6-2014

Homormophic Images and their Isomorphism Types

Diana Herrera
herrd300@csusb.edu

Homomorphic Images and their Isomorphism Types

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Diana Herrera Gil

June 2014

Homomorphic Images and their Isomorphism Types

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Diana Herrera Gil

June 2014

Approved by:

Dr.Zahid Hasan , Committee Chair | Date

Dr. Belisario Ventura , Committee Member

Dr. J.Paul Vicknair, Committee Member

Dr. Peter Williams, Chair,
Department of Mathematics

Dr. Charles Stanton
Graduate Coordinator,
Department of Mathematics

Abstract

In this thesis we have presented original homomorphic images of permutations and monomial progenitors. In some cases we have used the double coset enumeration technique to construct the images and for all of the homomorphic images that we have discovered, the isomorphism type of each group is given. The homomorphic images discovered include Linear groups, Alternating groups, and two sporadic simple groups J_1 and $J_2 \times 2$ where J_1 is the smallest Janko group and J_2 is the second Janko sporadic group.

Acknowledgements

I would like to thank Dr.Zahid Hasan for his patience, encouragement, and support. He's been a phenomal person and has shown me that anything is possible if we work hard and put the time into it. I would also like to thank my committee, Dr.Belisario Ventura and Dr.J.Paul Vicknair. Thank you both for helping me reach my academic goal. My gratitude goes to Dr.Charles Stanton, Dr.Peter Williams, and the Mathematics department for helping me thru the M.A. program. Last but not least, I would like to thank my husband for all of his understanding and words of encouragement, my friends, and family for their moral support.

Gracias a mis padres por todo el apoyo que me han brindando ya que ellos son y seran mi mas grande motivación para seguir adelante.

Table of Contents

List of Tables

List of Figures

Chapter 1

Background Information

The following terminology and theorems will be used throughout this thesis:

1.1 Groups

Definition 1.1 [Rot95] *A* **group** *is a nonempty set, G, equipped with an associative operation, $*$, such that:*

(i) $e * a = a = a * e$ for all $a \in G$.

(ii) for every $a \in G$, there is an element $b \in G$ with $a * b = e = b * a$.

To avoid any confussion, we will write G instead of $(G, *)$ and we'll keep in mind that $*$ continues to exist.

Definition 1.2 [Rot95] *Let G and H be groups. A function $f : G \mapsto H$ is a* **homomorphism** *if, for all $a, b \in G$,*

$$f(ab) = f(a)f(b)$$

An **isomorphism** *is a homomorphism that is also a bijection. We say that G is isomorphic to H, denoted by $G \cong H$, if there exist an isomorphism $f : G \mapsto H$.*

1.2 Group Action

A group action is a way to describe elements of a group acting on elements of a set in certain ways. Its a very useful abstraction, and is used in many fields, such as geometry and algebra itself. The following definitions and theorems are used:

Definition 2.1 [Rot95] *If χ is a set and G is a group, then χ is a G-**set** if there is a function $\alpha : G \times \chi \mapsto \chi$ (called an **action**), denoted by $\alpha : (g, x) \mapsto gx$, such that:*

(i) $1a = a$ for all $a \in \chi$; and

(ii) $g(ha) = (gh)a$ for all $g, h \in G$ and $a \in \chi$.

One also says that G **acts** on χ. And $|\chi| = n$, the order of χ, then n is called the **degree** of the G-set χ.

Definition 2.2 [Rot95] The set of all permutations of n letters is called the *symmetric group* on n letters, and is denoted by S_n.

Definition 2.3 [Rot95] *If χ is a G-set and $x \in \chi$, then the G-**orbit** of x is*

$$\vartheta(x) = \{gx : g \in G\} \subset \chi.$$

Throughout this thesis we will be refering to the G-orbit as orbit.

Definition 2.4 [Rot95] *If χ is a G-set and $x \in \chi$, then the **stabiliser** of x, denoted by G_x, is the subgroup*

$$G_x = \{g \in G : gx = x\} \leq G.$$

1.3 Normal Series

Definition 3.1 [Rot95] *Anormal series $G = H_0 \geq H_1 \geq ... \geq H_m = 1$ is a **refinement** of a normal series $G = G_0 \geq G_1 \geq ... \geq G_n = 1$ if $G_0, G_1, ..., G_n$ is a subsequence of $H_0, H_1, ..., H_m$.*

Definition 3.2 [Rot95] *If G has a composition series, then the factor groups of this series are called the **composition factors** of G.*

Chapter 2

Presentation of $2^5 : A_5$

We will show that $2^{*5} : A_5$, where $x \sim (01234)$ and $y \sim (421)$, by performing a double coset enumeration of G over A_5. A symmetric presentation of G is give by:

$$G =< x, y, t | x^5, y^3, (x * y)^2, t^2, (t, y), (t, x^2 * y * x^{-1}), (t * t^x)^2 > .$$

2.1 Relations

As mentioned above, we have the progenitor $2^{*5} : A_5$ being factored by the relation $t_0t_1t_0t_1 = 1$. We will utilize this relation to determine the equal cosets with words composed of t_1, t_2, t_3, and t_4. Simplifying our relation:

$$\begin{aligned} t_0t_1t_0t_1 &= 1 \\ t_0t_1t_0 &= t_1 \\ t_0t_1 &= t_1t_0 \end{aligned}$$

Hence, our relation in simplest terms is $t_0t_1 = t_1t_0$. We will use this relation to help us determine the equal cosets within words of length two. That is, we will conjugate $Nt_0t_1 = Nt_1t_0$ with every element contained in our control group A_5 to obtain the following:

$$\begin{array}{lll} t_0t_2 \sim t_2t_0 & t_1t_2 \sim t_2t_1 & t_2t_3 \sim t_3t_2 \\ t_0t_3 \sim t_3t_0 & t_1t_3 \sim t_3t_1 & t_2t_4 \sim t_4t_2 \\ t_0t_4 \sim t_4t_0 & t_1t_4 \sim t_4t_1 & t_3t_4 \sim t_4t_3 \end{array}$$

To obtain the relations for the words of length three we will use the relations found for words of length two and right multiply each relation to obtain longer relations. We'll look in detail as to how we can obtain a relation of length three using the relation $t_0t_1 \sim t_1t_0$, right multiply both sides by t_2 and use the above relations:

$$\begin{aligned} t_0t_1t_2 &= t_1t_0t_2 \\ &= t_1t_2t_0 \quad (\because t_0t_2 \sim t_2t_0) \\ &= t_2t_1t_0 \quad (\because t_1t_2 \sim t_2t_1) \\ &= t_2t_0t_1 \quad (\because t_1t_0 \sim t_0t_1) \\ &= t_0t_2t_1 \quad (\because t_2t_0 \sim t_0t_2) \end{aligned}$$

Applying this process we can figure out all the relations that will be useful for the words of length three. We will use the following notation for the relations found:

$$012 \sim 102 \sim 120 \sim 210 \sim 201 \sim 021$$

$$031 \sim 301 \sim 310 \sim 130 \sim 103 \sim 013$$

$$041 \sim 401 \sim 410 \sim 140 \sim 104 \sim 014$$

$$241 \sim 421 \sim 412 \sim 142 \sim 124 \sim 214$$

$$231 \sim 321 \sim 312 \sim 132 \sim 123 \sim 213$$

$$341 \sim 431 \sim 413 \sim 143 \sim 134 \sim 314$$

Similarly we can find the relations in regards to the words of length four and length five.

2.2 Double Coset Enumeration

NeN

We begin with NeN, the first double coset, which contains all the words of lenght zero. We have that $NeN = \{N\}$ and it will be denoted $[*]$. Also, $N =< x, y >\cong A_5$ and is of order 60. The number of elements in $[*]$ is $\frac{|N|}{|N|} = \frac{60}{60} = 1$. Hence $[*]$ consists of the single coset, N. This single coset contains the single orbit $\{0, 1, 2, 3, 4\}$. We then take an element from the orbit and right multiply it with the representative coset Ne to obtain Nt_0N. We now have a new double coset, Nt_0N, denoted as [0].

$\mathbf{Nt_0N}$

In this double coset we have the words of length one and the representative is Nt_0. We first find the coset stabilizer, $N^{(0)}$, which consists of all the permutations in N that fix the element 0 and permute $1, 2, 3, 4$. Hence, $N^{(0)} =< (142), (234) >$, is the coset stabilizer in N which contains 12 elements. Also, the number of single cosets in the double coset [0] are found by $\frac{|N|}{|N^{(0)}|} = \frac{60}{12} = 5$. Now, we find the orbits of $N^{(0)}$ on $\{0, 1, 2, 3, 4\}$ by taking the representative from the double coset [0], and conjugating it by the coset stabilizer. Since the element 0 is the only fixed, then $N^{(0)}$ has $\{0\}$ and $\{1, 2, 3, 4\}$ as its orbits. Next, we will take a representative from [0] and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

$$Nt_0 \cdot t_0 = N(t_0)^2 \in NeN$$

Since the orbit of 0 is of length one, then 1 element will collapse from [0] to $[*]$.

$$Nt_0 \cdot t_1 = Nt_0t_1 \in Nt_0t_1N$$

Since the orbit of 1 is of length four, then 4 elements will extend to the double coset Nt_0t_1N denoted as [01].

$\mathbf{Nt_0t_1N}$

We begin by finding the point stabilizer of 0 and 1, N^{01}. To find the point stabilizer we need to find elements that belong to N which fix 0 and 1 and permute $2, 3, 4$. Hence $N^{01} = \{(142), (132), e\}$. Also, since we now have words of length two, our relations will increase the stabiliser, $N^{(01)}$. Now to find the set stabiliser, $N^{(01)}$, we must find a relation in N such that when Nt_0t_1 is conjugated by such relation, Nt_0t_1 goes back to itself. Say that such relation is (10), then we have the following:

$$Nt_0t_1 = Nt_1t_0 \Rightarrow Nt_0t_1^{(10)} = Nt_1t_0 = Nt_0t_1$$

Hence, we have $(10) \in N^{(01)}$. So, $N^{(01)} \geq < N^{01}, (10) >$ and contains 6 elements. Also, we have that the number of single cosets in Nt_0t_1N is $\frac{|N|}{|N^{(01)}|} = \frac{60}{6} = 10$. Now, to find the orbits of $N^{(01)}$ on $\{0, 1, 2, 3, 4\}$ we'll conjugate Nt_0t_1, a representative from the double coset [01] and conjugating it by the point stabilizer to obtain the following:

$0^{N^{(01)}} = \{1\}$, $1^{N^{(01)}} = \{0\}$, and $2^{N^{(01)}} = \{2, 3, 4\}$.

Hence the orbits of $N^{(01)}$ are $\{0\}, \{1\}, \{2, 3, 4\}$. Next, we will take the representative Nt_0t_1 of [01] and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

$$Nt_0t_1 \cdot t_0 = Nt_0t_1t_0 = Nt_1t_0t_0 = Nt_1 \in Nt_0N$$

Since the orbit of 0 is of length one, then 1 element will collapse from [01] to [0].

$$Nt_0t_1 \cdot t_1 = Nt_0t_1t_1 = Nt_0 \in Nt_0N$$

Since the orbit of 1 is of length one, then 1 element will collapse from [01] to [0].

$$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$$

Since the orbit of 2 is of length three, then 3 elements will expand from [01] to [012]. Now $Nt_0t_1t_2N$ is a new double coset which will be represented as [012].

$\mathbf{Nt_0t_1t_2N}$

We begin by finding the point stabilizer of 0,1 and 2, N^{012}. To find the point stabilizer we need to find elements that belong to N which fix 0,1, and 2 and permute $3, 4$. Hence $N^{012} = \{e\}$. Also, since we now have words of length three, our relations will increase the stabiliser, $N^{(012)}$. Now to find the set stabiliser, $N^{(012)}$,we must find a realiton in N such that when $Nt_0t_1t_2$ is conjugated by such relation $Nt_0t_1t_2$ goes back to itself. Say that such relation is (012), then we have the following:

$$Nt_0t_1t_2 = Nt_1t_0t_2 \Rightarrow Nt_0t_1t_2^{(012)} = Nt_1t_2t_0 = Nt_0t_1t_2$$

So, we have $(012) \in N^{(012)}$. We also know that $|N^{(012)}| = 6$ and the number of single cosets in [012] is $\frac{|N|}{|N^{(012)}} = \frac{60}{6} = 10$.

Now, we find the orbits of $N^{(012)}$ on $\{0, 1, 2, 3, 4\}$ by taking the representative $Nt_0t_1t_2$ from the double coset [012] and conjugating it by the point stabilizer to obtain the following:

$0^{N^{(012)}} = \{0, 1, 2\}$, $3^{N^{(012)}} = \{3, 4\}$. Hence the orbits of $N^{(01)}$ are $\{0, 1, 2\}$ and $\{3, 4\}$. Next, we will take the representative $Nt_0t_1t_2$ and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

$$Nt_0t_1t_2 \cdot t_2 = Nt_0t_1t_2t_2 = Nt_0t_1 = Nt_1 \in Nt_0t_1N$$

Since the orbit of 2 is of length three, then 3 elements will collapse from [012] to [01].

$$Nt_0t_1t_2 \cdot t_3 = Nt_0t_1t_2t_3 \ \in \ Nt_0t_1t_2t_3N$$

Since the orbit of 3 is of length two, then 2 elements will extend from [012] to [0123]. Now $Nt_0t_1t_2t_3N$ is a new double coset which will be represented by [0123].

$\mathbf{Nt_0t_1t_2t_3N}$

We begin by finding the point stabilizer of 0,1,2, and 3, N^{0123}. To find the point stabilizer we need to find elements that belong to $N = S_5$ which fix 0,1,2,3 and permute 4. Hence $N^{0123} = \{e\}$. Also, since we now have words of length four, our relations will increase the stabiliser, $N^{(0123)}$. Now to find the set stabiliser, $N^{(0123)}$, we must find a realiton in N such that when $Nt_0t_1t_2t_3$ is conjugated by such relation $Nt_0t_1t_2t_3$ goes back to itself. Say that such relation is (012), then we have the following:

$$Nt_0t_1t_2t_3 = Nt_1t_0t_2t_3 \Rightarrow Nt_0t_1t_2t_3^{(012)} = Nt_1t_2t_0t_3 = Nt_1t_0t_2t_3 = Nt_0t_1t_2t_3$$

So, we have $(012) \in N^{(0123)}$. Also, notice that if we conjugate $Nt_0t_1t_2t_3$ by $(13)(02)$ we obtain the following:

$$Nt_0t_1t_2t_3^{(13)(02)} = Nt_2t_3t_0t_1 = Nt_2t_0t_3t_1 = Nt_0t_2t_3t_1 = Nt_0t_2t_1t_3 = Nt_0t_1t_2t_3$$

We can conclude that $(13)(02) \in N^{(0123)}$. We also know that $|N^{(0123)}| = 12$ and the number of single cosets in [0123] is $\frac{|N|}{|N^{(0123)}|} = \frac{60}{12} = 5$.

Now, we find the orbits of $N^{(0123)}$ on $\{0,1,2,3,4\}$ by taking the representative $Nt_0t_1t_2t_3$ from the double coset [0123] and conjugating it by the point stabilizer to obtain the following:

$$0^{N^{(0123)}} = \{0,1,2,3\},\ 4^{N^{(0123)}} = \{4\}$$

Hence the orbits of $N^{(0123)}$ are $\{0,1,2,3\}, \{4\}$. Next, we will take the representative $Nt_0t_1t_2t_3$ and conjugate it with a representative from each orbit to determine if the elements will extend or collapse:

$$Nt_0t_1t_2t_3 \cdot t_3 = Nt_0t_1t_2t_3t_3 = Nt_0t_1t_2 \in Nt_0t_1t_2N$$

Since the orbit of 3 is of length four, then 4 elements will collapse from [0123] to [012].

$$Nt_0t_1t_2t_3 \cdot t_4 = Nt_0t_1t_2t_3t_4 \in Nt_0t_1t_2t_3t_4N$$

Since the orbit of 4 is of length one, then 1 element will expand from [0123] to [01234]. Now $Nt_0t_1t_2t_3t_4N$ is a new double coset which will be represented as [01234].

$\mathbf{Nt_0t_1t_2t_3t_4N}$

We begin by finding the point stabilizer of 0,1,2,3, and 4, N^{01234}. To find the point stabilizer we need to find elements that belong to $N = S_5$ which fix 0,1,2,3,4. Hence $N^{01234} = \{e\}$. Also, since we now have words of length five, our relations will increase the stabiliser, $N^{(01234)}$. Now to find the stabiliser, $N^{(01234)}$, we must find a realiton in N such that when $Nt_0t_1t_2t_3t_4$ is conjugated by such relation, $Nt_0t_1t_2t_3t_4$ goes back to itself. Say that such relation is (0123), then we have the following:

$$Nt_0t_1t_2t_3t_4^{(0123)} = Nt_1t_2t_3t_0t_4 = Nt_1t_2t_0t_3t_4 = Nt_1t_0t_2t_3t_4 = Nt_0t_1t_2t_3t_4$$

Hence we have $(0123) \in N^{(01234)}$. Also, notice that if we conjugate $Nt_0t_1t_2t_3t_4$ by (12)(34) we obtain the following:

$$Nt_0t_1t_2t_3^{(12)(34)} = Nt_0t_2t_1t_4t_3 = Nt_0t_1t_2t_4t_3 = Nt_0t_1t_2t_3t_4$$

We also know that $|N^{(01234)}| = 60$ and the number of single cosets in [01234] is $\frac{|N|}{|N^{(01234)}|} = \frac{60}{60} = 1$.

