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# A Maple package to deal with the birationality of curves and surfaces parametrizations

Jorge Caravantes, Sonia Pérez-Díaz, and J. Rafael Sendra

Universidad de Alcalá,  
Dpto. Física y Matemáticas,  
Alcalá de Henares, Madrid, Spain.  
`jorge.caravantes@uah.es,sonia.perez@uah.es,rafael.sendra@uah.es`

**Abstract.** In this paper we present the Maple package `Luroth` for dealing with the birationality of curves and surfaces parametrizations. The procedures in the package decide whether a given, either curve or surface, parametrization is injective by computing its degree map. In addition, if the parametrization is not injective, it determines a birational reparametrization. For the curve case, the corresponding command always provides an optional answer. For the surface case, not all cases are covered. Nevertheless, we illustrate using Maple some new ideas on how to approach those surface cases not covered in the package.

**Keywords:** Maple · Symbolic computation · Birational (proper) parametrization · Algebraic Curves · Algebraic Surfaces.

## 1 Introduction

Algebraic varieties are definable as the zero-set of polynomials. Nevertheless, for some special cases, namely those that are unirational, they can also be represented by means of a tuple of rational functions; see [10], [11] for the case of radical parametric representations. It is well-known, and illustrated in the literature (see e.g. [3]), that depending on the particular problem to be approached a different representation might be more advisable. In this paper, we stay within the world of rational parametric representations, that is we deal with unirational varieties, and more specifically with unirational curves and surfaces. Even in this case, different computational and theoretical questions appear that affect to the feasibility of the applications of the parametric representations. More precisely, one may consider the injectivity and/or the surjectivity (see [1], [9]) of the parametric representation. We here deal with the injectivity.

The natural question in this context is whether a rationally parametrized variety (i.e. parametrized by means of a tuple of rational functions) can be parametrized birationally (i.e. the map being injective on a non-empty Zariski open subset of the parameter space); we will refer to a birational parametrization as a proper parametrization. This question can be reformulated in terms of field theory by using the field of rational functions of the variety. Then, it holds that any unirationally parametrized curve can be parametrized birationally for any field (see e.g. [12]) because of Lüroth's Theorem:

**Theorem 1.** (*Lüroth*) Consider the inclusion of fields  $\mathbb{K} \subset \mathbb{L} \subset \mathbb{K}(x)$ . Then, there exists  $f(x) \in \mathbb{K}(x)$  such that  $\mathbb{L} = \mathbb{K}(x)$ .

Equivalently, let  $\mathcal{P} \in \mathbb{K}(t)^n$  a (maybe non proper) curve parametrization. Then, the Zariski closure  $\mathcal{C}$  of  $\mathcal{P}(\mathbb{K})$  is a rational curve (i.e. there exists a birational parametrization of  $\mathcal{C}$ ).

*Example 1.* Consider the parametrization

$$t \mapsto (t^6 - 3t^5 - 3t^4 + 11t^3 + 5t^2 - 11t - 6, t^4 - 2t^3 - 3t^2 + 4t + 3). \quad (1)$$

Its image is a curve that, by Lüroth's Theorem, is rational. In fact, it is the nodal cubic  $y^3 - x^2 + y^2$ , that can be properly (not injectively) parametrized by  $t \mapsto (t^3 - t, t^2 - 1)$ . It is interesting to have means to find the proper parametrization from the map (1) and the relation between both of them.

However, for the surface case, the characterization is only possible when the field is algebraically closed (see [13]); for other dimensions, the situation is even more restrictive.

In this paper, we deal with the problem of deciding the properness, and computing birational parametrizations, for the case of curves and surfaces. For the curve case, there are constructive proofs of Lüroth's Theorem (see [12] and the references therein), but we will mainly use here the algorithmic approaches in [5] and [6]. For the surface case, one may proceed as follows: compute the implicit equation of the surface, using e.g. [7], and then apply a parametrization algorithm, e.g. Schicho's algorithm (see [8]). However, here, we want to approach the problem without implicitizing, that is by reparametrizing the original parametrization. For this purpose, we will use the algorithm in [6]. Results in [6] provide a wide, but partial, solution to the problem. But, up to our knowledge, there is no complete algorithmic (reparametrizing) answer for the surface case.

More precisely, we present a Maple package, that we call **Luroth** for dealing with the described problems. Furthermore, we also show some ideas on how to approach the cases not covered in [6] that we illustrate with the help of Maple. The paper is structured as follows. In Section 2 we briefly recall the results in [5] and [6] that we use in our implementations. In Section 3 we give an overview of the package and we show some examples that are additionally illustrated in the Appendix. The Maple package is available (see Section 3 for details) at the web site [http://www3.uah.es/jorge\\_caravantes/research.html](http://www3.uah.es/jorge_caravantes/research.html). In Section 4, some on-going working ideas to approach the general case using Maple are presented. Finally, in the appendix (Section 5), the Maple executions, corresponding to the examples in the Subsection 3.2, are shown.

## 2 Theoretical and algorithmic framework

In this section, we briefly recall some theoretical facts and algorithms that will be used in the implementation of the package. Throughout this section,  $\mathbb{K}$  is an algebraically close field and  $\mathbb{L}$  a subfield of  $\mathbb{K}$ ; usually the ground field where the parametrization is expressed.

## 2.1 The curve case

Let  $\mathcal{C} \subset \mathbb{K}^n$  be a curve, rationally parametrized by

$$\mathcal{P}(t) = (p_1(t), \dots, p_n(t)) \in \mathbb{L}(t)^n,$$

where  $\mathcal{P}$  is not necessarily proper. By Lüroth's Theorem, we know that there exists a rational function  $R \in \mathbb{L}(t)$  and a birational parametrization  $\mathcal{Q}(t) \in \mathbb{L}(t)^n$  of  $\mathcal{C}$  such that  $\mathcal{P}(t) = \mathcal{Q}(R(t))$ . In Algorithm 1, we outline the ideas presented in [5] for effectively computing  $\mathcal{Q}$  and  $R$ . For this purpose, we denote by  $\text{Res}$  the univariate resultant of two polynomials, by  $\text{Num}$  the numerator of a rational function expressed in reduced form, and by  $\text{Den}$  the denominator of a rational function expressed in reduced form.

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### Algorithm 1 Proper reparametrization of space curves

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**Input:**  $\mathcal{P}(t) = (p_1(t), \dots, p_n(t)) \in \mathbb{L}(t)^n \subset \mathbb{K}(t)^n$  a parametrization of  $\mathcal{C}$ .

