13)$
2	13	8	8	$PGL(3, 3)$

5.3 $2^{*6} : ((C_3 \times C_3) : C_2)$

$G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^2, b^3, c^3, b^a = b, c^a = c^2, c^b = c, t^2, (t, b^{-1}c^{-1}), (a*t)^i, (a*b^{-1}c^{-1}*t)^j, (c^{-1}*t^a*t)^k, (b^{-1}c^{-1}*t^a)^l, (a*b*c*t*t^c*t)^m \rangle;$

Table 5.3: $2^{*6} : ((C_3 \times C_3) : C_2)$

i	j	k	l	m	G
0	0	0	3	11	M_{12}
0	0	5	3	0	$3^\bullet S_8$
0	0	5	3	10	S_8

5.4 $2^{*7} : (C_7 : C_3)$

$G\langle a, b, t \rangle := \text{Group}\langle a, b, t \mid a^3, b^7, b^a = b^2, t^2, (t, a),$
 $(a * b * a^{-1} * t * t^{(b^2)})^i, (a^{-1} * t^b * t * t^b)^j, (b * a * t^{(b^{-1})})^k,$
 $(a * b^{-1} * a^{-1} * t * t^{(b^2)} * t^{(b^{-1})})^l$
 $\rangle;$

Table 5.4: $2^{*7} : (C_7 : C_3)$

i	j	k	l	G
0	2	0	0	$U(3, 3)$
0	0	5	0	$4 \times M_{22}$
0	7	5	8	M_{22}

5.5 $2^{*5} : ((C_5 : C_2) : C_2)$

$G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^2 = b, b^2, c^5, b^a = b, c^a = c^2,$
 $c^b = c^4, t^2, (t, a), (t, b), (b * c * t)^i, (a * c^{-1} * a * t)^j, (c * t)^k,$
 $(c^2 * t * t^{c^2})^l, (a^{-1} * t * t^{c * t})^m \rangle;$

Table 5.5: $2^{*5} : ((C_5 : C_2) : C_2)$

i	j	k	l	m	G
0	0	6	6	12	$(4 \times M_{12}) : 2$
0	0	7	0	7	$2^\bullet SZ(8)$
0	0	7	13	7	$SZ(8)$
0	0	8	5	7	$2^\bullet(PSL(3, 4) : 2)$

5.6 $2^{*12} : (2^2 : 3)$

$G\langle x, y, z, t \rangle := \text{Group}\langle x, y, z, t \mid x^3, y^2, z^2, y^x = z, \\ z^x = y * z, t^2, (x * t)^a, (x^2 * y * z * t^y)^b, \\ (y * x * z * x^2 * t * t^y)^c, ((x * y * z)^a * t * t^z)^d, (t * (t^z))^e \rangle;$

Table 5.6: $2^{*12} : (2^2 : 3)$

a	b	c	d	e	f	G
0	2	3	0	4	8	$PGL(2, 7)$
0	2	4	5	5	6	$((2^2 \times U(3, 5)) : 3) : 2$
2	0	3	7	0	0	$PSL(2, 13) \times PGL(2, 7)$
2	7	0	6	5	3	J_1
2	7	8	6	0	3	$PSL(2, 97)$
2	8	4	6	6	0	$3^\bullet : (((4 \times PSL(3, 3)) : 3) : 2)$
2	9	0	0	3	3	$PSL(2, 37)$
2	10	5	6	5	3	$6 \times M_{12}$

5.7 $2^{*11} : D_{22}$

$G\langle x, y, t \rangle := \text{Group}\langle x, y, t \mid x^{11}, y^2, (x^{-1} * y)^2, t^2, (t, y * x), \\ (y * t^{(x^{-1})})^i, (x * t)^j, (x^2 * t * t^y)^k, (x^5 * t)^l, \\ (x * t * t^y * t^x)^m \rangle;$

Table 5.7: $2^{*11} : D_{22}$

i	j	k	l	m	G
0	12	6	4	2	$(2 \times 11)^\bullet : (PGL(2, 11))$
0	5	5	0	2	$2 \times PSL(2, 89)$

5.8 $2^{*10} : D_{20}$

$G\langle x, y, t \rangle := \text{Group}\langle x, y, t \mid x^{10}, y^2, (x^{-1}y)^2, t^2, (t, y*x), (y*x^2*t)^i, (x^5*t)^j, (y*t*t^x*t^{(x^2)})^k, (y*t^{(x^2)})^l, (x*y*t^x)^m, (x^2*t)^n \rangle;$

Table 5.8: $2^{*10} : D_{20}$

i	j	k	l	m	n	G
0	0	0	2	4	6	$2 \times PGL(2, 11)$
0	0	6	2	4	7	$PGL(2, 29)$
0	0	0	3	4	5	$PGL(2, 16)$
0	0	0	3	6	3	$2^\bullet((A_5 \times A_5) : 2)$
0	0	0	3	4	4	$S_3 : PGL(2, 25)$
0	0	2	3	12	10	$(6 \times PSL(2, 11)) : 2$
0	0	2	4	7	10	$(2 \times A_7) : 2$
0	0	2	10	5	5	$(2 \times A_5 \times PSL(2, 7)) : 2$
0	0	3	2	0	5	$2^\bullet PSL(2, 59)$
0	0	3	2	7	0	$2^\bullet PSL(2, 29)$
0	0	6	2	4	7	$PSL(2, 29)$

5.9 $2^{*14} : D_{28}$

$G\langle x, y, t \rangle := \text{Group}\langle x, y, t \mid x^{14}, y^2, (x^{-1}y)^2, t^2, (t, y*x), (x*t)^i, (x*y*t^x)^j, (x^6*t)^k, (x^5*t)^l \rangle;$

Table 5.9: $2^{*14} : D_{28}$

i	j	k	l	G
0	0	4	3	$3^* : (((PSL(2, 7) \times PGL(2, 13)) : 2) : 2)$
0	6	4	3	$S_3 : PGL(2, 13)$
8	8	3	4	$2^\bullet((PSL(2, 7) \times PSL(2, 7) : 2) : 2)$
0	12	3	4	$6^\bullet : PGL(2, 10)$
3	4	15	14	$PGL(2, 29)$
3	8	3	7	$PGL(2, 7)$
3	9	3	7	$PSL(2, 8)$
3	13	3	7	$PSL(2, 13)$
4	8	3	8	$2^\bullet PGL(2, 49)$
4	3	0	0	$2^\bullet PSL(2, 43)$
7	0	4	3	$2^\bullet PSL(2, 13)$

5.10 $2^{*11} : L_2(7)$

```
G<a,b,t>:=Group<a,b,t|a^5,b^3,b*a^2*b^-1*a^2*b*a^-1*b^-1*a^-1,
(a*b*a*b*a)^2, t^2, (t,a^2 * b^-1 * a^-1 * b * a^-2),
(t,a^-1 * b * a^2 * b * a^2), (t,b * a^-2 * b * a^-1 * b * a),
(t, a * b^-1 * a * b^-1 * a^-1 * b * a),
(b*t*t^b*t^b^2)^i, (a*b^-1*t^b*a^4)^j, (b^-1*a^-1*t*t^a*t)^k,
(b^2*a*t*t^(a*b^-1)*t^a^4)^l, (a*t)^m,
(a^2 * b * a^-1 * b * a^-1 * b^-1 * a*t)^n>;
```

Table 5.10: $2^{*11} : L_2(7)$

i	j	k	l	m	n	G
0	0	0	6	0	4	$2^{11} : PSL(2, 11)$
0	0	0	5	19	5	J_1

5.11 $3^{*3} :_m S_4$

```
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^3,(t,x^2*y),t^y=t^2,
(y*t)^i,((x^2*y)^2*t^(x^3))^j,(y*x^2*t^x)^k, (y*t^x)^l,
(y*x*t^x)^m>;
```

Table 5.11: $3^{*3} :_m S_4$

i	j	k	l	m	G
2	0	0	0	5	$PGL(2, 11)$
0	2	6	5	13	$PSL(2, 25)$
0	0	12	6	8	$PGL(3, 3)$
0	2	6	10	7	$3^\bullet(A_7 : 2)$
0	2	6	0	12	$(3 \times PSL(3, 5)) : 2$
0	0	6	10	13	$2^\bullet PSL(2, 25)$
0	2	14	14	6	$(3 \times PSL(2, 13)) : 2$

5.12 $2^{*7} : (7 : 3)$

$G\langle a, b, t \rangle := \text{Group}\langle a, b, t \mid a^3, b^7, b^a = b^2, t^2, (t, a), (a * b * a^{-1} * t * t^{(b^2)})^i, (a^{-1} * t^b * t * t^b)^j, (b * a * t^{(b^{-1})})^k, (a * b^{-1} * a^{-1} * t * t^{(b^2)} * t^{(b^{-1})})^l \rangle;$

Table 5.12: $2^{*7} : (7 : 3)$

i	j	k	l	G
0	0	5	0	$6 \bullet M_{22}$
0	0	5	8	$4 \bullet M_{22}$
0	2	8	6	$U(3, 3) : 2$
0	7	5	8	M_{22}

5.13 $2^{*7} : ((7 : 3) : 2)$

$G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^2, b^3, c^7, b^a = b, c^a = c^6, c^b = c^2, t^2, (t, a), (t, b), (a * b * c^{-1} * b^{-1} * t)^i, (a * c * t)^j, (c * t^c * t^{(c^2)} * t^{(c * b^{-1})})^k, (a * b * t^c)^l, (b^{-1} * c * t)^m, (b * c * b^{-1} * t * t^{c^4 * t})^n \rangle;$

Table 5.13: $2^{*7} : ((7 : 3) : 2)$

i	j	k	l	m	n	G
0	0	0	7	7	6	J_2

5.14 $7^{*2} :_m D_{18}$

$G\langle x, y, t \rangle := \text{Group}\langle x, y, t \mid x^{-9}, y^2, (x^{-1} * y)^2, t^7, t^x = t^2, (y * t)^i, (y * t^x)^j, (y * t^{(x^2)})^k, (x * t)^l, (x * t * t^x * t)^m \rangle;$

Table 5.14: $7^{*2} :_m D_{18}$

i	j	k	l	m	G
0	0	3	0	3	$PSL(2, 7)$
0	0	4	9	0	$S_3 \times PSL(2, 7)$
0	5	7	9	12	A_7
0	6	7	3	9	$2^6 : PSL(2, 7)$
5	15	5	3	3	J_1
6	6	6	0	3	$(3 : PGL(3, 4)) : 2$

5.15 $2^{*10} : (2^\bullet(5 : 2) : 2)$

$G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^4, b^2, c^5, b^a = b, c^a = c^2, c^b = c^4, t^2, (t, a), (a * t)^i, (t * t^b)^j, (t * a * t)^k, (t^c * t)^l, (c * t^a)^m, (b * t * c)^n \rangle;$

Table 5.15: $2^{*10} : (2^\bullet(5 : 2) : 2)$

i	j	k	l	m	n	G
4	1	0	9	8	6	$(4 \times M_{12}) : 2$

5.16 $2^{*6} : (3^\bullet(3 : 2))$

$G\langle a, b, c, t \rangle := \text{Group}\langle a, b, c, t \mid a^2, b^3, c^3, b^a = b, c^a = c^2, c^b = c, t^2, (t, b^{-1} * c^{-1}), (a * t)^i, (a * b^{-1} * c^{-1} * t)^j, (c^{-1} * t^a * t)^k, (b^{-1} * c^{-1} * t^a)^l, (a * b * c * t * t^c * t)^m \rangle;$

Table 5.16: $2^{*6} : (3^\bullet(3 : 2))$

i	j	k	l	m	G
0	0	20	3	9	$4^\bullet : ((A_5)^3 : 3)$
0	0	0	3	10	$3^\bullet(A_7 : 2)$
0	0	6	3	11	M_{12}
0	0	5	3	10	$A_7 : 2$

Chapter 6

Manual Double Coset Enumeration

6.1 Definition for Double Coset Enumeration

Definition 6.1. Double Coset

[Cur07] Let H and K be subgroups of the group G and define a relation on G as follows:

$$x \sim y \iff \exists h \in H \text{ and } k \in K \text{ such that } y = hxk$$

where \sim is an equivalence relation and the equivalence classes are sets of the following form

$$HxK = \{hxk \mid h \in H, k \in K\} = \cup_{k \in K} Hxk = \cup_{h \in H} hxk$$

Such a subset of G is called a **double coset**.

Now we consider the double cosets of the form NxN , where $x = \pi w$ for some $n \in N$ and w is a reduced word in the t'_i s. Thus $NxN = N\pi wN = NwN = [w]$.

Definition 6.2. [Cur07] Let G be a group of permutations of a set S . For each $g, s \in S$, let $g^s = g$, then we call the set of $s \in S$ the **point stabilizer** of $g \in G$.

Definition 6.3. [Cur07] The **coset stabilizing group** of a coset Nw is defined as

$$N^{(w)} = \{\pi \in N \mid Nw\pi = Nw\}$$

where $n \in N$ and w a reduced word in the t'_i s.

Theorem 6.4. Number of single cosets in NwN [Cur07] The above definition gives,

$$\begin{aligned} N^{(w)} &= \{\pi \in N \mid Nw\pi = Nw\} \\ &= \{\pi \in N \mid (Nw)^\pi = Nw\} \\ &= N \cap N^w \end{aligned}$$

The number of single cosets in NwN is given by $[N : N^{(w)}]$

Definition 6.5. Let G be a group of permutations of a set S . For each s in S , let $\text{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\text{orb}_G(s)$ is a subset of S called the **orbits** of s under G . We use $|\text{orb}_G(s)|$ to denote the number of elements in $\text{orb}_G(s)$.

6.2 Double Coset Enumeration $2^{*5} : A_5$

We factor the progenitor $2^{*5} : A_5$ by a single relator, $t_3t_4t_1t_2t_5 = e$, and let $G \cong 2^{*5} : A_5$ be a symmetric presentation of G given by:

$$\langle x, t, y \mid x^2, y^3, (xy)^5, t^2, (t, x), (t, xyx^{?1}, xy^{?1}), (xyt)^5 \rangle$$

where $N \cong A_5 = \langle x, y \mid x^2, y^3, (xy)^5 \rangle$ and $x \sim (1, 2)(3, 4)y \sim (1, 3, 5)$. Our relation is $t_3t_4t_1t_2t_5 = e$. First, we are going to rearrange our relation,

$$\begin{aligned} t_3t_4t_1t_2t_5 &= e \\ \implies t_3t_4t_1 &= t_5t_2 \\ \implies t_3t_4 &= t_5t_2t_1 \end{aligned}$$

We will begin the manual double coset enumeration by looking at our first double coset. Note the definition of a double coset is as follows: $NwN = \{Nwn \mid n \in N\}$. For our first double coset we have, $NeN = \{Nen \mid n \in N\} = \{N\}$ denoted by $[*]$, which contains one single coset. N is transitive on $\{1, 2, 3, 4, 5\}$, so it has a single orbit $\{1, 2, 3, 4, 5\}$. Take a representative from the orbit, say 5, and find to which double coset Nt_5 belongs. This will create a new double coset, which we will label as $[5]$. Note $Nt_5N = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5\}$. Now consider the coset stabilizer $N^{(5)}$. Note that the coset stabilizer of Nt_5 is equal to the point stabilizer N^5 .

Thus $N^{(5)} = \langle (1, 2)(3, 4), (2, 3, 4), (1, 3, 4) \rangle$ and the number of the single cosets in the double coset, Nt_5N , is at most: $\frac{|N|}{|N^{(5)}|} = \frac{60}{12} = 5$. Looking at the generators of $N^{(5)}$, we can see that the orbits of $N^{(5)}$ on $\{1, 2, 3, 4, 5\}$ are $\{1, 2, 3, 4\}$ and $\{5\}$. We take a representative from each orbit, say 1 and 5 respectively and then determine to which double coset Nt_5t_5 , and Nt_5t_1 belong. All the t 's have order two, thus $Nt_5t_5 = N \in [*]$. Since the orbit $\{5\}$ contains one element, then one symmetric generator goes back to the double coset $[*]$. Now, $Nt_5t_1 \in Nt_5t_1N$ is a new double coset we have yet to see, therefore this will be our new double coset. Note four symmetric generators go to the next double coset which we will label [51].

We then consider the coset stabilizer of $N^{(51)}$. $N^{(51)} = N^{51} = \langle (2, 3, 4) \rangle$ and the number of single cosets in the double coset, Nt_5t_1N , is at most: $\frac{|N|}{|N^{(51)}|} = \frac{60}{3} = 20$. In order to find the distinct single cosets in [51], we must find the right cosets of $N^{(51)}$ in N . Without the loss of generality, they are $Nt_5t_1(e)$, $Nt_5t_1(1, 2)(3, 4)$, $Nt_5t_1(1, 3, 5)$, $Nt_5t_1(1, 2, 3, 4, 5)$, $Nt_5t_1(1, 4, 3, 5, 2)$, $Nt_5t_1(1, 5, 3)$, $Nt_5t_1(2, 4, 5)$, $Nt_5t_1(1, 2, 5, 3, 4)$, $Nt_5t_1(1, 5, 4, 3, 2)$, $Nt_5t_1(1, 3, 2, 5, 4)$, $Nt_5t_1(2, 3)(4, 5)$, $Nt_5t_1(1, 4, 2, 3, 5)$, $Nt_5t_1(1, 5, 2, 4, 3)$, $Nt_5t_1(1, 3, 2, 5, 4)$, $Nt_5t_1(1, 4, 2, 5, 3)$, $Nt_5t_1(1, 5)(2, 3)$, $Nt_5t_1(1, 4, 2)$, $Nt_5t_1(3, 4, 5)$, $Nt_5t_1(1, 3, 4)$, $Nt_5t_1(1, 2)(4, 5)$. Taking a representative from each of the cosets, we form the set of transversals, say T . Then, $T = \{(e), (1, 2)(3, 4), (1, 3, 5), (1, 2, 3, 4, 5), (1, 4, 3, 5, 2), (1, 5, 3), (2, 4, 5), (1, 2, 5, 3, 4), (1, 5, 4, 3, 2), (1, 3, 2, 5, 4), (2, 3)(4, 5), (1, 4, 2, 3, 5), (1, 5, 2, 4, 3), (1, 3, 2, 5, 4), (1, 4, 2, 5, 3), (1, 5)(2, 3), (1, 4, 2), (3, 4, 5), (1, 3, 4), (1, 2)(4, 5)\}$.

Conjugating the coset Nt_5t_1 by each of the elements in the set T , we get the other distinct cosets in Nt_5t_1N . Thus we will have the following cosets in the double coset [51]: $Nt_1t_3, Nt_5t_3, Nt_4t_1, Nt_5t_4, Nt_2t_4, Nt_1t_4, Nt_3t_4, Nt_3t_1, Nt_1t_2, Nt_5t_2, Nt_4t_2, Nt_2t_1, Nt_5t_1, Nt_1t_5, Nt_2t_5, Nt_3t_5, Nt_4t_3, Nt_3t_2, Nt_2t_3, Nt_4t_5$. Looking at the generators of $N^{(51)}$, we can see that the orbits of $N^{(51)}$ on $\{1, 2, 3, 4, 5\}$ are $\{2, 3, 4\}$, $\{1\}$ and $\{5\}$. We take a representative from each orbit, say $\{2\}$, $\{1\}$, and $\{5\}$ respectively and then determine to which double coset $Nt_5t_1t_2$, $Nt_5t_1t_1$ and $Nt_5t_1t_5$ belong. Since our t 's have order 2, $Nt_5t_1t_1 = N \in [5]$. The orbit containing 1 only has one symmetric generator which will be sent back to that double coset [5]. We have yet to see $Nt_5t_1t_5 \in Nt_5t_1t_5N$, a new double coset which we will label as [515]. The coset $Nt_5t_1t_2$ requires further investigation. Our relation is $t_3t_4t_1 = t_5t_2$, and to obtain all of the relations we conjugate our

relation by N . However, we want to know specifically where the coset $Nt_5t_1t_2$ goes. The conjugation of $t_3t_4t_1 = t_5t_2$ by $(1, 2, 3, 5, 4) \in N$ gives

$$t_3t_4t_1^{(1,2,3,5,4)} = t_5t_2^{(1,2,3,5,4)} \implies t_5t_1t_2 = t_1t_3. \text{ This implies } Nt_5t_1t_2 = Nt_1t_3 \in [51].$$

Therefore the coset $Nt_5t_1t_2$ will loop back to [51]. Since there are three symmetric generators in the orbit that contains 2, three symmetric generators will loop back into the double coset [51]. Continuing with our new double coset [515], we will compute the coset stabilizing group $N^{(515)}$. Note that the coset stabilizing group

$$N^{(515)} = N^{(51)} = \langle (2, 3, 4) \rangle. \text{ Although, this increases with our relation. We then con-}$$

jugate our new coset $Nt_5t_1t_5$ by all elements of N . It gives us a list of 20 single cosets, and

the set is given as follows: $\{Nt_1t_5t_1, Nt_2t_4t_2, Nt_3t_5t_3, Nt_2t_1t_2, Nt_3t_1t_3, Nt_4t_2t_4, Nt_5t_3t_5,$

$Nt_5t_4t_5, Nt_1t_4t_1, Nt_1t_2t_1, Nt_5t_1t_5, Nt_1t_3t_1, Nt_4t_5t_4, Nt_2t_5t_2, Nt_3t_4t_3, Nt_5t_2t_5, Nt_3t_2t_3,$

$Nt_2t_3t_2,$

$Nt_4t_3t_4\}$. We find that $Nt_5t_1t_5 = Nt_2t_1t_2 = Nt_3t_1t_3 = Nt_4t_1t_4$. These relations will

increase the elements in our coset stabilizer, since

$$Nt_5t_1t_5^{(2,4,5)} = Nt_2t_1t_2 = Nt_5t_1t_5N \implies (2, 4, 5) \in N^{(515)}$$

$$Nt_5t_1t_5^{(2,4,3)} = Nt_5t_1t_5 = Nt_5t_1t_5N \implies (2, 4, 3) \in N^{(515)}$$

$$Nt_5t_1t_5^{(2,3)(4,5)} = Nt_4t_1t_4 = Nt_5t_1t_5N \implies (2, 3)(4, 5) \in N^{(515)}$$

$$Nt_5t_1t_5^{(3,4,5)} = Nt_3t_1t_3 = Nt_5t_1t_5N \implies (3, 4, 5) \in N^{(515)}$$

Thus, $N^{(515)} \geq \langle (2, 4, 3), (2, 4, 5), (2, 5)(3, 4), (2, 3, 5) \rangle = \{e, (2, 4, 3), (2, 5)(3, 4), (2, 3, 5), (3, 4, 5), (2, 4, 5), (2, 5, 4), (2, 3, 4), (3, 5, 4), (2, 4)(3, 5), (2, 5, 3), (2, 3)(4, 5)\}$.

