ON THE MEROMORPHIC SOLUTIONS OF A CERTAIN TYPE OF NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATION

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Abstract. The main objective of this paper is to give the specific forms of the meromorphic solutions of the nonlinear difference-differential equation

$$f^n(z)+P_d(z,f)=p_1(z)\mathrm{e}^{\alpha_1(z)}+p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f)$ is a difference-differential polynomial in $f(z)$ of degree $d\leqslant n-1$ with small functions of $f(z)$ as its coefficients, p_1, p_2 are nonzero rational functions and α_1, α_2 are nonconstant polynomials. More precisely, we find out the conditions for ensuring the existence of meromorphic solutions of the above equation.

Keywords: nonlinear differential equation; differential polynomial; Nevanlinna's value distribution theory

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1. Introduction, definitions and results

In the paper, a meromorphic function means a function meromorphic in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna theory, e.g., $N(r,f)$, $m(r,f)$, $T(r,f)$, $N(r,a;f)$, $\overline{N}(r,a;f)$, $m(r,a;f)$, etc. (see [2]). We denote by $S(r,f)$ a quantity, not necessarily the same at each of its occurrence, that satisfies the condition $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ except possibly a set of finite linear measure.

A meromorphic function $a=a(z)$ is called a small function of a meromorphic function f if $T(r,a)=S(r,f)$. Let us denote by $S(f)$ the class of all small functions of f. Clearly $\mathbb{C}\subset S(f)$ and if f is a transcendental function, then every rational function is a member of $S(f)$.

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The order and hyper-order of a meromorphic function $f(z)$ are denoted and defined by

$$\varrho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\quad\text{and}\quad\varrho_2(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r},$$

respectively. It is clear that if $\varrho(f)<\infty$, then $\varrho_2(f)=0$.

Let $k\in\mathbb{N}$ and $a\in\mathbb{C}\cup\{\infty\}$. We use the notations $N_{k)}(r,a;f)$ and $N_{(k+1}(r,a;f)$ to denote the counting function of a-points of f with multiplicity not greater than k and the counting function of a-points of f with multiplicity greater than k, respectively. Similarly, $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ are their reduced functions, respectively.

By a differential polynomial $P_d(z,f)$ in $f(z)$ of degree d, we mean a polynomial in $f(z)$ and its derivatives of a total degree at most d with small functions of $f(z)$ as coefficients. When the coefficients are polynomials, we call $P_d(z,f)$ an algebraic differential polynomial.

By a difference-differential polynomial $P_d(z,f)$ in $f(z)$ of degree d, we mean a polynomial in $f(z)$, its shifts and their derivatives of a total degree at most d with small functions of $f(z)$ as coefficients.

It is always an interesting and quite difficult problem to prove the existence of the entire or meromorphic solutions $f(z)$ of a given differential equation and to find out the solutions if they exist. A special type of nonlinear differential equation

$$f^n(z)+P_d(z,f)=h(z),$$

where $h(z)$ is a given entire or meromorphic function and $P_d(z,f)$ is a differential polynomial in $f(z)$ of degree d, has become a matter of increasing interest among the researchers.

It is easy to show that the function $f_1(z)=\sin z$ is a solution of the nonlinear differential equation $4f^3(z)+3f''(z)=-\sin 3z$. In [3], it was proved that $f_2(z)=-\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$ is also a solution of this equation. In 2004, Yang and Li (see [10]) proved that this equation admits exactly three entire solutions, namely $f_1(z)$, $f_2(z)$ and $f_3(z)=\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$. Since the function $-\sin 3z$ is a linear combination of $\mathrm{e}^{\mathrm{i}3z}$ and $\mathrm{e}^{-\mathrm{i}3z}$, so it is interesting to find all entire solutions of the general equation

$$f^n(z)+P_d(z,f)=p_1\mathrm{e}^{\lambda z}+p_2\mathrm{e}^{-\lambda z}, \tag{1.1}$$

where p_1, p_2 and λ are nonzero constants and $P_d(z,f)$ denotes a differential polynomial in $f(z)$ of degree $d\leqslant n-1$.

In 2004, Yang and Li (see [10]) answered the above question partially and obtained the following result.

Theorem A ([10]). *Let $n \in \mathbb{N} \setminus \{1, 2\}$, $P_d(z, f)$ be a differential polynomial in f of degree $d \leqslant n-3$, $b \in S(f)$ and λ, p_1, p_2 be three nonzero constants. Then the differential equation*

$$f^n(z) + P_d(z, f) = b(z)(p_1 \mathrm{e}^{\lambda z} + p_2 \mathrm{e}^{-\lambda z})$$

has no transcendental entire solution $f(z)$.

In 2006, Li and Yang (see [6]) derived similar conclusion when the term on the right-hand side of equation (1.1) was replaced by $p_1(z)\mathrm{e}^{\alpha_1 z} + p_2(z)\mathrm{e}^{\alpha_2 z}$, where $p_1(z)$, $p_2(z)$ are nonzero polynomials, α_1, α_2 are two constants with $\alpha_1/\alpha_2 \notin \mathbb{Q}$, and presented their result as follows.

Theorem B ([6]). *Let $n \in \mathbb{N}\setminus\{1, 2, 3\}$ and $P_d(z, f)$ denote an algebraic differential polynomial in $f(z)$ of degree $d \leqslant n-3$. Let $p_1(z)$, $p_2(z)$ be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \notin \mathbb{Q}$. Then the differential equation*

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1 z} + p_2(z)\mathrm{e}^{\alpha_2 z}$$

has no transcendental entire solutions.

In 2011, Li derived the possible forms of solutions of equation (1.1) when $d \leqslant n-2$, and obtained the following result (see [5]).

Theorem C ([5]). *Let $n \in \mathbb{N} \setminus \{1\}$, $P_d(z, f)$ be a differential polynomial in $f(z)$ of degree $d \leqslant n-2$ and p_1, p_2, α_1, α_2 be nonzero constants and $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the equation*

$$f^n(z) + P_d(z, f) = p_1 \mathrm{e}^{\alpha_1 z} + p_2 \mathrm{e}^{\alpha_2 z}$$

satisfying $N(r, \infty; f) = S(r, f)$, then one of the following holds:

(i) $f(z) = c_0(z) + c_1 \mathrm{e}^{\alpha_1/nz}$,

(ii) $f(z) = c_0(z) + c_2 \mathrm{e}^{\alpha_2/nz}$,

(iii) $f(z) = c_1 \mathrm{e}^{\alpha_1/nz} + c_2 \mathrm{e}^{\alpha_2/nz}$ *and* $\alpha_1 + \alpha_2 = 0$,

where $c_0 \in S(f)$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_i^n = p_i$, $i = 1, 2$.

In 2013, Liao, Yang and Zhang (see [7]) extended the above results by considering that $h(z)$ is a meromorphic function of integer order and improved the results of Theorems B and C. Actually, they obtained the following result.

Theorem D ([7]). *Let $n \in \mathbb{N} \setminus \{1, 2\}$ and $P_d(z, f)$ be a differential polynomial in $f(z)$ of degree d with rational functions as its coefficients. Suppose that p_1, p_2 are nonzero rational functions and α_1, α_2 are polynomials. If $d \leqslant n-2$, the differential equation*

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}$$

admits a meromorphic function $f(z)$ with finitely many poles. Then α_1'/α_2' is a rational number. Furthermore, only one of the following four cases holds:

(1) *$f(z) = q(z)\mathrm{e}^{p(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = 1$, where $q(z)$ is a nonzero rational function and $p(z)$ is a polynomial with $np'(z) = \alpha_1'(z) = \alpha_2'(z)$;*

(2) *$f(z) = q(z)\mathrm{e}^{p(z)}$ and either $\alpha_1'(z)/\alpha_2'(z) = k/n$ or $\alpha_1'(z)/\alpha_2'(z) = n/k$, where $q(z)$ is a nonzero rational function, $k \in \mathbb{N}$ with $1 \leqslant k \leqslant d$ and $p(z)$ is a polynomial with $np'(z) = \alpha_1'(z)$ or $np'(z) = \alpha_2'(z)$;*

(3) *$f(z)$ satisfies the first order linear differential equation $f'(z) = n^{-1}(p_2'(z)/p_2(z) + \alpha_2'(z))f(z) + \psi(z)$ and $\alpha_1'(z)/\alpha_2'(z) = (n-1)/n$ or $f(z)$ satisfies the first order linear differential equation $f'(z) = n^{-1}(p_1'(z)/p_1(z) + \alpha_1'(z))f(z) + \psi(z)$ and $\alpha_1'(z)/\alpha_2'(z) = n/(n-1)$, where $\psi(z)$ is a rational function;*

(4) *$f(z) = \gamma_1(z)\mathrm{e}^{\beta_1(z)} + \gamma_2(z)\mathrm{e}^{-\beta_1(z)}$ and $\alpha_1'(z)/\alpha_2'(z) = -1$, where $\gamma_1(z)$, $\gamma_2(z)$ are nonzero rational functions and $\beta_1(z)$ is a polynomial with $n\beta_1'(z) = \alpha_1'(z)$ or $n\beta_1'(z) = \alpha_2'(z)$.*

Now it is interesting to find out all the meromorphic solutions of the following nonlinear differential-difference equation:

$$f^n(z) + P_d(z, f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}, \tag{1.2}$$

where $P_d(z, f)$ is a differential-difference polynomial in $f(z)$ of degree $d \leqslant n-1$ with small functions of $f(z)$ as its coefficients, $p_1(z)$, $p_2(z)$ are nonzero rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant polynomials.