Now, we find the orbits of $N^{(01234)}$ on $\{0,1,2,3,4\}$ by taking the representative $Nt_0t_1t_2t_3t_4$ from the double coset [01234] and conjugating it by the point stabilizer to obtain the following:

$$Nt_0t_1t_2t_3t_4^{N^{(01234)}} = \{0,1,2,3,4\}$$

Hence the single orbit of $N^{(01234)}$ is $\{0,1,2,3,4\}$. Next, we will take the representative $Nt_0t_1t_2t_3t_4$ from [01234] and conjugate it with a representative from the orbit found to determine if it'll extend or collapse.

$$Nt_0t_1t_2t_3t_4 \cdot t_4 = Nt_0t_1t_2t_3t_4t_4 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N$$

Since the orbit of 4 is of length five, then 5 elements will collapse from [01234] to [0123]. Since the set of right cosets is closed under right coset multiplication, the double coset enumeration is now complete. Thus, we have summarize all of our work in the Cayley graph.

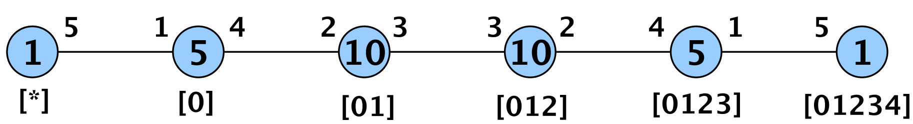


Figure 2.1: Cayley Graph of $2^5 : A_5$ over A_5

Chapter 3

$L_2(49)$ as a Homomorphic Image of $7^{*8} :_m L_2(7)$

We'll begin with the group $L_2(7)$, generated by $x \sim (3,6,7)(4,5,8)$ and $y \sim (1,6,2)(3,8,7)$. We want to induce a linear character from a subgroup H up to $L_2(7)$. We will induce a linear character of H to get an irreducible character of $L_2(7)$ of degree 3. To do so, we find the largest index from the character table of $L_2(7)$. We will induce from a subgroup of index 8. To make the notation easier, let $L_2(7) = G$. Now we must find the character tables for G and H, using MAGMA we have the following:

Now, using Magma, the character table of H is as follows and $\mathbb{Z}_1 = \frac{-1+i\sqrt{7}}{2}$:

Table 3.1: Character Table of H

Conjugacy Classes	C_1	C_2	C_3	C_4	C_5	C_6
Order	1	2	3	4	7	7
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	$\mathbb{Z}_1$	$\mathbb{Z}_1\#3$
χ_3	3	-1	0	1	$\mathbb{Z}_1\#3$	$\mathbb{Z}_1$
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

Also using Magma, the character table of G is as follows, with $\mathbb{Z}_1 = \frac{-1+i\sqrt{7}}{2}$

and J is a root of unity 3:

Table 3.2: Character Table of $L_2(7)$

Conjugacy Classes	C_1	C_2	C_3	C_4	C_5
Order	1	3	3	7	7
χ_1	1	1	1	1	1
χ_2	1	J	$-1-J$	1	1
χ_3	1	$-1-J$	J	1	1
χ_4	3	0	0	$\mathbb{Z}_1$	$\mathbb{Z}_1\#3$
χ_5	3	0	0	$\mathbb{Z}_1\#3$	$\mathbb{Z}_1$

Now, we will induce the second character of H up to G. To do so, we first need to find the right transversals of G. The transversals of H in G are: $\{e,(3,6,7)(4,5,8),$ $(3,7,6)(4,8,5),(1,6,3,2)(4,5,7,8),(1,2,6,8,4,5,3),(1,2,7,3)(4,8,5,6),$ $(1,4,5,2,7,8,3),(1,6,4,3,2,8,5)\}$. Then we will label the transversals as t_1, t_2,...,t_8 respectively. Each transversal will represent a t_i, since there are eight transversals, we will have an 8×8 matrix representation. and we have:
$G = Ht_1 \cup Ht_2 \cup Ht_3 \cup Ht_4 \cup Ht_5 \cup Ht_6 \cup Ht_7 \cup Ht_8$.

Then the matrices $A(xx)$ and $A(yy)$ are a representation of G induced from the representative of H. Hence the general form for the matrices $A(xx)$ and $A(yy)$ are as follows:

$$A(x) = \begin{bmatrix} B(t_1\,x\,t_1^{-1}) & B(t_1\,x\,t_2^{-1}) & B(t_1\,x\,t_3^{-1}) & \dots & \dots & B(t_1\,x\,t_7^{-1}) & B(t_1\,x\,t_8^{-1}) \\ B(t_2\,x\,t_1^{-1}) & B(t_2\,x\,t_2^{-1}) & B(t_2\,x\,t_3^{-1}) & \dots & \dots & B(t_2\,x\,t_7^{-1}) & B(t_2\,x\,t_8^{-1}) \\ B(t_3\,x\,t_1^{-1}) & B(t_3\,x\,t_2^{-1}) & B(t_3\,x\,t_3^{-1}) & \dots & \dots & B(t_3\,x\,t_7^{-1}) & B(t_3\,x\,t_8^{-1}) \\ B(t_4\,x\,t_1^{-1}) & B(t_4\,x\,t_2^{-1}) & B(t_4\,x\,t_3^{-1}) & \dots & \dots & B(t_4\,x\,t_7^{-1}) & B(t_4\,x\,t_8^{-1}) \\ B(t_5\,x\,t_1^{-1}) & B(t_5\,x\,t_2^{-1}) & B(t_5\,x\,t_3^{-1}) & \dots & \dots & B(t_5\,x\,t_7^{-1}) & B(t_5\,x\,t_8^{-1}) \\ B(t_6\,x\,t_1^{-1}) & B(t_6\,x\,t_2^{-1}) & B(t_6\,x\,t_3^{-1}) & \dots & \dots & B(t_6\,x\,t_7^{-1}) & B(t_6\,x\,t_8^{-1}) \\ B(t_7\,x\,t_1^{-1}) & B(t_7\,x\,t_2^{-1}) & B(t_7\,x\,t_3^{-1}) & \dots & \dots & B(t_7\,x\,t_7^{-1}) & B(t_7\,x\,t_8^{-1}) \\ B(t_8\,x\,t_1^{-1}) & B(t_8\,x\,t_2^{-1}) & B(t_8\,x\,t_3^{-1}) & \dots & \dots & B(t_8\,x\,t_7^{-1}) & B(t_8\,x\,t_8^{-1}) \end{bmatrix}$$

We begin to compute the first entry, $B(t_1\,x\,t_1^{-1})$ by substituting in the values $t_1 = e$ and $x \sim (3,6,7)(4,5,8)$. Substituting the corresponding values we obtain: $B(e\,(3,6,7)(4,5,8)\,e) = B((3,6,7)(4,5,8))$, now we look for the element $(3,6,7)(4,5,8)$

in the characer table of H along the second row; since $(3,6,7)(4,5,8)$ is not in it $B((3,6,7)(4,5,8)) = 0$. Let's look at the next entry on the matrix: $B(t_1\,x\,t_2^{-1}) = B(e\,(3,6,7)(4,5,8)\,((3,6,7)(4,5,8))^{-1}) = B(e)$ since (e) is in H under the column corresponding to 1 then $B(e) = 1$. Continuing with the above process we obtain the matrix:

$$A(xx) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

We do a similar process to obtain the matrix $A(yy)$:

$$A(yy) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

Now $A(xx)$ and $A(yy)$ are a faithful representation of $L_2(7)$, since $|x| = |A(xx)| = 3$, $|y| = |A(yy)| = 3$ and $|xx * yy| = |A(xx) \cdot A(yy)| = 4$. Now, we can use the matrices to find permutation representations to use for the progenitor along with the following rule: If $a_{ij} = 1$ then $t_i \to t_j$ and if $a_{ij} = -1$ then $t_i \to t_j^{-1}$.

To find the permutation representation for $A(xx)$ and $A(yy)$, we will label the t_i's as follows:

Table 3.3: Labeling t_i's

1. t_1	7. t_2	13. t_3	19. t_4	25. t_5	31. t_6	37. t_7	43. t_8
2. t_1^2	8. t_2^2	14. t_3^2	20. t_4^2	26. t_5^2	32. t_6^2	38. t_7^2	44. t_8^2
3. t_1^3	9. t_2^3	15. t_3^3	21. t_4^3	27. t_5^3	33. t_6^3	39. t_7^3	45. t_8^3
4. t_1^4	10. t_2^4	16. t_3^4	22. t_4^4	28. t_5^4	34. t_6^4	40. t_7^4	46. t_8^4
5. t_1^5	11. t_2^5	17. t_3^5	23. t_4^5	29. t_5^5	35. t_6^5	41. t_7^5	47. t_8^5
6. t_1^6	12. t_2^6	18. t_3^6	24. t_4^6	30. t_5^6	36. t_6^6	42. t_7^6	48. t_8^6

Let's begin with matrix $A(xx)$. Say we begin with entry $a_{12} = 1$ using the relation as mentioned, it implies that $t_1 \to t_2$ using the labeling we have $1 \to 7$. Having found this relation it implies that all the powers of t_1 go to the corresponding powers of t_2. The relations for the remaining powers of t_1 and t_2 using the labeling are: $2 \to 8$, $3 \to 9$, $4 \to 10$, $5 \to 11$ and $6 \to 12$. Similarly, for entry $a_{24} = 1$ implies that $t_2 \to t_4$, with the labeling we obtain $7 \to 19$. Having found this relation we know that all the powers of t_2 go to the powers of t_4. Continuing with the process for the remaining powers we have the remaining relations: $8 \to 20$, $9 \to 21$, $10 \to 22$, $11 \to 23$, $12 \to 24$. Continuing with the pattern, the results are summarized on the following table:

Table 3.4: Permutation of the t_i's using $A(xx)$

t_1	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6	t_2	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6	t_3	t_3^2	t_3^3	t_3^4	t_3^5
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
7	8	9	10	11	12	19	20	21	22	23	24	25	26	27	28	29
t_3^6	t_4	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6	t_5	t_5^2	t_5^3	t_5^4	t_5^5	t_5^6	t_6	t_6^2	t_6^3	t_6^4
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
30	1	2	3	4	5	6	31	32	33	34	35	36	13	14	15	16
t_6^5	t_6^6	t_7	t_7^2	t_7^3	t_7^4	t_7^5	t_7^6	t_8	t_8^2	t_8^3	t_8^4	t_8^5	t_8^6			
35	36	37	38	39	40	41	42	43	44	45	46	47	48			
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
17	18	40	37	41	38	42	39	44	46	48	47	45	47			

Hence, the permutation representation for $A(xx)$ is: $x \sim (1,7,19)(2,8,20)$ $(3,9,21)(4,10,22)(5,11,23)(6,12,24)(13,25,31)(14,26,32)(15,27,33)(16,28,34)$ $(17,29,35)(18,30,36)(37,40,38)(39,41,42)(43,44,46)(45,48,47)$.

Now, for matrix $A(yy)$, we'll be using the same labeling of the t_i's used for $A(xx)$. Along with the conditions, if $a_{ij} = 1$ then $t_i \to t_j$ and if $a_{ij} = -1$ then $t_i \to t_j^{-1}$.

We'll begin with the entry $a_{13} = 1$ which implies that $t_1 \to t_3$, using the labeling we have $1 \to 13$. Continuing with the same process, with all 48 t_i's, the following table summarizes our findings:

Table 3.5: Permutation of the t_i's using $A(yy)$

t_1	t_1^2	t_1^3	t_1^4	t_1^5	t_1^6	t_2	t_2^2	t_2^3	t_2^4	t_2^5	t_2^6	t_3	t_3^2	t_3^3	t_3^4	t_3^5
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
13	14	15	16	17	18	7	10	11	8	12	9	25	26	27	28	29
t_3^6	t_4	t_4^2	t_4^3	t_4^4	t_4^5	t_4^6	t_5	t_5^2	t_5^3	t_5^4	t_5^5	t_5^6	t_6	t_6^2	t_6^3	t_6^4
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
30	20	22	3	4	5	6	31	32	33	34	35	36	13	14	15	16
t_6^5	t_6^6	t_7	t_7^2	t_7^3	t_7^4	t_7^5	t_7^6	t_8	t_8^2	t_8^3	t_8^4	t_8^5	t_8^6			
35	36	37	38	39	40	41	42	43	44	45	46	47	48			
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$			
17	18	40	37	41	38	42	39	44	46	48	47	45	47			

The permutation representation for $A(yy)$ is:$y \sim (1,13,25)(2,14,26)(3,15,27)$ $(4,16,28)(5,17,29)(6,18,30)(7,10,8)(9,11,12)(19,20,22)(21,24,23)(31,37,43)$ $(32,38,44)(33,39,45)(34,40,46)(35,41,47)(36,42,48)$.

Now the progenitor for our group G is as follows:

$$Group < x, y, t | x^3, y^3, (x*y)^4, (x,y)^3, (x^2*y)^7, t^7, < t >^N = < t >>$$

where $< t >^N = < t >$ is the normaliser of $< t >$ in N. The normalizer of the subgroup $< t_1 >$ in N is $\{g \in N | < t_1 > n = n < t_1 >\}$. The normaliser of $< t_1 >$ is the stabiliser of all powers of t_1 in N. The permutations that fix all powers of t_1 are: $(1,2,4)(3,6,5)(7,10,8)(9,11,12)(13,31,19)(14,32,20)(15,33,21)(16,34,22)(17,35,23)$ $(18,36,24)(25,46,38)(26,43,40)(27,47,42)(28,44,37)(29,48,39)(30,45,41)$ and $(7,26,44,34,38,14,19)(8,28,46,31,40,16,20)(9,30,48,35,42,18,21)$ $(10,25,43,32,37,13,22)(11,27,45,36,39,15,23)(12,29,47,33,41,17,24)$.

To be able to use the above permutations in our progenitor we need to represent them in terms of x and y. Using the Schreier System we are able to find out what each of the above look like: $xy^{-1}x^{-1}$ and $y^{-1}x^{-1}y^{-1}$. Hence, we have that $< t_1 > = < xy^{-1}x^{-1}, y^{-1}x^{-1}y^{-1} >$. So far we have $7^{*8} :_m L_2(7)$, where 7^{*8} denotes the order of the matrices, 7, and the number of our t_i's, being 8. Also, to complete the representation of our progenitor we find relations for which t conjugates with. Since we're dealing with a monomial representation, from 7^{*8}, we are working with eight symmetric generators, denoted by $\{t_1, t_2, t_3, t_4, t_5. t_6, t_7, t_8\}$. The elements x and y act on the generators by conjugation, so a presentation for the progenitor is:

$$7^{*8} :_m L_2(7) = < x^3, y^3, (x*y)^4, (x,y)^3, (x^2*y)^7, t^7, t^{(x*y^{-1}*x^{-1})} = t^2, (t, y^{-1}*x^{-1}*y^{-1}), (x^2*y*t*t^x)^2 > .$$

Having completed this monomial progenitor, we can now look at its composition factors. MAGMA prints the composition factors of this progenitor as:

$$A(1,49) = L(2,49)$$

which is a computer-based proof that we will verify by constructing the following group.