The number of single cosets in the double coset, $Nt_5t_1t_5N$, is at most $\frac{|N|}{|N^{(515)}|} = \frac{60}{12} = 5$. In

order to find the different cosets in [515], we find the right cosets of $N^{(515)}$ in N . The right

cosets are as follows: $Nt_5t_1t_5(e), Nt_5t_1t_5(1, 2)(3, 4), Nt_5t_1t_5(1, 3, 5), Nt_5t_1t_5(1, 4, 3, 5, 2),$

$Nt_5t_1t_5(1, 5, 3)$. Taking a representative from each of the cosets, we form the set of

transversals, $T = \{(e), (1, 2)(3, 4), (1, 3, 5), (1, 4, 3, 5, 2), (1, 5, 3)\}$. Conjugating the coset

$Nt_5t_1t_5N$ by each of the elements in the set T , we get the other distinct cosets in

$Nt_5t_1t_5N$. Thus we will have the following cosets in the double coset [515] with their

equal names: $515 \sim 212 \sim 313 \sim 414$

$$\begin{aligned}
\{515 \sim 212 \sim 313 \sim 414\}^{(1,2)(3,4)} &= \{525 \sim 121 \sim 424 \sim 323\} \\
\{515 \sim 212 \sim 313 \sim 414\}^{(1,3,5)} &= \{131 \sim 232 \sim 535 \sim 343\} \\
\{515 \sim 212 \sim 313 \sim 414\}^{(1,4,3,5,2)} &= \{242 \sim 141 \sim 545 \sim 343\} \\
\{515 \sim 212 \sim 313 \sim 414\}^{(1,5,3)} &= \{353 \sim 252 \sim 151 \sim 454\}
\end{aligned}$$

Looking at the generators of $N^{(515)}$, we can see that the orbits of $N^{(515)}$ on $\{1, 2, 3, 4, 5\}$ are $\{2, 3, 4, 5\}$ and $\{1\}$. We take a representative from each orbit, say 5 and 1 respectively, and then determine to which double coset $Nt_5t_1t_5t_1$, and $Nt_5t_1t_5t_5$ belong. Since our t 's have order 2 $Nt_5t_1t_5t_5 \in Nt_5t_1 = [51]$. The orbit that contains 5 has three other symmetric generators thus a total of four symmetric generators will be sent back to the double coset $[51]$. We have yet to see $Nt_5t_1t_5t_1 \in Nt_5t_1t_5t_1N = [5151]$. Now the coset stabilizing group of $N^{(5151)} = N^{(51)} = \langle (2, 3, 4) \rangle$. However, as before this increases with our relation. Proceeding as we did with the double coset $[515]$ we conjugate the coset $Nt_5t_1t_5t_1$ by all the elements of N . It gives us a list of 20 different single cosets and they are shown below. $\{Nt_1t_5t_1t_5, Nt_2t_4t_2t_4, Nt_3t_5t_3t_5, Nt_2t_1t_2t_1, Nt_3t_1t_3t_1, Nt_4t_2t_4t_2, Nt_5t_3t_5t_3, Nt_5t_4t_5t_4, Nt_1t_4t_1t_4, Nt_1t_2t_1t_2, Nt_5t_1t_5t_1, Nt_1t_3t_1t_3, Nt_4t_5t_4t_5, Nt_2t_5t_2t_5, Nt_3t_4t_3t_4, Nt_5t_2t_5t_2, Nt_3t_2t_3t_2, Nt_2t_3t_2t_3, Nt_4t_3t_4t_3\}$. We have that, $Nt_1t_5t_1t_5 = Nt_2t_4t_2t_4 = Nt_3t_5t_3t_5 = Nt_2t_1t_2t_1 = Nt_3t_1t_3t_1 = Nt_4t_2t_4t_2 = Nt_5t_3t_5t_3 = Nt_5t_4t_5t_4 = Nt_1t_4t_1t_4 = Nt_2t_1t_2t_1 = Nt_5t_1t_5t_1 = Nt_1t_3t_1t_3 = Nt_4t_5t_4t_5 = Nt_2t_5t_2t_5 = Nt_3t_4t_3t_4 = Nt_5t_2t_5t_2 = Nt_3t_2t_3t_2 = Nt_2t_3t_2t_3 = Nt_4t_3t_4t_3$. These relations will increase the elements in our coset stabilizer, since

$$\begin{aligned}
Nt_5t_1t_5t_1^{(2,4,5)} &= Nt_2t_1t_2t_1 &= Nt_5t_1t_5t_1N &\implies (2, 4, 5) \in N^{(5151)} \\
Nt_5t_1t_5t_1^{(2,4,3)} &= Nt_5t_1t_5t_1 &= Nt_5t_1t_5t_1N &\implies (2, 4, 3) \in N^{(5151)} \\
Nt_5t_1t_5t_1^{(1,3,2,5,4)} &= Nt_4t_1t_4t_1 &= Nt_5t_1t_5t_1N &\implies (1, 3, 2, 5, 4) \in N^{(5151)} \\
Nt_5t_1t_5t_1^{(1,3)(4,5)} &= Nt_4t_3t_4t_3 &= Nt_5t_1t_5t_1N &\implies (1, 3)(4, 5) \in N^{(5151)} \\
Nt_5t_1t_5t_1^{(2,3,5)} &= Nt_2t_1t_2t_1 &= Nt_5t_1t_5t_1N &\implies (2, 3, 5) \in N^{(5151)} \\
Nt_5t_1t_5t_1^{(1,3,5,4,2)} &= Nt_4t_3t_4t_3 &= Nt_5t_1t_5t_1N &\implies (1, 3, 5, 4, 2) \in N^{(5151)}
\end{aligned}$$

Thus $N^{(5151)} \geq \langle (2, 4, 3), (2, 4, 5), (2, 5)(3, 4), (2, 3, 5), (1, 3, 2, 5, 4), (1, 3, 5, 4, 2), (1, 3)(4, 5) \rangle$. The number of single cosets in the double cose $Nt_5t_1t_5t_1N$, is at most $\frac{|N|}{|N^{(5151)}|} = \frac{60}{60} = 1$. Also note that there is only one single orbit of $N^{(5151)}$ on $\{1, 2, 3, 4, 5\}$, which is $\{1, 2, 3, 4, 5\}$. If we take a representative from the orbit say $\{1\}$, we can see that $Nt_5t_1t_5t_1t_1 \in Nt_5t_1t_5 = [515]$. Thus five symmetric generators take us back. Finally, our Cayley diagram is as follows.

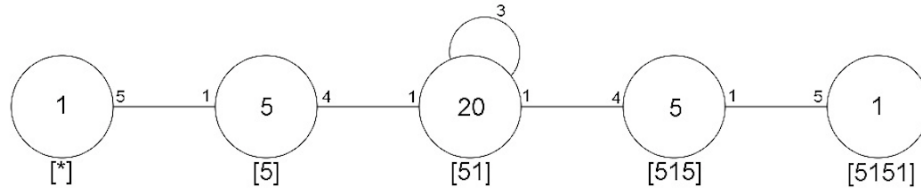


Figure 6.1: Cayley Diagram of $2^*5 : A_5$

6.3 Finding the Center of the Cayley Diagram

From the diagram above, it is clear that G contains a center. We first gather the stabilizer of a coset in G that fixes another coset at a maximum distance from the first. The blocks of imprimitivity are of size two, thus $|Z(G)| = 2$ where $Z(G) = \langle nw \rangle$. Let $\psi : G \rightarrow S_{32}$ and let $G = \langle \psi(x), \psi(y), \psi(t_5) \rangle$. Given G is a G -set, then by definition a block is a subset B of G such that, for each $g \in G$, either $gB = B$ or $gB \cap B = \emptyset$. Suppose that $\{1, 32\}$ is a block of G . From above, $\{1, 32\}^{\psi(x), \psi(y), \psi(t_5)} = \{1, 32\}$ or $\{1, 32\}^{\psi(x), \psi(y), \psi(t_5)} = \emptyset$:

$$\begin{aligned}
 & \{1, 32\}^{(3,5)(4,7)(6,9)(8,12)(10,17)(11,18)(13,22)(14,24)(15,25)(16,27)(19,28)(20,30)(21,26)(23,31)} \\
 &= \{1, 32\} \\
 & \{1, 32\}^{(2,3,4)(6,10,11)(8,13,14)(9,15,16)(12,20,21)(18,27,26)(19,29,23)(22,25,30)} = \{1, 32\} \\
 & \{1, 32\}^{(1,2)(3,6)(4,8)(5,9)(7,12)(10,17)(11,19)(13,23)(14,24)(15,26)(16,20)(18,28)(21,25)(22,31)(27,30)(29,32)} \\
 &= \{2, 29\}
 \end{aligned}$$

Now consider the block $\{2, 29\}$ and proceed as we have above. Then,

$$\begin{aligned}
& \{2, 29\}^{(3,5)(4,7)(6,9)(8,12)(10,17)(11,18)(13,22)(14,24)(15,25)(16,27)(19,28)(20,30)(21,26)(23,31)} \\
&= \{1, 32\} \\
& \{2, 29\}^{(2,3,4)(6,10,11)(8,13,14)(9,15,16)(12,20,21)(18,27,26)(19,29,23)(22,25,30)} = \{3, 23\} \\
& \{2, 29\}^{(1,2)(3,6)(4,8)(5,9)(7,12)(10,17)(11,19)(13,23)(14,24)(15,26)(16,20)(18,28)(21,25)(22,31)(27,30)(29,32)} \\
&= \{1, 32\}
\end{aligned}$$

It can also be shown that $\{3,23\}$, $\{4,19\}$, $\{5,31\}$, $\{6,13\}$, $\{7,28\}$, $\{8,11\}$, $\{9,22\}$, $\{10,14\}$, $\{12,18\}$, $\{15,25\}$, $\{16,30\}$, $\{17,24\}$, $\{20,27\}$, and $\{21,26\}$ are also blocks of G . Therefore the blocks of G are: $\{\{1,32\}, \{2,29\}, \{3,23\}, \{4,19\}, \{5,31\}, \{6,13\}, \{7,28\}, \{8,11\}, \{9,22\}, \{10,14\}, \{12,18\}, \{15,25\}, \{16,30\}, \{17,24\}, \{20,27\}, \{21,26\}\}$. Since G contains nontrivial blocks, G is imprimitive. We must now find the non-identity element, say z , of the center. From the Cayley diagram in the previous example, we note that the last double coset, $[5151]$, has one single coset which was $Nt_5t_1t_5t_1$. We then set $Nt_5t_1t_5t_1 = e \implies nt_5t_1t_5t_1 = e$ where $n \in N$, since $z \in G = \frac{2^{*5}:A_5}{t_3t_4t_1t_2t_5=e}$ and $z = nw$, where $n \in N$ and w is a word in the t 's. To factor G by the center, we need to consider $nt_5t_1t_5t_1 = e \in G$. Thus, $t_5t_1t_5t_1 = n^{-1}$. We let $m = n^{-1} \implies t_5t_1t_5t_1 = m$. We now compute m by its action on the cosets $\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5\}$. Computing the action of m on the coset Nt_1 we get:

$$\begin{aligned}
Nt_1^m &= Nt_1^{t_5t_1t_5t_1} \\
&= N(t_5t_1t_5t_1)^{-1}t_1t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_1t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5\underline{t_5t_4t_3t_2t_1}t_1t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_4t_3t_2t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_1t_5t_3t_2t_4t_4t_3t_2t_5t_1t_5t_1 \\
&= Nt_1t_3t_2t_3t_2t_5t_1t_5t_1 \\
&= Nt_1t_3t_2t_1t_4t_1t_5t_1 \\
&= Nt_1t_3t_2t_1t_4t_4t_2t_3t_5t_1t_1t_5t_1 \\
&= Nt_1t_3t_2t_1t_2t_3t_1 \\
&= Nt_1t_3t_2t_1t_5t_4 \\
&= Nt_1
\end{aligned}$$

We now compute the action of m on Nt_2 :

$$\begin{aligned}
Nt_2^m &= Nt_2^{t_5t_1t_5t_1} \\
&= N(t_5t_1t_5t_1)^{-1}t_2t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5\underline{t_2t_5t_1}t_5t_1 \\
&= Nt_1t_5t_1t_5t_4t_3t_1t_5t_1 \\
&= Nt_1t_5t_1t_1t_2t_5t_1 \\
&= Nt_1t_5t_2t_5t_1 \\
&= Nt_1t_1t_3t_4t_2t_5t_5t_2t_5t_1 \\
&= Nt_3t_4t_5t_1 \\
&= Nt_2
\end{aligned}$$

The action of m on the coset Nt_3 :

$$\begin{aligned}
Nt_3^m &= Nt_3^{t_5t_1t_5t_1} \\
&= N(t_5t_1t_5t_1)^{-1}t_3t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_3t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_2t_4t_1t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_2t_4t_5t_1 \\
&= Nt_1t_5t_1t_1t_3t_5t_1 \\
&= Nt_1t_5t_3t_5t_1 \\
&= Nt_1t_1t_4t_2t_3t_5t_5t_3t_5t_1 \\
&= Nt_4t_2t_5t_1 \\
&= Nt_3
\end{aligned}$$

The action of m on the coset Nt_4 :

$$\begin{aligned}
Nt_4^m &= Nt_4^{t_5t_1t_5t_1} \\
&= N(t_5t_1t_5t_1)^{-1}t_4t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_4t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_3t_2t_1t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_3t_2t_5t_1 \\
&= Nt_1t_5t_1t_1t_4t_5t_1 \\
&= Nt_1t_5t_4t_5t_1 \\
&= Nt_1t_1t_2t_3t_4t_5t_5t_4t_5t_1 \\
&= Nt_2t_3t_5t_1 \\
&= Nt_4
\end{aligned}$$

Next, we need to compute the action of m on the coset Nt_5 . To show this we will first

show that $t_5t_1t_5t_1 = t_1t_5t_1t_5$. Our original relation is $t_3t_4t_1t_2t_5 = e$

$$\begin{aligned}
\underline{t_3t_4t_1t_2t_5} &= e \\
\implies t_5t_2t_1t_1t_2t_5 &= e \\
\implies t_5t_1t_5t_4t_3t_1t_1t_2t_5 &= e \\
\implies t_5t_1t_5t_4t_3t_2t_5 &= e \\
\implies t_5t_1t_5t_1t_2t_5t_3t_3t_2t_5 &= e \\
\implies t_5t_1t_5t_1t_2t_5t_2t_5 &= e \\
\implies t_5t_1t_5t_1 &= t_2t_5t_2t_5
\end{aligned}$$

Thus,

$$\begin{aligned}
Nt_5^m &= Nt_5^{t_5t_1t_5t_1} \\
&= N(t_5t_1t_5t_1)^{-1}t_5t_5t_1t_5t_1 \\
&= Nt_1t_5t_1t_5t_1t_5t_1 \\
&= Nt_1t_2t_5t_2t_5t_5t_1 \\
&= Nt_1t_2t_5t_2t_1 \\
&= Nt_1t_2t_5\underline{t_5t_3t_4t_1t_2t_2t_1} \\
&= Nt_1t_2t_3t_4 \\
&= Nt_5
\end{aligned}$$

Finally this tells us, $Nt_1^m = Nt_1$, $Nt_2^m = Nt_2$, $Nt_3^m = Nt_3$, $Nt_4^m = Nt_4$, $Nt_5^m = Nt_5$. Thus $m = e$, and $t_5t_1t_5t_1 = e$ is the generator of the center. Now we factor $G = \frac{2^{*5} \cdot A_5}{t_3t_4t_1t_2t_5=e}$ by the additional center relation. However, we first determine whether or not $t_5t_1t_5t_1 = e$, implies the original relation. Now,

$$\begin{aligned}
& t_5 t_1 t_5 t_1 = e \\
\implies & t_5 t_1^{(1,3)(4,5)} = t_1 t_5^{(1,3)(4,5)} \\
& \implies t_4 t_3 = t_3 t_4 \\
\implies & t_4 t_3 t_1 t_2 t_5 = t_3 t_4 t_1 t_2 t_5
\end{aligned}$$

Note $t_3 t_4 t_1 t_2 t_5 = e \implies t_3 t_4 t_1 t_2 t_5^{(1,2)(3,4)} = e^{(1,2)(3,4)} \implies t_4 t_3 t_1 t_2 t_5 = e$. Therefore we have $t_3 t_4 t_1 t_2 t_5 = e$. Hence, G factored by $t_5 t_1 t_5 t_1 = e$ is

$$G \cong \frac{2^{*5}:A_5}{t_3 t_4 t_1 t_2 t_5 = e, t_5 t_1 t_5 t_1 = e} \cong \frac{2^{*5}:A_5}{t_5 t_1 t_5 t_1 = e}.$$

Now we will begin manual double coset enumeration by first looking at our two relations. Our relations are $t_3 t_4 t_1 t_2 t_5 = e$ and $t_5 t_1 t_5 t_1 = e$. Now, $t_3 t_4 t_1 t_2 t_5 = e \iff t_3 t_4 t_1 = t_5 t_2 \iff t_3 t_4 = t_5 t_2 t_1$ $t_5 t_1 t_5 t_1 = e \iff t_5 t_1 = t_1 t_5$. Our first double coset, $N e N = \{N e n | n \in N\} = \{N\}$ denoted by $[*]$, contains one single coset. N is transitive on $\{1,2,3,4,5\}$, so it has a single orbit $\{1,2,3,4,5\}$. We take a representative from the orbit, say $\{5\}$, and find to which double coset $N t_5$ belongs. This will create a new double coset, which we will label as $[5]$. Note $N t_5 N = \{N t_1, N t_2, N t_3, N t_4, N t_5\}$. Now consider the coset stabilizer $N^{(5)}$. Note that the coset stabilizer of $N t_5$ is equal to the point stabilizer N^5 . Thus $N^{(5)} = \langle (1,2)(3,4), (1,4)(2,3), (1,4,3) \rangle$ and the number of the single cosets in the double coset, $N t_5 N$, is at most $\frac{|N|}{|N^{(5)}|} = \frac{60}{12} = 5$. Looking at the generators of $N^{(5)}$, we can see that the orbits of $N^{(5)}$ on $\{1,2,3,4,5\}$ are $\{1,2,3,4\}$ and $\{5\}$. We take a representative from each orbit, say 1 and 5 respectively and then determine to which double coset $N t_5 t_5$, and $N t_5 t_1$ belong. All the t 's have order two, thus $N t_5 t_5 = N \in [*]$. Since the orbit $\{5\}$ contains one element, then one symmetric generator goes back to the double coset $[*]$. Now, $N t_5 t_1 \in N t_5 t_1 N$ is a new double coset we have yet to see. Note four symmetric generators go to the next double coset which we will label as $[51]$. We then consider the coset stabilizer of $N^{(51)}$. $N^{(51)} = N^{51} = \langle (2,3,4) \rangle$. Although, this increases with our relation. From our relation we have, $t_5 t_1 t_5 t_1 = e \iff t_5 t_1 = t_1 t_5 \implies N t_5 t_1 N = N t_1 t_5 N$. Therefore $(1,5)(2,4), (1,5)(3,4), (1,5)(2,3) \in N^{(51)}$ since

$$\begin{aligned}
Nt_5t_1^{(1,5)(2,4)} &= Nt_1t_5 = Nt_5t_1N \implies (1,5)(2,4) \in N^{(51)} \\
Nt_5t_1^{(1,5)(3,4)} &= Nt_1t_5 = Nt_5t_1N \implies (1,5)(3,4) \in N^{(51)} \\
Nt_5t_1^{(1,5)(2,3)} &= Nt_1t_5 = Nt_5t_1N \implies (1,5)(2,3) \in N^{(51)}
\end{aligned}$$

Thus $N^{(51)} = \langle (2,3,4), (1,5)(2,4), (1,5)(3,4), (1,5)(2,3) \rangle$ and the number of single cosets in the double coset, Nt_5t_1N , is at most $\frac{|N|}{|N^{(51)}|} = \frac{60}{6} = 10$. In order to find the distinct single cosets in [51], we must find the 51 right cosets of $N^{(51)} \in N$. Without loss of generality, the are $Nt_5t_1(e), Nt_5t_1(1,2,3), Nt_5t_1(2,5,3), Nt_5t_1(1,2,5), Nt_5t_1(1,2)(4,5), Nt_5t_1(1,5,4), Nt_5t_1(3,5,4), Nt_5t_1(1,3)(4,5)$. Taking a representative from each of the cosets, we form the set of transversals, T . Then, $T = \{(e), (1,2,3), (2,5,3), (1,2,5), (1,2)(4,5), (1,5,3), (1,2)(3,5), (1,5,4), (3,5,4), (1,3)(4,5)\}$. Conjugating the coset Nt_5t_1 by each of the elements in the set T , we get the other distinct cosets in Nt_5t_1N . Thus we will have the following cosets in the double coset [51] with their equal names: $51 \sim 15$.

$$\begin{aligned}
51 \sim 15^{(1,2,3)} &= 52 \sim 25 \\
51 \sim 15^{(2,5,3)} &= 31 \sim 13 \\
51 \sim 15^{(1,2,5)} &= 12 \sim 21 \\
51 \sim 15^{(1,2)(4,5)} &= 42 \sim 24 \\
51 \sim 15^{(1,5,3)} &= 35 \sim 53 \\
51 \sim 15^{(1,2)(3,5)} &= 32 \sim 23 \\
51 \sim 15^{(1,5,4)} &= 45 \sim 54 \\
51 \sim 15^{(3,5,4)} &= 41 \sim 14 \\
51 \sim 15^{(1,3)(4,5)} &= 43 \sim 34
\end{aligned}$$

Looking at the generators of $N^{(51)}$, we can see that the orbits of $N^{(51)}$ on $\{1,2,3,4,5\}$ are $\{1,5\}$ and $\{2,3,4\}$. We take a representative from each orbit, say 1, and 2 respectively and then determine to which double coset $Nt_5t_1t_1$ and $Nt_5t_1t_2$ belong. Since the t 's have order 2, $Nt_5t_1t_1 = N \in [5]$. The orbit containing 1 has two symmetric

generators which will be sent back to that double coset [5]. The coset $Nt_5t_1t_2$ requires further investigation. Our relation is $t_3t_4t_1 = t_5t_2$, and to obtain all of the relations we conjugate our relation by N . However, we want to know specifically where the coset $Nt_5t_1t_2$ goes. The conjugation of $t_3t_4t_1 = t_5t_2$ by $(1, 2, 3, 5, 4) \in N$ gives $t_3t_4t_1^{(1,2,3,5,4)} = t_5t_2^{(1,2,3,5,4)} \implies t_5t_1t_2 = t_1t_3$. This implies $Nt_5t_1t_2 = Nt_1t_3 \in [51]$. Therefore the coset $Nt_5t_1t_2$ will loop back to the double coset [51]. Since there are three symmetric generators in the orbit that contains 2, three symmetric generators will loop back into the double coset [51]. This completes our double coset enumeration and our Cayley diagram is as follows.