In 2018, Lü, Wu, Wang and Yang (see [8]) derived the possible forms of the solutions of equation (1.2) when $n = 3$, $d = 1$, and obtained the following result.

Theorem E ([8]). *Let $P_d(z, f)$ denote a difference-differential polynomial in $f(z)$ of degree one with small functions as its coefficients such that $P_d(z, 0) \equiv 0$ and let p_1, p_2, α_1, α_2 be nonzero constants such that $\alpha_1 \neq \alpha_2$. If $f(z)$ is an entire solution with $\varrho_2(f) < 1$ to equation*

$$f^3(z) + P_d(z, f) = p_1\mathrm{e}^{\alpha_1 z} + p_2\mathrm{e}^{\alpha_2 z},$$

then one of the following relations holds:

(1) $f(z) = c_1 \exp(\frac{1}{3}\alpha_1 z) + c_2 \exp(\frac{1}{3}\alpha_2 z)$, *where* $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *satisfying* $c_1^3 = p_1$, $c_2^3 = p_2$ *and* $\alpha_1 + \alpha_2 = 0$,
(2) $f^3(z) = (p_1 - c_1)\exp(\alpha_1 z)$ *and* $P_d(z, f) = c_1 \exp(\alpha_1 z) + p_2 \exp(\alpha_2 z)$, *where* c_1 *is a constant,*
(3) $f^3(z) = (p_2 - c_2)\exp(\alpha_2 z)$ *and* $P_d(z, f) = p_1 \exp(\alpha_1 z) + c_2 \exp(\alpha_2 z)$, *where* c_2 *is a constant.*

For further study, it is quite natural to ask the following questions.

Question 1. What happens if $f^3(z)$ is replaced by $f^n(z)$, where $n \in \mathbb{N}$, in Theorem E?

Question 2. What will happen if we delete the condition $P_d(z, 0) \equiv 0$ in Theorem E?

Question 3. How to find the solutions of equation (1.2) under the condition $n \geqslant d + 2$?

The main objective of this paper is to find out the possible answers to the above questions. The following theorem is the main result of the paper.

Theorem 1.1. *Let* $P_d(z, f)$ *be a difference-differential polynomial in* $f(z)$ *of degree* $d \in \mathbb{N} \cup \{0\}$ *with small functions of* $f(z)$ *as its coefficients and* $n \in \mathbb{N}$ *such that* $n \geqslant d + 2$. *Suppose that* $p_1(z)$, $p_2(z)$ *are nonzero rational functions and* $\alpha_1(z)$, $\alpha_2(z)$ *are non-constant polynomials. If* $f(z)$ *is a meromorphic solution to the difference-differential equation*

$$f^n(z) + P_d(z, f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} \tag{1.3}$$

satisfying $\varrho_2(f) < 1$ *and* $N(r, \infty; f) = S(r, f)$, *then one of the following cases holds:*
(1) $f(z) = q(z)e^{\alpha_2(z)/n}$ *and* $\alpha_1'(z) \equiv \alpha_2'(z)$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = c_0 p_2(z)$, *where* $c_0 \in \mathbb{C} \setminus \{0\}$;
(2) $f(z) = q(z)e^{\alpha_1(z)/n}$ *and* $\alpha_1' \equiv \alpha_2'(z)$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = p_1(z) + c_1 p_2(z)$, *where* $c_1 \in \mathbb{C}$;
(3) $T(r, e^{(k\alpha_1 - n\alpha_2)/(n+1)}) = S(r, f)$, *where* $k \in \{0, 1, 2, \ldots, d\}$. *In this case,* $f(z) = q(z)e^{\alpha_1(z)/n}$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = p_1(z)$;
(4) $T(r, e^{(k\alpha_2 - n\alpha_1)/(n+1)}) = S(r, f)$, *where* $k \in \{0, 1, 2, \ldots, d\}$. *In this case,* $f(z) = q(z)e^{\alpha_2(z)/n}$, *where* $q(z)$ *is a nonzero rational function such that* $q^n(z) = p_2(z)$;
(5) $T(r, e^{(n-1)\alpha_1 - n\alpha_2}) = S(r, f)$. *In this case,* $f(z) = u_1(z)e^{\alpha_1(z)/n} - v_1(z)$, *where* $u_1(z)$ *and* $v_1(z)$ *are nonzero small functions of* $f(z)$ *such that* $u_1^n(z) = p_1(z)$;
(6) $T(r, e^{(n-1)\alpha_2 - n\alpha_1}) = S(r, f)$. *In this case,* $f(z) = u_2(z)e^{\alpha_2(z)/n} - v_2(z)$, *where* $u_2(z)$ *and* $v_2(z)$ *are nonzero small functions of* $f(z)$ *such that* $u_2^n(z) = p_2(z)$;

(7) $T(r, \mathrm{e}^{\alpha_1-\alpha_2}) = S(r,f)$. *In this case,* $f(z) = q(z)\mathrm{e}^{\alpha_1/n}$ *and* $P_d(z,f) \equiv 0$, *where* $q(z)$ *and* $\varphi(z)$ *are nonzero small functions of* $f(z)$ *such that* $q^n(z) = p_1(z) + \varphi(z)p_2(z)$;

(8) $T(r, \mathrm{e}^{\alpha_1+\alpha_2}) = S(r,f)$. *In this case,* $f(z) = \delta_1(z)\mathrm{e}^{\gamma(z)} + \delta_2(z)\mathrm{e}^{-\gamma(z)}$, *where* $\delta_1(z)$, $\delta_2(z)$ *are nonzero small functions of* $f(z)$ *and* $\gamma(z)$ *is a non-constant polynomial such that either* $\mathrm{e}^{n\gamma(z)+\alpha_1(z)}$ *is a small function of* $f(z)$ *or* $\mathrm{e}^{n\gamma(z)+\alpha_2(z)}$ *is a small function of* $f(z)$.

From Theorem 1.1 we have the following corollary.

Corollary 1.1. *Equation* (1.2) *does not have any meromorphic solution* $f(z)$ *satisfying* $N(r,\infty;f) = S(r,f)$, $\varrho(f) = \infty$ *and* $\varrho_2(f) < 1$.

Remark 1.1. It is easy to see that conclusions (5) and (6) in Theorem 1.1 can not be removed by the following examples.

Example 1.1. Let us consider the difference-differential equation

$$f^3(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) = -\frac{1}{3}f'(z) - \frac{2}{27}$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 3z$ and $\alpha_2(z) = 2z$. Here $n = 3$ and $d = 1$. One can easily verify that $f(z) = u_1(z)\mathrm{e}^{\alpha_1(z)/3} - v_1(z)$, where $u_1(z) = 1$, $v_1(z) = \frac{1}{3}$ is a solution of the given difference-differential equation.

Example 1.2. Let us consider the difference-differential equation

$$f^4(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) = f^2(z+c) - 3(f'(z))^2 - 4f''(z)f(z) - 2f(z+c)$, $p_1(z) = 1$, $p_2(z) = 4$, $\alpha_1(z) = 4z$, $\alpha_2(z) = 3z$ and $c \in \mathbb{C}\setminus\{0\}$ such that $\mathrm{e}^c = 1$. Here $n = 4$ and $d = 2$. One can easily verify that $f(z) = u_2(z)\mathrm{e}^{\alpha_2(z)/4} - v_2(z)$, where $u_2(z) = 1$ and $v_2(z) = -1$ is a solution of the given difference-differential equation.

Remark 1.2. It is easy to see that conclusion (8) in Theorem 1.1 cannot be removed by the following examples.

Example 1.3. Let us consider the difference-differential equation

$$f^2(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) \equiv -2$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 2z$ and $\alpha_2(z) = -2z$. Here $n = 2$ and $d = 0$. One can easily verify that $f(z) = \delta_1(z)\mathrm{e}^{\gamma(z)} + \delta_2(z)\mathrm{e}^{-\gamma(z)}$ is a solution of the given difference-differential equation, where $\delta_1(z) = \delta_2(z) = 1$ and $\gamma(z) = z$. Also we see that $\mathrm{e}^{n\gamma(z)+\alpha_2(z)}$ is a small function of $f(z)$.

