THESIS

DEFINING PERSISTENCE DIAGRAMS FOR COHOMOLOGY OF A COFILTRATION INDEXED OVER A FINITE LATTICE

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ABSTRACT

DEFINING PERSISTENCE DIAGRAMS FOR COHOMOLOGY OF A COFILTRATION INDEXED OVER A FINITE LATTICE

Persistent homology and cohomology are important tools in topological data analysis, allowing us to track how homological features change as we move through a filtration of a space. Original work in the area focused on filtrations indexed over a totally ordered set, but more recent work has been done to generalize persistent homology. In one avenue of generalization, McCleary and Patel prove functoriality and stability of persistent homology of a filtration indexed over any finite lattice. In this thesis, we show a similar result for persistent cohomology of a cofiltration. That is, for P a finite lattice and $F : P \to \nabla K$ a cofiltration, the nth persistence diagram is defined as the Möbius inversion of the nth birth-death function. We show that, much like in the setting of persistent homology of a filtration, this composition is functorial and stable with respect to the edit distance. With a general definition of persistent cohomology, we hope to discover whether duality theorems from 1-parameter persistence generalize to more general lattices.

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Chapter 1

Introduction

Persistent homology and cohomology aim to answer the following question: how do the homological features of a topological space change as we move through a filtration? For example, in topological data analysis, one can turn a discrete set of data points into a simplicial complex by constructing the Vietoris-Rips or Cêch complex [1]. However, this construction relies on the choice of radius. Persistent (co)homology provides us with the ability us to consider various choices of radii and track when (co)cyles appear and disappear.

Original results in persistence theory focused primarily on 1-parameter filtrations [2–4]. Ample research has been done, and continues to be done, to understand persistence and its applications in the 1-parameter setting. However, one other avenue of current research is in generalizing persistence diagrams. For example, one may wish to consider multidimensional filtrations [5–7], or consider filtered chain complexes with coefficients in certain "nice" categories [8].

In this thesis, we are interested in persistent cohomology specifically. Persistent cohomology is of interest in topological data analysis for various reasons: for one, the persistent cohomology algorithm is faster than the persistent homology algorithm [4, 9]. Further, cohomology allows one to find circular coordinates for data when applicable [9]. Finally, the ring structure on cohomology allows for one to define richer invariants, such as the persistent cup-length [10]. These uses for cohomology inspired us to apply generalizations of persistent homology to persistent cohomology.

This thesis will be organized as follows: in Chapter 2, we introduce important preliminary information. Specifically, we will introduce basic ideas from category theory, lattice theory, and homological algebra, all of which will help us discuss previous work done in persistence theory and generalized persistence diagrams. In Chapter 3, we will define the category of cofiltrations. The definitions from Chapter 3 will allow us to define the persistence diagram arising from cohomology of a cofiltration in Chapter 4 and prove that it is functorial and, in Chapter 5, stable with respect to the edit distance. Finally, in Chapter 6, we will outline potential future directions for research.

Chapter 2

Preliminaries

Generalized persistent homology and cohomology utilize tools from various fields of math, including category theory, combinatorics, and algebraic topology. We will include brief introductions to these fields and the definitions and propositions utilized in our formulation of persistent cohomology of a cofiltration indexed over a finite lattice.

2.1 Category Theory

We begin by introducing basic tools and terminology from category theory. Category theory, initially developed by Eilenberg and Mac Lane, provides a coherent language and theoretical framework with which to describe various different areas of mathematics. This is particularly helpful when studying algebraic topology, as we will wish to translate between topological spaces and maps to algebraic structures and homomorphisms.

Definition 2.1.1. (Definition 1.1.1 in [11]) A **category** $\mathcal{C}$ consists of (1) a collection of objects $\mathcal{C}_0$, and (2) a collection of morphisms $\mathcal{C}_1$ with domain and codomain objects in $\mathcal{C}$. For $f \in \mathcal{C}_1$, we write $f : X \to Y$, where X is the domain and Y is the codomain of f. These collections are subject to the following conditions: for $f : X \to Y$ and $g : Y \to Z$, there exists a composite $gf : X \to Y \to Z$. This composition must be associative: meaning, if we can compose the triple f, g, h, then $h(gf) = (hg)f$. Finally, for every $X \in \mathcal{C}_0$, there is an identity morphism $1_X : X \to X$ such that, for any morphism $f : X \to Y$, $f = f1_X = 1_Y f$.

Examples of categories pertinent to this thesis are as follows:

Example 2.1.1. *Any **poset** P is a category, where the objects are poset elements and there is a morphism $a \to b$ if $a \leq b$ in P. We will discuss this example more in Section 2.2.*

Example 2.1.2. *The **category of $\mathbb{F}$-vector spaces**, denoted $\mathrm{Vect}_{\mathbb{F}}$, has as objects $\mathbb{F}$-vector spaces and morphisms linear maps.*

Example 2.1.3. *The* ***catgeory of topological spaces****, denoted* Top*, has as objects topological spaces and morphisms contiuous maps.*

Example 2.1.4. *The* ***catgeory of simplicial complexes****, denoted* Simp*, has as objects simplicial complexes and morphisms simplicial maps. See section 2.3 for more details on these objects and morphisms.*

Example 2.1.5. *The* ***category of groups****, denoted* Grp*, has as objects groups and morphisms group homomorphisms. The* ***category of ableian groups*** *is similar: it is denoted* Ab *and has as objects abelian groups ad morphisms group homomorphisms.*

For any category, we may "reverse the arrows" and get another category. This notion is formalized in the definition below, and is the philosophy behind *duality*.

Definition 2.1.2. (Definition 1.2.1 in [11]) For any catgeory $\mathcal{C}$, there exists an **opposite category** $\mathcal{C}^{\mathrm{op}}$. The objects of $\mathcal{C}^{\mathrm{op}}$ are identical to those of $\mathcal{C}$, but for any morphism $f : X \to Y \in \mathcal{C}_1$, we have a morphism $f^{\mathrm{op}} : Y \to X$.

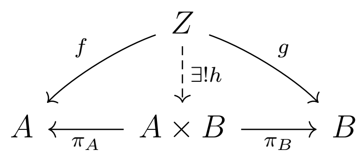
Now, we move on to discussing certain universal constructions in a category $\mathcal{C}$. These constructions are not guaranteed to exist, but often categories with them are "nicer", in a sense.

Definition 2.1.3. (Definitions 3.1.11 and 3.1.23 in [11]) A **initial object** in a category $\mathcal{C}$ is an object I such that, for any other $X \in \mathcal{C}_0$, there exists a unique morphism $I \to X$. A **final object** in a category $\mathcal{C}$ is an object F such that, for any other $X \in \mathcal{C}_0$, there exists a unique morphism $X \to F$. A **terminal object** is either initial or final. If an object $0 \in \mathcal{C}_0$ is both initial and final, we call it a **zero object**.

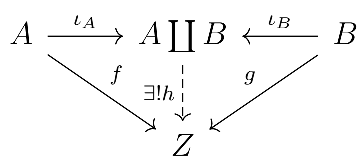
Example 2.1.6. $\mathrm{Vect}_{\mathbb{F}}$ *has a zero object; namely, the 0 vector space.*

Example 2.1.7. *In a bounded lattice (see Section 2.2), the top element is the final object and the bottom element is the initial element.*

Definition 2.1.4. (Definition 3.1.9 in [11]) Given two objects A, B of a category $\mathcal{C}$, their **product** (if it exists) is an object $A \times B \in \mathcal{C}_0$ together with morphisms $A \times B \xrightarrow{\pi_A} A$ and $A \times B \xrightarrow{\pi_B} B$. Further, $A \times B$ must satisfy the **universal property of products**: for any other $Z \in \mathcal{C}_0$ and morphisms $Z \xrightarrow{f} A$, $Z \xrightarrow{g} B$, there exists a unique $Z \xrightarrow{h} A \times B$ such that the following diagram commutes:



Definition 2.1.5. (Definitions 3.1.9 and 3.1.23 in [11]) Given two objects A, B of a category $\mathcal{C}$, their **coproduct** (if it exists) is an object $A \coprod B \in \mathcal{C}_0$ together with morphisms $A \xrightarrow{\iota_A} A \coprod B$ and $B \xrightarrow{\iota_B} A \coprod B$. Further, $A \coprod B$ must satisfy the **universal property of coproducts**: for any other $Z \in \mathcal{C}_0$ and morphisms $A \xrightarrow{f} Z$, $B \xrightarrow{g} Z$, there exists a unique $A \coprod B \xrightarrow{h} Z$ such that the following diagram commutes:



Although we will not prove it here, if the product or coproduct of two objects in $\mathcal{C}$ exists, then it is unique up to isomorphism.

Oftentimes, we want to "move from one category to another". Examples of such constructions are bountiful in mathematics: we move from the category of sets to the category of groups by creating the *free group generated by a set*. We move from the category of topological spaces to the category of groups by constructing the *fundamental group*. The definition below, that of a *functor*, makes this idea mathematically precise:

Definition 2.1.6. (Definition 1.3.1 in [11]) A **functor** is a map between categories, written $F : \mathcal{C} \to \mathcal{D}$ such that $FX \in \mathcal{D}_0$ for each object $X \in \mathcal{C}$ and $Ff : FX \to FY \in \mathcal{D}_1$ for each morphism $f : X \to Y$. For any composition $gf \in \mathcal{C}_1$, $FgFf = F(gf)$, and for any $X \in C_0$, $F1_X = 1_{FX}$.

Functors will play an important role in the definitions and chapters to come.

2.2 Posets and Lattices

Now, we will discuss the combinatorial objects and tools that will help us define generalized persistence diagrams for cohomology of a cofiltration. Further, we will notice that category theoretic language and thinking will be beneficial throughout this (and subsequent) section(s).

Combinatorics plays an important role in persistence theory. Tools from the theories of posets and, for this thesis specifically, lattices will be beneficial for definitions we make and propositions we prove in future chapters. In this section, we will introduce posets in general and discuss certain derivations and examples; namely, the opposite poset, the interval poset, and finite lattices. We also describe maps between posets and the incidence algebra of a finite poset.

First, however, we start with a definition.

Definition 2.2.1. A **partially ordered set**, or a **poset**, is a set P endowed with an order relation $\leq$. For all $a, b, c \in P$, the order relation $\leq$ must satisfy the following:

- (Reflexive Law) $a \leq a$.
- (Antisymmetric Law) If $a \leq b$ and $b \leq a$, then $a = b$.
- (Transitive Property) If $a \leq b$ and $b \leq c$, then $a \leq c$.

We write $(P, \leq)$, or simply P whenever the order relation is clear.

Notice that the three properties above show that P is a category, where the objects are elements of P and there is a morphism $a \to b$ for each relation $a \leq b$. As a category, each poset P has an opposite poset.

Definition 2.2.2. For each poset $(P, \leq)$, there is an **opposite poset** $(P^{\text{op}}, \geq)$. That is, for each $a \leq b$ in P, there is a relation $a \geq b$ in P^{op}.

Viewing a poset as a category, we may ask about the existence of products and coproducts, or about the existence of initial and final objects. First, we introduce terminology for these objects and provide a definition for when a poset P has all (finite) products and coproducts as well as terminal objects.

Definition 2.2.3. Let $a, b \in P$. The **meet** of a and b is the unique maximal element $a \wedge b \in P$ such that $a \wedge b \leq a$ and $a \wedge b \leq b$. That is, if $z \in P$ such that $z \leq a, b$, then $z \leq a \wedge b$.