**Output:** A proper parametrization  $\mathcal{Q}(t) \in \mathbb{L}(t)^n$  and  $R(t) \in \mathbb{L}(t)$  such that  $\mathcal{Q}(R(t)) = \mathcal{P}(t)$ .

1: Determine

$$S(s, t) = \gcd(\text{Num}(p_1(s) - p_1(t)), \dots, \text{Num}(p_n(s) - p_n(t))).$$

Let us say that  $S(s, t) = C_m(t)s^m + \dots + C_0(t)$ .

2: **if**  $\deg_t(S) = 1$  **then**

3:     **return**  $\mathcal{Q}(t) = \mathcal{P}(t)$ , and  $R(t) = t$ .

4: **end if**

5: Choose  $C, D \in \{C_0, \dots, C_m\}$  such that  $\gcd(C, D) = 1$  and  $C/D, C \cdot D \notin \mathbb{L}$  (see Section 2 in [5] for further details). Take  $R(t) = C/D$ .

6: **for**  $i = 1, \dots, n$  **do**

7:     Compute

$$L_i(s, x_i) = \text{Res}_t(\text{Num}(x_i - p_i(t)), \text{Num}(s - R(t))).$$

It holds that  $L_i$  is of the form  $L_i = (b_i(s)x_i - a_i(s))^{\deg(R)}$ .

8: **end for**

9: **return** the rational function  $R(t)$ , and the proper parametrization

$$\mathcal{Q}(t) = (a_1(t)/b_1(t), \dots, a_n(t)/b_n(t)) \in \mathbb{L}(t)^n.$$


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We illustrate Algorithm 1 by an example.

*Example 2.* Let  $\mathcal{C}$  be a rational space curve over  $\mathbb{C}$  defined by the parametrization

$$\mathcal{P}(t) = \left( \frac{(t+1)^2(t^2+1)}{2(t^4+4t^2+1+2t^3+2t)}, \frac{2(t^2+t+1)}{(t+1)^2}, \frac{-(t^2+1)(t^2+1+4t)}{(t+1)^4} \right).$$

In Step 1 of the algorithm, we get

$$\text{Num}(p_1(t) - p_1(s)) = -(s-t)(ts-1)(ts^2+t^2s+2ts+s+t)$$

$$\text{Num}(p_2(t) - p_2(s)) = 2(s-t)(ts-1),$$

$$\text{Num}(p_3(t) - p_3(s)) = 4(ts-1)(s-t)(ts^2+t^2s+s+4ts+t).$$

Thus, we get  $S(s, t) = C_0(t) + C_1(t)s + C_2(t)s^2$ , where  $C_0(t) = t$ ,  $C_1(t) = -t^2 - 1$ , and  $C_2(t) = t$ . Since  $\deg_t(S^{\mathcal{P}}) > 1$ , the parametrization is not proper (observe that, for general  $t$ , two values of  $s$  vanish  $S$ , so give the same image by  $\mathcal{P}$ ), so we go to Step 2 where we choose  $C$  as  $C_1$  and  $D$  as  $C_0$ . Therefore,  $R(t) = -(t^2 + 1)/t$ . Note that  $\gcd(C_0, C_1) = 1$ . Now, we compute the polynomials

$$\begin{aligned} L_1(s, x_1) &= (4x_1 + 4sx_1 - 2s - s^2 + 2s^2x_1)^2, \\ L_2(s, x_2) &= (2x_2 - 2 - 2s + sx_2)^2, \\ L_3(s, x_3) &= (4x_3 + 4s + 4sx_3 + s^2 + s^2x_3)^2. \end{aligned}$$

Finally, in Step 5, the algorithm outputs the proper parametrization  $\mathcal{Q}(t)$ , and the rational function  $R(t)$ :

$$\mathcal{Q}(t) = \left( \frac{t(t+2)}{2(2+2t+t^2)}, \frac{2(t+1)}{2+t}, \frac{-t(t+4)}{4+4t+t^2} \right) \in \mathbb{C}(t)^3, \quad R(t) = -\frac{t^2+1}{t} \in \mathbb{C}(t).$$

It is not difficult to see that  $\mathcal{P}(t) = \mathcal{Q}(R(t))$ .

## 2.2 The surface case: a partial solution

Let  $\mathcal{P}(\bar{t})$  be a rational affine parametrization over  $\mathbb{L}$  of an algebraic rational surface  $\mathcal{V}$ . We express  $\mathcal{P}$  as

$$\mathcal{P}(\bar{t}) = (p_1(\bar{t}), p_2(\bar{t}), p_3(\bar{t})) \in \mathbb{L}(\bar{t})^3, \quad p_i(\bar{t}) = p_{i,1}(\bar{t})/p_{i,2}(\bar{t}), \quad (2)$$

where  $\gcd(p_{i,1}, p_{i,2}) = 1$ ,  $i = 1, 2, 3$ , and  $\bar{t} = (t_1, t_2) \in \mathbb{K}^2$ .

The degree of the rational map induced by  $\mathcal{P}$  is denoted by  $\text{MapDeg}(\mathcal{P})$ ; see e.g. [2] pp.80, or [13] pp.143 for details. We recall that the properness of  $\mathcal{P}(\bar{t})$  is characterized by  $\text{MapDeg}(\mathcal{P})$ . More precisely,  $\mathcal{P}(\bar{t})$  is *proper* if and only if  $\text{MapDeg}(\mathcal{P}) = 1$  (see [2] and [13]). Also, the mapping degree is the cardinality of the fibre of a generic element (see Theorem 7, pp. 76 in [13]). That is,

$$\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{\bar{t} \in \mathbb{K}^2 \mid \mathcal{P}(\bar{t}) = P\},$$

where  $\mathcal{F}_{\mathcal{P}}(P)$  is the fibre of a point  $P \in \mathcal{V}$ . Associated with the parametrization  $\mathcal{P}$ , we consider the polynomials

$$G_i^{\mathcal{P}}(\bar{t}, \bar{s}) = p_{i,1}(\bar{t})p_{i,2}(\bar{s}) - p_{i,2}(\bar{t})p_{i,1}(\bar{s}) \in (\mathbb{K}[\bar{s}][\bar{t}]), \quad i = 1, 2, 3,$$

and

$$\begin{aligned} S_1^{\mathcal{P}}(t_1, \bar{s}) &= \text{PrimPart}_{\bar{s}}(\text{Content}_Z(\text{Res}_{t_2}(G_1, G_2 + ZG_3))), \\ S_2^{\mathcal{P}}(t_2, \bar{s}) &= \text{PrimPart}_{\bar{s}}(\text{Content}_Z(\text{Res}_{t_1}(G_1, G_2 + ZG_3))), \end{aligned}$$

where  $\bar{s} = (s_1, s_2) \in \mathbb{K}^2$ ,  $Z$  is an auxiliary variable, and  $\text{Content}_x(p)$  and  $\text{PrimPart}_x(p)$  are the content and the primitive part of a polynomial  $p$  with respect to the variable  $x$ . Let  $\mathbb{F} = \overline{\mathbb{K}(\bar{s})}$  be the algebraic closure of  $\mathbb{K}(\bar{s})$ .