E x a m p l e 1.4. Let us consider the difference-differential equation

$$f^3(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)},$$

where $P_d(z,f) = zf''(z) - f'(z) - (4z^3+3)f(z)$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 3z^2$ and $\alpha_2(z) = -3z^3$. Here $n = 3$ and $d = 1$. One can easily verify that $f(z) = \delta_1(z)\mathrm{e}^{\gamma(z)} + \delta_2(z)\mathrm{e}^{-\gamma(z)}$ is a solution of the given difference-differential equation, where $\delta_1(z) = \delta_2(z) = 1$ and $\gamma(z) = z^2$. Also we see that $\mathrm{e}^{n\gamma(z)+\alpha_2(z)}$ is a small function of $f(z)$.

2. Lemmas

The following lemmas are needful in the proof of our main result.

Lemma 2.1 ([4]). *Let $f(z)$ be a transcendental meromorphic function and $f^n(z)P(z,f) = Q(z,f)$, where $P(z,f)$ and $Q(z,f)$ are polynomials in $f(z)$ and its derivatives with meromorphic coefficients, say $\{a_\lambda(z)\colon \lambda \in I\}$ such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree of $Q(z,f)$ as a polynomial in $f(z)$ and its derivatives is less than or equal to n, then $m(r,P(z,f)) = S(r,f)$.*

Lemma 2.2 ([2]). *Let $f(z)$ be a non-constant meromorphic function and let $a_i \in S(f)$, $i = 1,2$. Then $T(r,f) \leqslant \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f)$.*

Lemma 2.3 ([9]). *Let $f(z)$ be a non-constant meromorphic function and let $a_n\,(\not\equiv 0), a_{n-1},\dots,a_0 \in S(f)$. Then $T\Big(r,\sum\limits_{i=0}^{n} a_i f^i\Big) = nT(r,f) + S(r,f)$.*

Lemma 2.4 ([11]). *Let f be a non-constant meromorphic function and $k \in \mathbb{N}$. Then*

$$m\Big(r,\frac{f^{(k)}}{f}\Big) = S(r,f)$$

and if f is of finite order of growth, then

$$m\Big(r,\frac{f^{(k)}}{f}\Big) = O(\log r).$$

Lemma 2.5 ([1]). *Let $c \in \mathbb{C}\setminus\{0\}$, $\varepsilon > 0$ and $f(z)$ be a non-constant meromorphic function such that $\varrho_2(f) < 1$. Then*

$$m\Big(r,\frac{f(z+c)}{f(z)}\Big) = o\Big(\frac{T(r,f)}{r^{1-\varrho_2(f)-\varepsilon}}\Big)$$

outside of an exceptional set of finite logarithmic measure.

Lemma 2.6. *Let $n \in \mathbb{N}$ and $P_d(z,f)$ be a difference-differential polynomial in $f(z)$ of degree $d \leqslant n-1$ with small functions of $f(z)$ as its coefficients. Suppose that $p_1(z)$, $p_2(z)$ are nonzero rational functions and $\alpha_1(z)$, $\alpha_2(z)$ are non-constant polynomials. If $f(z)$ is a meromorphic solution to the nonlinear difference-differential equation*

$$f^n(z) + P_d(z,f) = p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)} \tag{2.1}$$

satisfying $\varrho_2(f) < 1$ and $N(r,\infty;f) = S(r,f)$, then $f(z)$ is a transcendental meromorphic function of finite order.

P r o o f. Let $f(z)$ be a rational function satisfying the differential-difference equation (2.1). Then clearly $p_1(z)\mathrm{e}^{\alpha_1(z)}+p_2(z)\mathrm{e}^{\alpha_2(z)}$ is a rational function, say $R_1(z)$, and so $-p_1(z)\mathrm{e}^{\alpha_1(z)} = p_2(z)\mathrm{e}^{\alpha_2(z)}-R_1(z)$. This shows that $p_2(z)\mathrm{e}^{\alpha_2(z)}-R_1(z)$ has finitely many zeros. But from Lemma 2.2, one can easily conclude that $p_2(z)\mathrm{e}^{\alpha_2(z)} - R_1(z)$ has infinitely many zeros. Therefore we arrive at a contradiction. Consequently, any non-constant meromorphic solution of the difference-differential equation (2.1) must be transcendental.

A difference-differential polynomial $P_d(z,f)$ in $f(z)$ can be expressed as

$$P_d(z,f) = \sum_{\mu} b_\mu(z) G_\mu(z,f),$$

where $b_\mu \in S(f)$ and

$$\begin{aligned} G_\mu(z,f) = {} & (f(z))^{p_0^\mu}(f'(z))^{p_1^\mu}\dots(f^{(k)}(z))^{p_k^\mu}(f(z+c_0))^{q_0^\mu}(f(z+c_1))^{q_1^\mu}\dots(f(z+c_k))^{q_k^\mu} \\ & \times (f(z+c_\mu))^{l_0^\mu}(f'(z+c_\mu))^{l_1^\mu}\dots(f^{(k)}(z+c_\mu))^{l_k^\mu}, \end{aligned}$$

$p_0^\mu, p_1^\mu, \dots, p_k^\mu, q_0^\mu, q_1^\mu, \dots, q_k^\mu, l_0^\mu, l_1^\mu, \dots, l_k^\mu \in \mathbb{N} \cup \{0\}$ such that $\sum\limits_{j=0}^{k} p_j^\mu + \sum\limits_{j=0}^{k} q_j^\mu + \sum\limits_{j=0}^{k} l_j^\mu = \mu \leqslant d$. Therefore we have

$$P_d(z,f) = \sum_{\mu} b_\mu(z) \frac{G_\mu(z,f)}{f^\mu(z)} f^\mu(z). \tag{2.2}$$

Now by Lemmas 2.4 and 2.5, we derive

$$\begin{aligned} & m\Big(r, b_\mu(z)\frac{G_\mu(z,f)}{f^\mu(z)}\Big) \\ & \quad = m\Big(r, b_\mu(z)\Big(\frac{f'(z)}{f(z)}\Big)^{p_1^\mu}\dots\Big(\frac{f^{(k)}(z)}{f(z)}\Big)^{p_k^\mu}\dots\Big(\frac{f(z+c_\mu)}{f(z)}\Big)^{l_0^\mu}\dots\Big(\frac{f^{(k)}(z+c_\mu)}{f(z)}\Big)^{l_k^\mu}\Big) \\ & \quad = S(r,f). \end{aligned}$$

Therefore (2.2) takes the form

$$P_d(z,f) = c_d(z)f^d(z) + c_{d-1}(z)f^{d-1}(z) + \ldots + c_0(z),$$

where $c_d(z) \not\equiv 0$ and $m(r, c_i(z)) = S(r,f)$ for $i = 0,1,2,\ldots,d$. Now by using the mathematical induction, it follows that $m(r, P_d(z,f)) \leqslant dm(r,f) + S(r,f)$. Since $N(r,\infty;f) = S(r,f)$, it follows that

$$T(r, P_d(z,f)) \leqslant dT(r,f) + S(r,f). \tag{2.3}$$

Now from (2.1) and (2.3) we have

$$T(p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}) = T(r, f^n(z) + P_d(z,f)) = nT(r,f) + S(r,f) \tag{2.4}$$

and

$$\begin{aligned} T(p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}) &= T(r, f^n(z) + P_d(z,f)) \\ &\geqslant T(r, f^n(z)) - T(r, P_d(z,f)) \\ &\geqslant (n-d)T(r,f) + S(r,f). \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$(n-d)T(r,f) + S(r,f) \leqslant T(p_1(z)\mathrm{e}^{\alpha_1(z)} + p_2(z)\mathrm{e}^{\alpha_2(z)}) \leqslant nT(r,f) + S(r,f),$$

which implies that $\varrho(f) < \infty$. This completes the proof. □

Lemma 2.7 ([5]). *Suppose that $f(z)$ is a transcendental meromorphic function and $q_1, q_2, q_3, a \in S(f)$ such that $q_3 a \not\equiv 0$. If*

$$q_1 f^2 + q_2 f f' + q_3 (f')^2 = a,$$

then

$$q_3(q_2^2 - 4q_1q_3)\frac{a'}{a} + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q_3' \equiv 0.$$

Lemma 2.8 ([2]). *Let $f(z)$ be a non-constant meromorphic function and $n \in \mathbb{N}$. Suppose that*

$$g(z) = f^n(z) + P_{n-1}(z,f),$$

where $P_{n-1}(z,f)$ is a differential polynomial in $f(z)$ of degree at most $n-1$ with small functions of $f(z)$ as its coefficients and

$$N(r,f) + N\Big(r, \frac{1}{g}\Big) = S(r,f).$$

Then $g(z) = (f(z) + \gamma(z))^n$, where $\gamma \in S(f)$.

