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DIFFUSION APPROXIMATION FOR RETRIAL QUEUE WITH COLLISIONS AND NON-PERSISTENT CUSTOMERS

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This paper is devoted to the analysis of retrial queue with an arbitrary distribution of service times, collisions, and non-persistent customers. Our aim is to investigate the number of customers in the orbit of the system. To this end, we use the asymptotic-diffusion method to build a diffusion approximation for the steady-state distribution of the number of customers in the orbit.

Keywords: *Retrial queue, collision, non-persistent customer, diffusion approximation.*

Introduction

Retrial queues arose as models of communication systems. The basic phenomenon of such systems is the retrial behavior of customers: if the server is busy upon arrival, the customer enters the orbit and repeats the attempt to access the server after a random amount of time.

There are several modifications of retrial queues that reflect the system features such as collisions and non-persistent customers, which appear in various switching communication systems and CSMA-based networks [1]. In recent years, queueing systems with collisions are of interest due to the reborn of IEEE 802.11 wireless LANs. In paper [2], authors describe the markovian retrial queue with collisions and shows applications of persistence to modeling CSMA-CD protocols. In paper [3], the author consider similar markovian model and takes into account the impatience of customers.

Nazarov and Sztrik with their research group have considered several models of finite-source retrial queues with collisions [4, 5, 6]. The phenomena of non-persistent customers in retrial queues was considered in [7]. Lakaour and his colleagues have considered markovian models with collisions, transmission errors and unreliable server [8, 9].

We consider retrial queue with arbitrary distribution of service times, collisions and r-persistent customers. We build diffusion approximation for the number of customers in the orbit and construct the approximation of its probability distribution under the limit condition of growing delay in the orbit.

1. Model Description and Problem Definition

We consider a retrial queue with an arbitrary distribution of service times defined by the distribution function $B(x)$. The input is stationary Poisson process with rate λ. If the server is idle upon arrival, the incoming customer occupies it for service. Otherwise, the collision occurs and one of the customers joins the orbit. The other customer can also join the orbit with probability r or leave the system with probability $(1-r)$.

At the orbit, a customer waits for some random time and tries again to occupy the server. The duration of delay follows an exponential distribution with rate σ.

Let $k(t)$ denote the state of the server at instant t: 0, if the server is idle; 1, if the server is busy. Let $i(t)$ denote the number of customers in the orbit at instant t. We also introduce process $z(t)$, which represents the residual service time. Thus, process $\{k(t),\ i(t),\ z(t)\}$ has variable number of components and exhaustively describes the system state. We denote the probability distribution of process $\{k(t),\ i(t),\ z(t)\}$ as follows:

$$P_0(i,t) = P\{k(t)=0, i(t)=i\},\ P_1(i,z,t) = P\{k(t)=1, i(t)=i, z(t)<z\},$$

and introduce the partial characteristic functions

$$H_0(u,t) = \sum_{i=0}^{\infty} e^{jui} P_0(i,t),\ H_1(u,z,t) = \sum_{i=0}^{\infty} e^{jui} P_1(i,z,t),$$

where j is the imaginary unit. The Kolmogorov system of differential equations for the partial characteristic functions has the following form:

$$\frac{\partial H_0(u,t)}{\partial t} = -\lambda H_0(u,t) + j\sigma \frac{\partial H_0(u,t)}{\partial u} + \frac{\partial H_1(u,0,t)}{\partial z} +$$

$$+\lambda e^{ju}(1+r(e^{ju}-1))H_1(u,t) - j\sigma(1+r(e^{ju}-1))\frac{\partial H_1(u,t)}{\partial u},$$

$$\frac{\partial H_1(u,z,t)}{\partial t} = \frac{\partial H_1(u,z,t)}{\partial z} - \frac{\partial H_1(u,0,t)}{\partial z} - \lambda H_1(u,z,t) +$$

$$+j\sigma \frac{\partial H_1(u,z,t)}{\partial u} + \lambda H_0(u,t)B(z) - j\sigma e^{-ju}\frac{\partial H_0(u,t)}{\partial u}B(z). \qquad (1)$$

After that, we sum up the equations of system (2). Taking the limit by $z \to \infty$, we obtain

$$\frac{\partial H(u,t)}{\partial t} = (e^{ju}-1)\times$$

$$\times\left\{j\sigma e^{-ju}\frac{\partial H_0(u,t)}{\partial u}+\lambda(1+re^{ju})H_1(u,t)-j\sigma r\frac{\partial H_1(u,t)}{\partial u}\right\}. \quad (2)$$

Solving system (2) and equation (4) in the limit by $\sigma \to 0$, we derive drift and diffusion coefficients of approximating diffusion process.

