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Restrict Nearly Semiprime Submodule

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المقاسات الجزئية شبه الاولية القريبة المقصورة

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ABSTRACT

During this search all rings are commutative and all modules are unitary. In this search we introduced the concept of Restrict Nearly semi-prime Sub-modules as generalization of semi-prime Sub-modules and give some basic properties, examples and charactarizations of this concepts and stablished some sufficient conditions on Restrict Nearly semi-prime to be semi-prime.

Key words: Semi-prime Sub-module, Restrict Nearly semi-prime Sub-module, (distributive, semi simple and injective) module.

الخلاصة

أثناء هذا البحث، تكون جميع الحلقات تبادلية وجميع الوحدات أحادية. في هذا البحث، قدمنا مفهوم تقييد الوحدات الفرعية شبه الأولية تقريبًا كتعميم للوحدات الفرعية شبه الأولية وإعطاء بعض الخصائص الأساسية والأمثلة والتشخيصات لهذه المفاهيم ووضعنا بعض الشروط الكافية على تقييد شبه أولي شبه أولي ليكون شبه رئيس.

الكلمات المفتاحية:

الوحدة الفرعية شبه الأولية، تقييد الوحدة الفرعية شبه الأولية تقريبًا ، (التوزيع ، شبه البسيط والحقن).

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مـجلــة جـــامعة بــــابــل للعلــــــوم الصـــرفــة والتطبيقية مـجلــة جـــامعة بــــابــل للعلــــــوم الصـــرفــة والتطبيقية مـجلــة جـــامعة بــــابــل للعلــــــوم الصـــرفــة والتطبيقية

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INTRODUCTION

In the beginning, the later concept is a generalization of the concept of prime Sub-module. This concept was generalized to semi-prime by Athab in 1996 where a proper Sub-module $\mathcal{A}$ of an $\mathcal{R}-module\ \mathcal{W}$ is named semi-prime if whenever $\mathcal{S}^{\kappa}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{W}$, κ is a positive integer, it means that $\mathcal{S}\omega \in \mathcal{A}$[2]. Recently Nuhad 2015 gives a new generalization of a semi-prime Sub-modules, namely Nearly Semi prime where a proper Sub-module $\mathcal{A}$ of an $\mathcal{R}-module$ $\mathcal{W}$ is named Nearly semi-prime if whenever $\mathcal{S}^{\kappa}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{W}$, $\kappa \in Z^+$, it means that $\mathcal{S}\omega \in \mathcal{A} + J(\mathcal{W})$[6]. Where $J(\mathcal{W})$ is the Jacobson radical of $\mathcal{W}$ defined to be the intersaction of all maximall Sub-module of $\mathcal{W}$ (if $\mathcal{W}$ has no maximal Submodules then $J(\mathcal{W})$=$\mathcal{W}$) or the sum of all small Submodule of $\mathcal{W}$[4]. In 2019, a new generalizations of semi-prime Submodules where introduced by Ali namely approximaitly semi-prime where a proper Sub-module $\mathcal{A}$ of an $\mathcal{R}-module\ \mathcal{W}$ is an approximaitly semi-prime Sub-module of $\mathcal{W}$, if whenever $\mathcal{S}^{\kappa}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{W}$, $\kappa \in Z^+$, it means that $\mathcal{S}\omega \in \mathcal{A} + Soc(\mathcal{W})$[1]. Where $Soc(\mathcal{W})$ is the Socal of $\mathcal{W}$ defined to be the intersection of all essential Sub-module of $\mathcal{W}$ [3], where a nonzero Sub-module $\mathcal{A}$ of $\mathcal{W}$ is named an essential if $\mathcal{A} \cap \beta \neq 0$ for each nonzero Sub-module β of $\mathcal{W}$ [3]. In this work we introduce the concept "Restrict Nearly semi-prime Sub-module as a new generalization of a semi-prime Sub-modules and we give some basic properties examples of this new concept and stablished some sufficient conditions on Restrict Nearly semiprime to be semiprime.Submodule. We denoted to the Sub-module by S-module.

2. Restrict Nearly Semiprime S-modules

We recalled the definition of Restrict Nearly semi-prime S-module as new generalizetion of semi-prime S-module.

