THESIS

GENERALIZED RSK FOR ENUMERATING PROJECTIVE MAPS FROM n-POINTED CURVES

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ABSTRACT

GENERALIZED RSK FOR ENUMERATING PROJECTIVE MAPS FROM n-POINTED CURVES

Schubert calculus has been studied since the 1800s, ever since the mathematician Hermann Schubert studied the intersections of lines and planes. Since then, it has grown to have a plethora of connections to enumerative geometry and algebraic combinatorics alike. These connections give us a way of using Schubert calculus to translate geometric problems into combinatorial ones, and vice versa. In this thesis, we define several combinatorial objects known as Young tableaux, as well as the well-known RSK correspondence between pairs of tableaux and sequences. We also define the Grassmannian space, as well as the Schubert cells that live inside it. Then, we describe how Schubert calculus and the Littlewood-Richardson rule allow us to turn problems of intersecting geometric spaces into ones of counting Young tableaux with particular characteristics.

We give a combinatorial proof of a recent geometric result of Farkas and Lian on linear series on curves with prescribed incidence conditions. The result states that the expected number of degree-d morphisms from a general genus g, n-marked curve C to $\mathbb{P}^r$, sending the marked points on C to specified general points in $\mathbb{P}^r$, is equal to $(r+1)^g$ for sufficiently large d. This computation may be rephrased as an intersection problem on Grassmannians, which has a natural combinatorial interpretation in terms of Young tableaux by the classical Littlewood-Richardson rule. We give a bijection, generalizing the well-known RSK correspondence, between the tableaux in question and the $(r+1)$-ary sequences of length g, and we explore our bijection's combinatorial properties.

We also apply similar methods to give a combinatorial interpretation and proof of the fact that, in the modified setting in which $r = 1$ and several marked points map to the same point in $\mathbb{P}^1$, the number of morphisms is still 2^g for sufficiently large d.

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Chapter 1

Introduction

Many problems in combinatorics are inspired by, motivated by, or otherwise come from problems in related fields. Particularly, the development of *Schubert calculus* has yielded many connections between algebraic geometry and algebraic combinatorics. Schubert calculus is the study of the intersections of linear spaces, and is part of a larger branch of algebraic geometry known as intersection theory. Many enumerative problems in intersection theory are not solely about linear spaces. In this thesis, we will find another connection between combinatorics and intersection theory from a question relating to the study of curves.

Many questions in intersection theory have answers that are either zero (How many lines pass through three points in general position?) or infinity (How many lines pass through a given point and intersect a given line?). It is the remaining cases, where the answer is nonzero and finite, where we will focus our attention.

In [2], Farkas and Lian answer geometrically an enumerative question in the study of curves. Specifically, suppose C is a complex curve of genus g, and $\mathbb{P}^r$ the (complex) projective space of r dimensions. Additionally, suppose that we have n distinct marked points $x_1, x_2, \ldots, x_n$ on our curve C, and n distinct points $y_1, y_2, \ldots, y_n$ in $\mathbb{P}^r$. Then, let $L_{g,r,d}$ be the number of degree d morphisms $f : C \to \mathbb{P}^r$, such that $f(x_i) = y_i$ for all $i = 1, 2, \ldots, n$. See Figure 1.1 for a diagram of an example. The question answered by Farkas and Lian is to find explicit formulas for $L_{g,r,d}$ in terms of the parameters g, r, and d.

Remark. Note that we do not list n as a parameter here. This is because this number $L_{g,r,d}$ is both finite and nonzero precisely when $n = \frac{1}{r}(dr + d + r - rg)$.

Farkas and Lian answered this question for three separate cases:

- For $d \geq rg + r$
- For $r = 1$

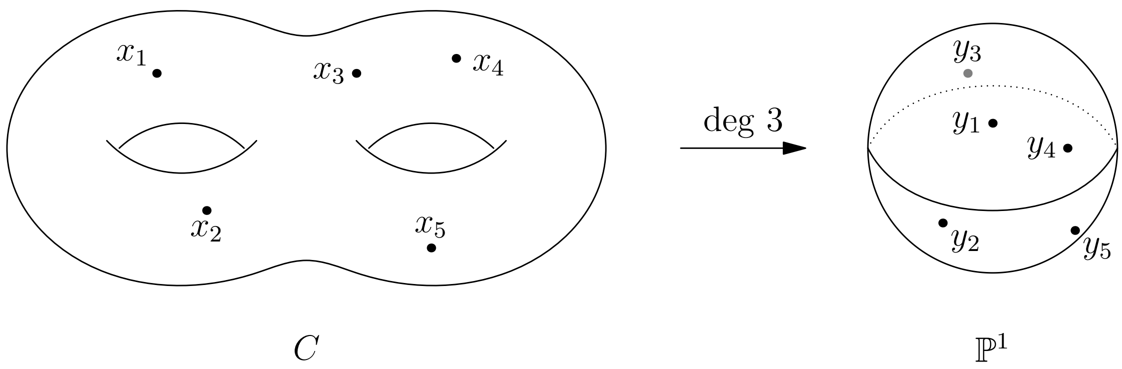


Figure 1.1: Case of $g = 2$, $r = 1$, and $d = 3$. In this case, $n = \frac{1}{1}(3+3+1-2) = 5$.

- For $d = r + \frac{rg}{r+1}$

They found that in the first two cases, the enumeration is $L_{g,r,d} = (r+1)^g$. In the third case, they note that the enumeration matches a formula of Castelnuovo [1]:

$$L_{g,r,d} = g! \cdot \frac{1! \cdot 2! \cdots r!}{s! \cdot (s+1)! \cdot \cdots \cdot (s+r)!},$$

where $s = \frac{g}{r+1}$.

Additionally, they considered a slight variation of this problem. To get this modified problem, first, set $r = 1$. Second, identify some number k of the points y_i. That is, set $y_1 = y_2 = \cdots = y_k$ for some integer $1 \leq k \leq \min\{n, d\}$. Let $L'_{g,d,k}$ be the number of maps $f : C \to \mathbb{P}^1$ such that $f(x_i) = y_i$ for all i. Again, the question is to express $L'_{g,d,k}$ in terms of its parameters g, d, and k. Here, Farkas and Lian answer the question with an intersection formula on the Grassmannian.

Both problems can be interpreted in terms of intersections of Schubert varieties, which as we will see in Chapter 2 can be converted into questions about Young tableaux. In the first problem, Farkas and Lian do this already. They define the structure called an L-tableau with parameters g, r, and d, and show that the number of the above maps equals the number of these tableaux. See Figure 1.2 for an example of an L-tableau, which is defined in Definition 3.1.

2	4	1	3	3	3
1	3	4	1	2	2
1	2	3	0	1	1
1	2	3	4	0	0

Figure 1.2: An example of an L-tableau with parameters $g = 4$, $r = 3$, and $d = 9$.

It is shown in [2] that whenever either $d \geq rg + r$, $r = 1$, or $d = r + \frac{rg}{r+1}$, we have

$$L_{g,r,d} = \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^{g} \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i} \right) \right], \tag{1.1}$$

and that in the first two cases, this formula equals $(r+1)^g$.

Here the notation σ_{1^r} is shorthand for $\sigma_{(1,1,1,\ldots,1)}$ where the tuple $(1,1,\ldots,1)$ has length r, and σ_{α_i} is shorthand for $\sigma_{(\alpha_i)}$. The integral indicates that the sum of products of Schubert cycles in question expands in the Schubert basis as a constant multiple of

$$\sigma_{(d-r)^{r+1}} := \sigma_{(d+1,d+1,\ldots,d+1)},$$

and the integral is defined to be this constant coefficient. The integral in equation (1.1) is equal to the coefficient of $s_{(d-r)^{(r+1)}}$ in the expansion

$$s_{1^r}^{g} \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg} \left(\prod_{i=0}^{r} s_{\alpha_i} \right) \right] \tag{1.2}$$

However, their proof is purely geometric in nature, and they leave finding a combinatorial proof as an open problem. Our first result is a combinatorial proof to a slightly stronger version of the above fact.

Theorem 1.1. *The number of L-tableaux with parameters (g,r,d) is $(r+1)^g$ whenever $d \geq g + r$.*

Our result is a stronger version of theirs, because our proof only requires $d \geq g + r$, as opposed to $d \geq rg + r$, and we do not require that d be a multiple of r. We prove this by defining an intermediary object, as well as a bijection between the L-tableaux and these intermediary objects. We then apply the RSK correspondence to form a bijection between this intermediary and $(r+1)$-ary sequences of length g, which are enumerated by $(r+1)^g$. In the case where $r=1$, our first bijection is trivial, and the result follows from the RSK correspondence alone.

