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Research Article

On the hyperbolic Horadam matrix functions

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Abstract

In this study, we introduce a new class of the hyperbolic matrix functions which are called symmetrical hyperbolic Horadam sine and cosine matrix functions and we present some hyperbolic and recursive properties of these new matrix functions. In addition, we introduce quasi-sine Horadam matrix function and also define the matrix form of the metallic shofars that related to the hyperbolic Horadam sine and hyperbolic Horadam cosine matrix functions.

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1. Introduction

As is known, the differential equation systems are used to develop many problems of applied sciences, so the differential systems have many applications in science and technology. Since the matrix functions exist in the solutions of the differential equation systems, the theory of the matrix functions is widely used in these areas. In particular, the matrix exponential and trigonometric matrix functions have received the most attention [1,5,6,8,13,14]. These matrix functions play a fundamental role in the differential equation systems. For example, the differential problem

$$X''(t) + DX(t) = 0, \qquad X(0) = X_0, \qquad X' = X_1$$

has the solution

$$X(t) = \cos\sqrt{D}tX + (\sqrt{D})^{-1}\sin(\sqrt{D}t)X_1,$$

where $\sqrt{D}$ denotes any square root of a non-singular matrix D. There exist many studies [6–10, 13] on the solutions of the more general problems that involve the hyperbolic and trigonometric matrix functions. Some hyperbolic and trigonometric matrix functions of a real or complex square matrix D are defined as [5–8, 10]:

$$\sinh(D) = \frac{e^D - e^{-D}}{2}, \qquad \cosh(D) = \frac{e^D + e^{-D}}{2},$$

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$$\sin(D) = \frac{e^{iD} - e^{-iD}}{2i}, \qquad \cos(D) = \frac{e^{iD} - e^{-iD}}{2}.$$

Some identities that are analogous to scalar case are valid for the matrix case under the assumption on that commutable matrices [9,10]. For example,

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

for the commutable matrices A and B. Also for $D \in \mathbb{C}^{n\times n}$ and the n-dimensional identity matrix I,

$$\sin(D) = \cos(D - \frac{\pi}{2}I) \quad \text{and} \quad \cos(D - k\pi I) = (-1)^k \cos(D) \tag{1.1}$$

where $k \in \mathbb{Z}$ [9,10].

In this study, we will introduce hyperbolic Horadam sine and cosine matrix functions and will give some of their properties. So, here are some information about the Horadam numbers and hyperbolic Horadam Functions.

2. Hyperbolic Horadam functions

Horadam number sequence $W_n(a, b; p, q)$ is generated by the rule $W_{n+1} = pW_n + qW_{n-1}$ where $n \geq 1$, $W_0 = a$, $W_1 = b$, p and q are non-zero real numbers [11]. The characteristic polynomial of W_n is $t^2 - pt - q = 0$ and the roots of this equation are $\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}$ and $\beta = \frac{p - \sqrt{p^2 + 4q}}{2}$ [11,12]. The Binet formula for W_n is

$$W_n = \frac{A\alpha^n - (-1)^n Bq^n\alpha^{-n}}{\sqrt{p^2 + 4q}} = \begin{cases} \dfrac{A\alpha^n + Bq^n\alpha^{-n}}{\sqrt{p^2 + 4q}}, & n \text{ is odd} \\[2ex] \dfrac{A\alpha^n - Bq^n\alpha^{-n}}{\sqrt{p^2 + 4q}}, & n \text{ is even}, \end{cases} \tag{2.1}$$

where $A = b - a\beta$ and $B = b - a\alpha$, also p and q are non-zero real numbers such that $p^2 + 4q \neq 0$ [11]. The Horadam number W_n is reduced to the Fibonacci number F_n and Lucas number L_n. That is,

$$F_n = W_n(0, 1; 1, 1) = \begin{cases} \dfrac{\alpha^n + \alpha^{-n}}{\sqrt{5}}, & n \text{ is odd} \\[2ex] \dfrac{\alpha^n - \alpha^{-n}}{\sqrt{5}}, & n \text{ is even}, \end{cases} \tag{2.2}$$

$$L_n = W_n(2, 1; 1, 1) = \begin{cases} \alpha^n + \alpha^{-n}, & n \text{ is odd} \\ \alpha^n - \alpha^{-n}, & n \text{ is even}. \end{cases} \tag{2.3}$$