Defintion 2.1

A proper S-module $\mathcal{A}$ of an $\mathcal{R}-module\ \mathcal{W}$ is named a Restrict Nearly semi-prime (for short RNSP) S-module of $\mathcal{W}$, if whenever $\mathcal{S}^{\kappa}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{W}$, κ is a positive integer, it means that $\mathcal{S}\omega \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$.

Remark 2.2

Every semiprime S-module of an $\mathcal{R}-module\ \mathcal{W}$ is a RNSP S-module, but not conversely.

Proof

Assume that $\mathcal{S}^{\kappa}\omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{W}$ and κ is a positive integer. But $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$, it follows that $\mathcal{S}\omega \in \mathcal{A} \subseteq \mathcal{A} + Soc(\mathcal{W}) \cap J(\mathcal{W})$. Hence $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$.

The opposite in the next example:

Example 2.3

Consider the $Z-module\ Z_{24}$, the S-module $\mathcal{A} = \langle\bar{8}\rangle$ of Z_{24} is not semiprime, since $2^2.\bar{2} \in \mathcal{A} = \langle\bar{8}\rangle$, for $2 \in Z$, $\bar{2} \in Z_{24}$, but $2.\bar{2} = \bar{4} \notin \langle\bar{8}\rangle$. On the other hand $<\bar{8}>$ is a RNSP S-module

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of the Z-module Z_{24} since $Soc(Z_{24}) = \langle\bar{4}\rangle$ and $J(Z_{24}) = \langle\bar{6}\rangle$. Hence $Soc(Z_{24}) \cap J(Z_{24}) = \langle\bar{4}\rangle \cap \langle\bar{6}\rangle = \langle\overline{12}\rangle$. whenever $\mathcal{S}^2 \omega \in \mathcal{A}$, for $\mathcal{S} \in Z$, $\omega \in Z_{24}$, it means that $\mathcal{S}\omega \in \mathcal{A} + (Soc(Z_{24}) \cap J(Z_{24})) = \langle\bar{4}\rangle$. That is if $2^2.\bar{2} \in \mathcal{A}$, it means that $2.\bar{2} = \bar{4} \in \mathcal{A} + (Soc(Z_{24}) \cap J(Z_{24})) = \langle\bar{8}\rangle + \langle\overline{12}\rangle = \langle\bar{4}\rangle$.

Remark 2.4

If $\mathcal{A}_1, \mathcal{A}_2$ are S-modules of an $\mathcal{R}-module\ \mathcal{W}$ such that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. If $\mathcal{A}_2$ is a RNSP S-module of $\mathcal{W}$, but $\mathcal{A}_1$ need not to be RNSP S-module of $\mathcal{W}$.

The following example demonstrates this:

We need to remember the next remark before we present the Next example:

Remark 2.5 [2, Rems. and Exaps. (1.3)]

Every prime S-module of an $\mathcal{R}-module\mathcal{W}$ is a semi-prime.

Example 2.6

In the Z-module Z_{16}, we see that the essential S-modules of Z_{16} are Z_{16} itself and the S-modules $\langle\bar{2}\rangle$, $\langle\bar{4}\rangle$ and $\langle\bar{8}\rangle$ so $Soc(Z_{16}) = Z_{16} \cap \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle = \langle\bar{8}\rangle$. And the only maximal S-modules of Z_{16} is $\langle\bar{2}\rangle$, so $J(Z_{16}) = Z_{16} \cap \langle\bar{2}\rangle = \langle\bar{2}\rangle$. Hence $Soc(Z_{16}) \cap J(Z_{16}) = \langle\bar{8}\rangle \cap \langle\bar{2}\rangle = \langle\bar{8}\rangle$. Now $\langle\bar{2}\rangle$ is prime S-module then by remark (**2.5**) $\langle\bar{2}\rangle$ is a semiprime S-module, now by remark (**2.2**) $\langle\bar{2}\rangle$ is a RNSP S-module and $\langle\bar{8}\rangle$ is not RNSP S-module as in the following $2^2.\bar{2} \in \langle\bar{8}\rangle$, implies that $2.\bar{2} = \bar{4} \notin \langle\bar{8}\rangle + \left(Soc(Z_{16}) \cap J(Z_{16})\right) = \langle\bar{8}\rangle + \langle\bar{8}\rangle = \langle\bar{8}\rangle$.The S-modules
$\mathcal{A}_1 = \langle\bar{8}\rangle$, $\mathcal{A}_2 = \langle\bar{2}\rangle$ of the Z-module Z_{16}, for $\langle\bar{8}\rangle \subseteq \langle\bar{2}\rangle$,we show that from above $\langle\bar{2}\rangle$ is RNSP S-module of Z_{16}. But $\langle\bar{8}\rangle$ is not RNSP S-module of Z_{16}.