The corresponding result from [2] for $L'_{g,d,k}$ is that for $k+g \le 2d+1$ and $2 \le k \le d$, we have

$$L'_{g,d,k} = \int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \sigma_{k-1} \left(\sum_{i+j=2d-g-k-1} \sigma_i \sigma_j \right) - \int_{\mathrm{Gr}(2,d)} \sigma_1^g \sigma_{k-2} \left(\sum_{i+j=2d-g-k-2} \sigma_i \sigma_j \right). \tag{1.3}$$

They leave the enumeration formula for $L'_{g,d,k}$ as the difference of two Schubert class intersection formulas shown in equation (1.3). For our second main result, we first provide a combinatorial interpretation of this expression, and then give an explicit enumeration for this interpretation.

Theorem 1.2. *If $d \ge g+k$, we have $L'_{g,d,k} = 2^g$.*

Our interpretation makes use of similar structures to the L-tableaux above that we call L'-tableaux (see Definitions 4.1 and 4.3). Our proof shows that those objects corresponding to this difference can be reduced to the $r=1$ case of our first result.

This thesis is structured as follows. In Chapter 2, we will cover the needed background material on tableaux, the RSK correspondence, projective space, Schubert varieties, and the Littlewood-Richardson rule. In Chapter 3 we translate the geometric formulas for $L_{g,r,d}$ into a Young tableaux enumeration problem and prove Theorem 1.1. We also show that our bijection reduces to ordinary RSK in the case $r=1$ (Section 3.3), and allows us to recover a classical theorem of Castelnuovo in the case $d = r + \frac{rg}{r+1}$ (Section 3.4). In Chapter 4 we define L'-tableaux for $L'_{g,d,k}$ and prove Theorem 1.2. Finally, in Chapter 5, we more fully explore the combinatorial properties of our constructions. We also note that Chapters 3, 4, and 5 are adapted from [5], a previous paper by the author and Maria Gillespie.

Chapter 2

Background

2.1 Young tableaux

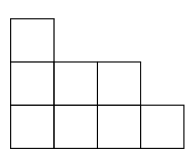
Young tableaux are a well known structure in algebraic combinatorics. We start by first introducing the related idea of partitions.

Definition 2.1. A **partition** of an integer n is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, with the properties that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$, and that $\lambda_i \geq \lambda_{i+1}$ for all i. We call r the number of *parts* of the partition. We sometimes write $\lambda \vdash n$.

Although tuples are one possible way to denote a partition, it can be useful to have a more visual depiction of partitions. Using Young diagrams is one common choice.

Definition 2.2. A **Young diagram** is a drawing of bottom- and left-justified unit squares in the first quadrant. If n is the total number of boxes (also known as the *size* of the diagram), we can think of a Young diagram in terms of the corresponding partition λ of n, where λ_i is the number of boxes in the ith row from the bottom of the diagram. [1]

Example 2.3. Below is an example of a Young diagram. It corresponds to the partition $\lambda = (4,3,1)$, and has size $4+3+1 = 8$.



One reason we often denote partitions as collections of boxes is so that we can label the boxes. Such a labeling is called a *tableau*.

[1] This is known as *French notation*. Some authors choose instead to use *English notation*, where the boxes are top- and left-justified.

Definition 2.4. A **semistandard Young tableau** is a Young diagram where each box is filled with a nonnegative integer, with the restriction that the entry in each box must be strictly greater than the entry in the box below, and weakly greater than the entry in the box to the left.

In general, a Young tableau may be filled using any alphabet, but in this paper we will always use nonnegative integers. The semistandard condition allows us to repeat entries as we move left to right across rows, but not as we move bottom to top along columns. We will often refer to the Young diagram underlying a Young tableau as the *shape* of that tableau.

Example 2.5. Here are two different semistandard Young tableaux of the same shape as the Young diagram in Example 2.3. Note how we have repeated entries in some of the rows, but not in any of the columns.

$$\begin{array}{|c|ccc}\cline{1-1} 3 & & & \\ \hline 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \\ \hline 0 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|ccc}\cline{1-1} 4 & & & \\ \hline 3 & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{8} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

We refer the multiset of our choice of box labels the *content* of the tableau.

Definition 2.6. Given a tuple $\mu = (\mu_1, \mu_2, \ldots)$, we say that a tableau has **content** μ if it has μ_1 boxes labeled 1, μ_2 boxes labeled 2, and so on.

Remark. Note that the content of a tableau is not always a partition. For example, the tableau

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

has content $(1,2,1)$, which is not a partition.

In addition to semistandard tableaux, we also require the related concept of a *standard* tableau. This is a special case of a semistandard tableau, which we define by imposing additional conditions.

Definition 2.7. A **standard** Young tableau is a semistandard tableau with the additional requirements that the entries also be strictly increasing across rows, and that each integer from 1 to n be used exactly once, where n is the total number of boxes.

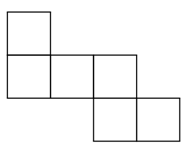
Example 2.8. Here we have a standard Young tableaux of the same shape as used in the previous examples. Each integer $1, 2, \ldots, 8$ occurs once.

7			
2	5	8	
1	3	4	6

Additionally, we will consider filling other shapes besides full partitions. One way we can vary the shape of a filling is by removing the shape of a smaller partition from that of a larger one.

Definition 2.9. Let λ and μ be two partitions such that $\lambda_i \geq \mu_i$ for all i. Then, a **skew tableau** of shape λ/μ is a semistandard filling of the shape formed by superimposing the shapes of λ and μ and deleting the boxes corresponding to the entries of μ.

Example 2.10. Let $\lambda = (4, 3, 1)$ and $\mu = (2)$. Then, the skew shape λ/μ is the shape below on the left, and one possible skew tableau of shape λ/μ is given on the right.



1			
0	2	2	
		1	3

A canonical way of reading off the entries of a tableau is known as the *reading word.*

Definition 2.11. The **reading word** of a tableau is formed by concatenating the entries of each row from top to bottom.

Thus, we read off the entries of a tableau in the same way as one would the words on a page of a book.

Example 2.12. The reading word of the tableau in Example 2.8 is 72581346. The reading word of the skew tableaux in Example 2.10 is 102213.

A particular type of tableau that will need when we work with intersections of Schubert varieties is that of *Littlewood-Richardson tableaux.* To define them, we first need to define what it means for a reading word to be *Yamanouchi.*

Definition 2.13. A word (or sequence) $w = w_1 w_2 \dots w_{n-1} w_n$ (whose entries are positive integers) is **Yamanouchi** if every subword of the form $w_k w_{k+1} \dots w_{n-1} w_n$ has the property that there are at least as many instances of i as there are of $i+1$ for every $i \in \mathbb{N}$.

Example 2.14. The word 2123121 is Yamanouchi, while the word 1223121 is not. This is because the subword 223121 contains three 2s, but only two 1s.

Combining these last two definitions, we can now define a Littlewood-Richardson tableau.

Definition 2.15. A (skew) semistandard tableau is a **Littlewood-Richardson tableau** if its reading word is Yamanouchi.

Remark. Regarding the Yamanouchi condition, in particular consider the subword that is the entire reading word itself. The condition that there be at least as many total is as total $(i+1)$s means that $\mu_i \geq \mu_{i+1}$, where μ is the content of the tableau. Thus, all Littlewood-Richardson tableaux have partitions as content.

Example 2.16. Consider the following two skew tableaux of shape $(4,3,3,2,1,1)/(3,3,2)$.

$$\begin{array}{|c|c|c|c|} \hline 3 & & & \\ 2 & & & \\ 1 & 2 & & \\ & & 1 & \\ & & & \\ & & & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 3 & & & \\ 2 & & & \\ 1 & 1 & & \\ & & 2 & \\ & & & \\ & & & 1 \\ \hline \end{array}$$

They have reading words 321212 and 321121, respectively. The second is Yamanouchi, while the first is not. So, only the second one is a Littlewood-Richardson tableau.

2.2 The RSK correspondence

The RSK correspondence is a famous bijection in combinatorics. It is named for the mathematicians Robinson, Schensted, and Knuth. There are several different variations of the correspondence. The most general is between two-line arrays and pairs of semistandard Young tableaux. Here, we will use use a special case in which one of the two tableaux is standard, and the pairs are in bijection with general sequences of length n. We state this version below, without proof. (See [3] or [10, Ch. 7] for a proof of the correspondence.)