If we think of a poset as a category, then the meet of two elements (if it exists) is their product.

Definition 2.2.4. Let $a, b \in P$. The **join** of a and b is the unique minimal element $a \vee b \in P$ such that $a \leq a \vee b$ and $b \leq a \vee b$. That is, if $z \in P$ such that $a, b \leq z$, then $a \vee b \leq z$.

If we think of a poset as a category, then the join of two elements (if it exists) is their coproduct.

If all (finite) meets and joins exist, then there is a nice algebraic structure on a poset P with $\vee$ and $\wedge$ as binary operations.

Definition 2.2.5. A **lattice** is a poset with all finite meets and joins. A **bounded lattice** is a lattice that has a top and a bottom element, denoted $\top$ and $\bot$ respectively. Thinking of P as a category, these are the *final* and *initial* objects. A **finite lattice** has only finitely many elements (in fact, any finite lattice is necessarily bounded [6]). We often write $(P, \top, \bot, \vee, \wedge)$, or just P when the terminal objects and operations are obvious or implied.

When dealing with multiple lattices, we sometimes write $\bot_P$, $\top_P$, $\vee_P$, and $\wedge_P$ to clarify that we are talking about the bottom/top elements or operations for P, specifically.

Definition 2.2.6. (Section 3.1 in [6]) A **bounded lattice map** is a map $\alpha : P \to Q$ between bounded lattices P and Q such that for any $a, b \in P$, $\alpha(a \vee_P b) = \alpha(a) \vee_Q \alpha(b)$, $\alpha(a \wedge_P b) = \alpha(a) \wedge_Q \alpha(b)$, $\alpha(\top_P) = \top_Q$, and $\alpha(\bot_P) = \bot_Q$.

That is, a map between bounded lattices must respect the algebraic structure of the lattices.

Proposition 2.2.1. *(Proposition 3.1 in [6]) Let $\alpha : P \to Q$ be a bounded lattice map. Then, for any $q \in Q$, $\alpha^{-1}[\bot, q]$ has a unique maximal element.*

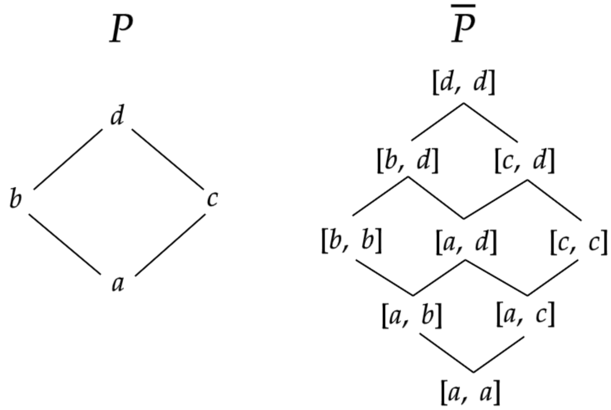


Figure 2.1: A lattice P and its corresponding interval lattice $\overline{P}$.

Proof. Since $\alpha(\bot) = \bot$, $\alpha^{-1}[\bot, q] \neq \emptyset$. Since P is finite, so is $\alpha^{-1}[\bot, q]$. Take $a, b \in \alpha^{-1}[\bot, q]$. Then, $\alpha(a \vee b) = \alpha(a) \vee \alpha(b) \leq q \vee q = q$ and $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b) \leq q \wedge q = q$. By definition, $\alpha(a \vee b), \alpha(a \wedge b) \geq \bot$ as well. Thus, $\alpha^{-1}[\bot, q]$ contains all meets and joins, so it is a lattice. As a finite lattice, it must have a unique maximal element. □

2.2.1 Möbius Inversions

Patel first observed that the persistence diagram is closely related to Rota's combinatorial Möbius inversion [8]. Thus, in persistence theory, lattices are not only useful for indexing a filtration (see Sections 2.4 and 2.5), but the theory of the incidence algebra and Möbius inversions are central to persistence.

For this section, let P be a finite poset.

Definition 2.2.7. For a poset P, denote by $\overline{P}$ its **poset of intervals**. That is, for any $a \leq b$ in P, define the **interval** $[a, b] = \{c \in P \mid a \leq c \leq b\}$. Then, we define $\overline{P} = \{[a, b] \mid a \leq b \in P\}$. The ordering $\leq$ on $\overline{P}$ is given as follows: $[a, b] \leq [c, d] \iff a \leq c$ and $b \leq d$ in P.

If P is a lattice, then so is $\overline{P}$. We define $[a, b] \vee [c, d] = [a \vee c, b \vee d]$, which is unique since joins are unique in P. A similar argument applies for meets. If P is a finite lattice, then so is $\overline{P}$. After all, the bottom element is $[\bot, \bot]$ and the top element is $[\top, \top]$.

Proposition 2.2.2. *Let* $\alpha : P \to Q$ *be a bounded lattice map. Then,* $\bar{\alpha} : \overline{P} \to \overline{Q}$ *is a bounded lattice map, where* $\bar{\alpha}[a, b] = [\alpha(a), \alpha(b)]$.

Proof. It is clear that $\bar{\alpha}$ preserves top and bottom elements. To see that $\bar{\alpha}$ preserves joins,

$$\begin{aligned}\bar{\alpha}\left([a, b] \vee [c, d]\right) = \bar{\alpha}\left([a \vee c, b \vee d]\right) &= [\alpha(a) \vee \alpha(c), \alpha(b) \vee \alpha(d)] \\ &= [\alpha(a), \alpha(b)] \vee [\alpha(c), \alpha(d)] = \bar{\alpha}[a, b] \vee \bar{\alpha}[c, d].\end{aligned}$$

A similar argument applies for meets. □

In this thesis, we are particularly interested in integer-valued functions defined on $\overline{P}$. For $f, g : \overline{P} \to \mathbb{Z}$, we may add $(f + g)$ and multiply by scalars ($z \cdot f$, for $z \in \mathbb{Z}$), and the result is another integer-valued function. We may also define *multiplication* of $f, g : \overline{P} \to \mathbb{Z}$ as *convolution* $f * g$, defined as

$$f * g[a, c] = \sum_{a \leq b \leq c} f[a, b]g[b, c].$$

There is a multiplicative identity, δ, defined as follows: $\delta[x, y] = 0$ if $x \neq y$ and $\delta[x, x] = 1$. After all, notice that $f * \delta[a, c] = \sum_{a \leq b \leq c} f[a, b]g[b, c] = f[a, c]\delta[c, c] = f[a, c]$.

Definition 2.2.8. (Page 344 in [12]) The **incidence algebra** of a finite poset P is the $\mathbb{Z}$-algebra of integer-valued functions $f : \overline{P} \to \mathbb{Z}$, with binary operations addition and convolution.

Example 2.2.1. *(Page 344 in [12]) The* ***zeta function*** $\zeta : \overline{P} \to \mathbb{Z}$ *is defined by* $\zeta[a, b] = 1$ *for all* $[a, b] \in \overline{P}$.

The zeta function, although very simple, is an important element in the incidence algebra. For one, it is *invertible* - see Proposition 1 in [12] for a proof.

Definition 2.2.9. (Proposition 1 in [12]) The inverse of the zeta function, denoted by μ, is the **Möbius function**. It is defined inductively as follows:

$$\mu[x, y] = -\sum_{x \leq z < y} \mu[x, z],$$

where the base case is given by $\mu[x,x] = 1$.

Given an integer-valued map $f : P \to \mathbb{Z}$ defined on a poset P, we may define the combinatorial derivative of f, otherwise known as the Möbius inversion.

Definition 2.2.10. (Proposition 2 in [12]) Let $f : P \to \mathbb{Z}$ be an integer-valued function with domain some finite poset P. Then, the **Möbius inversion** of f is the unique function $\sigma : P \to \mathbb{Z}$ such that, for all $b \in P$,

$$f(b) = \sum_{a \leq b \in P} \sigma(a).$$

Rota proves that, if σ is the Möbius inversion of f, then $\sigma(b) = \sum_{a \leq b} f(a)\mu[a,b]$, connecting the Möbius inversion of f to the Möbius function (Proposition 2 in [12]).

2.2.2 Metric Lattices

Before completing our discussion on lattice theory, we define a *metric lattice*. The definitions and propositions in this section will help us prove stability in Section 5.

Definition 2.2.11. (Section 3.2 in [6]) A **finite (extended) metric lattice** is a pair (P, d_P), where P is a lattice equipped with a metric $d_P : P \times P \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. We simply write P when the metric is clear.

Given a bounded lattice map $\alpha : P \to Q$, we may ask how α *distorts* the distances d_P and d_Q, if P and Q are metric lattices.

Definition 2.2.12. (Section 3.2 in [6]) A **metric lattice map** is a bounded lattice map $\alpha : P \to Q$ where (P, d_P) and (Q, d_Q) are finite metric lattices. The **distortion** of a morphism $\alpha : (P, d_P) \to (Q, d_Q)$ is

$$||\alpha|| = \max_{a,b \in P} \left| d_P(a,b) - d_Q(\alpha(a), \alpha(b)) \right|.$$

Example 2.2.2. *The most basic example of a metric lattice (P, d_P) is any finite lattice P with $d_P(x,y)$ defined as the length of the shortest path between x and y. That is, if we draw the Hasse diagram of P, $d_P(x,y)$ is the minimal number of edges between x and y. See Figure 2.2.*

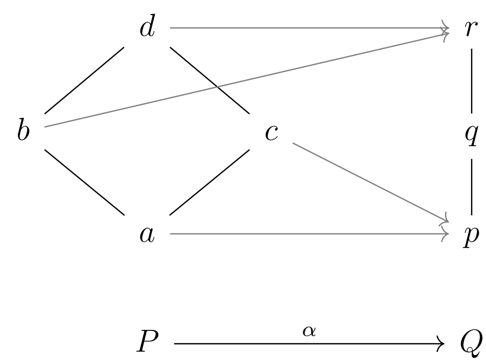


Figure 2.2: Let d_P and d_Q be the length of the shortest path between two lattice elements, as in Example 2.2.2. Then, $\alpha : P \to Q$ is a bounded lattice map with distortion $||\alpha|| = |d_P(a, c) - d_Q(\alpha(a), \alpha(c))| = |1 - 0| = 1$.

Further, if P is a finite metric lattice, then so is its lattice of intervals $\overline{P}$. After all, given d_P, we may define a metric $d_{\overline{P}}$ on $\overline{P}$ as follows:

$$d_{\overline{P}}([a, b], [c, d]) = \max\{d_P(a, c), d_P(b, d)\}.$$

We know from Proposition 2.2.2 that $\alpha : P \to Q$ induces a map $\bar{\alpha} : \overline{P} \to \overline{Q}$ on interval lattices. We may wonder, how does $||\bar{\alpha}||$ compare to $||\alpha||$?

Lemma 2.2.1. *(Proposition 3.4 in [6]) Let $\alpha : P \to Q$ be a metric lattice map, and let $\bar{\alpha} : \overline{P} \to \overline{Q}$ be the induced bounded lattice map on intervals. Then, $||\bar{\alpha}|| = ||\alpha||$.*

See [6] for the proof of Lemma 2.2.1.

2.3 Simplicial (Co)Homology

Now, we begin our discussion on algebraic topology. Specifically, we focus on homology and cohomology, as they are the algebraic invariants of use in this thesis.