3.1 Presentation of $L_2(49)$

We will be showing that $L_2(49) \cong \frac{7^{*8}:_m L_2(7)}{t_2 t_4^2 t_4^4 = t_2 t_1}$, where $x \sim (1,7,19)(2,8,20)(3,9,21)(4,10,22)(5,11,23)(6,12,24)(13,25,31)(14,26,32)(15,27,33)(16,28,34)(17,29,35)(18,30,36)(37,40,38)(39,41,42)(43,44,46)(45,48,47)$ and $y \sim (1,13,25)(2,14,26)(3,15,27)(4,16,28)(5,17,29)(6,18,30)(7,10,8)(9,11,12)(19,20,22)(21,24,23)(31,37,43)(32,38,44)(33,39,45)(34,40,46)(35,41,47)(36,42,48)$, by performing a double coset enumeration of $L_2(49)$ over $L_2(7)$. Hence the progenitor is:

$$L_2(49) = < x^3, y^3, (x*y)^4, (x,y)^3, (x^2*y)^7, t^7, t^{(x*y^{-1}*x^{-1})} = t^2, (t, y^{-1}*x^{-1}*y^{-1}), (x^2*y*t*t^x)^2 > .$$

As mentioned above, we have the progenitor $7^{*8} :_m L_2(7)$ being factored by the relation $[x^2 y t_1 t_1^x]^2 = e$. Simplifly the relation by conjugating t_1 by x we have $t_1 \to t_2$. Our relation now becomes: $[x^2 y t_1 t_2]^2 = e$. Let $x^2 y = \pi$ then $\pi = (1,20,7,13,37,44,31)(2,22,8,14,38,46,32)(3,24,9,15,39,48,33)(4,19,10,16,40,43,34)$

$(5, 21, 11, 17, 41, 45, 35)(6, 23, 12, 18, 42, 47, 36)$. The relation $(\pi t_1 t_2)^2 = e$ yields:

$$\begin{aligned} \pi t_1 t_2 \pi t_1 t_2 &= e \\ \pi \pi \pi{-}1 t_1 t_2 \pi t_1 t_2 &= e \\ \pi^2 (t_1 t_2)^\pi t_1 t_2 &= e \\ \pi^2 t_4^2 t_4^4 t_1 t_2 &= e \\ t_2 t_4^2 t_4^4 t_1 t_2 &= e, \end{aligned}$$

if we use the operation of right hand multiplication , we can simplify this relation to

$$t_2 t_4^2 t_4^4 = t_2 t_1.$$

3.2 Double Coset Enumeration

We will consider the following labeling for our t_i's to facilitate the double coset enumeration notation:
$1 = t_1$, $1^2 = t_1^2$, $1^3 = t_1^3$, $1^4 = t_1^4$, $1^5 = t_1^5$, $1^6 = t_1^6$,
$2 = t_2$, $2^2 = t_2^2$, $2^3 = t_2^3$, $2^4 = t_2^4$, $2^5 = t_2^5$, $2^6 = t_2^6$,
$3 = t_3$, $3^2 = t_3^2$, $3^3 = t_3^3$, $3^4 = t_3^4$, $3^5 = t_3^5$, $3^6 = t_3^6$,
$4 = t_4$, $4^2 = t_4^2$, $4^3 = t_4^3$, $4^4 = t_4^4$, $4^5 = t_4^5$, $4^6 = t_4^6$,
$5 = t_5$, $5^2 = t_5^2$, $5^3 = t_5^3$, $5^4 = t_5^4$, $5^5 = t_5^5$, $5^6 = t_5^6$,
$6 = t_6$, $6^2 = t_6^2$, $6^3 = t_6^3$, $6^4 = t_6^4$, $6^5 = t_6^5$, $6^6 = t_6^6$,
$7 = t_7$, $7^2 = t_7^2$, $7^3 = t_7^3$, $7^4 = t_7^4$, $7^5 = t_7^5$, $7^6 = t_7^6$,
$8 = t_8$, $8^2 = t_8^2$, $8^3 = t_8^3$, $8^4 = t_8^4$, $8^5 = t_8^5$, $8^6 = t_8^6$,

NeN

We begin with NeN, the first double coset, which contains all the words of lenght zero. We have that $NeN = \{N\}$ and it will be denoted [∗]. Also,
$N =< x, y >\cong L_2(7)$ and is of order 168. The number of elements in [∗] is $\frac{|N|}{|N|} = \frac{168}{168} = 1$, hence [∗] consists of the single coset, N. The single coset contains two orbits: $\{1, 1^2, 1^4, 2, 2^2, 2^4, 3, 3^2, 3^4, 4, 4^2, 4^4, 5, 5^2, 5^4, 6, 6^2, 6^4, 7, 7^2, 7^4, 8, 8^2, 8^4\}$ and $\{1^3, 1^5, 1^6, 2^3, 2^5, 2^6, 3^3, 3^5, 3^6, 4^3, 4^5, 4^6, 5^3, 5^5, 5^6, 6^3, 6^5, 6^6, 7^3, 7^5, 7^6, 8^3, 8^5, 8^6\}$. Now take a representative element from each orbit, say t_1 and t_1^3 and right multiply each with the representative coset Ne:

$$Ne \cdot t_1 = Nt_1 \in [1]$$

$$Ne \cdot t_1^3 = Nt_1^3 \in [1^3]$$

$\mathbf{Nt_1N}$

The double coset Nt_1N denoted by [1]. To find the elements in [1], we find the point stabiliser of 1, denoted as N^1. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and permutes the rest of the t_i's. Thus, we have that $|N^1| = 7$. We also have that the set stabiliser is the same as the set point stabiliser, $N^{(1)} = N^1$. Hence $|N^{(1)}| = 7$.

The orbits of $N^{(1)}$ are $\{1\}$, $\{1^2\}$, $\{1^3\}$, $\{1^4\}$, $\{1^5\}$,$\{1^6\}$, $\{2, 3^2, 4, 5^2, 6^4, 7^2, 8^2\}$, $\{2^2, 3^4, 4^2, 5^4, 6, 7^4, 8^4\}$, $\{2^3, 3^6, 4^3, 5^6, 6^5, 7^6, 8^6\}$, $\{2^4, 3, 4^4, 5, 6^2, 7, 8\}$, $\{2^5, 3^3, 4^5, 5^3, 6^6, 7^3, 8^3\}$, and $\{2^6, 3^5, 4^6, 5^5, 6^3, 7^5, 8^5\}$. The number of single cosets in the double coset [1] is $\frac{|N|}{|N^{(1)}|} = \frac{168}{7} = 24$.

Next we must take a representative from each orbit and right multiply it with the representative Nt_1, to determine if the t_i's will expand or collapse:

$$\begin{array}{ccc}
Nt_1 \cdot t_1 = Nt_1^2 \in [1] & Nt_1 \cdot t_1^5 = Nt_1^6 \in [1^3] & Nt_1 \cdot t_2^3 = Nt_1\,t_2^3 \in [1, 2^3] \\
Nt_1 \cdot t_1^2 = Nt_1^3 \in [1^3] & Nt_1 \cdot t_1^6 = Nt_1^7 = Ne \in [*] & Nt_1 \cdot t_2^4 = Nt_1\,t_2^4 \in [1, 2^4] \\
Nt_1 \cdot t_1^3 = Nt_1^4 \in [1] & Nt_1 \cdot t_2 = Nt_1\,t_2 \in [1, 2] & Nt_1 \cdot t_2^5 = Nt_1\,t_2^5 \in [1, 2^5] \\
Nt_1 \cdot t_1^4 = Nt_1^5 \in [1^3] & Nt_1 \cdot t_2^2 = Nt_1\,t_2^2 \in [1, 2^2] & Nt_1 \cdot t_2^6 = Nt_1\,t_2^6 \in [1, 2^6]
\end{array}$$

$\mathbf{Nt_1^3N}$

The double coset Nt_1^3N denoted by $[1^3]$. To find the elements in $[1^3]$, we find the point stabiliser of 1^3, denoted as N^{1^3}. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1^3 and permutes the rest of the $t_i's$. Thus, we have that $|N^{1^3}| = 7$. We also have that the set stabiliser is the same as the set point stabiliser, $N^{(1^3)} = N^{1^3}$. Hence $|N^{(1^3)}| = 7$.

The orbits of $N^{(1^3)}$ are $\{1\}$, $\{1^2\}$, $\{1^3\}$, $\{1^4\}$, $\{1^5\}$,$\{1^6\}$, $\{2, 3^2, 4, 5^2, 6^4, 7^2, 8^2\}$, $\{2^2, 3^4, 4^2, 5^4, 6, 7^4, 8^4\}$, $\{2^3, 3^6, 4^3, 5^6, 6^5, 7^6, 8^6\}$, $\{2^4, 3, 4^4, 5, 6^2, 7, 8\}$, $\{2^5, 3^3, 4^5, 5^3, 6^6, 7^3, 8^3\}$, and $\{2^6, 3^5, 4^6, 5^5, 6^3, 7^5, 8^5\}$. The number of single cosets in the double coset $[1^3]$ is obtained by the quotient $\frac{|N|}{|N^{(1^3)}|} = \frac{168}{7} = 24$.

To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(1^3)}$ in N. Then, we conjugate Nt_1^3, a representative of the coset, with the transversals to obtain the 24 single cosets in $[1^3]$. Next, take a representative from each orbit of $N^{(1^3)}$ and right multiply each with Nt_1^3, to determine if the elements will expand or collapse:

$$Nt_1^3 \cdot t_1 = Nt_1^3 t_1 = Nt_1^4 \in [1]$$
$$Nt_1^3 \cdot t_1^2 = Nt_1^3 t_1^2 = Nt_1^5 \in [1^3]$$
$$Nt_1^3 \cdot t_1^3 = Nt_1^3 t_1^3 = Nt_1^6 \in [1^3]$$
$$Nt_1^3 \cdot t_1^4 = Nt_1^3 t_1^4 = Nt_1^7 = Ne \in [*]$$
$$Nt_1^3 \cdot t_1^5 = Nt_1^3 t_1^5 = Nt_1 \in [1]$$
$$Nt_1^3 \cdot t_1^6 = Nt_1^3 t_1^6 = Nt_1^2 \in [1]$$
$$Nt_1^3 \cdot t_2 = Nt_1^3 t_2 \in [1, 2^3]$$
$$Nt_1^3 \cdot t_2^2 = Nt_1^3 t_2^2 \in [1, 2^6]$$
$$Nt_1^3 \cdot t_2^3 = Nt_1^3 t_2^3 \in [1, 2^2]$$
$$Nt_1^3 \cdot t_2^4 = Nt_1^3 t_2^4 \in [1^3, 2^4]$$
$$Nt_1^3 \cdot t_2^5 = Nt_1^3 t_2^5 \in [1, 2]$$
$$Nt_1^3 \cdot t_2^6 = Nt_1^3 t_2^6 \in [1, 2^4]$$

$\mathbf{Nt_1t_2N}$

The double coset $Nt_1 t_2N$ denoted by $[1, 2]$. To find the elements in $[1, 2]$, we find the point stabiliser of $\{1, 2\}$, denoted as N^{12}. The point stabiliser is made up of the permutations in $N = S_{48}$ that fix 1 and 2 and permutes the rest of the t_i's. Thus, we have that $|N^{12}| = 3$. We also have that the set stabiliser, $N^{(12)}$ has 3 elements.The number of single cosets in the double coset $[1, 2]$ is obtained by the quotient $\frac{|N|}{|N^{(1,2)}|} = \frac{168}{3} = 56$.

The orbits of $N^{(12)}$ are $\{1, 2^2, 3^4\}$, $\{1^2, 2^4, 3\}$, $\{1^3, 2^6, 3^5\}$, $\{1^4, 2, 3^2\}$, $\{1^5, 2^3, 3^6\}$, $\{1^6, 2^5, 3^3\}$, $\{4, 6, 8^4\}$, $\{4^2, 6^2, 8\}$, $\{4^3, 6^3, 8^5\}$, $\{4^4, 6^4, 8^2\}$, $\{4^5, 6^5, 8^6\}$, $\{4^6, 6^6, 8^3\}$, $\{5, 5^2, 5^4\}$,$\{5^3, 5^5, 5^6\}$,$\{7, 7^2, 7^4\}$, and $\{7^3, 7^5, 7^6\}$.

To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(12)}$ in N. Then we conjugate $Nt_1 t_2$, a representative of the coset, with the transversals to obtain the single cosets in $[1, 2]$ which total to 56. Next we take a representative from each orbit of $N^{(12)}$ and right multiply each with $Nt_1 t_2$, to determine if the elements will expand or collapse:

$$Nt_1 t_2 \cdot t_1 = Nt_1 t_2 t_1 \in [1, 2^3]$$
$$Nt_1 t_2 \cdot t_1^2 = Nt_1 t_2 t_1^2 \in [1, 2^5]$$
$$Nt_1 t_2 \cdot t_1^3 = Nt_1 t_2 t_1^3 \in [1]$$
$$Nt_1 t_2 \cdot t_1^4 = Nt_1 t_2 t_1^4 \in [1, 2^2]$$
$$Nt_1 t_2 \cdot t_1^5 = Nt_1 t_2 t_1^5 \in [1, 2^4]$$
$$Nt_1 t_2 \cdot t_1^6 = Nt_1 t_2 t_1^6 \in [1, 2^6]$$
$$Nt_1 t_2 \cdot t_4 = Nt_1 t_2 t_4 \in [1^3]$$
$$Nt_1 t_2 \cdot t_4^2 = Nt_1 t_2 t_4^2 \in [1, 2^6]$$
$$Nt_1 t_2 \cdot t_4^3 = Nt_1 t_2 t_4^3 \in [1^3, 2^4]$$
$$Nt_1 t_2 \cdot t_4^4 = Nt_1 t_2 t_4^4 \in [1, 2^4]$$

$$Nt_1\,t_2\cdot t_4^5 = Nt_1\,t_2\,t_4^5 \in [1,2^3] \qquad Nt_1\,t_2\cdot t_5^3 = Nt_1\,t_2\,t_5^3 \in [1,2]$$
$$Nt_1\,t_2\cdot t_4^6 = Nt_1\,t_2\,t_4^6 \in [1,2^2] \qquad Nt_1\,t_2\cdot t_7 = Nt_1\,t_2\,t_7 \in [1,2]$$
$$Nt_1\,t_2\cdot t_5 = Nt_1\,t_2\,t_5 \in [1,2] \qquad Nt_1\,t_2\cdot t_7^3 = Nt_1\,t_2\,t_7^3 \in [1,2]$$

$\mathbf{Nt_1t_2^2N}$

The double coset $Nt_1\,t_2^2N$ denoted by $[1,2^2]$. To find the elements in $[1,2^2]$, we find the point stabiliser of $\{1,2^2\}$, denoted as N^{12^2}. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and 2^2 and permutes with the remaining t_i's. Thus, we have that $|N^{12^2}| = 4$. We also have that the set stabiliser is $N^{(12^2)}$ also contains 4 elements. The number of single cosets in the double coset $[1,2^2]$ is obtained by the quotient $\frac{|N|}{|N^{(1,2^2)}|} = \frac{168}{4} = 42$.

The orbits of $N^{(12^2)}$ are $\{1,4,7^4,8^2\}$, $\{1^2,4^2,7,8^4\}$, $\{1^3,4^3,7^5,8^6\}$, $\{1^4,4^4,7^2,8\}$, $\{1^5,4^5,7^6,8^3\}$,$\{1^6,4^6,7^3,8^5\}$, $\{2,3,5^2,6^2\}$, $\{2^2,3^2,5^4,6^4\}$, $\{2^3,3^3,5^6,6^6\}$, $\{2^4,3^4,5,6\}$, $\{2^5,3^5,5^3,6^3\}$, and $\{2^6,3^6,5^5,6^5\}$. Next we must take a representative from each orbit of $N^{(12^2)}$ and conjugate each with $Nt_1t_2^2$, a representative of the coset, to determine if the elements will expand or collapse:

$$Nt_1\,t_2^2\cdot t_1 = Nt_1\,t_2^2\,t_1 \in [1,2] \qquad Nt_1\,t_2^2\cdot t_2 = Nt_1\,t_2^2\,t_2 \in [1,2^3]$$
$$Nt_1\,t_2^2\cdot t_1^2 = Nt_1\,t_2^2\,t_1^2 \in [1^3] \qquad Nt_1\,t_2^2\cdot t_2^2 = Nt_1\,t_2^2\,t_2^2 \in [1,2^4]$$
$$Nt_1\,t_2^2\cdot t_1^3 = Nt_1\,t_2^2\,t_1^3 \in [1,2^6] \qquad Nt_1\,t_2^2\cdot t_2^3 = Nt_1\,t_2^2\,t_2^3 \in [1,2^5]$$
$$Nt_1\,t_2^2\cdot t_1^4 = Nt_1\,t_2^2\,t_1^4 \in [1^3,2^4] \qquad Nt_1\,t_2^2\cdot t_2^4 = Nt_1\,t_2^2\,t_2^4 \in [1,2^6]$$
$$Nt_1\,t_2^2\cdot t_1^5 = Nt_1\,t_2^2\,t_1^5 \in [1,2^4] \qquad Nt_1\,t_2^2\cdot t_2^5 = Nt_1\,t_2^2\,t_2^5 \in [1]$$
$$Nt_1\,t_2^2\cdot t_1^6 = Nt_1\,t_2^2\,t_1^6 \in [1,2^3] \qquad Nt_1\,t_2^2\cdot t_2^6 = Nt_1\,t_2^2\,t_2^6 \in [1,2]$$

$\mathbf{Nt_1t_2^3N}$

The double coset $Nt_1\,t_2^3N$ denoted by $[1,2^3]$. To find the elements in $[1,2^3]$, we find the point stabiliser of $\{1,2^3\}$, denoted as N^{12^3}. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and 2^3 and permutes the rest of the t_i's. Thus, we have that $|N^{12^3}| = 4$. We also have that the set stabiliser is $N^{(12^3)} =$. The number of single cosets in the double coset $[1,2^3]$ is obtained by the quotient $\frac{|N|}{|N^{(12^3)}|} = \frac{168}{4} = 42$.

