TECHNISCHE UNIVERSITÄT BERGAKADEMIE FREIBERG

Die Ressourcenuniversität. Seit 1765.

# A Quaternionic Version Theory related to Spheroidal Functions 

Von der Fakultät für Mathematik und Informatik der Technischen Universität Bergakademie Freiberg<br>genehmigte<br>Habilitation<br>zur Erlangung des akademischen Grades<br>doctor rerum naturalium habilitatus<br>(Dr. rer. nat. habil.)

vorgelegt von Dr. João Pedro Leitão da Cruz Morais<br>geboren am 10. September 1982 in S. Domingos de Benfica - Lisboa, Portugal<br>Gutachter: Prof. Dr. Swanhild Bernstein<br>Prof. Dr. Elias Wegert<br>Prof. Dr. Vladimir Souček

## Zusammenfassung

In dieser Arbeit wird eine neue Theorie der quaternionischen Funktionen vorgestellt, welche das Problem der Bestapproximation von Familien prolater und oblater sphäroidalen Funktionen im Hilberträumen behandelt.

Die allgemeine Theorie beginnt mit der expliziten Konstruktion von orthogonalen Basen für Räume, definiert auf sphäroidalen Gebieten mit beliebiger Exzentrizität, deren Elemente harmonische, monogene und kontragene Funktionen sind und durch die Form der Gebiete parametrisiert werden. Eine detaillierte Studie dieser grundlegenden Elemente wird in dieser Arbeit durchgeführt. Der Begriff der kontragenen Funktion hängt vom Definitionsbereich ab und ist daher keine lokale Eigenschaft, während die Begriffe der harmonischen und monogenen Funktionen lokal sind. Es werden verschiedene Umwandlungsformeln vorgestellt, die Systeme harmonischer, monogener und kontragener Funktionen auf Sphäroiden unterschiedlicher Exzentrizität in Beziehung setzen. Darüber hinaus wird die Existenz gemeinsamer nichttrivialer kontragener Funktionen für Sphäroide jeglicher Exzentrizität gezeigt.

Der zweite wichtige Beitrag dieser Arbeit betrifft eine quaternionische Raumfrequenztheorie für bandbegrenzte quaternionische Funktionen. Es wird eine neue Art von quaternionischen Signalen vorgeschlagen, deren Energiekonzentration im Raum und in den Frequenzbereichen unter der quaternionischen Fourier-Transformation maximal ist. Darüber hinaus werden diese Signale im Kontext der Spektralkonzentration als Eigenfunktionen eines kompakten und selbstadjungierteren quaternionischen Integraloperators untersucht und die grundlegenden Eigenschaften ihrer zugehörigen Eigenwerte werden detailliert beschrieben. Wenn die Konzentrationsgebiete beider Räume kugelförmig sind, kann der Winkelanteil dieser Signale explizit gefunden werden, was zur Lösung von mehreren eindimensionalen radialen Integralgleichungen führt.

Wir nutzen die theoretischen Ergebnisse und harmonische Konjugierten um Klassen monogener Funktionen in verschiedenen Räumen zu konstruieren. Zur Charakterisierung der monogenen gewichteten Hardy- und BergmanRäume in der Einheitskugel werden zwei konstruktive Algorithmen vorgeschla-
gen. Für eine reelle harmonische Funktion, die zu einem gewichteten Hardyund Bergman-Raum gehört, werden die harmonischen Konjugiert in den gleichen Räumen gefunden. Die Beschränktheit der zugrundeliegenden harmonischen Konjugationsoperatoren wird in den angegebenen gewichteten Räumen bewiesen. Zusätzlich wird ein quaternionisches Gegenstück zum Satz von Bloch für monogene Funktionen bewiesen.

## Abstract

This work presents a novel Quaternionic Function Theory associated with the best approximation problem in the setting of Hilbert spaces concerning families of prolate and oblate spheroidal functions.

The general theory begins with the explicit construction of orthogonal bases for the spaces of harmonic, monogenic, and contragenic functions defined in spheroidal domains of arbitrary eccentricity, whose elements are parametrized by the shape of the corresponding spheroids. A detailed study regarding the elements that constitute these bases is carried out in this thesis. The notion of a contragenic function depends on the domain, and, therefore, it is not a local property in contrast to the concepts of harmonic and monogenic functions. Various conversion formulas that relate systems of harmonic, monogenic, and contragenic functions associated with spheroids of differing eccentricity are presented. Furthermore, the existence of standard nontrivial contragenic functions is shown for spheroids of any eccentricity.

The second significant contribution presented in this work pertains to a quaternionic space-frequency theory for band-limited quaternionic functions. A new class of quaternionic signals is proposed, whose energy concentration in the space and the frequency domains are maximal under the quaternion Fourier transform. These signals are studied in the context of spatialfrequency concentration as eigenfunctions of a compact and self-adjoint quaternion integral operator. The fundamental properties of their associated eigenvalues are described in detail. When the concentration domains are spherical in both spaces, the angular part of these signals can be found explicitly, leading to a set of one-dimensional radial integral equations.

The theoretical framework described in this work is applied to the construction of classes of monogenic functions in different spaces via harmonic conjugates. Two constructive algorithms are proposed to characterize the monogenic weighted Hardy and Bergman spaces in the Euclidean unit ball. For a real-valued harmonic function belonging to a Hardy and a weighted Bergman space, the harmonic conjugates in the same spaces are found. The boundedness of the underlying harmonic conjugation operators is proven
in the given weighted spaces. Additionally, a quaternionic counterpart of Bloch's Theorem is established for monogenic functions.

## Dedication

This Dissertation is dedicated to my beloved parents.

## Acknowledgements

It is not possible to acknowledge adequately all those who have assisted in this work, which required many years of study, discussion, and collaboration. The author's primary source of inspiration for writing this dissertation has been Professor Wolfgang Sprößig of the Technische Universität Bergakademie Freiberg and Professor Klaus Gürlebeck of the Bauhaus-Universität Weimar of the range and originality of their ideas but also for continual advice and encouragement.

It is with great pleasure that I express my most appreciation to all those who have expressed support, enthusiasm, and encouragement in this adventure. I am forever indebted to my family and close friends for their patience, understanding, and support: Lucília and Mário Morais, Jesús Mario Lozano, José António Morais, Constança Sofia Morais, and Andreia Roque. The voyage to completing this dissertation would have been harder without this group of people. So, I dedicate this dissertation to them.

Gratefully I acknowledge the support from my home institution ITAM (Instituto Tecnológico Autónomo de México), for their hospitality, kindness, and support, during which parts of the thesis were written. I am immensely grateful to Dean Beatriz Rumbos, Vice-President Alejandro Hernández, and President Arturo Fernández. I do not have enough words of praise and gratitude for their constant professional help.

Mainly, I desire to express my sense of the value of the excellent work done by the readers, Carlos Bosch and Michael Porter, who carefully checked parts of the thesis.

Last, but far from least, my thanks go to Victor Abramov, Mahmoud Abul-Ez, Swanhild Bernstein, Irene Falcão, Milton Ferreira, Eckhard Hitzer, Sören Kraußhar, Helmuth Malonek, Inês Matos, Miguel Ángel Mota, Craig Nolder, Bernd Schmeikal, Seenith Siva, Claudia Gómez Wulschner, Hanaa Zayed, and Mohra Zayed.

## Introduction

The theory of functions of a complex variable has a wide range of applications in several branches of mathematics. This theory includes developing a classification for functions based upon precise definitions, according to whether the functions present certain properties such as continuity, differentiability, or integrability, and so on, throughout the domain of the variables. Unfortunately, given its inherent nature, one is mainly restricted to two-dimensional problems, which has led to an increasing need for higher-dimensional counterparts of complex variable theory.

There are two main approaches for extending the theory of holomorphic functions to three and four dimensions. On the one hand, we can consider the concept of holomorphy within the function theory of several complex variables; on the other hand, we can achieve another generalization by applying the theory of monogenic functions with values in the Quaternion algebra. The latter is what is nowadays called Quaternionic analysis. An essential difference between the two extensions lies in their algebraic structures. While in the case of several complex variables, commutativity still holds, this is no longer true in the Quaternionic case. At first glance, this can be perceived as a drawback. However, the Quaternion variable theory has several advantages when compared with the theory of several complex variables. For instance, while the Cauchy kernel function is associated with particular poly-domains in the latter case, its counterpart in Quaternionic analysis is universal. This also implies an essential advantage in the Quaternionic-analytical approach since it does not suffer from the geometrical restrictions of the several complex variable theory. Another advantage lies in the fact that it allows for the factorization of higher-order differential operators in terms of lower-order ones. For example, the classical Laplacian can be factorized into first-order partial differential operators similar to the complex case - independently of the dimension of the underlying Euclidean space. In this work, we are mainly concerned with the ordinary three-dimensional Euclidean space. In this way, we say that the central paradigm followed in this work originated in the theory of Quaternionic analysis.

Quaternionic analysis is now a well-established mathematical discipline offering both a generalization of the classical theory of holomorphic functions of one complex variable and refinement of classical Harmonic analysis. Some of the earlier and more recent works on Quaternionic analysis, and more generally of Clifford analysis, were obtained by Scheffers [293], Dixon [97], Lanczos [193], Moisil and Teodorescu [232], Fueter [126, 127, 128, 129, 130, 131], Melijhzon [228, Iftimie [174], Hestenes [163], Delanghe [89, 90, 92], Deavours [88], Sudbery [318], Brackx, Delanghe and Sommen [48], Malonek [214, 215, 216, [217, 218, Gürlebeck and Sprößig [145, 146], Gilbert and Murray [140], Delanghe, Sommen and Soucek [91], Kravchenko and Shapiro [189], Kravchenko [190], Gürlebeck, Habetha and Sprößig [148, 153], Colombo, Sabadini and Struppa [80], Gentili, Stoppato and Struppa [136], among others. Quaternionic analysis has become an active area of scientific research and is currently applied to several branches of modern analysis, both pure and applied. In recent years, methods of Quaternionic analysis combined with other classical and advanced analytical techniques (such as harmonic analysis, variational methods, or finite difference methods) have become powerful tools towards the treatment of problems in mathematical physics, signal and image processing, computer vision, robotics, and even in classical mechanics and engineering. Much progress has been made on this topic. The ensuing results led to the treatment of boundary value problems of essential systems of partial differential equations of first- and higher-orders by adequate numerical methods. Those equations include but are not limited to the Laplace, Helmholtz, Maxwell, Lamé, and Stokes (later Navier-Stokes) equations.

The study of the fundamental properties of holomorphic functions of one complex variable is linked to harmonic functions of two real variables through the Cauchy-Riemann equations. Laplace initiated the theory of harmonic functions in a study published in 1785 [202], in which a second-order partial differential equation was derived - known today as the Laplace equation. That same year, Legendre developed the theory of zonal spherical harmonics, which are solutions of Laplace's equation in spherical coordinates with axial symmetry. Laplace himself solved his equation in spherical geometry without any symmetry, thus introducing the concept of tesseral spherical harmonics. The foundations of the theory of spherical harmonics began its active development in the book "Treatise on Natural Philosophy" [323] by Thomson and Tait. This work introduced the solid spherical harmonics that are homogeneous polynomial solutions of the Laplace equation. Green constructed the first solutions of the Laplace equation related to the ellipsoid in the same manner that spherical harmonics are related to the sphere in his memoir "On the Determination of the Exterior and Interior Attractions of Ellipsoids of Variable Densities" [142]. Green generated harmonic functions using only

Cartesian and spherical coordinates. Lamé solved the problem of determining a harmonic function in the interior of an ellipsoid having prescribed values on the boundary of the region, connected with the theory of heat conduction. This study led to the discovery of the functions that bear his name today [192]. The Lamé functions are commonly known as ellipsoidal harmonics [85, 170, 259]. In his Dissertation [159], Heine dealt with the same problem and showed for the first time that the functions that occur in the solution are the associated Legendre functions of the first kind. His main results on the solution of the external problem were published in the paper "Über einige Aufgaben, welche auf partielle Differentialgleichungen führen" [160], and in the treatise "Handbuch der Kugelfunktionen" [161]. Heine first introduced the Legendre functions of the second kind and the associated functions in this connection. But some characteristic properties of these functions as we understand them today have been stressed and applied since the beginning of the century. Excellent contributions to this subject have been made by Legendre [206], Liouville [212], Hilbert [164], Niven [259], Klein [181], Lindemann [211], Hill [165], Stieltjes [314], Darwin [85], Ferrers [118, 119], Féjer [117], Whittaker and Watson [330], Hobson [170], Szegö [320], Byerly [61], Sansone [292], Dixon and Lacroix [98], Dassios [86], and others.

One of the primary problems in our work is the explicit construction and computation of orthogonal bases of harmonic functions. The internal and external spherical harmonics are embedded in one-parameter families of spheroidal harmonics. The original impetus behind the study of orthogonal bases of polynomials for the spaces of square-integrable harmonic functions defined in spheroids of the form

$$
\left\{\mathbf{x} \in \mathbb{R}^{3}: \frac{x_{0}^{2}}{\cosh ^{2} \alpha}+\frac{x_{1}^{2}+x_{2}^{2}}{\sinh ^{2} \alpha}=1\right\}
$$

and

$$
\left\{\mathbf{x} \in \mathbb{R}^{3}: \frac{x_{0}^{2}}{\sinh ^{2} \alpha}+\frac{x_{1}^{2}+x_{2}^{2}}{\cosh ^{2} \alpha}=1\right\}
$$

called prolate and oblate, respectively, was developed by Garabedian in [132]. The orthogonality was taken in different norms, each of which led to the discussion of a partial differential equation through the kernel of the orthogonal system corresponding to that norm. The prescribed spheroids become more spherical as $\alpha \rightarrow \infty$ (since $\tanh \alpha \rightarrow 1$ ), but the radii also tend to infinity, so that the Euclidean ball is not included in the class of functions considered in [132]. Besides, this work does not cover those harmonics that vanish at infinity, which are, perhaps, from the point of view of physical application, the more interesting class. Some aspects of generating harmonic functions orthogonal in a region outside a prolate spheroid were discussed in [250].

However, useful properties such as the relationships between internal and external harmonics systems associated with spheroids of differing eccentricity were not studied. Furthermore, internal and external spherical harmonics were not considered part of these kinds of systems. The theoretical framework, extensively described in this thesis, provides a general answer to these questions.

Over the last few years, there has been a growing interest in various aspects of the time-frequency concentration problem, including applications in communication engineering. One such application aims to find the signals with maximum energy concentration simultaneously in time and frequency domains. In the 60s, Slepian, Landau, and Pollak [198, 199, 299, 300] found that the Prolate Spheroidal Wave Functions (hereafter abbreviated as PSWFs) are the most optimal energy concentration functions in a Euclidean space of finite dimension. The PSWFs are band-limited and exhibit interesting orthogonal relations. They are normalized versions of the solutions to the scalar Helmholtz equation in prolate spheroidal coordinates. Because of this, considerable effort has been put into developing applications for the PSWFs. They are often regarded as somewhat mysterious functions, showing no explicit or standard representation in terms of elementary functions, and are too difficult to compute numerically. PSWFs usually appear in Dirichlet problems in spheroidal domains arising in hydrodynamics, elasticity, and electromagnetism. Spheroidal functions are frequently encountered as solutions to boundary value problems of radiation, scattering, and propagation of acoustic signals and electromagnetism waves radiated by sources with spheroidal shapes.

Another topic that has been catching the interest of the scientific community is the development of numerical methods using PSWFs as basis functions [47, 75, 326, 327, 328. The PSWFs provide an optimal tool for approximating band-limited functions [46, 47, 234, 285, 295, 334, 335, 341, and they are preferable to classical polynomial bases (such as Legendre and Chebychev polynomials). The PSWFs were used in the sampling and reconstruction of band-limited signals in [329]. In [329], Walter et al. showed that the PSWFs could replace the sinc function in the sampling formula. An advantage of this is that the PSWFs are more concentrated on finite intervals than the sinc function. Applying the theory of reproducing kernel Hilbert spaces, Moumni et al. [253] extended the PSWFs to derive a sampling formula for a general class of functions that are band-limited to the unit cube and the unit ball in the Euclidean space. This extends Walter and Shen's Sampling Theorem to higher dimensions and bounds the truncation errors. In [262, [334], PSWFs were applied to construct quadratures, interpolation, and differentiation formulas for band-limited signals. The resulting numerical algorithms
were satisfactory [262]. These applications have stimulated a surge of new ideas and methods, both theoretical and applied. PSWFs have reawakened an interest in spectral analysis, signal processing, optical system analysis, approximation theory, potential theory, partial differential equations, and so forth.

Slepian first studied multi-dimensional PSWFs supported on the unit hyperball in [300] (cf. [116]). The author provided many of their analytical properties in great detail, including those that support the construction of numerical schemes. These functions are formally known as Slepian functions. Slepian was able to find a self-adjoint second-order differential operator that commutes with a particular finite convolution integral operator. In this way, the problem was converted into an eigenvalue problem for an ordinary differential equation. Slepian also showed that the more general problem of a hyperball could be entirely reduced to the two-dimensional case. Later, Morrison [235] successfully extended the domain of definition to hyperellipsoids and found differential equations for the eigenfunctions thus obtained. Slepian also proposed and developed the discrete version of the PSWFs in [300]. Shkolnisky et al. developed many of the properties of the two-dimensional circular PSWFs in [295, [296], which go in one limit over into the Zernike circle polynomials (this topic is further discussed in Section 4.4. Circular PSWFs were also used to study confocal laser modes and wave aberrations [182, 309]. In Astrophysics, the circular PSWFs were applied in stellar coronagraphy [309]. Based on an approximation scheme for band-limited functions concentrated in a disk, the theory developed in [194, 195 brought images in cryo-electron microscopy to a graphic representation. The PSWFs were extended to other domains, such as their spherical counterparts, with applications in geophysics [297]. More general PSWFs were studied in [294]. The generalized PSWFs have recently played a very active role, demonstrated by the multitude of problems arising from their applications in physical sciences and engineering, such as wave scattering, signal processing, and antenna theory.

Analytical properties of the PSWFs were proposed in the general context of various function spaces, such as quaternionic and Cliffordian spaces [185, 251, 350], and under different integral transforms [328, 341]. The frequency-domain was considered not only under the Fourier transform but also under the more general fractional Fourier transform (FFT) [268] and the linear canonical transform (LCT) [348, 349]. The generalized PSWFs associated with the FFT and the LCT are relevant when analyzing the status of energy preservation of optical systems, self-imaging phenomenon, and the resonance phenomenon of finite-sized one-stage and multiple-stage optical systems [266].

Nearly a century after Hamilton discovered the quaternions [155], the Swiss mathematician Fueter introduced the concept of quaternion-valued regular functions [126] (known today as monogenic functions) employing an ana$\log$ of a generalized Cauchy-Riemann system (cf. [127, [128, 129, 130, 131). As was mentioned by Sudbery in [318], this generalization is the only appropriate way to construct a broad class of functions that generalize the class of holomorphic functions of one complex variable. A technique for obtaining monogenic function systems uses the factorization of the Laplace operator and takes an appropriate set of harmonic functions as a starting point. In analogy with the one-dimensional case, we aim to reach a system with a simple structure in the sense that the underlying functions can be explicitly calculated, and the numerical costs grow only slightly. Bearing in mind the application of some differential and integral operators to the elements that constitute such sets and their extensions to more general functions via Fourier expansions, one must necessarily consider orthogonal bases. A basis can then be found for approximating monogenic functions or solutions of more general differential equations by series expansions in terms of monogenic polynomials via a continuous extension. At the same time, it would ensure the numerical stability of the best approximations.

The remarks above cover only a part of the problem (at least from the theoretical viewpoint). We must also consider the following facts:

1. The determination of harmonic and monogenic functions for the space interior and exterior to a spheroidal structure when their values on the surface are prescribed;
2. Monogenic functions (and, eventually, their associated scalar and vector parts) of different degree and order are orthogonal in the $L_{2}$-Hilbert space structure;
3. All hypercomplex derivatives and (monogenic) primitives of the functions deliver elements of the same structure again.

The problem of approximating a monogenic quaternion-valued function by polynomials or other systems of functions has a long history. The first ideas on how to characterize monogenic functions using power series expansions in terms of monogenic polynomials are mainly due to Fueter [126, 131]. This was done employing the notion of hypercomplex variables. Later, in [48] and [215], it was shown that a monogenic function could be developed locally as a Taylor series in terms of the so-called Fueter polynomials based on those variables. Following this line of reasoning and based on these polynomials, Leutwiler, in [208], constructed a basis in the real-linear Hilbert space of
reduced quaternion-valued homogeneous monogenic polynomials in $\mathbb{R}^{3}$. His results were generalized to arbitrary dimensions in a Clifford algebra framework by Delanghe in [94]. The approach, followed by both authors, relies on harmonic conjugates. The main difficulty in carrying out this technique is that the Fueter polynomials and their associated scalar parts are generally not orthogonal for the scalar inner product (we refer to [62] and [236] Ch. 2] for a particular approach). A naive technique applies the Gram-Schmidt procedure for the normalization of these polynomials. Unfortunately, this orthonormalization procedure is not easy to handle, and the numerical calculations are highly unstable. Consequently, one has to consider a more suitable basis.

Unrelated to previous research, a different effort was made by Ryan in [287]. The author built a complete orthonormal system of homogeneous monogenic polynomials for even dimensions. However, that system is not appropriate for the case presented in this thesis since we will only consider functions defined in domains of the Euclidean space $\mathbb{R}^{3}$ of odd dimension. In this context, it is worth mentioning the works of Brackx, Delanghe, and Sommen in [48] and Gürlebeck in [144]. The authors studied shifted systems of Cauchy kernels to approximate a monogenic function. Although the rational systems constructed constitute complete sets of functions, the major drawback is that they do not carry the orthogonality property. Moreover, in these systems, the construction of an orthogonal series expansion is not possible, and hypercomplex derivatives of the basis functions are not even finite linear combinations of the original basis functions.

Some examples of the research developed in the ' 90 s include [7, 8, 4, 10, 11, 12] by Abul-Ez and Constales. The authors studied a set of special monogenic polynomials involving only products of a hypercomplex variable and its hypercomplex conjugate. This research extends the basic sets of polynomials of one complex variable, which appeared in the '30s in Whittaker's work and was resumed later in his book [333]. Since the authors only constructed one polynomial for each degree, it was not enough to form a basis for the space of square-integrable monogenic functions. Note that the prescribed monogenic polynomials are deliberately similar, up to a rescaling factor, to those studied by Falcão and Malonek in [112, 113, 114. However, at the time of publication of [7], the concept of hypercomplex differentiability or the corresponding use of the hypercomplex derivative was not used to prove the fundamental characteristic Appell property of these polynomials as it is done at present. A range of significant results was achieved by applying these polynomials in the study of several elementary functions within Clifford analysis [5, 68, 69, 72, 83, 113, 219, 220, generalized Joukowski transformations in Euclidean spaces of arbitrary higher dimension [18, 82], as well as in other
topics. These polynomials were recently used to prove a Clifford counterpart of Hadamard's three-hyperballs Theorem [13].

A different but equally important effort was made by Cação, which resulted in her Ph.D. thesis [63] and follow-up papers [64, 65, 66]. Cação et al. researched complete orthogonal systems of homogeneous monogenic quaternion-valued polynomials in the Euclidean ball of $\mathbb{R}^{3}$. The resulting polynomials carry the property of having hypercomplex derivatives (resp. monogenic primitives) within the same basis one degree lower (resp. upper), contrary to the sets referred above. In [38, fundamental recursion formulas were obtained for the elements of these bases. These results were used in the book [148]. A unified and explicit construction of monogenic Appell bases in dimensions 2, 3, and 4 was given in [41]. These functions played a fundamental role in the study of quaternionic counterparts of the wellknown Bohr Theorem [150, 151, Borel-Carathéodory's Theorem [149], and Hadamard's Real-Part Theorems on the majorant of a Taylor series [236, 240], as well as Bloch's Theorem [241]. The standard domain in which all these works were developed is the Euclidean ball. In [239, 242], the present author studied a general theory of prolate spheroidal monogenics that, in particular cases, contains the prescribed spherical monogenic functions. In [243], it was shown that the underlying prolate spheroidal monogenics play an essential role in studying the monogenic Szegö kernel function for prolate spheroids. In [244, Remark 3.2], it was explained how one could generate oblate spheroidal monogenics from prolate spheroidal monogenics and viceversa. It was pointed out that only the radial part of a monogenic spheroidal function plays a role in the underlying transformation of variables. Associated with this topic, the authors in [250] exploited a complete orthogonal system of oblate spheroidal monogenics and found some recurrence formulas. It was shown that in the case of an oblate spheroid, a complete system could only be either an orthogonal or an Appell system. Long sought in this line of work are orthogonal bases of monogenic spheroidal functions that accommodate two classes: prolate and oblate spheroidal monogenics, which can be reduced to the internal and external solid spherical monogenics when the underlying eccentricity parameter tends to zero. It is toward the achievement of this goal that the current work is directed. The present theory gives a general basis for the study of each particular case and allows for a more significant simplification of many of the proofs involved.

This thesis is divided into five chapters, and its outline is detailed in the following. Chapter 1 gives some preliminaries concerning the algebra of Quaternions, including its insertion as a particular case of a Clifford algebra. The concept of monogenicity is introduced, and relevant results from the books [146] and [148] are summarized and applied in the subsequent chapters.

Since monogenic functions are harmonic in all of their components, we also summarize elementary facts from harmonic analysis, which will be needed throughout this thesis.

Chapter 2 presents single one-parameter families of internal and external spheroidal harmonics, including the spherical harmonic functions as limiting cases. The chapter proceeds to find relationships among the systems of spherical and spheroidal harmonic functions from which explicit conversion formulas that relate systems of harmonic functions associated with spheroids of differing eccentricity can be obtained.

Chapter 3 applies the spheroidal harmonics studied in the previous chapter to construct single one-parameter orthogonal bases of internal and external spheroidal monogenics, whose elements are parametrized by the shape of the corresponding spheroids. The principal point of interest is that the orthogonality of the elements that constitute the two bases does not depend on the eccentricity of the spheroids. Using expressions of change of basis calculated in the previous chapter, conversion formulas that relate different spheroidal monogenic systems are obtained. By selecting specific options among the spheroidal monogenic polynomials of the constructed bases, we derive an orthogonal basis in the quaternionic Hilbert space $L_{2}$ over spheroids of arbitrary eccentricity. Furthermore, the Bergman kernel function for the space of monogenic and square-integrable functions defined in a spheroid of arbitrary eccentricity is derived from the theory presented previously. The second part of the chapter focuses on constructing bases for the collection of spheroidal monogenic constants and spheroidal ambigenic polynomials. This allows reaching an explicit construction of a graded basis for the space of square-integrable contragenic functions. The elements of this basis, which are inhomogeneous polynomials of three spatial variables, depend polynomially on the eccentricity of the prescribed spheroids. To conclude the chapter, we investigate the relationships between the contragenic function systems for spheroids of different eccentricities, showing that there are common contragenic functions to all spheroids of all eccentricities.

