EXISTENCE AND STABILITY RESULTS OF NONLINEAR HIGHER-ORDER WAVE EQUATION WITH A NONLINEAR SOURCE TERM AND A DELAY TERM

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Abstract. We consider the initial-boundary value problem for a nonlinear higher-order nonlinear hyperbolic equation in a bounded domain. The existence of global weak solutions for this problem is established by using the potential well theory combined with Faedo-Galarkin method. We also established the asymptotic behavior of global solutions as $t \to \infty$ by applying the Lyapunov method.

Keywords: nonlinear higher-order hyperbolic equation; nonlinear source term; global existence

MSC 2020: 35B40, 35L75, 35L05

1. Introduction

In this paper we consider the following coupled problem of the nonlinear higher-order hyperbolic equation with nonlinear source term and delay term:

$$
\begin{cases}
u_{tt}(x,t) + \mathcal{A}u(x,t) + \mu_1 g_1(u_t(x,t)) & \\
\qquad\qquad + \mu_2 g_2(u_t(x,t-\tau)) = a|u|^{p-2}u & \text{in } \Omega \times \,]0,\infty[, \\
D^\alpha u(x,t) = 0, \ |\alpha| \leqslant m-1 & \text{on } \partial\Omega \times [0,\infty[, \\
u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{in } \Omega, \\
u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times \,]0,\tau[,
\end{cases}
\tag{1.1}
$$

where $\mathcal{A} = (-\Delta)^m$, $m \geqslant 1$, is a natural number, $\mu_1, \mu_2 > 0$ and $p > 1$ is a real number, Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, Δ is

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the Laplace operator in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $D = \partial_i^\alpha / \partial x_i^{\alpha_i} = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n})$, $x = (x_1, x_2 \ldots x_n)$, a, μ_1 and μ_2 are positive real numbers, g_1 and g_2 are two functions, $\tau > 0$ is a time delay, and the initial data (u_0, u_1, f_0) are in a suitable function space.

When $m = 1$, Liu and Zuazua (see [6], [15]) considered the equation

$$
\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a u_t(x, t-\tau) = 0 & \text{in } \Omega \times]0, \infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega. \end{cases} \tag{1.2}
$$

It is well-known, in the absence of delay ($a = 0$, $a_0 > 0$), that this system is exponentially stable. In the presence of delay ($a > 0$), Nicaise and Pignotti (see [9]) examined system (1.2) and proved, under the assumption that the weight of the feedback with delay is smaller than the one without delay (i.e., $0 < a < a_0$), that the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is unstable.

In the case for $m = 1$, Benaissa and Louhibi (see [2]) studied the following problem:

$$
\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 & \text{in } \Omega \times]0, \infty[, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau) & \text{in } \ \Omega \times]0, \tau[. \end{cases}
$$

They showed global existence of weak solutions using the Faedo-Galerkin method, and obtained general stability estimates by introducing multiplier method and general weighted integral inequalities.

For the initial-boundary value problem of a single higher order nonlinear hyperbolic equation

$$
\begin{cases} u_{tt}(x,t) + \mathcal{A}u(x,t) + a|u_t|^{r-2}u_t = b|u|^{p-2}u & \text{in } \Omega \times]0, \infty[, \\ D^\alpha u(x,t) = 0,\ |\alpha| \leqslant m-1 & \text{on } \partial\Omega \times [0, \infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1.3}
$$

Nakao (see [7]) has used Galerkin's method to present the existence and uniqueness of the bounded solutions, almost periodic solutions to problem (1.3) as the dissipative term is a linear function νu_t. Nakao and Kuwahara (see [8]) study decay estimates of global solutions to problem (1.3) with the degenerate dissipative term $a(x)u_t$ by using a different inequality.

In the case of $m \geqslant 1$ and $\mu_1 = \mu_2 = 0$, problem (1.1) becomes the following initial-boundary value problem:

$$
\begin{cases}
u_{tt}(x,t) + (-\Delta)^m u(x,t) = a|u|^{p-2}u & \text{in } \Omega \times]0,\infty[,\\
D^\alpha u(x,t) = 0,\ |\alpha| \leqslant m-1 & \text{on } \partial\Omega \times [0,\infty[,\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega.
\end{cases} \tag{1.4}
$$

Brenner and von Wahl (see [3]) proved the existence and uniqueness of classical solutions to (1.4) in a Hilbert space. Moreover, Ye (see [14]) proved that this system is stable polynomial type with decay rates depending on the smoothness of initial data. Pecher in [11] investigated the existence and uniqueness of Cauchy problem for the equation in (1.4) by use of the potential well method due to Payne and Sattinger (see [10]). Wang in [12] showed that the scattering operators map a band in H^s into H^s if the nonlinearities have critical or subcritical powers in H^s.

Yanbing et al. in [13] studied solutions to

$$
\begin{cases}
u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u(x,t) - \alpha\Delta u_t(x,t) = f(u) & \text{in } \Omega \times]0,\infty[,\\
\Delta u(x,t) = u(x,t) = 0,\ |\alpha| \leqslant m-1 & \text{on } \partial\Omega \times [0,\infty[,\\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega,
\end{cases} \tag{1.5}
$$

and proved a global well-posedness result, asymptotic behavior and finite time blow up for a strongly damped nonlinear wave equation.

In this article, we prove the global existence of solutions for problem (1.1) by applying the potential well theory and Faedo-Galerkin method. Meanwhile, we study the asymptotic behavior of global solutions by the Lyapunov method.

This article is organized as follows: in the next section, we give some preliminaries and main results. Then Section 3 contains the proofs of the global existence and general decay results.

2. Preliminaries and main results

To state and prove our result, we use the following assumptions:

(A1) $g_1\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing function of class C^1 and $H\colon \mathbb{R}_+ \to \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$ satisfying

$$
\begin{gathered}
H(0) = 0 \text{ and } H \text{ is linear on } [0,\varepsilon] \text{ or}\\
H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon] \text{ such that}\\
c_1'|s| \leqslant |g_1(s)| \leqslant c_1|s| \text{ if } |s| \geqslant \varepsilon,\\
s^2 + g_1^2(s) \leqslant H^{-1}(sg_1(s)) \text{ if } |s| \leqslant \varepsilon,
\end{gathered} \tag{2.1}
$$

where H^{-1} denotes the inverse function of H and ε, c_1, c_1' are positive constants.

(A2) $g_2\colon \mathbb{R}\to\mathbb{R}$ is an odd nondecreasing function of class $C^1(\mathbb{R})$ such that there exist $c_2, \alpha_1, \alpha_2 > 0$,

$$|g_2'(s)| \leqslant c_2, \tag{2.2}$$

$$\alpha_1 s g_2(s) \leqslant G(s) \leqslant \alpha_2 s g_1(s), \tag{2.3}$$

where $G(s) = \int_0^s g_2(r)\,\mathrm{d}r$.

(A3) $\alpha_2\mu_2 < \alpha_1\mu_1$.

(A4) $m \geqslant 1$ is a natural number, p satisfies $2 \leqslant p < \infty$ if $n \leqslant 2m$ and $2 \leqslant p \leqslant 2(n-m)/(n-2m)$ if $n > 2m$.

