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Title:

Architectural form-finding through Parametric Geometry

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Abstract:

This contribution briefly describes the main philosophy of the new book *Parametric Geometry of Curves and Surfaces: Architectural Form-Finding* (Birkhäuser, 2021) and how this philosophy is supported by concepts in the book. The mathematical investigations of such concepts are examined in detail while explaining their novelty and leading application in architectural elements. In addition to this, we focus on the importance of parametric objects from a mathematical point of view in contemporary architecture, and on understanding the geometry underlying the curves and surfaces which conform architectural manifestations. The essence of the book is illustrated by a couple of examples, together with a brief overview of its contents.

Differential geometry of curves and surfaces is a classical field of study in mathematics which has been developed since the eighteenth century. This branch of mathematics makes use of tools such as implicit equations and parametrizations which allow us to understand the essence behind the shape of a curve, or serve to create better knowledge of a certain surface. In fact, these tools may characterize a whole geometric object, whose DNA turns out to be completely described, and can therefore be studied from its mathematical description. For example, the parametrization which associates each number t to the point in the plane with coordinates $(t,-t^2)$ allows us to draw the parabola $y=-x^2$ by assigning different values to the parameter t.

The development of computers in the last decades made it possible to store, extract or find the information required to describe a geometric object within seconds. Returning to the previous example, a huge number of values can be given to the parameter in a very small interval of time in such a way that the points of the parabola which have been plotted look like the parabola itself. As a matter of fact, CAD programs have these mathematical tools incorporated in them, and one just needs to know how to ask the machine to "plot a parabola". Our knowledge of the mathematical tools involved seems to be unnecessary for the final result! However, the mathematical nature is really crucial to understand the geometry behind a geometric object.

Observe, for example, the similarity between the parabola $y=-x^2$ and the classic curve known as catenary $y=-\alpha\cosh(x/\alpha)+\alpha$, for $\alpha=0.7$ in Fig. 1, near their vertex.

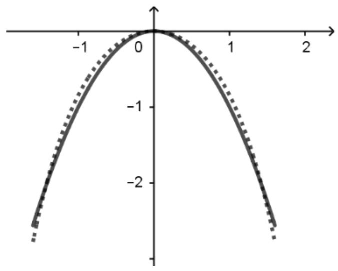

Fig. 2. Parabola y=-x² (solid line) and the catenary y=-αcosh(x/α)+α, for α=1/2 (dotted line)

The DNA, however, of both curves is radically different. The form of a catenary is determined by the physical law that all the horizontal stresses are compensated. Therefore, in an ideal catenary arch, lateral thrusts are minimized because the arch supports itself, which has been used in architecture in many manifestations, see Fig. 3.

Fig. 4. *Gateway Arch in St. Louis, USA, by Eero Saarinen. Source: Photo by Johnson Liu on Unsplash.*
Link: https://unsplash.com/photos/C3SEO9ORkMg

The previous physical property related to the catenary curve can also be transferred to surfaces. Consider the catenary curve in Fig. 3 (left, dotted line) in which *α=4/5*, and rotate it around its symmetry axis. The resulting surface surprisingly resembles an igloo, which is not fortuitous. Observe the shape of the catenary is quite similar to a semi-circle or the parabola $y=-0.7x^2$, and therefore the surface can be approximated by a semi-sphere, or a paraboloid, although their intrinsic geometry substantially differs.

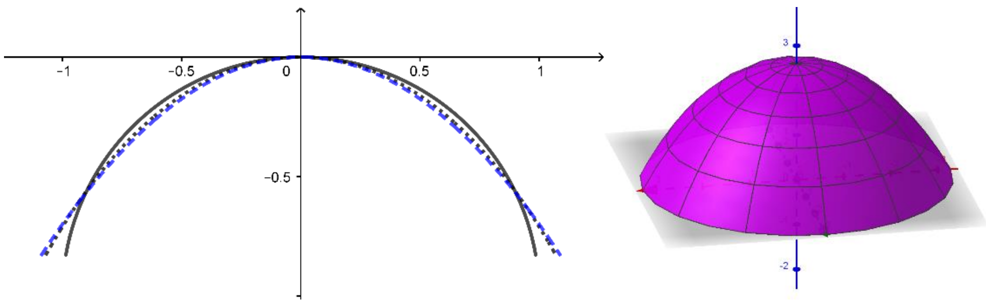

Fig.3. Catenary curve (dotted line) semicircle (solid line) and parabola (dashed blue line) (left). Surface of revolution (right)