In this thesis, we work with a certain type of topological space called a *simplicial complex*. Simplicial complexes are useful for two main reasons: (1) simplicial (co)homology is much easier to compute than singular (co)homology, and (2) simplicial complexes are particularly useful in topological data analysis, where one may work with a Vietoris-Rips complex or another similar construction [1].

In this section, we define a simplicial complex and discuss the basics of simplicial homology and cohomology. For a more in-depth discussion, see [13].

Definition 2.3.1. (Page 103 in [13]) An **n-simplex** is the convex hull of $n + 1$ distinct points $v_0, v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$ in general position. Given an orientation,[1] we denote the *n-simplex* as $\sigma = [v_0, v_1, \ldots, v_n]$. A k-simplex $\tau = [u_0, u_1, \ldots, u_k]$ is a face of σ if $\{u_0, u_1, \ldots, u_k\} \subset \{v_0, v_1, \ldots, v_n\}$.

Definition 2.3.2. A **simplicial complex** K is a set of simplicies such that if τ is a face of $\sigma \in K$, then $\tau \in K$. We also require that if $\sigma_1, \sigma_2 \in M$, then $\sigma_1 \cap \sigma_2$ is either empty or a face of both σ_1 and σ_2.

We may also define a simplicial complex combinatorially, in contrast to the geometric definition given above.

Definition 2.3.3. An **abstract simplicial complex** Δ is a collection of subsets of a set S such that, if $X \in \Delta$ and $Y \subseteq X$, then $Y \in \Delta$. We call Y a face of X. We refer to S as the vertex set.

Because simplicial complexes are given more structure than general topological spaces, maps between simplicial complexes ought to preserve this structure.

Definition 2.3.4. A **simplicial map** $f : K \to J$ between two simplicial complexes K, J is a map on the vertex sets of K and J such that simplices are mapped to simplices.

Example 2.3.1. *Let K be a subcomplex of J. Then, the inclusion $\iota : K \hookrightarrow J$ is a simplicial map.*

In algebraic topology, we wish to study a topological space (in this case, a simplicial complex) by studying certain algebraic invariants and descriptors. A popular example is the *homology* of K, which we define in this section.

---

[1] Usually, the orientation is inherited by the ordering on the indices.

Definition 2.3.5. (Page 104 in [13]) The nth **chain group** of K with coefficients in a field $\mathbb{F}$, denoted $C_n(K;\mathbb{F})$, is the free abelian group generated by the set of n simplices of K.

For every dimension n, there is a boundary map $\partial_n : C_n(K;\mathbb{F}) \to C_{n-1}(K;\mathbb{F})$. For each n simplex (that is, basis element) $[v_0, \cdots, v_n]$, we define

$$\partial_n([v_0, \cdots, v_n]) = \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \ldots, v_n]$$

and extend linearly. Here, $\hat{v}_i$ means we remove the vertex v_i, resulting in an $n-1$ simplex.

Lemma 2.3.1. *(Lemma 2.1 in [13]) For every n, $\partial_{n-1}\partial_n = 0$.*

Proof. We need only check this on a general basis element. Let $[v_0, \ldots, v_n]$ be an n simplex in K. Then,

$$\begin{aligned}
\partial_{n-1}\partial_n[v_0, \ldots, v_n] &= \partial_{n-1} \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \ldots, v_n] \\
&= \sum_{i<j} (-1)^i (-1)^j [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] \\
&\quad\quad + \sum_{i>j} (-1)^i (-1)^{j-1} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] \\
&= 0.
\end{aligned}$$

Since every basis element goes to 0, the composition of boundary maps sends any n-chain to 0. □

Definition 2.3.6. (Page 106 in [13]) A **chain complex** is a sequence of vector spaces V_n and homomorphisms $f_n : V_n \to V_{n-1}$ such that $\mathrm{im} f_{n+1} \subseteq \ker f_n$ for all n. This condition is equivalent to saying $f_{n-1}f_n = 0$.

In particular, Lemma 2.3.1 proves that the sequence of simplicial chain groups with boundary maps is a chain complex:

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(K;\mathbb{F}) \xrightarrow{\partial_{n+1}} C_n(K;\mathbb{F}) \xrightarrow{\partial_n} C_{n-1}(K;\mathbb{F}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(K;\mathbb{F}) \xrightarrow{\partial_0} 0.$$

Using this exact sequence, we may make the following definition.

Definition 2.3.7. Elements of $\ker \partial_n$ (often denoted $Z_n(K;\mathbb{F})$) are called n-**cycles**, and elements of $\mathrm{im}\partial_{n+1}$ (often denoted $B_n(K;\mathbb{F})$) are called n-**boundaries**. Since $\mathrm{im}\partial_{n+1} \subseteq \ker \partial_n$, we define the nth **homology group** of the chain complex $C_\bullet(K;\mathbb{F})$ as the quotient

$$H_n(K;\mathbb{F}) = \frac{\ker \partial_n}{\mathrm{im}\partial_{n+1}} = \frac{Z_n(K;\mathbb{F})}{B_n(K;\mathbb{F})}.$$

Given a simplicial map $f : K \to J$, there is an induced linear map $\hat{f}_n : C_n(K;\mathbb{F}) \to C_n(J;\mathbb{F})$ for every n. The induced maps commute with the boundary maps; that is, $\partial_n \circ \hat{f}_n = \hat{f}_{n-1} \circ \partial_n$ for all n. We call a collection of maps that satisfy this property **chain maps**, and denote them $f_\# : C_\bullet(K;\mathbb{F}) \to C_\bullet(J;\mathbb{F})$. Thus, f induces a chain map $f_\#$.

We can go further. Each linear map $\hat{f}_n : C_n(K;\mathbb{F}) \to C_n(J;\mathbb{F})$ induces a linear map on homology groups, $f_n : H_n(K,\mathbb{F}) \to H_n(J;\mathbb{F})$. We prove this assertion below.

Lemma 2.3.2. *A chain map $\hat{f}_n : C_n(K;\mathbb{F}) \to C_n(J;\mathbb{F})$ induces a linear map on homology groups, $f_n : H_n(K,\mathbb{F}) \to H_n(J;\mathbb{F})$.*

Proof. We need only show that $\hat{f}_n$ maps cycles to cycles and boundaries to boundaries. First, let $c \in C_n(K;\mathbb{F})$ be a cycle. Thus, $\partial c = 0$. Since $f_\#$ is a chain map, $\partial \hat{f}_n(c) = \hat{f}_{n-1}\partial(c) = 0$. Thus, $\hat{f}_n(c) \in Z_n(J;\mathbb{F})$. Now, let $c \in C_n(K;\mathbb{F})$ be a boundary. Thus, $\exists b \in C_{n+1}(K;\mathbb{F})$ such that $\partial b = c$. Then, $\partial \hat{f}_{n+1}(b) = \hat{f}_n \partial(b) = \hat{f}_n(c)$. That is, $\hat{f}_n(c) \in B_n(J;\mathbb{F})$. □

In summary, given a simplicial complex K and dimension n, we may define a vector space $H_n(K,\mathbb{F})$. Further, for a simplicial map $f : K \to J$, there is a linear map f_n. That is, after checking a couple other properties, we have shown the following:

Theorem 2.3.1. Homology is functorial. That is, $H_n(-;\mathbb{F}) : \mathrm{Simp} \to \mathrm{Vect}_\mathbb{F}$ where $K \mapsto H_n(K;\mathbb{F})$ is a functor.[2]

[2] I have stated the theorem in terms of simplicial homology. It is also true that singular homology is a functor $H_n(-;\mathbb{F}) : \mathrm{Top} \to \mathrm{Vect}_\mathbb{F}$.

Since each $C_n(K;\mathbb{F})$ is a $\mathbb{F}$-vector space, we may look at its vector space dual. That is, we may consider the vector space of all possible maps $C_n(K;\mathbb{F}) \to \mathbb{F}$, denoted $\text{Hom}(C_n(K;\mathbb{F}),\mathbb{F})$.

Definition 2.3.8. The n**th cochain group** is $C^n(K;\mathbb{F}) = \text{Hom}\,(C_n(K;\mathbb{F}),\mathbb{F})$.

The boundary maps between chain groups induce coboundary maps between cochain groups $\delta^n : C^n(K;\mathbb{F}) \to C^{n+1}(K;\mathbb{F})$, given by pre-composition with ∂_{n+1}. That is, for $c^* \in C^n(K;\mathbb{F})$, $\delta^n(c^*) = c^* \circ \partial_{n+1}$.

Lemma 2.3.3. *For every n, $\delta^{n+1}\delta^n = 0$.*

Proof. Take any $c^* \in C^n(K;\mathbb{F})$. Then, $\delta^{n+1}\delta^n(c^*) = c^* \circ \partial_{n+1} \circ \partial_{n+2} = 0$ by Lemma 2.3.1. □

In fact, by Lemma 2.3.3, the cochain groups equipped with the coboundary maps give rise to a **cochain complex**:

$$\cdots \xleftarrow{\delta^{n+1}} C^{n+1}(K;\mathbb{F}) \xleftarrow{\delta_n} C^n(K;\mathbb{F}) \xleftarrow{\delta^{n-1}} C^{n-1}(K;\mathbb{F}) \xleftarrow{\delta^{n-2}} \cdots \xleftarrow{\delta^0} C^0(K;\mathbb{F}) \leftarrow 0.$$

Definition 2.3.9. Elements of $\ker\delta^n$ (often denoted $Z^n(K;\mathbb{F})$) are called n-**cocycles**, and elements of $\text{im}\delta^{n-1}$ (often denoted $B^n(K;\mathbb{F})$) are called n-**coboundaries**. Since $\text{im}\delta^{n-1} \subseteq \ker\delta^n$, we define the nth **cohomology group** of the cochain complex $C^\bullet(K;\mathbb{F})$ as the quotient

$$H^n(K;\mathbb{F}) = \frac{\ker\delta^n}{\text{im}\delta^{n-1}} = \frac{Z^n(K;\mathbb{F})}{B^n(K;\mathbb{F})}.$$

We already know that, given a simplicial map $f : K \to J$, there is an induced map on chain groups $\hat{f}_n$ for every n. Thus, f also induces a map $\hat{f}^n : C^n(J,\mathbb{F}) \to C^n(K,\mathbb{F})$ on cochain groups for every n. After all, given any $c^* \in C^n(J,\mathbb{F})$, define $\hat{f}^n(c^*) = c^* \circ \hat{f}_n$. These induced maps also respect the coboundary maps: namely, $\hat{f}^n \circ \delta^{n-1} = \delta^{n-1} \circ \hat{f}^{n-1}$. We call a collection of maps that satisfies this property a **cochain map**, denoted $f^\# : C^\bullet(J,\mathbb{F}) \to C^\bullet(K,\mathbb{F})$.

For each n, $\hat{f}^n$ induces a linear map on cohomology $f^n : H^n(J;\mathbb{F}) \to H^n(K;\mathbb{F})$.

Again, notice that cohomology assigns to every simplicial complex a vector space and to every simplicial map a linear map. That is,

Theorem 2.3.2. Cohomology is a contravariant functor. That is, $H^n(-;\mathbb{F}) : \mathrm{Simp}^{\mathrm{op}} \to \mathrm{Vect}_{\mathbb{F}}$ where $K \mapsto H^n(K;\mathbb{F})$ is a functor.