Chapter 4 discusses a theory of functions with quaternionic values and three real variables determined by a Moisil-Teodorescu type operator with non-constant quaternionic coefficients, and it is intimately related to the theory of PSWFs. We proceed to study the relationship between two closed subspaces of $L_{2}$ : the subspace $\mathcal{D}(\mathbf{T})$ of all functions supported in the spatial domain $\mathbf{T}$ and the subspace $\mathcal{B}(\mathbf{W})$ of all functions whose Fourier transforms are supported in the frequency domain $\mathbf{W}$. We also analyze the composition $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$, where $D_{\mathbf{T}}$ and $B_{\mathbf{W}}$ are the projections onto $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$, respectively, and its spectrum and find that the eigenfunctions of the compact and self-adjoint integral operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ are a particular class of
band-limited quaternionic functions: the $c$-Quaternionic Prolate Spheroidal Wave Functions (hereafter abbreviated as $c$-QPSWFs). On the other hand, using these specific eigenfunctions as a basis for the band-limited functions in $L_{2}$, we can prove a series of results concerning the approximate concentration of functions in the spatial and the frequency domains and a quaternionic counterpart of Donoho and Stark's uncertainty principle. This is followed by a discussion regarding the relevancy of the reproducing kernel in extremum problems. Such reproducing kernel is obtained from the inverse quaternion Fourier transform of the characteristic function of the support of the quaternionic signals in the transformed domain. Furthermore, it is proven that the $c$-QPSWFs are orthogonal and complete over two different bounded domains along with the Euclidean space $\mathbb{R}^{3}$ under the assumption of a certain kind of symmetry, namely the space of square-integrable quaternionic functions on a cube and the reproducing kernel Hilbert space of band-limited quaternionic signals. In the second part of the chapter, the $c$-QPSWFs are used to examine the energy concentration of a signal in the spatial and frequency domains. Applying these results leads to calculating the $c$-QPSWFs restricted in the spatial domain to the Euclidean unit ball and frequency domain to the ball of radius $c>0$. Some of their fundamental properties are established.

Finally, Chapter 5 presents some applications and discusses two constructive approaches for generating Riesz systems of harmonic conjugates. Then some examples of function spaces illustrating the techniques involved are presented. More specifically, we discuss the weighted Hardy and Bergman spaces consisting of monogenic functions in the Euclidean unit ball. Moreover, we prove the boundedness of the underlying harmonic conjugation operators in specific weighted spaces. To conclude this chapter, the generalization of Bloch's Theorem for monogenic functions in the three-dimensional Euclidean space is presented, as well as an explicit computation of a lower bound for the Bloch constant.

## Contents

1 Basic Concepts of Quaternionic Analysis ..... 25
1.1 Quaternions and Clifford Algebras ..... 26
1.1.1 The Skew-Field $\mathbb{H}$ of Quaternions ..... 26
1.1.2 The Real Clifford Algebra ..... 29
1.2 Operators on Hilbert spaces over $\mathbb{H}$ ..... 30
1.2.1 Linear Spaces of $\mathbb{H}$-valued functions ..... 30
1.2.2 The Riesz Representation Theorem in HI ..... 35
1.2.3 Spectral Representations ..... 37
$1.3 \mathbb{H}$-valued functions on Spatial Domains ..... 38
1.3.1 Monogenic Functions ..... 38
1.3.2 Hypercomplex Derivatives and Primitives of Monogenic Functions ..... 42
1.4 The associated Legendre Functions ..... 44
1.5 The Prolate Spheroidal Wave Functions ..... 49
1.5.1 The Helmholtz Equation in Spheroidal Coordinates ..... 49
1.5.2 The Original Approach of Landau, Pollak, and Slepian ..... 52
1.6 The Quaternion Fourier Transform ..... 56
2 Solutions of Laplace's equation in spheroidal coordinates ..... 63
2.1 Harmonics in Spheroidal Coordinates ..... 63
2.2 Conversions among Spheroidal Harmonics ..... 75
2.3 Orthogonal Families of Spheroidal Harmonics ..... 83
3 Monogenics and Contragenics on Spheroidal Domains ..... 93
3.1 Orthogonal Bases of Spheroidal Monogenics ..... 94
3.1.1 Internal Monogenic Spheroidal Polynomials ..... 94
3.1.2 The Monogenic Bergman Kernel on Spheroids ..... 109
3.1.3 External Spheroidal Monogenic Functions ..... 113
3.2 Contragenics on Spheroidal Domains ..... 122
3.2.1 Ambigenic Spheroidal Polynomials ..... 122
3.2.2 Contragenic Spheroidal Polynomials ..... 127
3.2.3 Relations among Contragenic Functions ..... 136
4 The $c$-Quaternionic Prolate Spheroidal Wave Functions ..... 143
4.1 The PSWFs vs. the $c$-Hyperholomorphicity ..... 144
4.2 The $c$-QPSWFs ..... 149
4.2.1 Space-Limited and Band-Limited Quaternionic Signals ..... 149
4.2.2 Definition and Properties of the $c$-QPSWFs ..... 162
4.2.3 Extrapolation of a Band-limited QuaternionicFunction by the $c$-QPSWFs172
4.3 The $c$-QPSWFs vs. the Energy Extremal Problem ..... 176
4.4 Constructing $c$-QPSWFs on the Ball ..... 188
5 Applications ..... 195
5.1 On Conjugate Harmonic Functions in $\mathbb{R}^{3}$ ..... 195
5.1.1 Monogenic weighted Hardy spaces ..... 204
5.1.2 Monogenic weighted Bergman spaces ..... 210
5.2 A Bloch-type theorem for monogenic functions ..... 213
5.2.1 Estimates for monogenic functions bounded with re-spect to their hypercomplex derivatives213
5.2.2 The Bloch Theorem ..... 217
6 Conclusions and Suggestions for Further Study ..... 221

## List of Figures

$3.1 \quad l=1$ ..... 113
$3.2 \quad l=3$ ..... 113
$3.3 \quad l=5$ ..... 114
$3.4 \quad l=10$ ..... 114
$3.5 \quad l=13$ ..... 114
$3.6 \quad l=15$ ..... 114
$4.1 \quad c=0.1$ ..... 147
$4.2 \quad c=2$ ..... 147
$4.3 \quad c=3$ ..... 148
$4.4 \quad c=4$ ..... 148

| 4.5 | $\left\|\phi_{0.0}\right\|^{2}$ |
| :---: | :---: | ..... 148


| 4.6 | $\left.\phi_{1.0}\right\|^{2}$ |
| :--- | :--- |
| .7 | $\phi_{2}$ | ..... 148

$4.7 \quad\left|\phi_{2.0}\right|^{2}$ ..... 148

| 4.8 | $\left.\phi_{3.0}\right\|^{2}$ |
| :--- | :--- | ..... 148


| 4.9 | $\left.\phi_{4.0}\right\|^{2}$ |
| :--- | :--- | ..... 148

$\left.4.10 \quad \phi_{5,0}\right|^{2}$ ..... 148
4.11 The relationship between $\alpha(\mathbf{T})$ and $\beta(\mathbf{W})$ for $c=1$. ..... 188

## List of Tables

3.1 Basic spheroidal monogenic polynomials of degree $l=0,1,2$, parametrized by the eccentricity $\mu$. . . . . . . . . . . . . . . . 98
3.2 Basic spheroidal monogenic polynomials of degree $l=3$, parametrized by the eccentricity $\mu$. . . . . . . . . . . . . . . . . . . . . . . . 99
3.3 Spheroidal monogenic basis polynomials of degree $l=0,1,2,3$, parametrized by the eccentricity $\mu$. . . . . . . . . . . . . . . . 111
3.4 Dimensions of spaces of polynomials $(l \geq 0)$. . . . . . . . . . . 127
3.5 Spheroidal contragenic polynomials of low degree, parametrized by the eccentricity $\mu$. . . . . . . . . . . . . . . . . . . . . . . . 131

## 1

## Basic Concepts of Quaternionic Analysis


#### Abstract

Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements; in technical language it may be said to be "time plus space", or "space plus time": and in this sense it has, or at least involves a reference to, four dimensions.

And how the One of Time, of Space the Three, Might in the Chain of Symbols girdled be.

Quoted in R. P. Graves, Life of Sir William Rowan Hamilton.


The first part of this chapter is devoted to the exposition of the basic definitions and terminology that are to be used throughout this dissertation. We begin with a quick review of the properties of quaternion numbers and their embedding in more general systems of Clifford numbers. Then we introduce the generalized Cauchy-Riemann operator $\bar{\partial}$, which generalizes the well-known two-dimensional Cauchy-Riemann operator to quaternionic analysis. The null-solutions of this operator are called monogenic. The noncommutative structure of the Hamiltonian quaternion algebra makes it essential to distinguish between an application of the operator $\bar{\partial}$ from the left-hand side and right-hand side of a quaternion-valued function. It is possible to factor the Laplace operator in terms of operator $\overline{\bar{\partial}}$ and its quaternionic conjugate, similar to the complex case. This factorization gives the possibility to generate classes of monogenic functions from harmonic ones. Correspondingly, we proceed with the definition and some basic properties of the associated Legendre functions of the first and second kinds, which are used to build the classical systems of internal and external spherical,
and more generally, of spheroidal harmonics. We move forward to discussing the fundamental properties of the quaternion Fourier transform, including linearity, space shift, and frequency shift properties, the Riemann-Lebesgue Lemma, inversion, Plancherel, and Parseval identities.

The second part of the chapter begins with a brief review of the two contexts that give rise to the PSWFs of order zero and collect some of their essential properties.

### 1.1 Quaternions and Clifford Algebras

### 1.1.1 The Skew-Field $\mathbb{H}$ of Quaternions

The Hamilton Quaternions were devised on the 16th of October 1843 by the mathematical physicist William Rowan Hamilton. The original motivation behind this research was to build a set of hypercomplex numbers related to the three-dimensional space in the same way as complex numbers are related to the two-dimensional plane. But what Hamilton found was a number with one real component and three distinct imaginary components, and where all of the squares of the imaginary components are -1 . According to legend, Hamilton was inspired to reach this generalization during a walk along the Royal Canal in Dublin with his wife. While the quaternions are noncommutative, they are associative and form an algebra over the real field of dimension 4 [155]. The idea of using multiplication of 4 -vectors similar to quaternion multiplication predates Hamilton and can be found in the works of Euler, Gauss, and Rodrigues. In 1748 Euler discovered the four-square identity and used a quaternion representation to describe motions in the Euclidean space. This fact was rediscovered in 1957 by Blaschke and was mentioned in his speech on Euler's 250th anniversary in Berlin. It is also worth noting that Rodrigues used a quaternion multiplication technique to describe a parametrization of general rotations via the so-called Euler-Rodrigues parameters. Gauss derived similar results to those of Hamilton and Rodrigues in 1819, but they remained unpublished during his lifetime and, therefore, unknown to both of them.

We use the original notation $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for the standard basis of the Hamiltonian quaternions. The imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ obey the following multiplication rules:

$$
\begin{align*}
& \mathbf{i}^{2}=\mathbf{j}^{2}=\mathbf{k}^{2}=-1 \\
& \mathbf{i j}=\mathbf{k}=-\mathbf{j} \mathbf{i}, \quad \mathbf{j} \mathbf{k}=\mathbf{i}=-\mathbf{k j}, \quad \mathbf{k} \mathbf{i}=\mathbf{j}=-\mathbf{i} \mathbf{k} \tag{1.1.1}
\end{align*}
$$

and the usual componentwise defined addition. A quaternion is a number of the form

$$
\mathbf{p}=p_{0}+\mathbf{i} p_{1}+\mathbf{j} p_{2}+\mathbf{k} p_{3},
$$

where the $p_{i}=[p]_{i}$ are real numbers. The set of quaternions is denoted by $\mathbb{H}$, in honor of its discoverer.

Using relations (1.1.1), we define the multiplication of two quaternions $\mathbf{p}=p_{0}+\mathbf{i} p_{1}+\mathbf{j} p_{2}+\mathbf{k} p_{3}$ and $\mathbf{q}=q_{0}+\mathbf{i} q_{1}+\mathbf{j} q_{2}+\mathbf{k} q_{3}$ as follows:

$$
\begin{aligned}
\mathbf{p q}= & \left(p_{0} q_{0}-p_{1} q_{1}-p_{2} q_{2}-p_{3} q_{3}\right)+\mathbf{i}\left(p_{1} q_{0}+p_{0} q_{1}+p_{2} q_{3}-p_{3} q_{2}\right) \\
& +\mathbf{j}\left(p_{2} q_{0}+p_{0} q_{2}+p_{2} q_{1}-p_{1} q_{3}\right)+\mathbf{k}\left(p_{3} q_{0}+p_{0} q_{3}+p_{1} q_{2}-p_{2} q_{1}\right) .
\end{aligned}
$$

We denote by

$$
\mathrm{Sc}(\mathbf{p})=p_{0}
$$

the scalar part of $\mathbf{p}$ and by

$$
\operatorname{Vec}(\mathbf{p})=\mathbf{i} p_{1}+\mathbf{j} p_{2}+\mathbf{k} p_{3}
$$

its vector part. The real numbers are precisely those with zero vector part, and if $\mathbf{p}=\operatorname{Vec}(\mathbf{p})$, then $\mathbf{p}$ is called a pure quaternion.

The conjugate of a quaternion $\mathbf{p}$ is

$$
\overline{\mathbf{p}}=\operatorname{Sc}(\mathbf{p})-\operatorname{Vec}(\mathbf{p})=p_{0}-\mathbf{i} p_{1}-\mathbf{j} p_{2}-\mathbf{k} p_{3} .
$$

The quaternion conjugation can also be useful to extract the scalar and vector parts of $\mathbf{p} \in \mathbb{H}$ :

$$
\mathrm{Sc}(\mathbf{p})=\frac{1}{2}(\mathbf{p}+\overline{\mathbf{p}})
$$

and

$$
\operatorname{Vec}(\mathbf{p})=\frac{1}{2}(\mathbf{p}-\overline{\mathbf{p}})
$$

The following lemma can now be proved:
Lemma 1.1.1. For all $\mathbf{p}, \mathbf{q} \in \mathbb{H}$, the quaternion conjugation has the following properties:
(i) $\overline{\mathbf{p}+\mathbf{q}}=\overline{\mathbf{p}}+\overline{\mathbf{q}}$;
(ii) $\overline{\mathbf{p q}}=\overline{\mathbf{q}} \overline{\mathbf{p}}$;
(iii) $\overline{\overline{\mathbf{p}}}=\mathbf{p}$;
(iv) $\mathbf{p} \in \mathbb{R}$, if and only if $\mathbf{p}=\overline{\mathbf{p}} ; \mathbf{p}$ is a pure quaternion, if and only if $\mathbf{p}=-\overline{\mathbf{p}}$.

Following what has been introduced above, a cyclic multiplication symmetry holds

$$
\begin{equation*}
\mathrm{Sc}(\mathbf{p q r})=\mathrm{Sc}(\mathbf{r q p})=\mathrm{Sc}(\mathbf{q} \mathbf{p r}) \tag{1.1.2}
\end{equation*}
$$

for all $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{H}$.
The (algebraic) norm of a quaternion $\mathbf{p}=p_{0}+\mathbf{i} p_{1}+\mathbf{j} p_{2}+\mathbf{k} p_{3}$ is defined by

$$
|\mathbf{p}|:=(\mathbf{p} \overline{\mathbf{p}})^{1 / 2}=(\overline{\mathbf{p}} \mathbf{p})^{1 / 2}=\left(p_{0}^{2}+p_{1}^{2}+p_{2}^{2}+p_{3}^{2}\right)^{1 / 2}
$$

and it coincides with the corresponding Euclidean norm of $\mathbf{p}$, as a vector in $\mathbb{R}^{4}$. If $|\mathbf{p}|=1$, then $\mathbf{p}$ is called a unit quaternion. Moreover, it is easily seen that $|\mathbf{p q}|=|\mathbf{p} \| \mathbf{q}|$ and that every nonzero quaternion $\mathbf{p}$ possesses an inverse defined by $\mathbf{p}^{-1}:=\overline{\mathbf{p}} /|\mathbf{p}|^{2}$, such that $\mathbf{p p}^{-1}=\mathbf{p}^{-1} \mathbf{p}=1$. It follows that the inverse of a unit quaternion is its conjugate. Furthermore, it holds $|\mathbf{p}|^{-1}=\left|\mathbf{p}^{-1}\right|$.

As a consequence of the existence of inverse, the quaternions form a noncommutative division algebra, the skew-field $\mathbb{H}$ of quaternions. The quaternions remain the most straightforward algebra after the real and complex numbers. Due to the famous Theorem of Frobenius [124], the real numbers, the complex numbers, and the quaternions are the only associative division algebras [178]. Amongst these, the quaternions are the most general. For a detailed historical survey and an extended list of references on the real algebra of quaternions, we refer to [126, 145, 146, 148, 189, 190, 247, 318] and elsewhere.

We will now introduce a proper quaternion exponential function, which has many similarities with the complex exponential function. Jamison asserted the following definition in [176]:

Definition 1.1.2. Let $\mathbf{u}$ be any pure quaternion such that $\mathbf{u}^{2}=-1$, and let $\vartheta$ be any real number. The quaternion exponential function $\exp (\mathbf{u} \vartheta)$ is defined as

$$
\begin{equation*}
\exp (\mathbf{u} \vartheta)=\cos \vartheta+\mathbf{u} \sin \vartheta \tag{1.1.3}
\end{equation*}
$$

We then have the following lemma.
Lemma 1.1.3. The function defined by (1.1.3) has the following properties:
(i) $\exp \left(\mathbf{u} \vartheta_{1}\right) \exp \left(\mathbf{u} \vartheta_{2}\right)=\exp \left[\mathbf{u}\left(\vartheta_{1}+\vartheta_{2}\right)\right]$;
(ii) $\overline{\exp (\mathbf{u} \vartheta)}=\exp (-\mathbf{u} \vartheta)$;
(iii) $|\exp (\mathbf{u} \vartheta)|=1$;
(iv) $\frac{d}{d \vartheta} \exp (\mathbf{u} \vartheta)=\mathbf{u} \exp (\mathbf{u} \vartheta)$.

### 1.1.2 The Real Clifford Algebra

We link the previous definition of the quaternion algebra $\mathbb{H}$ with the more general one of universal Clifford algebra. Inspired by the work of Hamilton and combining ideas of geometric algebra developed by Grassmann, Clifford introduced the notion of what is now known as the Clifford algebra in 1878. Said algebra includes generalizations of the scalar and vector products to higher dimensions [79]. A Clifford algebra is an associative but usually noncommutative algebra over the real or the complex field. For more details about Clifford algebras, please refer, e.g., to [48, 148, 213, 273).

We henceforth consider the universal real Clifford algebra of signature $(0, n)$ denoted by $\mathcal{C} \ell_{0, n}$ and let $\left\{\mathbf{e}_{0}, \mathbf{e}_{1}, \mathbf{e}_{2}, \ldots, \mathbf{e}_{n}\right\}$ stand for the canonical basis of the Euclidean vector space $\mathbb{R}^{n+1}$. The basis elements satisfy the following multiplication rules:

$$
\begin{aligned}
& \mathbf{e}_{k} \mathbf{e}_{l}+\mathbf{e}_{l} \mathbf{e}_{k}=-2 \delta_{k, l} \mathbf{e}_{0}, \\
& \mathbf{e}_{0} \mathbf{e}_{k}=\mathbf{e}_{k} \mathbf{e}_{0}=\mathbf{e}_{k}, \quad(k, l=1,2, \ldots, n),
\end{aligned}
$$

where $\delta_{k, l}$ denotes the Kronecker delta and the element $\mathbf{e}_{0}$ is regarded as the usual unit, that is, $\mathbf{e}_{0}=1$.

A basis for $\mathcal{C} \ell_{0, n}$ is given by the elements $\mathbf{e}_{A}=\mathbf{e}_{i_{1}} \mathbf{e}_{i_{2}} \cdots \mathbf{e}_{i_{k}}$, where $A=$ $\left\{i_{1}, i_{2}, \ldots, i_{k}\right\} \subseteq\{1,2, \ldots, n\}$ is such that $1 \leq i_{1}<i_{2}<\cdots<i_{k} \leq n$ and $\mathbf{e}_{\emptyset}=\mathbf{e}_{0}=1$. It then follows that the dimension of $\mathcal{C} \ell_{0, n}$ is $2^{n}$. As $\mathcal{C} \ell_{0, n}$ is isomorphic to $\mathbb{R}^{2^{n}}$, we may provide it with the $\mathbb{R}^{2^{n}}$-norm $|\mathbf{a}|$, and one can verify that for any $\mathbf{a}, \mathbf{b} \in \mathcal{C} \ell_{0, n},|\mathbf{a} \mathbf{b}| \leq 2^{n / 2}|\mathbf{a}||\mathbf{b}|$, where $\mathbf{a}=$ $\sum_{A \subseteq\{1, \ldots, n\}} \mathbf{e}_{A} \mathbf{a}_{A}$ and $\mathbf{b}=\sum_{A \subseteq\{1, \ldots, n\}} \mathbf{e}_{A} \mathbf{b}_{A}$. The addition and multiplication of elements of $\mathcal{C} \ell_{0, n}$ by real numbers are defined componentwise. In this way, the multiplication between two elements of $\mathcal{C} \ell_{0, n}$ turns out to be associative, anticommutative, and has distributive properties.

Now, let $\mathcal{C} \ell_{0, n}^{k}$ be the real linear subspace of $\mathcal{C} \ell_{0, n}$, defined as

$$
\mathcal{C} \ell_{0, n}^{k}=\left\{\mathbf{a} \in \mathcal{C} \ell_{0, n}: \mathbf{a}=\sum_{|A|=k} \mathbf{e}_{A} \mathbf{a}_{A}\right\},
$$

where $|A|$ denotes the cardinality of the set $A$. The elements of $\mathcal{C} \ell_{0, n}^{2}$ are called bivectors, while the elements of $\mathcal{C} \ell_{0, n}^{3}$ are called pseudoscalars. We define the even subalgebra $\mathcal{C} \ell_{0,3}^{+}$as

$$
\mathcal{C} \ell_{0, n}^{+}=\underset{k \text { even }}{\bigoplus} \mathcal{C} \ell_{0, n}^{k} .
$$

We shall remark that $\mathcal{C} \ell_{0, n}^{+}$is again a Clifford algebra, but not a universal one. From the considerations above adduced, it is seen that $\mathbb{H}$ can be interpreted as

Clifford algebras in two different ways. On the one hand, $\mathbb{H}$ is isomorphic to the four-dimensional, even subalgebra $\mathcal{C} \ell_{0,3}^{+}$with the identification $-\mathbf{e}_{1} \mathbf{e}_{2} \rightarrow \mathbf{i}$, $-\mathbf{e}_{1} \mathbf{e}_{3} \rightarrow \mathbf{j}, \mathbf{e}_{2} \mathbf{e}_{3} \rightarrow \mathbf{k}$. On the other hand, $\mathbb{H}$ can be realized as the universal Clifford algebra $\mathcal{C} \ell_{0,2}=\left\langle\left\{1, \mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{1} \mathbf{e}_{2}\right\}\right\rangle$ with the identification $\mathbf{e}_{1} \rightarrow \mathbf{i}$, $\mathbf{e}_{2} \rightarrow \mathbf{j}, \mathbf{e}_{1} \mathbf{e}_{2} \rightarrow \mathbf{k}$.

We proceed to consider the subset of $\mathbb{H}$, defined as

$$
\begin{equation*}
\mathcal{A}:=\left\{x_{0}+\mathbf{i} x_{1}+\mathbf{j} x_{2}+\mathbf{k} x_{3} \in \mathbb{H}: x_{i} \in \mathbb{R}, x_{3}=0\right\} . \tag{1.1.4}
\end{equation*}
$$

The elements of $\mathcal{A}$ are known as reduced quaternions. The elements of $\mathbb{R}^{3}$ can be identified with elements of $\mathcal{A}$ by considering $\left(x_{0}, x_{1}, x_{2}\right) \in \mathbb{R}^{3}$. To this end, throughout the text, we will often use the same symbol $\mathbf{x}$ to be regarded as a point in $\mathbb{R}^{3}$ and to represent the corresponding reduced quaternion. There are other ways of embedding $\mathbb{R}^{3}$ in $\mathbb{H}$, for example, using the subspace of pure quaternions, i.e., by considering $\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$. Furthermore, since $\mathcal{A}$ is not closed under the quaternionic multiplication, it is clear that $\mathcal{A}$ is only a real vector subspace and not a subalgebra of $\mathbb{H}$.

### 1.2 Operators on Hilbert spaces over $\mathbb{H}$

### 1.2.1 Linear Spaces of $\mathbb{H}$-valued functions

Throughout the text, let $\Omega$ denote an open set of $\mathbb{R}^{3}$ with a piecewise smooth boundary. Here adopted, the notations for the boundary and the closure of $\Omega$ are, respectively, $\partial \Omega$ and $\bar{\Omega}$. We will use the notation $\Omega_{0}$ to denote the Euclidean unit ball.

A quaternion-valued function or, briefly, an $\mathbb{H}$-valued function is a mapping of the form $\boldsymbol{f}: \Omega \rightarrow \mathbb{H}$ such that

$$
\boldsymbol{f}(\mathbf{x})=[\boldsymbol{f}(\mathbf{x})]_{0}+\mathbf{i}[\boldsymbol{f}(\mathbf{x})]_{1}+\mathbf{j}[\boldsymbol{f}(\mathbf{x})]_{2}+\mathbf{k}[\boldsymbol{f}(\mathbf{x})]_{3},
$$

where $\mathbf{x} \in \Omega$ and the $[\boldsymbol{f}]_{i}$ are real-valued functions defined in $\Omega$. By abuse of notation, we shall use $[\boldsymbol{f}]_{0}$ and $\operatorname{Sc}(\boldsymbol{f})$ interchangeably. It is clear that if $[\boldsymbol{f}(\mathbf{x})]_{3}=0$ for all $\mathbf{x}$, then $\boldsymbol{f}$ is itself an $\mathcal{A}$-valued function. Properties such as continuity, differentiability, or integrability ascribed to $\boldsymbol{f}$, are defined coordinatewise.

Due to the noncommutativity of quaternions, it is necessary to distinguish between two types of linear spaces over $\mathbb{H}$, namely left-linear and right-linear spaces.

Definition 1.2.1. A left-linear space $\mathcal{L}$ over $\mathbb{H}$ is an additive abelian group in which there is defined operation of scalar multiplication by elements of $\mathbb{H}$.