The polynomials  $S_j^{\mathcal{P}}$  play an important role in deciding the properness of a parametrization  $\mathcal{P}$ . More precisely, in [4] the following theorem is proved.

**Theorem 2.** *The following statements hold:*

1.  $\mathcal{P}^{-1}(\mathcal{P}(\bar{s})) = \{\bar{t} \in \mathbb{F}^2 \mid G_i^{\mathcal{P}}(\bar{t}, \bar{s}) = 0, i = 1, 2, 3\}$  and, for generic  $\bar{s} \in \mathbb{K}^2$ ,  $\text{MapDeg}(\mathcal{P}) = \text{Card}(\mathcal{P}^{-1}(\mathcal{P}(\bar{s})))$ .
2. The polynomial  $S_i^{\mathcal{P}}$  defines the  $t_i$ -coordinates of the points in  $\mathcal{P}^{-1}(\mathcal{P}(\bar{s}))$ .
3. Considering  $\bar{s}$  as a couple of variables,  $\text{MapDeg}(\mathcal{P}) = \deg_{t_1}(S_1^{\mathcal{P}}(t_1, \bar{s})) = \deg_{t_2}(S_2^{\mathcal{P}}(t_2, \bar{s}))$ .

Therefore, by means of resultants and gcd's one can determine the degree map of a surface parametrization, and hence decide whether it is birational or not. We will refer to the algorithms in [4] for this purpose.

Now, let us assume that the given surface parametrization  $\mathcal{P}$  is not birational. Then, by Castelnuovo's Theorem, since we are working over an algebraically closed field, there exists a proper parametrization of the same surface. In general, this birational parametrization may require the extension of the ground field  $\mathbb{L}$ . One possibility, as already mentioned in the introduction, could be an implicitation-parametrization approach. Nevertheless, the idea here is to solve the problem staying within the parametric representation of the variety. For this purpose, we can apply the results in [6] that, although do not provide a complete answer, cover many of the cases.

The method in [6] is based on the application of Algorithm 1 to some *partial* parametrizations associated to  $\mathcal{P}$ , namely the parametrizations  $\mathcal{P}_i(t_j) := \mathcal{P}(\bar{t}) \in (\mathbb{K}(t_i))(t_j)^3$  (that is,  $\mathcal{P}$  is seen over  $\mathbb{K}(t_i)$ ), for  $i, j \in \{1, 2\}$  and  $i \neq j$ . Observe that the partial parametrization  $\mathcal{P}_i(t_j)$  ( $i \neq j$ ) defines a space curve over  $\mathbb{K}(t_i)$ . Hence, the goal of Algorithm 2 is to properly reparametrize the partial parametrizations by applying Algorithm 1. The algorithm outputs a rational parametrization  $\mathcal{Q}(\bar{t}) \in \mathbb{K}(\bar{t})^3$  of  $\mathcal{V}$ , and  $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$  such that  $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$ , and  $\text{MapDeg}(\mathcal{Q}) < \text{MapDeg}(\mathcal{P})$ . In fact, it is proved that  $\text{MapDeg}(\mathcal{P}) = \deg_{t_1}(S) \deg_{t_2}(T) \text{MapDeg}(\mathcal{Q})$  (see Theorem 4 in [6]). Furthermore, if some additional properties hold, then  $\mathcal{Q}$  is proper. These ideas are described in Algorithm 2 and we illustrate it by an example.

*Example 3.* Let  $\mathcal{V}$  be a rational surface defined over the field of the complex numbers,  $\mathbb{C}$ , by the parametrization

$$\begin{aligned} \mathcal{P}(\bar{t}) = & \left( -(3t_2^8 t_1^4 + 2t_2^6 t_1^6 + 2t_2^{10} t_1^2 + t_2^4 t_1^8 + t_2^{12} - t_2^4 t_1^2 - t_2^2 t_1^4 - t_2^6 + 2t_2^2) / t_2^2, \right. \\ & \left. -t_1^2 t_2^2 - t_1^4 - t_2^4 + 3t_1^4 t_2^4 + 2t_1^6 t_2^2 + 2t_1^2 t_2^6 + t_1^8 + t_2^8 + t_2^2 + t_2^6, 3 + t_1^2 t_2^2 + t_1^4 + t_2^4 \right). \end{aligned}$$

We apply Algorithm 2. For this purpose, in Step 1, we apply Algorithm 1, and we find that

$$S^{\mathcal{P}^2}(t_1, s_1) = (s_1 - t_1)(s_1 + t_1)(s_1^2 + t_2^2 + t_1^2) \in (\mathbb{C}[t_2])[t_1, s_1]$$

which implies that  $\mathcal{P}_2(t_1)$  is not proper; in fact,  $\text{MapDeg}(\mathcal{P}_2) = \deg_{t_1}(S^{\mathcal{P}_2}) = 4$ . Thus, we go to Step 2 and we apply Algorithm 1 to  $\mathcal{P}_2$ . We obtain

$$S_2(t_1) = -t_1^2 t_2^2 - t_1^4 \in (\mathbb{C}[t_2])[t_1].$$

Furthermore, we determine the polynomials

$$L_i(s_1, t_2, x_i) = \text{Res}_{t_1}(G_i^{\mathcal{P}_2}(\bar{t}, x_i), s_1 - S_2(t_1)) = (m_{i,2}(s_1, t_2)x_i - m_{i,1}(s_1, t_2))^{\deg_{t_1}(S_2)},$$

where  $G_i^{\mathcal{P}_2}(\bar{t}, x_i) = x_i p_{i,2}(\bar{t}) - p_{i,1}(\bar{t})$ , for  $i = 1, 2, 3$ , and we get  $\mathcal{M}(\bar{t}) =$

$$(-2 - t_2^{10} + t_2^4 + 2t_1 t_2^6 - t_1 - t_1^2 t_2^2, -t_2^4 + t_2^8 + t_2^2 + t_2^6 + t_1 - 2t_2^4 t_1 + t_1^2, 3 + t_2^4 - t_1)$$

(we rename  $s_1$  as  $t_1$ ).