Lemma 2.1. *Let q be a real number with $2 \leqslant q < \infty$ if $n \leqslant 2m$ and $2 \leqslant q \leqslant 2n/(n-2m)$ if $n > 2m$. Then there is a constant C_s depending on Ω and q such that*

$$\|u\|_q \leqslant C_s\|\mathcal{A}^{1/2}u\|_2 \quad \forall\, u \in H_0^m(\Omega).$$

Remark 2.2. Let us denote by Φ^* the conjugate function of the differentiable convex function Φ, i.e.,

$$\Phi^*(s) = \sup_{t\in\mathbb{R}^+}(st - \Phi(t)).$$

Then Φ^* is the Legendre transform of Φ, which is given by (see Arnold [1], pages 61–62)

$$\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi[(\Phi')^{-1}(s)] \quad \text{if } s \in (0, \Phi'(r)],$$

and Φ^* satisfies the generalized Young inequality

$$AB \leqslant \Phi^*(A) + \Phi(B) \quad \text{if } A \in (0, \Phi'(r)],\ B \in (0, r]. \tag{2.4}$$

We introduce, as in Nicaise and Pignotti (see [9]), the new variable

$$z(x,\varrho,t) = u_t(x, t-\varrho\tau), \quad x \in \Omega,\ \varrho \in (0,1),\ t > 0.$$

Then we have

$$\tau z_t(x,\varrho,t) + z_\varrho(x,\varrho,t) = 0 \quad \text{in } \Omega\times(0,1)\times(0,\infty). \tag{2.5}$$

Therefore, problem (1.1) is equivalent to

$$
(2.6)\qquad \begin{cases} u_{tt}(x,t)+\mathcal{A}u(x,t)+\mu_1 g_1(u_t(x,t)) \\ \qquad\qquad +\mu_2 g_2(z(x,1,t)) = a|u|^{p-2} & \text{in } \Omega\times\,]0,\infty[,\\ \tau z_t(x,\varrho,t)+z_\varrho(x,\varrho,t)=0 & \text{in } \Omega\times\,]0,1[\,\times\,]0,\infty[,\\ D^\alpha u(x,t)=0,\ |\alpha|\leqslant m-1 & \text{on } \partial\Omega\times[0,\infty[,\\ z(x,0,t)=u_t(x,t) & \text{on } \Omega\times[0,\infty[,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega,\\ z(x,\varrho,0)=f_0(x,-\varrho\tau) & \text{in } \Omega\times\,]0,1[. \end{cases}
$$

We first define the following functionals:

$$
(2.7)\qquad \begin{aligned} I(t)=I(u(t)) &= \|\mathcal{A}^{1/2}u\|_2^2 - a\|u\|_p^p,\\ J(t)=J(u(t)) &= \frac{1}{2}\|\mathcal{A}^{1/2}u\|_2^2-\frac{a}{p}\|u\|_p^p. \end{aligned}
$$

We denote the total energy by

$$
(2.8)\qquad \begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\mathcal{A}^{1/2}u(t)\|_2^2+\xi\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x-\frac{a}{p}|u|_p^p\\ &= \frac{1}{2}\|u_t\|_2^2+\xi\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x+J(u(t)), \end{aligned}
$$

where ξ is a positive constant such that

$$
\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}
$$

and

$$
E(0)=\frac{1}{2}\|u_1\|_2^2+\frac{1}{2}\|\mathcal{A}^{1/2}u_0\|_2^2+\xi\int_\Omega\int_0^1 G(f_0(x,-\varrho\tau))\,\mathrm{d}\varrho\,\mathrm{d}x-\frac{a}{p}|u_0|_p^p.
$$

Then for problem (2.6) we can define the stable set as

$$
\mathcal{W}=\{u\setminus u\in H_0^m(\Omega),\ I(u)>0\}\cup\{0\}.
$$

We give an explicit formula for the derivative of the energy.

Lemma 2.3. *Let* (u,z) *be a solution to problem* (2.6)*. Then the energy functional defined by* (2.8) *satisfies*

$$
(2.9)\qquad \begin{aligned} E'(t) \leqslant & -\Big(\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x\\ & -\Big(\frac{\xi\alpha_1}{\tau}-\mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,\mathrm{d}x\\ \leqslant & \ 0. \end{aligned}
$$

P r o o f. By multiplying the first equation in (2.6) by u_t, integrating over Ω and using integration by parts, we obtain

$$
\begin{aligned}
(2.10)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|\mathcal{A}^{1/2}u(t)\|_2^2-\frac{a}{p}\|u(t)\|_p^p\Big)\\
&+\mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x\\
&+\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,\mathrm{d}x=0.
\end{aligned}
$$

We multiply the second equation in (2.6) by $\xi g_2(z)$, we integrate the result over $\Omega\times(0,1)$ to obtain

$$
\begin{aligned}
\xi\int_\Omega\int_0^1 z_t(x,\varrho,t)g_2(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 z_\varrho(x,\varrho,t)g_2(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x\\
&= -\frac{\xi}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\varrho}(G(z(x,\varrho,t)))\,\mathrm{d}\varrho\,\mathrm{d}x\\
&= -\frac{\xi}{\tau}\int_\Omega (G(z(x,1,t))-G(z(x,0,t)))\,\mathrm{d}x.
\end{aligned}
$$

Hence

$$
(2.11)\quad \xi\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x=-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,\mathrm{d}x+\frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,\mathrm{d}x.
$$

Combining (2.10) and (2.11), we obtain

$$
\begin{aligned}
E'(t)=&-\mu_1\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,\mathrm{d}x\\
&-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,\mathrm{d}x+\frac{\xi}{\tau}\int_\Omega G(u_t(x,t))\,\mathrm{d}x,
\end{aligned}
$$

and recalling (2.3), we obtain

$$
\begin{aligned}
(2.12)\qquad E'(t)\leqslant&-\Big(\mu_1-\frac{\xi\alpha_2}{\tau}\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x\\
&-\mu_2\int_\Omega u_t(x,t)g_2(z(x,1,t))\,\mathrm{d}x-\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,\mathrm{d}x.
\end{aligned}
$$

From the definition of G and by using Remark 2.2, we obtain

$$
G^*(s)=sg_2^{-1}(s)-G(g_2^{-1}(s))\quad\forall\,s\geqslant 0.
$$

Hence

$$
\begin{aligned}
G^*(g_2(z(x,1,t)))&=z(x,1,t)g_2(z(x,1,t))-G(z(x,1,t))\\
&\leqslant(1-\alpha_1)z(x,1,t)g_2(z(x,1,t)).
\end{aligned}
$$

Using (2.3) and (2.4) with $A = g_2(z(x,1,t))$ and $B = u_t(x,t)$, from (2.12) we obtain

$$\begin{aligned}
E'(t) \leqslant & -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau}\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,\mathrm{d}x \\
& + \mu_2\int_\Omega (G(u_t(x,t)) + G^*(g_2(z(x,1,t))))\,\mathrm{d}x \\
\leqslant & -\Big(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x \\
& -\Big(\frac{\xi\alpha_1}{\tau} - \mu_2(1-\alpha_1)\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,\mathrm{d}x \\
\leqslant & \ 0.
\end{aligned}$$

□

Theorem 2.4 (Local existence)**.** *Assume that* (A1)–(A4) *hold. If* $u_0 \in H^{2m}(\Omega)\cap H_0^m(\Omega)$, $u_1 \in H_0^m(\Omega)$ *and* $f_0 \in H_0^m(\Omega, H^m(0,1))$ *satisfy the compatibility condition* $f(\cdot,0) = u_1$, *then there exists* $T > 0$ *such that problem* (1.1) *has a unique local solution* $u(t)$ *which satisfies*

$$u \in \mathcal{C}([0,\infty); H_0^m(\Omega)), \quad u_t \in \mathcal{C}([0,\infty); L^2(\Omega)).$$

Now we have the existence of a global solution.

Theorem 2.5. *Let*

$$u_0 \in H^{2m}(\Omega)\cap\mathcal{W}, \quad u_1 \in H_0^m(\Omega)\cap L^2(\Omega) \quad \text{and} \quad f_0 \in H_0^m(\Omega, H^m(0,1))$$

satisfy the compatibility condition $f(\cdot,0) = u_1$. *Assume that* (A1)–(A4) *hold. Then* (1.1) *admits a global weak solution* $u(x,t)$ *such that*

$$u \in L^\infty([0,\infty); H^{2m}(\Omega)\cap H_0^m(\Omega)), \quad u_t \in L^\infty([0,\infty); H_0^m(\Omega)\cap L^2(\Omega)),$$
$$u_{tt} \in L^2([0,\infty); L^2(\Omega)).$$

Also we have a uniform decay rates for the energy.