The orbits of $N^{(12^3)}$ are $\{1,6^4,7,8\}$, $\{1^2,6,7^2,8^2\}$, $\{1^3,6^5,7^3,8^3\}$, $\{1^4,6^2,7^4,8^4\}$, $\{1^5,6^6,7^5,8^5\}$,$\{1^6,6^3,7^6,8^6\}$, $\{2,3^4,4,5^4\}$, $\{2^2,3,4^2,5\}$, $\{2^3,3^5,4^3,5^5\}$, $\{2^4,3^2,4^4,5^2\}$, $\{2^5,3^6,4^5,5^6\}$, and $\{2^6,3^3,4^6,5^3\}$. Next, we take a representative from each orbit of $N^{(12^3)}$ and conjugate each with $Nt_1t_2^3$, a representative of the coset, to determine if the elements will expand or collapse:

$Nt_1\,t_2^3\cdot t_1 = Nt_1\,t_2^3\,t_1 \in [1,2]$	$Nt_1\,t_2^3\cdot t_2 = Nt_1\,t_2^3\,t_2 \in [1,2^4]$
$Nt_1\,t_2^3\cdot t_1^2 = Nt_1\,t_2^3\,t_1^2 \in [1,2^6]$	$Nt_1\,t_2^3\cdot t_2^2 = Nt_1\,t_2^3\,t_2^2 \in [1,2^5]$
$Nt_1\,t_2^3\cdot t_1^3 = Nt_1\,t_2^3\,t_1^3 \in [1,2^4]$	$Nt_1\,t_2^3\cdot t_2^3 = Nt_1\,t_2^3\,t_2^3 \in [1,2^6]$
$Nt_1\,t_2^3\cdot t_1^4 = Nt_1\,t_2^3\,t_1^4 \in [1,2^2]$	$Nt_1\,t_2^3\cdot t_2^4 = Nt_1\,t_2^3\,t_2^4 \in [1]$
$Nt_1\,t_2^3\cdot t_1^5 = Nt_1\,t_2^3\,t_1^5 \in [1^3]$	$Nt_1\,t_2^3\cdot t_2^5 = Nt_1\,t_2^3\,t_2^5 \in [1,2]$
$Nt_1\,t_2^3\cdot t_1^6 = Nt_1\,t_2^3\,t_1^6 \in [1^3,2^4]$	$Nt_1\,t_2^3\cdot t_2^6 = Nt_1\,t_2^3\,t_2^6 \in [1,2^2]$

$\mathbf{Nt_1t_2^4N}$

The double coset $Nt_1\,t_2^4N$ denoted by $[1,2^4]$. To find the elements in $[1,2^4]$, we find the point stabiliser of $\{1,2^4\}$, denoted as N^{12^4}. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and 2^4 and permutes the rest of the t_i's. Thus, we have that $|N^{12^4}| = 3$. We also have that the order of the set stabiliser is $|N^{(12^4)}| = 3$. The number of single cosets in the double coset $[1,2^4]$ is obtained by the quotient $\frac{|N|}{|N^{(12^4)}|} = \frac{168}{3} = 56$. The orbits of $N^{(12^4)}$ are $\{1,4^4,6^2\}$, $\{1^2,4,6^4\}$, $\{1^3,4^5,6^6\}$, $\{1^4,4^2,6\}$, $\{1^5,4^6,6^3\}$, $\{1^6,4^3,6^5\}$, $\{2,5,8^2\}$, $\{2^2,5^2,8^4\}$, $\{2^3,5^3,8^6\}$, $\{2^4,5^4,8\}$, $\{2^5,5^5,8^3\}$, $\{2^6,5^6,8^5\}$, $\{3,3^2,3^4\}$, $\{3^3,3^5,3^6\}$, $\{7,7^2,7^4\}$, and $\{7^3,7^5,7^6\}$. Next, we take a representative from each orbit of $N^{(12^4)}$ and conjugate each with $Nt_1t_2^4$, a representative of the coset, to determine if the elements will expand or collapse:

$Nt_1\,t_2^4\cdot t_1 = Nt_1\,t_2^4\,t_1 \in [1,2^3]$	$Nt_1\,t_2^4\cdot t_2^3 = Nt_1\,t_2^4\,t_2^3 \in [1]$
$Nt_1\,t_2^4\cdot t_1^2 = Nt_1\,t_2^4\,t_1^2 \in [1,2^2]$	$Nt_1\,t_2^4\cdot t_2^4 = Nt_1\,t_2^4\,t_2^4 \in [1,2]$
$Nt_1\,t_2^4\cdot t_1^3 = Nt_1\,t_2^4\,t_1^3 \in [1,2]$	$Nt_1\,t_2^4\cdot t_2^5 = Nt_1\,t_2^4\,t_2^5 \in [1,2^2]$
$Nt_1\,t_2^4\cdot t_1^4 = Nt_1\,t_2^4\,t_1^4 \in [1^3]$	$Nt_1\,t_2^4\cdot t_2^6 = Nt_1\,t_2^4\,t_2^6 \in [1,2^3]$
$Nt_1\,t_2^4\cdot t_1^5 = Nt_1\,t_2^4\,t_1^5 \in [1,2^6]$	$Nt_1\,t_2^4\cdot t_3 = Nt_1\,t_2^4\,t_3 \in [1,2^4]$
$Nt_1\,t_2^4\cdot t_1^6 = Nt_1\,t_2^4\,t_1^6 \in [1^3,2^4]$	$Nt_1\,t_2^4\cdot t_3^3 = Nt_1\,t_2^4\,t_3^3 \in [1,2^4]$
$Nt_1\,t_2^4\cdot t_2 = Nt_1\,t_2^4\,t_2 \in [1,2^5]$	$Nt_1\,t_2^4\cdot t_7 = Nt_1\,t_2^4\,t_7 \in [1,2^4]$
$Nt_1\,t_2^4\cdot t_2^2 = Nt_1\,t_2^4\,t_2^2 \in [1,2^6]$	$Nt_1\,t_2^4\cdot t_7^3 = Nt_1\,t_2^4\,t_7^3 \in [1,2^4]$

$\mathbf{Nt_1t_2^5N}$

The double coset $Nt_1\,t_2^5N$ denoted by $[1,2^5]$. To find the elements in $[1,2^5]$, we find the point stabiliser of $\{1,2^5\}$, denoted as N^{12^5}. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and 2^5 and permutes the rest of the t_i's. Thus, we have that $|N^{12^5}| = 1$. We also have that the set stabiliser is $|N^{(12^5)}| = 7$. The number of single cosets in the double coset $[1, 2^5]$ is obtained by the quotient $\frac{|N|}{|N^{(12^5)}|} = \frac{168}{7} = 24$. The orbits of $N^{(12^5)}$ are $\{8\}$, $\{8^2\}$, $\{8^3\}$, $\{8^4\}$, $\{8^5\}$,$\{8^6\}$, $\{1, 2^2, 3, 4^4, 5^2, 6^4, 7^4\}$, $\{1^2, 2^4, 3^2, 4, 5^4, 6, 7\}$, $\{1^3, 2^6, 3^3, 4^5, 5^6, 6^5, 7^5\}$, $\{1^4, 2, 3^4, 4^2, 5, 6^2, 7^2\}$, $\{1^5, 2^3, 3^5, 4^6, 5^3, 6^6, 7^6\}$, and $\{1^6, 2^5, 3^6, 4^3, 5^5, 6^3, 7^3\}$. Next, we take a representative from each orbit of $N^{(12^5)}$ and conjugate each with $Nt_1t_2^5$, to determine if the elements will expand or collapse:

$$Nt_1\,t_2^5 \cdot t_1 = Nt_1\,t_2^4\,t_1 \in [1]$$
$$Nt_1\,t_2^5 \cdot t_1^2 = Nt_1\,t_2^4\,t_1^2 \in [1, 2^2]$$
$$Nt_1\,t_2^5 \cdot t_1^3 = Nt_1\,t_2^4\,t_1^3 \in [1, 2^4]$$
$$Nt_1\,t_2^5 \cdot t_1^4 = Nt_1\,t_2^4\,t_1^4 \in [1, 2^6]$$
$$Nt_1\,t_2^5 \cdot t_1^5 = Nt_1\,t_2^4\,t_1^5 \in [1, 2]$$
$$Nt_1\,t_2^5 \cdot t_1^6 = Nt_1\,t_2^4\,t_1^6 \in [1, 2^3]$$
$$Nt_1\,t_2^5 \cdot t_8 = Nt_1\,t_2^4\,t_8 \in [1^3, 2^4]$$
$$Nt_1\,t_2^5 \cdot t_8^2 = Nt_1\,t_2^4\,t_8^2 \in [1, 2^5]$$
$$Nt_1\,t_2^5 \cdot t_8^3 = Nt_1\,t_2^4\,t_8^3 \in [1, 2^5]$$
$$Nt_1\,t_2^5 \cdot t_8^4 = Nt_1\,t_2^5\,t_8^4 \in [1, 2^5, 8^4]$$
$$Nt_1\,t_2^5 \cdot t_8^5 = Nt_1\,t_2^4\,t_8^5 \in [1^3, 2^4]$$
$$Nt_1\,t_2^5 \cdot t_8^6 = Nt_1\,t_2^4\,t_8^6 \in [1^3, 2^4]$$

$\mathbf{Nt_1t_2^6N}$

The double coset $Nt_1\,t_2^6N$ denoted by $[1, 2^6]$. To find the elements in $[1, 2^6]$, we find the point stabiliser of $\{1, 2^6\}$, denoted as N^{12^6}. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1 and 2^6 and permutes the remaining t_i's. Thus, we have that $|N^{12^6}| = 1$. We also have that the set stabiliser is $|N^{(12^6)}| = 3$. The number of single cosets in the double coset $[1, 2^6]$ is obtained by the quotient $\frac{|N|}{|N^{(12^6)}|} = \frac{168}{3} = 56$. The orbits of $N^{(12^6)}$ are $\{1, 1^2, 1^4\}$, $\{1^3, 1^5, 1^6\}$, $\{2, 3, 5^4\}$, $\{2^2, 3^2, 5\}$, $\{2^3, 3^3, 5^5\}$,$\{2^4, 3^4, 5^2\}$, $\{2^5, 3^5, 5^6\}$, $\{2^6, 3^6, 5^3\}$, $\{4, 6^2, 7^4\}$, $\{4^2, 6^4, 7\}$, $\{4^3, 6^6, 7^5\}$, $\{4^4, 6, 7^2\}$, $\{4^5, 6^3, 7^6\}$, $\{4^6, 6^5, 7^3\}$,$\{8, 8^2, 8^4\}$ and $\{8^3, 8^5, 8^6\}$. Next, we take a representative from each orbit of $N^{(12^6)}$ and conjugate each with $Nt_1t_2^6$, to determine if the elements will expand or collapse:

$$Nt_1\,t_2^6 \cdot t_1 = Nt_1\,t_2^6\,t_1 \in [1, 2^6]$$
$$Nt_1\,t_2^6 \cdot t_1^3 = Nt_1\,t_2^6\,t_1^3 \in [1, 2^6]$$
$$Nt_1\,t_2^6 \cdot t_2 = Nt_1\,t_2^6\,t_2 \in [1]$$
$$Nt_1\,t_2^6 \cdot t_2^2 = Nt_1\,t_2^6\,t_2^2 \in [1, 2]$$
$$Nt_1\,t_2^6 \cdot t_2^3 = Nt_1\,t_2^6\,t_2^3 \in [1, 2^2]$$
$$Nt_1\,t_2^6 \cdot t_2^4 = Nt_1\,t_2^6\,t_2^4 \in [1, 2^3]$$
$$Nt_1\,t_2^6 \cdot t_2^5 = Nt_1\,t_2^6\,t_2^5 \in [1, 2^4]$$
$$Nt_1\,t_2^6 \cdot t_2^6 = Nt_1\,t_2^6\,t_2^6 \in [1, 2^5]$$
$$Nt_1\,t_2^6 \cdot t_4 = Nt_1\,t_2^6\,t_4 \in [1, 2^4]$$
$$Nt_1\,t_2^6 \cdot t_4^2 = Nt_1\,t_2^6\,t_4^2 \in [1, 2^2]$$

$$Nt_1\, t_2^6 \cdot t_4^3 = Nt_1\, t_2^6\, t_4^3 \in [1^3]$$
$$Nt_1\, t_2^6 \cdot t_4^4 = Nt_1\, t_2^6\, t_4^4 \in [1^3, 2^4]$$
$$Nt_1\, t_2^6 \cdot t_4^5 = Nt_1\, t_2^6\, t_4^5 \in [1, 2^3]$$
$$Nt_1\, t_2^6 \cdot t_4^6 = Nt_1\, t_2^6\, t_4^6 \in [1, 2]$$
$$Nt_1\, t_2^6 \cdot t_8 = Nt_1\, t_2^6\, t_8 \in [1, 2^6]$$
$$Nt_1\, t_2^6 \cdot t_8^3 = Nt_1\, t_2^6\, t_8^3 \in [1, 2^6]$$

$\mathbf{Nt_1^3t_2^4N}$

The double coset $Nt_1^3\, t_2^4 N$ denoted by $[1^3, 2^4]$. To find the elements in $[1^3, 2^4]$, we find the point stabiliser of $\{1^3, 2^4\}$, denoted as $N^{1^3 2^4}$. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1^3 and 2^4 and permutes the remaining t_i's. Thus, we have that $|N^{1^3 2^4}| = 1$. We also have that the set stabiliser is $|N^{(1^3 2^4)}| = 7$. The number of single cosets in the double coset $[1^3, 2^4]$ is obtained by the quotient $\frac{|N|}{|N^{(1^3 2^4)}|} = \frac{168}{7} = 24$. The orbits of $N^{(1^3 2^4)}$ are $\{4\}$, $\{4^2\}$, $\{4^3\}$, $\{4^4\}$,$\{4^5\}$,$\{4^6\}$, $\{1, 2, 3^2, 5^4, 6^2, 7^4, 8\}$, $\{1^2, 2^2, 3^4, 5, 6^4, 7, 8^2\}$, $\{1^3, 2^3, 3^6, 5^5, 6^6, 7^5, 8^3\}$, $\{1^4, 2^4, 3, 5^2, 6, 7^2, 8^4\}$, $\{1^5, 2^5, 3^3, 5^6, 6^3, 7^6, 8^5\}$, and $\{1^6, 2^6, 3^5, 5^3, 6^5, 7^3, 8^6\}$. Next, we take a representative from each orbit of $N^{(1^3 2^4)}$ and conjugate each with $Nt_1^3 t_2^4$, a representative of the coset, to determine if the elements will expand or collapse:

$$Nt_1^3\, t_2^4 \cdot t_4 = Nt_1\, t_2^4\, t_4 \in [1, 2^5]$$
$$Nt_1^3\, t_2^4 \cdot t_4^2 = Nt_1\, t_2^4\, t_4^2 \in [1, 2^5]$$
$$Nt_1^3\, t_2^4 \cdot t_4^3 = Nt_1\, t_2^4\, t_4^3 \in [1, 2^5, 8^4]$$
$$Nt_1^3\, t_2^4 \cdot t_4^4 = Nt_1\, t_2^4\, t_4^4 \in [1^3, 2^4]$$
$$Nt_1^3\, t_2^4 \cdot t_4^5 = Nt_1\, t_2^4\, t_4^5 \in [1^3, 2^4]$$
$$Nt_1^3\, t_2^4 \cdot t_4^6 = Nt_1\, t_2^4\, t_4^6 \in [1, 2^5]$$
$$Nt_1^3\, t_2^4 \cdot t_1 = Nt_1\, t_2^4\, t_1 \in [1, 2]$$
$$Nt_1^3\, t_2^4 \cdot t_1^2 = Nt_1\, t_2^4\, t_1^2 \in [1, 2^4]$$
$$Nt_1^3\, t_2^4 \cdot t_1^3 = Nt_1\, t_2^4\, t_1^3 \in [1^3]$$
$$Nt_1^3\, t_2^4 \cdot t_1^4 = Nt_1\, t_2^4\, t_1^4 \in [1, 2^3]$$
$$Nt_1^3\, t_2^4 \cdot t_1^5 = Nt_1\, t_2^4\, t_1^5 \in [1, 2^6]$$
$$Nt_1^3\, t_2^4 \cdot t_1^6 = Nt_1\, t_2^4\, t_1^6 \in [1, 2^2]$$