Functoriality of homology and cohomology will play an important role in future chapters.

2.4 Persistence

Homology and cohomology are useful tools for topological data analysis. Of specific interest is the persistence diagram. In this section, we will discuss the traditional, 1-parameter setting for persistent homology and cohomology. However, persistence theory is a booming area of research, and we will only be scratching the surface of the interesting theorems and results in the field.

In this section, we will describe certain landmark findings and definitions in persistence theory; that is, we will describe filtrations, the rank function, and the persistence diagram, focusing specifically on category-theoretic definitions and the language of posets.

Let K be a simplicial complex. A filtration of K is a nested sequence of inclusions

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$

where each K_i is a subcomplex of K. For those inclined to category theory, a filtration is a functor as follows. Let $[n]$ denote the category consisting of objects $\{0, 1, \cdots, n\}$ and morphisms $i \to j$ for all $i \leq j$. Notice that $[n]$ is a *poset* (in fact, it is a totally ordered set). Let ΔK denote the category of subcomplexes of K, with morphisms inclusion maps.

Definition 2.4.1. A **filtration** of K is a functor $F : [n] \to \Delta K$.[3]

Now, recall that Homology is functorial. That is, we may choose a field $\mathbb{F}$ and desired dimension n, then compose $H_n(-;\mathbb{F}) \circ F$. The result is a special case of a more general *persistence module*:

[3]See [14] for how to move from a discrete filtration, which is the definition we provided, to a continuous filtration indexed over $\mathbb{R}$.

Definition 2.4.2. (Definition 3.2 in [2]) A **persistence module** is a family of $\mathbb{F}$-vector spaces $\{V_i\}_i$ together with linear transformations $\varphi_{i,j} : V_i \to V_j$ for $i \leq j$ and $\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k}$ whenever $i \leq j \leq k$.

In the special case where we apply homology, each V_i is the homology group of $F(i)$, for $i \in [n]$. Further, each $\varphi_{i,j}$ is the inclusion map $F(i) \to F(j)$ for $i \leq j$ in $[n]$.

By studying persistent homology, we aim to answer the question "how does the homology of a space K change as I move through the filtration F"? In other words, we want to track when homological features appear and disappear as we move through the filtration F. Below, we discuss the mathematics used to answer that question.

Let $F : [n] \to \Delta K$ be a filtration, and recall that $H_n(-;\mathbb{F}) \circ F$ sends each step in our filtration to its corresponding homology group. Because homology is functorial, each $a \leq b$ in $[n]$ induces a linear map $H_n(F(a);\mathbb{F}) \to H_n(F(b);\mathbb{F})$. We denote this map by $H_n(F(a \leq b);\mathbb{F})$, or simply $H_n(F(a \leq b))$.

Definition 2.4.3. (Definition 2.3 in [14]) Given a filtration $F : [n] \to \Delta K$ and persistence module $H_n(-;\mathbb{F}) \circ F$, the **rank function** $\text{rk} : \overline{[n]} \to \mathbb{Z}$ is defined as follows: for each $a \leq b \in [n]$, $\text{rk}_n[a, b] = \text{rk} H_n(F(a \leq b);\mathbb{F})$.

Definition 2.4.4. (Theorem 4.1 in [8]) The nth **persistence diagram** is the Möbius inversion of the rank function.

Specifically, the nth persistence diagram is also an integer-valued function $\text{PH}_n : \overline{[n]} \to \mathbb{Z}$. In the case of 1-parameter filtrations, the Möbius inversion may be computed $i \neq j$ using the following simple formula for [4, 14]:

$$\text{PH}_n[i, j] = \text{rk}_n[i, j-1] - \text{rk}_n[i-1, j-1] - \text{rk}_n[i, j] + \text{rk}_n[i-1, j].$$

Recall that in our general definition of a persistence module the use of homology was not necessary. We may also define a persistence module using cohomology groups. That is, we may define a persistence module where $H^n(-;\mathbb{F}) \circ F$ maps each $a \in [n]$ to the cohomology group $H^n(F(a);\mathbb{F})$, and $a \leq b \in [n]$ corresponds to a linear map $H^n(F(b);\mathbb{F}) \to H^n(F(a);\mathbb{F})$ (recall that cohomology is contravariant). Although the arrows are reversed, we may still use the rank function and the Möbius inversion to obtain the persistence diagram.

In [3], de Silva, Morozov, and Vejdemo-Johansson construct an isomorphism between persistent homology and persistent cohomology.

2.5 Generalized Persistence Diagrams

In the previous section, we briefly introduced persistent homology. Notice that our discussion on filtrations focused on totally ordered sets. Recently, much research was done to generalize persistence diagrams, and defining persistence for multiparameter filtrations is one avenue of generalization.

One solution to establishing a theory of multiparameter persistence was done by Alex McCleary and Amit Patel in [6]. We will briefly summarize their findings here, but we refer you to their paper for in-depth and rigorous discussions of morphisms, stability, proofs of functoriality, and the like. Many of the proofs in Section 4 will look very similar to the proofs in [6] (in fact, this is no surprise [3]).

For this section, let P be a finite metric lattice, K be a finite simplicial complex, and ΔK the category of subcomplexes of K with morphisms inclusion maps.

Definition 2.5.1. (Definition 4.1 in [6]) A **filtration** of K indexed by P is a functor $F : P \to \Delta K$ such that $F(\top) = K$. That is, for all $a \leq b$ in P, $F(a), F(b)$ are subcomplexes of K and there is an inclusion map $F(a) \to F(b)$.

Definition 2.5.2. (Definition 4.2 in [6]) A **filtration preserving morphism** is a triple (F, G, α) where $F : P \to \Delta K$ and $G : Q \to \Delta K$ are filtrations and $\alpha : P \to Q$ is a bounded lattice map such that, for all $q \in Q$, $Q(q) = F\left(\max \alpha^{-1}[\perp, q]\right)$.[4]

[4] $\max \alpha^{-1}[\perp, q]$ is guaranteed to exist by Proposition 2.2.1.

In [6], the authors show that the composition of filtration preserving morphisms is itself a filtration-preserving morphism. Thus, we are justified in making the following definition:

Definition 2.5.3. (Definition 4.7 in [6]) Denote by $\mathrm{Fil}(K)$ the **category of filtrations** of a simplicial complex K, with objects filtrations and morphisms filtration-preserving morphisms.

Given a filtration F, we are interested in tracking the birth and the death of homological features. We achieve this by defining the *birth-death* function associated to a filtration. First, however, we establish the category in which these functions live.

Definition 2.5.4. (Page 9 in [6]) Let P be a finite metric lattice. A **monotone integer function** is a function $f : \overline{P} \to \mathbb{Z}$ such that, whenever $I \leq J \in \overline{P}$, $f(I) \leq f(J)$.

Definition 2.5.5. (Definition 5.1 in [6]) A **monotone-preserving morphism** is a triple $(f, g, \overline{\alpha})$ where $f : \overline{P} \to \mathbb{Z}$, $g : \overline{Q} \to \mathbb{Z}$ are monotone integer functions defined on interval lattices and $\overline{\alpha} : \overline{P} \to \overline{Q}$ is a bounded lattice map that satisfies the following: for all $I \in \overline{Q}$, $g(I) = f(\max \overline{\alpha}^{-1}[\perp, I])$.

Again, see [6] for the proof that we may compose monotone-preserving morphisms. Thus, we may define the following category.

Definition 2.5.6. (Definition 5.5 in [6]) Let Mon be the **category of monotone integer functions**, with morphisms monotone-preserving functions.

Definition 2.5.7. (Definition 5.6 in [6]) Take a filtration $F : P \to \Delta K$. Its nth **birth-death function** is the monotone function $f_n : \overline{P} \to \mathbb{Z}$ defined by

$$\mathrm{ZB}_n[a, b] = \begin{cases} \dim\left(Z_n(a) \cap B_n(b)\right) & \text{if } b \neq \top \\ \dim\left(Z^n(a)\right) & \text{if } b = \top. \end{cases}$$

It turns out that the assignment of each filtration $F : P \to \Delta K$ to its birth-death function in functorial. That is,

Definition 2.5.8. (Definition 5.9 in [6]) The nth **birth-death functor** $\mathrm{ZB}_n : \mathrm{Fil}(K) \to \mathrm{Mon}$ is the functor that assigns each filtration to its nth birth-death function.

In [6], the authors show that, for 1-parameter filtrations, the birth-death function agrees with the rank function from Definition 2.4.3.

Just as in Section 2.4, we may use the *Möbius inversion* of the birth-death functor to define the persistence diagram. By definition, the Möbius inversion is an integer-valued function (not necessarily monotone); thus, we want to study the category of integer-valued (or, "integral") functions. To do so, we must first define a morphism of integral functions.

Definition 2.5.9. (Definition 6.1 in [6]) Let P, Q be finite metric lattices, and let $\sigma : \overline{P} \to \mathbb{Z}$ and $\tau : \overline{Q} \to \mathbb{Z}$ be two integral functions. A **charge-preserving morphism** is a triple $(\sigma, \tau, \overline{\alpha})$ where $\alpha : \overline{P} \to \overline{Q}$ is a bounded lattice map such that for all $I \neq [q, q] \in \overline{Q}$,

$$\tau(I) = \sum_{J \in \bar{\alpha}^{-1}(I)} \sigma(J).$$

if $\bar{\alpha}^{-1}(I) = \emptyset$, then we define $\tau(I) = 0$.

A composition of charge-preserving morphisms is again charge-preserving, so we may define the corresponding category.

Definition 2.5.10. (Definition 6.5 in [6]) Let Fnc be the **category of integer-valued functions** with morphisms charge-preserving morphisms.

McCleary and Patel show that the assignment of a monotone integer function to its Möbius inversion is functorial.

Definition 2.5.11. (Definition 5.9 in [6]) The **Möbius inversion functor** is the functor $\mathrm{MI} : \mathrm{Mon} \to \mathrm{Fnc}$ that assigns each monotone integer function its Möbius inversion.

Definition 2.5.12. (Definition 8.1 in [6]) The nth **persistence diagram** of a filtration $F : P \to \Delta K$ is the integral function given by $\mathrm{PH}^n(F) = \mathrm{MI} \circ \mathrm{ZB}_n(F)$.

That is, persistence diagrams are given by the following composition of functors:

$$\mathrm{Fil}(K) \xrightarrow{\mathrm{ZB}_*} \mathrm{Mon} \xrightarrow{\mathrm{MI}} \mathrm{Fnc}.$$

In [6], the authors show that these persistence diagrams are stable with respect to the edit distance, which we define in Section 5.1.

Chapter 3

Cofiltrations

Suppose we want to study persistent cohomology by applying the same functorial pipeline as in [6] and discussed in Section 2.5. That is, let $P \xrightarrow{F} \Delta K$ be a filtration, and let $a \leq b$ in P. The inclusion $F(a) \xrightarrow{\iota} F(b)$ induces a cochain map $\iota^{\#}$:

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & C^{n+1}(F(a)) & \xleftarrow{\delta^n} & C^n(F(a)) & \xleftarrow{\delta^{n-1}} & C^{n-1}(F(a)) & \longleftarrow \cdots \\
& \iota^{\#}\uparrow & & \iota^{\#}\uparrow & & \iota^{\#}\uparrow & \\
\cdots \longleftarrow & C^{n+1}(F(b)) & \xleftarrow{\delta^n} & C^n(F(b)) & \xleftarrow{\delta^{n-1}} & C^{n-1}(F(b)) & \longleftarrow \cdots
\end{array}$$

In order to study generalized cohomology, we want to understand what $\iota^{\#}$ does to cochains and coboundaries. Unfortunately, unlike in the homology setting, the maps between cochains and coboundaries are not necessarily injective nor surjective, even when looking at a 1-parameter filtration. For an easy counter-example, see Figure 3.1. This makes defining a birth-death functor for cohomology more complicated.