The next propositions are characterizations of RNSP S-modules.

Proposition 2.7

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ be a S-module of $\mathcal{W}$. Then $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ iff $\mathcal{J}^{\kappa}\beta \subseteq \mathcal{A}$ for every S-module β of $\mathcal{W}$ and every ideal $\mathcal{J}$ of $\mathcal{R}$, κ is a positive integer, it means that $\mathcal{J}\beta \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$.

Proof

$\Longrightarrow$ Assume that $\mathcal{S} \in \mathcal{J}\beta$, it means that $\mathcal{S} = \omega_1\mathcal{S}_1 + \omega_2\mathcal{S}_2 + \omega_3\mathcal{S}_3 + \ldots + \omega_t\mathcal{S}_t$, for $\omega_i \in \mathcal{J}, \mathcal{S}_i \in \beta_i$=1, 2, …, t, it follows that ${\omega_i}^t\ \mathcal{S}_i \in \mathcal{J}^{\kappa}\ \beta \subseteq \mathcal{A}$ that is ${\omega_i}^t\ \mathcal{S}_i \in \mathcal{A}$, since $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, it follows that $\omega_i\mathcal{S}_i \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$, for each i=1, 2, …, t. thus $\omega_1\mathcal{S}_1 + \omega_2\mathcal{S}_2 + \omega_3\mathcal{S}_3 + \ldots + \omega_t\mathcal{S}_t \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$,
therefore $\mathcal{S} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$. Hence $\mathcal{J}\ \beta \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$.

$\Longleftarrow$ Assume that $\mathcal{S}^{\kappa}\ \omega \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\omega \in \mathcal{W}$ and κ is a positive integer. It follows that $\langle\mathcal{S}\rangle^{\kappa}\mathcal{R}\ \omega \subseteq \mathcal{A}$. And so by our assumption we have $\langle\mathcal{S}\rangle\mathcal{R}\ \omega \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$ that is $\mathcal{S}\ \omega \in \langle\mathcal{S}\rangle\mathcal{R}\ \omega \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$, it means that $\mathcal{S}\omega \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$. Hence $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$.

As a direct result of proposition (**2.7**) we get the next corollaries.

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Corollary 2.8

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ be a S-module of $\mathcal{W}$. Then $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ iff $\mathcal{J}^2\beta \subseteq \mathcal{A}$ for every ideal $\mathcal{J}$ of $\mathcal{R}$, it means that $\mathcal{J}\beta \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$.

Corollary 2.9

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ be a S-module of $\mathcal{W}$. Then $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ iff $\mathcal{J}^{\kappa}\mathcal{W} \subseteq \mathcal{A}$ for every ideal $\mathcal{J}$ of $\mathcal{R}$, κ is a positive integer, it means that $\mathcal{J}\mathcal{W} \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$.

Corollary 2.10

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ be a S-module of $\mathcal{W}$. Then $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ iff $\mathcal{J}^{\kappa}\text{w} \subseteq \mathcal{A}$ for every ideal $\mathcal{J}$ of $\mathcal{R}$, and $\text{w} \in \mathcal{W}$, κ is a positive integer, it means that $\mathcal{J}\text{w} \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$.

Recall that the module $\mathcal{W}$ is said distributive if $\mathcal{A} \cap (\beta + C) = (\mathcal{A} \cap \beta) + (\mathcal{A} \cap C)$ or $\mathcal{A} + (\beta \cap C) = (\mathcal{A} + \beta) \cap (\mathcal{A} + C)$, for all S-modules $\mathcal{A}$, β and C.

Proposition 2.11

Let $\mathcal{W}$ be a distributive $\mathcal{R}-module$, and $\mathcal{A}$, β are S-modules of $\mathcal{W}$ such that $\beta \subseteq \mathcal{A}$. If $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, then $\frac{\mathcal{A}}{\beta}$ is a RNSP S-module of $\frac{\mathcal{W}}{\beta}$.