Lemma 2.9. *Let $f(z)$ be a non-constant meromorphic function and $n \in \mathbb{N}$. Suppose that*

$$g(z) = f^{n+1}(z) + P_{n-1}(z, f), \tag{2.6}$$

where $P_{n-1}(z, f)$ is a differential polynomial in $f(z)$ of degree at most $n-1$ with small functions of $f(z)$ as its coefficients and

$$N(r, f) + N\Big(r, \frac{1}{g}\Big) = S(r, f).$$

Then $g(z) = f^{n+1}(z)$ and $P_{n-1}(z, f) \equiv 0$.

P r o o f. Firstly, from Lemma 2.8 we have $g(z) = (f(z) + \gamma(z))^{n+1}$, where $\gamma \in S(f)$. If possible, suppose that $\gamma \not\equiv 0$. Now from (2.6) we have

$$(f(z) + \gamma(z))^{n+1} = f^{n+1}(z) + P_{n-1}(z, f)$$

and so

$$(n+1)\gamma(z) f^n(z) + Q_{n-1}(z, f) = P_{n-1}(z, f),$$

where $Q_{n-1}(z, f)$ is a differential polynomial in $f(z)$ of degree at most $n-1$ with small functions of $f(z)$ as its coefficients. Therefore we have

$$f^{n-1}(z)(n+1)\gamma(z) f(z) = P_{n-1}(z, f) - Q_{n-1}(z, f).$$

Now by Lemma 2.1, we conclude that $m(r, f) = S(r, f)$. Since $N(r, \infty; f) = S(r, f)$, it follows that $T(r, f) = S(r, f)$, which is impossible. Hence $\gamma \equiv 0$. Consequently, $g(z) = f^{n+1}(z)$ and $P_{n-1}(z, f) \equiv 0$. This completes the proof. $\square$

3. Proof of the theorem

P r o o f of Theorem 1.1. By the given condition, we have

$$f^n + P_d = p_1 \mathrm{e}^{\alpha_1} + p_2 \mathrm{e}^{\alpha_2}, \tag{3.1}$$

where $P_d = P_d(z, f)$. Let f be a meromorphic solution of equation (3.1). Then by Lemma 2.6, we can conclude that f is a transcendental meromorphic function of finite order. Now differentiating both sides of (3.1) once, we get

$$n f^{n-1} f' + P_d' = (p_1 \alpha_1' + p_1') \mathrm{e}^{\alpha_1} + (p_2 \alpha_2' + p_2') \mathrm{e}^{\alpha_2}. \tag{3.2}$$

Now by eliminating e^{α_2} from (3.1) and (3.2), we have

$$f^{n-1}(n p_2 f' - (p_2 \alpha_2' + p_2') f) + p_2 P_d' - (p_2 \alpha_2' + p_2') P_d = A_1 \mathrm{e}^{\alpha_1}, \tag{3.3}$$

where $A_1 = p_2(p_1\alpha_1' + p_1') - p_1(p_2\alpha_2' + p_2')$. Again by eliminating e^{α_1} from (3.1) and (3.2), we have

$$f^{n-1}(np_1f' - (p_1\alpha_1' + p_1')f) + p_1P_d' - (p_1\alpha_1' + p_1')P_d = -A_1e^{\alpha_2}. \tag{3.4}$$

Suppose that $A_1 \equiv 0$. Then we have $\alpha_1' - \alpha_2' = p_2'/p_2 - p_1'/p_1$ and so $\alpha_1' \equiv \alpha_2'$. Now from (3.3) we have

$$f^{n-1}(np_2f' - (p_2\alpha_2' + p_2')f) = (p_2\alpha_2' + p_2')P_d - p_2P_d'. \tag{3.5}$$

Suppose that $np_2f' - (p_2\alpha_2' + p_2')f \not\equiv 0$. Then by Lemma 2.1, we have

$$\begin{cases} m(r, np_2f' - (p_2\alpha_2' + p_2')f) = S(r,f), \\ m(r, np_2ff' - (p_2\alpha_2' + p_2')f^2) = S(r,f). \end{cases} \tag{3.6}$$

Since $N(r,\infty;f) = S(r,f)$, from (3.6) we conclude that

$$T(r,f) \leqslant T(r, np_2f'f - (p_2\alpha_2' + p_2')f^2) + T(r, np_2f' - (p_2\alpha_2' + p_2')f) + O(1) = S(r,f),$$

which is impossible. Therefore $np_2f' - (p_2\alpha_2' + p_2')f \equiv 0$ and so by integration, we get $f^n = c_0p_2e^{\alpha_2}$, where $c_0 \in \mathbb{C} \setminus \{0\}$. Therefore we let $f(z) = q(z)e^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = c_0p_2(z)$.

Next we suppose that $A_1(z) \not\equiv 0$. Now differentiating (3.3) once, we get

$$\begin{aligned} f^{n-2}(-(p_2\alpha_2' + p_2')'f^2 - np_2\alpha_2'ff' + (n-1)np_2(f')^2 + np_2ff'') + Q_d' \\ = (A_1' + A_1\alpha_1')e^{\alpha_1}, \end{aligned} \tag{3.7}$$

where

$$Q_d = p_2P_d' - (p_2\alpha_2' + p_2')P_d. \tag{3.8}$$

Eliminating e^{α_1} from (3.3) and (3.7), we get

$$f^{n-2}(h_{21}f^2 + h_{22}ff' + h_{23}(f')^2 + h_{24}ff'') = R_d, \tag{3.9}$$

where

$$\begin{cases} R_d = (A_1' + A_1\alpha_1')Q_d - A_1Q_d', \\ h_{21} = (p_2\alpha_2' + p_2')(A_1' + A_1\alpha_1') - A_1(p_2\alpha_2' + p_2')', \\ h_{22} = -n(\alpha_1' + \alpha_2')p_2A_1 - np_2A_1', \\ h_{23} = n(n-1)p_2A_1 \not\equiv 0, \\ h_{24} = np_2A_1 \not\equiv 0. \end{cases} \tag{3.10}$$

Clearly, h_{2j} are rational functions for $j = 1,2,3,4$.

First we suppose that $h_{21} \equiv 0$. Then we have

$$\frac{(p_2\alpha_2' + p_2')'}{p_2\alpha_2' + p_2'} - \frac{A_1'}{A_1} \equiv \alpha_1'$$

and so by integration we have $p_2\alpha_2' + p_2' = c_1 A_1 \mathrm{e}^{\alpha_1}$, where $c_1 \in \mathbb{C} \setminus \{0\}$. This shows that $A_1\mathrm{e}^{\alpha_1} \in S(f)$. Then from (3.3) we have

$$f^{n-1}(np_2 f' - (p_2\alpha_2' + p_2')f) = (p_2\alpha_2' + p_2')P_d - p_2 P_d' + A_1\mathrm{e}^{\alpha_1}. \tag{3.11}$$

In this case, one can also easily conclude that $f(z) = q(z)\mathrm{e}^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = c_1 p_2(z)$, where $c_1 \in \mathbb{C} \setminus \{0\}$.

Next we suppose that $h_{21} \not\equiv 0$. Let

$$h_{21}f^2 + h_{22}ff' + h_{23}(f')^2 + h_{24}ff'' = a. \tag{3.12}$$

Now we consider the following two cases.