2. Asymptotic-Diffusion Analysis

In system (2) and equation (4), we introduce the following notations:

$$\sigma=\varepsilon,\ u=\varepsilon w,\ \tau=\varepsilon t,$$

$$H_0(u,t)=F_0(w,\tau,\varepsilon),\ H_1(u,z,t)=F_1(w,z,\tau,\varepsilon), \quad (3)$$

and obtain the system of equations

$$\varepsilon\frac{\partial F_0(w,\tau,\varepsilon)}{\partial\tau}=-\lambda F_0(w,\tau,\varepsilon)+j\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w}+\frac{\partial F_1(w,0,\tau,\varepsilon)}{\partial z}+$$

$$+\lambda e^{jw\varepsilon}(1+r(e^{jw\varepsilon}-1))F_1(w,\tau,\varepsilon)-j(1+r(e^{jw\varepsilon}-1))\frac{\partial F_1(w,\tau,\varepsilon)}{\partial w},$$

$$\varepsilon\frac{\partial F_1(w,z,\tau,\varepsilon)}{\partial\tau}=\frac{\partial F_1(w,z,\tau,\varepsilon)}{\partial z}-\frac{\partial F_1(w,0,\tau,\varepsilon)}{\partial z}-\lambda F_1(w,z,\tau,\varepsilon)+$$

$$+j\frac{\partial F_1(w,z,\tau,\varepsilon)}{\partial w}+\lambda F_0(w,\tau,\varepsilon)B(z)-je^{-jw\varepsilon}\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w}B(z),$$

$$\varepsilon\frac{\partial F(w,\tau,\varepsilon)}{\partial\tau}=(e^{jw\varepsilon}-1)\times$$

$$\times\left\{je^{-jw\varepsilon}\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w}+\lambda(1+re^{jw\varepsilon})F_1(w,\tau,\varepsilon)-jr\frac{\partial F_1(w,\tau,\varepsilon)}{\partial w}\right\}. \quad (4)$$

We solve system (1) in the limit by $\varepsilon \to 0$ and formulate the following theorem.

Theorem 1. In considered retrial queue, under the limit condition $\sigma \to 0$, the following equality holds:

$$\lim_{\sigma\to 0}\mathbb{E}\left\{\sigma i\left(\frac{\tau}{\sigma}\right)\right\}=x(\tau),$$

where $x(\tau)$ is a solution of differential equation

$$x'(\tau)=-x(\tau)r_0+[\lambda+(\lambda+x(\tau))r]r_1, \quad (5)$$

values r_0, r_1 have the following form:

$$r_0=\frac{1}{2-B^*(\lambda+x)},\ r_1=\frac{1-B^*(\lambda+x)}{2-B^*(\lambda+x)}. \quad (6)$$

Here $B^*(s)$ is the Laplace-Stieltjes transform (LST) of the distribution function of the service times $B(x)$.

From (4), we denote function

$$a(x) = -xr_0 + (\lambda + (\lambda + x)r)r_1. \qquad (7)$$

For the second step of analysis, we make the following substitutions in equations (2)-(4):

$$H_0(u,t) = e^{j\frac{u}{\sigma}x(\sigma t)} H_0^{(2)}(u,t),\ H_1(u,z,t) = e^{j\frac{u}{\sigma}x(\sigma t)} H_1^{(2)}(u,z,t).$$