Scalar multiplication is assumed to obey the following laws for all $\mathbf{x}, \mathbf{y} \in \mathcal{L}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{H}:$
(i) $\boldsymbol{\alpha}(\mathbf{x}+\mathbf{y})=\boldsymbol{\alpha} \mathbf{x}+\boldsymbol{\alpha} \mathbf{y}$;
(ii) $(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathbf{x}=\boldsymbol{\alpha} \mathbf{x}+\boldsymbol{\beta} \mathbf{x}$;
(iii) $(\boldsymbol{\alpha} \boldsymbol{\beta}) \mathbf{x}=\boldsymbol{\alpha}(\boldsymbol{\beta} \mathbf{x})$.

Unless stated otherwise, all $\mathbb{H}$-linear spaces here are assumed to be left spaces over the quaternions; the right space is defined similarly. We shall adopt $\mathcal{L}(\Omega, \mathbb{H})$ for linear spaces consisting of $\mathbb{H}$-valued, or more particularly, of $\mathcal{A}$-valued functions. The $\mathbb{R}$-linearity of the space $\mathcal{L}(\Omega, \mathcal{A})$ needs not be confused with the prescribed (left) $\mathbb{H}$-linearity of $\mathcal{L}(\Omega, \mathbb{H})$. The context will usually distinguish between the two without clarification. In cases where ambiguity may occur, we shall denote the $\mathbb{R}$-linear spaces of $\mathcal{A}$-valued functions by $\mathcal{L}(\Omega, \mathcal{A})$ and the (left-) linear spaces of $\mathbb{H}$-valued functions by $\mathcal{L}(\Omega, \mathbb{H})$.

It should be noted that the fundamental theory of finite-dimensional linear spaces over associative division algebras is well-established [175]. The definitions of basis, dimension, subspace, etc., are the same as those in the complex case and will not be given here.

Definition 1.2.2. Let $\mathcal{L}$ be a linear space over $\mathbb{H}$. A mapping $F: \mathcal{L} \rightarrow \mathbb{H}$ is called a left-linear functional if $F(\mathbf{x}+\mathbf{y})=F(\mathbf{x})+F(\mathbf{y})$ and $F(\boldsymbol{\alpha} \mathbf{x})=\boldsymbol{\alpha} F(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{L}, \boldsymbol{\alpha} \in \mathbb{H}$.

Definition 1.2.3. A linear space over $\mathbb{H}$ is called a normed linear space if there exists a function $\|\cdot\|: \mathcal{L} \rightarrow \mathbb{R}$ with the following properties:
(i) $\|\mathbf{x}\| \geq 0$, for all $\mathbf{x} \in \mathcal{L}$ and $\|\mathbf{x}\|=0$, if and only if $\mathbf{x}=\mathbf{0}$;
(ii) $\|\boldsymbol{\alpha} \mathbf{x}\|=|\boldsymbol{\alpha}|\|\mathbf{x}\|$, for all $\mathbf{x} \in \mathcal{L}$ and $\boldsymbol{\alpha} \in \mathbb{H}$;
(iii) $\|\mathbf{x}+\mathbf{y}\| \leq\|\mathbf{x}\|+\|\mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{L}$.

Definition 1.2.4. Let $\mathcal{L}$ be a normed linear space (with respective norm $\|\cdot\|$ ) over $\mathbb{H}$. A left-linear functional $F$ is called bounded if $|F(\mathbf{x})| \leq k_{F}\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathcal{L}$, where $k_{F}>0$ and depends only on $F$.

As usual, we denote the partial derivative of a function with respect to the variable $x_{i}$ by $\frac{\partial}{\partial x_{i}}, i \in\{0,1,2\}$ and the partial derivatives of higher-order by

$$
\partial^{\boldsymbol{\lambda}}=\frac{\partial^{|\lambda|}}{\partial x_{0}^{\lambda_{0}} \partial x_{1}^{\lambda_{1}} \partial x_{2}^{\lambda_{2}}},
$$

where $\boldsymbol{\lambda}=\left(\lambda_{0}, \lambda_{1}, \lambda_{2}\right)$ is a multi-index of nonnegative integers such that $|\boldsymbol{\lambda}|=\lambda_{0}+\lambda_{1}+\lambda_{2}$.

We introduce the following quaternionic spaces, which will be of use in further discussion:

Definition 1.2.5. We denote by
(i) $C(\Omega, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions that are continuous in $\Omega$;
(ii) $C^{m}(\Omega, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions $\boldsymbol{f}$ such that $\partial^{\lambda} \boldsymbol{f} \in C(\Omega, \mathbb{H})$ whenever $|\boldsymbol{\lambda}| \leq m$;
(iii) $C^{\infty}(\Omega, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions that belong to $C^{m}(\Omega, \mathbb{H})$ for every $m \in \mathbb{N}$.

Definition 1.2.6. Let $1 \leq p<\infty$. The $L_{p}(\Omega, \mathbb{H})$ space is defined to be the class of all Lebesgue measurable $\mathbb{H}$-valued functions defined on $\Omega$ such that $|\boldsymbol{f}|^{p} \in L_{1}(\Omega)$ for all $\boldsymbol{f} \in L_{p}(\Omega, \mathbb{H})$; that is,

$$
L_{p}(\Omega, \mathbb{H})=\left\{\boldsymbol{f}: \Omega \rightarrow \mathbb{H} \text { measurable }:\left(\int_{\Omega}|\boldsymbol{f}(\mathbf{x})|^{p} d \mathbf{x}\right)^{1 / p}<\infty\right\}
$$

where $d \mathbf{x}$ denotes the Lebesgue measure on $\Omega$.
From the above definition, it is clear that if $\boldsymbol{f} \in L_{p}(\Omega, \mathbb{H})$, then $\boldsymbol{\alpha} \boldsymbol{f}$ is also in $L_{p}(\Omega, \mathbb{H})$ for all $\boldsymbol{\alpha} \in \mathbb{H}$. Since $|\boldsymbol{f}+\boldsymbol{g}|^{p} \leq 2^{p}\left(|\boldsymbol{f}|^{p}+|\boldsymbol{g}|^{p}\right), L_{p}(\Omega, \mathbb{H})$ is also closed under addition. Accordingly, $L_{p}(\Omega, \mathbb{H})$ is a left-linear space over $\mathbb{H}$. It can further be shown that the space $L_{p}(\Omega, \mathbb{H})(p \geq 1)$ is complete [176].

Next, we will consider the primary space $L_{2}(\Omega, \mathbb{H})$ endowed with the (left) quaternionic inner product defined below.

Definition 1.2.7. The $L_{2}(\Omega, \mathbb{H})$-inner product is defined by

$$
\begin{equation*}
\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\int_{\Omega} \boldsymbol{f}(\mathrm{x}) \overline{\boldsymbol{g}}(\mathrm{x}) d \mathrm{x} \tag{1.2.1}
\end{equation*}
$$

for any $\boldsymbol{f}, \boldsymbol{g} \in L_{2}(\Omega, \mathbb{H})$, which satisfies the following properties:
(i) $\langle\boldsymbol{f}, \boldsymbol{f}\rangle_{L_{2}(\Omega, \mathbb{H})}>0, \boldsymbol{f} \neq \mathbf{0}$;
(ii) $\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\overline{\langle\boldsymbol{g}, \boldsymbol{f}\rangle_{L_{2}(\Omega, H)}}$;
(iii) $\langle\boldsymbol{f}+\boldsymbol{g}, \boldsymbol{h}\rangle_{L_{2}(\Omega, \mathbb{H})}=\langle\boldsymbol{f}, \boldsymbol{h}\rangle_{L_{2}(\Omega, \mathbb{H})}+\langle\boldsymbol{g}, \boldsymbol{h}\rangle_{L_{2}(\Omega, \mathbb{H})}$;
(iv) $\langle\boldsymbol{\alpha} \boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\boldsymbol{\alpha}\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}, \quad\langle\boldsymbol{f}, \boldsymbol{\alpha} \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})} \overline{\boldsymbol{\alpha}}$.

Definition 1.2.8. Two functions $\boldsymbol{f}, \boldsymbol{g}$ of $L_{2}(\Omega, \mathbb{H})$ are called orthogonal in the $L_{2}$-sense if $\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\mathbf{0}$.

Definition 1.2.9. Let $\mathcal{B}$ be any subset of $L_{2}(\Omega, \mathbb{H})$. Then $\mathcal{B}^{\perp}$ is defined to be $\mathcal{B}^{\perp}=\left\{\boldsymbol{g}:\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\mathbf{0}\right.$ for every $\left.\boldsymbol{f} \in \mathcal{B}\right\}$.

In accordance with (1.1.4), we endow the space $L_{2}(\Omega, \mathcal{A})$ with a positive definite symmetric bilinear form (also known as the scalar inner product), defined by

$$
\begin{align*}
\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{0, L_{2}(\Omega, \mathcal{A})} & :=\frac{1}{2}\left[\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}+\langle\boldsymbol{g}, \boldsymbol{f}\rangle_{L_{2}(\Omega, \mathbb{H})}\right] \\
& =\operatorname{Sc} \int_{\Omega} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{g}}(\mathbf{x}) d \mathbf{x} . \tag{1.2.2}
\end{align*}
$$

It is clear that $(1.2 .2)$ does not define an inner product in $L_{2}(\Omega, \mathbb{H})$ seen as an $\mathbb{H}$-linear space because it is not homogeneous for quaternionic scalars. The scalar inner product (1.2.2) appears, for example, in [87] in the context of complex vector spaces, in [145] for spaces of $\mathbb{H}$-valued functions, and in [48] for spaces of $\mathcal{C} \ell_{0,2}$-valued functions.

Definition 1.2.10. For $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$, the $L_{2}$-norm induced by the (left) quaternionic inner product (1.2.1) is defined by

$$
\begin{equation*}
\|\boldsymbol{f}\|_{L_{2}(\Omega)}=\left(\langle\boldsymbol{f}, \boldsymbol{f}\rangle_{L_{2}(\Omega, H)}\right)^{1 / 2}=\left(\int_{\Omega}|\boldsymbol{f}(\mathrm{x})|^{2} d \mathrm{x}\right)^{1 / 2} \tag{1.2.3}
\end{equation*}
$$

The space $L_{2}(\Omega, \mathbb{H})$ furnished with the quaternionic inner product (1.2.1) is a (left) quaternionic Hilbert space, and the norm 1.2.3) turns $L_{2}(\Omega, \mathbb{H})$ into a Banach space [176].

Thus, the $L_{2}$-norm induced by the quaternionic inner product 1.2 .1 coincides with the quadratic form associated with the bilinear form (1.2.2) and further, with the $L_{2}$-norm of $\boldsymbol{f}$ considered as a vector-valued function. Therefore, convergence results are independent of the choice of the inner product. Accordingly, from now on, we will denote each of these norms by $\|\cdot\|_{L_{2}(\Omega)}$.

The next lemma is known as the Cauchy-Bunyakovsky-Schwarz inequality [322] (cf. [176]).

Lemma 1.2.11. If $\boldsymbol{f}, \boldsymbol{g} \in L_{2}(\Omega, \mathbb{H})$, then

$$
\begin{equation*}
\left|\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}\right| \leq\|\boldsymbol{f}\|_{L_{2}(\Omega)}\|\boldsymbol{g}\|_{L_{2}(\Omega)} . \tag{1.2.4}
\end{equation*}
$$

It should be observed that, as was shown in [48], the significant inequality given in the previous lemma is valid in much more broad circumstances. In particular, the inequality holds for a class of modules over the general (real) Clifford algebras.

With the aid of Definition (1.2.7), the preceding investigations allow us to define the angle between two functions in $L_{2}(\Omega, \mathbb{H})$; in virtue of the CauchySchwarz inequality (1.2.4), it follows that

$$
\left|\operatorname{Sc}\left(\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, H)}\right)\right| \leq\|\boldsymbol{f}\|_{L_{2}(\Omega)}\|\boldsymbol{g}\|_{L_{2}(\Omega)} .
$$

Thus, for all nonzero $\boldsymbol{f}, \boldsymbol{g} \in L_{2}(\Omega, \mathbb{H})$,

$$
-1 \leq \frac{\operatorname{Sc}\left(\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}\right)}{\|\boldsymbol{f}\|_{L_{2}(\Omega)}\|\boldsymbol{g}\|_{L_{2}(\Omega)}} \leq 1
$$

This discussion leads to the following definition:
Definition 1.2.12. The angle $\arg (\boldsymbol{f}, \boldsymbol{g})$ between two nonzero functions $\boldsymbol{f}, \boldsymbol{g} \in$ $L_{2}(\Omega, \mathbb{H})$ is defined by

$$
\begin{equation*}
\arg (\boldsymbol{f}, \boldsymbol{g})=\arccos \left(\frac{\operatorname{Sc}\left(\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}\right)}{\|\boldsymbol{f}\|_{L_{2}(\Omega)}\|\boldsymbol{g}\|_{L_{2}(\Omega)}}\right) . \tag{1.2.5}
\end{equation*}
$$

It should be remarked that the extreme values 0 and $\pi$ for $\arg (\boldsymbol{f}, \boldsymbol{g})$ can only be reached if $\boldsymbol{f}$ and $\boldsymbol{g}$ are proportional (so that equality holds in (1.2.4) and $\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}$ is real. It can also be seen that the $\arg (\boldsymbol{f}, \boldsymbol{g})$ defined in (1.2.5) equals $\pi / 2$, if and only if $\boldsymbol{f}$ and $\boldsymbol{g}$ are orthogonal (over $\mathbb{R}$ ). These observations being made, we proceed to consider a further aspect of an angle between two $\mathbb{H}$-valued functions, which is substantially the one given by Rao in [276] and adopted by Gustafson and Rao in [154, p.56]. Because this result is proved in the same manner as in complex Hilbert spaces, we state the technical lemma without proof.

Lemma 1.2.13. For all nonzero $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in L_{2}(\Omega, \mathbb{H})$ the angle in 1.2.5) satisfies the triangle inequality:

$$
\begin{equation*}
\arg (\boldsymbol{f}, \boldsymbol{g}) \leq \arg (\boldsymbol{f}, \boldsymbol{h})+\arg (\boldsymbol{g}, \boldsymbol{h}) \tag{1.2.6}
\end{equation*}
$$

Before proceeding to investigate the spectral theorem for quaternionic compact self-adjoint operators, it will be convenient to discuss the elementary properties of bounded, compact, and self-adjoint operators on infinitedimensional Hilbert spaces over the quaternions. To proceed further, we need to make the following definitions:

Definition 1.2.14. An operator $T: L_{2}(\Omega, \mathbb{H}) \rightarrow L_{2}(\Omega, \mathbb{H})$ is called
(i) additive if $T(\boldsymbol{f}+\boldsymbol{g})=T(\boldsymbol{f})+T(\boldsymbol{g})$ for all $\boldsymbol{f}, \boldsymbol{g} \in L_{2}(\Omega, \mathbb{H})$;
(ii) left-homogeneous if $T(\boldsymbol{\alpha} \boldsymbol{f})=\boldsymbol{\alpha} T(\boldsymbol{f})$ for all $\boldsymbol{\alpha} \in \mathbb{H}$;
(iii) left-linear if both (i) and (ii) hold.

Definition 1.2.15. An operator $T: L_{2}(\Omega, \mathbb{H}) \rightarrow L_{2}(\Omega, \mathbb{H})$ is called bounded if there exists a constant $M>0$ such that $\|T \boldsymbol{f}\|_{L_{2}(\Omega)} \leq M\|\boldsymbol{f}\|_{L_{2}(\Omega)}$ for all $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$. The norm of such an operator $T$ is defined by

$$
\|T\|=\sup _{0 \neq \boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})} \frac{\|T \boldsymbol{f}\|_{L_{2}(\Omega)}}{\|\boldsymbol{f}\|_{L_{2}(\Omega)}} .
$$

For the sake of clarity, it should be emphasized that the class of bounded left-linear operators on a Hilbert space over $\mathbb{H}$ is generally not a (left-) linear space over $\mathbb{H}$. This is because the quaternions are noncommutative.

Definition 1.2.16. Let $T$ be a left-linear operator on $L_{2}(\Omega, \mathbb{H})$. $T$ is called compact if for each bounded sequence $\left\{\boldsymbol{f}_{n}\right\}$ from $L_{2}(\Omega, \mathbb{H})$, the sequence $\left\{T \boldsymbol{f}_{n}\right\}$ contains a convergent subsequence in $L_{2}(\Omega, \mathbb{H})$.

The following lemma is a well-known result for compact operators on complex Hilbert spaces. The proof can be easily carried over to the case of quaternionic Hilbert spaces and will not be given.

Lemma 1.2.17. If $T_{1}$ is a compact operator on $L_{2}(\Omega, \mathbb{H})$ and $T_{2}$ is a bounded operator on $L_{2}(\Omega, \mathbb{H})$, then $T_{1} T_{2}$ and $T_{2} T_{1}$ are compact operators.

### 1.2.2 The Riesz Representation Theorem in $\mathbb{H}$

This section discusses the fundamental concepts of completeness and closure of orthonormal sets in Hilbert spaces over $\mathbb{H}$. Also, we revisit a quaternionic version of the Riesz Representation Theorem and the notion of a Reproducing Kernel Quaternion Hilbert Space (hereafter abbreviated as RKQHS).

Definition 1.2.18. Let $\Lambda$ be an index set and $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ a subset of $L_{2}(\Omega, \mathbb{H})$. $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ is called an orthonormal set if $\left\langle\boldsymbol{f}_{i}, \boldsymbol{f}_{j}\right\rangle_{L_{2}(\Omega, \mathbb{H})}=\delta_{i, j}$.

Definition 1.2.19. An orthonormal set $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ in $L_{2}(\Omega, \mathbb{H})$ is called complete if it is maximal in the partially ordered set of all orthonormal sets for $L_{2}(\Omega, \mathbb{H})$. This class is ordered by inclusion. A complete orthogonal (orthonormal) set in $L_{2}(\Omega, \mathbb{H})$ is called an orthogonal (orthonormal) basis in $L_{2}(\Omega, \mathbb{H})$.

Definition 1.2.20. A set $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ is called closed in $L_{2}(\Omega, \mathbb{H})$ if for every element $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$, it follows from $\left\langle\boldsymbol{f}, \boldsymbol{f}_{i}\right\rangle_{L_{2}(\Omega, \mathbb{H})}=\mathbf{0}(\forall i \in \Lambda)$ that $\boldsymbol{f}=\mathbf{0}$.

The following theorems were proved in [322], and they are essential to much that follows:

Theorem 1.2.21. Every nonzero Hilbert space over $\mathbb{H}$ contains an orthonormal basis.

Theorem 1.2.22. Let $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ be an orthonormal set in $L_{2}(\Omega, \mathbb{H})$. The following conditions are equivalent:
(i) $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ is complete;
(ii) $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ is closed in $L_{2}(\Omega, \mathbb{H})$;
(iii) if $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$, then $\boldsymbol{f}=\sum\left\langle\boldsymbol{f}, \boldsymbol{f}_{i}\right\rangle_{L_{2}(\Omega, \mathbb{H})} \boldsymbol{f}_{i}$;
(iv) if $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$, then $\|\boldsymbol{f}\|_{L_{2}(\Omega)}^{2}=\sum\left|\left\langle\boldsymbol{f}, \boldsymbol{f}_{i}\right\rangle_{L_{2}(\Omega, \mathbb{H})}\right|^{2}$. (Parseval's identity)

The considerations above adduced apply to classical problems of best approximation in Hilbert spaces over $\mathbb{H}$. In accordance with the condition (iii) of the above theorem and the fact that the $L_{2}$-norm defined by $(1.2 .3)$ is strictly convex, it should be observed that the prescribed best approximation of $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$ by elements of the orthonormal set $\left\{\boldsymbol{f}_{i}\right\}_{i \in \Lambda}$ in $L_{2}(\Omega, \mathbb{H})$ exists and is unique.

To sum up these results, we discuss the Quaternion Riesz Representation Theorem, which is one of the essential structure theorems for Hilbert spaces over $\mathbb{H}$ [176].

Theorem 1.2.23. For every bounded left-linear functional $F$ defined on $L_{2}(\Omega, \mathbb{H})$, there exists a unique $\boldsymbol{g} \in L_{2}(\Omega, \mathbb{H})$ such that $F(\boldsymbol{f})=\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}$ for all $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$.

We are thus led to the definition of RKQHS.
Definition 1.2.24. We say that $L_{2}(\Omega, \mathbb{H})$ is a RKQHS if the (left-)linear evaluation functional $F_{\mathbf{x}}: L_{2}(\Omega, \mathbb{H}) \rightarrow \mathbb{H}$, defined by $F_{\mathbf{x}}(\boldsymbol{f})=\boldsymbol{f}(\mathbf{x})$, is bounded for all $\mathbf{x} \in \Omega$.

### 1.2.3 Spectral Representations

As regards Definition 1.2.15, let $T$ be a bounded left-linear operator on $L_{2}(\Omega, \mathbb{H})$. Let $\boldsymbol{f}, \boldsymbol{g} \in L_{2}(\Omega, \mathbb{H})$ and define the functional $F(\boldsymbol{f}):=\langle T \boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}$. According to the Quaternion Riesz Representation Theorem 1.2.23, there exists a unique $\boldsymbol{h} \in L_{2}(\Omega, \mathbb{H})$, which depends on the $\boldsymbol{g}$ chosen initially, such that $F(\boldsymbol{f})=\langle\boldsymbol{f}, \boldsymbol{h}\rangle_{L_{2}(\Omega, \mathbb{H})}$ for every $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$. To emphasize this, let $\boldsymbol{h}$ be written as $\boldsymbol{h}=T^{*} \boldsymbol{g}$, where $T^{*}$ is called the adjoint of $T$. The operator $T^{*}$ which thus fulfills $\langle T \boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}(\Omega, \mathbb{H})}=\left\langle\boldsymbol{f}, T^{*} \boldsymbol{g}\right\rangle_{L_{2}(\Omega, \mathbb{H})}$, is unique by virtue of the properties of the quaternionic inner product (1.2.1). It can further be shown that $T^{*}$ is a bounded left-linear operator on $L_{2}(\Omega, \mathbb{H})$.

Just as in the case of complex Hilbert spaces, the following theorem may be established [176]:

Theorem 1.2.25. The adjoint operation $T \rightarrow T^{*}$ has the following properties:
(i) $\left(T_{1}+T_{2}\right)^{*}=T_{1}^{*}+T_{2}^{*}$;
(ii) $\left(T_{1} T_{2}\right)^{*}=T_{2}^{*} T_{1}^{*}$;
(iii) $\left(T^{*}\right)^{*}=T$;
(iv) $\left\|T^{*}\right\|=\|T\|$;
(v) $\left\|T^{*} T\right\|=\|T\|^{2}$.

The adjoint operation introduced above may be used to define certain types of operators on (left) quaternionic Hilbert spaces. We proceed with the following definitions:

Definition 1.2.26. Let $T$ be a left-linear operator on $L_{2}(\Omega, \mathbb{H})$. $T$ is called
(i) self-adjoint if $T=T^{*}$;
(ii) an orthogonal projection if it is self-adjoint and $T^{2}=T$;
(iii) positive if $\langle T \boldsymbol{f}, \boldsymbol{f}\rangle_{L_{2}(\Omega, \mathbb{H})}>0$ for all $\boldsymbol{f} \in L_{2}(\Omega, \mathbb{H})$.

Definition 1.2.27. If $T$ is a left-linear operator on $L_{2}(\Omega, \mathbb{H})$ and $\boldsymbol{f} \in$ $L_{2}(\Omega, \mathbb{H})(\boldsymbol{f} \neq \mathbf{0})$ for which $T \boldsymbol{f}=\boldsymbol{\lambda} \boldsymbol{f}$ for some $\boldsymbol{\lambda} \in \mathbb{H}$, then $\boldsymbol{f}$ is called an eigenfunction of $T$ and $\boldsymbol{\lambda}$ the eigenvalue of $T$ corresponding to $\boldsymbol{f}$.

Definition 1.2.28. Let $T$ be a left-linear operator on $L_{2}(\Omega, \mathbb{H})$ and let $\boldsymbol{\lambda}$ be an eigenvalue of $T$. The eigenmanifold, associated with the eigenvalue $\boldsymbol{\lambda}$, is the set of all elements $\boldsymbol{f}$ in $L_{2}(\Omega, \mathbb{H})$ such that $T \boldsymbol{f}=\boldsymbol{\lambda} \boldsymbol{f}$.

The following theorem, which will be of use in further discussion, is a well-known result [322] (cf. [176]).

Theorem 1.2.29. (i) The eigenvalues of a self-adjoint operator are real;
(ii) the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

The Spectral Theorem for quaternionic compact, self-adjoint operators may now be established [176]:

Theorem 1.2.30. Let $T$ be a (nonzero) self-adjoint and compact left-linear operator on $L_{2}(\Omega, \mathbb{H})$. Then there exists a countable infinite set $\left\{\boldsymbol{\psi}_{n}\right\}$ consisting of eigenfunctions of $T$ and a corresponding set of real numbers $\left\{\boldsymbol{\lambda}_{n}\right\}$ consisting of eigenvalues of $T$ such that $\left|\boldsymbol{\lambda}_{0}\right| \geq\left|\boldsymbol{\lambda}_{1}\right| \geq\left|\boldsymbol{\lambda}_{2}\right| \geq \cdots$, which may or not be finite. If the sequence $\left\{\boldsymbol{\lambda}_{n}\right\}$ is infinite, then $\left|\boldsymbol{\lambda}_{n}\right| \rightarrow 0$ as $n \rightarrow \infty$. Each nonzero eigenvalue occurs in the sequence $\left\{\boldsymbol{\lambda}_{n}\right\}$. Further, the eigenmanifold corresponding to a particular $\boldsymbol{\lambda}_{i}$ is finite-dimensional, and its dimension is exactly the number of times this particular eigenvalue is repeated in the set.

## 1.3 $\mathbb{H}$-valued functions on Spatial Domains

### 1.3.1 Monogenic Functions

In this section, we introduce the basic facts of monogenic functions and their associated quaternionic function theory. For detailed information, we refer, e.g., to [145, 146, 148, 318].

The generalized Cauchy-Riemann (or Fueter) operator is defined as

$$
\begin{equation*}
\bar{\partial}=\frac{\partial}{\partial x_{0}}+\mathbf{i} \frac{\partial}{\partial x_{1}}+\mathbf{j} \frac{\partial}{\partial x_{2}} . \tag{1.3.1}
\end{equation*}
$$

In the same manner, we define the conjugate generalized Cauchy-Riemann operator as

$$
\begin{equation*}
\partial=\frac{\partial}{\partial x_{0}}-\mathbf{i} \frac{\partial}{\partial x_{1}}-\mathbf{j} \frac{\partial}{\partial x_{2}} . \tag{1.3.2}
\end{equation*}
$$

Suggested by the complex case, we will focus on particular classes of $\mathbb{H}$-valued functions analogous to complex holomorphic and antiholomorphic functions. The particular classes we are interested in are the following.
Definition 1.3.1. A function $\boldsymbol{f} \in C^{1}(\Omega, \mathbb{H})$ is called
(i) left- (resp. right-) monogenic in $\Omega$ if $\bar{\partial} \boldsymbol{f}=\mathbf{0}$ (resp. $\boldsymbol{f} \bar{\partial}=\mathbf{0}$ ) identically in $\Omega$;
(ii) left- (resp. right-) antimonogenic in $\Omega$ if $\partial \boldsymbol{f}=\mathbf{0}$ (resp. $\boldsymbol{f} \partial=\mathbf{0}$ ) identically in $\Omega$.