Proposition 2.17 (The RSK correspondence for words)**.** *Let $A(n)$ be the set of all sequences of length n and $B(n)$ the set of all pairs of tableaux (P,Q), such that P and Q have the same shape of size n, P is semistandard, and Q is standard. Then, there exists a constructive bijection between $A(n)$ and $B(n)$.*

If we place some restriction on the content of the sequence in $A(n)$, that imposes the same restriction on the content of P. This observation gives us the following corollary:

Corollary 2.18. *Let $A(r,n)$ be the set of all sequences of length n from the alphabet $\{0,1,\dots,r\}$. Let $B(r,n)$ the set of all pairs of tableaux (P,Q), such that P and Q have the same shape of size n, P is semistandard with content from the set $\{0,1,\dots,r\}$, and Q is standard. Then, there exists a constructive bijection between $A(r,n)$ and $B(r,n)$.*

We now describe and illustrate the RSK correspondence.

Algorithm 2.19. Let $w = w_1 w_2 \dots w_n$ be an arbitrary sequence of length n. Set P and Q to each be the empty tableau. Then, for each letter w_i in order, we do the following:

1. Set $r := 1$, and attempt to insert w_i into row r of P.

2. To insert w_i into row r, check if w_i is at least as large as the largest entry of row r, or that row r is empty. If so, add a new box to the end of row r labeled w_i, and go to step 4.

3. Otherwise, find the leftmost entry b in row r strictly larger than w_i, and 'bump' b up one row. That is, replace the label b with w_i, and then repeat the insertion process of step 2 with the entry b and row $r+1$.

4. Add a new box to Q in the same position as that of the new box that was added to P, and label it i.

At the end, this will result in two tableaux P and Q with the same shape, and P and Q will be semistandard and standard, respectively.

Suppose we start with the sequence $0, 2, 1, 1, 0, 3, 0, 0, 1$. Then, since P is currently empty, we can freely add the first 0 to the first row of P, also creating a box in Q with a 1.

$$\begin{array}{|c|}\hline 0 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

The next entry of the sequence is a 2, so since $2 > 0$, we add a box to the right of the 0 and fill it with 2. Since this is the second entry, we do the same in Q.

$$\begin{array}{|c|c|}\hline 0 & 2 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

Next, we try to insert a 1 into the first row of P, but since 1 is larger than 2, we instead 'bump' out the 2. That is, we replace it with 1, and add the 2 to the first column of the second row. In Q, that same box is labeled with a 3.

$$\begin{array}{|c|c|}\cline{1-1} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c}{} \\ \hline 0 & 1 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|}\cline{1-1} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array}$$

The fourth entry is a 1, and even though this is not larger than the last entry in the first row of P, repeated entries are fine, so we add on a second 1 to the end of the row.

$$\begin{array}{|c|c|c|}\cline{1-1} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 0 & 1 & 1 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|c|}\cline{1-1} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline\end{array}$$

Then, we have another 0, so we bump the first 1 out of the first row. Since this 1 is smaller than the 2 in the second row, it in turn bumps the 2 up into the third row.

$$\begin{array}{|c|c|c|}\cline{1-1} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 0 & 0 & 1 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|c|}\cline{1-1} \multicolumn{1}{|c|}{5} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline\end{array}$$

We continue on in the same manner described above:

$$\begin{array}{llll} 2 & & & \\ 1 & & & \\ 0 & 0 & 1 & 3 \end{array} \qquad \begin{array}{llll} 5 & & & \\ 3 & & & \\ 1 & 2 & 4 & 6 \end{array}$$

$$\begin{array}{llll} 2 & & & \\ 1 & 1 & & \\ 0 & 0 & 0 & 3 \end{array} \qquad \begin{array}{llll} 5 & & & \\ 3 & 7 & & \\ 1 & 2 & 4 & 6 \end{array}$$

$$\begin{array}{llll} 2 & & & \\ 1 & 1 & 3 & \\ 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{llll} 5 & & & \\ 3 & 7 & 8 & \\ 1 & 2 & 4 & 6 \end{array}$$

$$\begin{array}{lllll} 2 & & & & \\ 1 & 1 & 3 & & \\ 0 & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{lllll} 5 & & & & \\ 3 & 7 & 8 & & \\ 1 & 2 & 4 & 6 & 9 \end{array}$$

Thus, we are left with the above pair (P, Q), which do indeed have the same shape, with P being semistandard and Q being standard.

Remark. It is well known (see [3, Ch. 3]) that in the RSK bijection, the number of rows corresponds to the length of the longest increasing subsequence, and the number of columns to the length of the longest nondecreasing subsequence. We will use this fact later.

2.3 The Grassmannian and projective space

Definition 2.20. The **(complex) Grassmannian**, denoted $\mathrm{Gr}(k,n)$ is the set of all k-dimensional subspaces of $\mathbb{C}^n$.

One example of a Grassmannian is projective space.

Definition 2.21. The n-dimensional (complex) **projective space** $\mathbb{P}^n$ is the quotient of the space $\mathbb{C}^{n+1} \setminus \{(0,0,\ldots,0)\}$ by the equivalence relation $(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$, for any $\lambda \in \mathbb{C} \setminus \{0\}$.

In other words, we can view projective space as the complex space one dimension higher, where we identify points on the same line through the origin. In this way, we can view $\mathbb{P}^n$ as the Grassmannian $\mathrm{Gr}(1, n+1)$. Since we do not care about scaling, we can also view projective space as the set of all ratios on $n+1$ coordinates, and will sometimes denote elements accordingly by $(x_0 : x_1 : \cdots : x_n)$.

Example 2.22. Consider the space $\mathbb{P}^2 \cong \mathrm{Gr}(1,3)$. We can express each element in exactly one of the following three forms: $(1 : a : b)$, $(0 : 1 : c)$, and $(0 : 0 : 1)$, where $a, b, c \in \mathbb{C}$. As seen here, it is common practice to express elements of projective space in the form where the first non-zero entry is 1.

We can in turn view the Grassmannian in the context of projective space, but to do so we need to introduce one more concept.

Definition 2.23. A **variety** is the set of all common zeros to some collection of polynomials, usually denoted $\mathbb{V}(f_1, f_2, \ldots, f_r)$. In this work, we are interested only in **projective varieties**. That is, for given homogeneous polynomials $f_1, f_2, \ldots, f_r \in \mathbb{P}[x_1, x_2, \ldots, x_n]$,

$$\mathbb{V}(f_1, f_2, \ldots, f_r) = \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n \mid f_i(a_0 : a_1 : \cdots : a_n) = 0 \,\forall\, i\}.$$

A particular element of the Grassmannian $\mathrm{Gr}(k, n)$ is a k-dimensional subspace of $\mathbb{C}^n$, so we can think of it in terms of being the span of k vectors of length n. We can arrange these into a $k \times n$ matrix. This allows us to view the Grassmannian as a projective variety via what is known as the **Plücker embedding**.

Given an element of $\mathrm{Gr}(k, n)$, we can compute the determinants of all $\binom{n}{k}$ of the $k \times k$ square submatricies. Given a particular k-element subset of $[n]$ $\{i_1, i_2, \ldots i_k\}$, the determinant of the matrix given by choosing columns $i_1, i_2, \ldots, i_k$ we call $x_{i_1, i_2, \ldots, i_k}$. Then, for a fixed ordering of these k-element subsets $K_1, K_2, \ldots, K_m$ (with $m = \binom{n}{k}$), we get the **Plücker coordinate** $(x_{K_1} : x_{K_2} : \cdots : x_{K_m}) \in \mathbb{P}^{m-1}$. Since rescaling any row of our original matrix rescales each determinant equally, this embedding is indeed well-defined.

Example 2.24. Consider the element of $\mathrm{Gr}(2,4)$ given by

$$\begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & 3 & 2 \end{bmatrix}.$$

We then have $x_{12} = 1$, $x_{13} = 3$, $x_{14} = -2$, $x_{23} = -4$, $x_{24} = 2$, and $x_{34} = 2$. Then, this element has Plücker coordinate $(x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34}) = (1 : 3 : -2 : -4 : 2 : 2)$.

2.4 Schubert cells and Schubert varieties

As noted above, each point in the Grassmannian may be represented by a $k \times n$ matrix. However, these representatives need not be unique. So, to ensure uniqueness, we take by convention the matrix that is in reduced row echelon form. Additionally, we are often only concerned with where the pivots are, and will say that two matrices have the same reduced row echelon form when they have the same pivots and only differ in their not-necessarily 0 or 1 entries.