Case 1. Suppose that $a \equiv 0$. Then from (3.12) we have

$$-h_{21}f^2 \equiv h_{22}ff' + h_{23}(f')^2 + h_{24}ff''. \tag{3.13}$$

Let z_1 be a zero of f of order l_1 such that $h_{2i}(z_1) \neq 0, \infty$ for $i = 1, 2, 3, 4$. Clearly, z_1 is a zero with multiplicity $2l_1$ of the left-hand side of equation (3.13) and a zero with multiplicity $2l_1 - 2$ of the right-hand side of equation (3.13). Therefore we arrive at a contradiction from (3.13). Now from (3.13) we can easily conclude that $N(r, 0; f) = O(\log r)$. Since $a \equiv 0$, from (3.9) and (3.10) we have

$$R_d \equiv 0, \quad \text{i.e., } (A_1' + A_1\alpha_1')Q_d \equiv A_1 Q_d'. \tag{3.14}$$

First we suppose that $Q_d \equiv 0$. Then from (3.8) we have

$$(p_2\alpha_2' + p_2')P_d \equiv p_2 P_d'. \tag{3.15}$$

If $P_d \equiv 0$, then from (3.1) and (3.3) we have, respectively,

$$f^n = p_1\mathrm{e}^{\alpha_1} + p_2\mathrm{e}^{\alpha_2} \tag{3.16}$$

and

$$f^{n-1}(np_2 f' - (p_2\alpha_2' + p_2')f) = A_1\mathrm{e}^{\alpha_1}. \tag{3.17}$$

Now (3.17) gives

$$np_2\frac{f'}{f} - (p_2\alpha_2' + p_2') = A_1\frac{\mathrm{e}^{\alpha_1}}{f^n}. \tag{3.18}$$

Using Lemma 2.4, one can easily conclude from (3.18) that $m(r, \mathrm{e}^{\alpha_1}/f^n) = O(\log r)$. Since $N(r, 0; f) = O(\log r)$, we have $T(r, \mathrm{e}^{\alpha_1}/f^n) = O(\log r)$. Then by the first fundamental theorem, we have $T(r, f^n/\mathrm{e}^{\alpha_1}) = O(\log r)$. Also from (3.16) we have

$$f^n \mathrm{e}^{-\alpha_1} = p_1 + p_2 \mathrm{e}^{\alpha_2 - \alpha_1}.$$

This shows that $T(r, \mathrm{e}^{\alpha_2 - \alpha_1}) = O(\log r)$ and so $\mathrm{e}^{\alpha_2 - \alpha_1}$ is a nonzero constant. Let $\mathrm{e}^{\alpha_2 - \alpha_1} = c_2 \in \mathbb{C} \setminus \{0\}$. Clearly $\alpha' \equiv \alpha_2'$. Now from (3.16) we have $f^n = \varphi_1 \mathrm{e}^{\alpha_1}$, where $\varphi_1 = p_1 + c_1 p_2$ is a rational function. In this case we also have $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = p_1(z) + c_1 p_2(z)$.

Next we suppose that $P_d \not\equiv 0$. Then from (3.15) we have

$$\frac{P_d'}{P_d} \equiv \alpha_2' + \frac{p_2'}{p_2}. \tag{3.19}$$

Integrating, we get $P_d = c_3 p_2 \mathrm{e}^{\alpha_2}$, where $c_3 \in \mathbb{C} \setminus \{0\}$ and so from (3.1) we get

$$f^n + \Big(1 - \frac{1}{c_3}\Big) P_d = p_1 \mathrm{e}^{\alpha_1}.$$

If $c_3 \neq 1$, then by Lemma 2.9, we have $f^n = p_1 \mathrm{e}^{\alpha_1}$ and $P_d \equiv 0$, which contradicts the fact that $P_d \not\equiv 0$. Therefore $c_3 = 1$ and so $f^n = p_1 \mathrm{e}^{\alpha_1}$ and $P_d = p_2 \mathrm{e}^{\alpha_2} \not\equiv 0$. In this case also, we have $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = p_1(z)$. Note that

$$P_d(z, f) = \sum_{\mu} b_\mu(z) \frac{G_\mu(z, f)}{f^\mu(z)} f^\mu(z), \tag{3.20}$$

where $b_\mu \in S(f)$ and

$$\begin{aligned} G_\mu(z, f) = {} & (f(z))^{p_0^\mu} (f'(z))^{p_1^\mu} \ldots (f^{(k)}(z))^{p_k^\mu} \\ & \times (f(z + c_\mu))^{q_0^\mu} (f'(z + c_\mu))^{q_1^\mu} \ldots (f^{(k)}(z + c_\mu))^{q_k^\mu}, \end{aligned}$$

$p_0^\mu, p_1^\mu, \ldots, p_k^\mu, q_0^\mu, q_1^\mu, \ldots, q_k^\mu \in \mathbb{N} \cup \{0\}$ such that $\sum\limits_{j=0}^{k} p_j^\mu + \sum\limits_{j=0}^{k} q_j^\mu = \mu \leqslant d$. Now by Lemmas 2.4 and 2.5, we derive $m(r, G_\mu(z, f)/f^\mu(z)) = S(r, f)$. Since $N(r, \infty; f) + N(r, 0; f) = S(r, f)$, it follows that $T(r, G_\mu(z, f)/f^\mu(z)) = S(r, f)$. Therefore (3.20) takes the form $P_d(z, f) = c_d(z) f^d(z) + c_{d-1}(z) f^{d-1}(z) + \ldots + c_0(z)$, where $c_d(z) \not\equiv 0$ and $c_i \in S(f)$ for $i = 0, 1, 2, \ldots, d$. Now substituting $f(z) = q(z)\mathrm{e}^{\alpha_1(z)/n}$ into $P_d(z, f) = p_2(z)\mathrm{e}^{\alpha_2(z)}$, we get

$$\sum_{k=0}^{d} a_{2k}(z) \mathrm{e}^{k\alpha_1(z)/n} = p_2(z) \mathrm{e}^{\alpha_2(z)}, \tag{3.21}$$

where $a_{2k}(z)$ $(k = 0, 1, \ldots, d)$ are small functions of $f(z)$.

Since $T(r,f)=T(r,\mathrm{e}^{\alpha_1/n})+S(r,f)$, it follows that $a_{2k}(z)$, $k=0,1,\ldots,d$, are small functions of $\mathrm{e}^{\alpha_1/n}$ and so $a_{2k}(z)$, $k=0,1,\ldots,d$, are small functions of $\mathrm{e}^{k\alpha_1/n}$, where $k\in\{1,2,\ldots,d\}$. Since $p_2\not\equiv 0$, from (3.21) we conclude that there exists at least one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$. We now claim that there exists exactly one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$. If $d=0$, then our claim is true. Next we suppose that $d\geqslant 1$. If possible, suppose that there exist at least two values of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$. For the sake of simplicity we may assume that $a_{2k}\not\equiv 0$ for $k\in\{0,1,2,\ldots,d\}$. Now by Lemma 2.3 we have

$$T\Big(r,\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)=dT(r,\mathrm{e}^{\alpha_1/n})+S(r,\mathrm{e}^{\alpha_1/n}). \tag{3.22}$$

Also from (3.21) we have

$$N\Big(r,-a_{20};\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)=N(r,0;p_2)\leqslant S(r,\mathrm{e}^{\alpha_1/n}). \tag{3.23}$$

Now from Lemmas 2.2, 2.3, (3.22) and (3.23) we have

$$\begin{aligned}
dT(r,\mathrm{e}^{\alpha_1/n})&\leqslant \overline{N}\Big(r,0;\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+\overline{N}\Big(r,\infty;\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)\\
&\quad+\overline{N}\Big(r,-a_{20};\sum_{k=1}^{d}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+S(r,\mathrm{e}^{\alpha_1/n})\\
&\leqslant \overline{N}\Big(r,0;\sum_{k=0}^{d-1}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+S(r,\mathrm{e}^{\alpha_1/n})\\
&\leqslant T\Big(r,\sum_{k=0}^{d-1}a_{2k}\mathrm{e}^{k\alpha_1/n}\Big)+S(r,\mathrm{e}^{\alpha_1/n})\\
&=(d-1)T(r,\mathrm{e}^{\alpha_1/n})+S(r,\mathrm{e}^{\alpha_1/n}),
\end{aligned}$$

which is impossible. Therefore there exists exactly one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$ and so from (3.21) we conclude that there must exist exactly one value of $k\in\{0,1,2,\ldots,d\}$ such that $\mathrm{e}^{(k\alpha_1-n\alpha_2)/n}$ is a small function of f.

Next we suppose that $Q_d\not\equiv 0$. Then from (3.14) we have

$$\frac{Q_d'}{Q_d}\equiv\frac{A_1'}{A_1}+\alpha_1'. \tag{3.24}$$

Integrating, we get $Q_d=c_4A_1\mathrm{e}^{\alpha_1}$, where $c_4\in\mathbb{C}\setminus\{0\}$ and so from (3.3) we get

$$f^{n-1}(np_2f'-(p_2\alpha_2'+p_2')f)\equiv\Big(\frac{1}{c_4}-1\Big)Q_d.$$

Let $\varphi_3 = np_2f' - (p_2\alpha_2' + p_2')f$. If $c_4 \neq 1$, then by Lemma 2.1, we have $m(r,\varphi_3) = S(r,f)$ and $m(r,\varphi_3 f) = S(r,f)$. Since $N(r,\infty;f) = S(r,f)$, it follows that $T(r,\varphi_3) = S(r,f)$ and $T(r,\varphi_3 f) = S(r,f)$. Note that

$$T(r,f) \leqslant T(r,\varphi_3 f) + T\Big(r,\frac{1}{\varphi_3}\Big) + S(r,f) = S(r,f),$$

which is impossible. Hence $c_4 = 1$ and so $\varphi_3 \equiv 0$. Then we have

$$n\frac{f'}{f} = \frac{p_2'}{p_2} + \alpha_2'.$$

On integration, we get $f^n = c_5p_2\mathrm{e}^{\alpha_2}$, where $c_5 \in \mathbb{C}\setminus\{0\}$. If $c_5 \neq 1$, then from (3.1) we have

$$\Big(1 - \frac{1}{c_5}\Big)f^n + P_d = p_1\mathrm{e}^{\alpha_1}.$$

Now by Lemma 2.9, we conclude that $P_d \equiv 0$ and so $Q_d \equiv 0$, which contradicts the fact that $Q_d \not\equiv 0$. Hence $c_5 = 1$ and so $f^n = p_2\mathrm{e}^{\alpha_2}$. Also from (3.1) we have $P_d = p_1\mathrm{e}^{\alpha_1}$. In this case we have $f(z) = q(z)\mathrm{e}^{\alpha_2(z)/n}$, where $q(z)$ is a nonzero rational function such that $q^n(z) = p_2(z)$. Also there must exist exactly one $k \in \{0,1,2,\dots,d\}$ such that $\mathrm{e}^{(k\alpha_2-n\alpha_1)/n}$ is a small function of f.