An orthogonal rotation of the catenary would lead to a catenoid, a minimal surface with interesting properties coming from its geometry. These particular examples of curves and surfaces appear in many architectural manifestations in which the form is deliberately sought-after. A building may incorporate structures associated to certain physical properties: building acoustics, the configuration of loads, architectural lighting, etc. On the other hand, a form may itself be the final aim in an architectural element, or rather a blending of physical and/or aesthetic reasons gives rise to the final form of a building. In addition to this, a single geometric form can influence an author or a style to the point of becoming a seal of identity.

Having all this in mind, a conscious form-finding process leads us to the point of departure: the need for certain knowledge about how curves and surfaces are mathematically constructed, and their intrinsic mathematical properties underlying in their parametrizations or the equations defining them. *Parametric Geometry of Curves and Surfaces: Architectural Form-Finding* (Birkhäuser, 2021) (Fig. 4) intends to explain the classical theory of differential geometry of curves and surfaces through its application to forms and geometries in architectural elements. This is the philosophy of form-finding in the book which differs from other classical books on differential geometry that remain apart from architecture, and also from other books in architecture where the mathematics are described with less or no detail. In addition, several related mathematical techniques and algorithms used by CAD programs are explained from the mathematical point of view. This is the case of the construction of the walls of the Church of Cristo Obrero in Atlántida, Uruguay, by Eladio Dieste (Fig. 5). Each wall can be modelized as part of the ruled surface joining the points of a line on the floor plane and those in a sinusoidal curve at some positive height (Fig. 6), both being parametric curves. By the way, the use of a masonry vault in the Church of Cristo Obrero, with double curvature, is known to follow a Gaussian vault geometry. More precisely, this vault is constructed by means of (again) catenary arcs leaning on the walls, whose endpoints change in depth in order to fit the structure. Such examples, and many others in the book, suggest that, as a whole, the built environment can be modelled from the intrinsic forms governing architectural shapes, and this is only possible departing from a conscious understanding from the mathematics underlying, and more precisely, from the parametric geometry which encrypts all its DNA.

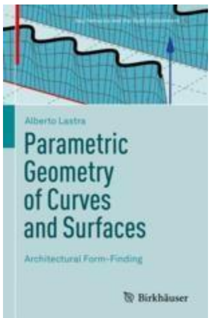

Fig. 4. *Parametric Geometry of Curves and Surfaces: Architectural Form-Finding*

Fig. 5. Detail of the Church of Cristo Obrero in Atlántida, Uruguay, by Eladio Dieste. Source: De Andrés Franchi Ugart, [CC BY-SA 3.0](CC BY-SA 3.0). Link: *https://commons.wikimedia.org/w/index.php?curid=54386268*