Now, in Step 2.2 of the algorithm, we apply Algorithm 1 to  $\mathcal{M}_1(t_2) \in (\mathbb{C}(t_1))(t_2)^3$ , and we find that

$$S^{\mathcal{M}_1}(t_2, s_2) = (t_2 - s_2)(t_2 + s_2) \in (\mathbb{C}[t_1])[t_2, s_2].$$

Thus, since  $\text{MapDeg}(\mathcal{M}_1) = \deg_{t_2}(S^{\mathcal{M}_1}) = 2$ , we get that  $\mathcal{M}_1$  is not proper. Then, we go to Step 2.3. We apply Algorithm 1 to  $\mathcal{M}_1$ , and we compute  $T_1(t_2) = t_2^2 \in (\mathbb{C}[t_1])[t_2]$ , and the polynomials

$$L_i(t_1, s_2, x_i) = \text{Res}_{t_2}(G_i^{\mathcal{M}_1}(\bar{t}, x_i), s_2 - T_1(t_2)) = (q_{i,2}(t_1, s_2)x_i - q_{i,1}(t_1, s_2))^{\deg_{t_2}(T_1)},$$

where  $G_i^{\mathcal{M}_1}(\bar{t}, x_i) = x_i m_{i,2}(\bar{t}) - m_{i,1}(\bar{t})$ , for  $i = 1, 2, 3$ . We obtain  $\mathcal{Q}(\bar{t}) =$

$$(-2 - t_1 - t_2 t_1^2 + t_2^2 + 2t_1 t_2^3 - t_2^5, t_1 + t_1^2 + t_2 - t_2^2 - 2t_1 t_2^2 + t_2^3 + t_2^4, 3 - t_1 + t_2^2)$$

(we rename  $s_2$  as  $t_2$ ). Finally, in Step 2.4 of the algorithm, we apply Algorithm 1 to  $\mathcal{Q}_2(t_1) \in (\mathbb{C}(t_2))(t_1)^3$ . We get that  $S^{\mathcal{Q}_2}(t_1, s_1) = s_1 - t_1 \in (\mathbb{C}[t_2])[t_1, s_1]$  which implies that  $\mathcal{Q}_2$  is proper. Therefore, Algorithm 2 outputs the parametrization  $\mathcal{Q}(\bar{t})$ , and

$$R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2)) = (-t_1^2 t_2^2 - t_1^4, t_2^2) \in \mathbb{C}(\bar{t})^2.$$

One may check that  $\text{MapDeg}(\mathcal{P}) = \deg_{t_1}(S)\deg_{t_2}(T) = 8$  and thus  $\mathcal{Q}$  is a proper reparametrization (we remind that  $\text{MapDeg}(\mathcal{P}) = \deg_{t_1}(S)\deg_{t_2}(T)\text{MapDeg}(\mathcal{Q})$ ).

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**Algorithm 2** Proper reparametrization of surfaces (partial case)
 

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**Input:** A rational parametrization

$$\mathcal{P}(\bar{t}) = (p_1(\bar{t}), p_2(\bar{t}), p_3(\bar{t})) \in \mathbb{K}(\bar{t})^3, \quad p_i(\bar{t}) = p_{i,1}(\bar{t})/p_{i,2}(\bar{t}),$$

$\gcd(p_{i,1}, p_{i,2}) = 1$ ,  $i = 1, 2, 3$ , of an algebraic surface  $\mathcal{V}$ .

**Output:** A rational parametrization

$$\mathcal{Q}(\bar{t}) = (q_1(\bar{t}), q_2(\bar{t}), q_3(\bar{t})) \in \mathbb{K}(\bar{t})^3, \quad q_i(\bar{t}) = q_{i,1}(\bar{t})/q_{i,2}(\bar{t}),$$

of  $\mathcal{V}$ , and  $R(\bar{t}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$  such that  $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$ , and  $1 \leq \text{MapDeg}(\mathcal{Q}) < \text{MapDeg}(\mathcal{P})$ .

- 1: **if**  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are proper (apply Algorithm 1) **then**
  - 2:     **return**
  - 3: **else if**  $\mathcal{P}_2$  is not proper **then**
  - 4:     Apply Algorithm 1 to  $\mathcal{P}_2$ . [It returns a parametrization  $\mathcal{M}(\bar{t}) \in \mathbb{K}(\bar{t})^3$ , and  $S(\bar{t}) \in \mathbb{K}(\bar{t})$  such that the partial parametrization associated to  $\mathcal{M}$ ,  $\mathcal{M}_2(t_1) \in (\mathbb{K}(t_2))(t_1)^3$ , is proper and  $S_2(t_1) \in (\mathbb{K}(t_2))(t_1)$  satisfies  $\mathcal{P}_2(t_1) = \mathcal{M}_2(S_2(t_1))$ ].
  - 5:     **if**  $\mathcal{M}_1(t_2) \in (\mathbb{K}(t_1))(t_2)^3$  is proper (apply Algorithm 1) **then**
  - 6:         **return**  $\mathcal{Q} := \mathcal{M}$ , and  $R(\bar{t}) := (S(\bar{t}), t_2)$ .
  - 7:     **else**
  - 8:         Apply Algorithm 1 to  $\mathcal{M}_1(t_2)$ . [It returns a parametrization  $\mathcal{Q}(\bar{t}) \in \mathbb{K}(\bar{t})^3$ , and  $T(\bar{t}) \in \mathbb{K}(\bar{t})$  such that the partial parametrization associated to  $\mathcal{Q}$ ,  $\mathcal{Q}_1(t_2) \in (\mathbb{K}(t_1))(t_2)^3$ , is proper and  $T_1(t_2) \in (\mathbb{K}(t_1))(t_2)$  satisfies  $\mathcal{M}_1(t_2) = \mathcal{Q}_1(T_1(t_2))$ ].
  - 9:     **end if**
  - 10:     **if** the partial parametrization associated to  $\mathcal{Q}$ ,  $\mathcal{Q}_2(t_1) \in (\mathbb{K}(t_2))(t_1)^3$ , is proper (apply Algorithm 1) **then**
  - 11:         **return**  $\mathcal{Q}$ , and  $R(\bar{t}) := (S(\bar{t}), T(S(\bar{t}), t_2))$ .
  - 12:     **else**
  - 13:         **return**  $\mathcal{Q}$ ,  $R(\bar{t}) := (S(\bar{t}), T(S(\bar{t}), t_2))$ , and the message “you may apply the algorithm again (Step 3 and so on) to  $\mathcal{Q}_2$ ”.
  - 14:     **end if**
  - 15: **else**
  - 16:     Apply Step 8 and the next one to  $\mathcal{P}$  and  $\mathcal{P}_1$  ( $\mathcal{P}_2$  is proper and  $\mathcal{P}_1$  is not).
  - 17: **end if**
- 