In this chapter, we resolve the issue above by introducing cofiltrations of a simplicial complex K. The tools established in this chapter will allow us to define the persistence diagram for cohomology of a cofiltration in Chapter 4.

3.1 Category of Cofiltrations

Before defining cofiltrations, we first introduce the notion of a "supcomplex".

Definition 3.1.1. Let σ be an n-simplex. A **coface** of σ is k-simplex τ such that σ is a face of τ (this condition implies $k \geq n + 1$).

Definition 3.1.2. Let K be a simplicial complex. A **supcomplex** of K is an open collection of simplices that is closed upwards. That is, if K' is a supcomplex of K and $\sigma \in K'$, then all cofaces of σ must also be in K'.

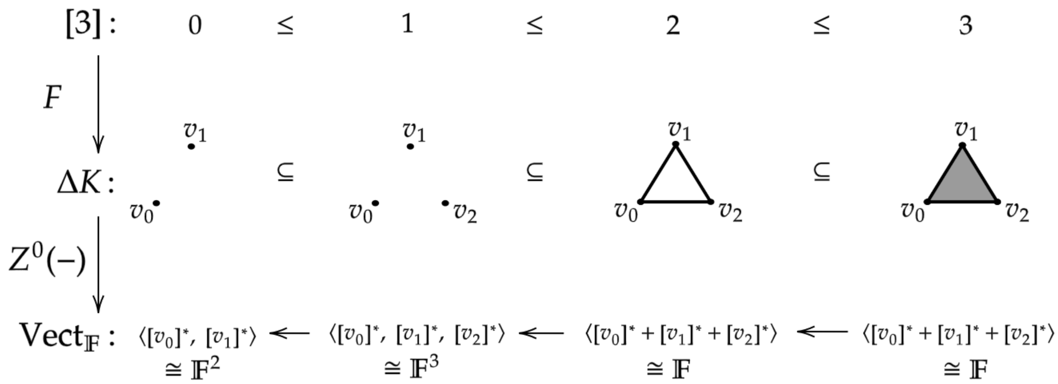


Figure 3.1: A filtration of the 2-simplex, indexed over the totally ordered lattice on 4 elements, $[3]$. We then compute the 0th cocyle group for each $F(i)$. Here, $[v_i]^*$ denotes the 0-cocyle that maps v_i to 1 and v_j to 0 for any $j \neq i$. Notice that $Z^0(F(1 \leq 2))$ is not surjective, and that $Z^0(F(0 \leq 1))$ is not injective.

Note that a supcomplex K' is not actually a simplicial complex, but $|K'|$ (the underlying topological space of K') is a subspace of $|K|$. Denote by ∇K the category of supcomplexes of K with morphisms inclusion maps. That is, if K', K'' are supcomplexes of K, then there is a morphism $K' \to K''$ when $K' \subseteq K''$.

Proposition 3.1.1. *∇K is a finite lattice.*

Proof. Since K is a finite simplicial complex, there are only finitely many supcomplexes. Further, $\top_{\nabla K} = K$ and $\bot_{\nabla K} = \emptyset$. The order is given by inclusion. Let $K', K'' \in \nabla K$. Meets are intersections. The intersection $K' \cap K''$ is, indeed, a supcomplex, since $\tau \in K' \cap K''$ means all cofaces of τ are in both K' and K'', therefore in $K' \cap K''$. Now, suppose $\exists J \in \nabla K$ such that $K' \cap K'' \subseteq J \subseteq K', K''$. Then, $\tau \in J$ means τ and all cofaces of τ are in both K' and K'', so $\tau \in K' \cap K'' \implies J \subseteq K' \cap K''$. Thus, $K' \cap K''$ is the unique maximal element such that $K' \cap K'' \subseteq K', K''$. Joins are unions. After all, $K' \cup K''$ is a supcomplex, since $\tau \in K' \cup K''$ means all cofaces of τ are in either K' or K'', therefore also in $K' \cup K''$. Now, suppose $\exists L$ such that $K', K'' \subseteq L \subseteq K' \cup K''$. Then, $\tau \in K' \cup K''$ means τ and all its cofaces are in either K'

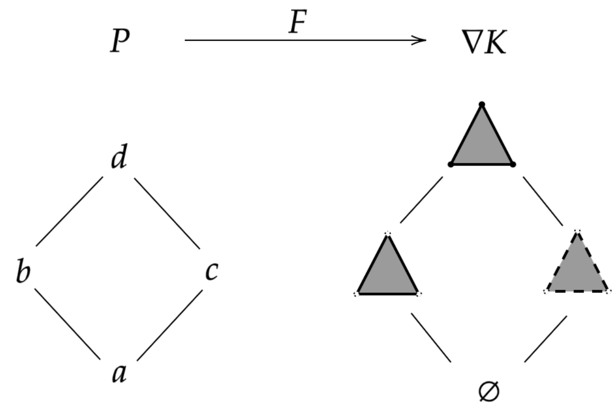


Figure 3.2: A cofiltration of the 2-simplex, indexed over a lattice P.

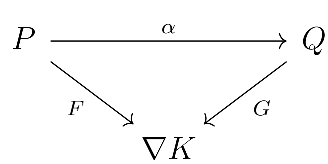
or K'', and therefore must be in L. So, $K' \cup K'' \subseteq L$, meaning $K' \cup K''$ is the unique minimal element such that $K', K'' \subseteq K' \cup K''$. $\square$

Definition 3.1.3. A **cofiltration** of K is a functor $F : P \to \nabla K$ where P is a finite metric lattice and $F(\top) = K$.

From Proposition 3.1.1, we may be tempted to call a cofiltration a bounded lattice map. This is not necessarily the case: we are not requiring that F preserves meets or joins. After all, in Figure 3.2, $F(b \wedge c) = F(a) = \emptyset \neq F(b) \cap F(c)$.

With a definition of cofiltrations, we now discuss maps between cofiltrations.

Definition 3.1.4. A **cofiltration-preserving morphism** is a triple (F, G, α), where $F : P \to \nabla K$ and $G : Q \to \nabla K$ are cofiltrations, and $\alpha : P \to Q$ is a bounded lattice map such that $G(q) = F(\max \alpha^{-1}[\perp, q])$.[5]



[5]We know $\max \alpha^{-1}[\perp, q]$ exists by Proposition 2.2.1.

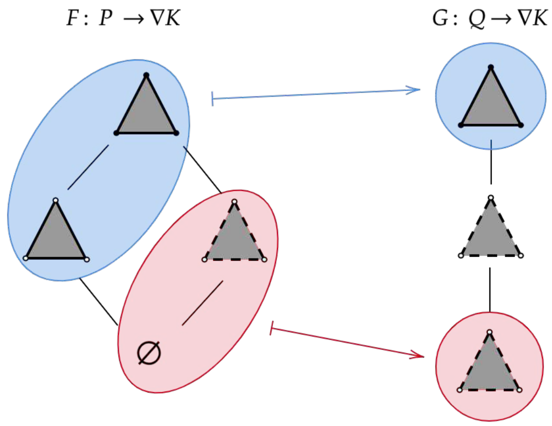


Figure 3.3: A cofiltration-preserving morphism (F, G, α), where α is the bounded lattice map as shown with the blue and red arrows.

Proposition 3.1.2. *If* (F, G, α) *and* (G, H, β) *are cofiltration-preserving morphisms, then so is the composition* $(F, H, \beta \circ \alpha)$.

$$\begin{array}{ccccc} P & \xrightarrow{\alpha} & Q & \xrightarrow{\beta} & R \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle G} & \swarrow{\scriptstyle H} & \\ & & \nabla K & & \end{array}$$

Proof. Since (F, G, α) and (G, H, β) are already cofiltration-preserving morphisms, we have the following:

$$G(q) = F(\max \alpha^{-1}[\bot_Q, q]) \quad \text{and} \quad H(r) = G(\max \beta^{-1}[\bot_R, r])$$

Thus,

$$\begin{aligned} H(r) &= G(\max \beta^{-1}[\bot_R, r]) = F(\max \alpha^{-1} \left[\bot_Q, (\max \beta^{-1}[\bot_R, r]), \right]) \\ &= F(\max \alpha^{-1}\beta^{-1} \left[\bot_R, r\right]) = F(\max(\beta \circ \alpha)^{-1} \left[\bot_R, r\right]). \end{aligned}$$

To see why the third equality holds, notice first that $\alpha^{-1} \left[\bot_Q, (\max \beta^{-1}[\bot_R, r])\right]$ and $\beta^{-1}[\bot_R, r]$ are both finite lattices themselves (Proposition 2.2.1). Particularly, $\max \alpha^{-1} \left[\bot_Q, (\max \beta^{-1}[\bot_R, r])\right]$ (the top element) must be mapped to $\max \beta^{-1}[\bot_R, r]$. $\square$

Thus, we are justified in making the following definition:

Definition 3.1.5. We denote by $\mathrm{CoFil}(K)$ the **category of cofiltrations** of K, having as objects cofiltrations and having as morphisms cofiltration-preserving morphisms.

Chapter 4

Persistent Cohomology of a Cofiltration

In this section, we use the tools described in Chapter 3 to discuss the persistent cohomology of a cofiltration. In summary, we will (1) define the birth-death functor for persistent cohomology and (2) apply the Möbius inversion functor. The composition of the two gives us the generalized persistence diagram.

4.1 Cohomology of Cofiltrations

Let $F : P \to \nabla K$ be a cofiltration. For every $a \in P$, we construct a chain complex $C_\bullet(F(a))$:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(F(a)) \xrightarrow{\partial_n} C_{n-1}(F(a)) \xrightarrow{\partial_{n-1}} \cdots$$

In the case of supcomplexes, we must slightly adjust the definition of the boundary map as follows. For $[v_0, \ldots, v_n]$ an n-simplex in $F(a)$, define $\partial_n[v_0, \ldots, v_n] = \sum_{i=0}^{n}(-c_i)^i[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ where $c_i = 1$ if $[v_0, \ldots, \hat{v}_i, \ldots, v_n] \in F(a)$ and $c_i = 0$ else. Indeed, this "modified" boundary map does satisfy $\partial^2 = 0$.

Further, for each $a \leq b$ in P, there is an inclusion map $\iota : F(a) \hookrightarrow F(b)$. In this section, we will study the map ι induces on chains and cochains.

First, we look at the induced map on chain groups:

Proposition 4.1.1. *The inclusion $\iota : F(a) \hookrightarrow F(b)$ induces a chain map $\iota_\#$:*

$$\begin{array}{ccccccccc} \cdots & \xleftarrow{\partial} & C_{n-1}(F(a)) & \xleftarrow{\partial} & C_n(F(a)) & \xleftarrow{\partial} & C_{n+1}(F(a)) & \xleftarrow{\partial} & \cdots \\ & & \uparrow\!\!\uparrow \iota_{n-1} & & \uparrow\!\!\uparrow \iota_n & & \uparrow\!\!\uparrow \iota_{n+1} & & \\ \cdots & \xleftarrow{\partial} & C_{n-1}(F(b)) & \xleftarrow{\partial} & C_n(F(b)) & \xleftarrow{\partial} & C_{n+1}(F(b)) & \xleftarrow{\partial} & \cdots \end{array}$$

Further, each $\iota_n : C_n(F(b)) \to C_n(F(a))$ is surjective.