Example 2.25. The matrices

$$\begin{bmatrix} 1 & 5 & 7 & 0 & 6 & 7 \\ 2 & 4 & 8 & 8 & 2 & 3 \\ -1 & 0 & -2 & 0 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 & 7 & 0 & 0 \\ 1 & -1 & 0 & 0 & 4 & 8 \\ 1 & 1 & 2 & 3 & 2 & -1 \end{bmatrix}$$

represent distinct points in $\mathrm{Gr}(3,6)$, but share the same reduced row echelon form

$$\begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix},$$

where $*$ represents an entry that can be any complex number.

Definition 2.26. A **Schubert cell** is the set of all points in the Grassmannian whose matrices have the same row reduced echelon form.

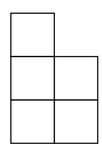
In this way, we can decompose the Grassmannian into the disjoint union of Schubert cells. We wish to classify the different Schubert cells of a particular Grassmannian. To do so, we will relate Schubert cells to partitions.

To determine the partition that indexes a particular Schubert cell, we consider the locations of the pivots. In general, the pivots will occur on the main diagonal, with a staircase of zeros underneath. However, these are the leftmost possible positions, as one or more pivots can occur further to the right. For a given row r, the number of spaces to the right of this expected position that the pivot occurs corresponds to the length of the $(k-r)$th row of the partition.

Example 2.27. The point in $\mathrm{Gr}(4,8)$ represented by the matrix

$$\begin{bmatrix} 1 & * & 0 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

corresponds to the Young diagram below:



Put another way, if we are given a matrix in the above form with entries that are 0, 1, or $*$, we can form the tableau by deleting all the pivot columns and turning each 0 entry into a box.

Our first remark is that this does allow for an empty tableau, in the case where the only 0s are in the bottom left staircase. Secondly, the largest possible tableau is a $k \times (n-k)$ box, which is known as the **ambient rectangle**, often called B. Then, it can be shown that a tableau corresponds to a nonempty Schubert cell if and only if it fits inside of the ambient rectangle.

We now revisit our definition of Schubert cells:

Definition 2.28. For a partition λ with at most k parts and $\lambda_i \leq n-k$ for each i, the **Schubert cell** Ω°_λ is the set of all points in the Grassmannian whose reduced row echelon matrices have forms corresponding to λ.

We can generalize this idea back to the concept of varieties:

Definition 2.29. The **Schubert variety** Ω_λ is the set

$$\Omega_\lambda = \{V \in \mathrm{Gr}(k,n) \mid \dim(V \cap \langle e_1, e_2, \dots, e_{n-k+i-\lambda_i}\rangle) \geq i\},$$

where the e_i are the standard unit vectors.

In other words, the Schubert variety Ω_λ is the disjoint union of its corresponding Schubert cell with all the Schubert cells whose tableaux fit inside of λ. In this way, the Schubert variety can be seen as the *closure* of its Schubert cell. That is, $\Omega_\lambda = \overline{\Omega^\circ_\lambda}$.

Finally, before we start relating these constructions, we need one last definition: that of flags.

Definition 2.30. A **(complete) flag** is a chain of subspaces

$$F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n,$$

where each space F_i has dimension i.

This allows us to define Schubert varieties relative to any basis of $\mathbb{C}^n$, not just the standard basis. So, we can generalize the definition of a Schubert variety to be

$$\Omega_\lambda(F_\bullet) = \{V \in \mathrm{Gr}(k,n) \mid \dim(V \cap F_{n-k+i-\lambda_i}) \geq i\},$$

for a given flag $F_\bullet$.

2.5 The Littlewood-Richardson rule for intersecting Schubert varieties

Recall our earlier discussion on intersection problems, and how we are in particular interested in when an intersection problem has an answer that is both positive and finite. In this case, the answer can be related to the intersections of Schubert varieties.

Example 2.31. This example is problem 3.8(a) from [4].

In $\mathbb{P}^4$, let 2-planes A and B intersect in a point X, and let P and Q be distinct points different from X. Let S be the set of all 2-planes C that contain both P and Q and intersect A and B each in a line. Express S as an intersection of Schubert varieties in $\mathrm{Gr}(3,5)$, when P is contained in A and Q is contained in B.

First, we let $F_\bullet^{(1)}$ and $F_\bullet^{(2)}$ be sufficiently general flags. Then, we will relate $F_\bullet^{(1)}$ to A and P, and $F_\bullet^{(2)}$ to B and Q. When we projectivize, P becomes 1-dimensional and A becomes 3-dimensional. So, we are looking for points $V \in \mathrm{Gr}(3,5)$ that intersect P in at least 1 dimension and A in at least 2. In other words, $V \cap F_3^{(1)} \geq 2$ and $V \cap F_1^{(1)} \geq 1$. The first equation gives us $i = 2$ and $5-3+2-\lambda_2 = 3$, so $\lambda_2 = 1$. The second equation gives us $i = 1$ and $5-3+1-\lambda_1 = 1$, so $\lambda_1 = 2$. Thus, the variety corresponding to A and P is $\Omega_{(2,1)}(F_\bullet^{(1)})$. Similarly, for B and Q we get $\Omega_{(2,1)}(F_\bullet^{(2)})$. To find S, we take the intersection of these two, $\Omega_{(2,1)}(F_\bullet^{(1)}) \cap \Omega_{(2,1)}(F_\bullet^{(2)})$.

For a given property to be true for "sufficiently general flags", it means that there must be a dense open subset of the flag variety for which the given property holds. In the 2-dimensional case above, it is sufficient to choose *transverse* flags. That is, choose $F_\bullet^{(1)}$ and $F_\bullet^{(2)}$ such that $F_i^{(1)}$ and $F_{n-i}^{(2)}$ intersect trivially for all i. We refer the reader to [4] for more details. In this work, we will always assume that our choice of flags is general enough to ensure that these properties always hold.

Now we have seen how these intersection questions relate to Schubert varieties, but when should we expect the answer to be interesting? That is, how can we tell from the varieties when the answer should be finite and positive? Or, put yet another way, when is an intersection of Schubert varieties of dimension zero? An intersection

$$\Omega_{\lambda^{(1)}}(F_\bullet^{(1)}) \cap \Omega_{\lambda^{(2)}}(F_\bullet^{(2)}) \cap \cdots \cap \Omega_{\lambda^{(r)}}(F_\bullet^{(r)})$$

is zero-dimensional when $\sum_{i=1}^{r} |\lambda^{(i)}| = k(n-k)$. Recall that $|B| = k(n-k)$, where B is our ambient rectangle.

We will now digress briefly into the world of algebraic topology. For more details, we refer the reader to [7].

First, we need the construction of a cell complex.

Definition 2.32. A **cell complex** is a topological space X constructed as follows:

1. Start with a set of points X^0, called the **0-skeleton**.

2. Given the $(n-1)$-skeleton X^{n-1}, we 'glue' to it some number of n-cells. That is, we take the disjoint union $X^{n-1} \bigsqcup_\alpha D^n_\alpha$, where D^n_α is some collection of n-cells, and then identify the boundary of each n-cell with X^{n-1}. The result is the ***n*-skeleton** X^n.

3. We either terminate after a finite number of iterations, in which case the cell complex is just $X := X^n$ for some n, or we continue infinitely, in which case $X := \bigcup\limits_n X^n$.

What we will want to do is to build a cell-complex representation of the Grassmannian, and then take its cohomology ring $H^*(\mathrm{Gr}(k,n)) = \bigoplus H^i(\mathrm{Gr}(k,n))$ as the setting where we compute the intersections.

Remark. When constructing a Grassmannian space, we only make use of even-dimension real cells (since $\mathbb{C}$ can be thought of as $\mathbb{R}^2$), so the odd cohomology groups are all 0, and the even cohomology groups are solely determined by the number of partitions λ of each size that fit inside the ambient rectangle B.

The way this construction works is that we first take X^0 to be Ω°_B. Then, we make X^2 by attaching the Schubert cell corresponding to removing 1 box from B, X^4 by attaching the cells corresponding to both ways of removing 2 boxes, and so on. The final skeleton $X^{2k(n-k)}$ is formed by attaching $\Omega^\circ_\varnothing$.

Then, the cohomology ring $H^*(\mathrm{Gr}(k,n))$, also sometimes called the *Chow ring*, will have as a basis the elements $\sigma_\lambda \in H^{2|\lambda|}(\mathrm{Gr}(k,n))$, where the σ_λ are indexed by λ that fit inside B.