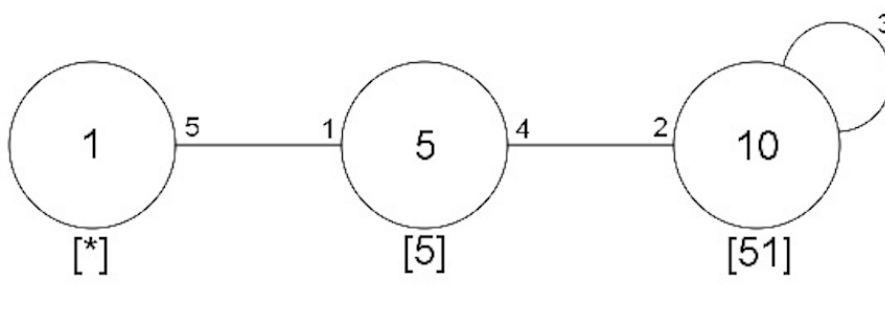


Figure 6.2: Cayley Diagram of $2^{*5} : A_5$ Factored by the Center

Chapter 7

Double Coset Enumeration over Maximal Subgroups

As we have seen before, double coset enumeration can get complex and difficult. Recall, the double coset enumeration process was done over a control group N that decomposed our group G into the form NwN , where w is a word in the t_i 's. It has been suggested that following the same algorithm over a maximal subgroup can provide the same information as does the enumeration over the smaller control group N . Thus, we will find a single coset decomposition of a group, G over M , where $N \leq M \leq G$. We will show that the double coset enumeration of G over M accomplishes the same task as the double coset enumeration of G over N . We will show this with a much smaller example, and then expand this concept to a much larger group.

7.1 Double Coset Enumeration of $S_5 \times 2$ over S_4

We start by factoring the progenitor $2^{*4} : S_4$ by a single relator $(1, 2, 4) = t_4 t_1 t_2 t_4$. Now let,

$$G \cong \frac{2^{*4}:S_4}{(1,2,4)=t_4 t_1 t_2 t_4}$$

The symmetric presentation of G is given by:

$G \langle x, y, t \rangle := \text{Group} \langle x, y, t \mid x^4, y^2, (x * y)^3, t^2, (t, y), (t, (xy)^{x^3}), (xy)^{x^2} t = t t^x t^{x^2} \rangle$ where,

$N \cong S_4 = \langle x, y | x^4, y^2, (x * y)^3 \rangle$ and our $x \sim (1, 2, 3, 4)$ $y \sim (1, 2)$ and our $t \sim t_4$.

Now we will follow the algorithm of double coset enumeration. First we note that $NeN = \{Ne^n | n \in N\} = N$. NeN will be labeled as $[*]$, which contains one single coset. $N \cong S_4$ which is 4-transitive. Thus N is transitive on $\{1, 2, 3, 4\}$, so we have only a single orbit $\{1, 2, 3, 4\}$. We take a representative from this orbit say $\{4\}$ and find to which double coset does Nt_2 belongs. Clearly, this will give us our new double coset, which we will label as $[4]$. Since,

$Nt_4 \in Nt_4N = \{Nt_4^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4\}$. Now we consider the coset stabilizer, denoted as $N^{(4)}$. Note that the coset stabilizer of Nt_4 is equal to the point stabilizer N^4 . Thus,

$N^{(4)} = N^4 = \langle (1, 2), (1, 3, 2), (2, 3) \rangle = \{e, (1, 2, 3), (1, 2), (1, 3, 2), (2, 3), (1, 3)\}$. Thus, the number of the single cosets in Nt_4N is at most: $\frac{|N|}{|N^{(4)}|} = \frac{24}{6} = 4$. The orbits of the coset stabilizing group can be found by simply looking at the generators. We can see that the orbits of $\{1, 2, 3, 4\}$ are $\{1, 2, 3\}$ and $\{4\}$. We take a representative from each orbit, say $\{2\}$ and $\{4\}$, respectively. Now we determine to which double coset Nt_4t_2 , and Nt_4t_2 belong. All t_i 's have order 2 thus, $Nt_4t_4 = N \in [*]$. Therefore, since the orbit $\{4\}$ contains one element, then one symmetric generator goes back to the double coset NeN , and Nt_4t_2 will send it forward to our next double coset. Note, three symmetric generators go to the next double coset $Nt_4t_2N \in [42]$. $[42]$ is the label we use for the double coset Nt_4t_2N . Continuing, we now consider the coset stabilizer $N^{(42)}$. The coset stabilizer of Nt_4t_2 is given by: $N^{(42)} \geq N^{42} = \{e, (1, 3)\}$. Elements that fix 4 and 2 point wise will also fix the coset Nt_4t_2N . Our goal is to find all permutations that stabilize the coset Nt_4t_2 . This is where we need to look at our relation, $(1, 2, 4) = t_4t_1t_2t_4$.

$$(1, 2, 4) = t_4t_1t_2t_4 \Rightarrow (1, 2, 4)t_4t_2 = t_4t_1$$

Taking N of both sides of the equation we see that the permutation $(1, 2, 4)$ will get absorbed by N , since $(1, 2, 4) \in N$. Thus we get $Nt_4t_2 = Nt_4t_1$. Recall the definition of coset stabilizer. The coset stabilizer is defined as $NwN = \{Nw^n = Nw | n \in N\}$. So we search for permutations that sends $4 \rightarrow 4$ and $2 \rightarrow 1$, since these will stabilize the coset Nt_4t_2 . Hence, the permutations $(1, 2) \in N^{(42)}$ and $(1, 3, 2) \in N^{(42)}$, since $Nt_4t_2^{(1,2)} = Nt_4t_1 = Nt_4t_2 \Rightarrow (1, 2) \in N^{(42)}$. $Nt_4t_2^{(1,3,2)} = Nt_4t_1 = Nt_4t_2 \Rightarrow (1, 3, 2) \in N^{(42)}$.

Now $N^{(4,2)} \geq \langle (1,3), (1,2), (1,3,2) \rangle = \{e, (1,2,3), (1,2), (1,3,2), (2,3), (1,3)\}$. Thus, $Nt_4t_2 = Nt_4t_1 = Nt_4t_3$. The number of the single cosets in Nt_4t_2N is at most: $\frac{|N|}{|N^{(4,2)}|} = \frac{24}{6} = 4$. In order to find the other three distinct cosets with three equal names for each in Nt_4t_2N , we find the right cosets of $N^{(4,2)}$ in N . We will take the following right cosets, Nt_4t_2e , $Nt_4t_2(1,2,3,4)$, $Nt_4t_2(1,3)(2,4)$, $Nt_4t_2(1,4,3,2)$. Taking a representative for each of the cosets we form the transversal, T . $T = \{e, (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$. By taking these representatives and conjugating the three different names we found above, we will get the other three distinct cosets in Nt_4t_2N . From above we found $42 \sim 41 \sim 43$.

$42 \sim 41 \sim 43$ conjugated by $(1,2,3,4)$ yields $13 \sim 12 \sim 14$

$42 \sim 41 \sim 43$ conjugated by $(1,3)(2,4)$ yields $24 \sim 23 \sim 21$

$42 \sim 41 \sim 43$ conjugated by $(1,4,3,2)$ yields $31 \sim 34 \sim 32$

Looking at the generators of $N^{(4,2)}$ on $\{1,2,3,4\}$, we can see that it will have two orbits $\{1,2,3\}$ and $\{4\}$. We take a representative from each orbit, say $\{2\}$ and $\{4\}$ respectively and see to which double coset $Nt_4t_2t_2$ and $Nt_4t_2t_4$ belong. Again, all t_i 's have order 2 thus, $Nt_4t_2t_2 = N \in [4]$. Therefore, since the orbit $\{2\}$ contains three elements, then three symmetric generators go back to the double coset Nt_4N , and $Nt_4t_2t_4$ will send it forward to our next double coset. Before we continue investigating this double coset, we will first return to our relation. As stated previously conjugating our relation(s) by N yields more relationships. Note, conjugating $(1,2,4) = t_4t_1t_2t_4$ by $(1,4)$ gives us $(4,2,1) = t_1t_4t_2t_1 \Rightarrow (4,2,1)t_1t_2 = t_1t_4$. Now, we need to look for the coset stabilizer of $Nt_4t_2t_4N$. From the last double coset we found that $42 \sim 41 \sim 43$. By multiplying t_4 to the right of each of these equal names, the equality still holds. Thus $424 \sim 414 \sim 434$. This implies that $N^{(4,2,4)} \geq N^{(4,2)} = \{e, (1,2,3), (1,2), (1,3,2), (2,3), (1,3)\}$. From our relation we see, $(1,2,4)t_4t_2 = t_4t_1 \Rightarrow (1,2,4)t_4t_2t_4 = t_4t_1t_4$.

From above, we can replace t_1t_4 by $(4, 2, 1)t_1t_2$ to achieve

$$\begin{aligned} (1, 2, 4)t_4t_2t_4 &= t_4\underline{t_1t_4} \\ &= t_4(4, 2, 1)t_1t_2 \\ &= t_2t_1t_2 \end{aligned}$$

Now taking N of both sides we see $Nt_4t_2t_4N = Nt_2t_1t_2$. So, we search for permutations that send $4 \rightarrow 2$ and $2 \rightarrow 1$, since these will stabilize the coset $Nt_4t_2t_4$. Thus the permutation $(4, 2, 1) \in N^{(424)}$, since $Nt_4t_2t_4^{(4,2,1)} = Nt_2t_1t_2 = Nt_4t_2t_4 \Rightarrow (4, 2, 1) \in N^{(424)}$. Notice when we add this element in to the set above we get all of S_4 , since $N^{(424)} \geq \langle (1, 3), (1, 2), (1, 3, 2), (4, 2, 1) \rangle = S_4$. Therefore, the number of the single cosets $Nt_4t_2t_4N$ is at most: $\frac{|N|}{|N^{(424)}|} = \frac{24}{24} = 1$. To achieve all the equal names of this double coset, we can conjugate $t_4t_2t_4$ by all of $N^{(424)}$. This gives us

$$313 \sim 343 \sim 323 \sim 242 \sim 232 \sim 212 \sim 131 \sim 121 \sim 141 \sim 424 \sim 414 \sim 434.$$

Again, by looking at the generators of $N^{(424)}$ on $\{1, 2, 3, 4\}$, we can see that it will have a single orbit of $\{1, 2, 3, 4\}$. We take a representative from this orbit, say $\{4\}$, and we note that $Nt_4t_2t_4t_4 = Nt_4t_2 \in [42]$. Therefore, four symmetric generators go back to the double coset Nt_4t_2N . Therefore, we have completed the double coset enumeration and it shows that the index of $N \cong S_4$ in G is at most: $1 + 4 + 4 + 1 = 10 \Rightarrow |G| \leq 10 * |N| = 10 * 24 = 240$.

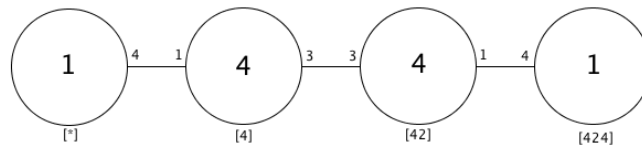


Figure 7.1: Cayley Diagram of S_5 over S_4

7.2 Double Coset Enumeration of H over N

Now let $H = \langle x, y, tt^x \rangle$ be a subgroup of G that is isomorphic to S_4 . We will perform a manual double coset enumeration of H over N . Thus,

$$\begin{aligned} H &= NeN \cup Nt_4t_1N \\ &= N \cup Nt_4t_1 \cup Nt_1t_3 \cup Nt_2t_4 \cup Nt_3t_1 \end{aligned}$$

7.3 Double Coset Enumeration of G over H

We begin this double coset enumeration as we do with N . We note that $Hen = \{He^n | n \in H\} = H$. HeN will be labeled as $[*]$, which contains one single coset. Now H is a subgroup of G that contains N and N is transitive on $\{1,2,3,4\}$. Thus, by taking a representative from this set, say 4, will give us our new double coset Ht_4N , which we will label as $[4]$. Now by definition, $Ht_4N = \{Ht_4^n | n \in N\} = \{Ht_4, Ht_1, Ht_2, Ht_3\}$. However, the order of H is 120. If $\{Ht_4, Ht_1, Ht_2, Ht_3\}$ are all distinct then this would allow us to say the order of $G > 240$ which is a contradiction. Thus we must show that the single cosets are not all distinct. Now,

$$\begin{aligned} Ht_1N &= Ht_2N \\ \iff Ht_1 &= Ht_2 \\ \iff &= Ht_1t_2 \in H \end{aligned}$$

Notice we let $H = \langle x, y, tt^x \rangle = \langle x, y, t_1t_2 \rangle$. Thus $Ht_1t_2 \in H$. Also, since N is 4-transitive a similar argument can be applied for t_2 and t_3 . Therefore, only one single coset exists in Ht_4N , namely Ht_4 . The double coset enumeration of G over H then becomes

$$\begin{aligned} G &= HeN \cup Ht_4N \\ &= H \cup Ht_4 \end{aligned}$$

7.4 Computing Double Coset Enumeration of G over N

Given the info above, we can compute the single coset decomposition of G over N .

$$\begin{aligned}
 G &= HeN \cup Ht_4N \\
 &= H \cup Ht_4t \\
 &= N \cup Nt_4t_1 \cup Nt_1t_3 \cup Nt_2t_4 \cup Nt_3t_1 \cup Nt_4 \cup \\
 &\quad Nt_4t_1t_4 \cup Nt_1t_3t_4 \cup Nt_2t_4t_4 \cup Nt_3t_1t_4
 \end{aligned}$$

Using the relations from G we know that $Nt_1t_3 = Nt_1t_4 \Rightarrow Nt_1t_3t_4 = Nt_1$.

Also, $Nt_3t_1 = Nt_3t_4 \Rightarrow Nt_3t_1t_4 = Nt_3$, and $Nt_2t_4t_4 = Nt_2$ since our t 's are of order 2.

Thus our final single coset decomposition is as follows:

$$\begin{aligned}
 G &= HeN \cup Ht_4N \\
 &= H \cup Ht_4t \\
 &= N \cup Nt_4t_1 \cup Nt_1t_3 \cup Nt_2t_4 \cup Nt_3t_1 \cup Nt_4 \cup \\
 &\quad Nt_4t_1t_4 \cup Nt_1 \cup Nt_2 \cup Nt_3
 \end{aligned}$$

From this example we have shown that by computing the double coset enumeration of both G over H and H over N we can form the double coset enumeration of G over N . This process allows us the ability to perform the double coset enumeration over a maximal subgroup of G and ultimately end up with the same result as the double coset enumeration of G over N , with a much smaller Cayley diagram.

7.4.1 Mathematical Insight

When we find the single coset decomposition of our maximal subgroup M over N we are expressing M as the following

$$M = \cup_{x \in T} Nx$$

where T represents the transversals for N in M .

Now computing the single coset decomposition for M in G we get a similar equation

$$G = \cup_{y \in T'} My$$

where T' represents the transversals for M in G .

By putting the two compositions together we arrive at the composition of G over N ,

$$G = \cup_{y \in T'} My = \cup_{x \in T, y \in T'} Nxy$$

7.5 Double Coset Enumeration of $U(3,3)$ over a Maximal Subgroup

Typically, double coset enumeration is done over the control group N , as seen in the previous examples. However, this process can get very complex and tedious. To allow for a much easier computation, we can accomplish the same goal by doing the process of double coset enumeration over a maximal subgroup of the image of our progenitor. Thus, we take a $N \leq M \leq G$ and achieve the single coset decomposition of $G = \cup Mt'_i sN$. We then compute the double coset enumeration of M over N (as shown in the above example). This leads us to the double coset enumeration of G over N .

We will now perform manual double coset enumeration of $U(3,3)$ over $M \cong PGL(2,7)$. The symmetric presentation $2^{*7} : (C_7 : C_3)$ is given by: $\langle a, b, t \mid a^3, b^7, b^9 = b^2, t^2, (t, a), (a^{-1}t^b t)t^b \rangle$, where $(C_7 : C_3) = \langle a, b \rangle$ and the action of N on the symmetric generators is given by $a \sim (2, 3, 4)(5, 7, 6)$ and $b \sim (1, 2, 3, 5, 4, 6, 7)$. We factor our progenitor by the relation $((2, 4, 3)(5, 6, 7)t_2 t_1 t_2)^2 = e$, which is equivalent to $t_3 t_1 t_3 t_2 t_1 t_2 = e$.

However, instead of the double coset enumeration over N , as usual, we will be performing the double coset enumeration over $M = \langle f(a), f(b), f(tbtba^{-1}tbtb^2) \rangle$ which is isomorphic to $PGL(2, 7)$.

First we need to look for the first double coset of our Cayley diagram. Since we are doing a double coset enumeration over M , our double coset definition changes to $MwN = \{Mw^n | n \in N\}$. So, for our first double coset we have, $MeN = \{Me^n | n \in N\} = \{Me | e \in N\} = \{M\}$. We will denote the first node as $[*]$, which contains one single coset. Our Cayley diagram this far is,

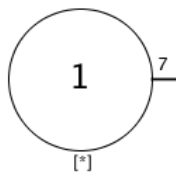


Figure 7.2: First Double Coset

Now N is transitive on $\{1, 2, 3, 4, 5, 6, 7\}$ so it has a single orbit $\{1, 2, 3, 4, 5, 6, 7\}$. Now taking a representative from this orbit, namely 1 and right multiplying it to the existing double coset we get a new double coset Mt_1N , which we will denote by $[1]$. Note, $Mt_1N = \{Mt_1^n | n \in N\} = \{Mt_1, Mt_2, Mt_3, Mt_4, Mt_5, Mt_6, Mt_7\}$. Now consider the coset stabilizer $M^{(1)}$, which is equal to the point stabilizer M^1 . $M^{(1)} = \langle (2, 3, 4)(5, 7, 6) \rangle = \{e, (2, 3, 4)(5, 7, 6), (2, 4, 3)(5, 6, 7)\}$. Then the number of single cosets of Mt_1N is at most $\frac{|N|}{|M^{(1)}|} = \frac{21}{3} = 7$. Now looking at the generators of $M^{(1)}$ we can see that the orbits on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1\}$, $\{2, 3, 4\}$ and $\{5, 6, 7\}$. We take a representative from each orbit, say $\{1\}$, $\{2\}$, and $\{5\}$ respectively. Now we determine to which double coset Mt_1t_1, Mt_1t_2 , and Mt_1t_5 belong. Since t_i 's have order 2 $Mt_1t_1 \in M \in [*]$. Thus one symmetric generator goes back. From our relation we have that $Mt_1t_2N = Mt_1t_5N$. Thus six symmetric generators send us to our new double coset Mt_1t_5 which we will denote as $[15]$.

Taking a look at our next double coset and using the definition we find that $Mt_1t_5N = \{Mt_1t_5^n | n \in N\} = \{Mt_1t_5, Mt_1t_7, Mt_2t_4, Mt_1t_6, Mt_2t_1, Mt_3t_2, Mt_3t_6, Mt_2t_7, Mt_3t_1, Mt_4t_3, Mt_5t_3, Mt_4t_5, Mt_5t_7, Mt_4t_1, Mt_5t_2, Mt_6t_5, Mt_7t_4, Mt_6t_4, Mt_7t_6, Mt_6t_2, Mt_7t_3\}$

Next we know that $M^{(15)} \geq M^{15} = \langle e \rangle$. Our relation will not increase the number of elements in the coset stabilizing group $M^{(15)}$. Thus, the number of single cosets Mt_1t_5N is at most: $\frac{|N|}{|M^{(15)}|} = \frac{21}{1} = 21$. The orbits on $\{1, 2, 3, 4, 5, 6, 7\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$, and $\{7\}$. Taking a representative from each orbit we must see to which double coset $Mt_1t_5t_1, Mt_1t_5t_2, Mt_1t_5t_3, Mt_1t_5t_4, Mt_1t_5t_5, Mt_1t_5t_6$, and $Mt_1t_5t_7$ belong. Clearly, $Mt_1t_5t_5 \in [1]$. From our relation we also have that $Mt_1t_5t_1 = Mt_5$. Thus two symmetric generators will go back to $[1]$. Through further investigation and many conjugations of our relation we find that

$$Mt_1t_5t_2 = Mt_4t_6$$

$$Mt_1t_5t_4 = Mt_7t_2$$

$$Mt_1t_5t_6 = Mt_7t_2$$

$$Mt_1t_5t_7 = Mt_4t_6$$

From these relations we find that four symmetric generators will loop back into the double coset $[15]$. Now, the only symmetric generator that sends us forward is $\{3\}$. Thus our new double coset is $Mt_1t_5t_3N$, which we will label as $[153]$.