Case 2. Suppose that $a \not\equiv 0$. Then by Lemma 2.1, we can conclude that a is a small function of f. Now from (3.12) we have

$$\frac{1}{f^2} = \frac{h_{21}}{a} + \frac{h_{22}}{a}\frac{f'}{f} + \frac{h_{23}}{a}\Big(\frac{f'}{f}\Big)^2 + \frac{h_{24}}{a}\frac{f''}{f}. \tag{3.25}$$

Therefore from Lemma 2.4 and (3.25) we conclude that $m(r,1/f^2) = S(r,f)$, i.e., $m(r,1/f) = S(r,f)$. Consequently, by the first fundamental theorem, we have $T(r,f) = N(r,0;f) + S(r,f)$. This shows that f has infinitely many zeros. Let z_2 be a multiple zero of f such that $h_{2i}(z_2) \neq 0,\infty$ for $i = 1,2,3,4$. Then from (3.12) we conclude that z_2 is a zero of a. Therefore $N_{(2}(r,0;f) \leqslant T(r,a) = S(r,f)$, i.e., $N_{(2}(r,0;f) = S(r,f)$. Consequently, f has infinitely many simple zeros. Differentiating (3.12) once, we have

$$\begin{aligned} a' = {} & h_{21}'f^2 + (2h_{21} + h_{22}')ff' + (h_{22} + h_{23}')(f')^2 + (h_{22} + h_{24}')ff'' \\ & + (2h_{23} + h_{24})f'f'' + h_{24}ff'''. \end{aligned} \tag{3.26}$$

Now from (3.12) and (3.26) we have

$$\begin{aligned} & (ah_{21}' - a'h_{21})f^2 + (2ah_{21} + ah_{22}' - a'h_{22})ff' + (ah_{22} + ah_{23}' - a'h_{23})(f')^2 \\ & + (ah_{22} + ah_{24}' - a'h_{24})ff'' + a(2h_{23} + h_{24})f'f'' + ah_{24}ff''' \equiv 0. \end{aligned} \tag{3.27}$$

Let z_3 be a simple zero of f which is not a zero or pole of the coefficients in (3.27). Now from (3.27) we see that z_3 is a zero of $(2ah_{23}+ah_{24})f''-(a'h_{23}-ah_{22}-ah'_{23})f'$. Let

$$\alpha=\frac{(2ah_{23}+ah_{24})f''-(a'h_{23}-ah_{22}-ah'_{23})f'}{f}. \tag{3.28}$$

Since $N(r,\infty;f)+N_{(2}(r,0;f)=S(r,f)$, from (3.28) we see that $N(r,\infty;\alpha)=S(r,f)$. Also by Lemma 2.4, we have $m(r,\alpha)=S(r,f)$ and so $T(r,\alpha)=S(r,f)$. This shows that α is a small function of f. Therefore from (3.28) we have

$$f''=\frac{a'h_{23}-ah_{22}-ah'_{23}}{2ah_{23}+ah_{24}}f'+\frac{\alpha}{2ah_{23}+ah_{24}}f. \tag{3.29}$$

Now from (3.12) and (3.29) we have

$$a=q_1f^2+q_2ff'+q_3(f')^2, \tag{3.30}$$

where

$$q_1=h_{21}-\frac{\beta}{2ah_{23}+ah_{24}},\quad q_2=h_{22}+\frac{a'h_{23}-ah_{22}-ah'_{23}}{2ah_{23}+ah_{24}}h_{24}\quad\text{and}\quad q_3=h_{23}$$

are small functions of f. Also from (3.10) we see that

$$\frac{q_2}{q_3}=-\frac{2}{2n-1}(\alpha'_1+\alpha'_2)-\frac{3}{2n-1}\frac{A'_1}{A_1}+\frac{1}{2n-1}\frac{a'}{a}-\frac{1}{2n-1}\frac{p'_2}{p_2}. \tag{3.31}$$

By Lemma 2.7, we have

$$q_3(q_2^2-4q_1q_3)\frac{a'}{a}+q_2(q_2^2-4q_1q_3)-q_3(q_2^2-4q_1q_3)'+(q_2^2-4q_1q_3)q'_3\equiv 0. \tag{3.32}$$

Let $\delta=q_2^2-4q_1q_3$. Clearly δ is a small function of f. Now we consider the following two sub-cases.

Sub-case 2.1. Suppose that $\delta=q_2^2-4q_1q_3\equiv 0$. Then from (3.30) we have

$$q_3\Big(f'+\frac{q_2}{2q_3}f\Big)^2=a.$$

This shows that $f'+q_2f/(2q_3)$ is a small function of f. Let $b=f'+q_2f/(2q_3)$. Since $a\not\equiv 0$, it follows that $b\not\equiv 0$. By substituting $f'=b-q_2f/(2q_3)$ into (3.3) and (3.4), we have, respectively,

$$f^n\Big(p_2\alpha'_2+p'_2+np_2\frac{q_2}{2q_3}\Big)-np_2bf^{n-1}+R_{1d}=A_1\mathrm{e}^{\alpha_1} \tag{3.33}$$

and

$$f^n\Big(p_1\alpha'_1+p'_1+np_1\frac{q_2}{2q_3}\Big)-np_1bf^{n-1}+R_{2d}=-A_1\mathrm{e}^{\alpha_2}, \tag{3.34}$$

where $R_{1d}=p_2P'_d-(p_2\alpha'_2+p'_2)P_d$ and $R_{2d}=p_1P'_d-(p_1\alpha'_1+p'_1)P_d$.

Let

$$\gamma_1 = p_2\alpha_2' + p_2' + np_2\frac{q_2}{2q_3} \quad \text{and} \quad \gamma_2 = p_1\alpha_1' + p_1' + np_1\frac{q_2}{2q_3}.$$

First we suppose that $\gamma_1 \equiv 0$. Then using (3.31), we get

$$\frac{p_2'}{p_2} + \alpha_2' = \frac{n}{2n-1}\Big(\alpha_1' + \alpha_2' + \frac{3}{2}\frac{A_1'}{A_1} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{p_2'}{p_2}\Big).$$

Therefore by integrating, we get

$$(p_2 e^{\alpha_2})^{2n-1} = c_6\frac{A_1^{3n/2} p_2^{n/2}}{a^{n/2}} e^{n(\alpha_1+\alpha_2)},$$

where $c_6 \in \mathbb{C}\setminus\{0\}$. This shows that $e^{(n-1)\alpha_2 - n\alpha_1}$ is a small function of f. Next we suppose that $\gamma_2 \equiv 0$. Then using (3.31), we get

$$\frac{p_1'}{p_1} + \alpha_1' = \frac{n}{2n-1}\Big(\alpha_1' + \alpha_2' + \frac{3}{2}\frac{A_1'}{A_1} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{p_2'}{p_2}\Big).$$

Therefore by integrating, we get

$$(p_1 e^{\alpha_1})^{2n-1} = c_7\frac{A_1^{3n/2} p_2^{n/2}}{a^{n/2}} e^{n(\alpha_1+\alpha_2)},$$

where $c_7 \in \mathbb{C}\setminus\{0\}$. This shows that $e^{(n-1)\alpha_1 - n\alpha_2}$ is a small function. Next we discuss the following four sub-cases.