### 3 The package Luroth

In this section, we present the creation of a package in the computer algebra system Maple, that we call **Luroth**. This package consists in several procedures that implement the algorithms described in Section 2. More precisely, the package deals with rational parametrizations, either of plane or space curves, or surfaces. It checks the injectivity giving the degree of the map. In addition, it provides birational reparametrizations of a given non-birational parametrization. For the case of curves, all cases are covered. For the case of surfaces, only those treated in [5] and [6] are considered. In Section 4, we show how, with the help of Maple, we approach the general case.



### 3.1 General description

The Maple package is initialized by the command:

```
> with(Luroth):
```

The main procedures in the package are:

- **IsTheCurveProper:**
  - i) *Feature:* This procedure checks whether a given rational curve parametrization, non necessarily planar, is birational. Briefly, the input and output of the procedure can be stated as follows:
    - ▷ **INPUT:**
      - A list, of length at least 2, whose entries are univariate rational functions, not all constant.
      - The variable of the rational functions
      - An option  $u \in \{\text{probabilistic, deterministic}\}$ .
    - ▷ **OUTPUT:** The command returns either *true* or *false*. *true* means that the parametrization is birational. If *false* then the input parametrization is not birational and the procedure returns also the degree of the map associated to the input parametrization.
  - ii) *Calling Sequence:* `> IsTheCurveProper(List,variable,option);`
  - iii) *Mathematical Argumentation:* the procedure is based on the results in [6] and implements the first steps of Algorithm 1.
- **CurveProperReparametrization:**
  - i) *Feature:* This procedure computes a birational parametrization of the curve defined by the input parametrization. Briefly, the input and output of the procedure can be stated as follows:
    - ▷ **INPUT:**
      - A list, of length at least 2, whose entries are univariate rational functions, not all constant.
      - The variable of the rational functions
    - ▷ **OUTPUT:** a proper curve parametrization of the input curve.
  - ii) *Calling Sequence:* `> CurveProperReparametrization(List,variable);`
  - iii) *Mathematical Foundation:* the procedure is based on the results in [6] and implements the first steps of Algorithm 1
- **IsTheSurfaceProper:**
  - i) *Feature:* This procedure checks whether a given rational surface parametrization is birational. Briefly, the input and output of the procedure can be stated as follows:
    - ▷ **INPUT:**
      - A list, of length at least 3, whose entries are bivariate rational functions, which generic Jacobian has rank 2.
      - The variables of the rational functions
      - An option  $u \in \{\text{probabilistic, deterministic}\}$ .
    - ▷ **OUTPUT:** The command returns either *true* or *false*. *true* means that the parametrization is birational. If *false* then the input parametrization is not birational and the procedure returns also the degree of the map associated to the input parametrization.

- ii) *Calling Sequence:*  
`> IsTheSurfaceProper(List,variable1, variable2,option);`
- iii) *Mathematical Foundation:* the procedure is based on the results in [4].
- **SurfaceProperReparametrization:**
  - i) *Feature:* This procedure tries to compute a birational parametrization of the surface defined by the input parametrization. Briefly, the input and output of the procedure can be stated as follows:
    - ▷ **INPUT:**
      - A list, of length at least 3, whose entries are univariate rational functions, not all constant.
      - The variables of the rational functions
    - ▷ **OUTPUT:** one of the following possibilities
      - A birational parametrization of the input surface.
      - A non-birational parametrization of the input surface with smaller map degree than the input parametrization.
      - A message informing that no improvement has been possible.
  - ii) *Calling Sequence:*  
`> SurfaceProperReparametrization(List,variable1,variable2);`
  - iii) *Mathematical Foundation:* the procedure is based on the results in [6] and implements the first steps of Algorithm 2.

There are other auxiliary procedures in the package that we do not mention here besides `Try`. Given a surface parametrization  $\mathcal{P} \in \mathbb{K}(t_1, t_2)^3$ , with  $\mathbb{K}$  a field of characteristic zero, `Try` decides whether the surface seen as a curve over the algebraic closure of  $\mathbb{K}(t_2)(t_1)$ , and parametrized by  $\mathcal{P}(t_1, t_2)$  as tuple in  $\mathbb{K}(t_1)(t_2)$ , is birational. In the affirmative case, it looks for a reparametrization function over the ground field  $\mathbb{K}(t_1)$ . Additionally, `FindR` executes Step 3 in Algorithm 1.

### 3.2 Illustrative examples

In this subsection, we illustrate the package by some examples run in Maple. In Example 4, we see the performance of the package for the case of curves. Examples 5, 6 and 7 are devoted to surfaces. In Example 5 the algorithm is able to provide a birational parametrization. Example 6 illustrates the case where the algorithm does not yield to an optimal answer but outputs a parametrization whose map degree has decreased. Finally Example 7 is devoted to the case where the algorithm does not provide any improvement. For additional information on how the execution is performed we will be referring to the tables in the Appendix.

The package and installation instructions can be found at

[http://www3.uah.es/jorge\\_caravantes/research.html](http://www3.uah.es/jorge_caravantes/research.html)

Once the package is installed, after executing the command `with(Luroth)`, the package is ready to be used (see Fig. 1).