$\mathbf{Nt_1t_2^5t_8^4N}$

The double coset $Nt_1\,t_2^5\,t_8^4N$ denoted by $[1,2^5,8^4]$. To find the elements in $[1,2^5,8^4]$, we find the point stabiliser of $\{1,2^5,8^4\}$, denoted as $N^{12^58^4}$. The point stabiliser is made up of the permutations in $N = L_2(7)$ that fix 1,2^5, and 8^4 and permutes with the remaining t_i's. Thus, we have that $|N^{12^58^4}| = 1$. We also have that the set stabiliser is $|N^{(12^58^4)}| = 168$. The number of single cosets in the double coset $[1,2^5,8^4]$ is obtained by the quotient $\frac{|N|}{|N^{(12^58^4)}|} = \frac{168}{168} = 1$. The orbits of $N^{(12^58^4)}$ are $\{1,1^2,1^4,2,2^2,2^4,3,3^2,3^4,4,4^2,4^4,5,5^2,5^4,6,6^2,6^4,7,7^2,7^4,8,8^2,8^4\}$ and $\{1^3,1^5,1^6,2^3,2^5,2^6,3^3,3^5,3^6,4^3,4^5,4^6,5^3,5^5,5^6,6^3,6^5,6^6,7^3,7^5,7^6,8^3,8^5,8^6\}$. Next, we take a representative from each orbit of $N^{(12^58^4)}$ and conjugate each with $Nt_1t_2^5t_8^4$, to determine if the elements will expand or collapse:

$$Nt_1\,t_2^5\,t_8^4\cdot t_1 = Nt_1\,t_2^6\,t_8^4\,t_1 \in [1^3,2^4]$$

$$Nt_1\,t_2^5\,t_8^4\cdot t_1^3 = Nt_1\,t_2^6\,t_8^4\,t_1^3 \in [1,2^5]$$

Hence, we summarize all of our work on the following Cayley graph:

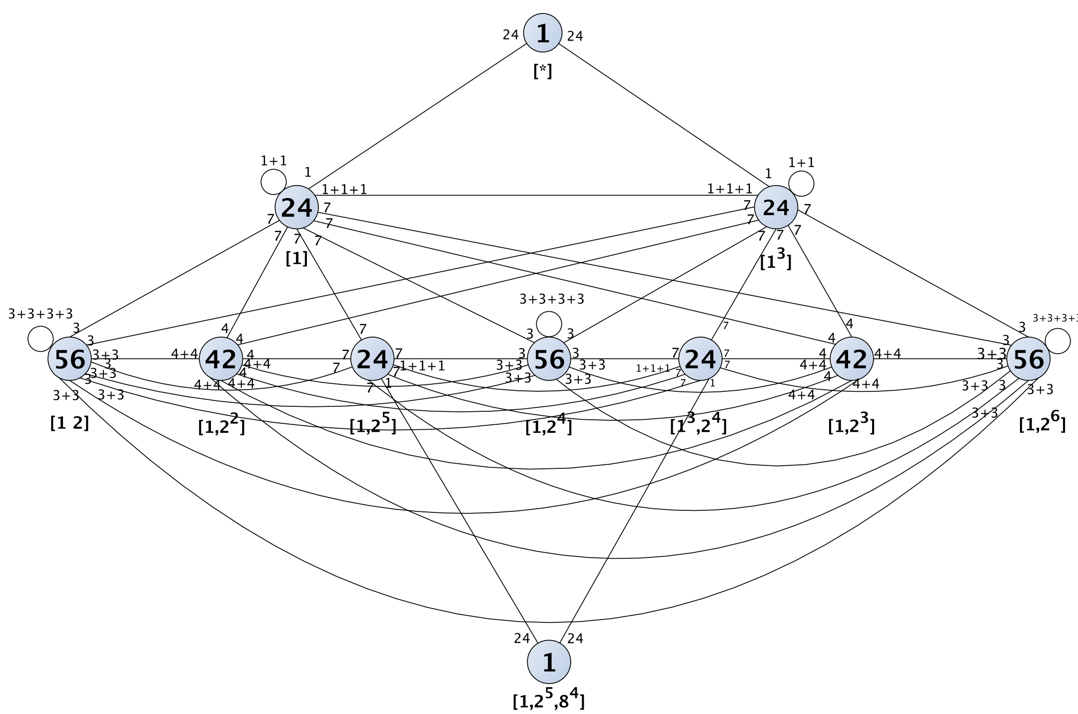


Figure 3.1: Cayley Graph of $L_2(49)$ over $L_2(7)$

Chapter 4

Composition Factors

4.1 Direct Products

We will be performing a composition series on a given group with the aid of MAGMA, a computer software that aids with the work on group theory. The progenitors that we investigated are:

$$< x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y) > \cong 2^{*15} : D_{15},$$

where $x \sim (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)$ and $y \sim (1, 12)(2, 11)(3, 10)(4, 9)$ $(5, 8)(6, 7)(13, 15)$.

$$< x, y, t | x^3, y^3, (x * y)^4, (x, y)^3, (x^2 * y)^7, t^7 > \cong L_2(7),$$

where $x \sim (1, 7, 19)(2, 8, 20)(3, 9, 21)(4, 10, 22)(5, 11, 23)(6, 12, 24)(13, 25, 31)(14, 26, 32)$ $(15, 27, 33)(16, 28, 34)(17, 29, 35)(18, 30, 36)(37, 40, 38)(39, 41, 42)(43, 44, 46)(45, 48, 47)$ and $y \sim (1, 13, 25)(2, 14, 26)(3, 15, 27)(4, 16, 28)(5, 17, 29)(6, 18, 30)(7, 10, 8)(9, 11, 12)$ $(19, 20, 22)(21, 24, 23)(31, 37, 43)(32, 38, 44)(33, 39, 45)(34, 40, 46)(35, 41, 47)(36, 42, 48)$.

$$< x, y, t | x^3, y^2, (xy)^2, t^2, (t, y) > \cong S_3,$$

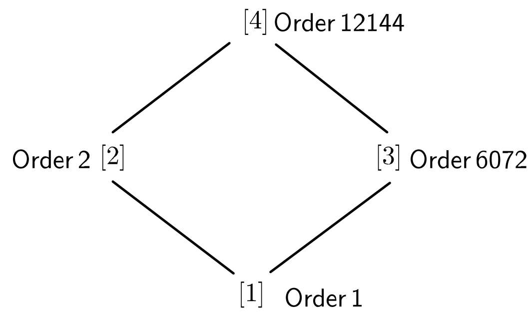
where $x \sim (1, 2, 3)$ and $y \sim (1, 2)$.

We will begin with small simple groups and work our way up to more complex groups. Let's begin with the group presentation:

$$G < x, y, t >:= Group < x, y, t | x^{11}, y^2, (x * y)^2, t^2, (t, y), (x * y * t^x)^3, (x^2 * y * t * t^x)^4 >$$

We have the composition factors of our group as well as its normal lattice. Both are used as a map/guide to find the structure of the composition series of our group:

```
    G
    |   A(1, 23) = L(2, 23)
G1  *
    |   Cyclic(2)
    1
```



Hence we begin to list the extensions from the composition factors: $G_1/1 \cong C_2$ which implies that $G_1 \cong C_2$, where C_2 is normal. Moving up the composition factors, we have $G/G_1 \cong PSL_2(23)$ which implies $G \cong G_1 \, ? \, PSL_2(23)$. Now, to solve the extension problem we need to figure out whether we have a direct product, or a semi-direct product.

We for a direct product first, that is, do we have $G \cong C_2 \, X \, PSL_2(23)$. Now, looking at the two minimal normal subgroups of G, one of order 2 and the other of order 6072, we can conlcude that we have $C_2 \, X \, PSL_2(23)$ with order 12144. Looking at the subgroup lattice, note that $NL[4]$ has order 12144 as well. We can verify that $C_2 \, X \, PSL_2(23)$ is indeed isomorphic to $NL[4]$. The progenitor for our group is:

$$G :< A, B, C, D | A^2, B^{23}, C^{11}, D^2, B^C * B^{-2}, (C * D)^2, (B * D)^3, (A, B), (A, C), (A, D) >$$

4.1.1 Example

By applying a similar process, the following can be shown:

1. Given:

$$G < x, y, t >:= Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^3, (x^2 * y * t^{t^x})^2 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C | A^2, B^2, C^3, (B * C)^5 >$$

and is of the form: $G \cong (C_2\, X\, Alt(5))$.

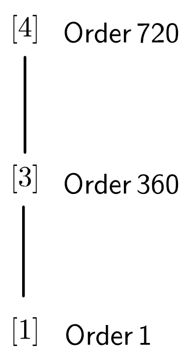
4.2 Semi-Direct Products

Our approach will begin the same way as we did on Section 4. Consider the following group presentation:

$$G < x, y, t > : = Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^5, (x^{y*t} * t^{x^2})^3, (y^t * t^{y*x})^8 > .$$

We have the composition factors of our group as well as its normal lattice. Both are used as a map/guide to find the structure of the composition series of our group:

```
    G
    |  Cyclic(2)
G1  *
    |  Alternating(6)
    1
```



Hence we begin to list the extensions from the composition factors: $G_1/1 \cong A(6)$ which implies that $G_1 \cong A(6)$, and $A(6)$ is normal. Moving up the composition factors, we have $G/G_1 \cong C_2$ which implies that $G \cong G_1 \,?\, C_2$, hence $G \cong A(6) \,?\, C_2$. To solve the extension problem we will check for a direct product first, that is, do we have $G \cong A(6)\, X\, C_2$. Looking at the minimal normal subgroup of G of order 360 we check the order of the group $A(6)$ and see that it is also 360. Looking at the subgroup lattice, note that $NL[2]$ has order 360 as well. So far this is our progenitor for G_1:

$$G_1 :< A, B | A^2, B^4, (A * B)^5, (A * B^2)^5 >$$

Back to our extension problem, in order to check if we have $G \cong A(6)\,X\,C_2$ we look at the normal lattice. Since we are at $NL[2]$, moving up the normal lattice, we can only move to $NL[3]$, since $NL[2]$ is the only maximal subgroup contained in $NL[3]$. Note that $NL[3]$ is isomorphic to G. Since the order of $NL[3]$ is twice the order of $NL[2]$ we look at the normal lattice of G to find a normal subgroup of order 2. Since there's no such normal subgroup then we know that we don't have a direct extension. Now we need to check if we have a semi-direct product. To do so, we will find an element C, in $NL[3]$ but not in $NL[2]$ of order 2. Hence, we have a semi-direct extension: $G \cong A(6) : C_2$. Since we are extending our group by the element C, once we determine how C affects A and B we can complete our progenitor:

$$G :< A, B, C | A^2, B^4, (A * B)^5, (A * B^2)^5, C^2, A^C = A * B^{-1} * A * B * A,$$
$$B^C = A * B^{-1} * A * B^{-1} * A * B * A >$$

Our progenitor is of order 720 and is isomorphic to our group G.

4.2.1 Example

By applying a similar process, the following can be shown:

1. Given:

$$G < x, y, t >:= Group < x, y, t | x^{11}, y^2, (x * y)^2, t^2, (t, y), (x^2 * y * t * t^x)^4, (x * t^x)^3 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C | A^2 = B^3 = (A * B)^{11} = (A, B * A * B * A * B)^2 = 1, C^2,$$
$$A^C = A * B * A * B^{-1} * A * B^{-1} * A * B^{-1} * A * B^{-1} * A * B * A * B^{-1} *$$
$$A * B^{-1}, B^C = A * B^{-1} * A * B * A * B * A * B^{-1} * A * B^{-1} * A * B^{-1} >$$

and is of the form: $G \cong (PSL(2, 11) : C_2)$.

2. Given:

$$G < x, y, t > : = Group < x, y, t | x^3, y^3, (x * y)^4, (x, y)^3, (x^2 * y)^7, t^7,$$
$$t^{x * y^{-1} * x^{-1}} = t^2, (t, y^{-1} * x^{-1} * y^{-1}), (y^x * t * t^{y * x})^2 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D, E|A^7, B^7, C^7, (A, B), (A, C), (B, C), D^2 = E^3 = (D * E)^7 = (D, E)^4 = 1, A^D = B^2 * A^{-2} * C^{-3}, A^E = C^3 * A^{-2} * B^{-3}, B^D = B * A^{-1} * C^{-3}, B^E = A^3 * C * B^{-1}, C^D = A^{-2} * B^{-3}, C^E = A * C^3 * B^{-1} >$$

and is of the form: $G \cong (7^3 : PSL(2, 7))$.

3. Given:

$$G < x, y, t >: = Group < x, y, t|x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, (x^{y*t} * t^{x^2})^2, (x^2 * y * t^{t^x})^2 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D, E|A^2, B^2, C^2, (A, B), (A, C), (B, C), D^3, A^D = A * C, B^D = B * C, C^D = B, E^2, A^E = A * C, B^E = B * C, C^E = C, D^E = D^{-1} * B >$$

and is of the form: $G \cong ((2^3 : C_3) : C_2)$.

4. Given:

$$G < x, y, t >: = Group < x, y, t|x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, (x^{y*t} * t^{x^2})^2, (x^2 * y * t^{t^x})^5 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D, E, F, K, L|A^5, B^5, C^5, D^5, (A, B), (A, C), (A, D), (B, C), (B, D), (C, D), E^2, A^E = A * B^2 * D^2, B^E = B^{-1}, C^E = C * B^{-2}, D^E = D^{-1}, F^2, A^F = E * A * C^2 * E, B^F = B, C^F = B^2 * C^{-1}, D^F = D^{-1}, E^F = E, K^3, A^K = E * A * E, B^K = C * B^{-1}, C^K = C * B^{-1} * D^{-1}, D^K = B^{-1}, E^K = C^2 * E * F, F^K = B^2 * E * D, L^2, A^L = E * A^{-1} * E, B^L = D, C^L = B * D * C^{-1}, D^L = B, E^L = D * E * B, F^L = E * C * F, K^L = A * K * A^{-1} * K >$$

and is of the form: $G \cong ((((5^4 : C_2) : C_2) : C_3) : C_2)$.

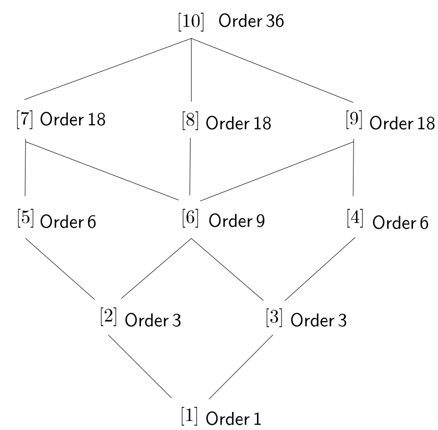
4.3 Semi-Direct and Direct Products

Given the following presentation:

$$G < x, y, t >:= Group < x, y, t | x^3, y^2, (x * y)^2, t^2, (t, y), (y * t * t^{x^2})^2 >$$

Listing the composition factors for which the group is composed of:

```
   G
   |  Cyclic(2)
G1 *
   |  Cyclic(2)
G2 *
   |  Cyclic(3)
G3 *
   |  Cyclic(3)
   1
```



The extensions from the composition factors: $G_3/1 \cong C_3$ imply that $G_3 \cong C_3$, and C_3 is normal. Moving up the composition factors, we have $G_2/G_3 \cong C_3$ which implies that $G_2/C_3 \cong C_3$, hence $G_2 \cong C_3 \,?\, C_3$. To solve the extension problem we will check for a direct product first, that is, do we have $G_2 \cong C_3 \, X \, C_3$. Now, we begin by looking at the two minimal normal subgroups of G, each of order 3. Hence we have $C_3 \, X \, C_3$ of order 9. Looking at the subgroup lattice, note that $NL[6]$ has order 9 as well. Note that $C_3 \, X \, C_3$ is Abelian and is isomorphic to $NL[6]$. Therefore we have $G_2 \cong C_3 \, X \, C_3$ and $G_2 \cong NL[6]$. The progenitor so far is:

$$G_2 :< A, B | A^3, B^3, (A, B) >$$

Continuing up the ladder of the composition factors, we have $G_1/G_2 \cong C_2$ which implies that $G_1/C_3 \, X \, C_3 \cong C_2$, so $G_1 \cong (C_3 \, X \, C_3) \,?\, C_2$. Now, to solve the extension problem let's check for a direct product first. Now, in reference to the normal lattice, we are located at $NL[6]$, to continue moving up the lattice we can either move to $NL[7]$ or $NL[9]$ since $NL[6]$ is contained in both $NL[7]$ and $NL[9]$. Say we choose to go up to $NL[9]$, since the order of $NL[9]$ is twice the order of $NL[6]$, we need to find a normal subgroup within $NL[9]$ of order 2. In order to look at the normal subgroups within $NL[9]$ we can generate the normal lattice of $NL[9]$.