Thus, we obtain the equations for the partial characteristic functions of centered number of customers in the orbit. After that, we introduce the following substitutions:

$$\sigma = \varepsilon^2,\ u = w\varepsilon,\ \tau = t\varepsilon^2,$$

$$H_0^{(2)}(u,t) = F_0^{(2)}(w,\tau,\varepsilon),\ H_1^{(2)}(u,z,t) = F_1^{(2)}(w,z,\tau,\varepsilon), \qquad (8)$$

and obtain the system of equations

$$\varepsilon^2 \frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + jw\varepsilon a(x) F_0^{(2)}(w,\tau,\varepsilon) = -(\lambda + x) F_0^{(2)}(w,\tau,\varepsilon) +$$
$$+ j\varepsilon \frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} + \frac{\partial F_1^{(2)}(w,0,\tau,\varepsilon)}{\partial z} +$$
$$+ (\lambda e^{jw\varepsilon} + x)(1 + r(e^{jw\varepsilon} - 1)) F_1^{(2)}(w,\tau,\varepsilon) -$$
$$- j\varepsilon(1 + r(e^{jw\varepsilon} - 1)) \frac{\partial F_1^{(2)}(w,\tau,\varepsilon)}{\partial w},$$

$$\varepsilon^2 \frac{\partial F_1^{(2)}(w,z,\tau,\varepsilon)}{\partial \tau} + jw\varepsilon a(x) F_1^{(2)}(w,z,\tau,\varepsilon) = \frac{\partial F_1^{(2)}(w,z,\tau,\varepsilon)}{\partial z} -$$
$$- \frac{\partial F_1^{(2)}(w,0,\tau,\varepsilon)}{\partial z} - (\lambda + x) F_1^{(2)}(w,z,\tau,\varepsilon) + j\varepsilon \frac{\partial F_1^{(2)}(w,z,\tau,\varepsilon)}{\partial w} +$$
$$+ (\lambda + x e^{-jw\varepsilon}) F_0^{(2)}(w,\tau,\varepsilon) B(z) - j\varepsilon e^{-jw\varepsilon} \frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} B(z),$$

$$\varepsilon^2 \frac{\partial F^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + jw\varepsilon a(x) F^{(2)}(w,\tau,\varepsilon) =$$
$$= (e^{jw\varepsilon} - 1) \left\{ j\varepsilon e^{-jw\varepsilon} \frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} - x e^{-jw\varepsilon} F_0^{(2)}(w,\tau,\varepsilon) + \right.$$

$$+(\lambda + r(\lambda e^{jw\varepsilon} + x))F_1^{(2)}(w,\tau,\varepsilon) - j\varepsilon r \frac{\partial F_1^{(2)}(w,\tau,\varepsilon)}{\partial w} \Bigg\}. \tag{9}$$

Solving system (9) in the limit by $\varepsilon \to 0$, we present Theorem 2.

Theorem 2. Function $\lim\limits_{\varepsilon\to 0} F_k^{(2)}(w,\tau,\varepsilon) = F_k^{(2)}(w,\tau)$ has the following form:

$$F_k^{(2)}(w,\tau) = \Phi(w,\tau) r_k,$$

where r_k is given by (6), function $\Phi(w,\tau)$ is the solution of equation

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = w \frac{\partial \Phi(w,\tau)}{\partial w} a'(x) + \frac{(jw)^2}{2} \Phi(w,\tau) b(x). \tag{10}$$

Function $a(x)$ is defined by (5), $b(x)$ is determined as follows:

$$b(x) = a(x) + 2[-(\lambda + x)(1+r)g_0 + x r_0 + r\lambda r_1], \tag{11}$$

where

$$g_0 = \frac{(a(x)+x)(1 - B^*(\lambda+x)) + (\lambda+x)a(x)B^{*\prime}(\lambda+x)}{(\lambda+x)(2 - B^*(\lambda+x))^2}.$$

Here equation (10) is the Fourier transform of the Fokker-Planck equation for the process approximating the number of customers in the orbit of considered retrial queue. If we make the inverse Fourier transform, we can see that the drift coefficient of the obtained diffusion limit is $a(x)$ and diffusion coefficient if $b(x)$.

Theorem 3. Discrete function $PD(i)$ is the approximation of the probability distribution of the number of customers in the orbit and has the following form:

$$PD(i) = \frac{D(i\sigma)}{\sum\limits_{n=0}^{\infty} D(n\sigma)}, \tag{12}$$

where

$$D(z) = \frac{1}{b(z)} \int\limits_0^z \frac{2}{\sigma} \frac{a(x)}{b(x)} dx.$$

3. Conclusion

We have considered the retrial queue with collisions and r-persistent customers. For the number of customers in the orbit, we have derived the approximation of the probability distribution (12). In future, we plan to investigate the stability mode in such system based on the obtained approximation.

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