*Example 4.* We consider the curve parametrization

$$\mathcal{P}(t) = \left( \frac{t^6}{(t^3 + t + 2)^2}, \frac{(t^3 + t + 2)^3}{t^9}, \frac{2t^6 + 2t^4 + 4t^3 + t^2 + 4t + 4}{(t^3 + t + 2)t^3} \right)$$

that parametrizes the space curve defined by

$$\{x^2y + xy - z, xz^2 - x^2 - 2x - 1, xyz - z^2 + x + 1, z^3 - xy - xz - y - 2z, xy^2 - yz + z^2 - x - 2\}.$$

Using the command `IsTheCurveProper`, one gets that the degree of the map induced by  $\mathcal{P}$  is 3 (see Fig. 2). Hence, the curve is traced three times when giving values to the parameter. Since the parametrization is not injective, we apply the command `CurveProperReparametrization` (see Fig. 3) to get the following birational parametrization of the same curve,

$$\mathcal{Q}(t) = \left( \frac{t^2}{(t-2)^2}, \frac{t^3 - 6t^2 + 12t - 8}{t^3}, 2 \frac{t^2 - 2t + 2}{t(t-2)} \right).$$

*Example 5.* We consider now the surface parametrization

$$\mathcal{P}(t_1, t_2) = \left( \frac{t_1 t_2 (t_1^2 - t_1 t_2 + t_2^2)}{(t_1 + t_2)^2}, t_2, \frac{t_2 (t_1^3 - t_2 t_1^2 + t_2^2 t_1 + t_1^2 + 2 t_1 t_2 + t_2^2)}{(t_1 + t_2)^2} \right).$$

Applying the command `IsTheSurfaceProper`, we get that the degree map is 3, and hence the  $\mathcal{P}(t_1, t_2)$  is not birational. We then apply the command

`ProperSurfaceReparametrization`

to get proper parametrization of the surface (see Fig. 4)

$$\mathcal{Q}(t_1, t_2) = \left( -\frac{t_2 (t_1 t_2 + 1)}{t_1}, t_2, -\frac{(1 + (t_2 - 1) t_1) t_2}{t_1} \right).$$

*Example 6.* We consider the following surface parametrization

$$\mathcal{P} = \left( -\frac{-93t_1^4 - 51t_1^2 t_2^2 + 90t_2^4 - 22t_1^2}{2601t_2^4 t_1^4 - 1530t_2^6 t_1^2 + 225t_2^8 + 2244t_2^2 t_1^4 - 660t_2^4 t_1^2 + 484t_1^4 - 2}, \right. \\ \left. -\frac{4743t_2^2 t_1^6 - 1395t_2^4 t_1^4 - 3825t_2^6 t_1^2 + 1125t_2^8 + 2046t_1^6 - 1650t_2^4 t_1^2 - 2}{-51t_1^2 t_2^2 + 15t_2^4 - 22t_1^2}, \right. \\ \left. -\frac{1}{3} \frac{8649t_1^8 - 13950t_2^4 t_1^4 + 5625t_2^8 + 51t_1^2 t_2^2 - 15t_2^4 + 22t_1^2 - 3}{-31t_1^4 + 25t_2^4} \right).$$

In this case, applying the command `IsTheSurfaceProper` we get that the parametrization has map degree 16. The command `SurfacePorperReparametrization`

does not get a proper parametrization but the procedure returns a new parametrization of the same surface, which is not proper, but where the degree map has decreased from 16 to 4 (see Fig. 5). The output parametrization is

$$\mathcal{Q} = \left( \frac{(22t_1 - 93)t_2^4 - 51t_1t_2^3 + 90t_1^2t_2^2}{(2t_1^2 - 484)t_2^4 + 2244t_2^3 + (-660t_1 - 2601)t_2^2 + 1530t_1t_2 - 225t_1^2}, \right. \\ \left. \frac{2t_1^3t_2^4 - 1650t_1^2t_2^2 + 2046t_2^4 - 1125t_1^3 + 3825t_1^2t_2 + 1395t_1t_2^2 - 4743t_2^3}{t_1^2t_2^2(22t_2^2 + 15t_1 - 51t_2)}, \right. \\ \left. \frac{(3t_2^4 + 15t_2^2 - 5625)t_1^4 + (22t_2^4 - 51t_2^3)t_1^3 + 13950t_1^2t_2^2 - 8649t_2^4}{75t_1^4t_2^2 - 93t_1^2t_2^4} \right).$$

*Example 7.* We consider the following surface parametrization

$$\mathcal{P} = \left( -\frac{4t_1^4t_2^2 - 4t_1^2t_2^3 - t_1^4 + t_2^4 + t_1^2t_2}{(2t_1^2 - t_2)^2(t_1^2 - t_2)}, -\frac{t_1t_2^2}{(2t_1^2 - t_2)(t_1^2 - t_2)}, \frac{t_2^4}{(t_1^2 - t_2)^2} \right).$$

The command `IsTheSurfaceProper` ensures that the map degree of  $\mathcal{P}$  is 3. However, The command `SurfacePorperReparametrization` does not get any parametrization with smaller map degree (see Fig. 6).

## 4 Approaching the general case.

As we have already mentioned, Lüroth's Theorem, for the case of surfaces, requires the field to be algebraically closed. This implies that, in general, the ground field  $\mathbb{L}$  of the parametrization needs to be extended. Observe that the curve case (see Algorithm 1) does not extend  $\mathbb{L}$ , and hence the surface partial approach behaves the same (see Algorithm 2). Therefore, a new strategy has to be considered. Here, we present some on-going working ideas to approach the general case. For this purpose, let  $\mathcal{P}(\bar{t}) = (p_1(\bar{t}), p_2(\bar{t}), p_3(\bar{t}))$  be as in (2). Let  $\mathcal{P}$  be non-birational and  $\Phi := \text{MapDeg}(\mathcal{P}) > 1$ . In this situation, we know that there exists a birational surface parametrization  $\mathcal{Q}$  of the same surface, and a dominant rational map  $R : \mathbb{K}^2 \dashrightarrow \mathbb{K}^2$  such that  $\mathcal{P} = \mathcal{Q} \circ R$ . Therefore, to find  $\mathcal{Q}$ , it is enough to find  $R$ . However, note that the possible pairs  $(\mathcal{Q}, R)$ , solving the problem, are not unique. Nevertheless, our idea is to observe that an answer can be achieved by looking for rational maps  $R$  with the same fiber as  $\mathcal{P}$ . One may proceed as follows:

1. Let  $\mathcal{I}$  be the ideal in  $\mathbb{K}[\bar{t}, \bar{s}, w]$  of the fibre of  $\mathcal{P}$ . That is  $\mathcal{I}$  is generated by

$$\{\text{Num}(p_i(\bar{t}) - p_i(\bar{s})), w \cdot \text{lcm}(\text{Den}(p_1), \text{Den}(p_2), \text{Den}(p_3)) - 1\}_{i=1,2}.$$

Note that  $\mathcal{I}$  is zero-dimensional with  $\Phi$  points.