The function $\boldsymbol{f}$ is called two-sided monogenic in $\Omega$ if it is both left- and right-monogenic in $\Omega$.

By Definition 1.3.1, an $\mathbb{H}$-valued function $\boldsymbol{f}$ is left-monogenic if it satisfies the following system of differential equations, known as the Moisil-Teodorescu system [232]:

$$
\left\{\begin{array}{l}
\frac{\partial[\boldsymbol{f}]_{0}}{\partial x_{0}}-\frac{\partial[\boldsymbol{f}]_{1}}{\partial x_{1}}-\frac{\partial[\boldsymbol{f}]_{2}}{\partial x_{2}}=0 \\
\frac{\partial[\boldsymbol{f}]_{1}}{\partial x_{0}}+\frac{\partial[\boldsymbol{f}]_{0}}{\partial x_{1}}+\frac{\partial[\boldsymbol{f}]_{3}}{\partial x_{2}}=0 \\
\frac{\partial[\boldsymbol{f}]_{2}}{\partial x_{0}}-\frac{\partial[\boldsymbol{f}]_{3}}{\partial x_{1}}+\frac{\partial[\boldsymbol{f}]_{0}}{\partial x_{2}}=0 \\
\frac{\partial[\boldsymbol{f}]_{3}}{\partial x_{0}}+\frac{\partial[\boldsymbol{f}]_{2}}{\partial x_{1}}-\frac{\partial[\boldsymbol{f}]_{1}}{\partial x_{2}}=0
\end{array}\right.
$$

or, in a more compact form:

$$
(\mathrm{MT})\left\{\begin{array}{cl}
\operatorname{div}\left([\boldsymbol{f}]_{3},[\boldsymbol{f}]_{2},-[\boldsymbol{f}]_{1}\right) & =0  \tag{1.3.3}\\
\nabla[\boldsymbol{f}]_{0}+\operatorname{curl}\left([\boldsymbol{f}]_{3},[\boldsymbol{f}]_{2},-[\boldsymbol{f}]_{1}\right) & =\mathbf{0}
\end{array}\right.
$$

To bring to light the essential distinction between classes of $\mathcal{A}$ - and $\mathbb{H}$-valued functions, as exhibited by the corresponding real vector space $\mathcal{A}$ defined in (1.1.4), we first of all remark that an $\mathcal{A}$-valued function $\boldsymbol{f}$ is left-monogenic if it satisfies the following system:

$$
\left\{\begin{aligned}
\frac{\partial[\boldsymbol{f}]_{0}}{\partial x_{0}}-\frac{\partial[\boldsymbol{f}]_{1}}{\partial x_{1}}-\frac{\partial[\boldsymbol{f}]_{2}}{\partial x_{2}} & =0 \\
\frac{\partial[\boldsymbol{f}]_{0}}{\partial x_{1}}+\frac{\partial[\boldsymbol{f}]_{1}}{\partial x_{0}} & =0 \\
\frac{\partial[\boldsymbol{f}]_{0}}{\partial x_{2}}+\frac{\partial[\boldsymbol{f}]_{2}}{\partial x_{0}} & =0 \\
\frac{\partial[\boldsymbol{f}]_{1}}{\partial x_{2}}-\frac{\partial[\boldsymbol{f}]_{2}}{\partial x_{1}} & =0
\end{aligned}\right.
$$

or, analogously, in a more compact form:

$$
\text { (R) }\left\{\begin{align*}
\operatorname{div}(\overline{\boldsymbol{f}}) & =0  \tag{1.3.4}\\
\operatorname{curl}(\overline{\boldsymbol{f}}) & =\mathbf{0}
\end{align*}\right.
$$

The 3-tuple $\overline{\boldsymbol{f}}$ in $(1.3 .4)$ is said to be a system of conjugate harmonic functions in the sense of Stein-Weiß [311, 312, and (R) is called the Riesz system [127, 283]. As was observed in [26, 27, 94, 95], the importance of the (R)- and (MT)-systems in physical applications has led to substantial generalizations, and some theoretical results can be consulted in [101, 278, 279, 288] and elsewhere.

Hereafter, the word monogenic (resp. antimonogenic) will always mean left-monogenic (resp. left-antimonogenic). All results obtained for leftmonogenic (resp. left-antimonogenic) functions can easily be adapted to right-monogenic (resp. right-antimonogenic) functions.

Definition 1.3.2. We denote by $\mathcal{M}(\Omega)$ the set of monogenic functions in $\Omega$ and by $\overline{\mathcal{M}}(\Omega)$ the set of antimonogenic functions in $\Omega$. Further, we write $\mathcal{M}_{2}(\Omega)=\mathcal{M}(\Omega) \cap L_{2}(\Omega, \mathbb{H})$ and $\overline{\mathcal{M}}_{2}(\Omega)=\overline{\mathcal{M}}(\Omega) \cap L_{2}(\Omega, \mathbb{H})$.

The notation in this definition does not indicate whether $\mathcal{M}(\Omega)$ (resp. $\overline{\mathcal{M}}(\Omega))$ is a space consisting of $\mathbb{H}$ - or $\mathcal{A}$-valued functions. When it is necessary to make clear this difference, we will write $\mathcal{M}(\Omega, \mathcal{B})($ resp. $\overline{\mathcal{M}}(\Omega, \mathcal{B})$ ), where $\mathcal{B}=\mathcal{A}$ or $\mathbb{H}$. We will use the same abbreviated form of notation for the spaces $\mathcal{M}_{2}(\Omega)$ and $\overline{\mathcal{M}}_{2}(\Omega)$.

As discussed in [236], it turns out that

$$
-\mathbf{k}(\bar{\partial} \boldsymbol{f}) \mathbf{k}=\partial \overline{\boldsymbol{f}}
$$

for every $\mathcal{A}$-valued function $\boldsymbol{f}$; and thus, it follows that an $\mathcal{A}$-valued function $\boldsymbol{f}$ is monogenic, if and only if $\overline{\boldsymbol{f}}$ is antimonogenic. This observation is analogous to the complex case and differs from the general situation of $\mathbb{H}$-valued monogenic functions.

As a matter of fact, it is proved that
Proposition 1.3.3. An $\mathcal{A}$-valued function is left-monogenic, if and only if it is right-monogenic. Further, the set of conjugates of $\mathcal{A}$-valued monogenic functions in $\Omega$ coincides with the set $\overline{\mathcal{M}}(\Omega)$.

A basic example of a two-sided $\mathcal{A}$-valued monogenic function is the socalled Cauchy-Fueter kernel, which is the fundamental solution of the operator (1.3.1):

Definition 1.3.4. The Cauchy-Fueter kernel is

$$
\begin{equation*}
\boldsymbol{q}(\mathrm{x})=\frac{1}{4 \pi} \frac{\overline{\mathbf{x}}}{|\mathbf{x}|^{3}} \tag{1.3.5}
\end{equation*}
$$

defined in $\mathbb{R}^{3} \backslash\{\mathbf{0}\}$.

As regards the above definition, the following analog of Cauchy's integral formula may be established [146, p.87-88]:
Theorem 1.3.5. Suppose $\boldsymbol{f}: \Omega \subset \mathbb{R}^{3} \rightarrow \mathbb{H}$ is monogenic in a neighborhood of the closure $\bar{\Omega}$ of a bounded, open set $\Omega$. Then, for each $\mathbf{x} \in \Omega$, we have that

$$
\begin{equation*}
\boldsymbol{f}(\mathrm{x})=\frac{1}{4 \pi} \int_{\partial \Omega} \boldsymbol{q}(\mathrm{x}-\mathbf{y}) \mathbf{n}(\mathbf{y}) \boldsymbol{f}(\mathrm{y}) d \sigma(\mathbf{y}) \tag{1.3.6}
\end{equation*}
$$

where $\mathbf{n}(\mathbf{y})$ is the outward pointing normal vector to $\Omega$ at $\mathbf{y}$, and $d \sigma$ is the Lebesgue measure on the surface $\partial \Omega$.

Moreover, by a simple calculation, one can verify that the operators (1.3.1) and (1.3.2) factor the Laplace operator in $\mathbb{R}^{3}$, in a sense, that

$$
\begin{equation*}
\Delta_{3}=\bar{\partial} \partial=\partial \bar{\partial} \tag{1.3.7}
\end{equation*}
$$

Accordingly, if $\boldsymbol{f}$ is an $\mathbb{H}$-valued function defined on $\Omega$, twice differentiable, monogenic, or antimonogenic, then $\boldsymbol{f}$ is harmonic in $\Omega$, and so are all its quaternionic components. The converse is not valid. The factorization (1.3.7) establishes a special relationship between quaternionic analysis and harmonic analysis in that monogenic and antimonogenic functions refine the properties of harmonic functions.

The notion of monogenicity provides a powerful generalization of complex analyticity to quaternionic analysis since many classical theorems of complex analysis can be generalized to higher dimensions following this approach. We refer, for instance, to [126, 127, 128, 129, 130, 131, 145, 146, 148, 153, 318] and elsewhere.

Before we proceed, we need to introduce some further notation.
Definition 1.3.6. We denote by $\mathcal{P}_{l}^{+}\left(\mathbb{R}^{3}\right)$ (resp. $\mathcal{P}_{l}^{-}\left(\mathbb{R}^{3}\right)$ ) the space of homogeneous polynomials of degree l in $\mathbb{R}^{3}$ (resp. homogeneous functions of degree $-(l+1)$ in $\left.\mathbb{R}^{3} \backslash\{\mathbf{0}\}\right)$ with real coefficients. The subspace of $\mathcal{P}_{l}^{+}\left(\mathbb{R}^{3}\right)$ (resp. $\left.\mathcal{P}_{l}^{-}\left(\mathbb{R}^{3}\right)\right)$ of those polynomials (resp. functions) that are harmonic is denoted by $\operatorname{Har}_{l}^{+}\left(\mathbb{R}^{3}\right)$ (resp. $\operatorname{Har}_{l}^{-}\left(\mathbb{R}^{3}\right)$ ). The set of $\mathbb{H}$-valued harmonic functions in $\Omega$ is denoted by $\operatorname{Har}(\Omega)$. Further,

$$
\begin{aligned}
& \operatorname{Har}_{2}(\Omega)=\operatorname{Har}(\Omega) \cap L_{2}(\Omega, \mathbb{H}), \\
& \mathcal{M}_{l}^{+}(\Omega)=\mathcal{M}(\Omega) \cap \mathcal{P}_{l}^{+}\left(\mathbb{R}^{3}\right), \\
& \mathcal{M}_{l}^{-}(\Omega)=\mathcal{M}\left(\mathbb{R}^{3} \backslash \bar{\Omega}\right) \cap \mathcal{P}_{l+2}^{-}\left(\mathbb{R}^{3}\right) .
\end{aligned}
$$

Definition 1.3.7. Let $0<p<\infty$. The $p$-integral mean of an $\mathbb{H}$-valued function $\boldsymbol{f}(\mathbf{x})=\boldsymbol{f}(\rho \boldsymbol{\zeta})$ in $\Omega_{0}(0 \leq \rho<1)$ is defined by

$$
\begin{equation*}
M_{p}(\boldsymbol{f} ; \rho)=\left(\int_{\partial \Omega_{0}}|\boldsymbol{f}(\rho \boldsymbol{\zeta})|^{p} d \sigma(\boldsymbol{\zeta})\right)^{1 / p} \tag{1.3.8}
\end{equation*}
$$

Definition 1.3.8. Let $1 \leq p<\infty$ and $\beta>0$. We define
(i) the harmonic weighted Hardy spaces as

$$
\begin{equation*}
H_{p, \beta}\left(\Omega_{0}\right)=\left\{h \in \operatorname{Har}\left(\Omega_{0}\right):\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)}=\sup _{0<p<1}(1-\rho)^{\beta} M_{p}(h ; \rho)<\infty\right\} ; \tag{1.3.9}
\end{equation*}
$$

(i) the monogenic weighted Hardy spaces as

$$
\begin{equation*}
\mathcal{H}_{p, \beta}\left(\Omega_{0}\right)=\left\{\boldsymbol{f} \in \mathcal{M}\left(\Omega_{0}\right):\|\boldsymbol{f}\|_{\mathcal{H}_{p, \beta}\left(\Omega_{0}\right)}=\sup _{0<\rho<1}(1-\rho)^{\beta} M_{p}(\boldsymbol{f} ; \rho)<\infty\right\} \tag{1.3.10}
\end{equation*}
$$

When $\beta=0$, (1.3.9) and (1.3.10) are the usual Hardy spaces of harmonic and monogenic functions denoted, respectively, by $H_{p}\left(\Omega_{0}\right)$ and $\mathcal{H}_{p}\left(\Omega_{0}\right)$.

Definition 1.3.9. Let $1<p<\infty$ and $\alpha>-1$. The weighted Bergman space of $\boldsymbol{f}$ on $\Omega_{0}$ is defined by

$$
\begin{aligned}
& L_{p, \alpha}\left(\Omega_{0}, \mathbb{H}\right)=\left\{\boldsymbol{f}: \Omega_{0} \rightarrow \mathbb{H}\right. \text { measurable : } \\
&\left.\|\boldsymbol{f}\|_{L_{p, \alpha}\left(\Omega_{0}\right)}=\left(\int_{\Omega_{0}}(1-|\mathbf{x}|)^{\alpha}|\boldsymbol{f}(\mathbf{x})|^{p} d \mathbf{x}\right)^{1 / p}<\infty\right\}
\end{aligned}
$$

Further, we denote the subspaces of $L_{p, \alpha}\left(\Omega_{0}, \mathbb{H}\right)$ consisting of harmonic and monogenic functions, respectively, by $B_{p, \alpha}\left(\Omega_{0}\right)$ and $\mathcal{B}_{p, \alpha}\left(\Omega_{0}\right)$.

### 1.3.2 Hypercomplex Derivatives and Primitives of Monogenic Functions

In this section, we study the existence of the quaternionic derivative and primitive of a monogenic function. Developments of the theory that go beyond those of which an account is given in the present subsection will be found in [147, 318].

At the end of the 19th century, some attempts were made to extend the concept of complex differentiability to $\mathbb{H}$-valued functions [293]. Due to the lack of commutativity of quaternions, two approaches were possible:
Definition 1.3.10. A function $\boldsymbol{f}: \Omega \subset \mathbb{R}^{3} \rightarrow \mathbb{H}$ is said to be quaternionic differentiable on the left (resp. right) at a point $\mathbf{q} \in \Omega$, if the limit

$$
\lim _{\mathbf{h} \rightarrow \mathbf{0}} \mathbf{h}^{-1}[\boldsymbol{f}(\mathbf{q}+\mathbf{h})-\boldsymbol{f}(\mathbf{q})] \quad\left(\text { resp. } \lim _{\mathbf{h} \rightarrow \mathbf{0}}[\boldsymbol{f}(\mathbf{q}+\mathbf{h})-\boldsymbol{f}(\mathbf{q})] \mathbf{h}^{-1}\right)
$$

exists, when $\mathbf{h}$ converges to zero along any direction in the quaternionic space.

However, and as was observed in [293], the only $\mathbb{H}$-valued functions that are quaternionic differentiable on the left (resp. right) in an open domain have the form $\boldsymbol{f}(\mathbf{q})=\mathbf{a}+\mathbf{q} \mathbf{b}(\operatorname{resp} . \boldsymbol{f}(\mathbf{q})=\mathbf{a}+\mathbf{b q})$, for some $\mathbf{a}, \mathbf{b} \in \mathbb{H}$. Since this builds a very restrictive class of functions, these approaches were abandoned. These statements were proved by Krylov in [191] and adopted by his student Melijhzon in [228]. Malonek developed characterization of the class of monogenic functions in terms of hypercomplex derivability in the sense of linear approximability in [214] (cf. also [93] and [148, Ch. 5]). A first attempt at characterizing a regular (monogenic) function $f: \mathbb{H} \rightarrow \mathbb{H}$ employing the existence of its quaternionic derivative was published by Sudbery in [318], in which the subject is treated from his point of view. In this framework, Mitelman and Shapiro in [230] considered the operator $(1 / 2) \partial$ a hypercomplex derivative by studying the generalized Martinelli-Bochner integral formulas for $\mathbb{H}$-valued functions and the directional derivability of the Cauchy-Fueter kernel. Following the idea of Sudbery's quaternion results, Gürlebeck and Malonek showed that monogenicity and hypercomplex derivability are equivalent in higher dimensions and that the hypercomplex linearization of a monogenic function $\boldsymbol{f}$ is precisely given by $(1 / 2) \partial \boldsymbol{f}$ [147]. In [216], Malonek provided details about the integral representation of the hypercomplex derivative and its corresponding mapping properties.

Definition 1.3.11. Let $\boldsymbol{f} \in C^{1}(\Omega, \mathbb{H})$. ( $\left.1 / 2\right) \partial \boldsymbol{f}$ is called the hypercomplex derivative of $\boldsymbol{f}$ in $\Omega$.

It is readily seen that the hypercomplex derivative of a monogenic function is again monogenic, namely

$$
\bar{\partial}\left(\frac{1}{2} \partial \boldsymbol{f}\right)=\frac{1}{2} \Delta_{3} \boldsymbol{f}=\mathbf{0} .
$$

Definition 1.3.12. We denote by $\mathcal{M}_{2,1}(\Omega)$ the Sobolev-type space of all functions from $\mathcal{M}_{2}(\Omega)$ whose hypercomplex derivatives also belong to $\mathcal{M}_{2}(\Omega)$.

Definition 1.3.13. A monogenic constant is a function of $C^{1}(\Omega, \mathbb{H})$, which is monogenic in $\Omega$ and has an identically vanishing hypercomplex derivative in $\Omega$.

As was observed in [63], monogenic constants do not depend on the variable $x_{0}$. They can be expressed as $\boldsymbol{f}=a_{0}+\mathbf{i}[\boldsymbol{f}]_{1}+\mathbf{j}[\boldsymbol{f}]_{2}$, where $a_{0} \in \mathbb{R}$ is a constant, and $[\boldsymbol{f}]_{1}-i[\boldsymbol{f}]_{2}$ is an ordinary holomorphic function of the complex variable $x_{1}+i x_{2}$.

We shall now consider the definition of monogenic primitive (or monogenic antiderivative) of a monogenic function. In the complex plane, we can
describe primitives easily by line integrals. It does not remain valid in higher dimensions as line integrals are generally path-dependent. In [318], it was proved that an $\mathbb{H}$-valued monogenic polynomial has a monogenic primitive, which is again a monogenic polynomial. Later in [48, monogenic primitives were discussed in the case of domains that are normal with respect to the $x_{0}$-direction. In [147], Gürlebeck et al. showed that the Fueter-regular polynomials have monogenic polynomial primitives, and for the first time, existence results for monogenic primitives were obtained. A different effort was made in [66] (cf. [63]), where an antiderivative operator was defined as the right-inverse of the hypercomplex derivative.

The following definition will be required [66]:
Definition 1.3.14. A function $\boldsymbol{F} \in C^{1}(\Omega, \mathbb{H})$ is called a monogenic primitive of a function $\boldsymbol{f} \in \mathcal{M}(\Omega)$, with respect to the hypercomplex derivative, if $\boldsymbol{F} \in \mathcal{M}(\Omega)$ and

$$
\begin{equation*}
\left(\frac{1}{2} \partial\right) \boldsymbol{F}=\boldsymbol{f} \tag{1.3.11}
\end{equation*}
$$

For a given $\boldsymbol{f} \in \mathcal{M}(\Omega)$, if such function $\boldsymbol{F}$ exists, we denote it by $\mathcal{P}(\boldsymbol{f})$.
Analogously to the complex case, Definition 1.3 .14 concerns the primitive of a given function $\boldsymbol{f}$ if it results from the application of the operator $\mathcal{P}$ to $\boldsymbol{f}$, without adding any monogenic constant. "Omitting" the constants means that we are looking for the unique primitive orthogonal to the monogenic constants [240, 241].

Note that the definition above can be extended to define primitives of a given function $\boldsymbol{f}$ as those functions $\boldsymbol{F}$ (not necessarily monogenic) that satisfy (1.3.11). As is well-known, the Teodorescu transform $T$ can also be interpreted as a primitive of a given square-integrable monogenic function. Bearing in mind that $\partial \bar{T}=I$, it follows that $\bar{T} \boldsymbol{f}$ is only an algebraic primitive of $\boldsymbol{f}$ for $\partial$, and this in accordance with Definition 1.3.11. Consequently, this algebraic primitive is harmonic but not monogenic for a given monogenic function [310]. Accordingly, $\bar{T}$ cannot be used as an analog of the complex integration of a holomorphic function.

### 1.4 The Associated Legendre Functions of the First and Second Kinds

The majority of functions used in technical and applied mathematics originated as the result of investigating practical problems. A relevant example is the class of functions known as the associated Legendre functions (also known
as the Ferrer's associated functions). The functions in question are solutions of the well-known Legendre's associated differential equation:

$$
\begin{equation*}
\left(1-t^{2}\right) y^{\prime \prime}(t)-2 t y^{\prime}(t)+\left(l(l+1)-\frac{m^{2}}{1-t^{2}}\right) y(t)=0 \tag{1.4.1}
\end{equation*}
$$

where $l$ and $m$ are nonnegative integers and $t \in \mathbb{R}$.
When $m=0$, 1.4.1 becomes the classical Legendre's differential equation:

$$
\begin{equation*}
\left(1-t^{2}\right) y^{\prime \prime}(t)-2 t y^{\prime}(t)+l(l+1) y(t)=0 . \tag{1.4.2}
\end{equation*}
$$

The fundamental system of solutions of (1.4.2) is given by two kinds of functions that we denote by $P_{l}(t)$ and $Q_{l}(t)$. The $P_{l}(t)$ is commonly referred to as the Legendre polynomial and the $Q_{l}(t)$ as the Legendre function of the second kind. For the reader's convenience and the sake of easy reference, we will follow mainly the notations introduced in [170].

The prescribed functions are defined by

$$
\left\{\begin{array}{l}
P_{l}(t)=\frac{1}{2^{l} l!} \frac{d^{l}}{d t^{l}}\left(t^{2}-1\right)^{l}, \quad t \in \mathbb{R}  \tag{1.4.3}\\
Q_{l}(t)=\frac{1}{2} P_{l}(t) \log \frac{t+1}{t-1}-\sum_{k=0}^{l-1} \frac{P_{k}(t) P_{l-k-1}(t)}{l-k}, \quad|t|>1
\end{array}\right.
$$

There are classical formulas that express the products of two Legendre polynomials in terms of a Legendre polynomial and two Legendre functions of the second kind in terms of a Legendre function of the second kind [30].

Proposition 1.4.1. Let $l_{1}, l_{2}$ be nonnegative integers such that $l_{1} \geq l_{2}$ and $|t|>1$. Then

$$
\begin{align*}
P_{l_{1}}(t) P_{l_{2}}(t)= & \sum_{r=0}^{l_{2}} \frac{A_{r} A_{l_{1}-r} A_{l_{2}-r}}{A_{l_{1}+l_{2}-r}}\left(\frac{2 l_{1}+2 l_{2}-4 r+1}{2 l_{1}+2 l_{2}-2 r+1}\right) P_{l_{1}+l_{2}-2 r}(t)  \tag{1.4.4}\\
Q_{l_{1}}(t) Q_{l_{2}}(t)= & \sum_{r=0}^{\infty} \frac{A_{r} A_{l_{1}+l_{2}+r+2}}{A_{l_{1}+r+1} A_{l_{2}+r+1}} \\
& \times\left(\frac{\left(l_{1}+l_{2}+r+2\right)\left(2 l_{1}+2 l_{2}+4 r+3\right)}{\left(l_{1}+r+1\right)\left(l_{2}+r+1\right)\left(2 l_{1}+2 l_{2}+2 r+3\right)}\right) Q_{l_{1}+l_{2}+2 r+1}(t) \tag{1.4.5}
\end{align*}
$$

where $A_{r}=(2 r-1)!!/ r!$.
We shall now formulate an essential relation between the above functions for different values of $l$, known as Neumann's formula [255, p. 24] (cf. [170, p. 63]).

Theorem 1.4.2. Let $l$ be a nonnegative integer and $|t|>1$. Then

$$
\begin{equation*}
Q_{l}(t)=\frac{1}{2} \int_{-1}^{1} \frac{P_{l}(u)}{t-u} d u \tag{1.4.6}
\end{equation*}
$$

The following is a fundamental technical lemma that generalizes the identity in [261, formula 18.17.19].

Lemma 1.4.3. Let $\mathbf{u}$ be any pure quaternion such that $\mathbf{u}^{2}=-1$. For each $l \geq 0$ and small $|s| \ll 1$,

$$
\frac{\mathbf{u}^{l}}{2} \int_{-1}^{1} P_{l}(t) \exp (-\mathbf{u} s t) d t=\sqrt{\frac{\pi}{2 s}} J_{l+1 / 2}(s)
$$

where $J_{l}(s)$ denotes the Bessel function of the first kind.
Proof. Using the orthogonality $(\sqrt{1.4 .24})$ of the Legendre polynomials and the well-known closed-form representation

$$
t^{l}=\sum_{k=l, l-2, \ldots} \frac{(2 k+1) l!}{2^{(l-k) / 2}((l-k) / 2)!(l+1+k)!!} P_{k}(t),
$$

it follows that

$$
\begin{aligned}
& \frac{\mathbf{u}^{l}}{2} \int_{-1}^{1} P_{l}(t) \exp (-\mathbf{u} s t) d t \\
& =\frac{\mathbf{u}^{l}}{2} \int_{-1}^{1} P_{l}(t)\left[\cdots+\frac{(-\mathbf{u} s t)^{l}}{l!}+\frac{(-\mathbf{u} s t)^{l+1}}{(l+1)!}+\cdots\right] d t \\
& =\frac{s^{l}}{(2 l+1)!!}+O\left(s^{l+1}\right)
\end{aligned}
$$

where the series in square brackets converges for any unit pure quaternion $\mathbf{u}$ since we have $|(-\mathbf{u} s t)|^{l} \leq|s|^{l}$. The right-hand side of the previous equality behaves like $[\pi /(2 s)]^{1 / 2} J_{l+1 / 2}(s)$ for the limiting case of small $s$.