Using the Binet formulas (2.2) and (2.3), hyperbolic Fibonacci and Lucas functions and their symmetrical forms have been defined [12, 15, 17]. Using the Binet formula (2.1), Bahşi and Solak [3] have defined hyperbolic Horadam sine and cosine functions for the continuous variable x:

$$sW(x) = \frac{A\alpha^{2x} - Bq^{2x}\alpha^{-2x}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cW(x) = \frac{A\alpha^{2x+1} + Bq^{2x+1}\alpha^{-2x-1}}{\sqrt{p^2 + 4q}} \tag{2.4}$$

respectively, where α is the positive root of the characteristic polynomial of W_n, p and q are non-zero real numbers such that $p^2 + 4q \neq 0$. Since these functions are not symmetrical with respect to the origin, the authors have defined symmetrical hyperbolic Horadam sine and cosine functions as:

$$sWs(x) = \frac{A\alpha^x - Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} \quad \text{and} \quad cWs(x) = \frac{A\alpha^x + Bq^x\alpha^{-x}}{\sqrt{p^2+4q}}, \tag{2.5}$$

respectively.

In the next section, we will define the matrix forms of the hyperbolic Horadam sine, cosine functions and their symmetrical forms by using the matrix $D \in \mathbb{C}^{n\times n}$ instead of $x \in \mathbb{R}$, also we will examine some properties of these new hyperbolic matrix functions.

3. Hyperbolic Horadam matrix functions

By using the matrix $D \in \mathbb{C}^{n\times n}$ instead of n in equations (2.2) and (2.3), Bahşi and Solak [2] have introduced the matrix forms of the symmetrical Fibonacci and Lucas functions as follows:

$$sFs(D) = \frac{\psi^D - \psi^{-D}}{\sqrt{5}}, \qquad \text{(Symmetrical Fibonacci sine matrix function)}$$

$$cFs(D) = \frac{\psi^D + \psi^{-D}}{\sqrt{5}}, \qquad \text{(Symmetrical Fibonacci cosine matrix function)}$$

$$sLs(D) = \psi^D - \psi^{-D}, \qquad \text{(Symmetrical Lucas sine matrix function)}$$

$$cLs(D) = \psi^D + \psi^{-D}, \qquad \text{(Symmetrical Lucas cosine matrix function)}$$

where $\psi = \dfrac{1+\sqrt{5}}{2}$ is the well known golden ratio. Then, they have given some properties of these matrix functions similar to that of the hyperbolic Fibonacci and Lucas functions. The matrix power of the golden ratio, ψ^D, has interesting identities similar to that of the matrix exponential [2]. In fact, these identities are satisfied for $p, q \in \mathbb{R}^+$, $D \in \mathbb{C}^{n\times n}$ and the n-dimensional identity matrix I:

(1) For $0 \in \mathbb{C}^{n\times n}$, $q^0 = I$.
(2) For $m, n \in \mathbb{Z}$, $q^I = qI$, $q^{mI} = (q^m)I$ and $q^m q^{nI} = q^{(m+n)I}$.
(3) $q^{(D^T)} = (q^D)^T$ where D^T is transpose matrix of D.
(4) The inverse of the matrix q^D is given by $(q^D)^{-1} = q^{-D}$.
(5) The derivative of the matrix q^{Dt} is $\dfrac{dq^{Dt}}{dt} = \ln(q)D\, q^{Dt}$.
(6) For $m \in \mathbb{Z}$, $(q^D)^m = q^{mD}$.
(7) For the commutable $n \times n$ matrices A and B, $q^{A+B} = q^A q^B$.
(8) For the commutable $n \times n$ matrices A and B, $p^A q^B = q^B p^A$.

The above identities immediately follow from the series expansion for $D \in \mathbb{C}^{n\times n}$ and $q \in \mathbb{R}^+$,

$$q^D = I + \ln(q)D + \frac{(\ln(q))^2 D^2}{2!} + \frac{(\ln(q))^3 D^3}{3!} + \ldots . \tag{3.1}$$

If we use the matrix $D \in \mathbb{C}^{n\times n}$ instead of $x \in \mathbb{R}$ in equations (2.4) and (2.5) we have:

Definition 3.1. Let α be the positive root of the characteristic polynomial of W_n. Then, the functions

$$sW(D) = \frac{A\alpha^{2D} - Bq^{2D}\alpha^{-2D}}{\sqrt{p^2+4q}} \quad \text{and} \quad cW(D) = \frac{A\alpha^{2D+I} + Bq^{2D+I}\alpha^{-2D-I}}{\sqrt{p^2+4q}} \tag{3.2}$$

are called hyperbolic Horadam sine and cosine matrix functions, respectively, where $D \in \mathbb{C}^{n\times n}$, p and q are non-zero real numbers such that $p^2+4q>0$. The symmetrical forms of these functions are defined as:

$$sWs(D)=\frac{A\alpha^D - Bq^D\alpha^{-D}}{\sqrt{p^2+4q}} \quad \text{and} \quad cWs(D)=\frac{A\alpha^D + Bq^D\alpha^{-D}}{\sqrt{p^2+4q}}.$$