Sub-case 2.1.1. Suppose that $\gamma_1 \equiv 0$ and $\gamma_2 \equiv 0$. Then both $e^{(n-1)\alpha_2 - n\alpha_1}$ and $e^{(n-1)\alpha_1 - n\alpha_2}$ are small functions of f. Clearly $e^{\alpha_1+\alpha_2}$ is a small function of f and so $e^{\alpha_2} = \varphi_4 e^{-\alpha_1}$, where φ_4 is a small function of f. Now from (3.33) and (3.34) we have, respectively,

$$-np_2 b f^{n-1} + R_{1d} = A_1 e^{\alpha_1} \tag{3.35}$$

and

$$-np_1 b f^{n-1} + R_{2d} = -A_1\varphi_4 e^{-\alpha_1}. \tag{3.36}$$

Eliminating e^{α_1} and $e^{-\alpha_1}$, from (3.35) and (3.36) we have

$$f^{2n-3}(n^2 b^2 p_1 p_2 f) + R_{3d} = -A_1^2\varphi_4, \tag{3.37}$$

where $R_{3d} = -np_2 b R_{2d} f^{n-1} - np_1 b R_{1d} f^{n-1} + R_{1d}R_{2d}$ is a differential polynomial in f of degree $\leqslant 2n-3$ with small functions as its coefficients. Then by applying Lemma 2.1, we get from (3.37) that $m(r,f) = S(r,f)$. Since $N(r,\infty;f) = S(r,f)$, it follows that $T(r,f) = S(r,f)$, which is impossible.

Sub-case 2.1.2. Suppose that $\gamma_1 \not\equiv 0$ and $\gamma_2 \equiv 0$. Since $\gamma_2 \equiv 0$, we have that $\mathrm{e}^{(n-1)\alpha_1 - n\alpha_2}$ is a small function of f and so

$$\mathrm{e}^{\alpha_2} = \varphi_5 \mathrm{e}^{(n-1)\alpha_1/n}, \quad \text{where } \varphi_5 \in S(f). \tag{3.38}$$

Now from (3.33) and Lemma 2.8, there exists a small function v_1 of f such that

$$(f + v_1)^n = \frac{A_1}{\gamma_1}\mathrm{e}^{\alpha_1}, \text{ i.e., } f = u_1 \mathrm{e}^{\alpha_1/n} - v_1, \tag{3.39}$$

where u_1 is a nonzero small function of f. Since f has infinitely many zeros, it follows that $v_1 \not\equiv 0$. Now from (3.1), (3.38) and (3.39) we have

$$(u_1 \mathrm{e}^{\alpha_1/n} - v_1)^n + P_d = p_1 \mathrm{e}^{\alpha_1} + c_5 p_2 \mathrm{e}^{(n-1)/n\alpha_1}.$$

Therefore by applying Lemma 2.4, we can conclude that $u_1^n(z) = p_1(z)$.

Sub-case 2.1.3. Suppose that $\gamma_1 \equiv 0$ and $\gamma_2 \not\equiv 0$. Since $\gamma_1 \equiv 0$, we have that $\mathrm{e}^{(n-1)\alpha_2 - n\alpha_1}$ is a small function of f and so $\mathrm{e}^{\alpha_1} = \varphi_6 \mathrm{e}^{(n-1)/n\alpha_2}$, where $\varphi_6 \in S(f)$. Now proceeding in the same way as in Sub-case 2.1.2, one can easily conclude that $f = u_2 \mathrm{e}^{\alpha_2/n} - v_2$, where u_2 and v_2 are nonzero small functions of f such that $u_2^n(z) = p_2(z)$.

Sub-case 2.1.4. Suppose that $\gamma_1 \not\equiv 0$ and $\gamma_2 \not\equiv 0$. Now from (3.33) and (3.34) and Lemma 2.8, there exist two small functions v_3 and v_4 of f such that

$$(f + v_3)^n = \frac{A_1}{\gamma_1}\mathrm{e}^{\alpha_1} \quad \text{and} \quad (f + v_4)^n = -\frac{A_1}{\gamma_2}\mathrm{e}^{\alpha_2}.$$

From these we have, respectively,

$$f = u_3 \mathrm{e}^{\alpha_1/n} - v_3 \quad \text{and} \quad f = u_4 \mathrm{e}^{\alpha_2/n} - v_4, \tag{3.40}$$

where $u_3^n = A_1/\gamma_1 \not\equiv 0$ and $u_4^n = -A_1/\gamma_2 \not\equiv 0$. Since f has infinitely many zeros, it follows that $v_3 \not\equiv 0$ and $v_4 \not\equiv 0$.

First we suppose that $\mathrm{e}^{\alpha_1 - \alpha_2}$ is a small function of f. Then clearly $\mathrm{e}^{\alpha_2} = \varphi_7 \mathrm{e}^{\alpha_1}$, where $\varphi_7 \in S(f)$. Now from (3.1) we have

$$f^n + P_d = p_5 \mathrm{e}^{\alpha_1}, \tag{3.41}$$

where $p_5 = p_1 + \varphi_7 p_2$. If $p_5 \equiv 0$, then from (3.41) we have $f^{n-1}f = -P_d$ and so by Lemma 2.1, we conclude that $m(r, f) = S(r, f)$. This shows that $T(r, f) = S(r, f)$, which is impossible. Next we suppose that $p_5 \not\equiv 0$. Then by Lemma 2.9, we conclude that $f^n = p_5 \mathrm{e}^{\alpha_1}$ and $P_d \equiv 0$. In this case we have $f(z) = q(z)\mathrm{e}^{\alpha_1/n}$, where $q(z)$ is a nonzero small function of $f(z)$ such that $q^n(z) = p_1(z) + \varphi_7(z)p_2(z)$.

Next we suppose that $e^{\alpha_1-\alpha_2}$ is not a small function of f. Note that $T(r,f) \leqslant T(r, e^{\alpha_1/n}) + S(r,f)$. Also

$$T(r, e^{\alpha_1/n}) \leqslant T(r, u_3e^{\alpha_1/n})+S(r,f) \leqslant T(r, u_3e^{\alpha_1/n}-v_3)+S(r,f) = T(r,f)+S(r,f).$$

Combining these, we get $T(r,f) = T(r, u_3e^{\alpha_1/n}) + S(r,f)$. Similarly, we have $T(r,f) = T(r, u_4e^{\alpha_2/n}) + S(r,f)$. These show that $S(r,f) = S(r, u_3e^{\alpha_1/n}) = S(r, u_4e^{\alpha_2/n})$. Clearly u_3, u_4, v_3 and v_4 are small functions of both $e^{\alpha_1/n}$ and $e^{\alpha_2/n}$. On the other hand, from (3.40) we have

$$u_3e^{\alpha_1/n} - u_4e^{\alpha_2/n} = v_3 - v_4. \tag{3.42}$$

We claim that $v_3 \equiv v_4$. If not, suppose that $v_3 \not\equiv v_4$. Now by Lemma 2.2, we get

$$\begin{aligned} T(r,f) = T(r, u_3e^{\alpha_1/n}) + S(r,f) &\leqslant \overline{N}(r,0; u_3e^{\alpha_1/n}) + \overline{N}(r,\infty; u_3e^{\alpha_1/n}) \\ &\quad + \overline{N}(r, v_3-v_4; u_3e^{\alpha_1/n}) + S(r, u_3e^{\alpha_1/n}) + S(r,f) \\ &= S(r,f), \end{aligned}$$

which is a contradiction. Hence, $v_3 \equiv v_4$ and so from (3.42) we have

$$u_3e^{\alpha_1/n} \equiv u_4e^{\alpha_2/n}.$$

This shows that $e^{(\alpha_1-\alpha_2)/n} = u_4/u_3$ and so $e^{\alpha_1-\alpha_2} = (u_4/u_3)^n$. Consequently, $e^{\alpha_1-\alpha_2}$ is a small function of f, which contradicts our assumption.

Sub-case 2.2. Suppose that $\delta = q_2^2 - 4q_1q_3 \not\equiv 0$. Then from (3.32) we have

$$\frac{q_2}{q_3} \equiv \frac{\delta'}{\delta} - \frac{q_3'}{q_3} - \frac{a'}{a}.$$

Therefore from (3.10) and (3.31) we have

$$2(\alpha_1' + \alpha_2') \equiv (2n-4)\frac{A_1'}{A_1} + (2n-2)\frac{a'}{a} + (2n-2)\frac{p_2'}{p_2} - (2n-1)\frac{\delta'}{\delta}.$$

Integrating, we get

$$e^{2(\alpha_1+\alpha_2)} = c_8\frac{A_1^{2n-4}a^{2n-2}p_2^{2n-2}}{\delta^{2n-1}},$$

where $c_8 \in \mathbb{C}$. This shows that $e^{\alpha_1+\alpha_2}$ is a small function of f and so $e^{\alpha_2} = \varphi_8 e^{-\alpha_1}$, where $\varphi_8 \in S(f)$. Now from (3.3) and (3.4), we have, respectively,

$$f^{n-1}(np_2f' - (p_2\alpha_2' + p_2')f) + R_{1d} = A_1e^{\alpha_1} \tag{3.43}$$

and

$$f^{n-1}(np_1f' - (p_1\alpha_1' + p_1')f) + R_{2d} = -\varphi_8 A_1e^{-\alpha_1}. \tag{3.44}$$