The solutions of (1.4.1) are the associated Legendre functions of the first and second kinds denoted, respectively, by $P_{l}^{m}(t)$ and $Q_{l}^{m}(t)$. The indices $l$ and $m$ are referred to as the degree and order of the associated Legendre functions. For nonnegative integer values of $m$, these functions are related to the Legendre functions for nonnegative integer $l$ by

$$
P_{l}^{m}(t)=\left\{\begin{array}{l}
(-1)^{m}\left(1-t^{2}\right)^{m / 2} \frac{d^{m} P_{l}(t)}{d t^{m}}, t \in[-1,1] \\
\left(t^{2}-1\right)^{m / 2} \frac{d^{m} P_{l}(t)}{d t^{m}},|t|>1,
\end{array}\right.
$$

and

$$
Q_{l}^{m}(t)=\left(t^{2}-1\right)^{m / 2} \frac{d^{m} Q_{l}(t)}{d t^{m}},|t|>1
$$

By the above definitions, we shall observe a slight difference between the variation of the indexes $l$ and $m$ in these functions. Although the $P_{l}^{m}(t)$ are only defined for nonnegative integer values of $m$, which are less than or equal to $l$, the functions $Q_{l}^{m}(t)$ are defined for all nonnegative integer values of $m$.

The associated Legendre functions are defined for negative integers $m$ by [170. Ch. III]

$$
P_{l}^{-m}(t)=\left\{\begin{array}{l}
(-1)^{m} \frac{(l-m)!}{(l+m)!} P_{l}^{m}(t), \quad t \in[-1,1]  \tag{1.4.7}\\
\frac{(l-m)!}{(l+m)!} P_{l}^{m}(t), \quad|t|>1,
\end{array}\right.
$$

and

$$
\begin{equation*}
Q_{l}^{-m}(t)=\frac{(l-m)!}{(l+m)!} Q_{l}^{m}(t), \quad|t|>1 \tag{1.4.8}
\end{equation*}
$$

We proceed to consider explicit expressions for the associated Legendre functions [284, p. 589].

Lemma 1.4.4. Let $l \geq 0$ and $0 \leq m \leq l$. Then

$$
P_{l}^{m}(t)=\left\{\begin{array}{c}
(-1)^{m}\left(1-t^{2}\right)^{m / 2} \sum_{s=m}^{l} \lambda_{s}^{l, m}(t-1)^{l-s}(t+1)^{s-m}  \tag{1.4.9}\\
\text { if } t \in[-1,1], \\
\left(t^{2}-1\right)^{m / 2} \sum_{s=m}^{l} \lambda_{s}^{l m}(t-1)^{l-s}(t+1)^{s-m} \\
\text { if }|t|>1,
\end{array}\right.
$$

where

$$
\begin{equation*}
\lambda_{s}^{l, m}=\frac{l!(l+m)!}{2^{l}(l+m-s)!(l-s)!(s-m)!s!} \tag{1.4.10}
\end{equation*}
$$

We further have from [271, p. 965]:
Lemma 1.4.5. Let $l$ and $m$ be nonnegative integer values and let $|t|>1$. Then

$$
\begin{equation*}
Q_{l}^{m}(t)=\frac{e^{i \pi m}(l+m)!\left(t^{2}-1\right)^{m / 2}}{(2 l+1)!!t^{l+m+1}}{ }_{2} F_{1}\left(\frac{l+m+2}{2}, \frac{l+m+1}{2} ; \frac{2 l+3}{2} ; \frac{1}{t^{2}}\right), \tag{1.4.11}
\end{equation*}
$$

where ${ }_{2} F_{1}$ denotes the classical Gaussian hypergeometric function.

We may now take the opportunity to consider the recurrence formulas for the associated Legendre functions, which will be used in the forthcoming chapters [170, 319].
Proposition 1.4.6. Let $l \geq 0,0 \leq m \leq l$ and let $t \in \mathbb{R}$. Then

$$
\begin{align*}
& \left(1-t^{2}\right)\left(P_{l+1}^{m}\right)^{\prime}(t)=(l+1+m) P_{l}^{m}(t)-(l+1) t P_{l+1}^{m}(t),  \tag{1.4.12}\\
& (l+1-m) P_{l+1}^{m}(t)=(2 l+1) t P_{l}^{m}(t)-(l+m) P_{l-1}^{m}(t) . \tag{1.4.13}
\end{align*}
$$

Proposition 1.4.7. Let $l \geq 0,0 \leq m \leq l$ and let $t \in[-1,1]$. Then

$$
\begin{align*}
& \left(t^{2}-1\right)\left(P_{l+1}^{m}\right)^{\prime}(t)=\left(1-t^{2}\right)^{1 / 2} P_{l+1}^{m+1}(t)+m t P_{l+1}^{m}(t),  \tag{1.4.14}\\
& \left(1-t^{2}\right)^{1 / 2} P_{l+1}^{m}(t)=\frac{1}{2 l+3}\left[-P_{l+2}^{m+1}(t)+P_{l}^{m+1}(t)\right],  \tag{1.4.15}\\
& 2 m t P_{l+1}^{m}(t) \\
& =-\left(1-t^{2}\right)^{1 / 2}\left[P_{l+1}^{m+1}(t)+(l+1+m)(l+2-m) P_{l+1}^{m-1}(t)\right],  \tag{1.4.16}\\
& \left(1-t^{2}\right)^{1 / 2} P_{l}^{m+1}(t)=(l-m) t P_{l}^{m}(t)-(l+m) P_{l-1}^{m}(t) . \tag{1.4.17}
\end{align*}
$$

Proposition 1.4.8. Let $l \geq 0,0 \leq m \leq l$ and let $|t|>1$. Then

$$
\begin{align*}
& \left(t^{2}-1\right)\left(P_{l+1}^{m}\right)^{\prime}(t)=\left(t^{2}-1\right)^{1 / 2} P_{l+1}^{m+1}(t)+m t P_{l+1}^{m}(t)  \tag{1.4.18}\\
& \left(t^{2}-1\right)^{1 / 2} P_{l+1}^{m}(t)=\frac{1}{2 l+3}\left[P_{l+2}^{m+1}(t)-P_{l}^{m+1}(t)\right]  \tag{1.4.19}\\
& 2 m t P_{l+1}^{m}(t) \\
& =\left(t^{2}-1\right)^{1 / 2}\left[-P_{l+1}^{m+1}(t)+(l+1+m)(l+2-m) P_{l+1}^{m-1}(t)\right] . \tag{1.4.20}
\end{align*}
$$

Proposition 1.4.9. Let $l$ and $m$ be nonnegative integer values and $|t|>1$. Then

$$
\begin{align*}
& \left(1-t^{2}\right)\left(Q_{l+1}^{m}\right)^{\prime}(t)=(l+1+m) Q_{l}^{m}(t)-(l+1) t Q_{l+1}^{m}(t),  \tag{1.4.21}\\
& (l+1-m) Q_{l+1}^{m}(t)=(2 l+1) t Q_{l}^{m}(t)-(l+m) Q_{l-1}^{m}(t),  \tag{1.4.22}\\
& \left(t^{2}-1\right)^{1 / 2} Q_{l}^{m+1}(t)=(l-m) t Q_{l}^{m}(t)-(l+m) Q_{l-1}^{m}(t), \tag{1.4.23}
\end{align*}
$$

with the initial values

$$
Q_{0}^{0}(t)=\frac{1}{2} \log \frac{t+1}{t-1}, \quad Q_{0}^{1}(t)=-\frac{1}{\sqrt{t^{2}-1}}, \quad Q_{0}^{2}(t)=\frac{t}{2} \log \frac{t+1}{t-1}-1 .
$$

Another essential property of the associated Legendre functions of the first kind is their orthogonality in $L_{2}([-1,1])$,

$$
\begin{equation*}
\int_{-1}^{1} P_{l_{1}}^{m_{1}}(t) P_{l_{2}}^{m_{2}}(t) d t=\frac{2\left(l_{1}+m_{1}\right)!}{\left(2 l_{1}+1\right)\left(l_{1}-m_{1}\right)!} \delta_{l_{1}, l_{2}} \delta_{m_{1}, m_{2}} \tag{1.4.24}
\end{equation*}
$$

For a detailed historical survey and an extended list of references on associated Legendre functions, we refer to [19, 34, 108, 170, 205, 254, 271, 275, [292, 332] and elsewhere.

### 1.5 The Prolate Spheroidal Wave Functions

### 1.5.1 The Helmholtz Equation in Spheroidal Coordinates

The PSWFs are solutions to the ordinary differential equations obtained from solving the scalar Helmholtz equation in prolate spheroidal coordinates by the usual procedure of separation of variables. Niven initially introduced the PSWFs in [259]. They were subsequently investigated by several authors such as Strutt [317], Stratton, Morse, Chu, Hutner, Little, and Corgbato [315, 316], Bouwkamp [44], Morse, and Feshbach [252], Meixner, Schäfke, and Wolf [226, 227], Flammer [120], Arscott [23], Hunter [171, 172], Hanish, Baier, Van Buren, and King [156, 157, 158, Abramowitz, and Stegun [6], Komarov, Ponomarev, and Slavyanov [183], Zhang, and Jin [346], Thompson [324], Olver, Lozier, Boisvert, and Clark [261] and others. In literature, the PSWFs are often regarded as "mysterious" functions of $L_{2}(\mathbb{R})$, showing no explicit or standard representation in terms of elementary functions and too challenging to compute numerically.

In three-dimensional Cartesian coordinates, the scalar Helmholtz equation (also called the reduced wave equation) has the form

$$
\begin{equation*}
\left(\Delta_{3}+k^{2}\right) u=\frac{\partial^{2} u}{\partial x_{0}^{2}}+\frac{\partial^{2} u}{\partial x_{1}^{2}}+\frac{\partial^{2} u}{\partial x_{2}^{2}}+k^{2} u=0, \quad k \in \mathbb{R}^{+} . \tag{1.5.1}
\end{equation*}
$$

The Helmholtz operator $\left(\Delta_{3}+k^{2}\right)$ acts on the space $\mathcal{C}^{2}\left(\mathbb{R}^{3}\right)$. For convenience, we consider this operator acting on $\mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right)$, where

$$
\Omega_{x_{0}, x_{1}, x_{2}}=\mathbb{R}^{3} \backslash\left\{\left(x_{0}, x_{1}, x_{2}\right) \in \mathbb{R}^{3}: x_{0} \in \mathbb{R}, x_{1}=x_{2}=0\right\} .
$$

In the sequel, consider a prolate spheroidal domain $\Omega_{\mu}$ in a copy of $\mathbb{R}^{3}$ with the prolate spheroidal coordinates $(\xi, t, \varphi)$ defined as follows [6, p. 752]:

$$
\begin{equation*}
x_{0}=\mu \xi t, x_{1}=\mu \sqrt{\left(\xi^{2}-1\right)\left(1-t^{2}\right)} \cos \varphi, x_{2}=\mu \sqrt{\left(\xi^{2}-1\right)\left(1-t^{2}\right)} \sin \varphi, \tag{1.5.2}
\end{equation*}
$$

where $\xi, t$ and $\varphi$ are parameters such that $1 \leq \xi \leq 1 / \mu,-1 \leq t \leq 1$ and $\varphi \in[0,2 \pi)$, and $\mu \in(0,1)$ is the eccentricity of the spheroid. Further details about spheroidal coordinates will be provided in Section 2.1.

Assume there exists a mapping

$$
\boldsymbol{\Phi}(\xi, t, \varphi):=\left(x_{0}=[\boldsymbol{\Phi}(\xi, t, \varphi)]_{1}, x_{1}=[\boldsymbol{\Phi}(\xi, t, \varphi)]_{2}, x_{2}=[\boldsymbol{\Phi}(\xi, t, \varphi)]_{3}\right),
$$

such that $\boldsymbol{\Phi} \in \mathcal{C}^{2}\left(\Omega_{\mu}\right)$ makes a one-to-one correspondence between the domains $\Omega_{\mu}$ and $\Omega_{x_{0}, x_{1}, x_{2}}$. Assume that $\Psi: \Omega_{x_{0}, x_{1}, x_{2}} \rightarrow \Omega_{\mu}$ is the inverse
map, i.e., it is such that $\boldsymbol{\Phi}\left(\boldsymbol{\Psi}\left(x_{0}, x_{1}, x_{2}\right)\right)=\left(x_{0}, x_{1}, x_{2}\right)$ for any $\left(x_{0}, x_{1}, x_{2}\right) \in$ $\Omega_{x_{0}, x_{1}, x_{2}}$, and $\boldsymbol{\Psi}(\boldsymbol{\Phi}(\xi, t, \varphi))=(\xi, t, \varphi)$ for any $(\xi, t, \varphi) \in \Omega_{\mu}$.

We proceed to introduce the linear operators under the change of variables (1.5.2):

$$
\begin{aligned}
& W_{\boldsymbol{\Phi}}: u \in \mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right) \mapsto u \circ \boldsymbol{\Phi}=: \widetilde{u} \in \mathcal{C}^{2}\left(\Omega_{\mu}\right), \\
& W_{\Psi}=W_{\boldsymbol{\Phi}}^{-1}: \widetilde{u} \in \mathcal{C}^{2}\left(\Omega_{\mu}\right) \mapsto \widetilde{u} \circ \boldsymbol{\Psi}=: u \in \mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right)
\end{aligned}
$$

It follows that $W_{\boldsymbol{\Phi}}$ is an isomorphism of $\mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right)$ onto $\mathcal{C}^{2}\left(\Omega_{\mu}\right)$, whereas $W_{\boldsymbol{\Psi}}$ is an isomorphism of $\mathcal{C}^{2}\left(\Omega_{\mu}\right)$ onto $\mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right)$.

Let $A$ be an arbitrary linear operator acting on $\mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right)$ and let $B$ be an arbitrary operator acting on $\mathcal{C}^{2}\left(\Omega_{\mu}\right)$. Define the operators $\widetilde{A}$ and $\widetilde{B}$ as $W_{\Phi} A W_{\Psi}=: \widetilde{A}$ and $W_{\Psi} B W_{\Phi}=: \widetilde{B}$. Obviously, $\widetilde{A}$ acts on $\mathcal{C}^{2}\left(\Omega_{\mu}\right)$ while $\widetilde{B}$ acts on $\mathcal{C}^{2}\left(\Omega_{x_{0}, x_{1}, x_{2}}\right)$.

Now, we take $A=\Delta_{3}+k^{2}$. Whence, $W_{\Phi} A W_{\Psi}=W_{\Phi} \Delta_{3} W_{\Psi}+W_{\Phi} k^{2} W_{\Psi}$, where

$$
\begin{aligned}
W_{\boldsymbol{\Phi}} \Delta_{3} W_{\boldsymbol{\Psi}}= & W_{\boldsymbol{\Phi}}\left(\frac{\partial^{2}}{\partial x_{0}^{2}}+\frac{\partial^{2}}{\partial x_{1}^{2}}+\frac{\partial^{2}}{\partial x_{2}^{2}}\right) W_{\boldsymbol{\Psi}} \\
= & W_{\boldsymbol{\Phi}} \frac{\partial^{2}}{\partial x_{0}^{2}} W_{\boldsymbol{\Psi}}+W_{\boldsymbol{\Phi}} \frac{\partial^{2}}{\partial x_{1}^{2}} W_{\boldsymbol{\Psi}}+W_{\boldsymbol{\Phi}} \frac{\partial^{2}}{\partial x_{2}^{2}} W_{\boldsymbol{\Psi}} \\
= & \left(W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{0}} W_{\boldsymbol{\Psi}}\right)\left(W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{0}} W_{\boldsymbol{\Psi}}\right)+\left(W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{1}} W_{\boldsymbol{\Psi}}\right)\left(W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{1}} W_{\boldsymbol{\Psi}}\right) \\
& +\left(W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{2}} W_{\boldsymbol{\Psi}}\right)\left(W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{2}} W_{\boldsymbol{\Psi}}\right) .
\end{aligned}
$$

For any $\widetilde{u} \in \mathcal{C}^{2}\left(\Omega_{\mu}\right)$, it can be seen that

$$
W_{\boldsymbol{\Phi}} k^{2} W_{\boldsymbol{\Psi}}[\widetilde{u}]=W_{\boldsymbol{\Phi}} k^{2}[u]=W_{\boldsymbol{\Phi}}\left[k^{2} u\right]=\widetilde{k^{2} u}=k^{2} \widetilde{u}
$$

It then follows that $W_{\boldsymbol{\Phi}} k^{2} W_{\Psi}=k^{2} I$, where $I$ denotes the identity operator. Applying all the above to the prolate spheroidal change of variables (1.5.2), we find

$$
\begin{equation*}
W_{\boldsymbol{\Phi}}\left(\Delta_{3}+k^{2}\right) W_{\Psi}=\frac{1}{h_{1}^{2}(\xi, t)}\left[\mathcal{W}_{\xi, t, \varphi}+c^{2}\left(\xi^{2}-t^{2}\right)\right] \tag{1.5.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{W}_{\xi, t, \varphi}:=\left(\xi^{2}-1\right) \frac{\partial^{2}}{\partial \xi^{2}}+\left(1-t^{2}\right) \frac{\partial^{2}}{\partial t^{2}}+2 \xi \frac{\partial}{\partial \xi}-2 t \frac{\partial}{\partial t}+\frac{h_{1}^{2}(\xi, t)}{h_{2}^{2}(\xi, t)} \frac{\partial^{2}}{\partial \varphi^{2}} \tag{1.5.4}
\end{equation*}
$$

with

$$
\begin{equation*}
h_{1}^{2}(\xi, t):=\mu^{2}\left(\xi^{2}-t^{2}\right), \quad h_{2}^{2}(\xi, t):=\mu^{2}\left(\xi^{2}-1\right)\left(1-t^{2}\right), \quad c:=\mu k . \tag{1.5.5}
\end{equation*}
$$

It may be seen that $\operatorname{ker}\left(\Delta_{3}+k^{2}\right)$ is isomorphic to $\operatorname{ker}\left(\mathcal{W}_{\xi, t, \varphi}+c^{2}\left(\xi^{2}-t^{2}\right)\right)$. Hereafter we assume that the operator $\mathcal{W}_{\xi, t, \varphi}+c^{2}\left(\xi^{2}-t^{2}\right)$ acts on $C^{2}\left(\Omega_{\xi, t, \varphi}\right)$, where

$$
\begin{equation*}
\Omega_{\xi, t, \varphi}=(1,1 / \mu] \times(-1,1) \times[0,2 \pi) . \tag{1.5.6}
\end{equation*}
$$

The PSWFs of degree $n$ and order zero are solutions of 1.5 .3 and can be represented in the form:

$$
\begin{equation*}
\Phi_{n, 0}(\xi, t):=S_{n, 0}(c, t) R_{n, 0}(c, \xi) . \tag{1.5.7}
\end{equation*}
$$

The separation of variables in 1.5.3 implies the following two coupled ordinary differential equations:

$$
\begin{align*}
& \left(1-t^{2}\right) \frac{d^{2} S}{d t^{2}}-2 t \frac{d S}{d t}+\left(\chi(c)-c^{2} t^{2}\right) S=0  \tag{1.5.8}\\
& \left(\xi^{2}-1\right) \frac{d^{2} R}{d \xi^{2}}+2 \xi \frac{d R}{d \xi}-\left(\chi(c)-c^{2} \xi^{2}\right) R=0 \tag{1.5.9}
\end{align*}
$$

where $\chi(c)$ is a parameter introduced during the method of separation of variables. Eqs. (1.5.8) and (1.5.9) are called, respectively, the angular and the radial prolate spheroidal equations. A detailed treatment of the separation of Eq. (1.5.3) in the prolate spheroidal coordinates (1.5.2) can be found in [227] (cf. [103]).

The solutions of 1.5 .8 ) form a countable sequence of angular prolate spheroidal functions $S_{n, 0}(c, t), n=0,1, \ldots$, each corresponding to a real positive eigenvalue in the set $\chi_{0}(c)<\chi_{1}(c)<\chi_{2}(c)<\cdots$, such that $\lim _{n \rightarrow \infty} \chi_{n}=\infty$. Each eigenfunction is real for real $t$ and can be extended to the whole complex plane as an entire function. It is thus seen that the $S_{n, 0}$ are orthogonal and complete in $L_{2}(-1,1)$. It can further be shown that $S_{n, 0}(c, t)$ has exactly $n$ zeros in $(-1,1)$ and is even or odd according to as $n$ is even or odd. Moreover, the eigenvalues $\chi_{n}(c)$ are continuous functions of $c$. It will be seen that the functions $S_{n, 0}(c, t)$ are also solutions of an integral equation involving the sinc function. When indexed by increasing values of $\chi$, they will agree with the notation of indexing by decreasing values of $\lambda$ (see Eq. 1.5.16 below).

In the particular case, $c=0$, Eq. 1.5.8 becomes the Legendre equation (1.4.2) having $\chi_{n}(0)=n(n+1), n=0,1,2, \ldots$, as eigenvalues, and $S_{n, 0}(c, t)=P_{n}(t)$, the Legendre polynomial of degree $n$, defined by (1.4.3), as the corresponding eigenfunctions. The functions $S_{n, 0}(c, t)$ have several essential properties that can be deduced from (1.5.8) [299].

The solutions of $(1.5 .9)$ for the same values of $\chi(c)$ are chosen as the radial prolate spheroidal functions of the first kind, which differ from the angular
functions only by a real scaling factor. The radial and angular spheroidal wave functions were implemented in Fortran [346] and C [324, 325]. Much progress has been made in this part of the subject, and implementations were obtained using double precision. Due to round-off errors, double precision might lead to large errors - especially for higher frequencies and modes. In [59, 60], the functions were implemented in Fortran using quadruple precision, leading to a better accuracy over a wide range of frequencies, modes, and argument values. Mathematica was used in [115, 207] to investigate complex frequencies and noninteger modes with arbitrary precision.

### 1.5.2 The Original Approach of Landau, Pollak, and Slepian

Denote by $\mathcal{B}(W)$ the subclass of $L_{2}(\mathbb{R})$ consisting of those signals whose FTs vanish, if $|\omega|>W$ for some fixed bandwidth parameter $W>0$. In other words,

$$
\begin{equation*}
\mathcal{B}(W)=\left\{f \in L_{2}(\mathbb{R}): \operatorname{supp} \mathcal{F}(f)(\omega) \subset[-W, W]\right\} \tag{1.5.10}
\end{equation*}
$$

where

$$
\mathcal{F}(f)(\omega)=\int_{-\infty}^{\infty} f(t) e^{-i \omega t} d t
$$

Each member $f(t)$ of (1.5.10 can be written as a finite-FT of a function integrable in the absolute square:

$$
\begin{equation*}
f(t)=\frac{1}{2 \pi} \int_{-W}^{W} \mathcal{F}(f)(\omega) e^{i \omega t} d \omega \tag{1.5.11}
\end{equation*}
$$

Members of 1.5 .10 are said to be of bandwidth $W$ or are said to be bandlimited to the band $[-W, W]$, and $\mathcal{B}(W)$ is referred to as the Paley-Wiener space of $W$-band-limited functions. Analogously, $f(t)$ is said to be timelimited if, for some $T>0, f(t)$ vanishes for all $|t|>T$.

According to the classical Paley-Wiener Theorem [264, 313] and (1.5.11), each member of $\mathcal{B}(W)$ admits an extension to an entire function of the complex variable $t$. That is, it has no singularities in the finite $t$-plane, it is infinitely differentiable everywhere, and has a Taylor series about every point with an infinite radius of convergence. Consequently, a nontrivial bandlimited signal $f$ cannot vanish on any interval of the $t$-axis. In other words, "no signal can be both time-limited and band-limited, except for the trivial case where $f$ is identically equal to zero" [123]. Thus, there is a dilemma in seeking signals that are somehow concentrated in both time and frequency domains.

A nonclassical statement of expressing the impossibility of simultaneous confinement of a signal and its FT is a version of Heisenberg's uncertainty principle given by Slepian et al. in [299], which is described as follows: Let the signal $f(t)$ have finite total energy; that is,

$$
\int_{-\infty}^{\infty}|f(t)|^{2} d t=\frac{1}{2 \pi} \int_{-\infty}^{\infty}|\mathcal{F}(f)(\omega)|^{2} d \omega<\infty
$$

If $\alpha^{2}(T)$ denotes the time-spread of $f(t)$ defined by

$$
\begin{equation*}
\alpha^{2}(T):=\frac{\int_{-T}^{T}|f(t)|^{2} d t}{\int_{-\infty}^{\infty}|f(t)|^{2} d t}, \tag{1.5.12}
\end{equation*}
$$

how large can $\alpha^{2}(T)$ in 1.5.12 be for $f(t) \in \mathcal{B}(W)$ for a given fixed interval $T$ ? To answer this question, we express $f(t)$ in (1.5.12) via 1.5.11) to obtain:

$$
\alpha^{2}(T)=\frac{\int_{-W}^{W} \int_{-W}^{W} \mathcal{F}(f)(\omega) \overline{\mathcal{F}}(f)\left(\omega^{\prime}\right) D_{T}\left(\omega, \omega^{\prime}\right) d \omega^{\prime} d \omega}{\int_{-W}^{W}|\mathcal{F}(f)(\omega)|^{2} d \omega}
$$

where the kernel

$$
D_{T}\left(\omega, \omega^{\prime}\right):=\frac{1}{2 \pi} \int_{-T}^{T} e^{-i t\left(\omega-\omega^{\prime}\right)} d t=\frac{\sin T\left(\omega-\omega^{\prime}\right)}{\pi\left(\omega-\omega^{\prime}\right)}
$$

is the scaled FT of the characteristic function of the interval $[-T, T]$.
From the theory of the eigenvalues of integral operators with real and symmetric kernels (see, for example, [81, pp.122-134]), $\alpha^{2}(T)$ attains its maximum value for fixed $T$, if and only if it is a solution of the following homogeneous Fredholm equation of the second kind:

$$
\begin{equation*}
\int_{-W}^{W} D_{T}\left(\omega, \omega^{\prime}\right) \mathcal{F}(f)\left(\omega^{\prime}\right) d \omega^{\prime}=\alpha^{2}(T) \mathcal{F}(f)(\omega) d \omega, \quad|\omega| \leq W \tag{1.5.13}
\end{equation*}
$$

With the appropriate change of variables, the quantities $\alpha^{2}(T)=\lambda$ may be seen to depend on the time-bandwidth product $2 T W$ (also known as the Shannon number), rather than on $T$ and $W$ separately (see Eq. 1.5.16) below). Physical applications of Eq. (1.5.13) can be found in diverse fields, such as stochastic processes [298], the determination of laser modes [45], the statistical theory of energy levels of complex systems [100, 135], antenna theory [280], and a variety of considerations of fundamental importance in communication theory [200, 269].

A rigorous exploration of this problem was initially considered jointly by Slepian, Pollak, and Landau in a series of papers [198, 199, 201, 299, 300, [301, 302, 303, 304], who discovered that the functions $\psi_{n}(c, t):=S_{n, 0}(c, t)$ are
the band-limited solutions of the maximum concentration energy problem in a fixed time interval. The $\left\{\psi_{n}(c, t)\right\}_{n=0}^{\infty}$ depend on the parameters $T$ and $W$. In most of the literature, this dependence is suppressed in the notation. We shall write $\psi_{n}(t)=\psi_{n}(c, t)$, where $c:=T W$ will remain fixed, and name it the Slepian frequency.