If $A=1$, $B=1$ and $p=1$, $q=1$ are taken in the formulas (3.2), then the functions convert the hyperbolic Fibonacci matrix functions. So, the hyperbolic Horadam sine and cosine matrix functions are the general forms of the hyperbolic Fibonacci matrix functions. By using the series expansion (3.1), we have the series expansions of $sWs(D)$ and $cWs(D)$ as:

$$\begin{aligned} sWs(D) &= (\sqrt{p^2+4q})^{-1}\Bigg\{A(I+\ln(\alpha)D+\frac{(\ln(\alpha))^2D^2}{2!}+\frac{(\ln(\alpha))^3D^3}{3!}+...) \\ &\qquad - B(I+\ln(\frac{q}{\alpha})D+\frac{(\ln(\frac{q}{\alpha}))^2D^2}{2!}+\frac{(\ln(\frac{q}{\alpha}))^3D^3}{3!}+...)\Bigg\}, \end{aligned}$$

$$\begin{aligned} cWs(D) &= (\sqrt{p^2+4q})^{-1}\Bigg\{A(I+\ln(\alpha)D+\frac{(\ln(\alpha))^2D^2}{2!}+\frac{(\ln(\alpha))^3D^3}{3!}+...) \\ &\qquad + B(I+\ln(\frac{q}{\alpha})D+\frac{(\ln(\frac{q}{\alpha}))^2D^2}{2!}+\frac{(\ln(\frac{q}{\alpha}))^3D^3}{3!}+...)\Bigg\}. \end{aligned}$$

The symmetrical hyperbolic Horadam sine and cosine matrix functions hold:

(1) For the zero matrix $cWs(0)=\dfrac{A+B}{\sqrt{p^2+4q}}I$, $sWs(0)=\dfrac{A-B}{\sqrt{p^2+4q}}I$,

(2) For the identity matrix $cWs(I)=\dfrac{A\alpha+Bq\alpha^{-1}}{\sqrt{p^2+4q}}I$, $sWs(I)=\dfrac{A\alpha-Bq\alpha^{-1}}{\sqrt{p^2+4q}}I$,

(3) For the transpose matrix $[cWs(D)]^T=cWs(D^T)$ and $[sWs(D)]^T=sWs(D^T)$.

Next two lemmas give the matrix forms of the identities $\alpha^2=p\alpha+q$, $q\alpha^{-2}=1-p\alpha^{-1}$, $\alpha-q\alpha^{-1}=p$ and $2q+\alpha^2+q^2\alpha^{-2}=p^2+4q$.

Lemma 3.2. *Let α be the positive root of the characteristic polynomial of W_n, p and q, non-zero real numbers such that $p^2+4q>0$, and I, the n-dimensional identity matrix. Then,*

$$a)\ \alpha^{2I}=p\alpha^I+qI, \qquad b)\ q^I\alpha^{-2I}=I-p\alpha^{-I} \quad \text{and} \quad c)\ \alpha^I-q\alpha^{-I}=pI.$$

Proof.

$$\begin{aligned} a)\ p\alpha^I+qI &= p\alpha I+qI=(p\alpha+q)I=\alpha^2I=\alpha^{2I}, \\ b)\ I-p\alpha^{-I} &= I-p\alpha^{-1}I=(1-\frac{p}{\alpha})I=q\alpha^{-2}I=qI\alpha^{-2}I=q^I\alpha^{-2I}, \\ c)\ \alpha^I-q\alpha^{-I} &= \alpha I-q\alpha^{-1}I=(\alpha-\frac{q}{\alpha})I=(\frac{\alpha^2-q}{\alpha})I=pI, \end{aligned}$$

where $p\alpha+q=\alpha^2$ and $1-\dfrac{p}{\alpha}=q\alpha^{-2}$. □

Lemma 3.3. *Let α be the positive root of the characteristic polynomial of W_n, p and q, non-zero real numbers such that $p^2+4q>0$, and I, the n-dimensional identity matrix. Then,*

$$2q^I+\alpha^{2I}+q^{2I}\alpha^{-2I}=(p^2+4q)I.$$

Proof.

$$\begin{aligned}
2q^{I}+\alpha^{2I}+q^{2I}\alpha^{-2I} &= 2qI+\alpha^{2}I+q^{2}\alpha^{-2}I\\
&=(2q+\alpha^{2}+q^{2}\alpha^{-2})I\\
&=(\alpha+\frac{q}{\alpha})^{2}I\\
&=(\frac{\alpha^{2}+q}{\alpha})^{2}I\\
&=(\frac{p\alpha+2q}{\alpha})^{2}I\\
&=(p^{2}+4q(\frac{p}{\alpha}+\frac{q}{\alpha^{2}}))I\\
&=(p^{2}+4q(\frac{p\alpha+q}{\alpha^{2}}))I\\
&=(p^{2}+4q)I,
\end{aligned}$$

where $p\alpha+q=\alpha^{2}$. □

Since we will often use Lemma 1, Lemma 2 and the properties (1)-(8) in page 3, these will not be specified in our proofs. Also, I denotes the identity matrix.