Proof

Let $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, and $\mathcal{S}^{\kappa}(\text{w} + \beta) = \mathcal{S}^{\kappa}\text{w} + \beta \in \frac{\mathcal{A}}{\beta}$, for $\mathcal{S} \in \mathcal{R}$, $\text{w} + \beta \in \frac{V}{\beta}$, $\text{w} \in \mathcal{W}$, κ is a positive integer. Then $\mathcal{S}^2\text{w} \in \mathcal{A}$, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, it means that $\mathcal{S}\text{w} \in \mathcal{A} + (\text{Soc}(\mathcal{W}) \cap J(\mathcal{W}))$. It follows that $\mathcal{S}\text{w} + \beta \in \frac{\mathcal{A}+(\text{Soc}(\mathcal{W})\cap J(\mathcal{W}))}{\beta}$, that is $\mathcal{S}\text{w} + \beta \in \frac{\mathcal{A}}{\beta} + \frac{\mathcal{A}+(\text{Soc}(\mathcal{W})\cap J(\mathcal{W}))}{\beta}$, but $\mathcal{W}$ is distributive module, then $\mathcal{S}\text{w} + \beta \in \frac{\mathcal{A}}{\beta} + \frac{\mathcal{A}+(\text{Soc}(\mathcal{W})\cap J(\mathcal{W}))}{\beta} = \frac{\mathcal{A}}{\beta} + (\frac{\mathcal{A}+\text{Soc}(\mathcal{W})}{\beta} \cap \frac{\mathcal{A}+J(\mathcal{W})}{\beta}) \subseteq \frac{\mathcal{A}}{\beta} + (\text{Soc}(\frac{\mathcal{W}}{\beta}) \cap J(\frac{\mathcal{W}}{\beta}))$, therefore $\mathcal{S}\text{w} + \beta \in \frac{\mathcal{A}}{\beta} + \text{Soc}\left(\frac{\mathcal{W}}{\beta}\right) \cap J\left(\frac{\mathcal{W}}{\beta}\right)$. Thus $\mathcal{S}(\text{w} + \beta) \in \frac{\mathcal{A}}{\beta} + (\text{Soc}\left(\frac{\mathcal{W}}{\beta}\right) \cap J\left(\frac{\mathcal{W}}{\beta}\right))$. Hence $\frac{\mathcal{A}}{\beta}$ is a RNSP S-module of $\frac{\mathcal{W}}{\beta}$.

We need to remember the next proposition and remark before we present the following results:

Proposition 2.12 $[4, \text{EX}(12), \text{P}, 239]$

Let $\mathcal{A}$ be a S-module of an $\mathcal{R}-module\mathcal{W}$ with $\mathcal{A}$ is a direct summand of $\mathcal{W}$, then $J(\frac{\mathcal{W}}{\mathcal{A}}) = \frac{J(\mathcal{W})+\mathcal{A}}{\mathcal{A}}$.

Remark 2.13 $[4, \text{ Ex.}(12)\text{C}]$

The module $\mathcal{W}$ is semi simple iff for each S-module $\mathcal{A}$ of $\mathcal{W}$, $Soc(\frac{\mathcal{W}}{\mathcal{A}}) = \frac{Soc(\mathcal{W})+\mathcal{A}}{\mathcal{A}}$.

Now, we give the convers of proposition (**2.11**).

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Proposition 2.14

Let $\mathcal{W}$ be a semi simple distributive $\mathcal{R}-module$, and $\mathcal{A},\beta$ are S-modules of $\mathcal{W}$ such that $\beta \subseteq \mathcal{A}$. If β and $\frac{\mathcal{A}}{\beta}$ are RNSP S-modules of $\mathcal{W}$ and $\frac{\mathcal{W}}{\beta}$ respectively, then $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$.