Proof. First, we show that each ι_n is surjective. It is sufficient to show that each basis element $\sigma \in C_n(F(a))$ has a preimage. As a basis element, σ is an n-simplex in $F(a)$. Since $\iota : F(a) \hookrightarrow F(b)$ is the inclusion map, $\sigma \in F(b)$ and is thus a generator for $C_n(F(b))$. Thus, $\iota_n(\sigma) = \sigma$. Indeed, ι_n is the restriction map: if $\tau \in F(b) \backslash F(a)$, then $\iota_n(\tau) = 0$.

Now, I claim that $\iota_{n-1} \circ \partial_n = \partial_n \circ \iota_n$ for every n. It suffices to show this for a general basis element of $C_n(F(b))$. Let $\sigma \in F(b)$, meaning $1 \cdot \sigma = \sigma$ is a basis element of $C_n(F(b))$. Suppose first that at least one face of σ is in $F(a)$, call it τ. Since $F(a) \subseteq F(b)$, we have that $\tau \in F(a) \implies \tau \in F(b)$. By the supcomplex condition, we know σ must be in both $F(a)$ and $F(b)$ as well. Thus, $\partial_n(\iota_n(\sigma))$ returns a signed sum of the faces of σ that are in $F(a)$. Now, consider $\partial_n(\sigma)$, which is a signed sum of the faces of σ that are in $F(b)$. The restriction map $\iota_{n-1}(\partial_n(\sigma))$ also returns a signed sum of the faces of σ that are in $F(a)$. Because the orientation on the simplices does not change throughout a cofiltration, $\iota_{n-1}(\partial_n(\sigma)) = \partial_n(\iota_n(\sigma))$.

Now, suppose the no faces of σ are in $F(a)$. That is, $\iota_{n-1}(\partial_n(\sigma)) = 0$. Consider the restriction $\iota_n(\sigma)$. If $\sigma \in F(a)$, then $\iota_n(\sigma) = \sigma$ and $\partial_n(\iota_n(\sigma)) = \partial_n(\sigma) = 0 = \iota_{n-1}(\partial_n(\sigma))$. If $\sigma \notin F(a)$, then $\partial_n(\iota_n(\sigma)) = \partial_n(0) = 0 = \iota_{n-1}(\partial_n(\sigma))$.

We have shown that $\iota_{n-1} \circ \partial_n(\sigma) = \partial_n \circ \iota_n(\sigma)$ for any basis element σ of $C_n(F(b))$, and therefore must be true for any chain. □

Now, we will use the fact that ι induces a map on chain groups to show that it also induces a map on cochain groups.

Proposition 4.1.2. *The inclusion $\iota : F(a) \hookrightarrow F(b)$ induces a cochain map $\iota^{\#}$:*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta} & C^{n-1}(F(a)) & \xrightarrow{\delta} & C^n(F(a)) & \xrightarrow{\delta} & C^{n+1}(F(a)) & \xrightarrow{\delta} & \cdots \\
 & & \big\downarrow \iota^{n-1} & & \big\downarrow \iota^{n} & & \big\downarrow \iota^{n+1} & & \\
\cdots & \xrightarrow{\delta} & C^{n-1}(F(b)) & \xrightarrow{\delta} & C^n(F(b)) & \xrightarrow{\delta} & C^{n+1}(F(b)) & \xrightarrow{\delta} & \cdots
\end{array}$$

Further, each $\iota^n : C^n(F(a)) \to C^n(F(b))$ is injective.

Proof. First, notice that each ι^n is injective; let $c^*, d^* \in C^n(F(a))$ and suppose $\iota^n(c^*) = \iota^n(d^*)$. That is, if ι_n is the induced map on chains, $c^* \circ \iota_n = d^* \circ \iota_n$. Since ι_n is surjective, $c^* = d^*$. Thus, ι^n is injective.

Now, I claim that $\iota^{n+1}\delta^n = \delta^n\iota^n$ for every n. After all,

$$\begin{aligned}\delta^n\iota^n(c^*) &= c^* \circ \iota_n \circ \partial_{n+1} = c^* \circ \partial_{n+1} \circ \iota_{n+1} \\ &= \iota^{n+1}(c^* \circ \partial_{n+1}) = \iota^{n+1}\partial^n(c^*).\end{aligned}$$

Thus, $\iota^{\#}$ is a cochain map. □

Since we are studying cohomology, we are particularly interested in where ι^n maps cocyles and coboundaries.

Proposition 4.1.3. *For every n, ι^n restricts to an injective map between n-cocycle spaces, $\iota^n : Z^n(F(a)) \hookrightarrow Z^n(F(b))$.*

Proof. First, show that ι^n does, indeed, send cocycles to cocyles. Let $c^* \in C^n(F(a))$ be a cocyle. That is, $\delta c^* = 0$. Since $\iota^{n+1}\delta = \delta\iota^n$ by Proposition 4.1.2, $\delta\iota^n(c^*) = \iota^{n+1}(\delta c^*) = \iota^{n+1}(0) = 0$. Thus, $\iota^n(c^*)$ is a cocyle. Injectivity follows from injectivity of ι^n on cochains. □

Proposition 4.1.4. *For every n, ι^n restricts to an injective map between n-coboundary spaces, $\iota^n : B^n(F(a)) \hookrightarrow B^n(F(b))$.*

Proof. Again, we first show that ι^n maps coboundaries to coboundaries. Let $c^* \in C^n(F(a))$ be a coboundary. That is, $c^* = \delta^{n-1}(b^*) = b^* \circ \partial_n$ for some $b^* \in C^{n-1}(F(a))$. Then, if ι_n is the induced map on chains,

$$\begin{aligned}\iota^n(c^*) = c^* \circ \iota_n = b^* \circ \partial \circ \iota_n &= b^* \circ (\iota_{n-1} \circ \partial) \\ &= (b^* \circ \iota_{n-1}) \circ \partial = \delta(b^* \circ \iota_{n-1}) \\ &= \delta(\iota^{n-1}(b^*)).\end{aligned}$$

Thus, $\iota^n(c^*)$ is a coboundary. Injectivity follows from injectivity of ι^n on cochains. $\square$

Specifically, for any $a \leq b \in P$, we get the following inclusions:

$$\begin{array}{ccccc} B^n(F(a)) & \hookrightarrow & B^n(F(b)) & & \\ \downarrow & & \downarrow & & \\ Z^n(F(a)) & \hookrightarrow & Z^n(F(b)) & \hookrightarrow & C^n(F(\top)) = C^n(K). \end{array}$$

4.2 Birth-Death Functor for Cohomology

From Section 4.1, we now understand the maps induced by $F(a \leq b)$ on chain and cochain groups. In this section, we utilize those results to define the birth-death function associated to a cofiltration F.

Definition 4.2.1. Let $F : P \to \nabla K$ be a cofiltration. For an interval $[a, b] \in \overline{P}$, let

$$\mathrm{ZB}^n[a,b] = \begin{cases} \dim\left(Z^n F(a) \cap B^n F(b)\right) & \text{if } b \neq \top \\ \dim\left(Z^n F(a)\right) & \text{if } b = \top. \end{cases}$$

where we take the intersection in $C^n(K)$. The nth **birth-death function** for cohomology of a cofiltration $F : P \to \nabla K$ is the map $f^n : \overline{P} \to \mathbb{Z}$, where $f^n[a,b] = \mathrm{ZB}^n[a,b]$.

Proposition 4.2.1. *Let $F : P \to \nabla K$ be a cofiltration, and let $f^n : \overline{P} \to \mathbb{Z}$ be its associated birth-death function. Then, f^n is a monotone integer function.*

Proof. Let $[a,b] \leq [c,d]$ in $\overline{P}$. Then, $c \leq a$ and $d \leq b$ in P. From Propositions 4.1.3 and 4.1.4, we have $Z^n F(d) \subseteq Z^n F(b)$ and $B^n F(c) \subseteq B^n F(a)$. Taking intersections gives us another inclusion $Z^n F(a) \cap B^n F(b) \subseteq Z^n F(c) \cap B^n F(d)$. Thus, we have shown $\mathrm{ZB}^n[a,b] = \dim\left(Z^n F(a) \cap B^n F(b)\right) \leq \dim\left(Z^n F(c) \cap B^n F(d)\right) = \mathrm{ZB}^n[c,d]$. $\square$

Proposition 4.2.2. *Let (F, G, α) be a cofiltration-preserving morphism. Let f^n, g^n be the nth birth-death functions for F, G. Then, $(f^n, g^n, \bar{\alpha})$ is a monotone-preserving morphism.*

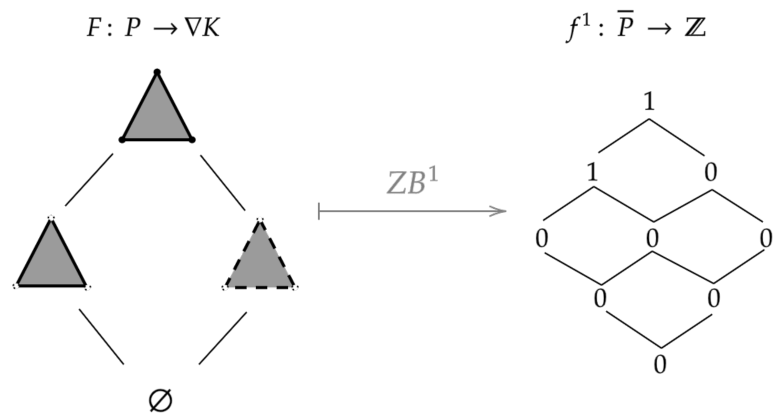


Figure 4.1: A cofiltration $F : P \to \nabla K$ and its corresponding birth-death function for 1-dimensional cohomology $f^1 : \overline{P} \to \mathbb{Z}$.

Proof. For an interval $I = [a, b] \in \overline{Q}$, let $I^\star = \max \bar{\alpha}^{-1}[\bot_{\overline{Q}}, I]$. Then, $g^n(I) = g^n[a, b] = f^n([\max \alpha^{-1}[\bot_Q, a], \max \alpha^{-1}[\bot_Q, b]]) = f^n(\max \bar{\alpha}^{-1}[\bot_{\overline{Q}}, I]) = f^n(I^\star)$. Thus, $(f^n, g^n, \bar{\alpha})$ is a monotone-preserving morphism. □

From Propositions 4.2.1 and 4.2.2, we have shown that the birth-death function is functorial.

Definition 4.2.2. Let $\mathrm{ZB}^n : \mathrm{CoFil}(K) \to \mathrm{Mon}$ be the functor that assigns to each cofiltration its birth-death function. We call ZB^n the nth **birth-death functor for cohomology**.