The intersection of Schubert varieties $\Omega_{\lambda^{(1)}}(F^{(1)}_\bullet) \cap \Omega_{\lambda^{(2)}}(F^{(2)}_\bullet) \cap \cdots \cap \Omega_{\lambda^{(r)}}(F^{(r)}_\bullet)$ is equivalent to taking the product $\sigma_{\lambda^{(1)}} \cdot \sigma_{\lambda^{(2)}} \cdot \cdots \cdot \sigma_{\lambda^{(r)}}$ in the cohomology ring. In the desired case where

$\sum_{i=1}^{r} |\lambda_i| = n(n-k)$, this product must live inside $H^{2k(n-k)}$, so it is some multiple of the generator σ_B. In other words,

$$\sigma_{\lambda^{(1)}} \cdot \sigma_{\lambda^{(2)}} \cdot \dots \cdot \sigma_{\lambda^{(r)}} = c^B_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}} \sigma_B,$$

for some coefficient $c^B_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}}$.

This coefficient $c^B_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}}$ is called a **Littlewood-Richardson coefficient**, and can be interpreted purely combinatorially.

Definition 2.33. A **chain** of skew tableaux is a sequence of skew tableaux $T_1, T_2, \dots, T_n$, where each pair T_i and T_j are disjoint, and $T_1 \cup T_2 \cup \dots \cup T_i$ is a partition shape for all i.

Proposition 2.34 (Littlewood-Richardson Rule). *Let $c^\nu_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}}$ denote the number of chains of Littlewood-Richardson tableaux with contents $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}$ and total shape ν. Then,*

$$\sigma_{\lambda^{(1)}} \cdot \sigma_{\lambda^{(2)}} \cdot \dots \cdot \sigma_{\lambda^{(r)}} = \sum_{\nu} c^\nu_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}} \sigma_\nu,$$

where ν varies over all tableaux that fit inside of the ambient rectangle.

Recall that we are interested in the case where $\sum |\lambda_i| = n(n-k)$. So, in this case we can only have $\nu = B$, which gives the following corollary.

Corollary 2.35. *Let $c^B_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}}$ denote the number of chains of Littlewood-Richardson tableaux with contents $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}$ and total shape B. Then,*

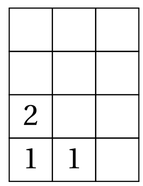
$$\sigma_{\lambda^{(1)}} \cdot \sigma_{\lambda^{(2)}} \cdot \dots \cdot \sigma_{\lambda^{(r)}} = c^B_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}} \sigma_B.$$

This aligns with our observation from earlier.

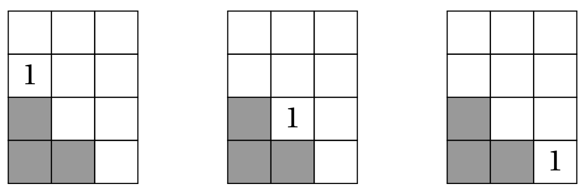
Example 2.36. Let $k = 4$ and $n = 7$. If $\lambda^{(1)} = (2,1)$, $\lambda^{(2)} = (1)$, $\lambda^{(3)} = (2,2)$, and $\lambda^{(4)} = (2,1,1)$, then there are three chains of Littlewood-Richardson tableaux with contents $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}$ that fill $B = (3,3,3,3)$. That is, $c^B_{\lambda^{(1)},\lambda^{(2)},\lambda^{(3)},\lambda^{(r)}} = 3$, so $\sigma_{(2,1)}\sigma_{(1)}\sigma_{(2,2)}\sigma_{(2,1,1)} = 3\sigma_B$.

To show this, we explicitly calculate the number of chains with contents $(2,1)$, (1), $(2,2)$, and $(2,1,1)$.

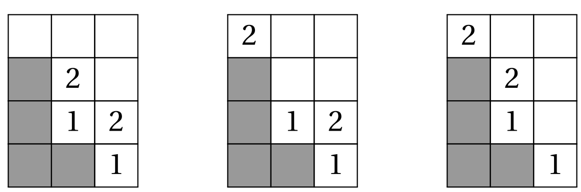
First, there are two semistandard tableau with content $(2,1)$. However, $\boxed{1}\boxed{1}\boxed{2}$ is not Yamanouchi, so so we fill the bottom of the ambient rectangle with the only valid tableau of content $(2,1)$:



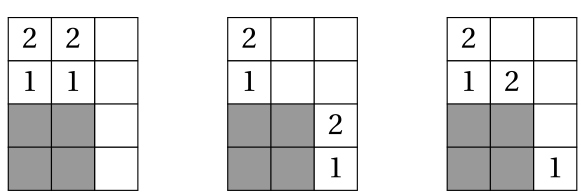
Next, there are three places to place a single box such that the shape is still a chain of skew tableaux:



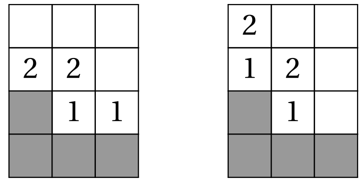
Then, we must fill in two 1s and two 2s. In the first case, there are three Yamanouchi ways to do this:



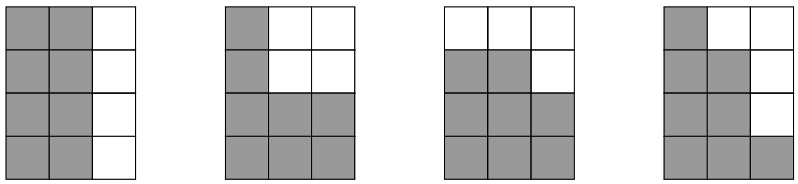
In the second case, there are also three Yamanouchi fillings with shape $(2,2)$:



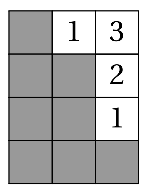
In the third case, there are only two Yamanouchi fillings:



Finally, for the tableaux in the chain, we can distill the above eight cases into four shapes:



Respectively, we have one instance of the first case above, two of the second and third, and three of the fourth. However, we cannot fill the first case with content $(2,1,1)$ at all, since we have two 1s. Additionally, the middle two cases can only be filled by $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 1 \\ \hline \end{array}$ and $\begin{array}{ccc} \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline & & \multicolumn{1}{|c|}{1} \\ \cline{3-3} \end{array}$, respectively, neither of which is Yamanouchi. Thus, all valid fillings are of the fourth case, which is filled by:



Thus, there are 3 total valid chains, and $c^{(3,3,3,3)}_{(2,1),(1),(2,2),(2,1,1)} = 3$.

We now introduce integral notation for 0-dimensional products of Schubert classes.

Definition 2.37. Let $\sum_{i=1}^{r} |\lambda^{(i)}| = k(n-k)$. Then, we define

$$\int_{\mathrm{Gr}(k,n)} \sigma_{\lambda^{(1)}} \sigma_{\lambda^{(2)}} \cdots \sigma_{\lambda^{(r)}} = C,$$

where C is the coefficient $c^{B}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(r)}}$ from Corollary 2.35.

Intuitively, what this tells us is that the intersection of the varieties $\Omega_{\lambda^{(i)}}(F_{\bullet}^{(i)})$ will have C distinct points. This allows us to discuss intersections of Schubert varieties in terms of generators of the cohomology ring, without having to make any particular choice of flags.

Example 2.38. For example, in the above problem, rather than saying that the intersection

$$\Omega_{2,1}(F_\bullet^{(1)}) \cap \Omega_1(F_\bullet^{(2)}) \cap \Omega_{2,2}(F_\bullet^{(3)}) \cap \Omega_{2,1,1}(F_\bullet^{(4)})$$

has three points in $\mathrm{Gr}(4,7)$, we can just write

$$\int_{\mathrm{Gr}(4,7)} \sigma_{2,1} \cdot \sigma_1 \cdot \sigma_{2,2} \cdot \sigma_{2,1,1} = 3.$$

Chapter 3

L-tableaux and enumeration by $(r+1)^g$

We now have the tools to show that the L-tableaux with parameters (g,r,d) do indeed enumerate the integrals $L_{g,r,d}$ in the Grassmannian, starting from equation (1.1). We will then show that the L-tableaux are enumerated by $(r+1)^g$.