Another significant discovery of Slepian and Pollak [299] is that the PSWFs described above are solutions of the following integral eigenvalue problem given by

$$
\begin{equation*}
\int_{-T}^{T} \psi_{n}(s) e^{i c t s} d s=\mu_{n} \psi_{n}(t), \quad t \in \mathbb{R} \tag{1.5.14}
\end{equation*}
$$

where $\mu_{n}=\mu_{n}(c) \in \mathbb{C}$ is a scaling factor up to which the PSWFs have the same shape as their FTs in the interval $[-T, T]$. This surprising connection is labeled a "lucky accident" by Slepian in [304] and makes the differential equation (1.5.8) and the integral equation (1.5.14) interchangeable when studying properties of the PSWFs.

By Eq. (1.5.14) it is thus seen that every even-numbered eigenfunctions $\psi_{n}$ are even, the odd-numbered ones are odd. All eigenvalues $\mu_{n}$ are nonzero and simple; the even-numbered ones are purely real, and the odd-numbered ones are purely imaginary.

It is worth mentioning that the integral equation (1.5.14) does not hold when $c=0$. In other words, the Legendre polynomials are not band-limited, which can also be seen from Lemma 1.4 .3 taking $\mathbf{u}=-i$ :

$$
\begin{equation*}
\int_{-1}^{1} P_{n}(t) e^{i s t} d t=i^{n} \sqrt{\frac{2 \pi}{s}} J_{n+1 / 2}(s), \quad s \neq 0 \tag{1.5.15}
\end{equation*}
$$

In correspondence with the previous identity, we may also show that the FT of $J_{n+1 / 2}(s)$ is, up to a constant multiple, the restriction of $P_{n}(t)$ to $[-1,1]$, so it is band-limited. Since a function and its FT cannot both have finite support, $P_{n}(t)$ is not band-limited, compared to the PSWFs.

Moreover, the eigenvalue problem (1.5.14) is equivalent to the following eigenvalue integral equation:

$$
\begin{align*}
\frac{W}{\pi} \int_{-T}^{T} \psi_{n}(t) K\left(\frac{W}{\pi}(s-t)\right) d t & =\lambda_{n} \psi_{n}(s),|s| \leq T  \tag{1.5.16}\\
\lambda_{n} & =\frac{c}{2 \pi}\left|\mu_{n}\right|^{2}
\end{align*}
$$

where

$$
\begin{equation*}
K(u):=\operatorname{sinc}(u) \quad\left(=D_{\pi}(u)\right) ; \tag{1.5.17}
\end{equation*}
$$

here, $\operatorname{sinc}(u)$ denotes the sinc function, defined as $\operatorname{sinc}(u):=\sin (\pi u) /(\pi u)$. Furthermore, it can be shown that the symmetric kernel $K$ belonging to $L_{2}\left([-T, T]^{2}\right)$ is positive definite so that according again to [81, pp.122-134], it follows that Eq. 1.5.16) has solutions in $L_{2}([-T, T])$ only for a discrete set of real positive values of $\lambda_{n}$, say $\lambda_{0}>\lambda_{1}>\lambda_{2}>\cdots$ bounded away from one, and such that $\lim _{n \rightarrow \infty} \lambda_{n}=0$. For any fixed $n$, it was shown by Fuchs in [125] that the eigenvalues $\lambda_{n}$ approach one exponentially in $c$. In [199], the authors proved that $\lambda_{[c]+1}$ is bounded away from one independently of $c$ and interpreted this to imply that the set of functions in $\mathcal{B}(W)$ whose energy is concentrated in $|t|<T$ has, in a well-defined sense, an approximate dimension bounded by $2 T W$. In [199], it was also shown that $\lambda_{[c]-1}$ is bounded away from zero independently of $c$.

The variational problem that led to (1.5.16) requires only that equation to hold for $|s| \leq T$. Nevertheless, the left-hand side of 1.5 .16 can be used to extend the range of definition of the $\psi_{n}$ 's. We set

$$
\psi_{n}(s):=\frac{W}{\pi \lambda_{n}} \int_{-T}^{T} \psi_{n}(t) K\left(\frac{W}{\pi}(s-t)\right) d t,|s|>T
$$

The eigenfunctions $\psi_{n}(s)$ are now defined for all $s$.
The considerations above adduced suggest that the PSWFs are closely related to their FTs. The FTs of $\psi_{n}$ are given by

$$
\mathcal{F}\left(\psi_{n}\right)(\omega)=(-1)^{n} \sqrt{\frac{2 \pi T}{W \lambda_{n}}} \psi_{n}\left(\frac{T \omega}{W}\right) \chi_{W}(\omega),
$$

where $\chi_{W}(\omega)$ is the characteristic function of $[-W, W]$.
In addition to Eq. 1.5.16), the $\left\{\psi_{n}\right\}_{n=0}^{\infty}$ satisfy an integral equation over $\mathbb{R}$ as well:

$$
\frac{W}{\pi} \int_{-\infty}^{\infty} \psi_{n}(t) K\left(\frac{W}{\pi}(s-t)\right) d t=\psi_{n}(s)
$$

with the same kernel.
This leads to double orthogonality [299]:

$$
\int_{-T}^{T} \psi_{n_{1}}(t) \psi_{n_{2}}(t) d t=\lambda_{n_{1}} \delta_{n_{1}, n_{2}},
$$

and

$$
\int_{-\infty}^{\infty} \psi_{n_{1}}(t) \psi_{n_{2}}(t) d t=\delta_{n_{1}, n_{2}} .
$$

With such a normalization, the $\lambda$ 's can be regarded as the indices of concentration of the signal's energy on the interval $[-T, T]$. Moreover, the PSWFs have the remarkable properties of forming an orthogonal basis of $L_{2}([-T, T])$,
an orthonormal system of $L_{2}(\mathbb{R})$, and, more importantly, an orthonormal basis of the subspace $\mathcal{B}(W)$ of $L_{2}(\mathbb{R})$. It was believed that no other system of classical orthogonal functions possesses these remarkable properties for a long time. Another example of such a system was found in [341].

Chapter 4 will generalize the Landau-Pollak-Slepian theory to infinitedimensional spaces for the cases of compact, self-adjoint operators over the quaternions. We will address all the above and explore new facts of the arising quaternionic function theory.

### 1.6 The Quaternion Fourier Transform

The works of Ernst et al. [109] and Delsuc [96] in the late 80s, as based upon Sommen's definition of a Clifford Fourier Transform (CFT) [306, 307], were the historical starting-point from which a significant part of the development of Quaternion Fourier Transforms (QFTs) originated. Ernst and Delsuc's two-dimensional quaternion transforms were put forward and applied to nuclear magnetic resonance imaging. The QFTs in question were of the following form:

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{f})(u, v)=\int_{\mathbb{R}^{2}} \boldsymbol{f}(x, y) e^{\mathbf{i} u x} e^{\mathbf{j} v y} d x d y ; \quad \boldsymbol{f}: \mathbb{R}^{2} \rightarrow \mathbb{H} \tag{1.6.1}
\end{equation*}
$$

where $(x, y),(u, v)$ are points in $\mathbb{R}^{2}$, and the quaternion exponential product $e^{\mathbf{i} u x} e^{\mathbf{j} v y}$ is a two-dimensional quaternion Fourier kernel.

This version of the QFT is merely a particular case of the CFT given by Brackx, Delanghe, and Sommen in [48]. Bülow, Felsberg, and Sommer followed a different approach to the CFTs in [58]. In [209], Li, McIntosh, and Qian extended the complex Fourier Transform (FT) holomorphically to a function of several complex variables. It is well to observe that with (1.6.1), the four QFT-components separate four symmetry cases in real signals instead of only two, as in the FT. In [222], Mawardi et al. investigated an uncertainty principle for the QFT of the form (1.6.1), which prescribes a lower bound on the product of the effective widths of quaternionic signals in the spatial and frequency domains; cf. also [166, 169, 221, 222]. In [137], Georgiev et al. used the form (1.6.1) to define the Quaternion FourierStieltjes Transform. The same type of QFT was employed in the extension of Bochner-Minlos Theorem within quaternionic analysis by Georgiev et al. in [138]. Two novel uncertainty principles were proposed recently in [339], commencing with a QFT in the form (1.6.1). Generalized sampling expansions of band-limited quaternionic signals associated with (1.6.1) were established in [77]. An account of the essential recent investigations which had their origin in the QFT (1.6.1) can be found in [55].

Because the exponentials in 1.6.1 do not commute, nor with the signal $\boldsymbol{f}$, it means that different formulations are possible for the two-dimensional QFT. In the meantime, an indication of a QFT with the two exponentials positioned on either side of the signal function was given by Ell in 1992 [104, 105]:

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{f})(u, v)=\int_{\mathbb{R}^{2}} e^{\mathrm{i} u x} \boldsymbol{f}(x, y) e^{\mathrm{j} v y} d x d y \tag{1.6.2}
\end{equation*}
$$

Zou et al. [350] recently used this version of the QFT to study a new class of two-dimensional quaternionic signals whose energy concentration is maximal in both time and frequency. For a given finite energy quaternionic signal, we found the possible proportions of its energy in a finite time-domain and a finite frequency-domain, including the signals that do the best job of simultaneous time and frequency concentration.

A different idea emerged in the late '90s in a paper by Sangwine and Ell [290]. Their definition of a QFT was mainly that of Jamison in [176]. Firstly, it consisted of considering a general pure quaternion $\mathbf{u}$, with negative square, rather than a quaternion basis unit (i, $\mathbf{j}$, or $\mathbf{k}$ ) and secondly, choosing a single exponential rather than two. Such a two-dimensional transform (1.6.3) suggests general applications in which the quaternionic signal has three or four independent components [246, 251]:

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{f})(u, v)=\int_{\mathbb{R}^{2}} \exp (\boldsymbol{u}\langle(u, v),(x, y)\rangle) \boldsymbol{f}(x, y) d x d y \tag{1.6.3}
\end{equation*}
$$

Another often used convention for two-dimensional QFTs is to split the factor $(2 \pi)^{-2}$ asymmetrically or equivalently, replace it with a factor $2 \pi$ in the exponents.

An essential part of the development of the QFTs is their applicability to signal and image processing. Much progress has been made on this topic and in applying QFTs to color images. These results can be found in image diffusion, electromagnetism, multi-channel processing, vector field processing, shape representation, linear scale-invariant filtering, fast vector pattern matching, phase correlation, analysis of nonstationary improper complex signals, flow analysis, partial differential systems, disparity estimation, and texture segmentation, as well as spectral representations for hypercomplex wavelet analysis (see [16, 29, 31, 32, 33, 51, 52, 53, 54, 57, 58, 102, 106, [221, 222, 223, 225, 267, 289, 291, 340 and elsewhere).

A discussion of the main properties of different types of QFTs, including (1.6.1)-(1.6.3), about linearity, shift, modulation, dilation, moments, inversion, derivatives, Plancherel and Parseval identities, and investigation of a convolution theorem can be found in [167]. Specific studies relating to
those particular cases of QFTs, and various general criteria were discussed in [107, 168] and [153, Ch. 11].

Let us now conceive a version of the right-sided QFT (1.6.3) of a threedimensional quaternionic signal in $L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ [106]. The treatment given here is a generalization of that provided by Jamison in [176]. We shall remark that all results can be performed straightforwardly to other types of QFTs [350], but we do not dwell further here on these structures.

Definition 1.6.1. Let $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. Let $\mathbf{u}$ be a fixed pure quaternion such that $\mathbf{u}^{2}=-1$ and let $\mathbf{x}, \boldsymbol{\omega}$ be points in $\mathbb{R}^{3}$. The steerable right-sided QFT of $\boldsymbol{f}$ is the function $\mathcal{F}(\boldsymbol{f}): \mathbb{R}^{3} \rightarrow \mathbb{H}$ defined as the quaternion-valued (Lebesgue) integral

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})=\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathrm{x}) \boldsymbol{E}(\boldsymbol{\omega}, \mathrm{x}) d \mathrm{x} \tag{1.6.4}
\end{equation*}
$$

with the quaternion Fourier kernel

$$
\begin{equation*}
\boldsymbol{E}: \mathbb{R}^{3} \times \mathbb{R}^{3} \rightarrow \mathbb{H}, \quad \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x}):=\exp (-\mathbf{u}\langle\boldsymbol{\omega}, \mathbf{x}\rangle) \tag{1.6.5}
\end{equation*}
$$

We refer to $\mathbf{x}$ as space-variables and $\boldsymbol{\omega}$ as angular-frequency variables.
Since $|\boldsymbol{f}(\mathbf{x}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x})|=|\boldsymbol{f}(\mathbf{x})|$ for $\mathbf{x}, \boldsymbol{\omega} \in \mathbb{R}^{3}$, it is clear that if $\boldsymbol{f}$ is absolutely integrable in $\mathbb{R}^{3}$, then the QFT given as (1.6.4) is defined, and the integral (1.6.4) converges absolutely and uniformly for $\boldsymbol{\omega}$ in $\mathbb{R}^{3}$. We shall observe that the order of the factors in (1.6.4) has to be written in a fixed order since the quaternion Fourier kernel (1.6.5) does not commute with every element of the algebra.

From (1.1.3), it follows that the QFT (1.6.4) has the representation

$$
\begin{align*}
\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})= & \int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \cos (\langle\boldsymbol{\omega}, \mathbf{x}\rangle) d \mathbf{x} \\
& +\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x})(-\mathbf{u}) \sin (\langle\boldsymbol{\omega}, \mathbf{x}\rangle) d \mathbf{x} . \tag{1.6.6}
\end{align*}
$$

Eq. 1.6.6) clearly shows how the QFT (1.6.4) separates real signals into four quaternionic components, i.e., the even and odd components of $\boldsymbol{f}$.

To make this work self-contained, we list and prove the elementary properties of the QFT (1.6.4) needed in the sequel.

Proposition 1.6.2. Let $\boldsymbol{f}, \boldsymbol{g} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. The QFT (1.6.4) satisfies the following properties:
(i) left-linearity: for any constants $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{H}$,

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{\alpha} \boldsymbol{f}+\boldsymbol{\beta} \boldsymbol{g})(\boldsymbol{\omega})=\boldsymbol{\alpha} \mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})+\boldsymbol{\beta} \mathcal{F}(\boldsymbol{g})(\boldsymbol{\omega}) \tag{1.6.7}
\end{equation*}
$$

(ii) space-shift: if $\boldsymbol{f}_{\boldsymbol{\alpha}}(\mathbf{x})=\boldsymbol{f}(\mathbf{x}-\boldsymbol{\alpha})$ with a fixed constant $\boldsymbol{\alpha} \in \mathbb{R}^{3}$, then

$$
\begin{equation*}
\mathcal{F}\left(\boldsymbol{f}_{\boldsymbol{\alpha}}\right)(\boldsymbol{\omega})=\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}) \boldsymbol{E}(\boldsymbol{\omega}, \boldsymbol{\alpha}) \tag{1.6.8}
\end{equation*}
$$

(iii) frequency-shift: if $\boldsymbol{h}(\mathbf{x})=\boldsymbol{f}(\mathbf{x}) \boldsymbol{E}(\boldsymbol{\sigma}, \mathbf{x})$ with a fixed constant $\boldsymbol{\sigma} \in \mathbb{R}^{3}$, then

$$
\mathcal{F}(\boldsymbol{h})(\boldsymbol{\omega})=\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}+\boldsymbol{\sigma}) ;
$$

(iv) the function $\mathcal{F}(\boldsymbol{f})$ is bounded for each $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$;
(v) the function $\mathcal{F}(\boldsymbol{f})$ is continuous for each $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$;
(vi) Riemann-Lebesgue: if $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, then $\boldsymbol{\mathcal { F }}(\boldsymbol{f})(\boldsymbol{\omega}) \rightarrow \mathbf{0}$ as $|\boldsymbol{\omega}| \rightarrow \infty$;
(vii) if $\boldsymbol{f}$ is a spherically symmetric function, then $\boldsymbol{\mathcal { F }}(\boldsymbol{f})$ is also spherically symmetric.

Proof. The properties (i), (ii), and (iii) are direct consequences of the definition (1.6.4) and, therefore, will be omitted. For the proof of property (iv), we observe that

$$
|\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})| \leq \int_{\mathbb{R}^{3}}|\boldsymbol{f}(\mathrm{x})| d \mathbf{x}=\|\boldsymbol{f}\|_{L_{1}\left(\mathbb{R}^{3}\right)} .
$$

To show the continuity of $\mathcal{F}(\boldsymbol{f})$ for each $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, we may proceed as follows:

$$
\begin{aligned}
|\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}+\boldsymbol{\sigma})-\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})| & =\left|\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x})[\boldsymbol{E}(\boldsymbol{\sigma}, \mathbf{x})-1] d \mathbf{x}\right| \\
& \leq \int_{\mathbb{R}^{3}}|\boldsymbol{f}(\mathbf{x})||\boldsymbol{E}(\boldsymbol{\sigma}, \mathbf{x})-1| d \mathbf{x}
\end{aligned}
$$

Since $|\boldsymbol{f}(\mathbf{x})||\boldsymbol{E}(\boldsymbol{\sigma}, \mathbf{x})-1| \leq 2|\boldsymbol{f}(\mathbf{x})|$ and $\lim _{\boldsymbol{\sigma} \rightarrow \mathbf{0}}|\boldsymbol{f}(\mathbf{x})||\boldsymbol{E}(\boldsymbol{\sigma}, \mathbf{x})-1|=0$, it follows from the Lebesgue's Dominated Convergence Theorem that

$$
\lim _{\boldsymbol{\sigma} \rightarrow 0} \mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}+\boldsymbol{\sigma})=\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})
$$

To prove the limit assertion (vi), we use a density argument as in the classical case. Let $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. We set $\mathbf{y}:=(\pi / 3)\left(1 / \omega_{1}, 1 / \omega_{2}, 1 / \omega_{3}\right)$; then by Lemma 1.1.3 and property (ii), it follows that

$$
\begin{aligned}
2|\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})| & =\left|\int_{\mathbb{R}^{3}}\left[\boldsymbol{f}(\mathbf{x})-\boldsymbol{f}_{\mathbf{y}}(\mathbf{x})\right] \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x}) d \mathbf{x}\right| \\
& \leq \int_{\mathbb{R}^{3}}\left|\boldsymbol{f}(\mathbf{x})-\boldsymbol{f}_{\mathbf{y}}(\mathrm{x})\right| d \mathbf{x},
\end{aligned}
$$

where $\boldsymbol{f}_{\mathbf{y}}(\mathrm{x}):=\boldsymbol{f}(\mathrm{x}+\mathbf{y})$. Let $\mathcal{C}_{\mathbf{c}}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ denote the linear space of $\mathbb{H}$-valued continuous and compactly supported functions ${ }^{1 /}$. Since functions in $\mathcal{C}_{\mathrm{c}}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ are dense in $L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, then for any $\epsilon>0$, we can choose a $\boldsymbol{g} \in C_{\mathrm{c}}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ so that $|\boldsymbol{f}-\boldsymbol{g}|<\epsilon$. Now, $\boldsymbol{f}-\boldsymbol{f}_{\mathbf{y}}=\boldsymbol{f}-\boldsymbol{g}+\boldsymbol{g}_{\mathrm{y}}-\boldsymbol{f}_{\mathrm{y}}+\boldsymbol{g}-\boldsymbol{g}_{\mathbf{y}}$. Because $\boldsymbol{g} \in C_{\mathrm{c}}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, when $\boldsymbol{\omega}$ is sufficiently large, while $\mathbf{y}$ becomes very small, $\left\|\boldsymbol{f}_{\mathbf{y}}-\boldsymbol{g}_{\mathbf{y}}\right\|_{L_{1}\left(\mathbb{R}^{3}\right)}=\|\boldsymbol{f}-\boldsymbol{g}\|_{L_{1}\left(\mathbb{R}^{3}\right)}<\epsilon$. Clearly, we have

$$
\left\|\boldsymbol{g}-\boldsymbol{g}_{y}\right\|_{L_{1}\left(\mathbb{R}^{3}\right)}=\int_{\mathbb{R}^{3}}\left|\boldsymbol{g}-\boldsymbol{g}_{y}\right| d \mathbf{x} \rightarrow 0
$$

as $\mathbf{y} \rightarrow \mathbf{0}$. The result follows.
We now prove (vii). Let $\mathbf{x}=\left(x_{0}, x_{1}, x_{2}\right) \in \mathbb{R}^{3}$. Suppose $\boldsymbol{f}(\mathbf{x})=\boldsymbol{f}(|\mathbf{x}|)$ is a spherically symmetric function (i.e., a function that depends only on the length of the vector $\mathbf{x}$ and not on its orientation) and let $\theta$ denote the angle between the vectors $\mathbf{x}$ and $\boldsymbol{\omega}$. We employ spherical coordinates $(\rho, \theta, \varphi)$ such that the $x_{2}$-axis is along the $\boldsymbol{\omega}$-vector and, with $|\mathbf{x}|=\rho$, we have

$$
\begin{aligned}
\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}) & =\mathcal{F}(\boldsymbol{f})(|\boldsymbol{\omega}|, \theta, \varphi) \\
& =\int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{2 \pi} \boldsymbol{f}(\rho) \exp (-\mathbf{u}|\boldsymbol{\omega}| \rho \cos \theta) \rho^{2} \sin \theta d \varphi d \theta d \rho \\
& =2 \pi \int_{0}^{\infty} \boldsymbol{f}(\rho)\left(\frac{\exp (\mathbf{u}|\boldsymbol{\omega}| \rho)-\exp (-\mathbf{u}|\boldsymbol{\omega}| \rho)}{\mathbf{u}|\boldsymbol{\omega}| \rho}\right) \rho^{2} d \rho \\
& =4 \pi \int_{0}^{\infty} \boldsymbol{f}(\rho) \frac{\sin (|\boldsymbol{\omega}| \rho)}{|\boldsymbol{\omega}| \rho} \rho^{2} d \rho .
\end{aligned}
$$

Since the above integral depends only on $|\boldsymbol{\omega}|$, it follows that the QFT of a spherically symmetric function is also spherically symmetric, completing the proof.

The following result is an immediate consequence of Eq. (1.6.6) and property (vi) of Proposition 1.6.2.

Corollary 1.6.3. If $\boldsymbol{f} \in L_{1}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, then

$$
\begin{aligned}
& \lim _{|\boldsymbol{\omega}| \rightarrow \infty} \int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \cos (\langle\boldsymbol{\omega}, \mathbf{x}\rangle) d \mathbf{x} \\
& =\lim _{|\boldsymbol{\omega}| \rightarrow \infty} \int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \sin (\langle\boldsymbol{\omega}, \mathbf{x}\rangle) d \mathbf{x}=\mathbf{0} .
\end{aligned}
$$

Under suitable conditions, the original signal $\boldsymbol{f}$ can be reconstructed from the QFT by the inverse transform [106, (176].

[^0]Definition 1.6.4. The inverse (right-sided) QFT of $\boldsymbol{g} \in L_{1} \cap L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ is the function $\mathcal{F}^{-1}(\boldsymbol{g}): \mathbb{R}^{3} \rightarrow \mathbb{H}$ defined by

$$
\begin{equation*}
\mathcal{F}^{-1}(\boldsymbol{g})(\mathbf{x})=\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}} \boldsymbol{g}(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \tag{1.6.9}
\end{equation*}
$$

where $\overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x})=\exp (\mathbf{u}\langle\boldsymbol{\omega}, \mathbf{x}\rangle)$ is called the inverse quaternion Fourier kernel.

Apart from the convention used in Definition 1.6 .4 with $1 /(2 \pi)^{3}$ in the inverse QFT (1.6.9), there are two other standard conventions: one is obtained by substituting $\boldsymbol{\omega} \rightarrow 2 \pi \boldsymbol{\omega}$ in 1.6.4. The other is obtained by evenly distributing the $2 \pi$ factors between the QFT and the inverse QFT, respectively,

$$
\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})=\frac{1}{(2 \pi)^{3 / 2}} \int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x}) d \mathbf{x}
$$

and

$$
\mathcal{F}^{-1}(\boldsymbol{g})(\mathrm{x})=\frac{1}{(2 \pi)^{3 / 2}} \int_{\mathbb{R}^{3}} \boldsymbol{g}(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathrm{x}) d \boldsymbol{\omega} .
$$

All calculations can easily be converted to these other conventions.
In our notation, Plancherel's Theorem is
Theorem 1.6.5. If $\boldsymbol{f}, \boldsymbol{g} \in L_{1} \cap L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, then

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathrm{x}) \overline{\boldsymbol{g}}(\mathrm{x}) d \mathrm{x}=\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}} \mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}) \overline{\mathcal{F}}(\boldsymbol{g})(\boldsymbol{\omega}) d \boldsymbol{\omega} \tag{1.6.10}
\end{equation*}
$$

In particular, Parseval's identity now reads as follows:
Corollary 1.6.6. If $\boldsymbol{f} \in L_{1} \cap L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, then

$$
\begin{equation*}
\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}=\frac{1}{(2 \pi)^{3}}\|\boldsymbol{\mathcal { F }}(\boldsymbol{f})\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} \tag{1.6.11}
\end{equation*}
$$

The quantity $E=\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}$ will be called the total energy of a quaternionic signal $\boldsymbol{f}$. In Chapter 4, we will restrict our attention to signals whose energy is finite.

According to 1.6.11), observe that the total signal energy calculated in the spatial-domain equals the total energy computed in the frequency-domain up to a constant. Parseval's identity (1.6.11) asserts that the QFT (1.6.4) is a bounded linear operator on the space $L_{1} \cap L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. In this way, standard density arguments allow us to extend the definition (1.6.4) in a unique way to the whole of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ [78]. In what follows, we always consider the QFT as an operator from $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ into $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$.

It will be convenient, before proceeding further, to observe the well-known fact that a counterpart of Parseval's identity fails for the two-sided QFT defined by (1.6.2) using the quaternionic inner product (1.2.1). As was observed in [167], this failure occurs since (1.2.1) does not obey the cyclic symmetry property (1.1.2). To obtain that the "energy" of a quaternionic signal is indeed invariant under 1.6.2, it is therefore either necessary to modify the definition of the inner product (1.2.1) or to modify the QFT itself. In [167], Hitzer changed the inner product (1.2.1) to the scalar inner product (1.2.2) and proved a quaternionic version of Parseval's identity. Chen et al. developed a different strategy in [76], which redefines the modulus of a quaternionic function in the frequency-domain, known as $Q$-modulus, while maintaining the original inner product in the time-domain. The latter approach was adopted by Zou et al. in [350], who investigated specific results of the quaternionic time-frequency spectral concentration problem that arises under this modulus. As the development of the $Q$-modulus is beyond the scope of the present work, reference is made to the account given in [76], where details of the subject can be found.