Since there's no normal subgroup of order 2, then G_1 is not a direct product . So, we check for a semi-direct product. To do so, we find an element, C, in $NL[9]$ but not in $NL[6]$ of order 2. $C = (1,2)(3,6)(4,5)$ is the required element. Hence, we have a semi-direct extension: $G_1 \cong (C_3\,X\,C_3) : C_2$. Since we are extending our group by the element C, we will determine how C affects A and B to elaborate on our progenitor:

$$G_1 :< A, B, C | A^3, B^3, (A, B), C^2, A^C = A^2 * B^2, B^C = A^3 * B >$$

The above progenitor is of order 18 and is isomorphic to $NL[9]$. Continuing with the composition factors we are now at $G/G_1 \cong C_2$ which implies $G/((C_3\,X\,C_3) : C_2) \cong C_2$. Our last extension problem is $G \cong ((C_3\,X\,C_3) : C_2)\,?\,C_2$. To solve it, we will repeat the process as we've been doing so far. Notice that on the normal lattice of G we are in $NL[9]$, from here the only place left to move up is to get to $NL[10]$ which represents our entire group, G. To be able to get there we need to find a normal subgroup of order 2 in $NL[10]$. Since $NL[10] = G$ then we refer back to the normal lattice belonging to G. Since there's no normal subgroup of order 2, then we don't have a direct product. So, we check for a semi-direct product. We need an element, D of order 2 in $NL[10]$ but not in $NL[9]$. The required element D is found and we conclude that $G \cong ((C_3\,X\,C_3) : C_2) : C_2$. We then check how D affects A, B, and C to be able to complete our progenitor. Hence the progenitor has order 36 and is isomorphic to G:

$$G :< A, B, C, D | A^3, B^3, (A, B), C^2, A^C = A^2 * B^2, B^C = A^3 * B, D^2,$$
$$A^D = A * B * C^2, B^D = B^2 * C^2, C^D = C >$$

4.3.1 Examples

By applying a similar process, the following can be shown:

1. Given:

$$G < x, y, t >:= Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^2 * y * t^{t^x}) >$$

   we can show that is isomorphic to the progenitor:

$$G :< A, B, C | A^2, B^2, (A, B), C^3, A^C = A * B, B^C = A, D^2, A^D = A * B,$$
$$B^D = B, C^D = C^{-1} * A >$$

and is of the form: $G \cong (((C_2 \, X \, C_2) \, : \, C_3) \, : \, C_2)$.

2. Given:

$$G < x, y, t > := Group < x, y, t | x^3, y^2, (x * y)^2, t^2, (t, y), (y * t * t^{x^2})^2 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D | A^3, B^3, (A, B), C^2, A^C = A^2 * B^2, B^C = A^3 * B, D^2, \\ A^D = A * B * C^2, B^D = B^2 * C^2, C^D = C >$$

and is of the form: $G \cong (((C_3 \, X \, C_3) \, : \, C_2) \, : \, C_2)$.

3. Given:

$$G < x, y, t > := Group < x, y, t | x^{11}, y^2, (x * y)^2, t^2, (t, y), (x^2 * y * t * t^x)^2 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C | A^{11}, B^{11}, (A, B), C^2, A^C = A^{10} * B^2, B^C = B, D^2, A^D = A * B^{\ 9} * \\ C^2, B^D = A^{11} * B^{10} * C^2, C^D = C >$$

and is of the form: $G \cong ((C_{11} \, X \, C_{11}) \, : \, C_2) \, : \, C_2$.

4. Given:

$$G < x, y, t > : = Group < x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, \\ (x^{y*t} * t^{x^2})^2, (x^2 * y * t^{t^x})^5 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D, E, F, K, L | A^5, B^5, C^5, D^5, (A, B), (A, C), (A, D), (B, C), \\ (B, D), (C, D), E^2, A^E = A * B^2 * D^2, B^E = B^{-1}, C^E = C * B^{-2}, \\ D^E = D^{-1}, F^2, A^F = E * A * C^2 * E, B^F = B, C^F = B^2 * C^{-1}, D^F = D^{-1}, \\ E^F = E, K^3, A^K = E * A * E, B^K = C * B^{-1}, C^K = C * B^{-1} * D^{-1}, \\ D^K = B^{-1}, E^K = C^2 * E * F, F^K = B^2 * E * D, L^2, A^L = E * A^{-1} * E, \\ B^L = D, C^L = B * D * C^{-1}, D^L = B, E^L = D * E * B, F^L = E * C * F, \\ K^L = A * K * A^{-1} * K >$$

and is of the form: $G \cong ((((5^4 \, : \, C_2) \, : \, C_2) \, : \, C_3) \, : \, C_2)$.

5. Given:

$$G < x, y, t > : = Group < x, y, t | x^{15}, y^2, (x*y)^2, t^2, (t, y), (x^{y^2} * t^y)^4,$$
$$(y^t * t^{y*x})^3 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D, E, F, J | A^5, B^5, C^5, (A, B), (A, C), (B, C), D^2, A^D = A^{-1},$$
$$B^D = B * A^{-2}, C^D = B^2 * A^{-2} * C^{-1}, E^2, A^E = A, B^E = A^2 * B^{-1},$$
$$C^E = A^2 * C^{-1}, D^E = B * C^{-1} * D, F^3, A^F = A * B^{-1}, B^F = A * C * B^{-2},$$
$$C^F = C * B^{-2}, D^F = B * D * E * C^2, E^F = A^2 * D, J^2, A^J = B * A^{-1},$$
$$B^J = B, C^J = B^2 * C^{-1}, D^J = F * D * F^{-1}, E^J = B * F^{-1} * D * F * E,$$
$$F^J = F^{-1} >$$

and is of the form: $((((5^3 : C_2) : C_2) : C_3) : C_2)$.

6. Given:

$$Group {=} < x, y, t | x^{15}, y^2, (x*y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, (y^t * t^{y*x})^6, (x^2 * y * t^{t^x})^2 >$$

we can show that is isomorphic to the progenitor:

$$G :< A, B, C, D, E, F, J, K | A^5, B^5, C^5, (A, B), (A, C), (B, C), D^2, A^D = A^{-1},$$
$$B^D = B^{-1}, C^D = C^{-1}, E^2, A^E = C^{-1}, B^E = C^2 * A^{-2} * B^{-1}, C^E = A^{-1},$$
$$D^E = B * C^2 * D * A, F^2, A^F = A * E * B * E, B^F = B^{-1}, C^F = C * B^{-1},$$
$$D^F = D * E * B * E, E^F = B * E * B^{-1}, J^3, A^J = B * C^{-1}, B^J = A^{-1} * C^{-1},$$
$$C^J = A^{-1}, D^J = D * A * C, E^J = C * F * E * B, F^J = E * B^2 * C, K^2,$$
$$A^K = C * B^{-1}, B^K = A^{-1} * C^{-1}, C^K = C^{-1}, D^K = A * B * C * D,$$
$$E^K = J * E * J^{-1}, F^K = B * E * B^{-1}, J^K = J^{-1} >$$

and is of the form: $((((C_5 \, X \, C_5 \, X \, C_5) : C_2) : C_2) : C_3) : C_2)$.

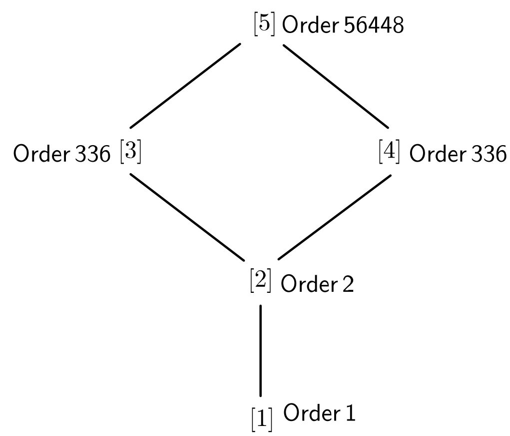
4.4 Central Extensions

The following examples are of a different type of extensions. The approach is very similar as we've done so far, but now we'll be dealing with central extensions, direct, and semi-direct products. We will begin with a small group and work our way up to a more complex group. Say we begin with the group presentation:

$$G < x, y, t >:= Group < x, y, t | x^{11}, y^2, (x * y)^2, t^2, (t, y), (x * y * t^x)^3, (x^2 * y * t * t^x)^4 >$$

Listing the normal lattice and the composition factors:

```
   G
   |  A(1, 7) = L(2, 7)
G1 *
   |  A(1, 7) = L(2, 7)
G2 *
   |  Cyclic(2)
   1
```



We begin to list the extensions from the composition factors: $G_2/1 \cong C_2$ which implies that $G_2 \cong C_2$, and C_2 is normal. Moving up the composition factors, we have $G_1/G_2 \cong PSL_2 7$ which implies that $G_1 \cong G_2 \,?\, PSL_2 7$, hence $G_1 \cong C_2 \,?\, PSL_2 7$. To solve the extension problem we check for a direct product first, that is, do we have $G_1 \cong C_2 \, X \, PSL_2 7$. Now, looking at the minimal normal subgroup of G of order 2 we can't conlcude that we have $C_2 \, X \, PSL_2 23$. Looking at the subgroup lattice, note that $NL[2]$ has order 2 just like C_2 has order 2. We refer to the normal lattice instead. Our choices are $NL[3]$ and $NL[4]$ since $NL[2]$ is the only maximal subgroup contained in each. Say we choose to move up to $NL[3]$. We need to look within $NL[3]$ for a normal subgroup of order 168, since the order of $NL[3]$ is 168 times the order of $NL[2]$. Looking within the normal lattice of $NL[3]$ we'll notice that there's no subgroup of order 168. Hence, we can now conclude that G_1 is not a direct product.

So, we now have to check if we have a semi-direct product. To do so, we will have to find an element, we'll call it B, in $NL[3]$ but not in $NL[2]$ of order 168. So, we are unable to find the required element B. Hence, we don't have a semi-direct product.

Now, we know that we don't have a direct or a semi-direct product. So, our next step is to check if we have a central extension. That is, does our group have a center and if it does we will attempt to factor our group G by the central element.

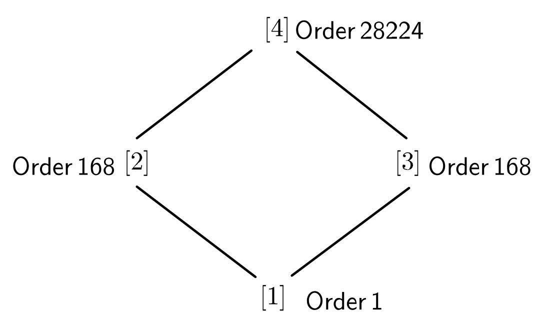
The center of G is a permutation group of order 2. Now, our task is to find which element in our lattice is the center of G. Since, the order of $NL[2]$ the center of G is indeed $NL[2]$. Now, we can factor G by its center. Doing so will generate a smaller group that we'll work with. Q, will be our new group for which we will apply the composition series approach as we've done so far.

So far this is how G looks like, where "$\bullet$" indicates that we've factored by the center:

$$G \cong 2^{\bullet}Q$$

Now, working with Q we will list the composition factors and the normal lattice of Q.

```
    Q
    |   A(1, 7) = L(2, 7)
 Q1 *
    |   A(1, 7) = L(2, 7)
    1
```



Listing the extensions from the composition factors: $Q_1/1 \cong PSL_2 7$ which implies that $Q_1 \cong PSL_2 7$, hence $PSL_2 7$ is normal. Moving up the composition factors, we have $Q/Q_1 \cong PSL_2 7$ which implies that $Q \cong Q_1 \, ? \, PSL_2 7$, hence $Q \cong PSL_2 7 \, ? \, PSL_2 7$. Now, to solve the extension problem we check for a direct product first, that is, do we have $Q \cong PSL_2 7 \, ? \, PSL_2 7$. Looking at the two minimal normal subgroups of Q each of order 168 we can conlcude that together they'll have order $(168)^2 = 28224$. Note that $nl[4]$ has order 28224 just like the order of the two minimal subgroups. Before we can conclude that we have a direct product. We check with MAGMA if $nl[2]$ and $nl[3]$ are a direct product and isomorphic to $nl[4]$:

```
D:=DirectProduct(nl[2],nl[3]);
s:=IsIsomorphic(D,Q);
s;
> true
```

Also, since $nl[4] \cong Q$, we know that Q is a direct product extension:
$Q \cong PSL_2 7 \, X \, PSL_2 7$. Now we can write the progenitor for Q:

$$Q :< A, B, C, D | A^2 = B^3 = (A*B)^8 = (A, B)^4 = 1, C^2 = D^3 = (C*D)^8 = (C, D)^4 = 1 >$$

Hence, our complete composition series for G is: $G \cong 2^{\bullet}(PSL_2 7 \, X \, PSL_2 7)$.

4.4.1 Examples

By applying a similar process, the following can be shown:

1. Given:

$$Group =< x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x^{y^2} * t^y)^4, (x^{y*t} * t^{x^2})^2 >$$

we will factor by the center and obtain a presentation for the group Q:

$$\begin{aligned} Q \; : < \; & A, B, C, D, E, F, J, K | A^5, B^5, C^5, D^5, (A, B), (A, C), (A, D), (B, C), (B, D), \\ & (C, D), E^2, A^E = A * C^2, B^E = B * D, C^E = C^{-1}, D^E = D^{-1}, F^2, \\ & A^F = E * A * B^2 * E, B^F = B^{-1} * D^{-1}, C^F = C^{-1} * D^{-1}, D^F = D, \\ & E^F = C^2 * D * E, J^3, A^J = A * C^2, B^J = C^{-1}, C^J = B * C^{-1}, D^J = C^2 * D, \\ & E^J = C * B^{-1} * F, F^J = C * E * F, K^2, A^K = A^{-1}, B^K = B^{-1} * C^{-1} * D^{-1}, \\ & C^K = C^{-1}, D^K = C^2 * D, E^K = C^2 * E, F^K = C * E * F, J^K = J^{-1} * F >, \end{aligned}$$

we can then conclude that our group is of the form $2^{\bullet}((((5^4 : C_2) : C_2) : C_3) : C_2)$.

We have investigated the progenitor $2^{*15} : D_{15}$ along with relations added to it, to be given as $< x, y, t | x^{15}, y^2, (x * y)^2, t^2, (t, y), (x * y * t^x)^a, (x * y * t^y)^b, (x^t)^c, (x * y^t)^d, (x^{y^2} * t^y)^e, (x^{y*t} * t^{x^2})^f, (y^t * t^{y*x})^g, (x^2 * y * t^{t^x})^h >$ where values for a, b, c, d, e, f, g, h where found:

Table 4.1: Homomorphic images of $2^{*15} : D_{15}$

Parameters									
a	b	c	d	e	f	g	h	Order of G	Isomorphic class
0	0	0	0	3	0	0	2	360	$2 \times A_5$
0	0	0	0	5	3	8	0	2160	$A_6 : 2$
0	0	0	0	4	2	0	2	24	$(2^3 : 3) : 2$
0	0	0	0	4	2	0	5	15000	$((((5^4 : 2) : 2) : 3 :)2)$
0	0	0	0	4	0	6	2	6000	$((((5^3 : 2) : 2) : 3 :)2)$
0	0	0	0	0	0	0	1	24	$((2^2 : 3) : 2)$
0	0	0	0	4	0	3	0	3000	$(((((5^3 : 2) : 2) : 2) : 3 :)2)$
0	0	0	0	4	2	0	0	30000	$2^{\bullet}((((5^4 : 2) : 2) : 3 :)2)$
0	0	0	0	0	3	5	0	7200	$PGL(2, 59)$
0	0	3	0	10	5	0	5	175560	J_1
0	0	3	0	0	9	7	0	178920	$PGL(2, 71)$
0	0	3	0	7	0	0	5	12180	$PGL(2, 29)$

Chapter 5

J_1 as a Homomorphic Image of $2^{*15} : D_{15}$

As mentioned in Chapter 4, the progenitor $2^{*15} : D_{15}$ has the Janko group, namely J_1, as its homomorphic image. We will investigate our progenitor to verify if it is a new presentation of J_1 In Curtis [Cur07], J_1 is given as a homomorphic image of $2^{*3} : S_3$ along with the relations: $[(0,1,2)t_0]^{10}$, $[(0,1)t_0]^{19}$, and $[(0,1)t_0t_2]^7$. We expand these relations below:

$$[(0,1,2)t_0]^{10} = 1 \Rightarrow (0,1,2)^{10}t_0t_2t_1t_0t_2 = t_0t_1t_2t_0t_1$$

$$[(0,1,2)t_0]^{10} = 1 \Rightarrow (0,1)^7t_0t_2t_1t_2t_0t_2t_1 = t_2t_0t_2t_1t_2t_0t_2$$

$$[(0,1)t_0]^{19} = 1 \Rightarrow (0,1)^{19}t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0 = t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1$$

Now, expand one of our relations $(x^t)^3 = x^3t_2t_{14} = 1$ which implies that $t_2 \sim t_{14}$. Having made this discovery, we can obtain the remaining equivalent t_i's:

$$t_1 \sim t_4 \sim t_7 \sim t_{10} \sim t_{13}$$

$$t_2 \sim t_5 \sim t_8 \sim t_{11} \sim t_{14}$$

$$t_3 \sim t_6 \sim t_9 \sim t_{12} \sim t_{15}$$

Having found the above relations we are able to reduce our presentation of $2^{*15} : D_{15}$ as a presentation of $2^{*3} : S_3$, and the progenitor now becomes

$$< x, y, t | x^3, y^2, (x*y)^2, t^2, (t,y), (x^{y^2} * t^y)^{10}, (x^{y*t} * t^{x^2})^5, (x^2 * y * t^{t^x})^5 >$$

where $x \sim (1,2,0)$ and $y \sim (1,2)$. We will now proceed to expand our remaining relations $(x^{y^2} * t^y)^{10}$,$(x^{y*t} * t^{x^2})^5$, and $(x^2 * y * t^{t^x})^5$:

$$(x^{y^2} * t^y)^{10} = 1 \Rightarrow (1,2,0)^{10} t_0t_2t_1t_0t_2 = t_0t_1t_2t_0t_1$$

$$(x^{y*t} * t^{x^2})^5 = 1 \Rightarrow (1,0,2)^5 t_1t_2t_1t_2t_0t_2t_0t_1 = t_2t_0t_2t_1t_2t_1t_0$$

$$(x^2 * y * t^{t^x})^5 = 1 \Rightarrow (1,3)^5 t_1t_0t_1t_0t_1t_0t_1t_0 = t_1t_0t_1t_0t_1t_0t_1$$

Comparing our findings to the expanded relations of Curtis, only one of our relations coincides. Because the other relations don't match then we will input our relations into the progenitor:

$$< x, y, t | x^3, y^2, (x*y)^2, t^2, (t,y)$$

where $x \sim (1,2,0)$ and $y \sim (1,2)$:

$$< x, y, t | x^3, y^2, (x*y)^2, t^2, (t,y), (xt)^{10}, (yxyt^{x^2}tt^{x^2})^5, (x^2yt^x tt^x)^5) > .$$

Once we ran it in MAGMA to verify the progenitor, it does give the homomorphic image of J_1. Hence, our relations from the progenitor $2^{*15} : D_{15}$ imply Curtis' relations.