3.1 The L-tableaux

Proposition 2.34, combined with the fact that the integral in equation (1.1) is the coefficient of $s_{(d-r)^{r+1}}$ in the corresponding product (1.2), shows that

$$L_{g,r,d} = \sum_{\alpha_0+\cdots+\alpha_r+rg=(r+1)(d-r)} c^{(d-r)^{r+1}}_{(1^r),(1^r),\ldots,(1^r),(\alpha_0),\ldots,(\alpha_r)}$$

where the subscripts on the coefficient contain g copies of (1^r). This summation is therefore the number of ways to form a transposed semistandard Young tableau using each of the numbers $1,2,\ldots,g$ exactly r times, and then extend it to fill the rest of the $(r+1)\times(d-r)$ grid with a semistandard Young tableau using the numbers $0,1,\ldots,r$ in some varying amounts $\alpha_0,\ldots,\alpha_r$ each.

We restate this in terms of our new notation here.

Definition 3.1. An ***L*-tableau with parameters (g,r,d)** is a way of filling an $(r+1)\times(d-r)$ rectangular grid with:

- (The 'red' tableau.) A transposed semistandard Young tableau having exactly r copies of each of the numbers $1,2,\ldots,g$. That is, its content is $(r^g)=(r,r,r,\ldots,r)$.
- (The 'blue' tableau.) A semistandard Young tableau on the remaining skew shape of boxes, with values from $\{0,1,\ldots,r\}$.

Example 3.2. The following is an L-tableau with parameters $(4,3,9)$. We write the "red" numbers as black font with a red shaded background for clarity.

2	4	1	3	3	3
1	3	4	1	2	2
1	2	3	0	1	1
1	2	3	4	0	0

Our discussion thus far, starting from Equation (1.1), has shown:

Proposition 3.3. *The number of L-tableau with parameters (g,r,d) is equal to $L_{g,r,d}$ whenever either $d \geq rg+r$, $r=1$, or $d = r + \frac{rg}{r+1}$, .*

We now show that we can "truncate" by removing some of the right-hand columns of the grid to reduce to a simpler case.

Lemma 3.4 (Truncation)**.** *For any g,r,d with $d \geq g+r$, the number of L-tableaux with parameters (g,r,d) is equal to the number of L-tableaux with parameters $(g,r,g+r)$.*

Proof. Suppose $d \geq g+r$. Notice that, since it is transposed semistandard, the red tableau has width at most g, since its bottom row is strictly increasing from left to right and uses only the numbers $1,2,\ldots,g$. Therefore, any column to the right of the g-th column is filled entirely with blue numbers, which strictly increase up the columns using the numbers $0,1,2,\ldots,r$, necessarily exactly once since the columns have height $r+1$.

It follows that there is only one way to fill each of the columns to the right of column g, and these columns therefore do not contribute to the enumeration. We therefore may remove the last $d-r-g$ columns and find that the number of L-tableaux with parameters (g,r,d) equals the number with parameters $(g,r,g+r)$. □

Lemma 3.4 tells us that in order to understand $L_{g,r,d}$ for $d \geq g+r$, it suffices to study the case $d = g+r$. We will restrict to this case throughout the remainder of this chapter.

Remark. When $d = g+r$, the rectangle containing the L-tableaux is size $(r+1)g = rg+g$. The red tableau has size rg and so the blue tableau has size g.

3.2 Enumeration by $(r+1)^g$

In this section we prove Theorem 1.1. We first define the following sets of tableaux.

Definition 3.5. Let $\mathrm{TrSSYT}(g,r)$ be the set of all transposed semistandard Young tableaux of content $(r^g) = (r, r, \ldots, r)$ and height $\le r+1$.

Note that $\mathrm{TrSSYT}(g,r)$ is the set of all possible 'red' tableaux in Definition 3.1. We will refer to them as **red tableaux** throughout this work.

Definition 3.6. Define a **180°-rotated SYT** to be the result of rotating a standard Young tableaux 180° in the plane. We write $\mathrm{SYT}^{180^\circ}(g,r)$ for the set of all 180°-rotated SYT of size g and height at most $r+1$.

We informally call such a tableau a **purple tableau**, as it will be used as an intermediate object relating the red and blue tableaux of Definition 3.1.

Example 3.7. Below is an example of a purple tableau in $\mathrm{SYT}^{180}(7,3)$.

$$\begin{array}{|c|c|c|}
\hline
7 & 3 & 1 \\
\hline
\multicolumn{1}{c|}{} & 6 & 2 \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 \\
\cline{3-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 5 \\
\cline{3-3}
\end{array}$$

Given a red tableau, note that each number $1, \ldots, g$ occurs once in every row except one. Relatedly, a purple tableau in the position of the blue tableau will have each number $1, \ldots, g$ in exactly one row. This leads us to define a bijection between the two as follows.

Definition 3.8 (Red to purple bijection)**.** Let $R \in \mathrm{TrSSYT}(g,r)$ be a red tableau. We define a 180°-rotated tableau $\varphi(R)$ in the upper right corner of a rectangle by the following iterative process. We add boxes labeled $1, 2, \ldots, g$ in order, where on the ith step we place a box labeled i as far to the right as possible in the unique row that does **not** contain an i in R.

Example 3.9. If R is the tableau at left below, $\varphi(R)$ is shown at right below.

2	4	5	6		
1	3	4	5	7	
1	2	3	5	6	7
1	2	3	4	6	7

7	3	1
	6	2
		4
		5

We now show φ is a bijection. We note that a generalized version of this map was shown to be a bijection in [8] (see also Section 5.1), but we include a direct proof here for the special case that we are considering, for the reader's convenience.

Lemma 3.10. *The map φ is a bijection from* $\mathrm{TrSSYT}(g,r)$ *to* $\mathrm{SYT}^{180^\circ}(g,r)$ *for all g, r. Moreover, for any $R \in$* $\mathrm{TrSSYT}(g,r)$*, the shapes of R and $\varphi(R)$ are complementary in an $(r+1)\times g$ rectangle.*

Proof. We show both statements by induction on g. For $g = 1$, the tableau R must be a column of 1's of height r, and $\varphi(R)$ is a single 1 in the top row. They are clearly complementary in an $(r+1)\times 1$ rectangle (column).

For $g > 1$, assume the statements hold for $g-1$ and let $R \in \mathrm{TrSSYT}(g,r)$. Consider the tableau R' formed by removing the vertical strip of g's from R. Let $T' = \varphi(R')$. Then T' and R' are complementary in an $(r+1)\times(g-1)$ rectangle by the inductive hypothesis.

By shifting T' one unit to the right, we form an empty vertical strip V of size $r+1$ between the two tableaux. Then all r of the g's in R must lie in this strip, and in fact the one remaining square x must be at the top of some column of V. Then, the square x is precisely the one that we label g to form $T = \varphi(R)$ starting from T', by Definition 3.8. Since the entries immediately above and to the right of x will have entries smaller than g (or x is on the right hand or top edge of the rectangle), this construction forms a 180°-rotated SYT T, so φ is well-defined and the resulting pair $(R, \varphi(R))$ is complementary.

Finally, note that by the induction hypothesis, φ is a bijection for $g-1$, and the possible squares we can add to T' to form g are precisely in bijection with the possible sub-strips of g's of the vertical strip V that we may add to R' to form R. Thus φ is a bijection for size g as well. $\square$

We now make precise the notion of a "blue tableau" (see Definition 3.1).

Definition 3.11. Define a **180°-rotated semistandard tableau**, or **blue tableau** (with parameters r, g), to be a filling of a 180°-rotated Young diagram of size g with numbers $0, 1, 2, \ldots, r$ such that the rows are weakly increasing from left to right and columns are strictly increasing from bottom to top.

Example 3.12. Below is an example of a blue tableau. It has the same shape as the purple tableau above in Example 3.9.

$$\begin{array}{|c|c|c|} \hline 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \multicolumn{2}{c|}{} & 1 \\ \cline{3-3} \multicolumn{2}{c|}{} & 0 \\ \cline{3-3} \end{array}$$

Lemma 3.13. *The pairs of blue and purple tableaux of the same shape correspond to $(r+1)$-ary sequences of length g bijectively, via inverting the entries of the blue tableau (that is, replacing each entry i by $r-i$), rotating both 180°, and applying RSK.*

Proof. Given a blue tableau S and purple tableau T of the same shape, rotate both by 180 degrees to form S^{180} and T^{180}. Then, T^{180} is a standard tableau, which we call Q. We also form a semistandard tableau P out of S^{180} by inverting its entries; that is, we replace each i in S^{180} with $r-i$ in P.