4. Some recursive and hyperbolic properties of the hyperbolic Horadam matrix functions

In this section, we give some properties that are analogous to properties of the hyperbolic Horadam function given in [10].

Proposition 4.1. ***(Recursive relation)***

$$a)\ sWs(D+2I)=pcWs(D+I)+qsWs(D),$$
$$b)\ cWs(D+2I)=psWs(D+I)+qcWs(D).$$

Proof.

$$\begin{aligned}
a)\ pcWs(D+I)+qsWs(D) &= p(\frac{A\alpha^{D+I}+Bq^{D+I}\alpha^{-D-I}}{\sqrt{p^{2}+4q}})+q(\frac{A\alpha^{D}-Bq^{D}\alpha^{-D}}{\sqrt{p^{2}+4q}})\\
&=\frac{A\alpha^{D}(p\alpha^{I}+qI)-Bq^{D+I}\alpha^{-D}(qq^{-I}-p\alpha^{-I})}{\sqrt{p^{2}+4q}}\\
&=\frac{A\alpha^{D}\alpha^{2I}-Bq^{D+I}\alpha^{-D}(I-p\alpha^{-I})}{\sqrt{p^{2}+4q}}\\
&=\frac{A\alpha^{D+2I}-Bq^{D+I}\alpha^{-D}q^{I}\alpha^{-2I}}{\sqrt{p^{2}+4q}}\\
&=\frac{A\alpha^{D+2I}-Bq^{D+2I}\alpha^{-D-2I}}{\sqrt{p^{2}+4q}}\\
&=sWs(D+2I).
\end{aligned}$$

$b)$ The proof is similar to the proof of (a). □

Proposition 4.2. ***(Cassini's Identity)***

$$a)\ [sWs(D)]^2 - cWs(D+I)cWs(D-I) = -ABq^{D-I},$$

$$b)\ [cWs(D)]^2 - sWs(D+I)sWs(D-I) = ABq^{D-I}.$$

Proof.

$$\begin{aligned} LHS &= \frac{(A\alpha^D - Bq^D\alpha^{-D})^2}{p^2+4q} - \frac{(A\alpha^{D+I} + Bq^{D+I}\alpha^{-D-I})(A\alpha^{D-I} + Bq^{D-I}\alpha^{-D+I})}{p^2+4q} \\ &= \frac{A^2\alpha^{2D} - 2ABq^D + B^2q^{2D}\alpha^{-2D}}{p^2+4q} \\ &\qquad - \frac{A^2\alpha^{2D} + ABq^{D-I}\alpha^{2I} + ABq^{D+I}\alpha^{-2I} + B^2q^{2D}\alpha^{-2D}}{p^2+4q} \\ &= \frac{-2ABq^D - ABq^{D-I}\alpha^{2I} - ABq^{D+I}\alpha^{-2I}}{p^2+4q} \\ &= \frac{-ABq^{D-I}(2q^I + \alpha^{2I} + q^{2I}\alpha^{-2I})}{p^2+4q} \\ &= -ABq^{D-I}, \end{aligned}$$

where LHS is left hand side of the equation.

b) The proof is similar to the proof of (a). □

Proposition 4.3. ***(Pythagorean Theorem)***

$$[cWs(D)]^2 - [sWs(D)]^2 = 4ABq^D(p^2+4q)^{-1}.$$

Proof.

$$\begin{aligned} LHS &= \frac{(A\alpha^D + Bq^D\alpha^{-D})^2 - (A\alpha^D - Bq^D\alpha^{-D})^2}{p^2+4q} \\ &= \frac{A^2\alpha^{2D} + 2ABq^D + B^2q^{2D}\alpha^{-2D} - A^2\alpha^{2D} + 2ABq^D - B^2q^{2D}\alpha^{-2D}}{p^2+4q} \\ &= 4ABq^D(p^2+4q)^{-1}, \end{aligned}$$