Now consider the coset stabilizer of $N^{(153)}$. We know that $N^{(153)} = N^{153} = \langle e \rangle$. After conjugating our relation and simplifying we have that $Mt_1t_5t_3 = Mt_3t_1t_5 = Mt_5t_3t_1$. Thus we look in N for permutations that send $1 \rightarrow 3, 5 \rightarrow 1$ and $3 \rightarrow 5$ and as well as permutations that send $1 \rightarrow 5, 5 \rightarrow 5$, and $3 \rightarrow 1$. Thus $Mt_1t_5t_3 \geq \langle (1, 3, 5)(2, 7, 6)(1, 5, 3)(2, 6, 7) \rangle$. Thus, the number of single cosets Mt_1t_5N is at most: $\frac{|N|}{|M^{(153)}|} = \frac{21}{3} = 7$. In order to find the other six distinct cosets with three equal names for each in $Nt_1t_5t_3N$ we find the right cosets of $M^{(153)}$ in N . We will take the following right cosets, and put them in a set T . So $T = \{e, (2, 3, 4)(5, 7, 6), (1, 5, 2)(3, 4, 6), (2, 4, 3)(5, 6, 7), (1, 7, 6, 4, 5, 3, 2), (1, 7, 4)(3, 5, 6), (1, 6, 2)(4, 7, 5)\}$. Taking these representatives and conjugating them by the three different names we found above, we will get the other six distinct cosets in $Nt_1t_5t_3N$ with their three names. From above we found $153 \sim 315 \sim 531$.

$$153 \sim 315 \sim 531 \text{ conjugated by } (2, 3, 4)(5, 7, 6) \text{ yields } 174 \sim 417 \sim 741$$

$$153 \sim 315 \sim 531 \text{ conjugated by } (1, 5, 2)(3, 4, 6) \text{ yields } 524 \sim 452 \sim 245$$

$$153 \sim 315 \sim 531 \text{ conjugated by } (2, 4, 3)(5, 6, 7) \text{ yields } 162 \sim 216 \sim 621$$

$$153 \sim 315 \sim 531 \text{ conjugated by } (1, 7, 6, 4, 5, 3, 2) \text{ yields } 732 \sim 273 \sim 327$$

$$153 \sim 315 \sim 531 \text{ conjugated by } (1, 7, 4)(3, 5, 6) \text{ yields } 765 \sim 576 \sim 657$$

$$153 \sim 315 \sim 531 \text{ conjugated by } (1, 6, 2)(4, 7, 5) \text{ yields } 643 \sim 364 \sim 436$$

Looking at the generators of $M^{(153)}$, we see that the orbits on $\{1,2,3,4,5,6,7\}$ are $\{4\}, \{1,3,5\}$, and $\{2,7,6\}$. Taking a representative from each orbit, say $\{4\}, \{3\}$, and $\{2\}$ respectively, we need to find to which double coset $Mt_1t_5t_3t_4$, $Mt_1t_5t_3t_3$, and $Mt_1t_5t_3t_2$ belong. Note $Mt_1t_5t_3t_3 \in [15]$, since all t_i 's are of order 2. Now using our relations we have that,

$$Mt_1t_5t_3t_4 = Mt_1t_3t_5$$

$$Mt_1t_5t_3t_2 = Mt_1t_3t_5.$$

From above, we see that a total of 3 symmetric generators go back to $[15]$ while the other 4 loop back into the current double coset. This completes the double coset enumeration of $U(3,3)$ over the maximal subgroup $M \cong PGL(2,7)$.

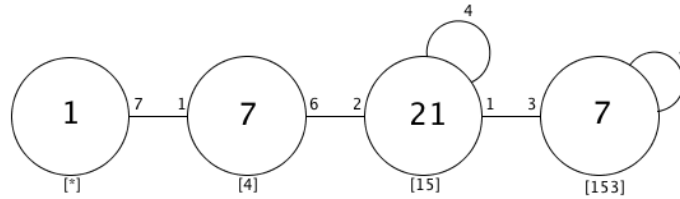


Figure 7.3: Caley Graph of $U(3,3)$

Now we will perform the double coset of $U(3, 3)$ over $H \cong \langle a, b, tbtba^{-1}btbb^2 \rangle$

$$\begin{aligned}
G &= HeN \cup Ht_1N \cup Ht_1t_5N \cup Ht_1t_5t_3N \\
&= H \cup Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4 \cup Ht_5 \cup Ht_6 \cup Ht_7 \cup \\
&Ht_1t_5 \cup Ht_1t_7 \cup Ht_2t_4 \cup Ht_1t_6 \cup Ht_2t_1 \cup Ht_3t_2 \cup \\
&Ht_3t_6 \cup Ht_2t_7 \cup Ht_3t_1 \cup Ht_4t_3 \cup Ht_5t_3 \cup Ht_4t_5 \cup \\
&Ht_5t_7 \cup Ht_4t_1 \cup Ht_5t_2 \cup Ht_6t_5 \cup Ht_7t_4 \cup Ht_6t_4 \cup \\
&Ht_7t_6 \cup Ht_6t_2 \cup Ht_7t_3 \cup Ht_1t_5t_3 \cup Ht_1t_7t_4 \cup \\
&Ht_5t_2t_4 \cup Ht_1t_6t_2 \cup Ht_7t_3t_2 \cup Ht_7t_6t_5 \cup Ht_6t_4t_3.
\end{aligned}$$

Similar to the example above, we will now compute the double coset enumeration of H over N .

$$\begin{aligned}
H &= NeN \cup Ntbtba^{-1}btbb^2N \\
&= N \cup Nt_6t_2t_4t_5t_3 \cup Nt_3t_1t_4t_6t_7 \cup Nt_7t_4t_3t_6t_2 \cup \\
&Nt_4t_1t_5t_3t_2 \cup Nt_3t_1t_6t_7t_1 \cup Nt_6t_1t_6t_1 \cup Nt_3t_4t_5t_2t_1 \cup \\
&Nt_4t_6t_4t_6 \cup Nt_7t_1t_7t_1 \cup Nt_5t_7t_3t_2t_1 \cup Nt_5t_3t_5t_3 \cup \\
&Nt_6t_7t_6t_7 \cup Nt_5t_2t_5t_2 \cup Nt_4t_1t_4t_1 \cup Nt_5t_4t_7t_6t_1
\end{aligned}$$

Combining the two decompositions above we could find the single coset decomposition of G over N which would result in 576 single cosets. If this were to be further investigated, one should compute the single cosets over N as needed.

Chapter 8

Use of Iwasawa's Theorem

8.1 Use of Iwasawa's

We consider

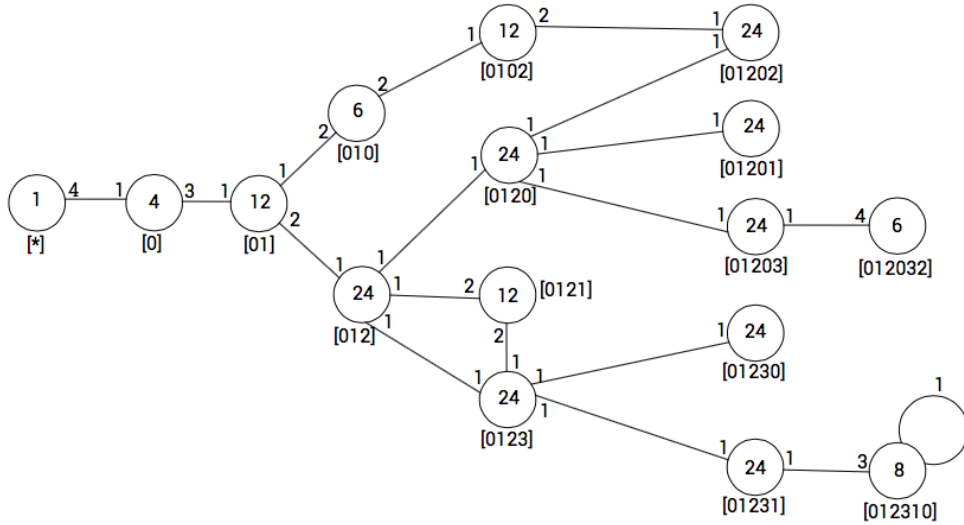
$$G \cong \frac{2^{*4}:S_4}{(xt_0)^{11}, (yxt_0)^{11}, y=(t_0t_1)^3}$$

A presentation for the group G is as follows:

$$\langle x, y, t | x^4, y^2, (xy)^3, t^2, (t, y), (t^x, y), (xt)^{11}, (yxt)^{11}, y = (tt^x)^3 \rangle$$

where $x \sim (0, 1, 2, 3)$ and $y \sim (2, 3)$.

The manual double coset enumeration was done by Lamies AlNazzal [Lam04], and the corresponding Cayley diagram is given as follows:

Figure 8.1: Caley Diagram of $L_2(23)$ over A_4

Iwasawa's lemma consists of three criterion to prove that a group G is simple. If G acts on X faithfully and primitively, G is perfect, and $\exists x \in X$ and a normal abelian subgroup K of G_x such that the conjugates of $K \in G$ generate G , then G is simple. We will first show that G acts on X faithfully and primitively.

Proof. Now from our Cayley diagram we have that G acts on

$$X = \{N, Nt_0N, Nt_0t_1N, Nt_0t_1t_0N, Nt_0t_1t_2N, Nt_0, t_1t_0t_2N, Nt_0t_1t_2t_0N, Nt_0t_1t_2t_1N, \\ Nt_0t_1t_2t_3N, Nt_0t_1t_2t_0t_2N, Nt_0t_1t_2t_0t_1N, Nt_0t_1t_2t_0t_3N, Nt_0t_1t_2t_3t_0N, \\ Nt_0t_1t_2t_3t_1N, Nt_0t_1t_2t_0t_3t_2N, Nt_0t_1t_2t_3t_1t_0N\}$$

G acts on $X \implies \exists$ a homomorphism $f : G \rightarrow S_{253}$. By the First Isomorphism Theorem then $G/\ker f \cong f(G)$. If the $\ker f = 1$ then f is faithful. Now N acts as the identity element of G , so we cannot have non identity elements of G be taken to non-identity elements of S_x . G acts on X faithfully if $gx = x, \forall x \in X$ exactly when $g = e$. From our cayley diagram above we see that $|G| \geq 6072$. However, theorem 2.21 gives $|G| = 253 * |G_1| = 253 \times 24 = 6072$. Therefore the $\ker f = 1$ and the action of G on X is faithful.

Now show that G is primitive by showing that G is transitive and there are no nontrivial blocks. Now from the Cayley diagram it is apparent that G is transitive on X . Since we will use this argument frequently, a proof of this statement is given below.

Theorem 8.1. *If a group G can be represented as a Cayley diagram, then G is transitive*

Proof. Given that G can be represented as a Cayley diagram, we see that every double coset is given a label, say NwN . Now transitivity would imply that there exist an element(s) of $g \in G$ such that you can go from one single coset to any other in the Cayley diagram. Assume we pick an arbitrary single coset, $N\bar{w}$ that lives inside the double coset NwN . Then by the definition of double coset, i.e. $Nwn = \{Nw^n | n \in N\}$, there exists an $n \in N$ such that $N\bar{w}^n = Nw$. Now by right multiplying by w^{-1} , we arrive at N . Clearly, from here we can then multiply by the appropriate w to then arrive at whichever double coset we desire. Since we can move through the Cayley diagram by multiplying by the correct elements of G , this shows that G is transitive. \square

Continuing we will show that G has no nontrivial blocks. Recall that the property of a block states that $|B|$ must divide $|X|$. Therefore, the only possible sizes for a block are 11 and 23. Let B be a nontrivial block, then transitive allows for $N \in B$. Then if $Nt_0 \in B$ then $Bt_0 = \{N, Nt_0\}$. By the definition of a block, if we take elements of G and multiply by B , we will see that B contains the entire double coset Nt_0N . ** As short hand, if one of the single cosets of G over N is in B , then the entire double coset containing that single coset is in B . Therefore, $B = \{N, Nt_0N\} = \{N, Nt_0, Nt_1, Nt_2, Nt_3\}$. Recall the definition of a block, $\forall g \in G, gB = B$ or $gB \cap B = \emptyset$. Now let $g = t_1$ and compute gB . Now

$B = \{Nt_1, Nt_0t_1, Nt_1t_1 = N, Nt_2t_1, Nt_3t_1\}$, since $N \in B \cap Bt_0, B = Bt_0$. Now from ** $B = \{N, Nt_0N, Nt_0t_1N\}$ and $|B| = 17$. So,

$B = \{N, Nt_0, Nt_1, Nt_2, Nt_3, Nt_0t_1, Nt_1t_2, Nt_1t_3, Nt_1t_0, Nt_2t_3, Nt_2t_0, Nt_2t_1, Nt_0t_2, Nt_0t_3, Nt_3t_0, Nt_3t_1, Nt_3t_2\}$. Now let $g = t_2$ and compute gB . We now see that

$B = \{Nt_2, \dots, Nt_2t_2, Nt_0t_1t_2, \dots\}$. Thus $Nt_0t_1t_2N \in B$. Since N is common and either $gB = B$ or $gB \cap B = \emptyset$, we have $gB = B$. Now $|B| = 41$, but $|X| = 253$ and the only divisors of 253 are 1, 11, 23, and 253. This concludes that if B is a nontrivial block and $N \in B$ then if $Nt_0 \in B$ $B = X$.

Through inspection, of the Cayley diagram, we see that the only time we can have a block of order 23 is if we include Nt_0 , however from above we see if we include this single coset the result turns out to be the entire set X . Thus since we cannot form a block of 11 or 23, G contains no non trivial blocks and is transitive, achieving G is primitive.

Next we show that $G = G'$. Now we know that $N = S_4 \subseteq G$ implying $S'_4 \subseteq G'$. The derived group of S_4 is given as $S'_4 = \langle (1, 2, 3), (2, 3, 4) \rangle$. Now $(1, 2, 3) = yx^2yxy$ and $(2, 3, 4) = x^2yx^3$. Thus we know $yx^2yxy, x^2yx^3 \in G'$. From above we see that G is generated by x, y, t but our relations give us $y = t_0t_1t_0t_1t_0t_1$ and $x^3t_0t_1t_2t_3t_0t_1t_2t_3t_0t_1t_2 = 1 \implies x = t_0t_1t_2t_3t_0t_1t_2t_3t_0t_1t_2$. Thus x, y can be written in terms of t 's. This allows us to say G is generated by the t_i 's, hence $G = \langle t_0, t_1, t_2, t_3 \rangle$.

Now,

$$yt_1t_0 = t_0t_1t_0t_1$$

$$yt_1t_0 = [0, 1]$$

Thus this implies $yt_1t_0 \in G'$. So far, $G' \geq \langle yt_0t_1, x^2yx^3, yx^2yxy \rangle$. Since $G' \trianglelefteq G$, then for any $a \in G'$ and $\forall g \in G, a^g \in G'$. So $yt_0t_1 = t_0t_1t_0t_1$ and we know $(1, 0, 2) \in S'_4 \subseteq G'$. Thus if $(1, 0, 2) \in G'$, then its inverse must also live there. Hence, $(1, 2, 0) \in G'$. Note that $(1, 2, 0) = t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$ and $yt_0t_1 = t_0t_1t_0t_1$.

Now,

$$\begin{aligned} (1, 2, 0)yt_0t_1 &= t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_0t_1t_0t_1 \\ &= t_1t_0t_2t_1t_0t_2t_1t_0t_2t_0t_1 \end{aligned}$$

Therefore $t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_0 t_1 \in G'$ Now conjugate the above element by $t_1 t_0 t_2 \in G$. Note this element will be in G' .

$$\begin{aligned} t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_0 t_1^{t_1 t_0 t_2} &= t_2 t_0 t_1 t_1 t_0 t_2 t_1 t_0 t_2 t_1 t_0 t_2 t_0 t_1 t_1 t_0 t_2 \\ &= t_1 t_0 t_2 t_1 t_0 \in G' \end{aligned}$$

Now we must multiply $yt_0 t_1 = t_0 t_1 t_0 t_1$ by the element above and since G' is a group the result will lie in G' .

So,

$$t_0 t_1 t_0 t_1 t_1 t_0 t_2 t_1 t_0 = t_0 t_1 t_2 t_1 t_0$$

Lastly, by conjugating this element by $t_0 t_1 \in G$ we achieve $t_0 t_1 t_2 t_1 t_0^{t_0 t_1} = t_1 t_0 t_0 t_1 t_2 t_1 t_0 t_0 t_1 = t_2$.

Now $t_2 \in G'$. Conjugating t_2 by x, x^2 , and x^3 gives t_3, t_0, t_1 respectively. However, $G \geq G' \geq \langle t_0, t_1, t_2, t_3 \rangle = G$. Thus $G = G'$ and G is perfect.

Finally, we require $x \in X$ and a normal abelian subgroup $K \leq G_x$ such that the conjugates of K generate G . Recall, $G_1 = N = S_4$. Now let

$K = \langle (1, 3)(2, 0), (1, 0)(2, 3) \rangle$. So, $(0, 2)(1, 3) = t_0 t_1 t_2 t_3 t_1 t_0 t_1 t_3 t_0 t_2 t_3 t_1$.

□

Chapter 9

Double Coset Enumeration of

$L_2(8)$ over D_{18}

A symmetric presentation of $2^{*9} : D_{18}$ is given by:

$G \langle x, y, t \rangle := \text{Group} \langle x, y, t | x^{-9}, y^2, (x^{-1} * y)^2, t^2, (t, y * x), (x^4 * t^x)^9, (y * t)^7, (x * t * t^{x^2} * t)^2 \rangle$; where $D_{18} = \langle x, y \rangle$, and the action of N on the symmetric generators is given by $x \sim 1, 2, 3, 4, 5, 6, 7, 8, 9$, $y \sim (1, 9)(2, 8)(3, 7)(4, 6)$. We factor G by using the following relations to obtain

$$\frac{2^{*9}:D_{18}}{(xt_1t_3t_1)^2, (x^4t_2)^9, (yt_1)^7} \cong L_2(8)$$

9.1 Manual Double Coset Enumeration

We begin by looking for our first double coset. Recall the definition of a double coset, $NwN = \{Nw^n | n \in N\}$. Thus we have, $NeN = \{Ne^n | n \in N\} = \{Nn | n \in N\} = \{N\}$. Standardly, we will denote NeN as $[*]$, and N contains one single coset. N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, so it has a single orbit $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now take a representative, say $\{1\}$, from this orbit, and find which double coset Nt_1 belong. This will generate a new double coset, Nt_1N , which we will label as $[1]$. By definition, $Nt_1N = \{Nt_1^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9\}$. Now we consider the coset stabilizer $N^{(1)}$. The coset stabilizer of Nt_1 is equal to the point stabilizer N^1 . Thus, $N^{(1)} = N^1 = \langle (2, 9)(3, 8)(4, 7)(5, 6) \rangle$. Then the number of single cosets in Nt_1N is at most $\frac{|N|}{|N^{(1)}|} = \frac{18}{2} = 9$. Observing the generators of $N^{(1)}$, we can see that the

orbits on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are $\{1\}$, $\{3, 8\}$, $\{2, 9\}$, $\{4, 7\}$, and $\{5, 6\}$. Taking a representative from each orbit, say $\{1\}$, $\{3\}$, $\{2\}$, $\{4\}$, and $\{5\}$ respectively we wish to determine to which double coset $Nt_1t_1, Nt_1t_3, Nt_1t_2, Nt_1t_4$, and Nt_1t_5 belong. Since our t 's are of order two we see that $Nt_1t_1 = N \in [*]$. Thus one symmetric generator will send us back to the double coset labeled $[*]$ Consider the following relation: $t_1t_3 = (1, 3)(4, 9)(5, 8)(6, 7)t_1$. By taking N to both sides of the equation, $Nt_1t_3 = Nt_1$. Since the orbit containing 3 has two elements then two symmetric generators will loop back into the double coset labeled $[1]$. Now we don't have any relations involving Nt_1t_2 , and Nt_1t_4 . However, through the use of MAGMA we find that $Nt_1t_4N = Nt_1t_5N$ thus the only two double cosets left to investigate are $[12]$, and $[14]$. Next, we will consider the double coset Nt_1t_2N . The coset stabilizer $N^{(1,2)} \geq N^{1,2} = \{e\}$. Considering our given relations we try to see if we can increase our coset stabilizing group. We have $t_1t_2 = (1, 9, 8, 7, 6, 5, 4, 3, 2)t_6t_5 \implies Nt_1t_2 = Nt_6t_5$. Thus $(1, 6)(2, 5)(3, 4)(7, 9) \in N^{(1,2)}$ since $Nt_1t_2^{(1,6)(2,5)(3,4)(7,9)} = Nt_6t_5$, but $Nt_1t_2 = Nt_6t_5$, hence $(1, 6)(2, 5)(3, 4)(7, 9)$ stabilizes the coset Nt_1t_2 . Thus the number of single cosets in Nt_1t_2N is at most $\frac{|N|}{|N^{(1,2)}|} = \frac{18}{2} = 9$. To get the other distinct cosets we must find the right distinct cosets of $N^{(1,2)}$. This will give us a set of transversals that will contain 2 equal names in each. The following set is the set of transversals, T : $T = \{e, (1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 4, 7)(2, 5, 8)(3, 6, 9), (1, 9, 8, 7, 6, 5, 4, 3, 2), (1, 3, 5, 7, 9, 2, 4, 6, 8), (1, 5, 9, 4, 8, 3, 7, 2, 6), (1, 8, 6, 4, 2, 9, 7, 5, 3), (1, 6, 2, 7, 3, 8, 4, 9, 5), (1, 7, 4)(2, 8, 5)(3, 9, 6)\}$. If we take a representative from the above set and conjugate $Nt_1t_2 = Nt_6t_5$, which we will write as $12 \sim 65$ (for easier notational purposes), we will gain the other eight distinct cosets in Nt_1t_2 .

$$\begin{aligned}
(12 \sim 65)^e &= 12 \sim 65 \quad (12 \sim 65)^{(1,2,3,4,5,6,7,8,9)} = 23 \sim 75 \\
(12 \sim 65)^{(1,4,7)(2,5,8)(3,6,9)} &= 45 \sim 98 \quad (12 \sim 65)^{(1,9,8,7,6,5,4,3,2)} = 91 \sim 54 \\
(12 \sim 65)^{(1,3,5,7,9,2,4,6,8)} &= 34 \sim 87 \quad (12 \sim 65)^{(1,5,9,4,8,3,7,2,6)} = 56 \sim 19 \\
(12 \sim 65)^{(1,8,6,4,2,9,7,5,3)} &= 89 \sim 43 \quad (12 \sim 65)^{(1,6,2,7,3,8,4,9,5)} = 67 \sim 21 \\
(12 \sim 65)^{(1,7,4)(2,8,5)(3,9,6)} &= 78 \sim 32
\end{aligned}$$

Looking at the generators of the coset stabilizing group of $N^{(1,2)}$, we can compute the orbits to be $\{8\}$, $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$, and $\{7, 9\}$. Taking a representative from each orbit, say $\{8\}$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{7\}$ respectively we wish to determine to which double coset $Nt_1t_2t_8, Nt_1t_2t_1, Nt_1t_2t_2, Nt_1t_2t_3$, and $Nt_1t_2t_7$ belong. Again, since our t 's are of order 2 then $Nt_1t_2t_2 \in [1]$. Since the orbit containing 2 has two elements, then two symmetric

generators go back to [1]. Through the use of *MAGMA* we obtain two relations: $t_1t_2t_1 = x^4t_1t_9$ and $t_1t_2t_7 = yx^4t_1t_2$. Taking N to right side of each of these relations we find that $Nt_1t_2t_1 = Nt_1t_9 \in Nt_1t_2N$ and $Nt_1t_2t_7 = Nt_1t_2 \in Nt_1t_2N$. Notice that each of the orbits containing 1 and 7 each have order 2. Thus 2 + 2 symmetric generators will loop back inside to [12]. In addition through the use of *MAGMA* we have found that $Nt_1t_2t_8N = Nt_1t_4N$ and $Nt_1t_2t_3N = Nt_1t_4N$. Therefore a total of 2 + 1 symmetric generators will send us to the double coset [14].