Then, (P, Q) is a pair in $B(r, g)$ (see Corollary 2.18), so by the RSK bijection on words, we have a bijection between these pairs (P, Q) and $A(r, g)$, which is precisely the set of $(r+1)$-ary sequences of length g. $\square$

Example 3.14. Consider the pair of blue and purple tableaux below.

$$\begin{array}{|c|c|c|} \hline 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \multicolumn{2}{c|}{} & 1 \\ \cline{3-3} \multicolumn{2}{c|}{} & 0 \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|} \hline 7 & 3 & 1 \\ \hline \multicolumn{1}{c|}{} & 6 & 2 \\ \cline{2-3} \multicolumn{2}{c|}{} & 4 \\ \cline{3-3} \multicolumn{2}{c|}{} & 5 \\ \cline{3-3} \end{array}$$

The corresponding pair (P, Q) is as follows:

3		
2		
1	2	
0	1	3

5		
4		
2	6	
1	3	7

Then, via RSK, this pair corresponds to the $(r+1)$-ary sequence $3,2,2,1,0,1,3$ where $r = 3$.

We now have the tools to produce a bijection between L-tableaux and $(r+1)$-ary sequences.

Proposition 3.15. *The L-tableaux with parameters $(g, r, g+r)$ are in bijection with the $(r+1)$-ary sequences of length g (with letters from the alphabet $\{0,1,2,\ldots,r\}$).*

Proof. Each such L-tableaux consists of a red tableau and a blue tableau. The bijection follows from combining Lemma 3.10 with Lemma 3.13, which provide bijections between red tableaux with purple tableaux, and between pairs of blue and purple tableaux with $(r+1)$-ary sequences of length g, respectively. □

Example 3.16. Below is an L-tableau with parameters $(7,3,10)$. From our previous examples, we see that it corresponds to the $(r+1)$-ary sequence $3,2,2,1,0,1,3$.

2	4	5	6	0	2	3
1	3	4	5	7	1	2
1	2	3	5	6	7	1
1	2	3	4	6	7	0

There are $(r+1)^g$ sequences of length g in the alphabet $0,1,2,\ldots,r$. Combining Proposition 3.15 with truncation (Lemma 3.4), we get as a corollary Theorem 1.1.

Theorem 1.1. The number of L-tableaux with parameters (g, r, d) is $(r+1)^g$ for all $d \geq r+g$.

We now analyze two special cases of our construction.

3.3 The case $r = 1$

We claim that at $r = 1$, the composition of bijections discussed above reduces to ordinary RSK. Indeed, in this case, the red tableau is simply a standard Young tableau on the numbers $1, 2, \ldots, g$ of height 2. The blue tableau consists of 0's and 1's, and when rotated 180 degrees is the same shape as the red tableau (and is semistandard after interchanging 0's and 1's). More precisely, we have the following.

Proposition 3.17. *When $r = 1$, the bijection φ (Definition 3.8) from red to purple tableaux reduces to* $180°$ *rotation.*

We illustrate this with an example. Consider the L-tableau with parameters $(5, 1, 6)$ below:

$$\begin{array}{|c|c|c|c|c|}\hline 3 & 0 & 1 & 1 & 1 \\ \hline 1 & 2 & 4 & 5 & 0 \\ \hline\end{array}$$

The bijection φ applied to the red tableau above gives:

$$\begin{array}{|c|cccc}\hline 3 & \multicolumn{1}{c}{} & & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 5 & 4 & 2 & 1 \\ \hline \multicolumn{3}{c|}{} & 3 \\ \cline{4-4}\end{array}$$

and combining the purple tableau on the right with the blue (after $180°$ rotation and switching 1's and 0's) gives the pair:

$$\begin{array}{|c|ccc}\hline 1 & \multicolumn{1}{c}{} & & \\ \hline 0 & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline\end{array} \qquad \begin{array}{|c|ccc}\hline 3 & \multicolumn{1}{c}{} & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline\end{array}$$

which corresponds under RSK to the binary sequence $0, 1, 0, 0, 1$.

3.4 The Castelnuovo case

In [2], the authors consider another special case, when $d = r + \frac{rg}{r+1}$, and show that their formula

$$L_{g,r,d} = \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^{g} \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i} \right) \right]$$

holds in this case as well. In fact, by construction we have $(r+1)(d-r)-rg=0$ and so the integral above reduces to the simple product $\sigma_{1^r}^g$ in the cohomology ring of the Grassmannian.

Due to work of Castelnuovo [1] and Griffiths and Harris [6], it is known that this quantity equals

$$g! \cdot \frac{1! \cdot 2! \cdots r!}{s! \cdot (s+1)! \cdot \cdots \cdot (s+r)!}$$

where $s = \frac{g}{r+1}$ (which must be an integer since $\frac{rg}{r+1} = d-r$ is an integer and r and $r+1$ are relatively prime). We observe here how this may be enumerated directly via a variant of L-tableaux, using our 'red to purple' bijection φ of Definition 3.8.

Indeed, the integral $\int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g$ is the coefficient of $s_{((d-r)^{r+1})}$ in the product $s_{(1^r)}^g$, which by Proposition 2.34 is the number of transposed semistandard Young tableaux having shape a $(r+1)\times(d-r)$ rectangle and exactly r of each letter $1,2,\ldots,g$. Note that $d-r=\frac{rg}{r+1}$ so the entire rectangle has

$$(r+1)\cdot\frac{rg}{r+1} = rg$$

boxes, and therefore it is completely filled by such a transposed tableau. In other words, we are counting the number of 'red' tableaux that precisely fill an $(r+1)\times rs$ rectangle where $s=\frac{g}{r+1}$. Note that such tableaux exist if and only if $r+1$ divides g.

Proposition 3.18. *The number of transposed semistandard Young tableaux of $(r+1)\times rs$ rectangle shape (where $s=\frac{g}{r+1}$) and content (r^g) is equal to*

$$g! \cdot \frac{1! \cdot 2! \cdots r!}{s! \cdot (s+1)! \cdot \cdots \cdot (s+r)!}. \tag{3.1}$$

Proof. These tableaux, under the bijection φ, correspond to the standard Young tableaux of rectangle shape $(r+1)\times s$. The classical 'hook length formula' (see [10, Ch. 7]) then results in the formula (3.1). $\square$

Example 3.19. Suppose $r=4$ and $g=10$, so $d=r+\frac{rg}{r+1}=12$ and $s=\frac{g}{r+1}=2$. Then one of the rectangular transposed tableaux enumerating $L_{g,r,d}$ is shown in red on the left below. Its image under φ is a 180°-rotated SYT filling an $(r+1) \times s$ box as shown in purple on the right below.

2	3	5	6	7	8	9	10
1	3	4	5	7	8	9	10
1	2	4	5	6	7	9	10
1	2	3	4	6	7	8	10
1	2	3	4	5	6	8	9

4	1
6	2
8	3
9	5
10	7

Chapter 4

L'-tableaux and enumeration by 2^g

We give a tableau interpretation of $L'_{g,d,k}$ in this chapter, starting from equation (1.3), and show that these tableaux are enumerated by 2^g to prove Theorem 1.2.

4.1 The L'-tableaux

Recall that equation (1.3) states that if $k+g \le 2d+1$ and $2 \le k \le d$:

$$L'_{g,d,k} = \int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \sigma_{k-1} \left(\sum_{i+j=2d-g-k-1} \sigma_i \sigma_j \right) - \int_{\mathrm{Gr}(2,d)} \sigma_1^g \sigma_{k-2} \left(\sum_{i+j=2d-g-k-2} \sigma_i \sigma_j \right).$$

We first give an interpretation of the left hand integral in the equation above. Recall that a **standard Young tableau** of size g is a semistandard Young tableau of size g in which the numbers $1,2,\ldots,g$ are each used exactly once.

Definition 4.1. A **positive L'-tableau with parameters (g,d,k)** is a way of filling a $2 \times (d-1)$ grid with:

- A standard Young tableau of size g in the lower left corner (shaded red),
- A shading of the $k-1$ rightmost boxes in the top row (gray),
- A skew semistandard Young tableau in two letters $0,1$ on the remaining squares (blue).

By rearranging so that the σ_{k-1} is last in each product, and applying Proposition 2.34, we see that the positive term in equation (1.3) is equal to the number of positive L'-tableaux.

Example 4.2. Here is an example of a positive L'-tableau with parameters $(3,7,4)$.

3	0	1			
1	2	0	0	1	1

The second term in (1.3), which we are subtracting, is similarly given by a set of smaller tableaux that we call *negative* tableaux.