Proof

Assume that β and $\frac{\mathcal{A}}{\beta}$ are RNSP S-modules of $\mathcal{W}$ and $\frac{\mathcal{W}}{\beta}$ respectively, and $\mathcal{S}^{\kappa}w \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $w \in \mathcal{W}$, κ is a positive integer. So $\mathcal{S}(w+\beta) = \mathcal{S}^{\kappa}w + \beta \in \frac{\mathcal{A}}{\beta}$. If $\mathcal{S}^{\kappa}w \in \beta$ and β is a RNSP S-module of $\mathcal{W}$ and we have $\beta \subseteq \mathcal{A}$, it means that $\mathcal{S}w \in \beta + (Soc(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$, therefore $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$. So, we may assume that $\mathcal{S}^{\kappa}w \notin \beta$. It follows that $\mathcal{S}^{\kappa}(w+\beta) \in \frac{\mathcal{A}}{\beta}$, but $\frac{\mathcal{A}}{\beta}$ is RNSP S-module of $\frac{\mathcal{W}}{\beta}$**,** it means that $\mathcal{S}w + \beta \in \frac{\mathcal{A}}{\beta} + (Soc\left(\frac{\mathcal{W}}{\beta}\right) \cap J\left(\frac{\mathcal{W}}{\beta}\right))$. Since $\mathcal{W}$ is a semi simple, then by remark (**2.13**) $Soc(\frac{\mathcal{W}}{\beta}) = \frac{\beta+Soc(\mathcal{W})}{\beta}$, and β is direct summand because $\mathcal{W}$ is a semi simple, then by propostion (**2.12**), $J(\frac{\mathcal{W}}{\beta}) = \frac{J(\mathcal{W})+\beta}{\beta}$, hence $\mathcal{S}w + \beta \in \frac{\mathcal{A}}{\beta} + (\frac{\beta+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\beta}{\beta})$**.** But $\mathcal{W}$ is distributive module, then $\frac{\mathcal{A}}{\beta} + (\frac{\beta+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\beta}{\beta})$ $=(\frac{\mathcal{A}}{\beta} + \frac{\beta+Soc(\mathcal{W})}{\beta}) \cap (\frac{\mathcal{A}}{\beta} + \frac{J(\mathcal{W})+\beta}{\beta})$ Since $\beta \subseteq \mathcal{A}$, it follows that $\beta + Soc(\mathcal{W}) \subseteq \mathcal{A} + Soc(\mathcal{W})$ and $\beta + J(\mathcal{W}) \subseteq \mathcal{A} + J(\mathcal{W})$ hence $\frac{\mathcal{A}}{\beta} + \frac{\beta+Soc(\mathcal{W})}{\beta} \subseteq \frac{\mathcal{A}}{\beta} + \frac{\mathcal{A}+Soc(\mathcal{W})}{\beta}$ and $\frac{\mathcal{A}}{\beta} + \frac{\beta+J(\mathcal{W})}{\beta} \subseteq \frac{\mathcal{A}}{\beta} + \frac{\mathcal{A}+J(\mathcal{W})}{\beta}$, therefore $\frac{\mathcal{A}}{\beta} + (\frac{\beta+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\beta}{\beta}) \subseteq \frac{\mathcal{A}}{\beta} + (\frac{\mathcal{A}+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\mathcal{A}}{\beta})$, since $\frac{\mathcal{A}}{\beta} \subseteq \frac{\mathcal{A}+Soc(\mathcal{W})}{\beta}$ and $\frac{\mathcal{A}}{\beta} \subseteq \frac{\mathcal{A}+J(\mathcal{W})}{\beta}$ then $\frac{\mathcal{A}}{\beta} \subseteq (\frac{\mathcal{A}+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\mathcal{A}}{\beta})$, it means that $\frac{\mathcal{A}}{\beta} + (\frac{\mathcal{A}+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\mathcal{A}}{\beta}) = (\frac{\mathcal{A}+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\mathcal{A}}{\beta})$. Thus $\mathcal{S}w + \beta \in (\frac{\mathcal{A}+Soc(\mathcal{W})}{\beta} \cap \frac{J(\mathcal{W})+\mathcal{A}}{\beta})$, it follows that $\mathcal{S}w \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$. Hence $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$.

We need to remember the next propositions and lemma before we present the following results:

Proposition 2.15 [4, Theo. (9.1.4)(a)]

Let $f: \mathcal{W} \longrightarrow \mathcal{A}$ be an $\mathcal{R}$-homomorphism, then $f(Soc(\mathcal{W})) \subseteq Soc(\mathcal{A})$.