2. We want  $R$  such that the ideal  $\mathcal{J}$  in  $\mathbb{K}[\bar{t}, \bar{s}, w]$  of its fibre is equal to  $\mathcal{I}$ . By Bézout's Theorem we know that a linear system of curves defining such  $\mathcal{R}$  must have degree greater than or equal to  $d = \lceil \sqrt{\Phi} \rceil$ .

3. Make an ansatz of unknown coefficients of a linear systems of degree  $d$ , and increase the degree by one until a  $R$  satisfying  $\mathcal{J} = \mathcal{I}$  is found. Since Castelnuovo's Theorem ensures the existence of  $R$ , the procedure terminates.

We illustrate these ideas by means of an example.

*Example 8.* We consider the parametrization  $\mathcal{P}$  of Example 7. Using Maple, we compute a Gröbner basis, with respect to the pure lex order with  $t_1 < t_2 < w$ , of the ideal  $\mathcal{I}$ . It contains 3 polynomials. Since  $\Phi := \text{MapDeg}(\mathcal{P}) = 3$  (see Example 7), a change of variables of degree greater than or equal to 2 is expected.

We start analyzing all the degree-2 transformations of  $\mathbb{C}^2$ . We consider a generic degree-2 transformation  $R$  and we require that  $\mathcal{J} \subset \mathcal{I}$ . For this purpose, we compute the normal forms w.r.t. to the Gröbner basis above, and we solve the system of equations derived from the vanishing conditions of the normal forms. This provides three different type of expressions for the coordinates of  $R$ , namely

$$\left\{ \begin{array}{l} \frac{2a_1t_1}{2b_4t_1^2 + 2b_1t_1 - b_4t_2}, \frac{a_5t_2^2}{b_4t_1^2 + b_5t_2^2 - b_4t_2}, \frac{2a_4t_1^2 + 2a_1t_1 - a_4t_2}{-4b_2t_1^2 + 2b_1t_1 + 2b_2t_2}, \\ \frac{-a_4t_1^2 - a_5t_2^2 + a_4t_2}{b_2t_1^2 - b_5t_2^2 - b_2t_2} \end{array} \right\}.$$

We now choose from above two shapes to be the entries of the transformation of the plane. For instance, take  $b_5 = 0, a_5 + b_4 = 0$  in the last entry, and  $a_4 = b_1 = 0, 2a_1 - b_4 = 0$  in the first entry. We get

$$R = (r_1, r_2) = \left( -\frac{t_2^2}{t_1^2 - t_2}, \frac{t_1}{2t_1^2 - t_2} \right).$$

From the equality  $\mathcal{P} = \mathcal{Q} \circ R$ , we get the parametrization

$$\mathcal{Q}(t_1, t_2) = (t_2^2 + t_1, t_1t_2, t_1^2).$$

Note that one may get  $\mathcal{Q}$ , for instance, using the idea that  $(r_1, r_2, p_i) \in \mathbb{L}(\bar{t})^3$  parametrizes the irreducible polynomial  $\text{Num}(q_i(x_1, x_2) - x_3)$  (for  $i = 1, 2, 3$ ), where we denote  $\mathcal{Q} = (q_1, q_2, q_3)$ ,  $q_i = q_{i,1}/q_{i,2}$ ,  $\text{gcd}(q_{i,1}, q_{i,2}) = 1$ . Thus, one only has to compute the implicit equations of the parametrizations  $(r_1, r_2, p_i)$  for  $i = 1, 2, 3$  (see e.g. [7]).

Finally, using the command `IsTheSurfaceProper` from the package `Luroth` one checks that  $\text{MapDeg}(\mathcal{Q}) = 1$ , and hence it is a birational transformation of the surface.

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## References

1. Caravantes, J., Sendra, J.R., Sevilla, D., Villarino, C. (2018). *On the Existence of Birational Surjective Parametrizations of Affine Surfaces*, J. of Algebra 501. pp. 206–214.
2. Harris, J. (1995). *Algebraic Geometry. A First Course*. Springer-Verlag.
3. Hoschek, J., Lasser, D. (1993). *Fundamentals of Computer Aided Geometric Design*. A.K. Peters, Ltd., Natick, MA, USA.
4. Pérez-Díaz, S., Sendra, J.R., (2004) *Computation of the Degree of Rational Surface Parametrizations*, Journal of Pure and Applied Algebra. 193/1-3. pp. 99–121.
5. Pérez-Díaz, S. (2006). *On the Problem of Proper Reparametrization for Rational Curves and Surfaces*, Computer Aided Geometric Design. Vol. 23(4). pp. 307–323.
6. Pérez-Díaz, S. (2013). *A Partial Solution to the Problem of Proper Reparametrization for Rational Surfaces*, Computer Aided Geometric Design. Vol. 30(8). pp. 743–759.
7. Pérez-Díaz, S., Sendra, J.R., (2008). *A univariate resultant-based implicitization algorithm for surfaces*. Journal of Symbolic Computation. Vol 43/2. pp 118–139.
8. Schicho, J. (1998). *Rational Parametrization of Surfaces*, Journal of Symbolic Computation. Vol. 26, pp. 1–9.
9. Sendra, J.R. (2002). *Normal Parametrization of Algebraic Plane Curves*. Journal of Symbolic Computation. Vol. 33. pp. 863–885.
10. Sendra, J.R., Sevilla, D. (2013). *First Steps Towards Radical Parametrization of Algebraic Surfaces*. Computer Aided Geometric Design. Vol. 30, Issue 4, pp. 374–388.
11. Sendra, J.R., Sevilla, D., Villarino, C. (2017). *Algebraic and Algorithmic Aspects of Radical Parametrizations*, Computer Aided Geometric Design. Vol. 55. pp. 1–14.
12. Sendra, J.R., Winkler, F., Pérez-Díaz, S. (2007). *Rational Algebraic Curves: A Computer Algebra Approach*. Series: Algorithms and Computation in Mathematics. Vol. 22. Springer Verlag.
13. Shafarevich, I.R. (1994). *Basic Algebraic Geometry 1 - Varieties in Projective Space*. Berlin New York: Springer-Verlag. Vol. 1.