Theorem 2.6. *Assume that* (A1)–(A4) *hold. Then there exist positive constants* w_1, w_2, w_3 *and* ε_0 *such that the solution of* (1.1) *satisfies*

$$E(t) \leqslant w_3 H_1^{-1}(w_1 t + w_2) \quad \forall\, t \geqslant 0,$$

where

(2.13)

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,\mathrm{d}s \quad \text{and} \quad H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0,\varepsilon'], \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0,\varepsilon']. \end{cases}$$

3. Proofs of main results

P r o o f of Theorem 2.5. Throughout this section we assume $u_0 \in H^{2m}(\Omega) \cap \mathcal{W}$, $u_1 \in H_0^m(\Omega) \cap L^2(\Omega)$ and $f_0 \in H_0^m(\Omega, H^m(0,1))$. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w^1, w^2, \ldots, w^k\}$, where the set $\{w^k, k \in \mathbb{N}\}$ is a basis of $H^{2m}(\Omega) \cap H_0^m(\Omega)$. Now, we define for $1 \leqslant j \leqslant k$, the sequence $\varphi^j(x, \varrho)$ as follows:

$$\varphi^j(x, 0) = w^j.$$

Then, we may extend $\varphi^j(x, 0)$ by $\varphi^j(x, \varrho)$ over $L^2(\Omega \times (0,1))$ so that $(\varphi^j)_j$ forms a base of $L^2(\Omega, H^m(0,1))$ and denote by Z_k the space generated by $\{\varphi^k\}$. We construct approximate solutions (u^k, z^k), $k = 1, 2, 3, \ldots$, in the form

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x), \quad z^k(t) = \sum_{j=1}^{k} d^{jk}(t) \varphi^j,$$

where c^{jk} and d^{jk} $(j = 1, 2, \ldots, k)$ are determined by the ordinary differential equations

$$\begin{aligned}(u_{tt}^k(t), w^j) + (\mathcal{A}^{1/2} u^k(t), \mathcal{A}^{1/2} w^j) + \mu_1(g_1(u_t^k), w^j)&\\ + \mu_2(g_2(z^k(\cdot, 1)), w^j) = a(|u^k|^{p-2} u^k, w^j)&, \end{aligned} \tag{3.1}$$

$$z^k(x, 0, t) = u_t^k(x, t), \tag{3.2}$$

$$u^k(0) = u_0^k = \sum_{j=1}^{k} (u_0, w^j) w^j \to u_0 \quad \text{in } H^{2m}(\Omega) \cap \mathcal{W} \text{ as } k \to \infty, \tag{3.3}$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^{k} (u_1, w^j) w^j \to u_1 \quad \text{in } H_0^m(\Omega) \cap L^2(\Omega) \text{ as } k \to \infty, \tag{3.4}$$

and

$$(\tau z_t^k + z_\varrho^k, \varphi^j) = 0, \quad 1 \leqslant j \leqslant k, \tag{3.5}$$

$$z^k(\varrho, 0) = z_0^k = \sum_{j=1}^{k} (f_0, \varphi^j) \varphi^j \to f_0 \quad \text{in } H_0^m(\Omega, H^m(0,1)) \text{ as } k \to \infty. \tag{3.6}$$

By virtue of the theory of ordinary differential equations, the systems (3.1)–(3.6) have a unique local solution which is extended to a maximal interval $[0, T_k]$ (with $0 < T_k < \infty$) by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that $u^k(t)$ is from the class C^2. In the next step, we obtain a priori estimates for the solution, so that it can be extended outside $[0, T_k]$ to obtain one solution defined for all $t > 0$. In order to use a standard compactness argument for the limiting procedure, it suffices to derive some a priori estimates for (u^k, z^k).

First estimate. Since the sequences u_0^k, u_1^k and z_0^k converge, the standard calculations, using (3.1)–(3.6), similar to those used to derive (2.9), yield

$$E^k(t) - E^k(0) \leqslant -\beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\,\mathrm{d}x\,\mathrm{d}s - \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\,\mathrm{d}x\,\mathrm{d}s,$$

where $\beta_1 = \mu_1 - \xi\alpha_2/\tau - \mu_2\alpha_2$ and $\beta_2 = \xi\alpha_1/\tau - \mu_2(1-\alpha_1)$. So we obtain

$$E^k(t) + \beta_1 \int_0^t \int_\Omega u_t^k g_1(u_t^k)\,\mathrm{d}x\,\mathrm{d}s + \beta_2 \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\,\mathrm{d}x\,\mathrm{d}s \leqslant E^k(0),$$

where

$$E^k(t) = \frac{1}{2}\|u_t^k\|_2^2 + \xi \int_\Omega \int_0^1 G(z^k(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x + J(u^k(t)),$$
$$J(u^k(t)) = \frac{1}{2}\|\mathcal{A}^{1/2}u^k\|_2^2 - \frac{a}{p}\|u^k\|_p^p$$

and

$$E^k(0) = \frac{1}{2}\|u_1^k\|_2^2 + \xi \int_\Omega \int_0^1 G(z^k(x,\varrho,0))\,\mathrm{d}\varrho\,\mathrm{d}x + \frac{1}{2}\|\mathcal{A}^{1/2}u_0^k\|_2^2 - \frac{a}{p}\|u_0^k\|_p^p \leqslant C_1.$$

For some C_1 independent of k we obtain the first estimate:

$$\|u_t^k\|_2^2 + \int_\Omega \int_0^1 G(z^k(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x + J(u^k(t)) + \int_0^t \int_\Omega u_t^k g_1(u_t^k)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_\Omega z^k(x,1,s) g_2(z^k(x,1,s))\,\mathrm{d}x\,\mathrm{d}s \leqslant C_1.$$

These estimates imply that the solution (u^k, z^k) exists globally in $[0,\infty[$.

(3.7) $\quad u^k$ is bounded in $L^\infty_{\mathrm{loc}}(0,\infty,H_0^m(\Omega))$,

(3.8) $\quad u_t^k$ is bounded in $L^\infty_{\mathrm{loc}}(0,\infty,L^2(\Omega))$,

(3.9) $\quad G(z^k(x,\varrho,t))$ is bounded in $L^\infty_{\mathrm{loc}}(0,\infty,L^1(\Omega\times(0,1)))$,

(3.10) $\quad u_t^k(t)g_1(u_t^k(t))$ is bounded in $L^1(\Omega\times(0,T))$,

(3.11) $\quad z^k(x,1,t)g_2(z^k(x,1,t))$ is bounded in $L^1(\Omega\times(0,T))$.

Second estimate. First, we estimate $u_{tt}^k(0)$. Taking $t=0$ in (3.1), we obtain

$$(u_{tt}^k(0), w^j) + (\mathcal{A}^{1/2}u^k(0), \mathcal{A}^{1/2}w^j) + \mu_1(g_1(u_t^k)(0), w^j) \\ + \mu_2(g_2(z^k(\cdot,1)(0), w^j)) = a(|u^k(0)|^{p-2}u^k(0), w^j),$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k,

$$(u_{tt}^k(0), u_{tt}^k(0)) + (\mathcal{A}u^k(0), u_{tt}^k(0)) + \mu_1(g_1(u_t^k)(0), u_{tt}^k(0)) \\ + \mu_2(g_2(z^k(\cdot,1)(0), u_{tt}^k(0)) = a(|u^k(0)|^{p-2}u^k(0), u_{tt}^k(0)).$$

Using Hölder's inequality, we have

$$\|u_{tt}^k(0)\| \leqslant \|\mathcal{A}u^k(0)\| + \mu_1\|g_1(u_1^k)\| + \mu_2\|g_2(z_1^k)\| + a\||u_0^k|^{p-2}u_0^k\|.$$

Since $g_1(u_1^k)$, $g_2(z_1^k)$ are bounded in $L^2(\Omega)$, (3.3), (3.4) and (3.6) yield

$$\|u_{tt}^k(0)\| \leqslant C,$$

where C is a positive constant independent of k.