The following code may be used to start a double coset enumeration:

```
S:=Sym(15);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15);
yy:=S!(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)(13, 15);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^10,(x^(y*t)*t^(x^2))^5,(x^2*y*t^(t^x))^5>;
Index(G,sub<G|x,y>);
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
/* creating the t_i s  */
ts:=[Id(G1):i in [1..15]];
ts[14]:=f(t);
ts[2]:=f(t^(x^3));
ts[3]:=f(t^(x^4));
ts[4]:=f(t^(x^5));
ts[5]:=f(t^(x^6));
```

```
ts[6]:=f(t^(x^7));
ts[7]:=f(t^(x^8));
ts[8]:=f(t^(x^9));
ts[9]:=f(t^(x^10));
ts[10]:=f(t^(x^11));
ts[11]:=f(t^(x^12));
ts[12]:=f(t^(x^13));
ts[13]:=f(t^(x^14));
ts[1]:=f(t^(x^2));
cst:=[null:i in [1..29260]] where null is [Integers() | ];
prodim:=function(pt,Q,I)
v:=pt;
for i in I do
v:=v^(Q[i]);
end for; return v; end function;
Dbl:=DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#Dbl;
/* Double coset [14] */
N14:=Stabiliser(N,14);
for g in N14 do g; end for;
S:={[14]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[14] eq g*(ts[(Rep(SSS[i]))[1]])
then print Rep(SSS[i]);
end if; end for; end for;
N14s:=N14;
for g in N do if [14]^g eq [3] then N14s:=sub<N|N14s,g>; end if;end for;
[14]^N14s;

/* List the elements that 14 is equal to  */
T14:=Transversal(N,N14);
for i in [1..#T14] do
ss:=[14]^T14[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..29260] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* List the orbits and the # of single cosets in [14]  */
Orbits(N14);
#N/#N14;
```

Chapter 6

Wreath Product of $\mathbb{Z}_5 \wr S_2$

Let H and K be permutation groups on X and Y, respectively. Let $Z = X \times Y$ define a permutation group on Z, called the *wreath product* of H by K, $H \wr K$.

Consider the following:
$X = \{1, 2, 3, 4, 5\}$, $Y = \{6, 7\}$, $H =< (1, 2, 3, 4, 5) >$, and $K =< (6, 7) >$.
Let $\gamma \in H$ and y be a fixed element of Y. Then, we define:

$$\gamma(y) = \begin{cases} (x, y) & \longmapsto ((x)\gamma, y) \\ (x, y) & \longmapsto (x, y_1) \text{ if } y_1 \neq y \end{cases}$$

As well as $k^* : (x, y) \longmapsto (x, (y)k$ where $k = \{6, 7\}$ and $k \in K$. Also, $Z = X \times Y = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (1, 7), (2, 7), (3, 7), (4, 7), (5, 7)\}$. Now, we also need to find $\gamma(6)$, $\gamma(7)$, and $\gamma(6, 7)^*$ where $\gamma = (1, 2, 3, 4, 5)$. The process is summarized in the following table:

Table 6.1: γ Function

$\gamma(6)$	$\gamma(7)$	$(6,7)^*$
$(1,6) \longmapsto (2,6)$	$(1,6) \longmapsto (1,6)$	$(1,6) \longmapsto (1,7)$
$(2,6) \longmapsto (3,6)$	$(2,6) \longmapsto (2,6)$	$(2,6) \longmapsto (2,7)$
$(3,6) \longmapsto (4,6)$	$(3,6) \longmapsto (3,6)$	$(3,6) \longmapsto (3,7)$
$(4,6) \longmapsto (5,6)$	$(4,6) \longmapsto (4,6)$	$(4,6) \longmapsto (4,7)$
$(5,6) \longmapsto (1,6)$	$(5,6) \longmapsto (5,6)$	$(5,6) \longmapsto (5,7)$
$(1,7) \longmapsto (1,7)$	$(1,7) \longmapsto (2,7)$	$(1,7) \longmapsto (1,6)$
$(2,7) \longmapsto (2,7)$	$(2,7) \longmapsto (3,7)$	$(2,7) \longmapsto (2,6)$
$(3,7) \longmapsto (3,7)$	$(3,7) \longmapsto (4,7)$	$(3,7) \longmapsto (3,6)$
$(4,7) \longmapsto (4,7)$	$(4,7) \longmapsto (5,7)$	$(4,7) \longmapsto (4,6)$
$(5,7) \longmapsto (5,7)$	$(5,7) \longmapsto (1,7)$	$(5,7) \longmapsto (5,6)$

Hence from the table above we have the relations:

$$\gamma(6) : (1,2,3,4,5)\,, \gamma(7) : (6,7,8,9,10),\ (6,7)^* : (1,6)(2,7)(3,8)(4,9)(5,10)$$

. Since $\gamma(y)$ and k^* are permutations of S_Z, we have $\gamma(6) \times \gamma(7) : Sym(2)$. We verify that our work is correct in MAGMA:

```
S:=Sym(10);
N:=sub<S|S!(1,2,3,4,5),S!(6,7,8,9,10),S!(1,6)(2,7)(3,8)(4,9)(5,10)>;
#N;
W:=WreathProduct(CyclicGroup(5),Sym(2));
#W;
s:=IsIsomorphic(N,W);
s;
```

Since the above prints true, we can now write the presentation for $\mathbb{Z}_5 \wr S_2$:

$$G < x,y,z > := Group < x,y,z | x^5, y^5, (x,y), z^2, x^z = y, y^z = x >$$

From here we can expand the progenitor by adding new relations hoping to discover new groups. After running the code in MAGMA the following group is obtained:

```
G<x,y,z,t>:=Group<x,y,z,t|x^5,y^5,(x,y),z^2,x^z=y,y^z=x,t^2,(t,y),
(x^2*t^y*z)^6, (t*t^z*t^x*y)^3 >;
f1,G1,k1:=CosetAction(G,sub<G|x,y,z>);
CompositionFactors(G1);
    G
    |  J2
    *
    |  Cyclic(2)
    1
```

6.1 $J_2 \times C_2$ as a Homomorphic Image of $\mathbb{Z}_5 \wr S_2$

The following code is used to begin the double coset enumeration of $J_2 \times C_2$. At the time or writing, the construction is not complete.

```
S:=Sym(10);
xx:=S!(1,2,3,4,5);
yy:=S!(6,7,8,9,10);
zz:=S!(1,6)(2,7)(3,8)(4,9)(5,10);
N:=sub<S|x,y,z>;
#N;
G<x,y,z,t>:=Group<x,y,z,t|x^5,y^5,(x,y),z^2,x^z=y,y^z=x,t^2,(t,y),
(x^2*t^y*z)^6,(t*t^z*t^x*y)^3 >;
f,G1,k1:=CosetAction(G,sub<G|x,y,z>);
Index(G,sub<G|x,y,z>);
IN:=sub<G1|f(x),f(y),f(z)>;
/* Creating the t_i s */
ts:=[Id(G1) : i in [1..10]];
ts[1]:=f(t);
ts[2]:=f(t^x);
ts[3]:=f(t^(x^2));
ts[4]:=f(t^(x^3));
ts[5]:=f(t^(x^4));
ts[6]:=f(t^z);
ts[7]:=f((t^x)^z);
ts[8]:=f(t^(x^2)^z);
ts[9]:= f(t^(x^3)^z);
ts[10]:=f(t^(x^4)^z);
cst:=[null:i in [1..24192]] where null is [Integers() | ];
prodim:=function(pt,Q,I)
v:=pt;
for i in I do
```

```
v:=v^(Q[i]);
end for;
return v;
end function;
```

Appendix A: Construction of $2^5 : A_5$

```
S:=Sym(5);
xx:=S!(5,1,2,3,4);
yy:=S!(4,2,1);
N:=sub<S|xx,yy>;
#N;
G<x,y,t>:=Group<x,y,t|x^5,y^3,(x*y)^2,t^2,(t,y),(t,x^2*y*x^1),
(t*t^x)^2>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1): i in [1..5]];
for i in [1..5] do ts[i]:=f(t^(x^i)); end for;
prodim := function(pt, Q, I) v := pt;
for i in I do v := v^(Q[i]); end for; return v;  end function;
cst := [null : i in [1 .. 90]] where null is [Integers() | ];
for i := 1 to 5 do
cst[prodim(1, ts, [i])] := [i]; end for;
m:=0;
for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for; m;
\\\\\\\\\\\\\\  [5]  \\\\\\\\\\\\\\
N5:=Stabiliser(N,5);
Orbits(N5);
N51:=Stabilizer(N5,1); S:={[5,1]};
SS:=S^N;
\\\\\\\\\\\\\\  [5 1]  \\\\\\\\\\\\\\
N51:=Stabilizer(N5,1);
S:={[5,1]};
SS:=S^N;
#N51;
Orbits(N51);
```

```
tr51 := Transversal(N, N51);
for i := 1 to #tr51 do ss := [5, 1]^tr51[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
\\\\\\\\\\\\\\\  [5 1 2]  \\\\\\\\\\\\\\\
N512:=Stabiliser(N51,2); S:={[5,1,2]};
SS:=S^N;
N512s:=N512;
for g in N do if [5,1,2]^g eq [1,5,2] then N512s:=sub<N|N512s,g>;
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [1,2,5] then N512s:=sub<N|N512s,g>;
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [2,1,5] then N512s:=sub<N|N512s,g>;
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [2,5,1] then N512s:=sub<N|N512s,g>;
end if;end for;
[5,1,2]^N512s;
for g in N do if [5,1,2]^g eq [5,2,1] then N512s:=sub<N|N512s,g>;
end if;end for;
[5,1,2]^N512s;
#N512s; #N/#N512s;
Orbits(N512s);
\\\\\\\\\\\\\\\  [5 1 2 3]  \\\\\\\\\\\\\\\
N5123:=Stabiliser(N512,3);
S:={[5,1,2,3]};
SS:=S^N;
N5123s:=N5123; \
for g in N do if [5,1,2,3]^g eq [1,5,2,3] then
N5123s:=sub<N|N5123s,g>;
end if;end for;
[5,1,2,3]^N5123s;
#N5123s;
for g in N do if [5,1,2,3]^g eq [1,2,5,3]
then N5123s:=sub<N|N5123s,g>;
end if;end for; [5,1,2,3]^N5123s;
#N5123s;
for g in N do if [5,1,2,3]^g eq [2,3,5,1]
then N5123s:=sub<N|N5123s,g>;
end if;end for; [5,1,2,3]^N5123s;
```

```
#N5123s;
#N/#N5123s;
Orbits(N5123s);
\\\\\\\\\\\\\\\  [5 1 2 3 4]  \\\\\\\\\\\\\\\
N51234:=Stabiliser(N5123,4);
S:={[5,1,2,3,4]};
SS:=S^N;
N51234s:=N51234;
for g in N do if [5,1,2,3,4]^g eq [1,5,2,3,4] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [1,2,5,3,4] then
N51234s:=sub<N|N51234s,g>;
end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [5,2,1,3,4] then
N51234s:=sub<N|N51234s,g>; end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [5,2,1,4,3] then
N51234s:=sub<N|N51234s,g>; end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
for g in N do if [5,1,2,3,4]^g eq [5,2,3,1,4] then
N51234s:=sub<N|N51234s,g>; end if;end for;
[5,1,2,3,4]^N51234s;
#N51234s;
#N/#N51234s;
```