where LHS is left hand side of the equation. □

Proposition 4.4. ***(nth derivatives)***

$$[cWs(Dt)]^{(n)} = \begin{cases} (D\ln(\alpha))^n sWs(Dt) + \dfrac{(\ln(\frac{q}{\alpha}))^n + (\ln(\alpha))^n}{\sqrt{p^2+4q}} BD^n q^{Dt}\alpha^{-Dt}, & n \text{ is odd} \\ (D\ln(\alpha))^n cWs(Dt) + \dfrac{(\ln(\frac{q}{\alpha}))^n - (\ln(\alpha))^n}{\sqrt{p^2+4q}} BD^n q^{Dt}\alpha^{-Dt}, & n \text{ is even} \end{cases}$$

and

$$[sWs(Dt)]^{(n)} = \begin{cases} (D\ln(\alpha))^n cWs(Dt) - \dfrac{(\ln(\dfrac{q}{\alpha}))^n + (\ln(\alpha))^n}{\sqrt{p^2+4q}} BD^n q^{Dt}\alpha^{-Dt}, & n \text{ is odd} \\ (D\ln(\alpha))^n sWs(Dt) - \dfrac{(\ln(\dfrac{q}{\alpha}))^n - (\ln(\alpha))^n}{\sqrt{p^2+4q}} BD^n q^{Dt}\alpha^{-Dt}, & n \text{ is even.} \end{cases}$$

Proof. We use the principle of mathematical induction on n. For $n = 1$ and $n = 2$,

$$\begin{aligned}
[cWs(Dt)]' &= (\frac{A\alpha^{Dt} + Bq^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}})' \\
&= \frac{A\ln(\alpha)D\alpha^{Dt} + BDq^{Dt}\alpha^{-Dt}(\ln(q) - \ln(\alpha))}{\sqrt{p^2+4q}} \\
&= (D\ln(\alpha))\frac{A\alpha^{Dt} - Bq^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}} + \frac{\ln(q)}{\sqrt{p^2+4q}} BDq^{Dt}\alpha^{-Dt} \\
&= (D\ln(\alpha))sWs(Dt) + \frac{(\ln(\dfrac{q}{\alpha}) + (\ln(\alpha))}{\sqrt{p^2+4q}} BDq^{Dt}\alpha^{-Dt}. \\
[[cWs(Dt)]'' &= [(D\ln(\alpha))sWs(Dt) + \frac{(\ln(\dfrac{q}{\alpha}) + (\ln(\alpha))}{\sqrt{p^2+4q}} BDq^{Dt}\alpha^{-Dt}]' \\
&= D\ln(\alpha)\frac{AD\ln(\alpha)\alpha^{Dt} - BDq^{Dt}\alpha^{-Dt}(\ln(q) - \ln(\alpha))}{\sqrt{p^2+4q}} \\
&\qquad + \frac{(\ln(\dfrac{q}{\alpha})) + (\ln(\alpha))}{\sqrt{p^2+4q}} BD^2q^{Dt}\alpha^{-Dt}(\ln(q) - \ln(\alpha)) \\
&= (D\ln(\alpha))^2(\frac{A\alpha^{Dt} + Bq^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}}) - \frac{BD^2q^{Dt}\alpha^{-Dt}\ln(q)\ln(\alpha)}{\sqrt{p^2+4q}} \\
&\qquad + \frac{BD^2q^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}}[\ln(\frac{q}{\alpha})(\ln(q) - \ln(\alpha)) + \ln(\alpha)(\ln(q) - \ln(\alpha))] \\
&= (D\ln(\alpha))^2 cWs(Dt) + \frac{(\ln(\dfrac{q}{\alpha}))^2 - (\ln(\alpha))^2}{\sqrt{p^2+4q}} BD^2q^{Dt}\alpha^{-Dt}.
\end{aligned}$$

Now, suppose that k is an odd number and the statement is true for $n = k$. We must show that the statement is true for the even number $n = k + 1$.