Next we investigate the double coset Nt_1t_4N , [14]. The coset stabilizer $N^{(1,4)} \geq N^{1,4} = \{e\}$. Considering our given relations we try to see if we can increase our coset stabilizing group. We have $t_1t_4 = (1, 9, 8, 7, 6, 5, 4, 3, 2)t_8t_5 \implies Nt_1t_4 = Nt_8t_5$. Thus $(1, 8)(2, 7)(3, 6)(4, 5) \in N^{(1,4)}$ since $Nt_1t_4^{(1,8)(2,7)(3,6)(4,5)} = Nt_8t_5$, but $Nt_1t_4 = Nt_8t_5$, hence $(1, 8)(2, 7)(3, 6)(4, 5)$ stabilizes the coset Nt_1t_4 . Thus the number of single cosets in Nt_1t_4N is at most $\frac{|N|}{|N^{(1,4)}|} = \frac{18}{2} = 9$. To get the other distinct cosets we must find the right distinct cosets of $N^{(1,4)}$. This will give us a set of transversals that will contain 2 equal names in each. The following set is the set of transversals, T : $T = \{e, (1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 9, 8, 7, 6, 5, 4, 3, 2), (1, 3, 5, 7, 9, 2, 4, 6, 8), (1, 8, 6, 4, 2, 9, 7, 5, 3), (1, 4, 7)(2, 5, 8)(3, 6, 9), (1, 7, 4)(2, 8, 5)(3, 9, 6), (1, 5, 9, 4, 8, 3, 7, 2, 6), (1, 6, 2, 7, 3, 8, 4, 9, 5)\}$. If we take a representative from the above set and conjugate $Nt_1t_4 = Nt_8t_5$, which we will write as $14 \sim 85$ again for easier notational purposes, we will gain the other eight distinct cosets in Nt_1t_4 .

$$\begin{aligned} (14 \sim 85)^e &= 14 \sim 85 \quad (14 \sim 85)^{(1,2,3,4,5,6,7,8,9)} = 25 \sim 96 \\ (14 \sim 85)^{(1,9,8,7,6,5,4,3,2)} &= 93 \sim 74 \quad (14 \sim 85)^{1,3,5,7,9,2,4,6,8} = 36 \sim 17 \\ (14 \sim 85)^{(1,8,6,4,2,9,7,5,3)} &= 82 \sim 63 \quad (14 \sim 85)^{(1,4,7)(2,5,8)(3,6,9)} = 47 \sim 28 \\ (14 \sim 85)^{(1,7,4)(2,8,5)(3,9,6)} &= 71 \sim 52 \quad (14 \sim 85)^{(1,5,9,4,8,3,7,2,6)} = 58 \sim 39 \\ (14 \sim 85)^{(1,6,2,7,3,8,4,9,5)} &= 69 \sim 41 \end{aligned}$$

Looking at the generators of the coset stabilizing group of $N^{(1,4)}$, we can compute the orbits to be $\{9\}$, $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, and $\{4, 5\}$. Taking a representative from each orbit, say $\{9\}$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$ respectively we wish to determine to which double coset $Nt_1t_4t_9$, $Nt_1t_4t_1$, $Nt_1t_4t_2$, $Nt_1t_4t_3$, and $Nt_1t_4t_4$ belong. Again, since our t 's are of order 2 then $Nt_1t_4t_4 \in [1]$. Since the orbit containing 2 has two elements, then two symmetric generators go back to [1]. However through the use of *MAGMA* we have that $t_1t_4t_1 = yx^2t_6 \in [1]$. Thus a total of 4 = 2 + 2 symmetric generators go back to the double

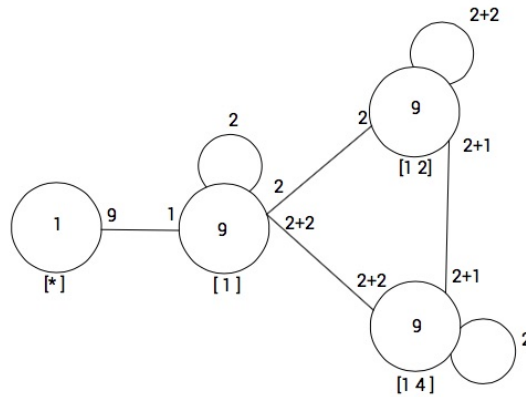


Figure 9.1: Cayley Diagram of $L_2(8)$

coset Nt_1N . In addition we also have the relations $t_1t_4t_9 = t_2t_3$, $t_1t_4t_2 = x^3yt_9t_1$, and $t_1t_4t_3 = x^4yt_2t_5$ and applying N to both sides to each of these relations we have the following:

$$Nt_1t_4t_9 = Nt_2t_3 \in [12] \quad Nt_1t_4t_2 = Nt_9t_1 \in [12] \quad Nt_1t_4t_3 = Nt_1t_4 \in [14]$$

This completes the double coset enumeration of $L_2(8)$ over D_{18} .

9.2 Iwasawa’s Lemma to Show $G \cong L_2(8)$

In order to prove that a group is simple using Iwasawa’s Lemma, we must show the following three criteria hold:

- (1) G acts faithfully and primitively X
- (2) G is perfect ($G = G'$)
- (3) There $\exists x \in X$ and a normal abelian subgroup K of G_x such that the conjugates of K generate G .

9.2.1 G Acts Faithfully X

Let G act on $X = \{N, Nt_1N, Nt_1t_2N, Nt_1t_4N\}$. G acts on $X \implies$ there exists a homomorphism

$$f : G \longrightarrow S_x \quad (|x| = 18)$$

By the First Isomorphism Theorem we have

$$\frac{G}{\ker f} \cong f(G)$$

If $\ker(f) = 1$ then we say f is faithful. Recall G_x represents the stabilizer of x in G . Now, $G_1 = N$, as routine, since the only elements of N . Thus by definition 2.23

$$\begin{aligned} |G| &= 28 * |G_1| \\ &= 28 * |N| \\ &= 28 * 18 \\ &= 504 \\ \implies |G| &= 504 \end{aligned}$$

From our Cayley diagram, $|G| \leq 504$. However, from above $|G| = 504$ implying that $\ker(f) = 1$. Since $\ker(f) = 1$ then G acts faithfully on $|X|$.

9.2.2 G Acts Primitively on X

To show that G is primitive, we must show that G is transitive on X and there exists no nontrivial blocks of X . From theorem 8.1 our Cayley diagram of G over N allows us to conclude that G is transitive. In addition, as stated in definition 2.26 if G is a transitive group on X and B is a nontrivial block then $|Bg| \mid |X|, \forall g \in G$. By observation of our Cayley diagram we see that the only divisors of $|X| = 28$ are 1, 2, 4, 7, 14 and 28 and we cannot create blocks of these sizes. Thus, we conclude that G is primitive.

9.2.3 G is Perfect, $G' = G$

Next we want to show that $G = G'$. Now $D_{18} \leq G$, so $D'_{18} \leq G'$. Now the derived group, $D'_{18} = \langle (1, 2, 3, 4, 5, 6, 7, 8, 9) \rangle \leq G'$, where $x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9)$. Also, G is generated by $\langle x, y, t \rangle$ but from our relations we see that $y = t_1 t_9 t_1 t_9 t_1 t_9 t_1$ and $x^2 = t_1 t_3 t_1 t_2 t_4 t_2 \implies x = (t_1 t_3 t_1 t_2 t_4 t_2)^5$. Since x, y can be written in terms of t 's then $G = \langle t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \rangle$.

Since $G' \trianglelefteq G$ then for any $a \in G'$ and $\forall g \in G$ then $a^g \in G'$. Now since, $N' \langle x \rangle \leq G$ then $x^2 \in G'$. Thus,

$$x^2 = t_1 t_3 t_1 t_2 t_4 t_2$$

Conjugating both sides by $t_1 t_2$ we get

$$\begin{aligned} (x^2)^{t_1 t_2} &= (t_1 t_3 t_1 t_2 t_4 t_2)^{t_1 t_2} \\ t_2 t_1 x^2 t_1 t_2 &= t_2 t_1 t_1 t_3 t_1 t_2 t_4 t_2 t_1 t_2 \\ x^2 t_4 t_3 t_1 t_2 &= t_2 t_1 t_1 t_3 t_1 t_2 t_4 t_2 t_1 t_2 \\ x^2 &= t_2 t_1 t_1 t_3 t_1 t_2 t_4 t_2 t_1 t_2 t_2 t_1 t_3 t_4 \end{aligned}$$

Since our t 's are of order 2 some t 's will cancel and we are left with,

$$x^2 = t_2 t_3 t_1 t_2 t_4 t_2 t_3 t_4$$

Recall that the derived group is generated by the commutators thus the commutator $[2, 4], [4, 3] \in G'$. Using this fact we simplify to achieve the following:

$$\begin{aligned} x^2 &= t_2 t_3 t_1 t_2 t_4 t_2 t_3 t_4 \\ &= t_2 t_3 t_1 [2, 4] t_4 t_3 t_4 \\ &= t_2 t_3 t_1 [2, 4] [4, 3] t_3 \in G' \end{aligned}$$

Now if we conjugate $t_2 t_3 t_1 [2, 4] [4, 3] t_3$ by t_3 and right multiply by the inverse of the commutators $[2, 4], [4, 3] \in G'$ we will have:

$$\begin{aligned} &t_3 t_2 t_3 t_1 \\ \implies &[3, 2] t_2 t_1 \in G' \end{aligned}$$

Now left multiplying by the inverse of the commutator $[3, 2]$ we obtain $t_2 t_1 \in G'$

From our double coset enumeration we also have relation $t_1t_2t_1 = x^4t_1t_9 \implies x^4 = t_1t_2t_1t_9t_1$. Since $x^4 \in N' \leq G'$ then $t_1t_2t_1t_9t_1 \in G'$. Now $[1, 9] \in G'$, so

$$\begin{aligned} t_1t_2t_1t_9t_1 &\in G' \\ \implies t_1t_2[1, 9]t_9 & \\ \implies t_9t_1t_2 &\in G' \end{aligned}$$

From our steps above we have to show that $t_9t_1t_2 \in G'$ and $t_2t_1 \in G'$. Since, G' is a group and is closed under multiplication then

$$(t_9t_1t_2)(t_2t_1) = t_9t_1t_2t_2t_1 = t_9 \in G'$$

Now conjugating $t_9 \in G'$ by all powers of x shows that $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \in G'$, but $G \geq G' \geq \langle t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \rangle = G$. Thus $G = G'$ and G is perfect.

9.2.4 Conjugates of a Normal Abelian Subgroup K Generate G

Now we require $x \in X$ and a normal abelian subgroup $K \leq G_x$ such that the conjugates of K generate G . Recall, $G_1 = N = D_{18}$. Let $K = \langle (1, 2, 3, 4, 5, 6, 7, 8, 9) \rangle$. Now since K is normal then for any $a \in K$ and $\forall g \in G$ then $a^g \in K$.

Since $x^2 \in K$ and $x^2 = t_1t_3t_1t_2t_4t_2$. Now, conjugating the relation $x^2 = t_1t_3t_1t_2t_4t_2$ by $t_1 \in G$ the result will lie in K , since K is normal. Thus,

$$\begin{aligned} x^2 &= t_1t_3t_1t_2t_4t_2 \\ \implies (x^2)^{t_1} &= (t_1t_3t_1t_2t_4t_2)^{t_1} \\ \implies t_1x^2t_1 &= t_1t_1t_3t_1t_2t_4t_2t_1 \\ \implies x^2t_3t_1 &= t_1t_1t_3t_1t_2t_4t_2t_1 \\ \implies x^2 &= t_1t_1t_3t_1t_2t_4t_2t_1t_1t_3 \end{aligned}$$

Since our t 's are of order 2 some t 's will cancel and we are left with,

$$x^2 = t_3t_1t_2t_4t_2t_3 \in K$$

Now conjugating $t_3t_1t_2t_4t_2t_3$ by t_3 gives us $t_1t_2t_4t_2 \in K$. Next if we conjugate $t_1t_2t_4t_2 \in K$ by $t_2t_4 \in G$ we get $(t_1t_2t_4t_2)^{t_2t_4} = t_4t_2t_1t_2 \in K$.

As a result from our double coset enumeration we also have the relation $x^3 = t_1t_2t_6t_1t_2$. Now, conjugating this relation by $t_1 \in G$ the result will again lie in K . Thus,

$$\begin{aligned} x^3 &= t_1t_2t_6t_1t_2 \\ \implies (x^3)^{t_1} &= (t_1t_2t_6t_1t_2)^{t_1} \\ \implies t_1x^3t_1 &= t_1t_1t_2t_6t_1t_2t_1 \\ \implies x^3t_4t_1 &= t_1t_1t_2t_6t_1t_2t_1t_4 \\ \implies x^3 &= t_2t_6t_1t_2t_4 \end{aligned}$$

Therefore we have $t_2t_6t_1t_2t_4 \in K$ and $t_4t_2t_1t_2 \in K$. Since, K is a group and is closed under multiplication then

$$(t_2t_6t_1t_2t_4)(t_4t_2t_1t_2) = t_2t_6t_2 \in K$$

Finally, conjugating $t_2t_6t_2$ by $t_2 \in G$ we see $t_2t_6t_2^{t_2} = t_6 \in K$. Now conjugating $t_6 \in K$ by all powers of x shows that $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \in K$, but $= G \langle t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \rangle$. Thus the conjugates of K generate G . Therefore by Iwasawa's Lemma, $G \cong L_2(8)$ is simple.

9.3 J_1 is Simple

We will now show that $G = J_1$, which was found from the progenitor $7^{*3} :_m S_3$ is simple. However, since J_1 is such a large group we will have to MAGMA verify all three conditions of the lemma.

To show G is transitive and primitive we normally would construct a Cayley diagram and show there exists no non-trivial blocks, where G acts on the set X consisting of the cosets from the double coset enumeration. Now completing the double coset enumeration over the maximal subgroup $2^{\bullet}A_5 \leq G$ is both transitive and primitive. The code below verifies the above statements.

```
> G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^2,(y*t)^15,
(y*t^2)^15,(x*y*t^3)^5>;
```

```

> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
  G
  | J1
  1
> M:=MaximalSubgroups(G1);
> M;
Conjugacy classes of subgroups
-----

[1]   Order 42           Length 4180
      Permutation group acting on a set of cardinality 29260
      Order = 42 = 2 * 3 * 7
[2]   Order 110          Length 1596
      Permutation group acting on a set of cardinality 29260
      Order = 110 = 2 * 5 * 11
[3]   Order 114          Length 1540
      Permutation group acting on a set of cardinality 29260
      Order = 114 = 2 * 3 * 19
[4]   Order 60           Length 2926
      Permutation group acting on a set of cardinality 29260
      Order = 60 = 2^2 * 3 * 5
[5]   Order 120          Length 1463
      Permutation group acting on a set of cardinality 29260
      Order = 120 = 2^3 * 3 * 5
[6]   Order 168          Length 1045
      Permutation group acting on a set of cardinality 29260
      Order = 168 = 2^3 * 3 * 7
[7]   Order 660          Length 266
      Permutation group acting on a set of cardinality 29260
      Order = 660 = 2^2 * 3 * 5 * 11
> C:=Conjugates(G1,M[5] 'subgroup);
> C:=SetToSequence(C);

```

Recall when we are performing double coset enumeration over a maximal subgroup we have to make sure that our N lies inside the maximal subgroup.

```

> for i in [1..#C] do if f(x) in C[i] and f(y) in C[i] then i;
end if; end for;
> 189
> C189:=C[189];
> f2,G2,k2:=CosetAction(G1,C189);
> IsPrimitive(G2);

```

```
true
```

Now we must show that G is perfect by showing $G = G'$. The code below confirms G is perfect.

```
> D:=DerivedGroup(G2);
> D eq G2;
true
```

Finally we must find a normal abelian subgroup $K \leq G_1 = N$ such that the conjugates of K generate G . This step is shown in the code given below:

```
> NL:=NormalLattice(C189);
> NL;
> IsAbelian(NL[2]);
true
> sub<G1|NL[2]^G1> eq G1;
true
```

By the use of MAGMA we have shown that J_1 is simple. For further investigation and completeness, one should perform the double coset enumeration over the maximal subgroup by hand.

Chapter 10

Conclusion

As we came to the end of our research, we began to concentrate our attention on Robert Curtis' [Cur07] example of how M_{24} was generated by seven involutions, mentioned in chapter 1. As a result of his findings we investigated numerous progenitors of all different types in the hope of finding sporadic groups. We wrote such progenitors on many groups with no thinking behind the choice of our control groups. We then searched for a method to find a more efficient way to choose a control group N , that upon writing our progenitor would generate a homomorphic image of a target sporadic group. We note that each sporadic group is simple and thus is generated by involutions (Feit Thompson Theorem). Knowing M_{24} was simple, [Cur07] used a maximal subgroup, M of M_{24} and found an element of order two that was not contained in M which then would generate M_{24} . Expanding on this idea we came up with the following observation:

Lemma 10.1. *If $G = \langle t_1, t_2, \dots, t_n \rangle$ where $|t_i| = 2$, for $1 \leq i \leq n$, and $N = \text{Normalizer}(G, \{\langle t_1 \rangle, \langle t_2 \rangle, \dots, \langle t_n \rangle\})$ where N acts transitively on $\{\langle t_1 \rangle, \langle t_2 \rangle, \dots, \langle t_n \rangle\}$, then G is a homomorphic image of the progenitor $2^{*n} : N$*

Curtis also proven the following theorem:

Theorem 10.2. *Any finite non-abelian simple group is an image of a progenitor of form $P = 2^{*n} : N$, where N is transitive subgroup of the symmetric group of S_n .*

The above theorem and lemma allowed us to prove a corollary to this theorem.

Corollary 10.3. *Let G be a non-abelian simple group with H is a proper subgroup of G and assume $\exists t \in G$ such that $|t| = 2$ and $G = \langle H, t \rangle$. Then G is a homomorphic image of $2^{*n} : H$, where H is a transitive subgroup of S_n . Moreover, H has a faithful permutation representation of the cosets of H over K , where K is the centralizer of t in H .*

Proof. Let G be non-abelian and simple. Let H be a proper subgroup of G with $t \in G$ such that $t \notin H$, $|t| = 2$ and $G = \langle H, t \rangle$. We will now show that

$G = \langle t^H \rangle$. $\langle t^H \rangle$ is normalized by H and t . Therefore,

$G = \langle H, t \rangle = \langle t^H \rangle$, otherwise $\langle t^H \rangle \neq 1 \trianglelefteq G$, but G is simple.

Thus $G = \{t_1, t_2, \dots, t_n\}$, $|t_i| = 2$ for $1 \leq i \leq n$. By theorem 2.7 we can define a homomorphism $\phi : 2^{*n} : H \rightarrow G$ given by $\phi(t_i) = t_i$ and $\phi(H) = H$. We note that $\phi(H) = H$, t_i has n conjugates under $\phi(H)$, and $\phi(H)$ acts as H on the n conjugates of t_i by conjugation implying that G is a homomorphic image of $2^{*n} : H$. To show that H is a transitive subgroup of S_n we must show H acts faithfully on the set $\{t_1, t_2, \dots, t_n\}$ by conjugation. Clearly, H is transitive on n letters, since $\{t_1, t_2, \dots, t_n\}$ was generated by t_i^H . Lastly, to show that H acts faithfully on $\{t_1, t_2, \dots, t_n\}$ then the only element that commutes with each t_i must be the identity element. Assume by contradiction, that $\exists h \in H \neq Id$ such that $t_i^H = t_i$ for $1 \leq i \leq n$. Therefore, $t_i h = h t_i$ for $1 \leq i \leq n$, but $G = \langle t_1, t_2, \dots, t_n \rangle$. Thus h commutes with g , $\forall g \in G$. Therefore, $h \in Z(G)$ but G is simple and $Z(G) \trianglelefteq G$ implies $Z(G) = G$, but G is non-abelian, a contradiction. Therefore, H is a transitive subgroup of S_n that acts faithfully.