Proposition 2.16 [4, Lem. (3.1.10)]

Let $f: \mathcal{W} \longrightarrow \mathcal{W}'$ be an $\mathcal{R}$-homomorphism, and $\{\mathcal{A}_i : i \in I\}$, $\{\mathcal{A}_i{}' : i \in I\}$ are families of S-modules of $\mathcal{W}$ and $\mathcal{W}'$ respectively then:

a) $f(\sum_{i\in I} \mathcal{A}_i) = \sum_{i\in I} f(\mathcal{A}_i)$; $f^{-1}(\bigcap_{i\in I} \mathcal{A}_i{}') = \bigcap_{i\in I} f^{-1}(\mathcal{A}_i{}')$.

b) $f^{-1}(\sum_{i\in I} \mathcal{A}_i{}') \supseteq \sum_{i\in I} f^{-1}(\mathcal{A}_i{}')$; $f(\bigcap_{i\in I} \mathcal{A}_i) \subseteq \bigcap_{i\in I} f(\mathcal{A}_i)$.

c) Let now $\mathcal{A}_i{}' \subseteq Im\, f$ for all $i \in I$, then $f^{-1}(\sum_{i\in I} \mathcal{A}_i{}') = \sum_{i\in I} f^{-1}(\mathcal{A}_i{}')$. And if $ker\, f \subseteq \mathcal{A}_i$ for all $i \in I$, then $f(\bigcap_{i\in I} \mathcal{A}_i) = \bigcap_{i\in I} f(\mathcal{A}_i)$.

Lemma 2.17 [4, Coro. (9.1.5)(a)]

If $f: \mathcal{W} \longrightarrow \mathcal{W}'$be an $\mathcal{R}$-epimorphism and $Ker\ f$ is a small S-module of $\mathcal{W}$, then $f(J(\mathcal{W})) = J(\mathcal{W}')$ and $f^{-1}(J(\mathcal{W}')) = J(\mathcal{W})$.

Proposition 2.18

Let $f: \mathcal{W} \to \mathcal{W}'$ be an $\mathcal{R}$-epimorphism and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}'$ with $ker\ f$ is a small . Then $f^{-1}(\mathcal{A})$ is a RNSP S-module of $\mathcal{W}$.

Proof

It is clear that $f^{-1}(\mathcal{A})$ is a proper S-module of $\mathcal{W}$. Now, assume that $\mathcal{S}^{\kappa}\text{w} \subseteq f^{-1}(\mathcal{A})$, for $\mathcal{S} \in \mathcal{R}$, $\text{w} \in \mathcal{W}$ and κ is a positive integer, then $\mathcal{S}^{\kappa} f(\text{w}) \in \mathcal{A}$. But $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}'$, it followse that $\mathcal{S}f(\text{w}) \in \mathcal{A} + (\text{Soc}(\mathcal{W}') \cap J(\mathcal{W}'))$. that is by lemma (**2.17**)
$\mathcal{S}\text{w} \in f^{-1}[\mathcal{A} + (\text{Soc}(\mathcal{W}') \cap J(\mathcal{W}'))] \subseteq f^{-1}(\mathcal{A}) + f^{-1}\big(\text{Soc}(\mathcal{W}') \cap J(\mathcal{W}')\big) = f^{-1}(\mathcal{A}) + f^{-1}(\text{Soc}(\mathcal{W}')) \cap f^{-1}(J(\mathcal{W}'))$, hence by using propostions (**2.15**) and (**2.16**) $\mathcal{S}\text{w} \in f^{-1}(\mathcal{A}) + (f^{-1}\big(\text{Soc}(\mathcal{W}')\big) \cap f^{-1}\big(\text{J}(\mathcal{W}')\big)) = f^{-1}(\mathcal{A}) + \big(\text{Soc}(\mathcal{W}) \cap J(\mathcal{W})\big)$. Thus $f^{-1}(\mathcal{A})$ is a RNSP S-module of $\mathcal{W}$.

Proposition 2.19

Let $f: \mathcal{W} \to \mathcal{W}'$ be an $\mathcal{R} - epimorphism$ and $\mathcal{A}$ be a RNSP S-module of $\mathcal{W}$ with $ker\ f \subseteq \mathcal{A}$ and $ker\ f$ is a small. Then $f(\mathcal{A})$ is a RNSP S-module of $\mathcal{W}'$.