$$\begin{aligned}
[[cWs(Dt)]^{(k)}]' &= [(D\ln(\alpha))^k \frac{A\alpha^{Dt} - Bq^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}} + \frac{(\ln(\frac{q}{\alpha}))^k + (\ln(\alpha))^k}{\sqrt{p^2+4q}} BD^k q^{Dt}\alpha^{-Dt}]' \\
&= (D\ln(\alpha))^k [\frac{AD\ln(\alpha)\alpha^{Dt} - BDq^{Dt}\alpha^{-Dt}(\ln(q) - \ln(\alpha))}{\sqrt{p^2+4q}}] \\
&\quad + \frac{(\ln(\frac{q}{\alpha}))^k + (\ln(\alpha))^k}{\sqrt{p^2+4q}} BD^{k+1} q^{Dt}\alpha^{-Dt}(\ln(q) - \ln(\alpha)) \\
&= (D\ln(\alpha))^{k+1} [\frac{A\alpha^{Dt} + Bq^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}}] - \frac{BD^{k+1}q^{Dt}\alpha^{-Dt}}{\sqrt{p^2+4q}} (\ln(\alpha))^k \ln(q) \\
&\quad + \frac{(\ln(\frac{q}{\alpha}))^{k+1} + (\ln(\alpha))^k \ln(\frac{q}{\alpha})}{\sqrt{p^2+4q}} BD^{k+1} q^{Dt}\alpha^{-Dt} \\
&= (D\ln(\alpha))^{k+1} cWs(Dt) \\
&\quad + \frac{(\ln(\frac{q}{\alpha}))^{k+1} + (\ln(\alpha))^k \ln(q) - (\ln(\alpha))^{k+1} - (\ln(\alpha))^k \ln(q)}{\sqrt{p^2+4q}} BD^{k+1} q^{Dt}\alpha^{-Dt} \\
&= (D\ln(\alpha))^{k+1} cWs(Dt) + \frac{(\ln(\frac{q}{\alpha}))^{k+1} - (\ln(\alpha))^{k+1}}{\sqrt{p^2+4q}} BD^{k+1} q^{Dt}\alpha^{-Dt}.
\end{aligned}$$

Thus, if k is an odd number the statement is true for $n = k+1$. If k is an even number, then it can be proved similarly that the statement is true for $n = k+1$. This completes the proof of $[cWs(Dt)]^{(n)}$. The proof of $[sWs(Dt)]^{(n)}$ is similar to the proof of $[cWs(Dt)]^{(n)}$. □

Proposition 4.5. ***(Moivre's Equation)***

$$a)\ [cWs(D) + sWs(D)]^n = [2A(\sqrt{p^2+4q})^{-1}]^{n-1}[cWs(nD) + sWs(nD)],$$

$$b)\ [cWs(D) - sWs(D)]^n = [2B(\sqrt{p^2+4q})^{-1}]^{n-1}[cWs(nD) - sWs(nD)].$$

Proof.

$$a)\ RHS = [2A(\sqrt{p^2+4q})^{-1}]^{n-1}[\frac{A\alpha^{nD}+Bq^{nD}\alpha^{-nD}}{\sqrt{p^2+4q}}+\frac{A\alpha^{nD}-Bq^{nD}\alpha^{-nD}}{\sqrt{p^2+4q}}]$$

$$= [\frac{2A}{\sqrt{p^2+4q}}]^{n-1}[\frac{2A\alpha^{nD}}{\sqrt{p^2+4q}}]$$

$$= [\frac{2A\alpha^{D}}{\sqrt{p^2+4q}}]^{n}$$

$$= [\frac{A\alpha^{D}+Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}}+\frac{A\alpha^{D}-Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}}]^{n}$$

$$= [cWs(D)+sWs(D)]^n,$$

where RHS is the right hand side of the identity.

$$b)\ RHS = [2B(\sqrt{p^2+4q})^{-1}]^{n-1}[\frac{A\alpha^{nD}+Bq^{nD}\alpha^{-nD}}{\sqrt{p^2+4q}}-\frac{A\alpha^{nD}-Bq^{nD}\alpha^{-nD}}{\sqrt{p^2+4q}}]$$

$$= [\frac{2B}{\sqrt{p^2+4q}}]^{n-1}[\frac{2Bq^{nD}\alpha^{-nD}}{\sqrt{p^2+4q}}]$$

$$= [\frac{2Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}}]^{n}$$

$$= [\frac{A\alpha^{D}+Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}}-\frac{A\alpha^{D}-Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}}]^{n}$$

$$= [cWs(D)-sWs(D)]^n,$$

where RHS is the right hand side of the identity. □

5. The quasi-sine Horadam matrix functions

Because of the equation $(-1)^n = \cos(n\pi)$ for $n \in \mathbb{Z}$, the formula (2.1) can be rewritten as

$$W_n = \frac{A\alpha^n - \cos(n\pi)Bq^n\alpha^{-n}}{\sqrt{p^2+4q}}.$$

Bahşi and Solak [3] have defined quasi-sine Horadam function:

$$WW(x) = \frac{A\alpha^x - \cos(\pi x)Bq^x\alpha^{-x}}{\sqrt{p^2+4q}},$$

for $x \in \mathbb{R}$ and have presented some properties of this function.

The next definition gives the matrix form of the quasi-sine Horadam functions.

Definition 5.1. For the matrix $D \in \mathbb{C}^{n\times n}$, the function

$$WW(D) = \frac{A\alpha^D - \cos(\pi D)Bq^D\alpha^{-D}}{\sqrt{p^2+4q}}$$

is called the matrix form of the quasi-sine Horadam function.