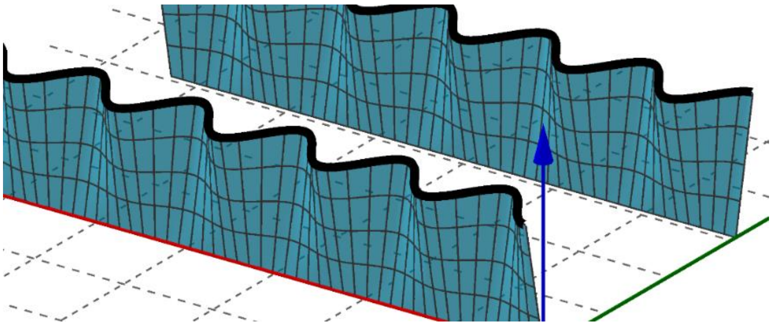

Fig.6. Walls of the Church of Cristo Obrero

The main purpose of the new book, *Parametric Geometry of Curves and Surfaces: Architectural Form-Finding*, is to investigate the shapes appearing in architecture, starting from a mathematical understanding of them. The mathematical knowledge of a geometric object is needed for its adequate manipulation and modification. It is true that nowadays any CAD software can create almost any geometry in seconds. However, one should be able to monitor such constructions, keeping track of the modifications and controlling the machine consciously, rather than just letting the machine do the job in a “Brave New World”. For all these reasons, *Parametric Geometry of Curves and Surfaces* can be relevant for an architectural audience. On the one hand, it can be of great interest for any researcher interested in design applications, showing the background of some of the main commands a CAD program can carry out, from the mathematical point of view. It can also be of help for researchers in architecture who aim to study, understand and control the geometry behind a form, and find them in specific architectural realizations. On the other hand, the book can also be interesting for future architects and technicians (especially those working with CAD programs) aiming to achieve technical knowledge about the mathematics behind design and architectural elements.

The contents of the book depart from classical ideas and well-known mathematical results on parametrizations of plane curves and surfaces, which lead to related geometric elements which have a great importance in architecture. Also, many mathematical techniques and algorithms used by CAD programs which appear to be related to the theory are explained from the mathematical point of view, such as the construction of an helix, extrusions, revolution or ruled surfaces, projections, intersections, approximations, and many others. In addition to this, novel mathematical algorithms and results are also presented. Therefore, the structure of the book can be seen as a tree, growing from a stem which is the classical mathematical theory, and with branches conformed with contemporary studies and further applications to elements related to architecture. All these branches are interspersed with architectural realizations throughout the book.

The insights in each chapter allow us to get an overall picture of the techniques and concepts used, rather than digging deeper into details which can easily lead to hard mathematical problems which still remain under study nowadays. In this concern, the novelty of the book in relation with other existing ones resides in the interdisciplinary flavor emerging from the theoretical mathematical aspects, in relation with past and current trends and realizations in architecture and with design tools.

In the first chapter, parametrizations and plane curves are studied. After briefly describing the main mathematical tools related to the classical theory from the point of view of differential geometry, we study some classic curves and relate them to architectural elements which have been inspired by them (cycloid, catenary, lemniscate, spirals, etc.). A whole section in this chapter is devoted to the study of conics, their classification and identification from algebraic components, algorithm of parametrization, and their use in classical and contemporary architecture in several manifestations. We also recall some algebraic tools on the implicitation and parametrization of curves, and the approximation and interpolation of curves, which are of great importance in CAD software, and their application in architecture through examples.

The study of space curves is constructed as an extension of that of plane curves. The applications to architecture focus on the helix and the twisted cubic. We also exploit the rigid and other more general transformations of the three-dimensional Euclidean space to particularize them in the case of a helix. In the third chapter, we provide an introduction and brief overview of the mathematical background of surfaces, with a focus on the points which appear afterwards in the applications to architectural elements. The appearance of classic surfaces in architecture, such as toroid structures or the more exotic non-orientable surfaces, are also included. We find that the knowledge and interpretation of the projection of surfaces onto planes is quite illustrative, helping the reader to understand the representation systems from a mathematical point of view, finding the equation of surfaces from their projections, etc. We also say some words on the intricated study of curves in surfaces and the intersection of surfaces, which can become hard problems from the theoretical and computational approach, through examples such as Seiffert's spiral or the geometry behind the form of a groin vault.

A final chapter is devoted to the study of particular families of surfaces of great importance in practice: ruled surfaces (Catalan surfaces and conoid structures among others), surfaces of revolution and quadrics. We conclude with the description of minimal and developable surfaces, which are of great importance in present-day architecture. A mathematical tool kit has been added as an appendix for the sake of completeness of the volume, together with the detailed solution to the suggested exercises, many of them being an extension of the mathematical theory or its applications to architecture. The book has many illustrations and QR codes linking to clarifying videos for a more comprehensive reading.

This new book is included in the series *Mathematics and the Built Environment*, published by Birkhäuser Basel. This series is focused on publishing high-standard recent research, of rigorous explorations of the many connections of mathematics and the built environment. In this direction, the book fits into this series, showing different connections not only between architecture and geometry, but also through design, following a conscious mathematical form-finding in architectural shapes. As such, form-finding in architecture and design is shown to be a living and evolving phenomenon, with mathematical tools standing as a core element. We hope this becomes clear while reading *Parametric Geometry of Curves and Surfaces*.