4.3 Generalized Persistence Diagram for Cohomology

Let $f : \overline{P} \to \mathbb{Z}$ be a monotone integer function. Recall from Section 2.2.1 that the Möbius inversion of f is the unique integral function $\sigma : \overline{P} \to \mathbb{Z}$ such that

$$f[a, b] = \sum_{[c,d] \in \overline{P}\colon\, [c,d] \leq [a,b]} \sigma[c, d].$$

Proposition 4.3.1. *(Proposition 6.6 in [6]) Let (f^n, g^n, α) be a morphism of monotone integer functions. Let σ, τ be the Möbius inversions of f^n, g^n. Then, $(\sigma, \tau, \bar{\alpha})$ is a morphism of integer functions.*

For the proof, see [6].

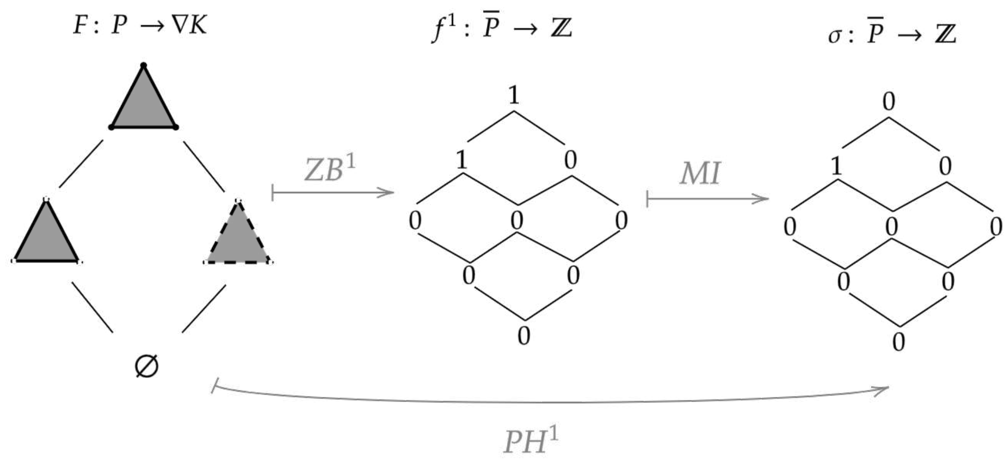


Figure 4.2: A cofiltration $F : P \to \nabla K$ and its corresponding 1-dimensional persistence diagram $\sigma : \overline{P} \to \mathbb{Z}$, which is the Möbius inversion of it's birth-death function for 1-dimensional cohomology $f^1 : \overline{P} \to \mathbb{Z}$.

By definition, the Möbius inversion of a monotone integral function is itself an integral function. Thus, by Proposition 4.3.1, the Möbius inversion is functorial. That is, $\mathrm{MI} : \mathrm{Mon} \to \mathrm{Fnc}$ is a functor.

Finally, we can define the persistence diagram arising from cohomology of a cofiltration:

Definition 4.3.1. Let K be a finite simplicial complex. For every $n \geq 0$ we have the following composition of functors:

$$\mathrm{CoFil}(K) \xrightarrow{\mathrm{ZB}^n} \mathrm{Mon} \xrightarrow{\mathrm{MI}} \mathrm{Fnc}.$$

We call this composition the nth **persistent cohomology functor**, denoted $\mathrm{PH}^n = \mathrm{MI} \circ \mathrm{ZB}^n$. Given a cofiltration $F : P \to \nabla K$, its nth **persistence diagram** is $\mathrm{PH}^n(F)$.

4.4 Comparing Persistence Diagrams

Now, with a general definition for persistent homology of a filtration and persistent cohomology of a cofiltration, it is natural to ask about the relationship between the two.

The most obvious way to construct a cofiltration from a filtration (or vice versa) is as follows: given $F : P \to \Delta K$ a filtration such that $F(\bot) = \emptyset$, define $G : P^{\mathrm{op}} \to \nabla K$ by $G(p) = K \backslash F(p)$. Indeed, G is a cofiltration. However, it turns out that this construction *is not functorial*. Specifically, it does not map filtration-preserving morphisms to cofiltration-preserving morphisms. With this warning in mind, we are still interested in studying the similarities and differences between F and G.

There are a few things we expect to happen. Since $F(\top_P) = G(\top_{P^{\text{op}}}) = K$ is a finite simplex, and since we are using field coefficients, we will see a bijection between the persistent cycles and cocyles. However, because the spaces we consider are inherently different (subcomplexes in the case of filtrations and supcomplexes in the case of cofiltrations), it is possible (perhaps even likely) that the two diagrams will differ. In this section, we compute the persistence diagrams for both homology of a filtration and cohomology of a cofiltration and compare the two.

Table 4.1 shows examples of a filtration $F : P \to \Delta K$ and the cofiltration $G : P^{\text{op}} \to \nabla K$ of the 2-simplex (denoted K) over a finite lattice P (or, it's opposite lattice P^{op}). Then, see Table 4.2 to see examples of the corresponding persistence diagrams for $n = 0, 1, 2$.

First, let's discuss the 0-dimensional persistence diagram. for F, we see 2 0-cycles born at $F(a)$ that die by $F(b)$ and 1 persistent 0-cycle born at $F(a)$. For G, we see 1 0-cocyle that is born at $G(\bot)$ and persists. This matches our intuition. Now, consider 1-dimensional persistence. For F, we see 1 1-cycle born at $F(b)$ that dies at $F(\top)$. For G, we see 2 1-cocyle born at $G(c)$ that die at $G(\bot)$. Overall, there are no persistent 1-cycles or 1-cocyles, again matching our intuition. Finally, for 2-dimensional persistence, we expect to see no persistent 2 cycles or cocyles: after all, for F, no 2-cycles are ever present in the filtration. For G, 1 2-cocyle is born at $G(b)$ and dies at $G(a)$, 1 2-cocyle is born at $G(c)$ and dies immediately at $G(c)$. By $G(a)$, we can see that these 2-cocyles are the same, hence the "-1" at $[a, a]$. Nonetheless, no 2-cocyles persist.

Table 4.1: A table showing examples of finite lattices P and P^{op}, the filtrations $F : P \to \Delta K$ and cofiltration $G : P^{\text{op}} \to \nabla K$, and the interval lattices $\overline{P}$ and $\overline{P^{\text{op}}}$. We use this information to compute the persistence diagrams shown in Table 4.2

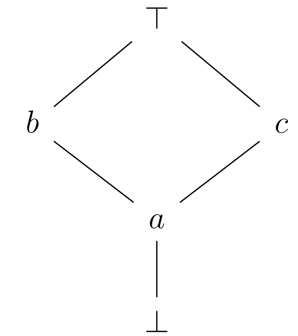
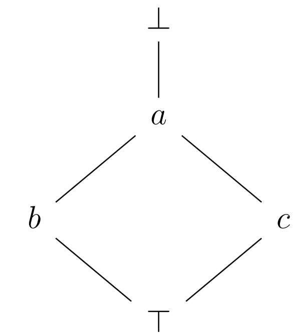
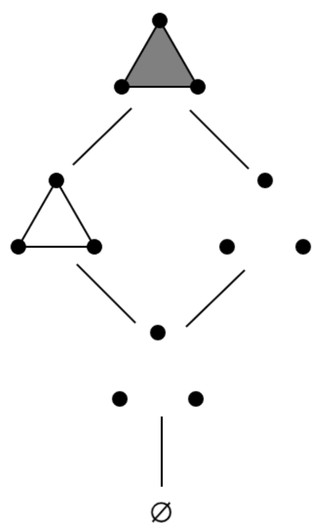
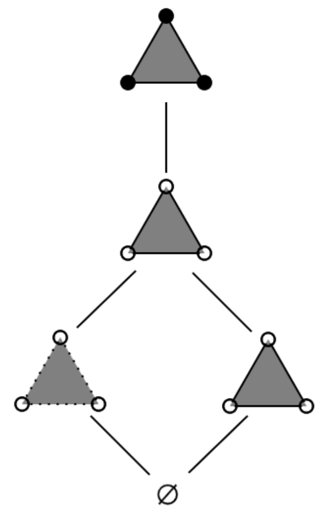
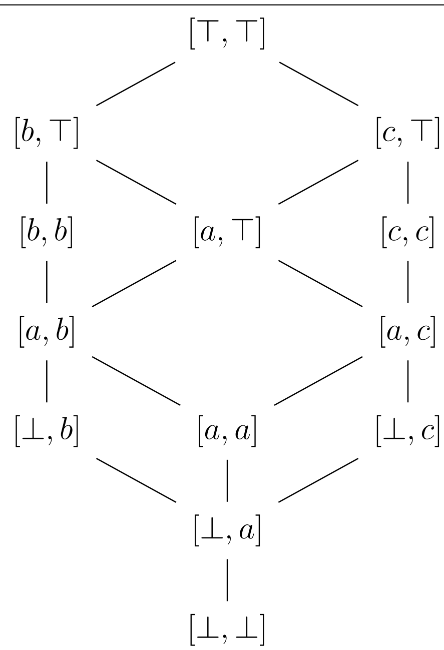
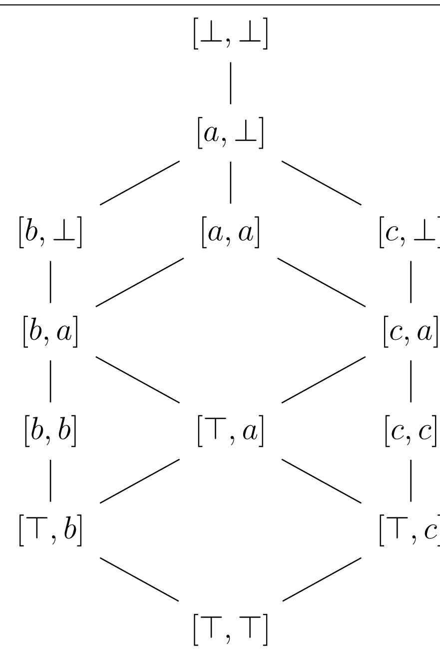
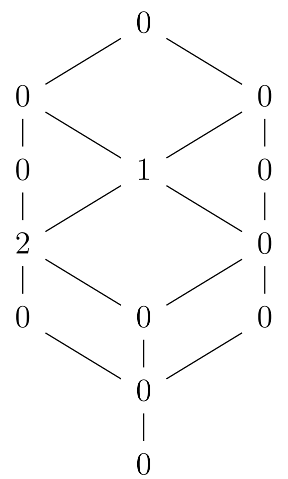
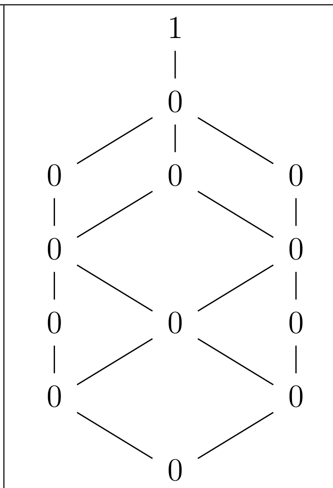
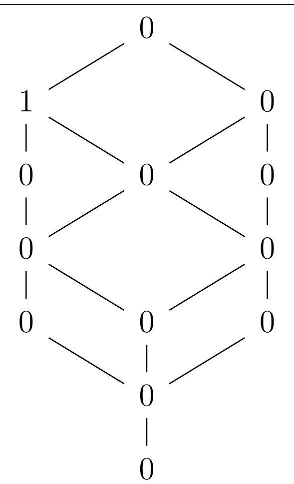
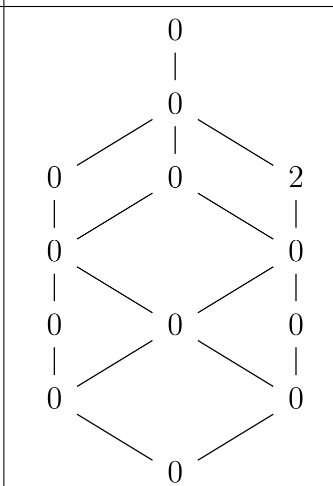
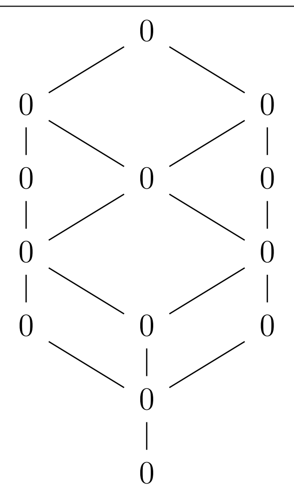
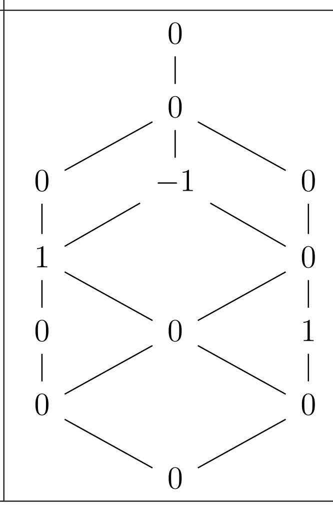
<table>
<tr><th>Lattices</th><th>$F : P \to \Delta K$</th><th>$G : P^{\text{op}} \to \nabla K$</th></tr>
<tr><td>P, P^{op}</td><td>
</td><td>
</td></tr>
<tr><td>$F(P), G(P^{\text{op}})$</td><td>
</td><td>
</td></tr>
<tr><td>$\overline{P}, \overline{P^{\text{op}}}$</td><td>
</td><td>
</td></tr>
</table>