Now, we give some properties of this new matrix function.

Proposition 5.2. ***(Recursive relation)***

$$WW(D+2I) = pWW(D+I) + qWW(D).$$

Proof. By using the equalities

$$\cos(\pi D + 2\pi I) = \cos(\pi D) = - \cos(\pi D + \pi I),$$

we have

$$\begin{aligned} RHS &= p(\frac{A\alpha^{D+I} - \cos(\pi(D+I))Bq^{D+I}\alpha^{-D-I}}{\sqrt{p^2+4q}}) + q(\frac{A\alpha^{D} - \cos(\pi D)Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}}) \\ &= \frac{A\alpha^{D}(p\alpha^{I} + qI) - \cos(\pi D + 2\pi I)Bq^{D+I}\alpha^{-D-I}(qIq^{-I}\alpha^{I} - pI)}{\sqrt{p^2+4q}} \\ &= \frac{A\alpha^{D+2I} - \cos(\pi(D+2I))Bq^{D+2I}\alpha^{-D-2I}}{\sqrt{p^2+4q}} \\ &= WW(D+2I), \end{aligned}$$

where RHS is the right hand side of the identity.

Proposition 5.3. ***(Cassini's identity)***

$$[WW(D)]^2 - WW(D+I)WW(D-I) = -ABq^{D-I}\cos(\pi D).$$

□

Proof. By using

$$\cos(\pi(D+I)) = \cos(\pi(D-I)) = -\cos(\pi D),$$

we have

$$\begin{aligned} LHS &= (\frac{A\alpha^{D} - \cos(\pi D)Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}})^2 \\ &\qquad - [(\frac{A\alpha^{D+I} - \cos(\pi(D+I))Bq^{D+I}\alpha^{-D-I}}{\sqrt{p^2+4q}}) \\ &\qquad\qquad \times (\frac{A\alpha^{D-I} - \cos(\pi(D-I))Bq^{D-I}\alpha^{-D+I}}{\sqrt{p^2+4q}})] \end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2\alpha^{2D} - 2A\alpha^D Bq^D\alpha^{-D}\cos(\pi D) + \cos^2(\pi D)B^2q^{2D}\alpha^{-2D}}{p^2+4q} \\
&\quad - \frac{A^2\alpha^{2D} - A\alpha^{D+I}Bq^{D-I}\alpha^{-D+I}\cos(\pi(D-I))}{p^2+4q} \\
&\quad + \frac{A\alpha^{D-I}Bq^{D+I}\alpha^{-D-I}\cos(\pi(D+I))}{p^2+4q} \\
&\quad - \frac{\cos(\pi(D+I))\cos(\pi(D-I))B^2q^{D+I}\alpha^{-D-I}q^{D-I}\alpha^{-D+I}}{p^2+4q} \\
&= \frac{-2ABq^D\cos(\pi D) + AB\alpha^{2I}q^{D-I}\cos(\pi(D-I))}{p^2+4q} \\
&\quad + \frac{AB\alpha^{-2I}q^{D+I}\cos(\pi(D+I))}{p^2+4q} \\
&= -ABq^{D-I}\cos(\pi D)\frac{(2q^I + \alpha^{2I} + q^{2I}\alpha^{-2I})}{p^2+4q} \\
&= -ABq^{D-I}\cos(\pi D).
\end{aligned}
$$

□

6. The matrix form of the Metallic Shofars

Three-dimensional Horadam spiral

$$CWW(x) = \frac{A\alpha^x - \cos(\pi x)Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} + i\frac{\sin(\pi x)Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} = \frac{A\alpha^x + ie^{i\pi(\frac{1}{2}-x)}Bq^x\alpha^{-x}}{\sqrt{p^2+4q}},$$

has been defined in [3], where $x \in \mathbb{R}$ and $i^2 = -1$. The matrix form of the three-dimensional Horadam spiral is:

Definition 6.1. The function

$$CWW(D) = \frac{A\alpha^D - \cos(\pi D)Bq^D\alpha^{-D}}{\sqrt{p^2+4q}} + i\frac{\sin(\pi D)Bq^D\alpha^{-D}}{\sqrt{p^2+4q}}$$

or

$$CWW(D) = \frac{A\alpha^D + ie^{i\pi(\frac{1}{2}I-D)}Bq^D\alpha^{-D}}{\sqrt{p^2+4q}}$$

is called the matrix form of the three-dimensional Horadam spiral, where $D \in \mathbb{C}^{n\times n}$ and $i^2 = -1$.