We note that our H is written on the same number of letters that G is written on, but we want to find a transitive and faithful permutation representation of H of degree n . Allowing $K \leq H$, with K equal to the centralizer of t in H , we find that the right cosets of H in K will always generate a transitive and faithful permutation representation. To show this we must first show that K is a subgroup of H . We note that K is not empty since $e \in K$, ($t_i^e = t_i$). Now let $h_1 \in K, h_2 \in K$ then show $h_1 * h_2^{-1} \in K$. Now if $h \in K$ then $h^{-1} \in K$ since

$$\begin{aligned}
t_i^h &= t_i \\
h^{-1}t_i h &= t_i \\
(h^{-1}t_i h)^{h^{-1}} &= t_i^{h^{-1}} \\
hh^{-1}t_i hh^{-1} &= t_i^{h^{-1}} \\
t_i &= t_i^{h^{-1}}
\end{aligned}$$

Now $h_1 \in K \implies t_i^{h_1} = t_i$ and $h_2 \in K \implies h_2^{-1} \in K$, from above. Thus $h_2^{-1} \in K \implies t_i^{h_2^{-1}} = t_i$. So,

$$t_i^{h_1} t_i^{h_2^{-1}} = t_i t_i = e \in K$$

Thus by the one step subgroup test K is a subgroup of H . By Theorem 2.9 we know that H over K is transitive on the n letters. It is left to show that the action of H on the cosets is faithful. We note that $Kh_i = Kh_j \iff t_i^{h_i} = t_i^{h_j}$, since if

$$\begin{aligned}
Kh_i &= Kh_j \\
\iff Kh_i h_j^{-1} &= K \\
\iff h_i h_j^{-1} &\in K \\
\iff t_i^{h_i h_j^{-1}} &= t_i \\
\iff t_i^{h_i} &= t_i^{h_j}
\end{aligned}$$

Thus, if $\exists h \in H$ such that $Kh_i h = Kh_i$ then $t_i^{h_i h} = t_i^{h_i} \implies [t_i^{h_i}]^h = t_i^{h_i}$. So $t_i^h = t_i$ for all $1 \leq i \leq n$ implies $h \in Z(G)$ since $G = \langle t_1, t_2, \dots, t_n \rangle$. Now G is simple gives $h = 1$. Therefore H acts faithfully on H over K . \square

In light of the above corollary Dustin Grindstaff and I have developed a program in *MAGMA* to find such control groups, H . Corollary 10.3 implies that given a subgroup, H of such a group G , we can always find a transitive and faithful permutation representation on the cosets of a subgroup K of H . The program is presented below.

```
load "Simple Group";
```

```

count:=0;
SG:=Subgroups(G);
for i in [1..#SG] do for t in G do
if Order(t) eq 2 and t notin SG[i]'subgroup and
sub<G|SG[i]'subgroup,t> eq G then

H:=SG[i]'subgroup;
K:=Centraliser(H,t);
f,N,k:=CosetAction(H,K);
"===== ";
"2 *", Index(H,K), ": N";
"t =", t;
"N = \n", N;
"\n", CompositionFactors(N);
"\n", FPGroup(N);
"\nStabiliser of 1 in N\n", Stabiliser(N,1);
"\n\n\n";
count:=count+1;
break; end if; end for; end for;
count;

```

From running the above program and creating their corresponding progenitors we have found the following groups.

While running the program to find homomorphic images of J_2 it produced a group $H \cong 2^\bullet(2^4 : 5)$ who which was transitive on 32 letters. As a result we wrote a permutation progenitor and found the following group. We would like to note that the relations that we used to find this group was a combination of first order relations and relations of our own with much thought in mind.

```

G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^2,f^2,(a*d)^2,
(b*d)^2,(c*d)^2,b*e^-1*a*e,d*e^-1*d*e,f*e^-1*c*e,a*b*a*b*d,
a*c*a*c*d,e*d*a*e^-1*f,c*a*b*a*c*b,e^-3*d*e^-2,e^-1*c*b*a*e*a*c,t^2,
(t,d*e),(f*e^-1*t)^i,(e*c*f*e*t*t^a)^j,
(a*b*t^c*t^b)^k,(f*a*e^-2*t*t^a*t)^l,(a*t^c)^m
>;

```

Table 10.1: $2^{*32} : (2^\bullet(2^4 : 5))$

i	j	k	l	m	G
0	0	5	0	3	$5^\bullet J_2$

As our research came to an end we didn't get to investigate the program as much as we would like. Often times if we ran the program looking for a particular sporadic group we didn't find the one we were looking for but we did find others of interest. In the following list of tables we first note the group we were trying to find but were unsuccessful, while listing the groups of much importance.

While looking for J_1 we ran the following two progenitors and this is what we found.

```
G<x,y,t>:=Group<x,y,t|x^3,x*y^-1*x^-1*y^2,t^2,(t,x),(y*x*t)^i,
(y*x*t^y*t*t^y)^j,(x^-1*y^-1*t^y*t*t^(y^4))^k,(y*t*t^(y^2)*t)^l,
(y^3*t*t^(x*y))^m
>;
```

Table 10.2: $2^{*7} : (7 : 3)$

i	j	k	l	m	G
0	0	5	5	0	$2^{\bullet}(6 \times M_{22})$
0	0	5	8	0	$4 \times M_{22}$
0	6	6	8	5	$PGL(3,4) \times 2$
0	0	7	6	6	A_8

```
G<x,y,t>:=Group<x,y,t|x^3,x*y^-1*x^-1*y^2,t^2,(t,x),(y*x*t)^i,
(y*x*t^y*t*t^y)^j,(x^-1*y^-1*t^y*t*t^(y^4))^k,(y*t*t^(y^2)*t)^l,
(y^3*t*t^(x*y))^m
>;
```

Table 10.3: $2^{*7} : (7 : 3)$

i	j	k	l	m	G
0	0	0	0	3	$3^2 \times (PSL(3,4) : 2)$
0	0	2	0	0	$U(3,3) : 2$

While looking for J_2 we ran the following two progenitors and what we found is listed below.

```
G<a,b,c,d,t>:=Group<a,b,c,d,t|a^4,b^3,c^4,d^2,a^-2*d,
c^-1*a^2*c^-1,c^-1*b*a*b^-1,a^-1*c^-1*a*c^-1,a^-1*b^-1*a^-1*b*c^-1,t^2,
(t,b),
(a*b*t)^i,
(b^-1*a^-1*t^c)^j,
(b*d*t*t^c)^k,
(b*d*t)^l,
(d*b^-1*t^a)^m,
(d*b^-1*t)^o
>;
```

Table 10.4: $2^{*7} : (7 : 2)$

i	j	k	l	m	n	o	G
0	0	3	0	0	8	3	$2^3 \times PSL(2, 3)$
0	0	4	0	6	0	0	S_8
0	0	4	0	8	2	3	$2^7 \times U(3, 3)$
0	0	5	0	6	8	3	M_{11}

```

G<a,b,c,d,t>:=Group<a,b,c,d,t|a^4,b^3,c^4,d^2,a^-2*d,
c^-1*a^2*c^-1,c^-1*b*a*b^-1,a^-1*c^-1*a*c^-1,a^-1*b^-1*a^-1*b*c^-1,t^2,
(t,b),
(d*b^-1*t^a)^i,
(b*d*t^a)^j,
(b*d*t^c*t^a*t)^k,
(d*a*t)^l
>;

```

Table 10.5: $2^{*7} : (7 : 2)$

i	j	k	l	G
0	6	3	6	$(2^2 \times M_{11}) : 3$

Lastly, we tried to find the Mathieu M_{24} group, however we were unsuccessful. Instead we found the groups listed below.

Table 10.6: $2^{*5} : (5 : 2)$

i	j	k	G
0	3	9	$PSL(2, 19)$
0	3	10	$2 \bullet PSL(2, 19)$
0	3	11	$PSL(2, 89)$

Appendix A

Double Coset Enumeration Codes

A.1 Double Coset Enumeration of $PSL(2, 8)$ over D_{18}

```

i:=0;j:=9;k:=7;l:=2;
G<x,y,t>:=Group<x,y,t|x^-9,y^2,(x^-1*y)^2,t^2,(t,y*x),
(x^3*t)^i,
(x^4*t^x)^j,
(y*t)^k,
(x*t*t^x^2*t)^l
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
S:=Sym(9);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9);
yy:=S!(1, 9)(2, 8)(3, 7)(4, 6);
N:=sub<S|xx,yy>;
H1:=sub<G|x,y>;
#DoubleCosets(G,H1,H1);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1):i in [1..9]];
ts[1]:=f(t);
ts[2]:=f(t)^f(x);
ts[3]:=f(t)^f(x^2);
ts[4]:=f(t)^f(x^3);
ts[5]:=f(t)^f(x^4);
ts[6]:=f(t)^f(x^5);
ts[7]:=f(t)^f(x^6);
ts[8]:=f(t)^f(x^7);
ts[9]:=f(t)^f(x^8);
prodim := function(pt, Q, I)

```

```

/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
return v;
end function;

cst := [null : i in [1 .. Index(G,sub<G|x,y>)]] where null is
[Integers() | ];
  for i := 1 to 9 do
    cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;
for i in [1..28] do if cst[i] ne [] then m:=m+1; end if; end for; m;
N1:=Stabiliser (N,[1]);
SSS:={[1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1; #N1;
T1:=Transversal(N,N1);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1);
//Checking Orbits//
for g in IN do for h in IN do if ts[1]*ts[2] eq g*(ts[1])^h then
"true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[3] eq g*(ts[1])^h then
"true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[4] eq g*(ts[1])^h then

```

```

"true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[5] eq g*(ts[1])^h then
  "true"; break; end if; end for; end for;
N12:=Stabiliser (N,[1,2]);
SSS:={[1,2]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[2]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12s:=N12;
for n in N do if 1^n eq 6 and 2^n eq 5 then N12s:=sub<N|N12s,n>;
end if; end for;
N12s; #N12s;
[1,2]^N12s;
T12:=Transversal(N,N12s);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for i in [1..#T12] do ([1,2]^N12s)^T12[i]; end for;
Orbits(N12s);
for g in IN do for h in IN do if ts[1]*ts[4] eq g*(ts[1]*ts[2])^h then
  "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[2]*ts[8] eq g*(ts[1]*ts[4])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[2]*ts[1] eq g*(ts[1]*ts[7])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[2]*ts[3] eq g*(ts[1]*ts[4])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[2]*ts[7] eq g*(ts[1]*ts[2])^h
then "true"; break; end if; end for; end for;
N14:=Stabiliser (N,[1,4]);
SSS:={[1,4]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);

```

```

Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[1]*ts[4]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N14s:=N14;
for n in N do if 1^n eq 8 and 4^n eq 5 then N14s:=sub<N|N14s,n>;
end if; end for;
N14s; #N14s;
[1,4]^N14s;
T14:=Transversal(N,N14s);
for i in [1..#T14] do
ss:=[1,4]^T14[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for i in [1..#T14] do ([1,4]^N14s)^T14[i]; end for;
Orbits(N14s);
for g in IN do for h in IN do if ts[1]*ts[5] eq g*(ts[1]*ts[4])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[4] eq g*(ts[1]*ts[2])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[4]*ts[9] eq g*(ts[1]*ts[2])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[4]*ts[1] eq g*(ts[1])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[4]*ts[2] eq g*(ts[1]*ts[2])^h
then "true"; break; end if; end for; end for;
for g in IN do for h in IN do if ts[1]*ts[4]*ts[3] eq g*(ts[1]*ts[4])^h
then "true"; break; end if; end for; end for;
xxx:=f(x);
yyy:=f(y);
ttt:=f(t);
N:=sub<G1|f(x),f(y),f(t)>;
ArrayP:=[Id(N): i in [1..#N]];
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xxx; end if;

```

```

if Eltseq(Sch[i])[j] eq -1 then P[j]:=xxx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yyy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=ttt; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do if ArrayP[i] eq N!(2, 4, 6, 10, 16, 15, 9, 5, 3)
(7, 13, 21, 27, 22, 14, 8, 12, 11)(17, 25, 19, 23,
28, 24, 20, 26, 18)
then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq N!(2, 16, 3, 10, 5, 6, 9, 4, 15)
(7, 22, 11, 27, 12, 21, 8, 13, 14)(17, 28, 18, 23,
26, 19, 20, 25, 24)then print Sch[i];
end if; end for;

```

A.2 Double Coset Enumeration of $\text{PSL}(2,23)$ to Find Relations

```

G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2,(t,y),(t^x,y),(x*t)^11,
(y*x*t)^11,y=(t*t^x)^3>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
S:=Sym(4);
xx:=S!(1,2,3,4);
yy:=S!(2,3);
N:=sub<S|xx,yy>;
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1):i in [1..4]];
ts[4]:=f(t);
ts[1]:=f(t)^f(x);
ts[2]:=f(t)^f(x^2);
ts[3]:=f(t)^f(x^3);
N4121:=Stabiliser (N,[4,1,2,1]);
SSS:={[4,1,2,1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do

```



```

for n in IN do
if ts[4]*ts[1]*ts[2]*ts[1]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N4121s:=N4121;
for n in N do if 4^n eq 3 and 1^n eq 2 and 2^n eq 1 and 1^n eq 2 then
N4121s:=sub<N|N4121s,n>; end if; end for;
T4121:=Transversal(N,N4121s);
for i in [1..#T4121] do ([4,1,2,1]^N4121s)^T4121[i]; end for;
So 3242~1424
for g in IN do if ts[3]*ts[2]*ts[4]*ts[2] eq g*(ts[1]*ts[4]*ts[2]*ts[4])
then g; end if; end for;
xx:=f(x);
yy:=f(y);
tt:=f(t);
N:=sub<G1|xx,yy,tt>;
NN<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2,(t,y),(t^x,y),(x*t)^11,
(y*x*t)^11,y=(t*t^x)^3>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=tt; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do if ArrayP[i] eq A then print Sch[i];
end if; end for;
//y * x^2 * y * x
x:=S!(1,2,3,4);
y:=S!(2,3);
N412432:=Stabiliser (N,[4,1,2,4,3,2]);
SSS:={[4,1,2,4,3,2]}; SSS:=SSS^N;

```

```

SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[4]*ts[1]*ts[2]*ts[4]*ts[3]*ts[2] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N412432s:=N412432;
for n in N do if 4^n eq 1 and 1^n eq 2 and 2^n eq 3 and 4^n eq 1 and 3^n
eq 4 and 2^n eq 3 then N412432s:=sub<N|N412432s,n>; end if; end for;
for n in N do if 4^n eq 2 and 1^n eq 3 and 2^n eq 4 and 4^n eq 2 and 3^n
eq 1 and 2^n eq 4 then N412432s:=sub<N|N412432s,n>; end if; end for;
for n in N do if 4^n eq 3 and 1^n eq 4 and 2^n eq 1 and 4^n eq 3 and 3^n
eq 2 and 2^n eq 1 then N412432s:=sub<N|N412432s,n>; end if; end for;
T412432:=Transversal(N,N412432s);
for i in [1..#T412432] do ([4,1,2,4,3,2]^N412432s)^T412432[i]; end for;
f(t*t^x*t^(x^2)*t^(x^3)*t^(x^2)*t^(x^3)*t*t^x*t^(x^3)*t^(x^2)*t^x);
for g in IN do if ts[4]*ts[1]*ts[3]*ts[4]*ts[2]*ts[3] eq
g*(ts[2]*ts[4]*ts[1]*ts[2]*ts[3]*ts[1]) then A:=g; end if; end for;
for i in [1..#N] do if ArrayP[i] eq A then print Sch[i];
end if; end for;
N414:=Stabiliser (N,[4,1,4]);
SSS:={[4,1,4]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[4]*ts[1]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N414s:=N414;
for n in N do if 4^n eq 1 and 1^n eq 4 and 4^n eq 1 then
N414s:=sub<N|N414s,n>; end if; end for;
T414:=Transversal(N,N414s);
for i in [1..#T414] do ([4,1,4]^N414s)^T414[i]; end for;
for g in IN do if ts[2]*ts[3]*ts[2] eq g*(ts[3]*ts[2]*ts[3]) then
A:=g; end if; end for;

```

```

N4142:=Stabiliser (N,[4,1,4,2]);
SSS:={[4,1,4,2]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[4]*ts[1]*ts[4]*ts[2]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*
ts[Rep(Seqq[i])[3]]*ts[Rep(Seqq[i])[4]]
then print Rep(Seqq[i]);
end if; end for; end for;
N4142s:=N4142;
for n in N do if 4^n eq 1 and 1^n eq 4 and 4^n eq 1
and 2^n eq 2
then N4142s:=sub<N|N4142s,n>; end if; end for;
T4142:=Transversal(N,N4142s);
for i in [1..#T4142] do ([4,1,4,2]^N4142s)^T4142[i];
end for;
for g in IN do if ts[3]*ts[1]*ts[3]*ts[1] eq
g*(ts[3]*ts[1]*ts[3])
then A:=g; end if; end for;
xxx:=S!(1,2,3,4);
yyy:=S!(2,3);
N412314:=Stabiliser (N,[4,1,2,3,1,4]);
SSS:={[4,1,2,3,1,4]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IN do
if ts[4]*ts[1]*ts[2]*ts[3]*ts[1]*ts[4] eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]*
ts[Rep(Seqq[i])[4]]*ts[Rep(Seqq[i])[5]]*ts[Rep(Seqq[i])[6]]
then print Rep(Seqq[i]);
end if; end for; end for;
N412314s:=N412314;
for n in N do if 4^n eq 1 and 1^n eq 3 and 2^n eq 2 and 3^n eq 4
and 1^n eq 3 and 4^n eq 1 then N412314s:=sub<N|N412314s,n>;
end if; end for;
for n in N do if 4^n eq 3 and 1^n eq 4 and 2^n eq 2 and 3^n eq 1
and 1^n eq 4 and 4^n eq 3 then N412314s:=sub<N|N412314s,n>;

```

```

end if; end for;
T412314:=Transversal(N,N412314s);
for i in [1..#T412314] do ([4,1,2,3,1,4]^N412314s)^T412314[i]; end for;
for g,h in IN do if ts[4]*ts[1]*ts[2]*ts[3]*ts[1]*ts[4] eq
g*(ts[4]*ts[1]*ts[2]*ts[3]*ts[1]*ts[4]*ts[2])^h then g,h; end if;
end for;
A:=G1!(2, 3, 4)(6, 13, 10)(7, 11, 14)(8, 9, 12)(15, 17, 19)(16, 40, 34)
(18, 20, 44)(21, 35, 38)(22, 37, 26)(23, 31, 46)(24, 33, 32)
(25, 36, 29)(27, 30, 28)(39, 42, 55)(41, 94, 92)(43, 56, 97)(45, 98, 68)
(47, 70, 99)(48, 50,53)(49, 109, 54)(51, 87, 111)(52, 107, 86)
(57, 103, 90)(58, 89, 78)(59, 75,119)(60, 85, 84)(61, 110, 63)
(62, 88, 81)(64, 82, 65)(66, 67, 71)(69, 83,108)(72, 77, 104)
(73, 80, 79)
(74, 91, 76)(93, 96, 127)(95, 184, 182)(100,102, 105)(101, 190, 126)
(106, 142, 189)(112, 113, 133)(114, 201, 206)(115, 117, 130)
(116, 212, 196)(118, 138, 188)(120, 122, 125)(121, 194, 139)
(123,175, 219)(124, 216, 173)(128, 214, 180)(129, 209, 166)
(131, 163, 231)(132,203, 200)(134, 199, 171)(135, 205, 137)
(136, 202, 169)(140, 186, 145)(141,187, 144)(143, 197, 210)
(146, 218, 156)(147, 192, 217)(148, 179, 178)(149,152, 224)
(150, 177, 215)(151, 181, 153)(154, 172, 193)(155, 176, 174)
(157,191, 158)(159, 170, 204)(160, 165, 198)(161, 168, 167)
(162, 195, 164)(183,185, 244)(207, 208, 239)(211, 213, 232)
(220, 221, 241)(222, 223, 243)(225,226, 240)(227, 253, 252)
(228, 230, 229)(233, 246, 251)(234, 236, 237)
(235, 250, 249)(242, 247, 248);
B:=G1!(2, 4)(6, 14)(7, 10)(8, 12)(11, 13)(15, 17)(16, 38)(20, 44)
(21, 34)(22, 46)(23, 26)(24, 36)(25, 33)(27, 28)(29, 32)(31, 37)
(35, 40)(39, 53)(41, 111)(42,50)(43, 54)(45, 99)(47, 68)(48, 55)
(49, 97)(51, 92)(52, 90)(56, 109)(57,86)(58, 108)(59, 63)(60, 104)
(61, 119)(62, 91)(64, 80)(65, 79)(66, 71)(69, 78)(70, 98)(72, 84)
(73, 82)(74, 88)(75, 110)(76, 81)(77, 85)(83, 89)(87,94)(93, 133)
(95, 206)(96, 113)(100, 102)(101, 189)(103, 107)(106, 126)
(112, 127)(114, 182)(115, 125)(116, 219)(117, 122)(118, 139)
(120, 130)(121,188)(123, 196)(124, 210)(128, 200)(129, 217)
(131, 156)(132, 180)(134,215)(135, 224)(136, 181)(137, 149)
(138, 194)(140, 187)(141, 186)(142,190)(143, 173)(144, 145)
(146, 231)(147, 166)(148, 204)(150, 171)(151,202)(152, 205)
(153, 169)(154, 198)(155, 195)(157, 168)(158, 167)(159, 178)
(160, 193)(161, 191)(162, 176)(163, 218)(164, 174)(165, 172)
(170, 179)(175, 212)(177, 199)(183, 239)(184, 201)(185, 208)
(192, 209)(197, 216)(203, 214)(207, 244)(211, 241)(213, 221)
(220, 232)(222, 230)(223,228)(225, 237)(226, 236)(227, 249)
(229, 243)(233, 248)(234, 240)(235, 252)(238, 245)(242, 251)

```

```

(246, 247)(250, 253);
for i in [1..#N] do if ArrayP[i] eq A then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq B then print Sch[i];
end if; end for;
xxx:=S!(1,2,3,4);
yyy:=S!(2,3);
for g,h in IN do if ts[4]*ts[1]*ts[2]*ts[1]*ts[4] eq
g*(ts[4]*ts[1]*ts[2]*ts[3])^h then g,h; end if; end for;
for i in [1..#N] do if ArrayP[i] eq g then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq h then print Sch[i];
end if; end for;

```

A.3 Double Coset Enumeration of $U(3,3)$ over a Maximal Subgroup

```

i:=0;j:=2;
G<a,b,t>:=Group<a,b,t|a^3,b^7,b^a=b^2,t^2,(t,a),
(a*b*a^-1*t*t^(b^2))^i,
(a^-1*t^b*t*t^b)^j
>;
f,G1,k:=CosetAction(G,sub<G|a,b>);
S:=Sym(5);
ww:=S!(2,3,4,5);
xx:=S!(2,4)(3,5);
yy:=S!(1,2,3,5,4);
N:=sub<S|ww,xx,yy>;
M:=MaximalSubgroups(G1);
C:=Conjugates(G1,M[1]'subgroup);
C:=Setseq(C);
for i in [1..#C] do if f(a) in C[i] and f(b) in C[i] then i;
end if; end for;
NumberOfGenerators(C[33]);
A:=C[28].1;
B:=C[28].2;
xx:=f(a);
yy:=f(b);
tt:=f(t);
N:=sub<G1|xx,yy,tt>;
#N;
NN<a,b,t>:=Group<a,b,t|a^3,b^7,b^a=b^2,t^2,(t,a),
(a^-1*t^b*t*t^b)^2>;