Definition 4.3. A **negative L'-tableau with parameters (g, d, k)** is a filling of a $2 \times (d-2)$ grid with:

- A standard Young tableau of size g in the lower left corner (shaded red),
- A shading of the $k-2$ rightmost boxes in the top row (gray),
- A skew semistandard Young tableau in two letters $0, 1$ on the remaining squares (blue).

Example 4.4. Here is an example of a negative L'-tableau with parameters $(3, 7, 4)$.

3	0	1		
1	2	0	0	1

Notice that there exist positive L'-tableaux if and only if $(k-1)+g \le 2(d-1)$, which is slightly stronger than the given condition $k+g \le 2d+1$. In particular if $k+g = 2d$ or $k+g = 2d+1$ we have $L'_{g,d,k} = 0$, so we restrict our attention to the case that $k+g \le 2d-1$.

4.2 Enumeration by 2^g

We now prove Theorem 1.2.

Definition 4.5. For fixed g, d, k, write L'_+ and L'_- for the set of positive and negative L' tableaux respectively of type (g, d, k). Also write $\psi : L'_- \to L'_+$ for the map that takes a negative tableau T and adds a blue 1 to the end of the bottom row and a gray box to the end of the top row.

Our above analysis shows that

$$L'_{g,d,k} = |L'_+| - |L'_-|,$$

and we analyze this difference combinatorially. The definitions above directly show that ψ is a well-defined injective map, and so

$$L'_{g,d,k} = |L'_+ \setminus \psi(L'_-)|. \tag{4.1}$$

The following proposition characterizes the image $\psi(L'_-)$.

Proposition 4.6. *A positive tableau T is equal to $\psi(S)$ for some negative tableaux S if and only if the bottom row of T contains a blue* 1.

Proof. By the definition of ψ, any tableau in its image has a blue 1 on the bottom right. Conversely, if the bottom row of T contains a blue 1, then by semistandardness of the blue tableau, the bottom-rightmost entry is a blue 1 as well, and removing the last column of T yields a negative tableau S for which $\psi(S) = T$. $\square$

Applying this proposition and equation (4.1), we obtain the following combinatorial interpretation of $L'_{g,d,k}$.

Corollary 4.7. *The quantity $L'_{g,d,k}$ is equal to the number of positive L'-tableaux with parameters (g,d,k) for which the bottom row contains no blue 1 (and hence the only blue numbers in the bottom row are 0's).*

For sufficiently large d, we can simplify this characterization even further.

Theorem 1.2. If $d \geq g+k$, we have $L'_{g,d,k} = 2^g$.

Proof. Suppose $d \geq g+k$. Then $d-1 \geq g+(k-1)$, so the red and gray boxes of any positive L'-tableau with parameters (g,d,k) cannot share a column. In particular, for any positive tableau T that has no blue 1 in the bottom row, there are all blue 0's under the gray squares, and moreover any remaining columns to the right of the red tableau are uniquely determined (having one 0 and one 1) as well. Thus the data determining T is entirely contained in its first g columns, which consists of a red standard tableau Q, and a binary tableau P of the same shape as Q, where P is obtained by rotating the blue numbers in these columns 180° and replacing all 0's with 1's and 1's with 0's.

By the RSK correspondence, these pairs (P,Q) are precisely in bijection with the binary sequences of length g, and so we have that $L'_{g,d,k} = 2^g$ as desired. $\square$

Example 4.8. The tableau below at left is a positive L'-tableau with parameters $(3,7,4)$ that is not the image of a negative one. Since $g = 3$, we restrict our attention to the first three columns (second image below), then consider the associated pair of tableaux of the same shape by rotating the blue tableaux and inverting the labels. Finally, this pair corresponds under RSK to a unique length 3 binary sequence.

$$\begin{array}{|c|c|c|c|c|c|}\hline 3 & 0 & 1 & & & \\ \hline 1 & 2 & 0 & 0 & 0 & 0 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 3 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline\end{array} \longrightarrow \left(\begin{array}{cc} \cline{1-1} \multicolumn{1}{|c|}{3} & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} , \begin{array}{cc} \cline{1-1} \multicolumn{1}{|c|}{1} & \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \right) \longrightarrow 110$$

Chapter 5

Further combinatorial observations

In this chapter we provide two generalizations/observations regarding the combinatorics discussed above. In particular we consider two variations of L-tableaux and explore their properties.

5.1 Generalizing the map φ

We note that the 'red to purple' bijection φ may be generalized to transposed tableaux with i of each entry (for any positive integer $i \le r$) as follows.

Definition 5.1. Let $\mathrm{TrSSYT}(g, r, i)$ be the set of all transposed semistandard Young tableaux of content $(i^g) = (i, i, \ldots, i)$ and height $\le r + 1$.

In particular, setting $i = r$ gives us the red tableaux defined in Definition 3.5, and setting $i = 1$ gives us the set of standard Young tableaux of size g and height $\le r + 1$.

Proposition 5.2. *There is a bijection $\varphi_i : \mathrm{TrSSYT}(g, r, i) \to \mathrm{TrSSYT}(g, r, r + 1 - i)$ for each i, that agrees with the bijection φ of Definition 3.8 at $i = r$ (up to a $180°$ rotation of the output).*

See Figure 5.1 for an example of this map for $i = 3$ and $r = 4$.

In fact, the map φ_i can be realized as a special case of an even more general map studied by Stanley [9]. It was later studied by Reiner and Shimozono [8], who call the map the *box complement* and study it on a generalization of partition diagrams called *%-avoiding shapes.* Proposition 5.2 may be proven using similar methods to our proof of Lemma 3.10, but we simply refer to [8] for an existing proof in the more general setting.

Remark. For $i = r$, the transposed semistandard Young tableaux of height at most $r + 1$ are in bijection with standard Young tableaux, and therefore may be enumerated by the hook length

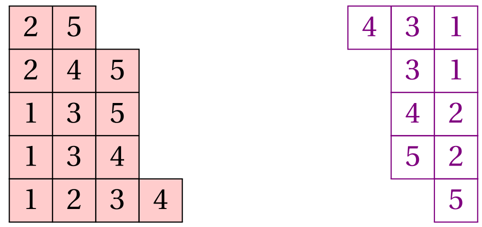


Figure 5.1: A tableau in TrSSYT$(5,4,3)$ and its image under φ_3 in TrSSYT$(5,4,2)$.

formula for any given shape. It would be interesting to investigate whether there is a hook-length-like formula enumerating transposed semistandard Young tableaux with content (i^g) and height at most $r+1$ for $1 < i < r$.

5.2 Restricting the alphabet

We now ask whether our bijective constructions can lead to related interesting enumeration problems. In particular, one natural variant we may consider is limiting the alphabet of the blue tableau (in the L-tableau setting) to a smaller size, so that under RSK we end up with words in a smaller alphabet.

Definition 5.3. Define a **restricted $\boldsymbol{L}$-tableau with parameters $\boldsymbol{(g,r,g+r,i)}$** to be an L-tableau of paramters $(g,r,g+r)$ where we restrict the alphabet of the blue integers to $\{0,1,\ldots,r-i\}$.

Note that the parameter $g+r$ is redundant, and we simply include it for consistency with the parameter d in our previous notation. For larger d there would be no restricted L-tableaux, because a full column of height $r+1$ cannot be filled by blue integers from $\{0,1,\ldots,r-i\}$ in a semistandard tableau.

It turns out that this restricted setting simply reduces to a smaller case of our usual L-tableaux.

Proposition 5.4. *The number of restricted L-tableaux with parameters $(g,r,g+r,i)$ is equal to the number of L-tableaux with parameters $(g,r-i,g+r)$.*

Proof. By Lemma 3.4 we may assume we are working with truncated tableaux. Since the blue integers are restricted to the alphabet $\{0,1,\ldots,r-i\}$, the blue tableau has height at most $r-i$.

So, the bottom i rows of the red tableau have width g. These rows must be filled with each integer $1, \ldots, g$. As there is a unique way to do this, the act of removing the bottom i rows of the $(r+1) \times (g)$ grid gives a bijection between the restricted L-tableaux with parameters (g, r, d, i) and the L-tableaux with parameters $(g, r-i, d)$. $\square$

Combining this proposition with Theorem 1.1 yields the following enumerative result.

Corollary 5.5. *There are exactly* $(r-i+1)^g$ *restricted* L*-tableaux with parameters* $(g, r, g+r, i)$.

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