Now, differentiating (3.1) with respect to t,

$$(u_{ttt}^k(t), w^j) + (\mathcal{A}u_t^k(t), w^j) + \mu_1(u_{tt}^k g_1'(u_t^k), w^j) \\ + \mu_2(z_t^k g_2'(z^k(\cdot,1)), w^j) = a(p-1)(|u^k|^{p-2}u_t^k, w^j),$$

multiplying by c_{tt}^{jk} and summing over j from 1 to k,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2) + \mu_1\int_\Omega (u_{tt}^k(t))^2 g_1'(u_t^k(t))\,\mathrm{d}x \\ + \mu_2\int_\Omega u_{tt}^k(t) z_t^k(x,1,t) g_2'(z^k(x,1,t))\,\mathrm{d}x \\ = a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)\,\mathrm{d}x. \tag{3.12}$$

We have from Hölder's inequality

$$a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)\,\mathrm{d}x \leqslant a(p-1)\|u^k(t)\|_{2(p-1)}^{p-2}\|u_t^k(t)\|_{2(p-1)}\|u_{tt}^k(t)\|_2,$$

where

$$\frac{p-2}{2(p-1)} + \frac{1}{2(p-1)} + \frac{1}{2} = 1.$$

Using Lemma 2.1, Young's inequality and (3.7), we have

$$\begin{aligned}(3.13)\qquad & a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)\,\mathrm{d}x \\ &\quad\leqslant a(p-1)C_s^{p-1}\|\mathcal{A}^{1/2}u^k(t)\|_2^{p-2}\|\mathcal{A}^{1/2}u_t^k(t)\|_2\|u_{tt}^k(t)\|_2 \\ &\quad\leqslant C(\varepsilon)\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2+\varepsilon\|u_{tt}^k(t)\|_2^2.\end{aligned}$$

Differentiating (3.5) with respect to t, we obtain

$$(\tau z_{tt}^k + z_{t\varrho}^k, \varphi^j) = 0.$$

Multiplying by d_t^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_t^k\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\varrho}\|z_t^k\|_2^2 = 0.$$

Integrating over $(0,1)$ with respect to ϱ, we obtain

$$(3.14)\qquad \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \|z_t^k\|_2^2\,\mathrm{d}\varrho + \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 - \frac{1}{2}\|u_{tt}^k(x,t)\|_2^2 = 0.$$

Taking the sum of (3.12) and (3.14), we obtain

$$\begin{aligned}(3.15)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \tau\int_0^1 \|z_t^k\|_2^2\,\mathrm{d}\varrho\Big) \\ &\qquad + \mu_1\int_\Omega (u_{tt}^k(t))^2 g_1'(u_t^k(t))\,\mathrm{d}x + \frac{1}{2}\|z_t^k(x,1,t)\|_2^2 \\ &\quad = a(p-1)\int_\Omega |u^k(t)|^{p-2}u_t^k(t)u_{tt}^k(t)\,\mathrm{d}x \\ &\qquad - \mu_2\int_\Omega u_{tt}^k(t)z_t^k(x,1,t)g_2'(z^k(x,1,t))\,\mathrm{d}x + \frac{1}{2}\|u_{tt}^k(x,t)\|_2^2.\end{aligned}$$

Using (2.2) and Young's inequality, we conclude

$$(3.16)\qquad \int_\Omega |u_{tt}^k(t)||z_t^k(x,1,t)||g_2'(z^k(x,1,t))|\,\mathrm{d}x \leqslant \varepsilon\|z_t^k(x,1,t)\|_2^2 + \frac{c_2^2}{4\varepsilon}\|u_{tt}^k\|_2^2.$$

A combination of (3.13), (3.15) and (3.16) then yields

$$\begin{aligned}(3.17)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \tau\int_0^1 \|z_t^k\|_2^2\,\mathrm{d}\varrho\Big) \\ &\qquad + \mu_1\int_\Omega (u_{tt}^k(t))^2 g_1'(u_t^k(t))\,\mathrm{d}x + \Big(\frac{1}{2}-\varepsilon\Big)\|z_t^k(x,1,t)\|_2^2 \\ &\quad \leqslant \Big(\varepsilon + \frac{c_3^2}{4\varepsilon} + \frac{1}{2}\Big)\|u_{tt}^k\|_2^2 + C(\varepsilon)\|A^{1/2}u_t^k(t)\|^2,\end{aligned}$$

integrating over $[0,t]$ for all $t \in [0,T]$ with arbitrary fixed T,

$$
\begin{aligned}
(3.18)\quad & \frac{1}{2}(\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \tau\|z_t^k(x,\varrho,t)\|^2_{L^2(\Omega\times(0,1))}) \\
&\qquad + \mu_1\int_0^t\int_\Omega (u_{tt}^k(t))^2 g_1'(u_t^k(t))\,\mathrm{d}x\,\mathrm{d}t + \Big(\frac{1}{2}-\varepsilon\Big)\int_0^t \|z_t^k(x,1,t)\|_2^2\,\mathrm{d}t \\
&\leqslant \frac{1}{2}(\|u_{tt}^k(0)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(0)\|_2^2 + \tau\|z_t^k(x,\varrho,0)\|^2_{L^2(\Omega\times(0,1))}) \\
&\qquad + \Big(\varepsilon + \frac{c_3^2}{4\varepsilon} + \frac{1}{2}\Big)\int_0^t \|u_{tt}^k\|_2^2\,\mathrm{d}t + C(\varepsilon)\int_0^t \|A^{1/2}u_t^k(t)\|^2\,\mathrm{d}t.
\end{aligned}
$$

Then from (3.18), after choosing ε small enough and using Gronwall's lemma, we obtain

$$
\begin{aligned}
&\|u_{tt}^k(t)\|_2^2 + \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \tau\|z_t^k(x,\varrho,t)\|^2_{L^2(\Omega\times(0,1))} \\
&\qquad + \mu_1\int_0^t\int_\Omega (u_{tt}^k(t))^2 g_1'(u_t^k(t))\,\mathrm{d}x\,\mathrm{d}t + \Big(\frac{1}{2}-\varepsilon\Big)\int_0^t \|z_t^k(x,1,t)\|_2^2\,\mathrm{d}t \leqslant M
\end{aligned}
$$

for all $t \in [0,T]$, where M is a positive constant independent of $k \in \mathbb{N}$. Therefore, we conclude that

$$(3.19)\qquad u_{tt}^k \text{ is bounded in } L^\infty_{\mathrm{loc}}(0,\infty,L^2(\Omega)),$$

$$(3.20)\qquad u_t^k \text{ is bounded in } L^\infty_{\mathrm{loc}}(0,\infty,H_0^m(\Omega)),$$

$$(3.21)\qquad z_t^k \text{ is bounded in } L^\infty_{\mathrm{loc}}(0,\infty,L^2(\Omega\times(0,1))).$$