## 5 Appendix

In this appendix, the Maple executions, corresponding to the examples in the Subsection 3.2, are shown.

```
> with(Luroth) ;
[Clean, CurveProperReparametrization, CurveProperReparametrizationAux, FindR,
 FindRAux, IsTheCurveProper, IsTheCurveProperAux, IsTheSurfaceProper,
 IsTheSurfaceProperAux, IsTheSurfaceProperDeterministic,
 IsTheSurfaceProperProbabilistic, SurfaceProperReparametrization, Try]
```

**Fig. 1.** Starting the package

```

P:=[t^6/(t^3+t+2)^2, (t^3+t+2)^3/t^9, (2*t^6+2*t^4+4*t^3+t^2+4*t+4)/
(t^3+t+2)*t^3];
P := [ \frac{t^6}{(t^3+t+2)^2}, \frac{(t^3+t+2)^3}{t^9}, \frac{2t^6+2t^4+4t^3+t^2+4t+4}{(t^3+t+2)t^3} ]
> IsTheCurveProper(P,t,probabilistic);
false
The map degree is
3

```

**Fig. 2.** It checks the properness of  $\mathcal{P}$  in Example 4. The same result is achieved with the option `deterministic`

```

> Q:=CurveProperReparametrization(P,t);
A proper parametrization is
Q := [ \frac{t^2}{(t-2)^2}, \frac{t^3-6t^2+12t-8}{t^3}, \frac{2(t^2-2t+2)}{t(t-2)} ]

```

**Fig. 3.** It computes a proper parametrization of the curve in Example 4

```

> IsTheSurfaceProper(P,t[1],t[2],deterministic);
false
The map degree is
3
> Q:=SurfaceProperReparametrization(P,t[1],t[2]);
A proper parametrization is
Q := [ -\frac{t_2(t_1 t_2 + 1)}{t_1}, t_2, -\frac{(1 + (t_2 - 1) t_1) t_2}{t_1} ]

```

**Fig. 4.** It checks the properness of  $\mathcal{P}$  in Example 5. The same result is achieved with the option `probabilistic`. Applying the command `SurfacePorperReparametrization` one gets a proper parametrization of the surface.

```

> P := [ -(-51*t[1]^2*t[2]^2+90*t[2]^4-22*t[1]^2-93*t[1]^4)/(2601*t
[2]^4*t[1]^4-1530*t[2]^6*t[1]^2+2244*t[2]^2*t[1]^4+225*t[2]^8
-660*t[2]^4*t[1]^2+484*t[1]^4-2), -(-3825*t[2]^6*t[1]^2+1125*t[2]^
^8-1650*t[2]^4*t[1]^2+4743*t[2]^2*t[1]^6-1395*t[2]^4*t[1]^4+2046*
t[1]^6-2)/(-51*t[1]^2*t[2]^2+15*t[2]^4-22*t[1]^2), -1/3*(51*t[1]^
^2*t[2]^2-15*t[2]^4+22*t[1]^2+5625*t[2]^8-13950*t[2]^4*t[1]^
^4+8649*t[1]^8-3)/(25*t[2]^4-31*t[1]^4) ];
P := [ -
          -93 t1^4 - 51 t1^2 t2^2 + 90 t2^4 - 22 t1^2
          -----,
          2601 t2^4 t1^4 - 1530 t2^6 t1^2 + 2244 t2^2 t1^4 - 660 t2^8 t1^2 + 484 t1^4 - 2
          -----,
          4743 t2^6 t1^2 - 1395 t2^8 t1^2 + 1125 t2^8 + 2046 t1^6 - 1650 t2^4 t1^2 - 2
          -----,
          -51 t1^2 t2^2 + 15 t2^4 - 22 t1^2
          ----- ]
          8649 t1^8 - 13950 t2^4 t1^4 + 5625 t2^8 + 51 t1^2 t2^2 - 15 t2^4 + 22 t1^2 - 3
          -----
          3 (-31 t1^4 + 25 t2^4)
> IsTheSurfaceProper (P, t[1], t[2], deterministic);
      false
      The map degree is
      16
> Q:=SurfaceProperReparametrization (P, t[1], t[2]);
      The algorithm does not get a proper parametrization but it gets
      [New Parametrization, Degree Map]
Q := [ [
          (22 t1 - 93) t2^4 - 51 t1 t2^2 + 90 t1^2 t2^2
          -----,
          (2 t1^2 - 484) t2^4 + 2244 t2^2 + (-660 t1 - 2601) t2^2 + 1530 t1 t2 - 225 t1^2
          -----,
          2 t1^3 t2^2 - 1650 t1^2 t2^2 + 2046 t2^4 - 1125 t1^3 + 3825 t1^2 t2 + 1395 t1 t2^2 - 4743 t2^3
          -----,
          t2^2 t1 (22 t2^2 + 15 t1 - 51 t2)
          -----,
          (3 t2^4 + 15 t2^2 - 5625) t1^4 + (22 t2^4 - 51 t2^2) t1^3 + 13950 t1^2 t2^2 - 8649 t2^4
          -----,
          75 t2^4 t1 - 93 t2^4 t1^2
          ----- ], 4 ]
    
```

**Fig. 5.** It checks the properness of  $\mathcal{P}$  in Example 6. The same result is achieved with the option `probabilistic`. Applying the command `SurfaceProperReparametrization` one gets a degree map 4 parametrization of the surface.

```

> P := [ -(4*t[1]^4*t[2]^2-4*t[1]^2*t[2]^3-t[1]^4+t[2]^4+t[1]^2*t[2]
)/(2*t[1]^2-t[2])^2*(t[1]^2-t[2]), -t[1]*t[2]^2/(2*t[1]^2-t[2]
)* (t[1]^2-t[2]), t[2]^4/(t[1]^2-t[2])^2);
P := [ -
          4 t1^4 t2^2 - 4 t1^2 t2^3 - t1^4 + t2^4 + t1^2 t2
          -----, -
          t1 t2^2
          -----,
          (2 t1^2 - t2)^2 (t1^2 - t2)
          -----,
          (2 t1^2 - t2) (t1^2 - t2)
          -----,
          t2^4
          ----- ]
          (t1^2 - t2)^2
> IsTheSurfaceProper (P, t[1], t[2], deterministic);
      false
      The map degree is
      3
> Q:=SurfaceProperReparametrization (P, t[1], t[2]);
      Q := The algorithm does not get any improvement
    
```

**Fig. 6.** It checks the properness of  $\mathcal{P}$  in Example 7. The same result is achieved with the option `probabilistic`. The command `SurfaceProperReparametrization` does not get any parametrization with smaller map degree.