```

```

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=tt; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do if ArrayP[i] eq A then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq B then print Sch[i];
end if; end for;

i:=0;j:=2;
G<a,b,t>:=Group<a,b,t|a^3,b^7,b^a=b^2,t^2,(t,a),
(a*b*a^-1*t*t^(b^2))^i,
(a^-1*t^b*t*t^b)^j
>;
f,G1,k:=CosetAction(G,sub<G|a,b>);

for g in C[28] do if sub<G1|f(a),f(b),g> eq C[28] then g;
end if; end for;

S:=Sym(576);
T:=S!(1, 506)(2, 431)(3, 337)(4, 428)(5, 220)(6, 513)(7, 503)
(8, 438)(9, 95)(10, 274)(11, 143)(12, 153)(13, 444)(14, 434)
(15, 525)(16, 217)(17, 249)(18, 267)(19, 275)(20, 512)(21, 157)
(22, 40)(23, 240)(24, 549)(25, 270)(26, 135)(27, 134)(28, 179)
(29, 86)(30, 163)(31, 71)(32, 138)(33, 158)(34, 89)(35, 385)
(36, 142)(37, 218)(38, 511)(39, 518)(41, 47)(42, 162)(43, 229)
(44, 419)(45, 221)(46, 137)(48, 122)(49, 140)(50, 288)(51, 542)
(52, 354)(53, 383)(54, 238)(55, 149)(56, 279)(57, 350)(58, 316)
(59, 441)(60, 455)(61, 423)(62, 517)(63, 480)(64, 550)(65, 104)

```

```

(66, 401)(67, 559)(68,308)(69, 439)(70, 362)(72, 370)(73, 183)
(74, 319)(75, 242)(76, 378)(77, 196)(78, 239)(79, 414)(80, 285)
(81, 489)(82, 351)(83, 457)(84, 338)(85, 376)(87, 359)(88, 191)
(90, 257)(91, 398)(92, 507)(93, 410)(94, 458)(96, 515)(97, 342)
(98, 384)(99, 287)(100, 462)(101, 417)(102, 309)(103, 348)(105,409)
(106, 231)(107, 449)(108, 538)(109, 203)(110, 466)(111, 305)
(112, 372)(113, 488)(114, 421)(115, 402)(116, 141)(117, 294)(118, 380)
(119, 358)(120, 422)(121, 554)(123, 301)(124, 181)(125, 253)(126, 151)
(127, 456)(128, 485)(129, 306)(130, 176)(131, 546)(132, 408)(133, 326)
(136,161)(139, 254)(144, 344)(145, 341)(146, 332)(147, 391)(148, 166)
(150,187)(152, 245)(154, 522)(155, 258)(156, 371)(159, 479)(160, 375)
(164, 207)(165, 280)(167, 227)(168, 339)(169, 277)(170, 300)(171, 430)
(172, 330)(173, 262)(174, 299)(175, 268)(177, 182)(178, 329)(180, 425)
(184, 474)(185, 519)(186, 562)(188, 266)(189, 420)(190, 505)(192, 404)
(193, 353)(194, 259)(195, 263)(197, 224)(198, 366)(199, 246)(200, 233)
(201,260)(202, 251)(204, 528)(205, 400)(206, 459)(208, 388)(209, 331)
(210,523)(211, 367)(212, 365)(213, 361)(214, 282)(215, 468)(216, 310)
(219,521)(222, 335)(223, 315)(225, 454)(226, 499)(228, 432)(230, 486)
(232,465)(234, 281)(235, 501)(236, 530)(237, 320)(241, 276)(243, 496)
(244,575)(247, 442)(248, 392)(250, 545)(252, 460)(255, 395)(256, 555)
(261,557)(264, 552)(265, 537)(269, 481)(271, 360)(272, 494)(273, 429)
(278,463)(283, 539)(284, 548)(286, 298)(289, 461)(290, 415)(291, 544)
(292,377)(293, 379)(295, 411)(296, 407)(297, 543)(302, 571)(303, 476)
(304,526)(307, 547)(311, 567)(312, 576)(313, 527)(314, 450)(317, 451)
(318,498)(321, 514)(322, 541)(323, 508)(324, 427)(325, 470)(327, 566)
(328,531)(333, 469)(334, 487)(336, 565)(340, 445)(343, 484)(345, 532)
(346,475)(347, 570)(349, 446)(352, 529)(355, 560)(356, 453)(357, 574)
(363,535)(364, 533)(368, 509)(369, 448)(373, 563)(374, 524)(381, 491)
(382,500)(386, 471)(387, 561)(389, 572)(390, 540)(393, 443)(394, 477)
(396,440)(397, 553)(399, 490)(403, 564)(405, 502)(406, 504)(412, 568)
(413, 558)(416, 569)(418, 467)(424, 536)(426, 472)(433, 520)(435, 447)
(436, 483)(437, 573)(452, 551)(464, 510)(473, 482)(478, 492)(493, 516)
(495, 556)(497, 534);
for i in [1..#N] do if ArrayP[i] eq T then print Sch[i];
end if; end for;
#sub<G1|f(a),f(b),f(t * b * t * b * t * a^-1 * b * t * b * t * b^2)>;
//////////DOUBLE COSET ENUMERATION//////////
i:=0;j:=2;
G<a,b,t>:=Group<a,b,t|a^3,b^7,b^a=b^2,t^2,(t,a),
(a*b*a^-1*t*t^(b^2))^i,
(a^-1*t^b*t*t^b)^j
>;
H1:=sub<G|a,b>;
H2:=sub<G|a,b,t * b * t * b * t * a^-1 * b * t * b * t * b^2>;

```

```

DoubleCosets(G,H2,H1);
f,G1,k:=CosetAction(G,sub<G|a,b>);
#k;
S:=Sym(7);
xx:=S!(2, 3, 4)(5, 7, 6);
yy:=S!(1, 2, 3, 5, 4, 6, 7);
N:=sub<S|xx,yy>;
#N;
IM:=sub<G1|f(a),f(b),
f(t * b * t * b * t * a^-1 * b * t * b * t * b^2)>;
IN:=sub<G1|f(a),f(b)>;
ts:=[Id(G1):i in [1..7]];
ts[1]:=f(t);
ts[2]:=f(t)^f(b);
ts[3]:=f(t)^f(b^2);
ts[4]:=f(t)^f(b^4);
ts[5]:=f(t)^f(b^3);
ts[6]:=f(t)^f(b^5);
ts[7]:=f(t)^f(b^6);
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
  v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
return v;
end function;

cst := [null : i in [1 .. Index(G,sub<G|a,b>)]] where null is
[Integers() | ];
  for i := 1 to 7 do
    cst[prodim(1, ts, [i])] := [i];
  end for;
m:=0;
for i in [1..576] do if cst[i] ne [] then m:=m+1; end if; end for; m;
N1:=Stabiliser (N,[1]);
SSS:={[1]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do

```



```

for n in IN do
if ts[1]eq
n*ts[Rep(Seqq[i])[1]]
then print Rep(Seqq[i]);
end if; end for; end for;
N1; #N1;
T1:=Transversal(N,N1);
for i in [1..#T1] do
ss:=[1]^T1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..576] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N1);
N15:=Stabiliser (N,[1,5]);
SSS:={[1,5]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[1]*ts[5]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N15; #N15;
T15:=Transversal(N,N15);
for i in [1..#T15] do
ss:=[1,5]^T15[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..576] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N15);
CHECKING WHERE EACH ORBIT GOES {1},and {5}
for g in IM do for h in IN do if ts[1]*ts[5]*ts[1] eq
g*(ts[1])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[5] eq
g*(ts[1])^h then
"true"; break; end if; end for; end for;
CHECKING WHERE EACH ORBIT GOES {2},{4},{6} and {7}
for g in IM do for h in IN do if ts[1]*ts[5]*ts[2] eq

```

```

g*(ts[1]*ts[2])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[4] eq
g*(ts[1]*ts[2])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[6] eq
g*(ts[1]*ts[2])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[7] eq
g*(ts[1]*ts[2])^h then
"true"; break; end if; end for; end for;
N12:=Stabiliser (N,[1,2]);
SSS:={ [1,2] }; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[1]*ts[2]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if; end for; end for;
N12; #N12;
T12:=Transversal(N,N12);
for i in [1..#T12] do
ss:=[1,2]^T12[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..576] do if cst[i] ne []
then m:=m+1; end if; end for; m;
Orbits(N12);
for g in IM do for h in IN do if ts[1]*ts[2]*ts[1] eq
g*(ts[1])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]*ts[2] eq
g*(ts[1])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]*ts[3] eq
g*(ts[1]*ts[5])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]*ts[4] eq
g*(ts[1]*ts[5])^h then
"true"; break; end if; end for; end for;

```

```

for g in IM do for h in IN do if ts[1]*ts[2]*ts[5] eq
g*(ts[1]*ts[5])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2]*ts[7] eq
g*(ts[1]*ts[5])^h then
"true"; break; end if; end for; end for;
N153:=Stabiliser (N,[1,5,3]);
SSS:={[1,5,3]}; SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
for i in [1..#SSS] do
for n in IM do
if ts[1]*ts[5]*ts[3]eq
n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if; end for; end for;
N153s:=N153;
for n in N do if 1^n eq 3 and 5^n eq 1 and 3^n eq 5 then
N153s:=sub<N|N153s,n>; end if; end for;
for n in N do if 1^n eq 5 and 5^n eq 3 and 3^n eq 1 then
N153s:=sub<N|N153s,n>; end if; end for;
N153s; #N153s;
[1,5,3]^N153s;
T153:=Transversal(N,N153s);
for i in [1..#T153] do
ss:=[1,5,3]^T153[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..364] do if cst[i] ne []
then m:=m+1; end if; end for; m;
for i in [1..#T153] do ([1,5,3]^N153s)^T153[i]; end for;
Orbits(N153s);
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[1] eq
g*(ts[1]*ts[5])^h then"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[2] eq
g*(ts[1]*ts[5]*ts[3])^h then "true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[3] eq
g*(ts[1]*ts[5])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[4] eq
g*(ts[1]*ts[5]*ts[3])^h then
"true"; break; end if; end for; end for;

```

```
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[5] eq
g*(ts[1]*ts[5])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[6] eq
g*(ts[1]*ts[5]*ts[3])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[5]*ts[3]*ts[7] eq
g*(ts[1]*ts[5]*ts[3])^h then
"true"; break; end if; end for; end for;
for g in IM do for h in IN do if ts[1]*ts[2] eq g*(ts[1]*ts[5])^h
then "true"; break;
end if; end for; end for;
```

Appendix B

J1 is Simple Using Iwasawa's

```

i:=15;j:=0;k:=15;l:=5;
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^2,
(y*t)^i,
(x*t*t^x*t^(x^2))^j,
(y*t^2)^k,
(x*y*t^3)^l
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
M:=MaximalSubgroups(G1);
C:=Conjugates(G1,M[5] 'subgroup);
C:=SetToSequence(C);
for i in [1..#C] do if f(x) in C[i] and f(y) in C[i] then i;
end if; end for;
C189:=C[189];
NL:=NormalLattice(C402);
NL;
sub<G1|NL[2]^G1> eq G1;
f2,G2,k2:=CosetAction(G1,C402);
IsPrimitive(G2);
#Stabiliser(G2,402);
#sub<G1|C402>;
A:=C189.1;
B:=C189.2;
C:=C189.3;
D:=DerivedGroup(G2);
D eq G2;

```

Appendix C

Solved Composition Factor of the

$$6^\bullet : (PSL(2, 4) : 2)$$

```

a:=0;b:=5;c:=10;d:=0;e:=10;
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2,(t,y),(t,y^x),(x*t)^a,
(x*t*t^x)^b,(x*y*t*t^x*t^(y))^c,(x^3*t*t^x*t^y)^d,(x*y*t)^e>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
IsAbelian(NL[4]);
q,ff:=quo<G1|NL[4]>;
D:=DirectProduct(NL[2],NL[3]);
s,t:=IsIsomorphic(D,NL[4]);
s;
H<a,b>:=Group<a,b|a^2,b^4,(a*b)^7,(a*b^2)^5,
(a*b*a*b^2)^7,(a*b*a*b*a*b^2*a*b^-1)^5>;
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
NL:=NormalLattice(q);
NL;
s,t:=IsIsomorphic(H1,NL[2]);
s;
for z1 in NL[3] do if Order(z1) eq 2 and z1 notin NL[2] and
NL[3] eq sub<q|NL[2],z1> then Z1:=z1;break; end if; end for;
s,t:=IsIsomorphic(H1,NL[2]);
A:=t(f1(a));
B:=t(f1(b));
N:=sub<q|A,B>;
NN<a,b>:=Group<a,b|a^2,b^4,(a*b)^7,(a*b^2)^5,

```

```

(a*b*a*b^2)^7, (a*b*a*b*a*b^2*a*b^-1)^5>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..#N] do if ArrayP[i] eq A^Z1 then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq B^Z1 then print Sch[i];
end if; end for;
H<a,b,c>:=Group<a,b,c|a^2,b^4,(a*b)^7,(a*b^2)^5,(a*b*a*b^2)^7,
(a*b*a*b*a*b^2*a*b^-1)^5,c^2,a^c=a*b*a*b^-1*a*b*a*b*\
a*b^-1*a*b^-1*a*b*a*b^-1*a*b^2*a*b^-1*a*b^2,b^c=a*b*a*\
b^-1*a*b*a*b^-1*a*b^-1*a*b*a*b*a*b^-1>;
f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(H2,q);
s;
NL:=NormalLattice(G1);
T:=Transversal(G1,NL[4]);
A:=t(f2(a));
B:=t(f2(b));
C:=t(f2(c));
for i in [1..#T] do if ff(T[i]) eq A then i; end if;end for;
for i in [1..#T] do if ff(T[i]) eq B then i; end if;end for;
for i in [1..#T] do if ff(T[i]) eq C then i; end if;end for;
A:=T[16890];
B:=T[6838];
C:=T[14720];
for d,e in NL[4] do if Order(e) eq 2 and Order(d) eq 3 and e^d eq e
then D:=d; E:=e; end if; end for;
Order(D);
sub<G1|E,D> eq NL[4];
for i in [0..3] do for j in [0..2] do if A^2 eq D^i*E^j
then i,j; break; end if;

```

```

end for; end for;
for i in [0..3] do for j in [0..2] do if  $B^4 = D^i E^j$ 
then i,j; break; end if;
end for; end for;
for i in [0..3] do for j in [0..2] do if  $(A*B)^7 = D^i E^j$ 
then i,j; break; end if;
end for; end for;
for i in [0..3] do for j in [0..2] do if  $(A*B^2)^5 = D^i E^j$ 
then i,j; break; end if;
end for; end for;
for i in [0..3] do for j in [0..2] do if  $(A*B*A*B^2)^7 = D^i E^j$ 
then i,j; break;
end if;end for; end for;
for i in [0..3] do for j in [0..2] do if
 $(A * B * A * B * A * B^2 * A * B^{-1})^5 =$ 
 $D^i E^j$  then i,j; break; end if;end for; end for;
for i in [0..3] do for j in [0..2] do if  $C^2 = D^i E^j$  then
i,j; break; end if;end for;
end for;
for i in [0..3] do for j in [0..2] do if  $A^C = D^i E^j$  then
i,j; break; end if;end for;
end for;
for i in [0..3] do for j in [0..2] do if  $B^C = D^i E^j$  then
i,j; break; end if;end for;
end for;
for i,k in [0..2] do for j in [0..4] do if  $A^D =$ 
 $A^i B^j C^k$  then i,j,k; break; end if;
end for;end for;
for i,k in [0..2] do for j in [0..4] do if  $A^E =$ 
 $A^i B^j C^k$  then i,j,k; break; end if;
end for; end for;
for i,k in [0..2] do for j in [0..4] do if  $B^E =$ 
 $A^i B^j C^k$  then i,j,k; break; end if;
end for; end for;
for i,k in [0..2] do for j in [0..4] do if  $B^D =$ 
 $A^i B^j C^k$  then i,j,k; break; end if;
end for; end for;
for i,k in [0..2] do for j in [0..4] do if  $C^D =$ 
 $A^i B^j C^k$  then i,j,k; break; end if;
end for; end for;
for l in [0..3] do for i,k,m in [0..2] do for j in [0..4]
do if  $C^D = A^i B^j C^k D^l$ 
 $E^m$  then i,j,k,l,m; break; end if; end for; end for;
end for;

```



```

for i,k in [0..2] do for j in [0..4] do if C^E eq
A^i*B^j*C^k then i,j,k; break; end if;
end for;end for;
H<a,b,c,d,e>:=Group<a,b,c,d,e|d^3,e^2,(d,e),a^2,
b^4,(a*b)^7=d*e,(a*b^2)^5,
(a*b*a*b^2)^7=d,(a*b*a*b*a*b^2*a*b^-1)^5\
=d^2,c^2,a^c=a*b*a*b^-1*a*b*a*b*a*b^-1*a*b^-1*
a*b*a*b^-1*a*b^2*a*b^-1*a*
b^2,b^c=a*b*a*b^-1*a*b*a*b^-1*a*b^-1*a*b*a*b\
*a*b^-1,a^d=a,a^e=a,b^d=b,b^e=b,c^d=c*d^2,c^e=c>;
#H;
H<a,b,c,d,e>:=Group<a,b,c,d,e|d^3,e^2,(d,e),a^2,b^4,
(a*b)^7=d*e,(a*b^2)^5,
(a*b*a*b^2)^7=d,(a*b*a*b*a*b^2*a*b^-1)^5\
=d^2,c^2,a^c=a*b*a*b^-1*a*b*a*b*a*b^-1*a*b^-1*a*b
*a*b^-1*a*b^2*a*b^-1*a*b^2,
b^c=a*b*a*b^-1*a*b*a*b^-1*a*b^-1*a*b*a*b\
*a*b^-1,a^d=a,a^e=a,b^d=b,b^e=b,c^d=c*d^2,c^e=c>;
#H;
#G1;
f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(H2,G1);
s;

```

Appendix D

Unsuccessful Progenitors and their Relations

D.1 Wreath Product $Z_2 \text{ Wr } A_5$

```

S:=Sym(10);
tt:=S!(1,6);
uu:=S!(2,7);
vv:=S!(3,8);
ww:=S!(4,9);
xx:=S!(5,10);
yy:=S!(1,2)(3,4)(6,7)(8,9);
zz:=S!(1,5,4)(6,10,9);
N:=sub<S|tt,uu,vv,ww,xx,yy,zz>;
#N;
NN<a,b,c,d,e,f,g>:=Group<a,b,c,d,e,f,g|a^2,b^2,c^2,d^2,
e^2,(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
f^2,g^3,(f*g)^5,a^f=b,b^f=a,c^f=d,d^f=c,e^f=e,a^g=e,
b^g=b,c^g=c,d^g=a,e^g=d>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=tt; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=uu; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=vv; end if;

```

```

if Eltseq(Sch[i])[j] eq 4 then P[j]:=ww; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 7 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -7 then P[j]:=zz^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
N1:=Stabiliser(N,[1]);
N1;
for i in [1..#N] do if ArrayP[i] eq N!(2, 7) then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq N!(3, 8) then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq N!(4, 9) then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq N!(5, 10) then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq
N!(2, 9)(3, 5)(4, 7)(8, 10) then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq
N!(3, 9, 10)(4, 5, 8) then print Sch[i];
end if; end for;
/*Progenitor */
G<a,b,c,d,e,f,g,t>:=Group<a,b,c,d,e,f,g,t|a^2,b^2,c^2,d^2,e^2,(a,b),
(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),f^2,g^3,
(f*g)^5,a^f=b,b^f=a,c^f=d,d^f=c,e^f=e,a^g=e,b^g=b,c^g=c,d^g=a,
e^g=d,t^2,(t,b),
(t,c),(t,d),(t,e),(t,b*d*g*f*g^-1),(t,c*e*g^-1*f*g*f*g^-1*f*g^-1)>;
C:=Centraliser(N,Stabiliser(N,[1,2]));
C;
for i in [1..#N] do if ArrayP[i] eq N!(3, 8)(4,9)(5,10) then
print Sch[i];end if; end for;
for i in [1..#N] do if ArrayP[i] eq
N!(1, 6)(2, 7)(3, 8)(4, 9)(5, 10) then print Sch[i];
end if; end for;
for i in [1..#N] do if ArrayP[i] eq N!(2, 7)(3, 8)(4, 9)(5, 10)
then print Sch[i];
end if; end for;
for i,j,k,l,m in [0..50] do

```

```

G<a,b,c,d,e,f,g,t>:=Group<a,b,c,d,e,f,g,t|a^2,b^2,c^2,d^2,e^2,(a,b),
(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),f^2,g^3,
(f*g)^5,a^f=b,b^f=a,c^f=d,d^f=c,e^f=e,a^g=e,b^g=b,c^g=c,d^g=a,
e^g=d,t^2,(t,b),(t,c),(t,d),(t,e),(t,b*d*g*f*g^-1),
(t,c*e*g^-1*f*g*f*g^-1*f*g^-1),
(f*t)^i,
(g*t*t^a*t^g*e)^j,
(b*c*d*e*t^f*t^f*b)^k,
(a*b*c*d*e*t)^l,
c*d*e=(t*t^f)^m>;
if Index(G,sub<G|a,b,c,d,e,f,g>) ge 2 then i,j,k,l,m,
Index(G,sub<G|a,b,c,d,e,f,g>), #G;
end if; end for;

```

D.2 Wreath Product $Z_2 \text{ Wr } S_4$

```

W:=WreathProduct(CyclicGroup(2),Sym(4));
W;
G<a,b,c,d,e,f>:=Group<a,b,c,d,e,f|a^2,b^2,c^2,d^2,(a,b),
(a,c),(a,d),(b,c),(b,d),(c,d),e^4,f^2,(e*f)^3,a^e=b,b^e=c,
c^e=d,d^e=a,a^f=b,b^f=a,c^f=c,d^f=d>;
#G;
/*First I tried this*/
for A,B,C,D,E,F in W do if Order(A) eq 2 and Order(B) eq 2 and
Order(C) eq 2 and Order(D) eq 2 and (A,B) eq Id(W) and (A,C) eq
Id(W) and (A,D) eq Id(W) and (B,C) eq Id(W) and (B,D) eq Id(W)
and (C,D) eq Id(W) and Order(E) eq 4 and Order(F) eq 2 and
Order((e*f)) eq 3 and A^E eq B and B^E eq C and C^E eq D and
D^E eq A and A^F eq B and B^F eq A and C^F eq C and D^F eq
D and W eq sub<W|A,B,C,D,E,F> then A,B,C,D,E,F; break; end if;
end for;
/*Now the Classes*/
CC:=Classes(W);
#CC;
C[2][1];
C[8][1];
for A,B,C,D,F in Class(W,CC[2][3]) join Class(W,CC[3][3]) join
Class(W,CC[4][3]) join Class(W,CC[5][3]) join Class(W,CC[6][3])
join Class(W,CC[7][3]) join Class(W,CC[8][3]) do for E in
Class(W,CC[11][3]) join Class(W,CC[12][3]) join Class(W,CC[13][3]) j
oin Class(W,CC[14][3]) join Class(W,CC[15][3]) join
Class(W,CC[16][3]) do if Order(E*F) eq 3 and (A,B) eq Id(W) and (A,C)
eq Id(W) and (A,D) eq Id(W) and (B,C) eq Id(W) and (B,D) eq Id(W) and