Third estimate. Replacing w^j by $\mathcal{A}w^j$ in (3.1), multiplying by c_t^{jk} and summing over j from 1 to k, it follows that

$$
\begin{aligned}
(3.22)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \|\mathcal{A}u^k(t)\|_2^2) + \mu_1\int_\Omega |\mathcal{A}^{1/2}u_t^k(t)|^2 g_1'(u_t^k)\,\mathrm{d}x \\
&\qquad + \mu_2\int_\Omega \mathcal{A}^{1/2}z^k(x,1,t)\mathcal{A}^{1/2}u_t^k g_2'(z^k(x,1,t))\,\mathrm{d}x \\
&= a\int_\Omega \mathcal{A}^{1/2}(|u^k|^{p-2}u^k)\mathcal{A}^{1/2}u_t^k\,\mathrm{d}x.
\end{aligned}
$$

Replacing φ^j by $\mathcal{A}\varphi^j$ in (3.5), multiplying by d^{jk} and summing over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{A}^{1/2}z_t^k\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\varrho}\|\mathcal{A}^{1/2}z_t^k\|_2^2 = 0.$$

We integrate over $(0,1)$ to find

$$(3.23)\qquad \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \|\mathcal{A}^{1/2}z^k(t)\|_2^2\,\mathrm{d}\varrho + \frac{1}{2}\|\mathcal{A}^{1/2}z_t^k(x,1,t)\|_2^2 - \frac{1}{2}\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 = 0.$$

Combining (3.22), (3.23), using (2.2), Cauchy-Schwarz and Young's inequalities produce the estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \|\mathcal{A}u^k(t)\|_2^2 + \int_0^1 \|\mathcal{A}^{1/2}z^k(x,\varrho,t)\|_2^2\,\mathrm{d}\varrho\Big)&\\
+\mu_1\int_\Omega |\mathcal{A}^{1/2}u_t^k(t)|^2 g_1'(u_t^k)\,\mathrm{d}x&\\
+\Big(\frac{1}{2}-\varepsilon\Big)\|\mathcal{A}^{1/2}z_t^k(x,1,t)\|_2^2&\\
\leqslant C(\varepsilon)\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + a\int_\Omega \mathcal{A}^{1/2}(|u^k|^{p-2}u^k)\mathcal{A}^{1/2}u_t^k\,\mathrm{d}x.&\end{aligned}$$

Integrating the last inequality over $(0,t)$, we have

$$\begin{aligned}\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \|\mathcal{A}u^k(t)\|_2^2 + \int_0^1 \|\mathcal{A}^{1/2}z^k(x,\varrho,t)\|_2^2\,\mathrm{d}\varrho&\\
+2\mu_1\int_0^t\int_\Omega |\mathcal{A}^{1/2}u_t^k(s)|^2 g_1'(u_t^k)\,\mathrm{d}x\,\mathrm{d}t&\\
+2\Big(\frac{1}{2}-\varepsilon\Big)\int_0^t \|\mathcal{A}^{1/2}z_t^k(x,1,s)\|_2^2\,\mathrm{d}t&\\
\leqslant A^k(0) + C(\varepsilon)\int_0^t \|\mathcal{A}^{1/2}u_t^k(s)\|_2^2\,\mathrm{d}t&\\
+c\int_0^t \|\mathcal{A}^{1/2}u^k(s)\|^{p-1}\|\mathcal{A}^{1/2}u_t^k(s)\|\,\mathrm{d}t&\\
\leqslant A^k(0) + C(\varepsilon)\int_0^t \|\mathcal{A}^{1/2}u_t^k(s)\|_2^2\,\mathrm{d}t&\\
+\frac{c}{2}\int_0^t \|\mathcal{A}^{1/2}u^k(s)\|^{2(p-1)}\,\mathrm{d}t + \frac{c}{2}\int_0^t \|\mathcal{A}^{1/2}u_t^k(s)\|\,\mathrm{d}t&\\
\leqslant A^k(0) + \max\Big\{\Big(C(\varepsilon)+\frac{c}{2}\Big), C(E^k(0))^p\Big\}&\\
\times\int_0^t (\|\mathcal{A}^{1/2}u_t^k(s)\|_2^2 + \|\mathcal{A}^{1/2}u^k(s)\|^2)\,\mathrm{d}t,&\end{aligned}$$

where

$$A^k(0) = \|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \|\mathcal{A}u^k(t)\|_2^2 + \|\mathcal{A}^{1/2}z^k(x,\varrho,t)\|_{L^2(\Omega\times(0,1))}^2,$$

and using Gronwall's lemma, we have

$$\|\mathcal{A}^{1/2}u_t^k(t)\|_2^2 + \|\mathcal{A}u^k(t)\|_2^2 + \int_0^1 \|\mathcal{A}^{1/2}z^k(x,\varrho,t)\|_2^2\,\mathrm{d}\varrho \leqslant A^k(0)\mathrm{e}^{cT} \tag{3.24}$$

for all $t \in \mathbb{R}_+$, therefore we conclude that

$$(3.25) \qquad u^k \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty,H^{2m}(\Omega)),$$

$$(3.26) \qquad z^k \text{ is bounded in } L^\infty_{\text{loc}}(0,\infty,H_0^m(\Omega,L^2(0,1))).$$

Passing to the limit. Applying Dunford-Petti's theorem, we conclude from (3.7)–(3.11), (3.19)–(3.21) and (3.25)–(3.26), after replacing the sequences u^k and z^k by subsequences if necessary, that

$$(3.27) \qquad u^k \rightharpoonup u \quad \text{weak-star in } L^\infty(0,\infty;H^{2m}(\Omega)),$$

$$(3.28) \qquad u_t^k \rightharpoonup u_t \quad \text{weak-star in } L^\infty(0,\infty;H_0^m(\Omega)),$$

$$(3.29) \qquad u_{tt}^k \rightharpoonup u_{tt} \quad \text{weak-star in } L^\infty(0,\infty;L^2(\Omega)),$$

$$(3.30) \qquad g_1(u_t^k) \rightharpoonup \chi \quad \text{weak-star in } L^2(\Omega\times(0,T)),$$

$$(3.31) \qquad z^k \rightharpoonup z \quad \text{weak-star in } L^\infty(0,\infty,H_0^m(\Omega,L^2(0,1))),$$

$$(3.32) \qquad z_t^k \rightharpoonup z_t \quad \text{weak-star in } L^\infty(0,\infty,L^2(\Omega\times(0,T))),$$

$$(3.33) \qquad g_2(u_t^k) \rightharpoonup \psi \quad \text{weak-star in } L^2(\Omega\times(0,T)).$$

On the other hand, from Aubin-Lions theorem (see Lions [5]), we deduce that there exists a subsequence $\{u^m\}$ of $\{u^k\}$ such that

$$(3.34) \qquad u^m \to u \quad \text{strongly in } L^2(0,T,L^2(\Omega)),$$

$$(3.35) \qquad u_t^m \to u_t \quad \text{strongly in } L^2(0,T,L^2(\Omega)),$$

which implies

$$u^m \to u \text{ almost everywhere in } \mathcal{B}$$

and

$$(3.36) \qquad u_t^m \to u_t \text{ almost everywhere in } \mathcal{B}.$$

Hence

$$(3.37) \qquad |u^m|^{p-2}u^m \to |u|^{p-2}u \text{ almost everywhere in } \mathcal{B},$$

where $\mathcal{B} = \Omega\times(0,T)$.

$$\int_{\mathcal{B}} (|u^m|^{p-2}u^m)^{p/(p-1)}\,dx\,dt \leqslant \int_{\mathcal{B}} |u^m|^p\,dx\,dt \leqslant C\|\mathcal{A}^{1/2}u^m\|^p_{L^2(\mathcal{B})},$$

using (3.7) we obtain

$$(3.38) \qquad \|u^m\|_{L^{p/(p-1)}(\mathcal{B})} \leqslant C.$$

Thus, using (3.37), (3.38) and Lions lemma, we derive

$$|u^m|^{p-2}u^m \rightharpoonup |u|^{p-2}u \text{ weakly in } L^{p/(p-1)}(\mathcal{B}), \tag{3.39}$$

and

$$z^m \to z \text{ strongly in } L^2(0,T,L^2(\Omega)),$$

which implies

$$z^m \to z \text{ almost everywhere in } \mathcal{B}.$$

Lemma 3.1. *For each $T>0$, $g_1(u_t), g_2(z(x,1,t)) \in L^1(\mathcal{B})$ and $\|g_1(u_t)\|_{L^1(\mathcal{B})} \leqslant K$, $\|g_2(z(x,1,t))\|_{L^1(\mathcal{B})} \leqslant K$, where K is a constant independent of t.*