Proof

It is clear that $f(\mathcal{A})$ is a proper S-module of $\mathcal{W}$. Now, assume that $\mathcal{J}^2\text{w}' \subseteq f(\mathcal{A})$, for $\mathcal{J}$ is an ideal of $\mathcal{R}$, $\text{w}' \in \mathcal{W}'$, then $\mathcal{J}^2 f(\text{w}) \subseteq f(\mathcal{A})$. Now f is epimorphism then $f(\mathcal{J}^2\text{w}) \subseteq f(\mathcal{A})$, it means that
$f(\mathcal{J}^2\text{w}) \subseteq f(t)$ for some non-zero $t \in \mathcal{A}$, it follows that $f(\mathcal{J}^2\text{w} - t) = 0$ that is $\mathcal{J}^2\text{w} - t \in Ker\ f \subseteq \mathcal{A}$, then $\mathcal{J}^2\text{w} \subseteq \mathcal{A}$, since $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, then by (**2.10**) it follows that $\mathcal{J}\text{w} \subseteq \mathcal{A} + (\text{Soc}(\mathcal{W}) \cap J(\mathcal{W}))$. thus $\mathcal{J}f(\text{w}) \subseteq f(\mathcal{A}) + f(\text{Soc}(\mathcal{W}) \cap J(\mathcal{W}))$, now by propostion (**2.16**) $\mathcal{J}f(\text{w}) \subseteq f(\mathcal{A}) + f(\text{Soc}(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{J}f(\text{w}) \subseteq f(\mathcal{A}) +$
$f(\text{Soc}(\mathcal{W})) \cap f(J(\mathcal{W}))$, hence by using propostion (**2.15**) $\mathcal{J}f(\text{w}) \subseteq f(\mathcal{A}) + (\text{Soc}(\mathcal{W}') \cap J(\mathcal{W}'))$. Therefore $f(\mathcal{A})$ is a RNSP S-module of $\mathcal{W}$.

3. Relationship between RNSP S-module and Semiprime S-module

In this part, we study the relationship RNSP S-module and Semiprime S-module

Proposition 3.1

Every semiprime S-module of an $\mathcal{R} - module\mathcal{W}$ is a RNSP S-module, but not conversely.
The proof is in remark (**2.2**) and Example by (**2.3**) .

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Proposition 3.2

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ with $(\mathrm{Soc}(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Proof

Assume that $\mathcal{A}$ be a RNSP S-module of $\mathcal{W}$, and $\mathcal{S}^{\kappa}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{W}$ and κ is a positive integer, it followsthat $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$, but we have $(\mathrm{Soc}(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A}$, then $\mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W})) = \mathcal{A}$, therefore $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A}$, thus $\mathcal{S}\mathrm{w} \in \mathcal{A}$. Hence $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Propostion 3.3

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ with either $Soc(\mathcal{W}) = 0$ or $J(\mathcal{W}) = 0$. Then $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Proof

Assume that $\mathcal{A}$ be a RNSP S-module of $\mathcal{W}$, and $\mathcal{S}^{\kappa}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{W}$ and κ is a positive integer. It follows that $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$, but we have if $Soc(\mathcal{W}) = 0$, then $\mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W})) = \mathcal{A} + (0 \cap J(\mathcal{W})) = \mathcal{A}$, therefore $\mathcal{S}\mathrm{w} \in \mathcal{A}$. Now if $J(\mathcal{W}) = 0$ by the same way we get $\mathcal{S}\mathrm{w} \in \mathcal{A}$. Hence $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

We need to remember the next propositions before we present the following results:

Propostion 3.4 $[3, \mathrm{Coro.}(9.9)]$

Let $\mathcal{A}$ be a S-module of an $\mathcal{R}-module$ $\mathcal{W}$, then $Soc(\mathcal{A}) = \mathcal{A} \cap Soc(\mathcal{W})$.

Propostion 3.5 $[4, prop, (9.1.4)b]$

If $\mathcal{A}$ is a S-module of an $\mathcal{R}-module$ $\mathcal{W}$ with $J(\frac{\mathcal{W}}{\mathcal{A}}) = 0$ then $J(\mathcal{W}) \subseteq \mathcal{A}$.

Propostion 3.6

Let $\mathcal{W}$ be an $\mathcal{R}-module$, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ with $J(\frac{M}{\mathcal{A}}) = \{0\}$ and $\mathrm{Soc}(\mathcal{A}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Proof

Assume that $\mathcal{S}^{\kappa}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{W}$ and κ is a positive integer. Since $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, it followsthat $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$. But we have $J(\frac{M}{\mathcal{A}}) = \{0\}$ then by propostion (**3.5**) $J(\mathcal{W}) \subseteq \mathcal{A}$ therefore $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A} + (\mathrm{Soc}(\mathcal{W}) \cap \mathcal{A})$, then by propostion (**3.4**) $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap \mathcal{A}) = \mathcal{A} + Soc(\mathcal{A}) = \mathcal{A}$. Hence $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Before we give the next result we need to recall the following remark:

Remark 3.7 $[4, EX(2)p.(217)]$

If an $\mathcal{R}-module\mathcal{W}$ has no maximal S-module then $J(\mathcal{W}) = \mathcal{W}$.