## 2

## Families of Harmonic Functions on Spheroidal Domains

In this chapter, we introduce two distinct single one-parameter orthogonal families of internal and external spheroidal harmonics, whose elements are parametrized by the shape of the corresponding spheroid. The main point of interest is that the orthogonality of the elements that constitute the two families does not depend on the eccentricity of the spheroids. A general expression for the basis changes between different systems of spheroidal harmonic functions is then calculated, obtaining conversion formulas that relate systems of harmonic functions associated with spheroids of arbitrary eccentricity.

### 2.1 Harmonics in Spheroidal Coordinates

Traditionally, spherical domains are considered a reference while studying realistic problems. Given this perfectly symmetrical property of the domains, theory and applications of the discussed methods become easier. However, in many cases, the spherical reference domain is inappropriate, and spheroidal domains are used instead.

A spheroid is a quadric surface that is generated by rotating an ellipse around its major axis. The analysis of harmonic functions on spheroids typically separates the prolate and oblate cases, which are parametrized in their respective confocal families

$$
\left\{\mathbf{x} \in \mathbb{R}^{3}: \frac{x_{0}^{2}}{\cosh ^{2} \alpha}+\frac{x_{1}^{2}+x_{2}^{2}}{\sinh ^{2} \alpha}=1\right\}
$$

and

$$
\left\{\mathbf{x} \in \mathbb{R}^{3}: \frac{x_{0}^{2}}{\sinh ^{2} \alpha}+\frac{x_{1}^{2}+x_{2}^{2}}{\cosh ^{2} \alpha}=1\right\}
$$

for $\alpha>0$. These domains do not include the case of a Euclidean ball, but they become rounder as they degenerate with $\alpha \rightarrow \infty$. In the sequel, we combine them into a single family of coaxial spheroidal domains $\Omega_{\mu}$, oriented so that their axes of rotational symmetric are along the $x_{0}$-axis and whose focal lengths equal $2\left(1-e^{2 \nu}\right)^{1 / 2}$ :

$$
\begin{equation*}
\Omega_{\mu}=\left\{\mathbf{x} \in \mathbb{R}^{3}: x_{0}^{2}+\frac{x_{1}^{2}+x_{2}^{2}}{e^{2 \nu}}=1\right\} \tag{2.1.1}
\end{equation*}
$$

where $\nu \in \mathbb{R}$ is arbitrary. The parameter $\mu=\mu(\nu)=\left(1-e^{2 \nu}\right)^{1 / 2}$ denotes the eccentricity of the generating ellipses all centered at the origin and lying in the $\left(x_{0}, x_{1}\right)$-plane, which by convention is in the interval $(0,1)$ when $\nu<0$ (prolate spheroid), and in $i \mathbb{R}^{+}$when $\nu>0$ (oblate spheroid); the intermediate value $\nu=0, \mu=0$ gives the Euclidean unit ball $\Omega_{0}=\left\{\mathbf{x} \in \mathbb{R}^{3}:|\mathbf{x}|^{2}<1\right\}$. It is significant to note that in the prolate case, we obtain the mentioned spheroid by setting $e^{\nu}=\tanh \alpha$ and rescaling $\mathbf{x}$ by a factor of $\mu^{-1}$, while for the oblate case, we set $e^{\nu}=\operatorname{coth} \alpha$ and rescale by a factor of $(\mu / i)^{-1}$.

The main objective of the present section is to develop families of internal and external harmonic functions with particular emphasis on those orthogonal in the $L_{2}$-Hilbert space structure, which are parametrized by the single value $\nu$. This cannot be done with models in which the ball only is approximated as a degenerate case. It requires a separate yet utterly analogous treatment for prolate and oblate spheroids [132, 170]. Garabedian in [132] developed the original impetus behind the study of orthogonal bases of polynomials for the spaces of square-integrable harmonic functions defined in a prolate or oblate spheroid. The orthogonality was taken for certain natural inner products, each of which leads to the discussion of a partial differential equation employing the kernel of the orthogonal system corresponding to that inner product. Some aspects of generating harmonic functions orthogonal in a region outside a prolate spheroid were recently considered in [250]. However, valuable properties, such as the relationships between systems of internal and external harmonics associated with the spheroids $\Omega_{\mu}$ to those related to the ball $\Omega_{0}$, were not studied. Also, internal and external spherical harmonics were not considered part of these kinds of systems.

In general, the boundary value problems involving prolate and oblate spheroidal bodies may be treated using prolate and oblate spheroidal coordinate systems. There are several equivalent ways to introduce spheroidal coordinates [6, Ch. XXI] and [205, Ch. VIII]. Suppose for the moment that $\nu<0$ (the other case $\nu>0$ will be explained below). In this way, we use
prolate spheroidal coordinates $(\eta, \vartheta, \varphi)$ related to Cartesian coordinates by the following transformation:

$$
\begin{equation*}
x_{0}=\mu \cosh \eta \cos \vartheta, x_{1}=\mu \sinh \eta \sin \vartheta \cos \varphi, x_{2}=\mu \sinh \eta \sin \vartheta \sin \varphi, \tag{2.1.2}
\end{equation*}
$$

where $\mu>0$, and $\eta \in\left[0, \eta_{\mu}\right]$ with $\eta_{\mu}=\operatorname{arctanh} e^{\nu}$ is the "radial" coordinate, $\vartheta \in[0, \pi]$ is the spheroidal colatitude angle, and $\varphi \in[0,2 \pi)$ is the azimuthal angle.

Let

$$
\Omega_{\mu}^{*}:=\Omega_{\mu} \cup\left\{(\eta, \vartheta, \varphi): \eta \in\left(\eta_{\mu}, \infty\right), \vartheta \in[0, \pi], \varphi \in[0,2 \pi)\right\},
$$

where $\nu<0$.
It is a relatively simple matter to verify that the Laplace equation in terms of the coordinates (2.1.2) becomes

$$
\begin{align*}
& \frac{1}{\mu^{2}\left(\sinh ^{2} \eta+\sin ^{2} \vartheta\right)}\left(\frac{\partial^{2} U[\mu]}{\partial \eta^{2}}+\frac{\partial^{2} U[\mu]}{\partial \vartheta^{2}}+\operatorname{coth} \eta \frac{\partial U[\mu]}{\partial \eta}+\cot \vartheta \frac{\partial U[\mu]}{\partial \vartheta}\right) \\
& +\frac{1}{\mu^{2} \sinh ^{2} \eta \sin ^{2} \vartheta} \frac{\partial^{2} U[\mu]}{\partial \varphi^{2}}=0 ; \quad U[\mu] \in \mathcal{C}^{2}\left(\Omega_{\mu}^{*}\right) . \tag{2.1.3}
\end{align*}
$$

The previous equation is separable in prolate spheroidal coordinates [233, Ch. I]. For $\mathbf{x} \in \Omega_{\mu}^{*}$ the corresponding solutions of (2.1.3), often referred to as the prolate spheroidal harmonics, are given by

$$
\begin{equation*}
U[\mu](\mathbf{x}):=\Xi(\eta) \Theta(\vartheta) \Phi(\varphi), \tag{2.1.4}
\end{equation*}
$$

where $\Xi(\eta)$ is the radial function, $\Theta(\vartheta)$ is the angular function, and $\Phi(\varphi)$ is the azimuthal function, which are solutions of the following ordinary differential equations:

$$
\begin{array}{r}
\frac{d^{2} \Xi}{d \eta^{2}}+\operatorname{coth} \eta \frac{d \Xi}{d \eta}-\left[\frac{m^{2}}{\sinh ^{2} \eta}+l(l+1)\right] \Xi=0 \\
\frac{d^{2} \Theta}{d \vartheta^{2}}+\cot \vartheta \frac{d \Theta}{d \vartheta}+\left[l(l+1)-\frac{m^{2}}{\sin ^{2} \vartheta}\right] \Theta=0 \\
\frac{d^{2} \Phi}{d \varphi^{2}}+m^{2} \Phi=0 \tag{2.1.5}
\end{array}
$$

where $l$ is a constant and $m$ is a parameter introduced during the method of separation of variables. Hence the solutions of the first two equations are, for $\Xi(\eta)$ and $\Theta(\vartheta)$, respectively, $P_{l}^{m}(\cosh \eta)$ or $Q_{l}^{m}(\cosh \eta)$, and $P_{l}^{m}(\cos \vartheta)$ or $Q_{l}^{m}(\cos \vartheta)$. For the typical applications, we take $P_{l}^{m}(\cos \vartheta)$, where $m$ and $l$ are positive integers, including zero, and suppose $m$ not greater than $l$; we shall accordingly confine ourselves at present to this case. Thus, solutions of equation 2.1.5) are either $\cos (m \varphi)$ or $\sin (m \varphi)$.

## 66 2. SOLUTIONS OF LAPLACE'S EQUATION IN SPHEROIDAL COORDINATES

Correspondingly, we define the required solutions (2.1.4 to be employed in the sequel as follows.

Definition 2.1.1. Let $l \geq 0$ and $0 \leq m \leq l$. For $\mu \neq 0$, the basic spheroidal harmonics of degree $l$ and order $m$ are

$$
\alpha_{l, m} \mu^{l} P_{l}^{m}(\cos \vartheta) P_{l}^{m}(\cosh \eta){ }_{\sin }^{\cos }(m \varphi),
$$

and $\quad \frac{\beta_{l, m}}{\mu^{l+1}} P_{l}^{m}(\cos \vartheta) Q_{l}^{m}(\cosh \eta){ }_{\sin }^{\cos }(m \varphi)$,
with

$$
\alpha_{l, m}=\frac{(l-m)!}{(2 l-1)!!}, \quad \beta_{l, m}=\frac{(2 l+1)!!}{(l+m)!} .
$$

It is of interest to remark that the second of these functions cannot be applied in a space that contains the origin $x_{0}=x_{1}=x_{2}=0$ because the functions $Q_{l}^{m}(\cosh \eta)$ become infinite when $\eta=0$. Hence, for the interior of the prescribed spheroids (2.1.1), the one-parameter family of spheroidal harmonics

$$
\begin{equation*}
U_{l, m}^{ \pm}[\mu](\mathbf{x}):=\alpha_{l, m} \mu^{l} P_{l}^{m}(\cos \vartheta) P_{l}^{m}(\cosh \eta) \Phi_{m}^{ \pm}(\varphi) \tag{2.1.6}
\end{equation*}
$$

will be taken. We have written

$$
\Phi_{m}^{+}(\varphi)=\cos (m \varphi), \quad \Phi_{m}^{-}(\varphi)=\sin (m \varphi)
$$

to unify the notation for the odd and even functions. We will not use $\Phi_{0}^{-}$ since it is identically zero.

On the other hand, we need functions that vanish at infinity for the exterior of the prescribed spheroids. Since $P_{l}^{m}(\cosh \eta)$ become infinite with $\eta$, the appropriate spheroidal harmonics employed for the space exterior $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ will be

$$
\begin{equation*}
\widehat{U}_{l, m}^{ \pm}[\mu](\mathbf{x}):=\frac{\beta_{l, m}}{\mu^{l+1}} P_{l}^{m}(\cos \vartheta) Q_{l}^{m}(\cosh \eta) \Phi_{m}^{ \pm}(\varphi) \tag{2.1.7}
\end{equation*}
$$

Versions of a capital letter with and without circumflex will always denote external and internal functions. The harmonics (2.1.7) are not polynomials but rather algebraic functions, which are homogeneous of degree $-(l+1)$.

We add that $U_{l, 0}^{-}[\mu]$ and $\widehat{U}_{l, 0}^{-}[\mu]$ vanish identically, as do all $U_{l, m}^{ \pm}[\mu]$ and $\widehat{U}_{l, m}^{ \pm}[\mu]$ for $m \geq l+1$. Therefore when we refer to the sets $\left\{U_{l, m}^{ \pm}[\mu]\right\}$ and $\left\{\hat{U}_{l, m}^{ \pm}[\mu]\right\}$, we always exclude the indices which apply to these trivial cases, even when we do not explicitly state $0 \leq m \leq l$ for the " + " case and $1 \leq m \leq l$ for the "-" case.

The spheroidal harmonics (2.1.6) and 2.1.7, except for the constant factors $\alpha_{l, m}$ and $\beta_{l, m}$ and the rescaling of the $\mathbf{x}$ variable, are the functions defined in [170, Ch. X]. We shall discuss their properties more fully hereafter. The motivation behind the choice for redefining these rescaling functions will be explained in detail later in this section (see Proposition 2.1 .2 below).

The ideas that led us here allow the discussion of internal and external harmonics for oblate spheroids in a similar manner. By equations (2.1.2), a direct computation shows that

$$
|\mathbf{x}|^{2}+\mu^{2}=\mu^{2}\left(\cosh ^{2} \eta+\cos ^{2} \vartheta\right),
$$

and also

$$
\mu^{2}(\cosh \eta \pm \cos \vartheta)^{2}=\left(x_{0} \pm \mu\right)^{2}+x_{1}^{2}+x_{2}^{2}
$$

Hence

$$
\cosh \eta=\frac{\omega(\mu)}{2 \mu}, \quad \cos \vartheta=\frac{2 x_{0}}{\omega(\mu)}
$$

where

$$
\begin{equation*}
\omega(\mu):=\sqrt{\left(x_{0}+\mu\right)^{2}+x_{1}^{2}+x_{2}^{2}}+\sqrt{\left(x_{0}-\mu\right)^{2}+x_{1}^{2}+x_{2}^{2}} \tag{2.1.8}
\end{equation*}
$$

is positive. In the considerations to follow, we will often omit the argument of (2.1.8) and write $\omega$ instead of $\omega(\mu)$. It is now evident that the oblate case $\nu>0$ is obtained from this by analytic continuation, thinking of $\mu \in i \mathbb{R}^{+}$ as being boundary values of the first quadrant in the complex plane. The following terms

$$
\zeta(\mu)=|\mathbf{x}|^{2}+\mu^{2}+2 x_{0} \mu, \quad \bar{\zeta}(\mu)=|\mathbf{x}|^{2}+\mu^{2}-2 x_{0} \mu
$$

inside the radicals in 2.1 .8 are now complex conjugates, where

$$
\begin{equation*}
|\zeta(\mu)|=\left|\mu^{2}-\left(x_{0}+i \sqrt{x_{1}^{2}+x_{2}^{2}}\right)^{2}\right| \tag{2.1.9}
\end{equation*}
$$

When it is necessary to particularize the variables involved in it, we shall write $\zeta(\mu, \mathbf{x})$.

It is interesting to note that the function defined by (2.1.9) describes a single-parametrical family of Cassini surfaces as it equals to the product of the distances from any point on the prescribed spheroids to the two foci $( \pm \mu, 0,0)$. This function will play an essential role in the forthcoming sections.

Moreover, from (2.1.8) it follows that

$$
\omega=\sqrt{2(\operatorname{Re} \zeta(\mu)+|\zeta(\mu)|)}
$$

## 68 2. SOLUTIONS OF LAPLACE'S EQUATION IN SPHEROIDAL COORDINATES

is real and slightly less than $2|\mathbf{x}|$ for $\mu / i$ small. We have, thus,

$$
\frac{2 x_{0}}{\omega}=x_{0} \sqrt{\frac{2}{\operatorname{Re} \zeta(\mu)+|\zeta(\mu)|}}, \quad \frac{\omega}{2 \mu}=i \sqrt{\frac{\operatorname{Re} \zeta(\mu)+|\zeta(\mu)|}{2\left(e^{2 \nu}-1\right)}},
$$

and from this, one can verify that $\left|2 x_{0} / \omega\right| \leq 1$ and $\operatorname{Im}(\omega / 2 \mu)$ takes values in the interval $[0, \infty)$. Thus, for the interior (resp. exterior) of an oblate spheroid we first replace $\eta$ with the value $\operatorname{arcsinh} \cosh \eta$ to retain the formula (2.1.2) through the relations

$$
\frac{\omega}{2 \mu}=i \sinh \eta, \quad \frac{2 x_{0}}{\omega}=\cos \vartheta
$$

where $\eta \in\left[0, \tilde{\eta}_{\mu}\right]$ (resp. $\left.\eta \in\left(\tilde{\eta}_{\mu}, \infty\right)\right)$ with $\tilde{\eta}_{\mu}=\operatorname{arccoth} e^{\nu}$ and $\vartheta \in[0, \pi]$, and then use expressions (2.1.6) and (2.1.7) again to define the oblate harmonics. The construction of the spheroidal harmonics in its extended signification being thus completed, no difficulties can occur in restricting particular cases.

The use of the coefficients $\alpha_{l, m}$ and $\beta_{l, m}$ in the expressions 2.1.6) and (2.1.7) is for the following.

Proposition 2.1.2. For all $\mathbf{x} \in \mathbb{R}^{3},|\mathbf{x}| \neq 0$, the limits

$$
\lim _{\mu \rightarrow 0} U_{l, m}^{ \pm}[\mu](\mathbf{x}), \quad \lim _{\mu \rightarrow 0} \widehat{U}_{l, m}^{ \pm}[\mu](\mathbf{x})
$$

exist and are given, respectively, by

$$
\begin{align*}
& U_{l, m}^{ \pm}[0](\mathbf{x})=|\mathbf{x}|^{l} P_{l}^{m}\left(\frac{x_{0}}{|\mathbf{x}|}\right) \Phi_{m}^{ \pm}(\varphi),  \tag{2.1.10}\\
& \widehat{U}_{l, m}^{ \pm}[0](\mathbf{x})=\frac{1}{|\mathbf{x}|^{l+1}} P_{l}^{m}\left(\frac{x_{0}}{|\mathbf{x}|}\right) \Phi_{m}^{ \pm}(\varphi), \tag{2.1.11}
\end{align*}
$$

where we employ spherical coordinates $x_{0}=\rho \cos \theta, x_{1}=\rho \sin \theta \cos \varphi$, and $x_{2}=\rho \sin \theta \sin \varphi$.

Proof. Since the variable $\varphi$ in (2.1.6 and 2.1.7 does not depend on the variable $x_{0}$, we examine the factors $P_{l}^{m}\left(2 x_{0} / \omega\right) P_{l}^{m}(\omega /(2 \mu))$ in 2.1.6) (resp. $P_{l}^{m}\left(2 x_{0} / \omega\right) Q_{l}^{m}(\omega /(2 \mu))$ in (2.1.7) $)$ with $\omega$ given by (2.1.8).

Since

$$
\sqrt{\left(x_{0} \pm \mu\right)^{2}+x_{1}^{2}+x_{2}^{2}}=|\mathbf{x}| \pm \frac{x_{0}}{|\mathbf{x}|} \mu+O\left(\mu^{2}\right)
$$

it follows that $\omega=2|\mathbf{x}|+O\left(\mu^{2}\right)$ as $\mu \rightarrow 0$. Furthermore, we have once more from (2.1.8) that

$$
\frac{2 x_{0}}{\omega}=\frac{x_{0}}{|\mathbf{x}|}+O(\mu)
$$

so $P_{l}^{m}\left(2 x_{0} / \omega\right) \rightarrow P_{l}^{m}\left(x_{0} /|\mathbf{x}|\right)$ as $\mu \rightarrow 0$. Using the explicit representation (1.4.9) valid for real $|t|>1$, and the fact that

$$
\left(\alpha_{l, m}\right)^{-1}=\frac{l!(l+m)!}{2^{l}} \sum_{s=m}^{l} \lambda_{s}^{l, m}
$$

with the constants $\lambda_{s}^{l, m}$ given by (1.4.10), we have the required asymptotic behavior $\alpha_{l, m} P_{l}^{m}(t) \simeq t^{l}$ as $t=\omega / 2 \mu$ tends to infinity, which corresponds to $\mu \rightarrow 0$ for fixed $\mathbf{x}$. In a similar way, according to formula (1.4.11) it follows that $\beta_{l, m} Q_{l}^{m}(t) \simeq 1 / t^{l+1}$ as $t=\omega / 2 \mu$ tends to infinity, corresponding to $\mu \rightarrow 0$ for fixed $\mathbf{x}$.

By the proposition just proved, it is observed that the internal and the external solid spherical harmonics defined, respectively, by (2.1.10) and (2.1.11) are embedded in the preceding one-parameter families of spheroidal harmonics. In contrast, in treatments such as [132] and [250], the spheroidal harmonics degenerate as the eccentricity of the spheroid decreases.

Unlike the spherical functions $U_{l, m}^{ \pm}[0](\mathbf{x})$ and $\widehat{U}_{l, m}^{ \pm}[0](\mathbf{x}), U_{l, m}^{ \pm}[\mu](\mathbf{x})$ and $\widehat{U}_{l, m}^{ \pm}[\mu](\mathbf{x})$ are generally not homogeneous.

We add a note concerning the orthogonality of the spheroidal harmonics with respect to two natural inner products. As was shown in [132], the family $\left\{U_{l, m}^{ \pm}[\mu]\right\}$ turns out to be orthogonal with respect to the Dirichlet inner product over $\Omega_{\mu}$, defined by

$$
\begin{equation*}
(f, g)_{\mu}=\int_{\eta=\eta_{\mu}} f(\mathbf{x}) \frac{\partial g}{\partial \mathbf{n}}(\mathbf{x}) d \sigma, \quad(\Delta g=0) \tag{2.1.12}
\end{equation*}
$$

where $\mathbf{n}$ denotes the unit outward normal vector to the boundary of $\Omega_{\mu}$ at the point $P^{*}:=\left(\eta_{\mu}, \vartheta, \varphi\right)$, with $\cosh \eta_{\mu}=1 / \mu$ :

$$
\begin{equation*}
\mathbf{n}=\frac{1}{\sqrt{1-\mu^{2} \cos ^{2} \vartheta}}\left[\sqrt{1-\mu^{2}} \cos \vartheta+(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) \sin \vartheta\right] \tag{2.1.13}
\end{equation*}
$$

Since the measure on the boundary is $d \sigma=\sqrt{1-\mu^{2}} \sin \vartheta d \varphi d \vartheta$, we can compute the outward normal derivative $\partial / \partial \mathbf{n}$ of $g$ at any point using (2.1.13) to obtain

$$
d \sigma \frac{\partial g}{\partial \mathbf{n}}(\vartheta, \varphi)=\sqrt{1-\mu^{2}} \sin \vartheta d \varphi d \vartheta \frac{\partial g}{\partial \eta}\left(P^{*}\right)
$$

Hence, we deduce from (2.1.12) that

$$
\begin{equation*}
(f, g)_{\mu}=\int_{0}^{\pi} \int_{0}^{2 \pi} f\left(P^{*}\right) \frac{\partial g}{\partial \eta}\left(P^{*}\right) \sqrt{1-\mu^{2}} \sin \vartheta d \varphi d \vartheta \tag{2.1.14}
\end{equation*}
$$

## 70 2. SOLUTIONS OF LAPLACE'S EQUATION IN SPHEROIDAL COORDINATES

We will now show similar orthogonality of the external functions (2.1.7) in the sense of the integral (2.1.12). Here and in the sequel, we introduce the notation $U_{l, m}^{ \pm}[\mu]=U_{l, m}[\mu] \Phi_{m}^{ \pm}$(resp. $\left.\widehat{U}_{l, m}^{ \pm}[\mu]=\widehat{U}_{l, m}[\mu] \Phi_{m}^{ \pm}\right)$for use when the factors $\Phi_{m}^{ \pm}$are not of interest.

Proposition 2.1.3. For fixed $\mu$, the harmonic functions $\widehat{U}_{l, m}^{ \pm}[\mu]$ are orthogonal over $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ in the sense of the Dirichlet integral (2.1.12).

Proof. We will assume that $\nu<0$, because the case $\nu>0$ is similar. When $m_{1} \neq m_{2}$, we have by the orthogonality of ordinary Fourier series

$$
\begin{aligned}
& \left(\widehat{U}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), \widehat{U}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right)_{\mu}=0, \\
& \left(\widehat{U}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), \widehat{U}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right)_{\mu}=0, \\
& \left(\widehat{U}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), \widehat{U}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right)_{\mu}=0, \\
& \left(\widehat{U}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), \widehat{U}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right)_{\mu}=0 .
\end{aligned}
$$

By definition of the integral (2.1.12) and using (2.1.14), for $m_{1}=m_{2}=m$ a direct computation shows that

$$
\begin{aligned}
& \left(\widehat{U}_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), \widehat{U}_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right)_{\mu} \\
= & \int_{0}^{\pi} \int_{0}^{2 \pi} \widehat{U}_{l_{1}, m}[\mu]\left(P^{*}\right) \frac{\partial \widehat{U}_{l_{2}, m}[\mu]}{\partial \eta}\left(P^{*}\right)\left(\Phi_{m}^{+}(\varphi)\right)^{2} \sqrt{1-\mu^{2}} \sin \vartheta d \varphi d \vartheta \\
= & \frac{\beta_{l_{1}, m} \beta_{l_{2}, m}}{\mu^{l_{1}+l_{2}+1}} \pi\left(1+\delta_{0, m}\right) Q_{l_{1}}^{m}(1 / \mu)\left[\sqrt{1-\mu^{2}} Q_{l_{2}}^{m+1}(1 / \mu)+m Q_{l_{2}}^{m}(1 / \mu)\right] \\
& \times \int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta d \vartheta \\
= & \frac{\left(\beta_{l_{1}, m}\right)^{2}}{\mu^{2 l_{1}+1}} \frac{2 \pi\left(l_{1}+m\right)!}{\left(2 l_{1}+1\right)\left(l_{1}-m\right)!}\left(1+\delta_{0, m}\right) \delta_{l_{1}, l_{2}} \\
& \times Q_{l_{1}}^{m}(1 / \mu)\left[\sqrt{1-\mu^{2}} Q_{l_{1}}^{m+1}(1 / \mu)+m Q_{l_{1}}^{m}(1 / \mu)\right] .
\end{aligned}
$$

The same value is obtained when we replace $\Phi_{m}^{+}(\varphi)$ by $\Phi_{m}^{-}(\varphi)$ throughout, $m>0$.

We turn next to a less obvious result, which asserts that the spheroidal harmonics $U_{l, m}^{ \pm}[\mu]$ (resp. $\widehat{U}_{l, m}^{ \pm}[\mu]$ ) are not necessarily orthogonal in the closed subspace $\operatorname{Har}_{2}\left(\Omega_{\mu}\right)\left(\right.$ resp. $\left.\operatorname{Har}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)\right)$ of $L_{2}\left(\Omega_{\mu}\right)\left(\right.$ resp. $\left.L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)\right)$ with respect to the ordinary $L_{2}$-inner product. It turns out that the collections $\left\{U_{l, m}^{ \pm}[\mu]: l \geq 0,0 \leq m \leq l\right\}$ and $\left\{\widehat{U}_{l, m}^{ \pm}[\mu]: l \geq 0,0 \leq m \leq l\right\}$ do not form orthogonal bases for the spaces of square-integrable harmonic functions, except for the unit ball, $\mu=0$.