Proof. By (A1) and (3.36), we have

$$g_1(u_t^m(x,t)) \to g_1(u_t(x,t)) \text{ almost everywhere in } \mathcal{B},$$
$$0 \leqslant u_t^k(x,t) g_1(u_t^m(x,t)) \to u_t(x,t) g_1(u_t(x,t)) \text{ almost everywhere in } \mathcal{B}.$$

Hence, by (3.10) and Fatou's lemma, we have

$$\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t)) \, \mathrm{d}x \, \mathrm{d}t \leqslant K_1 \quad \text{for } T>0. \tag{3.40}$$

Now, we can estimate $\int_0^T \int_\Omega |g_1(u_t(x,t))| \, \mathrm{d}x \, \mathrm{d}t$. By Cauchy-Schwarz inequality and using (3.40), we have

$$\begin{aligned}\int_0^T \int_\Omega |g_1(u_t(x,t))| \, \mathrm{d}x \, \mathrm{d}t &\leqslant c|\mathcal{B}|^{1/2} \Big(\int_0^T \int_\Omega u_t(x,t) g_1(u_t(x,t)) \, \mathrm{d}x \, \mathrm{d}t \Big)^{1/2} \\ &\leqslant c|\mathcal{B}|^{1/2} K_1^{1/2} \equiv K.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^T \int_\Omega |g_2(z(x,1,t))| \, \mathrm{d}x \, \mathrm{d}t &\leqslant c|\mathcal{B}|^{1/2} \Big(\int_0^T \int_\Omega z(x,1,t) g_2(z(x,1,t)) \, \mathrm{d}x \, \mathrm{d}t \Big)^{1/2} \\ &\leqslant c|\mathcal{B}|^{1/2} K_1^{1/2} \equiv K.\end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *$g_1(u_t^k) \to g_1(u_t)$ in $L^1(\Omega \times (0,T))$, $g_2(z^k) \to g_2(z)$ in $L^1(\Omega \times (0,T))$.*

P r o o f. Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \left\{ (x,t) \in E\colon\ |g_1(u_t^k(x,t))| \leqslant \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) = \inf\{|s|\colon\ s \in \mathbb{R} \text{ and } |g(s)| \geqslant r\}$,

$$\int_E |g_1(u_t^k)| \,\mathrm{d}x\,\mathrm{d}t \leqslant c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |u_t^k g_1(u_t^k)| \,\mathrm{d}x\,\mathrm{d}t.$$

By applying (3.10) we deduce that $\sup\limits_k \int_E |g_1(u_t^k)| \,\mathrm{d}x\,\mathrm{d}t \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that

$$g_1(u_t^k) \to g_1(u_t) \text{ in } L^1(\Omega \times (0,T)).$$

Similarly, we have

$$g_2(z^k) \to g_2(z) \text{ in } L^1(\Omega \times (0,T)).$$

This completes the proof of Lemma 3.2. □

Hence

$$g_1(u_t^k) \rightharpoonup g_1(u_t) \text{ weak in } L^2(\Omega \times (0,T)), \tag{3.41}$$

$$g_2(z^k) \rightharpoonup g_2(z) \quad \text{weak in } L^2(\Omega \times (0,T)). \tag{3.42}$$

By multiplying (3.1) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

$$\begin{aligned} \int_0^T (u_t^k(t), w^j)\theta'(t)\,\mathrm{d}t + \int_0^T (\mathcal{A}^{1/2}u^k(t), \mathcal{A}^{1/2}w^j)\theta(t)\,\mathrm{d}t& \\ + \mu_1 \int_0^T (g_1(u_t^k), w^j)\theta(t)\,\mathrm{d}t& \\ + \mu_2 \int_0^T (g_2(z^k(\cdot,1)), w^j)\theta(t)\,\mathrm{d}t& \\ = \int_0^T (|u^k|^{p-2}u^k, w^j)\theta(t)\,\mathrm{d}t&, \end{aligned} \tag{3.43}$$

and by multiplying (3.3) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T) \times (0,1)$, it follows that

$$\int_0^T \int_0^1 (\tau z_t^k + z_\varrho^k, \varphi^j)\theta(t)\,\mathrm{d}t\,\mathrm{d}\varrho = 0. \tag{3.44}$$

The convergences of (3.27)–(3.33), (3.39), (3.41) and (3.42) are sufficient to pass to the limit in (3.43) and (3.44) to obtain

$$\int_0^T (u_t^k(t), w)\theta'(t)\,\mathrm{d}t + \int_0^T (\mathcal{A}^{1/2}u^k(t), \mathcal{A}^{1/2}w)\theta(t)\,\mathrm{d}t + \mu_1 \int_0^T (g_1(u_t^k), w)\theta(t)\,\mathrm{d}t \\ + \mu_2 \int_0^T (g_2(z^k(\cdot,1)), w)\theta(t)\,\mathrm{d}t = \int_0^T (|u^k|^{p-2}u^k, w)\theta(t)\,\mathrm{d}t$$

and

$$\int_0^T \int_0^1 (\tau z_t + z_\varrho, \varphi)\theta(t)\,\mathrm{d}t\,\mathrm{d}\varrho = 0.$$

By integrating, we have

$$\int_0^T (u_{tt} + \mathcal{A}u^k(t) + \mu_1 g_1(u_t) + \mu_2 g_2(z(\cdot,1)), w)\theta(t)\,\mathrm{d}t = \int_0^T |u|^{p-2}u\theta(t)\,\mathrm{d}t.$$

This completes the proof of Theorem 2.5. □

4. Asymptotic behavior

Proof of Theorem 2.6. In this section, we prove the energy decay result by constructing a suitable Lyapunov functional.

Now we define the following functional

$$L(t) = NE(t) + N_1F_1(t) + F_2(t), \tag{4.1}$$

where

$$F_1(t) = \int_\Omega u u_t\,\mathrm{d}x, \tag{4.2}$$

$$F_2(t) = \int_\Omega \int_0^1 \mathrm{e}^{-2\tau\varrho} G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x \tag{4.3}$$

and we also need the following lemma:

Lemma 4.1. *Let* (u, z) *be a solution of problem* (2.6). *Then there exist two positive constants* λ_1, λ_2 *such that*

$$\lambda_1 E(t) \leqslant L(t) \leqslant \lambda_2 E(t), \qquad t \geqslant 0, \tag{4.4}$$

for N *sufficiently large and* N_1 *a positive real number to be chosen appropriately later.*

P r o o f. Let $L(t) = N_1F_1(t) + F_2(t)$,

$$|L(t)| \leqslant N_1 \int_\Omega |uu_t| \, \mathrm{d}x + \int_\Omega \int_0^1 \mathrm{e}^{-2\tau\varrho} G(z(x,\varrho,t)) \, \mathrm{d}\varrho \, \mathrm{d}x.$$

By Young's inequality, Lemma 2.1, (2.8) and the fact that $\mathrm{e}^{-2\tau\varrho} \leqslant 1$ for all $\varrho \in [0,1]$, we obtain

$$|L(t)| \leqslant \frac{N_1}{2} \|u_t\|_2^2 + \frac{N_1 C_s}{2} \|A^{1/2}u\|_2^2 + \int_\Omega \int_0^1 G(z(x,\varrho,t)) \, \mathrm{d}\varrho \, \mathrm{d}x \leqslant cE(t).$$