Appendix B: Construction of $7^{*8} :_m L_2(7)$

```
SS:=Sym(48);
xx:=SS!(1,7,19)(2,8,20)(3,9,21)(4,10,22)(5,11,23)(6,12,24)
(13,25,31)(14,26,32)(15,27,33)(16,28,34)(17,29,35)(18,30,36)
(37,40,38)(39,41,42)(43,44,46)(45,48,47);
yy:=SS!(1,13,25)(2,14,26)(3,15,27)(4,16,28)(5,17,29)(6,18,30)
(7,10,8)(9,11,12)(19,20,22)(21,24,23)(31,37,43)(32,38,44)(33,39,45)
(34,40,46)(35,41,47)(36,42,48);
N:=sub<SS|xx,yy>;
#N;
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y)^4,(x,y)^3,(x^2*y)^7,t^7,
t^(x * y^-1 * x^-1 )=t^2,(t,y^-1 * x^-1 * y^-1),
(x^2 * y * t * t^x)^2>;
Index(G,sub<G|x,y>);
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y)>;
#G;

/* gives the sequences of the double cosets  */
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
prodim := function(pt, Q, I)
v := pt;
for i in I do v:=v^(Q[i]);
end for; return v; end function;

/* Defining the t_i's  */
ts:=[Id(G1): i in [1..48] ];
ts[1]:=f(t);ts[2]:=(ts[1])^2;ts[3]:=(ts[1])^3;ts[4]:=(ts[1])^4;
```

```
ts[5]:=(ts[1])^5;ts[6]:=(ts[1])^6;ts[7]:=f(t^x);ts[8]:=(ts[7])^2;
ts[9]:=(ts[7])^3;ts[10]:=(ts[7])^4;ts[11]:=(ts[7])^5;
ts[12]:=(ts[7])^6;ts[13]:=f(t^y);ts[14]:=(ts[13])^2;
ts[15]:=(ts[13])^3;ts[16]:=(ts[13])^4;
ts[17]:=(ts[13])^5;ts[18]:=(ts[13])^6;ts[19]:=f(t^(x^2));
ts[20]:=(ts[19])^2;ts[21]:=(ts[19])^3;ts[22]:=(ts[19])^4;
ts[23]:=(ts[19])^5;ts[24]:=(ts[19])^6;ts[25]:=f(t^(y^2));
ts[26]:=(ts[25])^2;ts[27]:=(ts[25])^3;ts[28]:=(ts[25])^4;
ts[29]:=(ts[25])^5;ts[30]:=(ts[25])^6;ts[31]:=f(t^(y*x^-1));
ts[32]:=(ts[31])^2;ts[33]:=(ts[31])^3;ts[34]:=(ts[31])^4;
ts[35]:=(ts[31])^5;ts[36]:=(ts[31])^6;ts[37]:=f(t^(y*x^(-1)*y));
ts[38]:=(ts[37])^2;ts[39]:=(ts[37])^3;ts[40]:=(ts[37])^4;
ts[41]:=(ts[37])^5;ts[42]:=(ts[37])^6;ts[43]:=f(t^(y*x^(-1)*y^-1));
ts[44]:=(ts[43])^2;ts[45]:=(ts[43])^3;ts[46]:=(ts[43])^4;
ts[47]:=(ts[43])^5;ts[48]:=(ts[43])^6;

Orbits(N);
cst:= [null : i in [1 .. Index(G,sub<G|x,y>)]] where null is
 [Integers() | ]; for i in {1, 2, 4, 7, 8, 10, 13, 14, 16, 19, 20,
22, 25, 26, 28, 31, 32, 34, 37,38, 40, 43, 44, 46 }
do cst[prodim(1,ts,[i])]:=[i]; end for; m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for;
m;
for i in {3, 5, 6, 9, 11, 12, 15, 17, 18, 21, 23, 24, 27, 29, 30,
33, 35, 36, 39, 41, 42, 45, 47, 48 }
do cst[prodim(1,ts,[i])]:=[i]; end for; m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

/* Double coset [1]  */
N1:=Stabiliser(N,1);
for g in N1 do g;end for;
SSS:={[1]};
SSS:=SSS^N;
N1s:=N1;
SSS:=Setseq(SSS);
for i in [1..#SSS] do
for g in IN do if ts[1] eq g*(ts[(Rep(SSS[i]))[1]])
then print Rep(SSS[i]);
end if; end for; end for;
T1:=Transversal(N,N1);
for i in [1..#T1] do
ss:=[1]^T1[i];
```

```
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
T1;
Orbits(N1);
#N/#N1s;

/* Double coset [3]  */
N3:=Stabiliser(N,3);
for g in N3 do g;end for;
 SSS:={[3]};
 SSS:=SSS^N;
 N3s:=N3;
 SSS:=Setseq(SSS);
 for i in [1..#SSS] do
 for g in IN do if ts[3] eq g*(ts[(Rep(SSS[i]))[1]])
 then print Rep(SSS[i]);
 end if;
 end for; end for;

T3:=Transversal(N,N3);
for i in [1..#T3] do
ss:=[3]^T3[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
T3;
Orbits(N3s);
#N/#N3s;

/* double coset [1,7] */
N17:=Stabiliser(N,[1,7]);
SSS:={[1,7]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
```

```
for g in IN do if ts[1]*ts[7]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N17s:=N17;
for g in N do if 1^g eq 8 and 7^g eq 14 then N17s:=sub<N|N17s,g>;
end if; end for;
#N17s;
[1,7]^N17s;

T17:=Transversal(N,N17);
for i in [1..#T17] do
ss:=[1,7]^T17[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N17; Orbits(N17s);
#N/#N17s;

/* double coset [1,8] */

N18:=Stabiliser(N,[1,8]);
SSS:={[1,8]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[8]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N18s:=N18;
for g in N do if 1^g eq 19 and 8^g eq 34 then N18s:=sub<N|N18s,g>;
end if; end for;
#N18s;
[1,8]^N18s;
```

```
T18:=Transversal(N,N18);
for i in [1..#T18] do
ss:=[1,8]^T18[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N18; Orbits(N18s);
#N/#N18s;

/* double coset [1,9] */
N19:=Stabiliser(N,[1,9]);
SSS:={[1,9]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[9]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N19s:=N19;
for g in N do if 1^g eq 37 and 9^g eq 17 then N19s:=sub<N|N19s,g>;
end if; end for;
#N19s;
[1,9]^N19s;

T19:=Transversal(N,N19);
for i in [1..#T19] do
ss:=[1,9]^T19[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

 #N19s; Orbits(N19s);
```

```
#N/#N19s;

/* double coset [1,10] */
N110:=Stabiliser(N,[1,10]);
SSS:={[1,10]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[10]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N110s:=N110;
for g in N do if 1^g eq 22 and 10^g eq 43 then
N110s:=sub<N|N110s,g>;
end if; end for;
#N110s;
[1,10]^N110s;

T110:=Transversal(N,N110);
for i in [1..#T110] do
ss:=[1,10]^T110[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1; end if;
end for; m;

#N110s; Orbits(N110s);
#N/#N110s;

/* double coset [1,11] */
N111:=Stabiliser(N,[1,11]);
SSS:={[1,11]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;
```

```
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[11]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N111s:=N111;
for g in N do if 1^g eq 8 and 11^g eq 29
then N111s:=sub<N|N111s,g>;
end if; end for;
#N111s;
[1,11]^N111s;

T111:=Transversal(N,N111);
for i in [1..#T111] do
ss:=[1,11]^T111[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N111s; Orbits(N111s);
#N/#N111s;

/* double coset [1,12] */
N112:=Stabiliser(N,[1,12]);
SSS:={[1,12]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[12]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N112s:=N112;
for g in N do if 1^g eq 2 and 12^g eq 18
then N112s:=sub<N|N112s,g>;
```

```
end if; end for;
#N112s;
[1,12]^N112s;

T112:=Transversal(N,N112);
for i in [1..#T112] do
ss:=[1,12]^T112[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N112s; Orbits(N112s);
#N/#N112s;

/* double coset [3,10] */
N310:=Stabiliser(N,[3,10]);
SSS:={[3,10]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[3]*ts[10]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
then print Rep(Seqq[i]);
end if;end for;end for;

N310s:=N310;
for g in N do if 3^g eq 9 and 10^g eq 31

then N310s:=sub<N|N310s,g>;
end if; end for;
#N310s;
[3,10]^N310s;

T310:=Transversal(N,N310);
for i in [1..#T310] do
ss:=[3,10]^T310[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
```

```
m:=0;
for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

#N310s; Orbits(N310s);
#N/#N310s;

/* double coset [1,11,46] */
N11146:=Stabiliser(N,[1,11,46]);
SSS:={[1,11,46]};
SSS:=SSS^N;
SSS;
#(SSS);
Seqq:=Setseq(SSS);
Seqq;

for i in [1..#SSS] do
for g in IN do if ts[1]*ts[11]*ts[46]
eq g*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
then print Rep(Seqq[i]);
end if;end for;end for;

N11146s:=N11146;
for g in N do if 1^g eq 37 and 11^g eq 27 and 46^g eq 13 then
 N11146s:=sub<N|N11146s,g>;end if; end for;

for g in N do if 1^g eq 40 and 11^g eq 9 and 46^g eq 14 then
 N11146s:=sub<N|N11146s,g>;end if; end for;

for g in N do if 1^g eq 4 and 11^g eq 9 and 46^g eq 25 then
 N11146s:=sub<N|N11146s,g>;end if; end for;

#N11146s;
[1,11,46]^N11146s;
T11146:=Transversal(N,N11146);
for i in [1..#T11146] do
 ss:=[1,11,46]^T11146[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
 m:=0;
 for i in [1..350] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
 #N11146s; Orbits(N11146s);
 #N/#N11146s;
```

Appendix C: Images of $2^{*15} : D_{15}$

We have investigated the progenitor $2^{*15} : D_{15}$ along with relations added to it, to be given as $< x, y, t | x^{15}, y^2, (x*y)^2, t^2, (t, y), (x*y*t^x)^a, (x*y*t^y)^b, (x^t)^c, (x*y^t)^d, (x^{y^2}*t^y)^e, (x^{y*t}*t^{x^2})^f, (y^t*t^{y*x})^g, (x^2*y*t^{t^x})^h >$ where values for a, b, c, d, e, f, g, h where found as well as their composition factors:

```
/* 0 0 3 0 10 5 0 5 29260 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^10,(x^(y*t)*t^(x^2))^5,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  J1
    1

/* 0 0 3 0 8 8 0 5 322560 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^8,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);

    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
```

```
    *
    |  A(2, 4) = L(3, 4)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 8 10 0 4 50000 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^10,(x^2*y*t^(t^x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    1

/* 0 0 5 0 0 4 5 10 262144 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^5,(x^2*y*t^(t^x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
    |  Cyclic(2)
```

```
    *
    |  Cyclic(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 0 3 0 0 480 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^(y*t)*t^(x^2))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
```

```
   G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
```

```
/* 0 0 3 0 0 8 0 2 128*/
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^8,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
```

```
   G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

```
/* 0 0 3 0 10 10 10 0 41600 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^10,(x^(y*t)*t^(x^2))^10, (y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
```

```
    G
    |  Cyclic(2)
    *
    |  2A(2, 4) = U(3, 4)
    *
    |  Cyclic(2)
    1

/*0 0 3 0 8 6 0 4 3888*/
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^6,(x^2*y*t^(t^x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1

/*0 0 5 0 0 8 4 6 12288*/
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y*t)*t^(x^2))^8, (y^t*t^(y*x))^4,(x^2*y*t^(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
```

```
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 0 4 0 6 78336 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^4,(x^2*y*t^(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  A(1, 17) = L(2, 17)
    *
    |  Cyclic(2)
```

```
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 0 4 0 5 48720 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^4,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  A(1, 29) = L(2, 29)
    1

/* 0 0 3 0 0 9 7 0 29820 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^9, (y^t*t^(y*x))^7>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 71) = L(2, 71)
    1

/* 0 0 3 0 0 10 7 0 4060 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^10, (y^t*t^(y*x))^7>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 29) = L(2, 29)
    1

/* 0 0 3 0 7 0 0 5 2030 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^7,(x^2*y*t^(t^x))^5>;
```

```
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 29) = L(2, 29)
    1

/* 0 0 3 0 8 5 0 0 4960 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 31) = L(2, 31)
    1

/* 0 0 3 0 8 8 0 4 8192 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^8,(x^2*y*t^(t^x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
```

```
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 0 3 10 0 12960 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y*t)*t^(x^2))^3, (y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Alternating(5)
    *
    |  Cyclic(3)
    1

/* 0 0 5 0 9 9 4 5 35244  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^9,(x^(y*t)*t^(x^2))^9, (y^t*t^(y*x))^4,
(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 89) = L(2, 89)
    1

/* 0 0 5 0 0 8 4 6 12288 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y*t)*t^(x^2))^8, (y^t*t^(y*x))^4,(x^2*y*t^(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(5)
```

```
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1


/* 0 0 3 0 10 6 0 4 15120 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^10,(x^(y*t)*t^(x^2))^6,(x^2*y*t^(t^x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1

/* 0 0 3 0 9 10 10 0 1140 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^9,(x^(y*t)*t^(x^2))^10, (y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
```

```
CompositionFactors(g1);
   G
    |  A(1, 19) = L(2, 19)
    *
    |  Cyclic(2)
    1

/* 0 0 3 8 7 0 0 5 2030  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,(x*y^t)^8,
(x^(y^2)*t^y)^7,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 29) = L(2, 29)
    1

/* 0 0 3 2 7 8 0 0 3584 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,(x*y^t)^2,
(x^(y^2)*t^y)^7,(x^(y*t)*t^(x^2))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
    |  Cyclic(2)
    *
    |  A(1, 7) = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 9 0 10 0 3420  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^9,(y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
```

```
    G
    |  A(1, 19) = L(2, 19)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    1

/* 0 0 3 0 9 4 0 0 816 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3
,(x^(y^2)*t^y)^9,(x^(y*t)*t^(x^2))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 17) = L(2, 17)
    1

/* 0 0 3 0 0 7 8 0 11480 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
x^(y*t)*t^(x^2))^7, (y^t*t^(y*x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 41)= L(2, 41)
    1

/* 0 0 3 0 0 4 0 4 1024 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^0,(x^2*y*t^(t^x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
 G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
```

```
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

```
/* 0 0 3 0 0 6 8 0 5376 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  A(1, 7) = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

```
/* 0 0 3 0 0 4 10 10 440 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
```

```
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^10,(x^2*y*t^(t^x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 11) = L(2, 11)
    *
    |  Cyclic(2)
    1


/* 0 0 3 0 0 6 7 7 364 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^7,(x^2*y*t^(t^x))^7>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 13) = L(2, 13)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 6 0 0 8 288 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^6,,(x^2*y*t^(t^x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
```

```
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 0 5 0 3 570 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^5,(x^2*y*t^(t^x))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 19) = L(2, 19)
    1

/* 0 0 3 0 8 0 10 0 720 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^8, (y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    1

/* 0 0 3 0 8 4 0 8 256 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^4,(x^2*y*t^(t^x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
```

```
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 6 0 0 10 450 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^6,(x^2*y*t^(t^x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1

/* 0 0 3 0 6 0 0 6 162 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y^2)*t^y)^6,(x^2*y*t^(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
```

```
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1

/* 0 0 3 0 0 5 10 0 220 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^5, (y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 11)= L(2, 11)
    1

/* 0 0 3 0 0 4 8 0 112 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 7) = L(2, 7)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 0 7 7 0 84 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^7, (y^t*t^(y*x))^7>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 8)= L(2, 8)
    1

/* 0 0 3 0 8 2 0 8 16 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
```

```
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^2,(x^2*y*t^(t^x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 3 0 6 0 10 0 50 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^6,(y^t*t^(y*x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(5)
    *
    |  Cyclic(5)
    1

/* 0 0 3 0 0 6 5 0 20 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    1
```

```
/* 0 0 3 0 0 3 0 10 60 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^3,
(x^(y*t)*t^(x^2))^3,(x^2*y*t^(t^x))^10>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    1

/* 0 0 3 0 0 3 0 5 10 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^3,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    1

/* 0 0 3 0 0 4 0 2 32 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^4,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

```
/* 0 0 3 0 0 8 7 0 56 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y*t)*t^(x^2))^8, (y^t*t^(y*x))^7>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
     G
    |  Cyclic(2)
    *
    |  A(1, 7)= L(2, 7)
    1

/* 0 0 3 0 6 0 0 2 18 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^6,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
   G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1

/* 0 0 5 0 6 0 6 2 24 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^6, (y^t*t^(y*x))^6,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 5 6 5 5 96 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y^2)*t^y)^5,(x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^5,
```

```
(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 0 10 6 2 120 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y*t)*t^(x^2))^10, (y^t*t^(y*x))^6,(x^2*y*t^(t^x))^2>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 6 0 5 3 132 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5
,(x^(y^2)*t^y)^6, (y^t*t^(y*x))^5,(x^2*y*t^(t^x))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 11) = L(2, 11)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 4 0 0 8 144 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y), (x^t)^5,
(x^(y^2)*t^y)^4,(x^2*y*t^(t^x))^8>;
```

```
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 5 6 0 5 192 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^5,(x^(y*t)*t^(x^2))^6,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 5 2 0 3 10 8 216 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x*y^t)^2,(x^(y*t)*t^(x^2))^3,(y^t*t^(y*x))^10,(x^2*y*t^(t^x))^8>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Cyclic(3)
    1

/* 0 0 5 0 0 6 4 0 264 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
```

```
(x^(y*t)*t^(x^2))^6, (y^t*t^(y*x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 11) = L(2, 11)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 0 5 5 0 342 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y*t)*t^(x^2))^5, (y^t*t^(y*x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 19) = L(2, 19)
    1

/* 0 0 5 0 5 9 0 5 486 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^5,(x^(y*t)*t^(x^2))^9,(x^2*y*t^(t^x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Alternating(5)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1

/* 0 0 5 0 6 0 7 3 504  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^6, (y^t*t^(y*x))^7,(x^2*y*t^(t^x))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
```

```
    |  Cyclic(2)
    *
    |  Alternating(7)
    1

/* 0 0 5 0 8 4 5 0 512 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^8,(x^(y*t)*t^(x^2))^4, (y^t*t^(y*x))^5>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1


/* 0 0 5 0 5 0 6 6 660  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^5,(y^t*t^(y*x))^6,(x^2*y*t^(t^x))^6>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  A(1, 11) = L(2, 11)
    *
```

```
    |  Cyclic(2)
    *
    |  Cyclic(5)
    1

/* 0 0 5 0 4 0 0 9 684  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^4,(x^2*y*t^(t^x))^9>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 19) = L(2, 19)
    1

/* 0 0 3 0 10 4 0 0 1320  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^3,
(x^(y^2)*t^y)^10,(x^(y*t)*t^(x^2))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 11) = L(2, 11)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    1

/* 0 0 5 0 6 6 0 3 1512  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^6,(x^(y*t)*t^(x^2))^6,(x^2*y*t^(t^x))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(7)
    *
    |  Cyclic(3)
    1
```

```
/* 0 0 5 0 0 7 4 0 2436 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y*t)*t^(x^2))^7, (y^t*t^(y*x))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 29)= L(2, 29)
    1

/*  0 0 5 0 0 0 4 3 2976   */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(y^t*t^(y*x))^4,(x^2*y*t^(t^x))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  A(1, 31) = L(2, 31)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 4 4 0 0 3840  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^4,(x^(y*t)*t^(x^2))^4>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
```

```
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

/* 0 0 5 0 5 3 0 0 4320 */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^5,(x^(y*t)*t^(x^2))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    *
    |  Alternating(5)
    1

/* 0 0 5 0 6 3 0 0 22692  */
G<x,y,t>:=Group<x,y,t|x^15,y^2,(x*y)^2,t^2,(t,y),(x^t)^5,
(x^(y^2)*t^y)^6,(x^(y*t)*t^(x^2))^3>;
f1,g1,k1:=CosetAction(G,sub<G|x,y>);
CompositionFactors(g1);
CompositionFactors(g1);
    G
    |  Cyclic(2)
    *
    |  A(1, 61) = L(2, 61)
    1
```

Bibliography

[BCH96] J.N. Bray, R. T. Curtis, and A.M.A. Hamas. *A systematic approach to symmetric presentations.* Math Proc. Cambridge Philos. Soc., 1996.

[Bra97] John N. Bray. *Symmetric presentations of sporadic groups and related topics.* University of Birmingham, England, 1997.

[Cea05] J. Cannon et al. *MAGMA programming language (various versions up to version 2.12).* School of Mathematics and Statistics, University of Sydney, 1993-2005.

[CH96] R. T. Curtis and Z. Hasan. *Symmetric representation of elements of the Janko group* J_1. Symbolic Comput. 1996.

[Con71] J. H. Conway. *Three lectures on exceptionalgroups.* In Finite Simple Groups (Proc. Instructional Conf., Oxford,1969). Academic Press, London, 1971.

[Cur07] Robert T. Curtis. *Symmetric generation of groups.* volume 111 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2007.

[HK06] Z. Hasan and A. Kasouha. Symmetric representation of elements of finite groups. *arXiv:math/0612042*, 2006.

[Rot95] Joseph J. Rotman. *An introduction to the theory of groups*, volume 148 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.