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Propostion 3.8

Let $\mathcal{W}$ be an $\mathcal{R}-module$ and $\mathcal{W}$ has no maximal S-module, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ with $\mathrm{Soc}(\mathcal{W}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Proof

Assume that $\mathcal{S}^{\kappa}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{W}$ and κ is a positive integer. Since $\mathcal{A}$ is a RNSP S-module, it followsthat $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$. But we have $\mathrm{Soc}(\mathcal{W}) \subseteq \mathcal{A}$, therefore $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A} + (\mathcal{A} \cap J(\mathcal{W}))$, now $\mathcal{W}$ has no maximal S-module then by remark (**3.7**) $J(\mathcal{W}) = \mathcal{W}$, thus $\mathcal{S}\mathrm{w} \in \mathcal{A} + (\mathcal{A} \cap J(\mathcal{W})) = \mathcal{A} + (\mathcal{A} \cap \mathcal{W}) = \mathcal{A} + \mathcal{A} = \mathcal{A}$, it means that $\mathcal{S}\mathrm{w} \in \mathcal{A}$. Hence $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Before we give the next result we need to recall the following remark:

Propostion 3.9 [5, *Lemma*(2.3)]

Let $\mathcal{W}$ be injective $\mathcal{R}-module$, then $J(\mathcal{W}) = \mathcal{W}$.

Propostion 3.10

Let $\mathcal{W}$ be injective $\mathcal{R}-module$, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$ with $\mathrm{Soc}(\mathcal{W}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Proof

Assume that $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, and $\mathcal{S}^{\kappa}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{W}$ and κ is a positive integer. Since $\mathcal{A}$ is a RNSP S-module, it followsthat $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$. But we have $\mathrm{Soc}(\mathcal{W}) \subseteq \mathcal{A}$, therefore $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W})) \subseteq \mathcal{A} + (\mathcal{A} \cap J(\mathcal{W}))$, now $\mathcal{W}$ is injective S-module then by propostion (**3.9**) $J(\mathcal{W}) = \mathcal{W}$, thus $\mathcal{S}\mathrm{w} \in \mathcal{A} + (\mathcal{A} \cap J(\mathcal{W})) = \mathcal{A} + (\mathcal{A} \cap \mathcal{W}) = \mathcal{A} + \mathcal{A} = \mathcal{A}$, it means that $\mathcal{S}\mathrm{w} \in \mathcal{A}$. Hence $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Before we give the next result we need to recall the following remark:

Propostion 3.11 [4, *Theo*(9.2.1)(*a*)(*h*)]

If $\mathcal{W}$ is a semi simple, then $J(\mathcal{W}) = 0$.

Propostion 3.12

Let $\mathcal{W}$ be a semi simple $\mathcal{R}-module$, and $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$. Then $\mathcal{A}$ is a semiprime S-module of $\mathcal{W}$.

Proof

Assume that $\mathcal{A}$ is a RNSP S-module of $\mathcal{W}$, and $\mathcal{S}^{\kappa}\mathrm{w} \in \mathcal{A}$, for $\mathcal{S} \in \mathcal{R}$, $\mathrm{w} \in \mathcal{W}$ and κ is a positive integer. Since $\mathcal{A}$ is a RNSP S-module, then $\mathcal{S}\mathrm{w} \in \mathcal{A} + (Soc(\mathcal{W}) \cap J(\mathcal{W}))$). But $\mathcal{W}$ be a semi simple then by propostion (**3.11**) $J(\mathcal{W}) = 0$ it means that $\mathcal{S}\mathrm{w} \in \mathcal{A} + (\mathrm{Soc}(\mathcal{W}) \cap 0) = \mathcal{A} + 0 = \mathcal{A}$, then $\mathcal{S}\mathrm{w} \in \mathcal{A}$. Hence $\mathcal{A}$ is a semi prime S-module of $\mathcal{W}$.

Conflict of interests.

There are non-conflicts of interest.

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