Proposition 6.2. ***(Recursive relation)***

$$CWW(D+2I) = pCWW(D+I) + qCWW(D).$$

Proof. The equalities $\cos(\pi(D+2I)) = \cos(\pi D) = -\cos(\pi(D+I))$ and $\sin(\pi(D+2I)) = \sin(\pi D) = -\sin(\pi(D+I))$ are valid for $D \in \mathbb{C}^{n\times n}$ and the n-dimensional identity matrix

I. By using these equalities, we have

$$\begin{aligned}
RHS &= p[(\frac{A\alpha^{D+I}-\cos(\pi(D+I))Bq^{D+I}\alpha^{-D-I}}{\sqrt{p^2+4q}})+i(\frac{\sin(\pi(D+I))Bq^{D+I}\alpha^{-D-I}}{\sqrt{p^2+4q}})] \\
&\quad + q[(\frac{A\alpha^{D}-\cos(\pi D)Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}})+i(\frac{\sin(\pi D)Bq^{D}\alpha^{-D}}{\sqrt{p^2+4q}})] \\
&= \frac{A\alpha^D(p\alpha^I+qI)-\cos(\pi D+2\pi I)Bq^{D+I}\alpha^{-D-I}(qIq^{-I}\alpha^I-pI)}{\sqrt{p^2+4q}} \\
&\quad + i(\frac{\sin(\pi D+2\pi I))Bq^{D+I}\alpha^{-D-I}(qIq^{-I}\alpha^I-pI)}{\sqrt{p^2+4q}}) \\
&= \frac{A\alpha^{D+2I}-\cos(\pi(D+2I)Bq^{D+I}\alpha^{-D-I}(q^I\alpha^{-I})}{\sqrt{p^2+4q}} \\
&\quad + i(\frac{\sin(\pi(D+2I))Bq^{D+I}\alpha^{-D-I}(q^I\alpha^{-I})}{\sqrt{p^2+4q}}) \\
&= \frac{A\alpha^{D+2I}-\cos(\pi(D+2I)Bq^{D+2I}\alpha^{-D-2I}}{\sqrt{p^2+4q}}) \\
&\quad + i(\frac{\sin(\pi(D+2I))Bq^{D+2I}\alpha^{-D-2I}}{\sqrt{p^2+4q}} \\
&= CWW(D+2I),
\end{aligned}$$

where RHS is the right hand side of the identity. □

For the matrix $X \in \mathbb{C}^{n\times n}$, the system of matrix equations depends on the matrix form of the three-dimensional Horadam matrix spiral is:

$$y(X) - \frac{A\alpha^X}{\sqrt{p^2+4q}} = \frac{-\cos(\pi X)Bq^X\alpha^{-X}}{\sqrt{p^2+4q}}$$
$$z(X) = \frac{\sin(\pi X)Bq^X\alpha^{-X}}{\sqrt{p^2+4q}}.$$

By summing up the squares of equations of the above system and using the equation $\sin^2(\pi X)+\cos^2(\pi X) = I$, we have

$$(y(X) - \frac{A\alpha^X}{\sqrt{p^2+4q}})^2 + z^2(X) = (\frac{Bq^X\alpha^{-X}}{\sqrt{p^2+4q}})^2. \tag{6.1}$$

By considering the formulas of the matrix functions $cWs(X)$ and $sWs(X)$, the equation (6.1) is rewritten as

$$z^2(X) = [cWs(X) - y(X)][y(X) - sWs(X)].$$

The equation (6.1) corresponds to the matrix form of the Metallic Shofar [4]. If $a = 0$, $b = 1$, $p = m$ and $q = 1$ are taken in the definition of the Horadam number $W_n(a, b; p, q)$, the matrix form of the Metallic Shofar is expressed with the equation

$$(y(X) - \frac{\alpha^X}{\sqrt{m^2+4}})^2 + z^2(X) = (\frac{\alpha^{-X}}{\sqrt{m^2+4}})^2,$$

where α is the golden $(m, 1)$ proportion [12]. For the cases $m = 1, 2, 3$ the equation (6.1) corresponds to the matrix form of Golden Shofar [4, 16], the matrix form of Silver Shofar [4, 12] and the matrix form of Bronze Shofar [4, 12], respectively. That is, we have the equations

$$(y(X) - \frac{\varphi^X}{\sqrt{5}})^2 + z^2(X) = (\frac{\varphi^{-X}}{\sqrt{5}})^2,$$
$$(y(X) - \frac{\psi^X}{\sqrt{8}})^2 + z^2(X) = (\frac{\psi^{-X}}{\sqrt{8}})^2,$$
$$(y(X) - \frac{\tau^X}{\sqrt{13}})^2 + z^2(X) = (\frac{\tau^{-X}}{\sqrt{13}})^2,$$

where φ, ψ and τ are golden, silver and bronze proportions, respectively.

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