```

```
(C,D) eq Id(W) and A^E eq B and B^E eq C and C^E eq D and D^E eq A
and A^F eq B and B^F eq A and C^F eq C and D^F eq D and W eq
sub<W|A,B,C,D,E,F> then A,B,C,D,E,F; break; end if; end for; end for;
```

```
G<a,b,c,d,e,f>:=Group<a,b,c,d,e,f|a^2,b^2,c^2,d^2,(a,b),(a,c),(a,d),
(b,c),(b,d),(c,d),e^4,f^2,(e*f)^3,a^e=b,b^e=c,c^e=d,d^e=a,a^f=b,
b^f=a,c^f=c,d^f=d>;
```

```
#G;
```

```
f,N,k:=CosetAction(G,sub<G|Id(G)>);
```

```
A:=N.1;B:=N.2; C:=N.3; D:=N.4; E:=N.5; F:=N.6;
```

```
NN:=G;
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
```

```
ArrayP:= [Id(N): i in [1..384]];
```

```
for i in [2..384] do
```

```
P:= [Id(N): 1 in [1..#Sch[i]]];
```

```
for j in [1..#Sch[i]] do
```

```
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
```

```
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
```

```
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
```

```
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
```

```
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
```

```
if Eltseq(Sch[i])[j] eq -5 then P[j]:=E^-1; end if;
```

```
if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
```

```
end for;
```

```
PP:=Id(N);
```

```
for k in [1..#P] do
```

```
PP:=PP*P[k]; end for;
```

```
ArrayP[i]:=PP;
```

```
end for;
```

```
S:=Subgroups(N);
```

```
for i in [1..#S] do if Index(N,S[i]'subgroup) eq 8 then i; end if;
```

```
end for;
```

```
H:=S[171]'subgroup;
```

```
for i in [1..384] do if ArrayP[i] eq H.1 then Sch[i]; end if; end for;
```

```
for i in [1..384] do if ArrayP[i] eq H.2 then Sch[i]; end if; end for;
```

```
for i in [1..384] do if ArrayP[i] eq H.3 then Sch[i]; end if; end for;
```

```
for i in [1..384] do if ArrayP[i] eq H.4 then Sch[i]; end if; end for;
```

```
for i in [1..384] do if ArrayP[i] eq H.5 then Sch[i]; end if; end for;
```

```
G<a,b,c,d,e,f>:=Group<a,b,c,d,e,f|a^2,b^2,c^2,d^2,(a,b),(a,c),(a,d),
```

```
(b,c),(b,d),(c,d),e^4,f^2,(e*f)^3,a^e=b,b^e=c,c^e=d,d^e=a,a^f=b,
```

```
b^f=a,c^f=c,d^f=d>;
```

```
HH:=sub<G|a,a * b * c * d * e * f * e^2 * f,e^2 * f * e^-1,a*c,b*c>;
```

```
#HH;
```

```

f,G1,k:=CosetAction(G,HH);
#G1;
IsIsomorphic(G1,W);
S:=Sym(8);
A:=S!(5,6);
B:=S!(7,8);
C:=S!(1,2);
D:=S!(3,4);
E:=S!(1,3,6,7)(2,4,5,8);
F:=S!(5,8)(6,7);
N:=sub<S|A,B,C,D,E,F>;
NN:=G;
N1:=Stabiliser(N,[1]);
N1;
for i in [1..384] do if ArrayP[i] eq N!(3,6)(4,5) then print Sch[i];
end if; end for;
G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^2,(a,b),(a,c),
(a,d),(b,c),(b,d),(c,d),e^4,f^2,(e*f)^3,a^e=b,b^e=c,c^e=d,d^e=a,
a^f=b,b^f=a,c^f=c,d^f=d, t^2, (t,a), (t,b), (t,d), (t,f),(t,e*f*e^-1)>;
for i,j,k,l,m,n in [0..10] do
G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^2,(a,b),
(a,c),(a,d),(b,c),(b,d),(c,d),e^4,f^2,(e*f)^3,a^e=b,b^e=c,c^e=d,
d^e=a,a^f=b,b^f=a,c^f=c,d^f=d, t^2, (t,a), (t,b), (t,d), (t,f),
(t,e*f*e^-1), (a*t)^i,(e*t*t^(b))^j,(t*f*t)^k,(e*t*t^(d))^l,
(c*t^(f*c))^m,(c*t*e)^n>;if Order(G) ge 18 then i,j,k,l,m,n,
Order(G); end if;
end for;

```

D.3 Wreath Product $Z_2 \text{ Wr } A_4$

```

W:=WreathProduct(CyclicGroup(2),Alt(4));
#W;
G<a,b,c,d,e,f>:=Group<a,b,c,d,e,f|a^2,b^2,c^2,
d^2,(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),e^3,f^2,(e*f)^3,
(e,f)^2,a^e=b,b^e=c,c^e=a,d^e=d,a^f=b,b^f=a,c^f=d,
d^f=c>;
#G;
/*for A,B,C,D,E,F in W do if Order(A) eq 2 and
Order(B) eq 2 and Order(C) eq 2\
and Order(D) eq 2 and (A,B) eq Id(W) and (A,C) eq
Id(W) and (A,D) eq Id(W) and (B,C) eq Id(W) and
(B,D) eq Id(W) and (C,D) eq Id(W)
and Order(E) eq 3 and 0\

```

```

rder(F) eq 2 and Order((e*f)) eq 3 and Order((e,f)) eq 2
and A^E eq B and B^E eq C and C^E eq A and \
D^E eq D and A^F eq B and B^F eq A and C^F eq D and
D^F eq C and W eq sub<W|A,\
B,C,D,E,F> then A,B,C,D,E,F; break; end if; end for;*/
CC:=Classes(W);
for A,B,C,D,F in Class(W,CC[2][3]) join Class(W,CC[3][3]) join
Class(W,CC[4][3]) join Class(W,CC[5][3]) join
Class(W,CC[6][3]) do for E in Class(W,CC[7][3]) join
Class(W,CC[8][3]) do if Order(E*F) eq 3 and Order((E,F))
eq 2 and (A,B) eq Id(W) and (A,C) eq Id(W) and (A,D) eq
Id(W) and (B,C) eq Id(W) and (B,D) eq Id(W) and (C,D) eq
Id(W) and A^E eq B and B^E eq C and C^E eq A and
D^E eq D and A^F eq B and B^F eq A and C^F eq D and
D^F eq C and W eq sub<W|A,B,C,D,E,F> then A,B,C,D,E,F;
break; end if; end for; end for;
S:=Sym(8);
A:=S!(3,4);
B:=S!(7,8);
C:=S!(1,2);
D:=S!(5,6);
E:=S!(1,3,8)(2,4,7);
F:=S!(1,5)(2,6)(3,8)(4,7);
N:=sub<S|A,B,C,D,E,F>;
NN:=G;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..192]];
for i in [2..192] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=E; end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=E^-1; end if;
if Eltseq(Sch[i])[j] eq 6 then P[j]:=F; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..192] do if ArrayP[i] eq N!(3,4) then print Sch[i];

```

```

end if; end for;
for i in [1..192] do if ArrayP[i] eq N!(7,8) then print Sch[i];
end if; end for;
for i in [1..192] do if ArrayP[i] eq N!(5,6) then print Sch[i];
end if; end for;
for i in [1..192] do if ArrayP[i] eq N!(3,6,7,4,5,8) then
print Sch[i];
end if; end for;
G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^2,(a,b)
,(a,c),(a,d),(b,c),(b,d),(c,d),e^3,f^2,(e*f)^3,(e,f)^2,a^e=b,
b^e=c,c^e=a,d^e=d,a^f=b,b^f=a,c^f=d,d^f=c,t,(t,a),(t,b),(t,d),
(t,a*f*e*f)>;
for i,j,k,l,m,n in [0..10] do
G<a,b,c,d,e,f,t>:=Group<a,b,c,d,e,f,t|a^2,b^2,c^2,d^2,
(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),e^3,f^2,(e*f)^3,(e,f)^2,
a^e=b,b^e=c,c^e=a,d^e=d,a^f=b,b^f=a,c^f=d,d^f=c,t,
(t,a),(t,b),(t,d),(t,a*f*e*f),(a*t)^i,(e*t*t^(b))^j,(t*f*t)^k,
(e*t*t^(d))^l,(c*t^(f*c))^m,(c*t*e)^n>;
if Order(G) ge 18 then i,j,k,l,m,n, Order(G); end if; end for;

```

D.4 Wreath Product $Z_3 \text{ Wr } Z_2$

```

W:=WreathProduct(CyclicGroup(3),CyclicGroup(2));
G<a,b,c>:=Group<a,b,c|a^3,b^3,(a,b),c^2,a^c=b,b^c=a>;
#G;
for A,B,C in W do if Order(A) eq 3 and Order(B) eq 3 and
(A,B) eq Id(W) and Order(C) eq 2 and A^C eq B and B^C eq A
and W eq sub<W|A,B,C> then A,B,C; break; end if; end for;
S:=Sym(6);
A:=S!(4,5,6);
B:=S!(1,2,3);
C:=S!(1,4)(2,5)(3,6);
N:=sub<S|A,B,C>;
N eq W;
N1:=Stabiliser(N,1);
G<a,b,c,t>:=Group<a,b,c,t|a^3,b^3,(a,b),c^2,a^c=b,
b^c=a,t^2,(t,a)>;
for i,j,k,l,m,n in [0..10] do
G<a,b,c,t>:=Group<a,b,c,t|a^3,b^3,(a,b),c^2,a^c=b,b^c=a,t^2,
(t,a),(a*t)^i,(t*t^(b))^j,(t*a*t)^k,(t^c*t)^l,(c*t^a)^m,(b*t*c)^n>;
if Order(G) ge 18 then i,j,k,l,m,n, Order(G); end if; end for;

```


Appendix E

Difficult Extension Problems

E.1 Mixed Extension Using Database

```

a:=0;b:=0;c:=0;d:=3;e:=0;
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^3,t^2,(t,y),(t,y^x),(x*t)^a,
(x*t*t^x)^b,(x*y*t*t^x*t^(y))^c,(x^3*t*t^x*t^y)^d,(x*y*t)^e>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
D:=DerivedGroup(G1);
D;
s:=IsIsomorphic(D,NL[3]);
IsPerfect(D);
DerivedGroup(D) eq D;
DB := PerfectGroupDatabase();
"A5" in TopQuotients(DB);
ExtensionPrimes(DB, "A5");
ExtensionExponents(DB, "A5", 5);
ExtensionNumbers(DB, "A5", 5, 3);
H1:=Group(DB, "A5", 5, 3,1);
H1;
P1:=PermutationGroup(DB, "A5", 5, 3,1);
s:=IsIsomorphic(NL[3],P1);
s;
P2:=PermutationGroup(DB, "A5", 5, 3,2);
s:=IsIsomorphic(NL[3],P2);
s;
H2:=Group(DB, "A5", 5, 3,2);

```

```

H2;
q,ff:=quo<NL[3]|NL[2]>;
A:=q.1;
B:=q.2;
ff(NL[3].1) eq A;
ff(NL[3].2) eq B;
T:=Transversal(NL[3],NL[2]);
ff(T[2]) eq A;
ff(T[3]) eq B;
#T;
(A*B)^5;
(T[2]*T[3])^5 in NL[2];
(T[2]*T[3])^5;
X:=NL[2].2;
Y:=NL[2].3;
Z:=NL[2].4;
C:=T[2];
D:=T[3];
N:=sub<G1|X,Y,Z>;
NN<k,1,m>:=Group<k,1,m|k^5,1^5,m^5,(k,1),(k,m),(1,m)>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..125]];
for i in [2..125] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=X; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=X^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=Y; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=Y^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=Z; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=Z^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
X eq NL[3].3;
Y eq NL[3].4;
Z eq NL[3].5;
for i in [1..125] do if ArrayP[i] eq (C*D)^5 then Sch[i];
end if; end for;
A:=[Id(NN) : i in [1..6]];
for i in [1..125] do if X^C eq ArrayP[i] then A[1]:=Sch[i];

```

```

Sch[i]; end if; end for;
for i in [1..125] do if X^D eq ArrayP[i] then A[2]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..125] do if Y^C eq ArrayP[i] then A[3]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..125] do if Y^D eq ArrayP[i] then A[4]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..125] do if Z^C eq ArrayP[i] then A[5]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..125] do if Z^D eq ArrayP[i] then A[6]:=Sch[i];
Sch[i]; end if; end for;
A;
NN<a,b,k,l,m>:=Group<a,b,k,l,m|k^5,l^5,m^5,(k,l),(k,m),(1,m),
a^2,b^3,(a*b)^5=k * m * l^-2,
k^a=m^-1,k^b= k * m,l^a= m * k^-1 * l^-1,l^b= m,m^a= k^-1,
m^b= l^-1 * m^-1>;
#NN;
N1:=CosetAction(NN,sub<NN|Id(NN)>);
f1,N1,k1:=CosetAction(NN,sub<NN|Id(NN)>);
#N1;
s:=IsIsomorphic(N1,NL[3]);
s;
for r in G1 do if Order(r) eq 2 and r notin NL[3] and G1 eq
sub<G1|NL[3],r> then R:=r; break; end if; end for;
G1 eq sub<G1|NL[3],R>;
N:=sub<G1|X,Y,Z,C,D>;
NN<a,b,k,l,m>:=Group<a,b,k,l,m|k^5,l^5,m^5,(k,l),(k,m),(1,m),
a^2,b^3,(a*b)^5=k * m * l^-2,
k^a=m^-1,k^b= k * m,l^a= m * k^-1 * l^-1,l^b= m,m^a= k^-1,
m^b= l^-1 * m^-1>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..7500]];
for i in [2..7500] do
P:=[Id(N): 1 in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=X; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=X^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=Y; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=Y^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=Z; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=Z^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq 5 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq -5 then P[j]:=D^-1; end if;

```

```

end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
A:=[Id(NN) : i in [1..5]];
for i in [1..7500] do if X^R eq ArrayP[i] then A[1]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..7500] do if Y^R eq ArrayP[i] then A[2]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..7500] do if Z^R eq ArrayP[i] then A[3]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..7500] do if C^R eq ArrayP[i] then A[4]:=Sch[i];
Sch[i]; end if; end for;
for i in [1..7500] do if D^R eq ArrayP[i] then A[5]:=Sch[i];
Sch[i]; end if; end for;

```

E.2 Extension Problem $(2 \times 11)^\bullet : (PGL(2, 11))$

```

i:=0;j:=0;k:=0;l:=4;m:=2;
G<x,y,t>:=Group<x,y,t|x^11,y^2,(x^-1*y)^2,t^2,(t,y*x),
(y*t^(x^-1))^i,
(x*t)^j,
(x^2*t*t^y)^k,
(x^5*t)^l,
(x*t*t^y*t^x)^m
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
Center(G1);
NL:=NormalLattice(G1);
NL;
IsAbelian(NL[4]);
q,ff:=quo<G1|NL[4]>;
CompositionFactors(q);
s,t:=IsIsomorphic(q,PGL(2,11));
s;
FPGroup(q);
H<a,b,c>:=Group<a,b,c|a^-11,b^2,c^2,(a^-1*b)^2,(a*c*b)^2,
a^2*c*a^-2*c*a*b*c,\
(a*c*a*c*a^2)^2>;
#H;

```

```

#PGL(2,11);
f1,H1,k1:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(H1,q);
s;
A:=t(f1(a));
B:=t(f1(b));
C:=t(f1(c));
T:=Transversal(G1,NL[4]);
for i in [1..#T] do if ff(T[i]) eq A then i; end if; end for;
for i in [1..#T] do if ff(T[i]) eq B then i; end if; end for;
for i in [1..#T] do if ff(T[i]) eq C then i; end if; end for;
A:=T[367];
B:=T[694];
C:=T[495];
NL[4];
D:=DirectProduct(CyclicGroup(2),CyclicGroup(11));
s,t:=IsIsomorphic(D,NL[4]);
s;
for d,e in NL[4] do if Order(d) eq 2 and Order(e) eq 11 and d^e eq d
then D:=d; E:=e; end if; end for;
Order(D);
H;
for i in [0..1] do for j in [0..10] do if A^-11 eq D^i*E^j then
i,j; break; end if; end for; end for;

for i in [0..1] do for j in [0..10] do if B^2 eq D^i*E^j then
i,j; break; end if; end for; end for;

for i in [0..1] do for j in [0..10] do if C^2 eq D^i*E^j then
i,j; break; end if; end for; end for;

for i in [0..1] do for j in [0..10] do if (A^-1*B)^2 eq D^i*E^j
then i,j; break; end if; end for; end for;

for i in [0..1] do for j in [0..10] do if (A*C*B)^2 eq D^i*E^j
then i,j; break; end if; end for; end for;

for i in [0..1] do for j in [0..10] do if A^2*C*A^-2*C*A*B*C eq
D^i*E^j then i,j; break; end if; end for; end for;

for i in [0..1] do for j in [0..10] do if (A*C*A*C*A^2)^2 eq D^i*E^j
then i,j;
break; end if; end for; end for;

```

```

for i,b,c in [0..1] do for j in [0..10] do for a in [0..21] do if A^D eq
  A^a*B^b*C^c*D^i*E^j
then a,b,c,i,j; break; end if; end for; end for;end for;

```

```

for i,b,c in [0..1] do for j in [0..10] do for a in [0..21] do if B^D eq
  A^a*B^b*C^c*D^i*E^j
then a,b,c,i,j; break; end if; end for; end for;end for;

```

```

for i,b,c in [0..1] do for j in [0..10] do for a in [0..21] do if C^D eq
  A^a*B^b*C^c*D^i*E^j then a,b,c,i,j; break; end if; end for;
  end for;end for;

```

```

for i,b,c in [0..1] do for j in [0..10] do for a in [0..21] do
if A^E eq A^a*B^b*C^c*D^i*E^j then a,b,c,i,j; break; end if; end for;
end for;end for;

```

```

for i,b,c in [0..1] do for j in [0..10] do for a in [0..21] do if B^E
eq A^a*B^b*C^c*D^i*E^j then a,b,c,i,j; break; end if; end for; end for;
end for;

```

```

for i,b,c in [0..1] do for j in [0..10] do for a in [0..21] do if C^E eq
A^a*B^b*C^c*D^i*E^j then a,b,c,i,j; break; end if; end for; end for;
end for;

```

```

H<a,b,c,d,e>:=Group<a,b,c,d,e|a^-11=d,b^2,c^2,(a^-1*b)^2,(a*c*b)^2=
e^6,a^2*c*a^-2*c*a*b*c=e,(a*c*a*c*a^2)^2=e^10,d^2,e^11,(d,e),a^d=a,
b^d=b,c^d=c,a^e=a,b^e=b*e^2,c^e=c*e^2>;
f2,H2,k2:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(H2,G1);
s;

```

E.3 Extension Problem $2^\bullet(U(3,4) : 2)$

```

i:=2;j:=10;k:=10;l:=10;
G<x,y,t>:=Group<x,y,t|x^-9,y^2,(x^-1*y)^2,t^2,(t,y*x),
(x^3*t)^i,
(x^4*t^x)^j,
(y*t)^k,
(x*t*t^x^2*t)^l
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
NL:=NormalLattice(G1);

```

```

NL;
Center(G1);
q,ff:=quo<G1|NL[2]>;
CompositionFactors(q);
Center(q);
Permutation group acting on a set of cardinality 1600
Order = 1
nl:=NormalLattice(q);
nl;
//FROM ATLAS WE HAVE A PRESENTATION FOR U(3,4):2 //
H<a,b>:=Group<a,b|a^2,b^3,(a*b)^8,(a,b)^13,
(a,b*a*b*a*b*a*b^-1*a*b*a*b)^2,(a\
,b^-1*a*b*a*b)^5>;
#H;
f1,H1,k1:=CosetAction(H,sub<P|Id(H)>);
s,t:=IsIsomorphic(P1,q);
s;
T:=Transversal(G1,NL[2]);
Current total memory usage: 96.2MB, failed memory request: 13982.0MB
System error: Out of memory.
At this point we would of found elements from the above presentation
that can be written in terms of the center but Magma ran out of storage.

```

Bibliography

- [BC94] W. Bosma and J. Cannon. *Handbook of MAGMA Functions*. University of Sydney, 1994.
- [Cur07] Robert Curtis. *Symmetric Generation of Groups*. volume 111 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.
- [Gri15] Dustin Grindstaff. *Symmetric Presentations and Generation*. CSUSB, 2015.
- [Hum96] John F. Humphreys. *A course in Group Theory*. Oxford University Press, 1996.
- [Isa76] I. Martin Isaacs. *Character Theory of Finite Groups*. Academic Press, New York, 1976.
- [Lam04] AlNazzal Lamies. *Homomorphic images of semi-direct products*. CSUSB, 2004.
- [Led87] Walter Ledermann. *Introduction to group charters*, volume 2. Cambridge University Press, 1987.
- [Maj14] May Majid. *Construction of Finite Homomorphic Images*. CSUSB, 2014.
- [Rot95] Joseph J. Rotman. *An Introduction to the Theory of Groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [Tra13] Jesse Train. *Group Theory in Music*. CSUSB, 2013.
- [WB99] Robert Wilson and John Bray. *Atlas of finite group representations*. <http://brauer.maths.qmul.ac.uk/Atlas/v3/>, 1999.
- [Why06] Sophie Whyte. *Symmetric Generation: Permutation Images and Irreducible Monomial Representations*. The University of Birmingham, 2006.