Table 4.2: A table showing the persistence diagrams (for dimensions $n = 0, 1, 2$) arising from the filtration F and cofiltration G in Table 4.1.

<table>
<tr><th>n</th><th>$\mathrm{PH}_n(F)$</th><th>$\mathrm{PH}^n(G)$</th></tr>
<tr><td>0</td><td>
</td><td>
</td></tr>
<tr><td>1</td><td>
</td><td>
</td></tr>
<tr><td>2</td><td>
</td><td>
</td></tr>
</table>

Chapter 5

Stability

Stability of classical persistence diagrams with respect to the bottleneck distance was first proved by Cohen-Steiner, Edelsbrunner, and Harer in [4]. In [6], the authors show that generalized persistence diagrams are stable with respect to the edit distance. In this chapter, we prove the same stability theorem for the persistence diagram admitted by the Möbius inversion of the birth-death functor for cohomology of a cofiltration.

5.1 Edit Distance

Before making a statement about stability of persistence diagrams, we must first discuss the edit distance in each of our three main categories: $\mathrm{CoFil}(K)$, Mon, and Fnc.

Definition 5.1.1. (Section 7.1 in [6]) A **path between two cofiltrations** $F, G \in \mathrm{CoFil}(K)$ is a finite sequence of morphisms

$$F \overset{\alpha_1}{\leftrightarrow} H_1 \overset{\alpha_2}{\leftrightarrow} \cdots \overset{\alpha_{n-1}}{\leftrightarrow} H_{n-1} \overset{\alpha_n}{\leftrightarrow} G,$$

where $\leftrightarrow$ means the cofiltration preserving morphism is in either direction. The **length** of a path is the sum of the distortions of the bounded lattice maps, $\sum_{i=1}^{n} ||\alpha_i||$ (see Definition 2.2.12).

If the distortion $||\alpha_i||$ is infinite for some i, then the length of the path is also infinite. Further, any two cofiltrations are connected by a path, as the cofiltration $I : \{\star\} \to \nabla K$ is the final object in $\mathrm{CoFil}(K)$, where $\{\star\}$ is the lattice of one element.

Definition 5.1.2. (Definition 7.1 in [6]) The **edit distance** $d_{\mathrm{CoFil}(K)}(F, G)$ between two cofiltrations $F, G \in \mathrm{CoFil}(K)$ is the length of the shortest path between F and G.

Definition 5.1.3. (Section 7.2 in [6]) A **path between two monotone functions** $f, g \in \mathrm{Mon}$ is a finite sequence of morphisms

$$f \overset{\bar{\alpha}_1}{\leftrightarrow} h_1 \overset{\bar{\alpha}_2}{\leftrightarrow} \cdots \overset{\bar{\alpha}_{n-1}}{\leftrightarrow} h_{n-1} \overset{\bar{\alpha}_n}{\leftrightarrow} g,$$

where $\leftrightarrow$ means the monotone-preserving morphism is in either direction. The **length** of a path is the sum of the distortions of the bounded lattice maps, $\sum_{i=1}^{n} ||\bar{\alpha}_i||$.

Given two monotone integer functions $f : \overline{P} \to \mathbb{Z}$ and $g : \overline{Q} \to \mathbb{Z}$ such that $f[\top, \top] = g[\top, \top]$, there always exists a path between f and g. After all, suppose $f[\top, \top] = n = g[\top, \top]$, and let $e : \overline{\{\star\}} \to \mathbb{Z}$ be the monotone integer function given by $e[\star, \star] = n$. Then, there is a unique monotone-integral function from f to e, given by the bounded lattice map that sends all intervals in $\overline{P}$ to $[\star, \star]$. The same is true for g: that is, there is a unique monotone-integral function from g to e. Thus, we have the following path between f and g: $f \to e \leftarrow g$.

Definition 5.1.4. (Definition 7.2 in [6]) The **edit distance** $d_{\mathrm{Mon}}(f, g)$ between two monotone functions $f, g \in \mathrm{Mon}$ is the length of the shortest path between f and g. If there are no paths, then set $d_{\mathrm{Mon}}(f, g) = \infty$.

Lemma 5.1.1. *(Lemma 7.3 in [6]) Let* $F, G \in \mathrm{CoFil}(K)$*. Then, for every* n*,*

$$d_{\mathrm{Mon}}(\mathrm{ZB}^n F, \mathrm{ZB}^n G) \leq d_{\mathrm{CoFil}(K)}(F, G).$$

Proof. Let $d_{\mathrm{CoFil}(K)}(F, G) = \epsilon$. Then, there exists a path of length ϵ in $\mathrm{CoFil}(K)$ between F, G. Applying ZB^n to every morphism in the path gives a path in Mon between $\mathrm{ZB}^n F$ and $\mathrm{ZB}^n G$. By Lemma 2.2.1, the length of this path is also ϵ. Thus, $d_{\mathrm{Mon}}(\mathrm{ZB}^n F, \mathrm{ZB}^n G) \leq d_{\mathrm{CoFil}(K)}(F, G)$. □

Definition 5.1.5. (Section 7.3 in [6]) A **path between two integral functions** $\sigma, \tau \in \mathrm{Fnc}$ is a finite sequence of morphisms

$$\sigma \overset{\bar{\alpha}_1}{\leftrightarrow} \theta_1 \overset{\bar{\alpha}_2}{\leftrightarrow} \cdots \overset{\bar{\alpha}_{n-1}}{\leftrightarrow} \theta_{n-1} \overset{\bar{\alpha}_n}{\leftrightarrow} \tau,$$

where $\leftrightarrow$ means the charge-preserving morphism is in either direction. The **length** of a path is the sum of the distortions of the bounded lattice maps, $\sum_{i=1}^{n} ||\bar{\alpha}_i||$.

We also have that any two integral functions $\sigma : \overline{P} \to \mathbb{Z}$ and $\tau : \overline{Q} \to \mathbb{Z}$ are connected by a path. After all, let $\omega : \overline{\{\star\}} \to \mathbb{Z}$ be any integral function. From the definition of a charge-preserving morphism (Definition 2.5.9), we see that ω is terminal in the category Fnc.

Definition 5.1.6. (Definition 7.4 in [6]) The **edit distance** $d_{\mathrm{Fnc}}(\sigma, \tau)$ between two integral functions $\sigma, \tau \in \mathrm{Fnc}$ is the length of the shortest path between σ and τ.

Lemma 5.1.2. *(Lemma 7.5 in [6]) Let $f, g \in \mathrm{Mon}$. Then, for every n,*

$$d_{\mathrm{Fnc}}(\mathrm{MI}(f), \mathrm{MI}(g)) \leq d_{\mathrm{Mon}}(f, g).$$

See [6] for the proof of Lemma 5.1.2.

Theorem 5.1.1. (Stability, Theorem 8.4 in [6]) Given a finite simplicial K and two cofiltrations $F : P \to \nabla K$ and $G : Q \to \nabla K$, then

$$d_{\mathrm{Fnc}}(\mathrm{PH}^n(F), \mathrm{PH}^n(G)) \leq d_{\mathrm{CoFil}(K)}(F, G).$$

Proof. This theorem follows immediately from Lemmas 5.1.1 and 5.1.2. □

That is, with respect to the edit distance in the respective categories, the edit distance between two cofiltrations F and G is an upper bound for the edit distance between the persistence diagrams $\mathrm{PH}^n(F)$ and $\mathrm{PH}^n(G)$ for any n.

Chapter 6

Conclusion

In this thesis, we defined cofiltrations of a finite simplicial complex K indexed over a finite lattice. Then, we discussed the persistence diagram for cohomology of a cofiltration using the Möbius inversion of the birth-death function. In fact, these functions are functorial and stable with respect to the edit distance. In summary, we have provided a mathematical framework in which to study persistent cohomology of a cofiltartion. To finish, we will discuss the usefulness of persistent cohomology and outline potential directions for future work.

One reason for studying persistent cohomology is the ring structure obtained by equipping cohomology groups with the cup product. For example, the cohomology *groups* of the spaces $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and $\mathbb{T}^1$ are isomorphic. However, when endowed with the cup product, the cohomology ring structures of the two spaces are distinguishable. Work has been done to utilize the extra structure of cohomology in persistence. For example, the authors in [10] define the persistent cup-length formula and the persistent cup-length diagram. However, this definition requires the existence of a barcode, which we lose when considering multiparameter persistence. Nonetheless, we hope there is an analogous notion of the persistent cup-length for our more general notion of persistent cohomology of a cofiltration.

Further research could also be done in defining different diagrams for persistent cohomology as well. For example, in [7], the authors define and compare three different diagrams coming from Möbius inversions of the rank function, kernel function, and the birth-death function. We have discussed the last diagram in this thesis. Could we prove similar relationships between the three corresponding diagrams for persistent cohomology of a cofiltration? How would the diagrams utilizing cofiltrations compare to the diagrams arising from the persistence modules obtained by applying cohomology to a filtration?

Finally, it would be interesting to consider duality theorems pertaining to the work in this thesis. In [3], the authors present various duality theorems for persistent (co)homology in the 1-parameter setting. Which of these duality theorems, in any, hold in the multiparameter setting?

Overall, persistent cohomology of a 1-parameter filtartion has already been utilized in various applications, and there are interesting theoretical results describing the relationship between 1-parameter persistent homology and cohomology. In this thesis, we have defined persistent cohomology of a cofiltration indexed over a finite lattice. In the future, we are hopeful that multiparameter persistent cohomology will be useful in applications, and that interesting theoretical results arise from the definitions we have made.

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