Consequently, $|L(t) - NE(t)| \leqslant cE(t)$, which yields

$$(N-c)E(t) \leqslant L(t) \leqslant (N+c)E(t).$$

Choising N large enough, we obtain estimate (4.4). □

Lemma 4.2. *Let* (u,z) *be a solution of* (2.6). *Then the functional* $F_1(t)$ *defined by* (4.2) *satisfies for any* $\eta > 0$ *the estimate*

(4.5)
$$\begin{aligned} F_1'(t) \leqslant \|u_t\|_2^2 + a\|u\|_p^p - (1 - \eta C_s^2(\mu_1+\mu_2))\|\mathcal{A}^{1/2}u\|_2^2 \\ + \frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2 \, \mathrm{d}x + \frac{\mu_2 c_2}{4\eta}\int_\Omega z(x,1,t) g_2(z(x,1,t)) \, \mathrm{d}x. \end{aligned}$$

P r o o f. Taking the derivative of $F_1(t)$ with respect to t and using the first equation of (2.6), we obtain

(4.6)
$$\begin{aligned} F_1'(t) = \|u_t\|_2^2 + \int_\Omega u_{tt} u \, \mathrm{d}x = \|u_t\|_2^2 + a\|u\|_p^p - \|\mathcal{A}^{1/2}u\|_2^2 \\ - \mu_1 \int_\Omega u g_1(u_t(x,t)) \, \mathrm{d}x - \mu_2 \int_\Omega u g_2(z(x,1,t)) \, \mathrm{d}x. \end{aligned}$$

Now, we estimate the terms on the right hand side of (4.6) using Young's inequality and Lemma 2.1 and we obtain

(4.7)
$$\int_\Omega u g_1(u_t) \, \mathrm{d}x \leqslant \eta C_s^2 \|\mathcal{A}^{1/2}u\|_2^2 + \frac{1}{4\eta}\int_\Omega |g_1(u_t)|^2 \, \mathrm{d}x,$$

(4.8)
$$\int_\Omega u g_2(z(x,1,t)) \, \mathrm{d}x \leqslant \eta C_s^2 \|\mathcal{A}^{1/2}u\|_2^2 + \frac{1}{4\eta}\int_\Omega |g_2(z(x,1,t))|^2 \, \mathrm{d}x.$$

From (2.2), (4.8) becomes

(4.9)
$$\int_\Omega u g_2(z(x,1,t)) \, \mathrm{d}x \leqslant \eta C^2 \|\mathcal{A}^{1/2}u\|_2^2 + \frac{c_2}{4\eta}\int_\Omega z(x,1,t) g_2(z(x,1,t)) \, \mathrm{d}x.$$

Estimate (4.5) follows by substituting (4.7) and (4.9) into (4.6). □

Lemma 4.3. *Let* (u,z) *be the solution to* (2.6). *Then the functional* $F_2(t)$ *defined by* (4.3) *satisfies the estimate*

$$
\begin{aligned}
(4.10)\quad F_2'(t) \leqslant & -2\mathrm{e}^{-2\tau}\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x \\
& -\frac{\alpha_1\mathrm{e}^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,\mathrm{d}x+\frac{\alpha_2}{\tau}\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x.
\end{aligned}
$$

P r o o f. By differentiating (4.3) with respect to t and using (2.3) and (2.5), we obtain

$$
\begin{aligned}
F_2'(t) = & -\frac{1}{\tau}\int_\Omega\int_0^1 \mathrm{e}^{-2\tau\varrho}\frac{\partial}{\partial\varrho}G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x \\
= & -\frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\varrho}(\mathrm{e}^{-2\tau\varrho}G(z(x,\varrho,t)))\,\mathrm{d}x+2\tau\mathrm{e}^{-2\tau\varrho}G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x \\
= & -\frac{1}{\tau}\int_\Omega(\mathrm{e}^{-2\tau}G(z(x,1,t))-G(u_t(x,t)))\,\mathrm{d}x \\
& -2\int_\Omega\int_0^1 \mathrm{e}^{-2\tau\varrho}G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x \\
= & -\frac{1}{\tau}\int_\Omega \mathrm{e}^{-2\tau}G(z(x,1,t))\,\mathrm{d}x+\frac{1}{\tau}\int_\Omega G(u_t(x,t))\,\mathrm{d}x \\
& -2\int_\Omega\int_0^1 \mathrm{e}^{-2\tau\varrho}G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x \\
= & -2F_2(t)+\frac{1}{\tau}\int_\Omega G(u_t(x,t))\,\mathrm{d}x-\frac{\mathrm{e}^{-2\tau}}{\tau}\int_\Omega G(z(x,1,t))\,\mathrm{d}x \\
\leqslant & -2F_2(t)+\frac{\alpha_2}{\tau}\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x \\
& -\frac{\alpha_1\mathrm{e}^{-2\tau}}{\tau}\int_\Omega z(x,1,t)g_2(z(x,1,t))\,\mathrm{d}x.
\end{aligned}
$$

Since $-\mathrm{e}^{-2\tau\varrho}$ is an increasing function, we have $-\mathrm{e}^{-2\tau\varrho}\leqslant-\mathrm{e}^{-2\tau}$ for all $\varrho\in[0,1]$, we deduce

$$
-2F(t)\leqslant-2\mathrm{e}^{-2\tau}\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x.
$$

The proof of Lemma 4.3 is complete. □

Lemma 4.4. *Let* (u,z) *be a solution of* (2.6) *and assume that* (A1)–(A4) *hold. Then the functional defined by* (4.1) *satisfies*

$$
(4.11)\qquad L'(t)\leqslant -mE(t)+C_1(\|u_t\|_2^2+\|g_1(u_t)\|_2^2)
$$

for positive constants m *and* C_1.

P r o o f. By differentiating (4.1) and recalling (2.9), (4.5) and (4.10), we obtain

$$
\begin{aligned}
L'(t) &= NE'(t) + N_1F_1'(t) + F_2'(t) \\
&\leqslant -\Big(N\beta_1 - \frac{\alpha_2}{\tau}\Big)\int_\Omega u_t(x,t)g_1(u_t(x,t))\,\mathrm{d}x \\
&\quad - \Big(N\beta_2 - N_1\frac{\mu_2 c_2}{4\eta} + \frac{\alpha_1\mathrm{e}^{-2\tau}}{\tau}\Big)\int_\Omega z(x,1,t)g_2(z(x,1,t))\,\mathrm{d}x \\
&\quad - 2\mathrm{e}^{-2\tau}\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x + N_1\frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,\mathrm{d}x \\
&\quad + N_1\|u_t\|_2^2 + N_1a\|u\|_p^p - N_1(1-\eta C_s^2(\mu_1+\mu_2))\|\mathcal{A}^{1/2}u\|_2^2.
\end{aligned} \tag{4.12}
$$

We choose N large enough such that

$$N\beta_1 - \frac{\alpha_2}{\tau} > 0 \quad\text{and}\quad N\beta_2 - N_1\frac{\mu_2 c_2}{4\eta} > 0.$$

Thus, (4.12) becomes

$$
\begin{aligned}
L'(t) \leqslant{}& -2\mathrm{e}^{-2\tau}\int_\Omega\int_0^1 G(z(x,\varrho,t))\,\mathrm{d}\varrho\,\mathrm{d}x - N_1(1-\eta C^2(\mu_1+\mu_2))\|\mathcal{A}^{1/2}u\|_2^2 \\
&+ N_1a\|u\|_p^p + N_1\|u_t\|_2^2 + N_1\frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,\mathrm{d}x.
\end{aligned}
$$