Eliminating e^{α_1} and $e^{-\alpha_1}$, from (3.43) and (3.44) we have

$$f^{2n-2}(np_2f' - (p_2\alpha_2' + p_2')f)(np_1f' - (p_1\alpha_1' + p_1')f) + \mathcal{Q}_d^* = -\varphi_8 A_1^2, \tag{3.45}$$

where

$$\mathcal{Q}_d^* = f^{n-1}(np_2f' - (p_2\alpha_2' + p_2')f)R_{2d} + f^{n-1}(np_1f' - (p_1\alpha_1' + p_1')f)R_{1d} + R_{1d}R_{2d}$$

is a differential polynomial in f of degree $\leqslant 2n-2$ with small functions of f as its coefficients. Now by Lemma 2.1, we conclude that $((p_1\alpha_1' + p_1')f - np_1f') \times ((p_2\alpha_2' + p_2')f - np_2f') = b_{11}$, where b_{11} is a small function of f. If $b_{11} \equiv 0$, then we have either $(p_1\alpha_1' + p_1')f - np_1f' \equiv 0$ or $(p_2\alpha_2' + p_2')f - np_2f' \equiv 0$. Thus, in either case one can easily conclude that $N(r,0;f) = S(r,f)$, which is impossible here. Hence $b_{11} \not\equiv 0$. Therefore we can assume that

$$(p_2\alpha_2' + p_2')f - np_2f' = b_1e^{\gamma} \quad \text{and} \quad (p_1\alpha_1' + p_1')f - np_1f' = b_2e^{-\gamma}, \tag{3.46}$$

where b_1, b_2 are small functions of f such that $b_1b_2 = b_{11}$ and γ is an entire function. Since f is of finite order, it follows that γ is a polynomial.

First we suppose that γ is a constant. Then from (3.46) we have

$$f' = \frac{1}{n}\Big(\alpha_2' + \frac{p_2'}{p_2}\Big)f - \frac{b_1e^{\gamma}}{np_2} \quad \text{and} \quad f' = \frac{1}{n}\Big(\alpha_1' + \frac{p_1'}{p_1}\Big)f - \frac{b_2e^{-\gamma}}{np_1}.$$

These imply that

$$\Big(\alpha_1' - \alpha_2' + \frac{p_1'}{p_1} - \frac{p_2'}{p_2}\Big)f = \frac{b_2e^{-\gamma}}{p_1} - \frac{b_1e^{\gamma}}{p_2}. \tag{3.47}$$

If $\alpha_1' - \alpha_2' + p_1'/p_1 - p_2'/p_2 \equiv 0$, then by integration, we have $e^{\alpha_1 - \alpha_2} = c_9p_2/p_1$, where $c_9 \in \mathbb{C} \setminus \{0\}$ and so $\alpha_1 - \alpha_2$ is a constant. Since $e^{\alpha_2} = \varphi_8 e^{-\alpha_1}$, it follows that e^{α_2} is a small function of f. Certainly e^{α_1} is also a small function of f. Now from (3.1) and Lemma 2.1, we conclude that $m(r,f) = S(r,f)$ and so $T(r,f) = S(r,f)$, which is impossible here. Therefore $\alpha_1' - \alpha_2' + p_1'/p_1 - p_2'/p_2 \not\equiv 0$. Now from (3.47), it follows that f is a small function of f, which is absurd.

Next we suppose that γ is a non-constant polynomial. Now solving for f, we get from (3.46) that

$$(p_1p_2(\alpha_2' - \alpha_1') + p_1p_2' - p_1'p_2)f = p_1b_1e^{\gamma} - p_2b_2e^{-\gamma}. \tag{3.48}$$

Using a similar argument, one can easily prove that $p_1p_2(\alpha_2' - \alpha_1') + p_1p_2' - p_1'p_2 \not\equiv 0$. Now from (3.48) we get $f(z) = \delta_1(z)e^{\gamma(z)} + \delta_2(z)e^{-\gamma(z)}$, where

$$\delta_1 = \frac{p_1b_1}{p_1p_2' - p_1'p_2 - p_1p_2(\alpha_1' - \alpha_2')} \quad \text{and} \quad \delta_2 = \frac{-p_2b_2}{p_1p_2' - p_1'p_2 - p_1p_2(\alpha_1' - \alpha_2')}.$$

Equation (3.46) can be rewritten as

$$A_2f - np_2f' = b_1\mathrm{e}^{\gamma}, \tag{3.49}$$

where $A_2 = p_2\alpha_2' + p_2'$. Differentiating (3.49) once, we get

$$A_2'f + (A_2 - np_2')f' - np_2f'' = (b_1' + b_1\gamma')\mathrm{e}^{\gamma}. \tag{3.50}$$

Using (3.29), we get from (3.50) that

$$\Big(A_2' - n\frac{p_2\alpha}{2ah_{23} + ah_{24}}\Big)f + \Big(A_2 - np_2' - n\frac{a'h_{23} - ah_{22} - ah_{23}'}{2ah_{23} + ah_{24}}p_2\Big)f' = (b_1' + b_1\gamma')\mathrm{e}^{\gamma}. \tag{3.51}$$

Now from (3.10) and (3.51) we get

$$\begin{aligned}\Big(A_2' - \frac{1}{2n-1}\frac{\alpha}{aA_1}\Big)f + \Big(A_2 - np_2' - \frac{1}{2n-1}(\alpha_1' + \alpha_2')p_2 \\ - \frac{n(n-1)}{2n-1}\frac{a'}{a}p_2 + \frac{n(n-1)}{2n-1}p_2' + \frac{n(n-1)}{2n-1}\frac{A_1'}{A_1}p_2\Big)f' = (b_1' + b_1\gamma')\mathrm{e}^{\gamma}.\end{aligned} \tag{3.52}$$

Dividing (3.52) by (3.49), we get

$$\zeta_1 f + \zeta_2 f' \equiv 0, \tag{3.53}$$

where

$$\zeta_1 = A_2' - \frac{1}{2n-1}\frac{\alpha}{A_1} - A_2\Big(\frac{b_1'}{b_1} + \gamma'\Big)$$

and

$$\begin{aligned}\zeta_2 = A_2 - np_2' - \frac{1}{2n-1}(\alpha_1' + \alpha_2')p_2 - \frac{n(n-1)}{2n-1}\frac{a'}{a}p_2 \\ + \frac{n(n-1)}{2n-1}p_2' + \frac{n(n-1)}{2n-1}\frac{A_1'}{A_1}p_2 + n\Big(\frac{b_1'}{b_1} + \gamma'\Big)p_2.\end{aligned}$$

Since $ff' \not\equiv 0$, it follows from (3.53) that either $\zeta_1 \not\equiv 0$ and $\zeta_2 \not\equiv 0$ or $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 0$. First we suppose that $\zeta_1 \not\equiv 0$ and $\zeta_2 \not\equiv 0$. Then from (3.53), one can easily conclude that $N(r, 0; f) = S(r, f)$, which is a contradiction. Next we suppose that $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 0$. Now $\zeta_2 \equiv 0$ yields

$$\alpha_2' - \frac{(n-1)^2}{2n-1}\frac{p_2'}{p_2} - \frac{1}{2n-1}(\alpha_1' + \alpha_2') - \frac{n(n-1)}{2n-1}\frac{a'}{a} - \frac{n(n-1)}{2n-1}\frac{A_1'}{A_1} + n\frac{b_1'}{b_1} + n\gamma' \equiv 0,$$

which implies that $\mathrm{e}^{(2n-1)(n\gamma+\alpha_2)} = c_{10}p_2^{(n-1)^2}\mathrm{e}^{\alpha_1+\alpha_2}(aA_1)^{n(n-1)}b_1^{-n}$, where $c_{10} \in \mathbb{C}\setminus\{0\}$. Consequently, $\mathrm{e}^{n\gamma+\alpha_2}$ is a small function of f. Therefore $f(z) = \delta_1(z)\mathrm{e}^{\gamma(z)} + \delta_2(z)\mathrm{e}^{-\gamma(z)}$ and $\mathrm{e}^{\alpha_1(z)+\alpha_2(z)}$ is a small function of $f(z)$, where $\delta_1(z)$, $\delta_2(z)$ are nonzero small functions of $f(z)$ and $\gamma(z)$ is a non-constant polynomial such that either $\mathrm{e}^{n\gamma(z)+\alpha_2(z)}$ is a small function of $f(z)$ or $\mathrm{e}^{n\gamma(z)+\alpha_1(z)}$ is a small function of $f(z)$. $\square$

4. An open problem

For further study, one may raise the following question as an open problem:

Open Problem. What will happen if we remove the condition $\varrho_2(f) < 1$ from Theorem 1.1?

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