We choose η small enough so that $1-\eta C^2(\mu_1+\mu_2) > 0$. Noting by

$$m = \min\Big\{2N_1(1-\eta C^2(\mu_1+\mu_2)), \frac{2\mathrm{e}^{-2\tau}}{\xi}\Big\}$$

and choosing N_1 small enough so that $pN_1 \leqslant m$, we obtain

$$
\begin{aligned}
L'(t) &\leqslant -mE(t) + \frac{m}{2}\|u_t\|_2^2 + N_1\|u_t\|_2^2 + N_1\frac{\mu_1}{4\eta}\int_\Omega |g_1(u_t)|^2\,\mathrm{d}x \\
&\leqslant -mE(t) + c\Big(\|u_t\|_2^2 + \int_\Omega |g_1(u_t)|^2\,\mathrm{d}x\Big).
\end{aligned}
$$

This completes the proof of Lemma 4.4. □

As in Komornik (see [4]), we consider the following partition of Ω:

$$\Omega_1 = \{x\in\Omega\colon\ |u_t|\geqslant\varepsilon\}, \quad \Omega_2 = \{x\in\Omega\colon\ |u_t|\leqslant\varepsilon\}.$$

By using (2.1), we have

$$\int_{\Omega_1}|u_t|^2\,\mathrm{d}x + \int_{\Omega_1}|g_1(u_t)|^2\,\mathrm{d}x \leqslant (c_1'+c_1)\int_{\Omega_1}u_tg_1(u_t)\,\mathrm{d}x \leqslant -\mu_1E'(t) \tag{4.13}$$

and

$$\int_{\Omega_2}|u_t|^2\,\mathrm{d}x + \int_{\Omega_2}|g_1(u_t)|^2\,\mathrm{d}x \leqslant \int_{\Omega_2}H^{-1}(u_tg_1(u_t))\,\mathrm{d}x.$$

Case 1. H is linear on $[0,\varepsilon']$. In this case, one can easily check that there exists $\mu_2 > 0$ such that

$$\int_{\Omega_2} |u_t|^2 \,\mathrm{d}x + \int_{\Omega_2} |g_1(u_t)|^2 \,\mathrm{d}x \leqslant -\mu_2 E'(t). \tag{4.14}$$

Substitution of (4.13) and (4.14) into (4.11) gives

$$(L(t) + \mu E(t))' \leqslant m H_2(E(t)), \quad \text{where } \mu = C_1(\mu_1 + \mu_2). \tag{4.15}$$

Case 2. $H'(0) > 0$ and $H'' > 0$ on $]0,\varepsilon']$. We define

$$I_1(t) = \frac{1}{|\Omega_2|}\int_{\Omega_2} u_t g(u_t)\,\mathrm{d}x$$

and use Jensen's inequality and the concavity of H^{-1} to obtain

$$H^{-1}(I_1(t)) \geqslant \widetilde{C}\int_{\Omega_2} H^{-1}(u_t g(u_t))\,\mathrm{d}x.$$

By using (2.1), we obtain

$$\begin{aligned}\int_{\Omega_2} (|u_t|^2 + |g_1(u_t)|^2)\,\mathrm{d}x &\leqslant \int_{\Omega_2} H^{-1}(u_t g_1(u_t))\,\mathrm{d}x \\ &\leqslant \widetilde{C} H^{-1}(I_1(t)) \leqslant \widetilde{C} H^{-1}(-C_2 E'(t)).\end{aligned} \tag{4.16}$$

Combining (4.11), (4.13) and (4.16), we get

$$(L(t) + C_1\mu_1 E(t))' \leqslant -mE(t) + \widetilde{C} H^{-1}(-C_2 E'(t)). \tag{4.17}$$

By recalling that $E' \leqslant 0$, $H' > 0$, $H'' > 0$ on $(0,\varepsilon]$ and making use of (4.17), we obtain

$$\begin{aligned}&(H'(\varepsilon_0 E(t))(L(t) + C_1\mu_1 E(t)) + \widetilde{C} C_2 E(t))' \\ &\qquad = \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C_1\mu_1 E(t)) \\ &\qquad\quad + H'(\varepsilon_0 E(t))(L(t) + C_1\mu_1 E(t))' + \widetilde{C} C_2 E'(t) \\ &\qquad \leqslant -mH'(\varepsilon_0 E(t))E(t) + \widetilde{C} C_2 E'(t) \\ &\qquad\quad + \widetilde{C} H'(\varepsilon_0 E(t)) H^{-1}(-C_2 E'(t)),\end{aligned}$$

by using Remark 2.2 with H^*, the convex conjugate of H in the sense of Young, we obtain

$$(4.18)\qquad \begin{aligned}(H'(\varepsilon_0 E(t))(L(t)+C_1\mu_1 E(t))+\widetilde{C}C_2E(t))'\\ \leqslant\ &-mH'(\varepsilon_0E(t))E(t)+\widetilde{C}H^*(H'(\varepsilon_0E(t)))\\ \leqslant\ &-mH'(\varepsilon_0E(t))E(t)+\widetilde{C}\varepsilon_0H'(\varepsilon_0E(t))E(t)\\ \leqslant\ &-C_3H'(\varepsilon_0E(t))E(t)\\ =\ &-C_3H_2(E(t)).\end{aligned}$$

Let

$$\widetilde{L}(t)=\begin{cases}L(t)+\mu E(t) & \text{if } H \text{ is linear on } [0,\varepsilon],\\ H'(\varepsilon_0E(t))\{L(t)C+C_1\mu_1E(t))\}+\widetilde{C}C_2E(t) & \\ & \text{if } H'(0)>0 \text{ and } H''>0 \text{ on }]0,\varepsilon].\end{cases}$$

From (4.15) and (4.18), it follows

$$(4.19)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}(t)\leqslant -C_3H_2(E(t))\quad \forall\, t\geqslant t_0.$$

From Lemma 4.1, we have that $L(t)$ is equivalent to $E(t)$. So, $\widetilde{L}(t)$ is also equivalent to $E(t)$ for some positive constants $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$,

$$(4.20)\qquad \widetilde{\varepsilon}_1E(t)\leqslant \widetilde{L}(t)\leqslant \widetilde{\varepsilon}_2E(t).$$

Let

$$(4.21)\qquad L(t)=\frac{1}{\widetilde{\varepsilon}_2}\widetilde{L}(t).$$

Then we observe from (4.19) and (4.21) that

$$L'(t)\leqslant -\frac{C_3}{\widetilde{\varepsilon}_2}H_2(E(t))\leqslant -\frac{C_3}{\widetilde{\varepsilon}_2}H_2\Big(\frac{1}{\widetilde{\varepsilon}_2}\widetilde{L}(t)\Big)=-\frac{C_3}{\widetilde{\varepsilon}_2}H_2(L(t)).$$

Then

$$\frac{L'(t)}{H_2(L(t))}\leqslant -\frac{C_3}{\widetilde{\varepsilon}_2}.$$

By recalling (2.13), we deduce $H_2(t)=-1/H_1'(t)$, hence

$$L'(t)H_1'(L(t))\geqslant \frac{C_3}{\widetilde{\varepsilon}_2}.$$

A simple integration over $(0,t)$ yields

$$H_1(L(t))\geqslant H_1(L(0))+\frac{C_3}{\widetilde{\varepsilon}_2}t.$$

Exploiting the fact that H_1^{-1} is decreasing, we infer

$$L(t) \leqslant H_1^{-1}\Big(\frac{C_3}{\widetilde{\varepsilon}_2}t + H_1(L(0))\Big).$$

Consequently, the equivalence of L, $\widetilde{L}$ and E yields the estimate

$$E(t) \leqslant w_3 H_1^{-1}(w_1 t + w_2).$$

This completes the proof of Theorem 2.6. □

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