For simplicity, we assume again that $\nu<0$, with $0<\eta \leq \eta_{\mu}$ when we refer to $\Omega_{\mu}$ and $\eta_{\mu}<\eta<\infty$ when we refer to $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$, with $\cosh \eta_{\mu}=1 / \mu$. Now, let

$$
\begin{equation*}
\langle f, g\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=\int_{\Omega_{\mu}} f(\mathbf{x}) g(\mathbf{x}) d \mathbf{x} \tag{2.1.15}
\end{equation*}
$$

where $d \mathbf{x}=d x_{0} d x_{1} d x_{2}$.
Applying the coordinates (2.1.2), gives the infinitesimal volume element

$$
\begin{equation*}
d \mathbf{x}=\mu^{3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right) \sin \vartheta \sinh \eta d \vartheta d \eta \tag{2.1.16}
\end{equation*}
$$

It is clear that, when $m_{1} \neq m_{2}$, we have

$$
\begin{aligned}
& \left\langle U_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), U_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\langle U_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), U_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\langle U_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), U_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\langle U_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), U_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0,
\end{aligned}
$$

and similarly for $\widehat{U}_{l, m}^{ \pm}[\mu]$.
We now compute

$$
\begin{align*}
& \left\langle U_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), U_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)} \\
= & \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} U_{l_{1}, m}[\mu] U_{l_{2}, m}[\mu] \mu^{3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right) \sin \vartheta \sinh \eta d \vartheta d \eta \\
& \times \int_{0}^{2 \pi}\left(\Phi_{m}^{+}(\varphi)\right)^{2} d \varphi \\
= & \alpha_{l_{1}, m} \alpha_{l_{2}, m} \mu^{l_{1}+l_{2}+3}\left(1+\delta_{0, m}\right) \pi \\
& \times\left[\int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta d \vartheta\right. \\
& \times \int_{0}^{\eta_{\mu}} P_{l_{1}}^{m}(\cosh \eta) P_{l_{2}}^{m}(\cosh \eta) \sinh \eta \cosh ^{2} \eta d \eta \\
& -\int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta \cos ^{2} \vartheta d \vartheta \\
& \left.\times \int_{0}^{\eta_{\mu}} P_{l_{1}}^{m}(\cosh \eta) P_{l_{2}}^{m}(\cosh \eta) \sinh \eta d \eta\right] . \tag{2.1.17}
\end{align*}
$$

Furthermore, according to (1.4.13), it follows that

$$
\begin{aligned}
& \cos ^{2} \vartheta P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \\
& =\frac{\left(l_{1}+1-m\right)\left(l_{2}+1-m\right)}{\left(2 l_{1}+1\right)\left(2 l_{2}+1\right)} P_{l_{1}+1}^{m}(\cos \vartheta) P_{l_{2}+1}^{m}(\cos \vartheta) \\
& \quad+\frac{\left(l_{1}+1-m\right)\left(l_{2}+m\right)}{\left(2 l_{1}+1\right)\left(2 l_{2}+1\right)} P_{l_{1}+1}^{m}(\cos \vartheta) P_{l_{2}-1}^{m}(\cos \vartheta) \\
& \quad+\frac{\left(l_{1}+m\right)\left(l_{2}+1-m\right)}{\left(2 l_{1}+1\right)\left(2 l_{2}+1\right)} P_{l_{1}-1}^{m}(\cos \vartheta) P_{l_{2}+1}^{m}(\cos \vartheta) \\
& \quad+\frac{\left(l_{1}+m\right)\left(l_{2}+m\right)}{\left(2 l_{1}+1\right)\left(2 l_{2}+1\right)} P_{l_{1}-1}^{m}(\cos \vartheta) P_{l_{2}-1}^{m}(\cos \vartheta) .
\end{aligned}
$$

It may be found that the underlying functions are generally not orthogonal (when $\mu \neq 0$ ), neither when $l_{2}=l_{1}+2$ nor when $l_{2}=l_{1}-2$. For $l_{2}=l_{1}+2$, we obtain

$$
\begin{aligned}
& \left\langle U_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), U_{l_{1}+2, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)} \\
& =-\frac{\mu^{2 l_{1}+5} 2 \pi\left(1+\delta_{0, m}\right)}{\left(2 l_{1}+5\right)} \frac{\left(l_{1}+2+m\right)!}{\left(l_{1}+2-m\right)!}\left(\alpha_{l_{1}+2, m}\right)^{2} I_{l_{1}, m}(\mu)
\end{aligned}
$$

for $\mu>0$, where

$$
\begin{equation*}
I_{l_{1}, m}(\mu):=\int_{1}^{1 / \mu} P_{l_{1}}^{m}(t) P_{l_{1}+2}^{m}(t) d t \tag{2.1.18}
\end{equation*}
$$

The same values are obtained when we replace $\Phi_{m}^{+}(\varphi)$ by $\Phi_{m}^{-}(\varphi)$ throughout, $m>0$.

For the sake of simplicity, we will show here only that $I_{l, 0}(\mu) \neq 0$ for each $l=0,1, \ldots$ and fixed $\mu>0$. The general case $I_{l, m}(\mu)(m>0)$ can be treated analogously. Using (2.1.18), (1.4.4), and the well-known representation of the Legendre polynomials

$$
P_{l}(t)=2^{l} \sum_{k=0}^{l}\binom{l}{k}\binom{(l+k-1) / 2}{l} t^{k},
$$

we obtain

$$
\mu^{2 l+5} I_{l, 0}(\mu)=\sum_{r=0}^{l} \frac{A_{r} A_{l+2-r} A_{l-r}}{A_{2 l+2-r}}\left(\frac{4 l-4 r+5}{4 l-2 r+5}\right) \mu^{2 l+5} \int_{1}^{1 / \mu} P_{l_{r}}(t) d t,
$$

where $l_{r}=2 l+2-2 r$, and

$$
\mu^{2 l+5} \int_{1}^{1 / \mu} P_{l_{r}}(t) d t=2^{l_{r}} \sum_{k=0}^{l_{r}}\binom{l_{r}}{k}\binom{\left(l_{r}+k-1\right) / 2}{l_{r}} \mu^{2(l+2)-k}\left(1-\mu^{k+1}\right) .
$$

Note, in passing, that for the limiting case, $\mu=0$, we make use of the continuity and the asymptotic behavior of the Legendre functions of the first kind (see Proposition 2.1.2 above) to show that $\lim _{\mu \rightarrow 0} \mu^{2 l+5} I_{l, m}(\mu)=0$, for all $l=0,1, \ldots$ and therefore $\left\langle U_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), U_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}$ tends to zero. Analogously, we obtain that $\left\langle U_{l_{1}, m}[\mu] \Phi_{m}^{-}(\varphi), U_{l_{2}, m}^{m}[\mu] \Phi_{m}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)} \rightarrow$ 0 when $\mu$ tends to zero. Hence it follows that the internal solid spherical functions $U_{l, m}^{ \pm}[0]$ defined by 2.1 .10 are orthogonal with respect to the scalar product (2.1.15). Thus the collection $\left\{U_{l, m}^{ \pm}[0]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ forms an orthogonal basis of $\operatorname{Har}_{2}\left(\Omega_{0}\right)$ [292]. Moreover, by (2.1.17), we have the known result

$$
\begin{equation*}
\left\|U_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(\Omega_{0}\right)}^{2}=\frac{2 \pi\left(1+\delta_{0, m}\right)(l+m)!}{(2 l+1)(2 l+3)(l-m)!} \tag{2.1.19}
\end{equation*}
$$

From the above, we deduce the following general result:
Proposition 2.1.4. The set $\left\{U_{l, m}^{ \pm}[\mu]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ does not form an orthogonal basis of $\operatorname{Har}_{2}\left(\Omega_{\mu}\right)$, unless $\mu=0$.

As mentioned in [132], there are further orthogonality properties of the basic harmonics $U_{l, m}^{ \pm}[\mu]$, which do not depend on the shape of the prescribed spheroids. However, we make no pretense here at tabulating all possible orthogonal harmonic polynomials of this type [320] but proceed instead to apply the results to the construction of more suitable spheroidal harmonics, which are orthogonal in the usual $L_{2}$-sense.

As was before observed, it now remains to show that the functions $\widehat{U}_{l, m}^{ \pm}[\mu]$ do not form an orthogonal basis of $\operatorname{Har}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)$, unless $\mu=0$. We have then

$$
\begin{aligned}
& \left\langle\widehat{U}_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), \widehat{U}_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)} \\
& =- \\
& -\frac{\beta_{l_{1}, m} \beta_{l_{2}, m}}{\mu^{l_{1}+l_{2}-1}}\left(1+\delta_{0, m}\right) \pi \\
& \\
& \times\left[\int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta d \vartheta\right. \\
& \quad \times \int_{\eta_{\mu}}^{\infty} Q_{l_{1}}^{m}(\cosh \eta) Q_{l_{2}}^{m}(\cosh \eta) \sinh \eta \cosh ^{2} \eta d \eta \\
& \quad-\int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta \cos ^{2} \vartheta d \vartheta \\
& \left.\quad \times \int_{\eta_{\mu}}^{\infty} Q_{l_{1}}^{m}(\cosh \eta) Q_{l_{2}}^{m}(\cosh \eta) \sinh \eta d \eta\right]
\end{aligned}
$$

It may be found that, in general, the external functions are not orthogonal (when $\mu \neq 0$ ) with respect to the scalar product (2.1.15). For $l_{2}=l_{1}+2$, it
can be shown that

$$
\begin{aligned}
& \left\langle\widehat{U}_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), \widehat{U}_{l_{1}+2, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)} \\
& =-\frac{2 \pi\left(1+\delta_{0, m}\right)}{\mu^{2 l_{1}+1}\left(2 l_{1}+1\right)} \frac{\left(l_{1}+m\right)!}{\left(l_{1}-m\right)!}\left(\beta_{l_{1}, m}\right)^{2} \widehat{I}_{l_{1}, m}(\mu),
\end{aligned}
$$

where $\mu>0$, and

$$
\begin{equation*}
\widehat{I}_{l_{1}, m}(\mu):=\int_{1 / \mu}^{\infty} Q_{l_{1}}^{m}(s) Q_{l_{1}+2}^{m}(s) d s \tag{2.1.20}
\end{equation*}
$$

The same values are obtained when $\Phi_{m}^{+}(\varphi)$ is replaced by $\Phi_{m}^{-}(\varphi), m>0$.
Using (1.4.5), it can be shown that $\widehat{I}_{l_{1}, 0}(\mu) \neq 0$ for each $l_{1}=0,1, \ldots$ and fixed $\mu>0$.

When $\mu=0$, we use the continuity and the asymptotic behavior of the Legendre functions of the second kind (see Proposition 2.1.2 above) to show that $\lim _{\mu \rightarrow 0} \widehat{I}_{l, m}(\mu) / \mu^{2 l+1}=0$, for all $l=0,1, \ldots$ and therefore $\left\langle\widehat{U}_{l_{1, m}}[\mu] \Phi_{m}^{+}(\varphi), \widehat{U}_{l_{2, m}}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}$ tends to zero. Similarly, we can prove that $\left\langle\widehat{U}_{l_{1}, m}[\mu] \Phi_{m}^{-}(\varphi), \widehat{U}_{l_{2}, m}[\mu] \Phi_{m}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)} \rightarrow 0$ when $\mu$ tends to zero. It then follows that the external solid spherical functions defined by (2.1.11) are orthogonal with respect to the scalar product (2.1.15), and thus form an orthogonal basis of $\operatorname{Har}_{l}^{-}\left(\Omega_{0}\right)$ (cf. [170, Ch. IV]). Moreover, bearing in mind the orthogonal Hilbert space decomposition

$$
\operatorname{Har}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{0}\right)=\bigoplus_{l=0}^{\infty} \operatorname{Har}_{l}^{-}\left(\Omega_{0}\right)
$$

it follows that the collection $\left\{\widehat{U}_{l, m}^{ \pm}[0]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ constitutes an orthogonal basis of $\operatorname{Har}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{0}\right)$. This proves the following proposition:
Proposition 2.1.5. The set $\left\{\widehat{U}_{l, m}^{ \pm}[\mu]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ does not form an orthogonal basis of $\operatorname{Har}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)$, unless $\mu=0$.

The lack of orthogonality of the basis harmonics (2.1.6) and 2.1.7) over the interior, and exterior, of the prescribed spheroids means defining suitable orthogonal families of harmonic functions shall be handled with care. It is always possible to use an appropriate geometric weighting factor or to apply an orthogonalization process to the prescribed basis harmonics, for example, the Gram-Schmidt procedure that restores orthogonality. However, this orthogonalization process may be time-consuming and unstable. We rather discuss a constructive approach and show how it will be helpful not only from a function point of view in spheroidal domains but also for fast and stable computations.

With these observations at hand, we shall be concerned now with the proper spheroidal harmonics defined as follows:

$$
\begin{align*}
& V_{l, m}^{ \pm}[\mu](\mathbf{x})=\frac{\partial}{\partial x_{0}} U_{l+1, m}^{ \pm}[\mu](\mathbf{x}),  \tag{2.1.21}\\
& \widehat{V}_{l, m}^{ \pm}[\mu](\mathbf{x})=\frac{\partial}{\partial x_{0}} \widehat{U}_{l+1, m}^{ \pm}[\mu](\mathbf{x}) . \tag{2.1.22}
\end{align*}
$$

The variation of the indexes $l$ and $m$ in any of these functions is determined in greater detail in the next subsection. Bearing in mind that partial derivatives of harmonic functions are also harmonic, the construction above results in two new families of internal and external harmonics. They have, as will hereafter be seen, new properties.

We have an analogous result to Proposition 2.1.2 for the harmonics 2.1.21) and (2.1.22).

Proposition 2.1.6. For all $\mathbf{x} \in \mathbb{R}^{3},|\mathbf{x}| \neq 0$, the limits

$$
\lim _{\mu \rightarrow 0} V_{l, m}^{ \pm}[\mu](\mathbf{x}), \quad \lim _{\mu \rightarrow 0} \widehat{V}_{l, m}^{ \pm}[\mu](\mathbf{x})
$$

exist and are given, respectively, by

$$
V_{l, m}^{ \pm}[0](\mathbf{x})=\frac{\partial}{\partial x_{0}} U_{l+1, m}^{ \pm}[0](\mathbf{x}), \quad \widehat{V}_{l, m}^{ \pm}[0](\mathbf{x})=\frac{\partial}{\partial x_{0}} \widehat{U}_{l+1, m}^{ \pm}[0](\mathbf{x}) .
$$

In the following section, we shall accordingly proceed, after investigating general expressions for the proper harmonics (2.1.21) and (2.1.22), to prove that they form, respectively, orthogonal bases over the interior and exterior of the prescribed spheroids in the $L_{2}$-Hilbert space. Besides, we show the corresponding properties of orthogonality and completeness of the proper harmonics over the surface of the spheroids with respect to a suitable weight function.

### 2.2 Conversions among Spheroidal Harmonics

It is desirable to relate the proper spheroidal harmonics, (2.1.21) and (2.1.22) associated with one spheroid $\Omega_{\mu}$ to those for another spheroid. It is natural to use the ball $\Omega_{0}$ as a point of reference, which will be the case in the first results. We shall do it in the following manner. We will determine the
nonvanishing coefficients $\alpha, \widetilde{\alpha}$, and $\beta, \widetilde{\beta}$ of the following direct and inverse transformation formulas:

$$
\begin{array}{ll}
V_{l, m}[\mu]=\sum_{k} \alpha_{l+1, m, k} \mu^{2 k} V_{l-2 k, m}[0], & V_{l, m}[0]=\sum_{k} \widetilde{\alpha}_{l+1, m, k} \mu^{2 k} V_{l-2 k, m}[\mu], \\
\widehat{V}_{l, m}[\mu]=\sum_{k} \beta_{l+1, m, k} \mu^{2 k} \widehat{V}_{l+2 k, m}[0], & \widehat{V}_{l, m}[0]=\sum_{k} \widetilde{\beta}_{l+1, m, k} \mu^{2 k} \widehat{V}_{l+2 k, m}[\mu] .
\end{array}
$$

By referring to these expansions, we shall employ the constraints that the index $m$ is not involved in the summations, and the values of the same evenness restrict the index $k$ as a given $l$. It will then follow from symmetry considerations that the above relations will work for the " + " and " - " cases (cosines and sines) and, strikingly, for all values of $\mu$. So that the relationships between the proper internal and external harmonics in the form of linear combinations may be exhibited, it appears to be most convenient to start from already existing transformation formulas between the basic spheroidal harmonics $(2.1 .6)$ and (2.1.7), and their corresponding limiting cases, respectively, 2.1.10 and 2.1.11).

Various authors investigated harmonic series expansions in terms of spherical and spheroidal harmonics, the investigations usually resting on a more or less identical basis. For functions that are harmonic inside a prolate or an oblate spheroid, the transition from the expansion in internal spheroidal harmonics to that in internal spherical harmonics (and vice-versa) is worked out in [56] and [177], while the relation between the coefficients in the expansions in external oblate spheroidal and spherical harmonics is given in [20]. Some of these formulas are discussed thoroughly in [22]. In classic books [110, 170, 252, 258, these expansions are used separately without specifying relations between them.

Two of these fundamental formulas relevant in the sequel are reproduced in our notation below [56].

Proposition 2.2.1. Let $l \geq 0$ and $0 \leq m \leq l$. Then

$$
\begin{aligned}
& U_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l-m}{2}\right]} \alpha_{l, m, k} \mu^{2 k} U_{l-2 k, m}^{ \pm}[0], \\
& U_{l, m}^{ \pm}[0]=\sum_{k=0}^{\left[\frac{l-m}{2}\right]} \widetilde{\alpha}_{l, m, k} \mu^{2 k} U_{l-2 k, m}^{ \pm}[\mu],
\end{aligned}
$$

where as usual $[s]$ denotes the integer part of $s$, and the coefficients are given
by

$$
\begin{align*}
\alpha_{l, m, k} & =(-1)^{k} \frac{(l+m)!(2 l-1-2 k)!(l-1)!}{k!(l-1-k)!(l+m-2 k)!(2 l-1)!}  \tag{2.2.1}\\
\widetilde{\alpha}_{l, m, k} & =\frac{(l+m)!(2 l+1-4 k)!(l-k)!}{k!(l-2 k)!(2 l+1-2 k)!(l+m-2 k)!} \tag{2.2.2}
\end{align*}
$$

Direct and inverse transformation formulas between $\widehat{U}_{l, m}^{ \pm}[\mu]$ and $\widehat{U}_{l, m}^{ \pm}[0]$ can also be derived [20, 22].

Proposition 2.2.2. Let $l \geq 0$ and $0 \leq m \leq l$. Then

$$
\begin{aligned}
\widehat{U}_{l, m}^{ \pm}[\mu] & =\sum_{k=0}^{\infty} \beta_{l, m, k} \mu^{2 k} \widehat{U}_{l+2 k, m}^{ \pm}[0], \\
\widehat{U}_{l, m}^{ \pm}[0] & =\sum_{k=0}^{\infty} \widetilde{\beta}_{l, m, k} \mu^{2 k} \widehat{U}_{l+2 k, m}^{ \pm}[\mu],
\end{aligned}
$$

where the coefficients are given by

$$
\begin{align*}
& \beta_{l, m, k}=(-1)^{k} \frac{(l+2 k-m)!(l+k)!(2 l+1)!}{(l-m)!k!(2 l+1+2 k)!l!}  \tag{2.2.3}\\
& \widetilde{\beta}_{l, m, k}=\frac{(l+2 k-m)!(2 l+2 k)!(l+2 k)!}{(l-m)!k!(l+k)!(2 l+4 k)!} \tag{2.2.4}
\end{align*}
$$

Given the practical applicability of the $V_{l, m}^{ \pm}[\mu]$ and $\widehat{V}_{l, m}^{ \pm}[\mu]$, it might be interesting to know whether one can obtain similar transition formulas for these functions and their limiting configurations. Since $\partial / \partial x_{0}$ is a linear operator, we automatically have the corresponding transformation formulas for the proper spheroidal harmonics:

Corollary 2.2.3. Let $l \geq 0$ and $0 \leq m \leq l$. Then

$$
\begin{align*}
& V_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} \alpha_{l+1, m, k} \mu^{2 k} V_{l-2 k, m}^{ \pm}[0],  \tag{2.2.5}\\
& V_{l, m}^{ \pm}[0]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} \widetilde{\alpha}_{l+1, m, k} \mu^{2 k} V_{l-2 k, m}^{ \pm}[\mu], \tag{2.2.6}
\end{align*}
$$

where the constants $\alpha_{l, m, k}$ and $\widetilde{\alpha}_{l, m, k}$ are given by (2.2.1) and (2.2.2).

Corollary 2.2.4. Let $l \geq 0$ and $0 \leq m \leq l+1$. Then

$$
\begin{aligned}
& \widehat{V}_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\infty} \beta_{l+1, m, k} \mu^{2 k} \widehat{V}_{l+2 k, m}^{ \pm}[0], \\
& \widehat{V}_{l, m}^{ \pm}[0]=\sum_{k=0}^{\infty} \widetilde{\beta}_{l+1, m, k} \mu^{2 k} \widehat{V}_{l+2 k, m}^{ \pm}[\mu],
\end{aligned}
$$

where the constants $\beta_{l, m, k}$ and $\widetilde{\beta}_{l, m, k}$ are given by (2.2.3) and (2.2.4).
Following the notation already employed, we shall use $V_{l, m}^{ \pm}[\mu]=V_{l, m}[\mu] \Phi_{m}^{ \pm}$ (resp. $\widehat{V}_{l, m}^{ \pm}[\mu]=\widehat{V}_{l, m}[\mu] \Phi_{m}^{ \pm}$) when the factors $\Phi_{m}^{ \pm}$are not of interest. Before discussing the properties of the ansatz functions $V_{l, m}[\mu]$ and $\widehat{V}_{l, m}[\mu]$, it will be convenient to investigate their algebraical forms. Explicit expressions may be derived by applying different formulas involving Legendre functions of the first and second kinds, see Section 1.4. We will assume in the sequel that $\nu<0$ because the case $\nu>0$ is similar.

From differentiating (2.1.2),

$$
\frac{\partial}{\partial x_{0}}=\frac{1}{\mu\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}\left(\cos \vartheta \sinh \eta \frac{\partial}{\partial \eta}-\sin \vartheta \cosh \eta \frac{\partial}{\partial \vartheta}\right)
$$

from which the definition (2.1.21) gives

$$
\begin{aligned}
\frac{\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}{\alpha_{l+1, m} \mu^{l}} V_{l, m}[\mu]= & \cos \vartheta \sinh ^{2} \eta P_{l+1}^{m}(\cos \vartheta)\left(P_{l+1}^{m}\right)^{\prime}(\cosh \eta) \\
& +\sin ^{2} \vartheta \cosh \eta P_{l+1}^{m}(\cosh \eta)\left(P_{l+1}^{m}\right)^{\prime}(\cos \vartheta)
\end{aligned}
$$

So, by (1.4.12) and (1.4.13), it follows that

$$
\begin{align*}
V_{l, m}[\mu]= & \frac{\alpha_{l+1, m} \mu^{l}}{\cosh ^{2} \eta-\cos ^{2} \vartheta} \\
& \times\left(\operatorname { c o s h } \eta P _ { l + 1 } ^ { m } ( \operatorname { c o s h } \eta ) \left[(l+1+m) P_{l}^{m}(\cos \vartheta)\right.\right. \\
& \left.-(l+1) \cos \vartheta P_{l+1}^{m}(\cos \vartheta)\right] \\
& +\cos \vartheta P_{l+1}^{m}(\cos \vartheta)\left[(l+1) \cosh \eta P_{l+1}^{m}(\cosh \eta)\right. \\
& \left.\left.-(l+1+m) P_{l}^{m}(\cosh \eta)\right]\right) \\
= & \frac{\alpha_{l+1, m}(l+1+m) \mu^{l}}{\cosh ^{2} \eta-\cos ^{2} \vartheta}\left[\cosh \eta P_{l}^{m}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta)\right. \\
& \left.-\cos \vartheta P_{l+1}^{m}(\cos \vartheta) P_{l}^{m}(\cosh \eta)\right] \tag{2.2.7}
\end{align*}
$$

with the initial values

$$
V_{l, l}[\mu]=(2 l+1) U_{l, l}[\mu], \quad V_{l+1, l}[\mu]=2(l+1) U_{l+1, l}[\mu] .
$$

After what precedes, it seems worthy to research a similar expression for the proper functions $\widehat{V}_{l, m}[\mu]$. By (1.4.21) and (1.4.22), direct computations can be used to show that

$$
\begin{align*}
\widehat{V}_{l, m}[\mu]= & \frac{\beta_{l+1, m}(l+1+m)}{\mu^{l+3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}\left[\cosh \eta P_{l}^{m}(\cos \vartheta) Q_{l+1}^{m}(\cosh \eta)\right. \\
& \left.-\cos \vartheta P_{l+1}^{m}(\cos \vartheta) Q_{l}^{m}(\cosh \eta)\right] . \tag{2.2.8}
\end{align*}
$$

The following are some particular values:

$$
\widehat{V}_{l, l}[\mu]= \begin{cases}\frac{3}{2 \mu^{3}} \log \left(\frac{\cosh \eta+1}{\cosh \eta-1}\right)-\frac{3 \cosh \eta}{\mu^{3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)} & \text { if } l=0 \\ \frac{(2 l+1)(2 l+3)}{\mu^{2}}\left(\widehat{U}_{l, l}[\mu]-\frac{1}{(2 l-2)!!} \mu^{2} \cosh \eta \mathcal{V}_{l}[\mu]\right) & \text { if } l>0\end{cases}
$$

$$
\widehat{V}_{l+1, l}[\mu]=\left\{\begin{array}{cc}
\frac{45}{4 \mu^{3}} \cos \vartheta\left[\cosh \eta \log \left(\frac{\cosh \eta+1}{\cosh \eta-1}\right)-2\right]  \tag{2.2.9}\\
-\frac{15 \cos \vartheta}{2 \mu^{3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)} & \text { if } l=0 \\
\frac{(2 l+3)(2 l+5)}{2 \mu^{2}}\left(\widehat{U}_{l+1, l}[\mu]-\frac{2 l+1}{(2 l-2)!!} \mu \cos \vartheta \mathcal{V}_{l}[\mu]\right) & \text { if } l>0
\end{array}\right.
$$

where

$$
\begin{equation*}
\mathcal{V}_{l}[\mu]:=\frac{P_{l}^{l}(\cos \vartheta) Q_{l-1}^{l}(\cosh \eta)}{\mu^{l+3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)} \tag{2.2.10}
\end{equation*}
$$

We may observe in the first place that there is a slight difference between the two preceding proper functions $V_{l, m}[\mu]$ and $\widehat{V}_{l, m}[\mu]$. It is quite evident from the form (2.2.7) that $V_{l, l+1}[\mu]=0$ for all $l$. However, it can be seen that $\widehat{V}_{l, l+1}[\mu] \neq 0$. More specifically, we find

$$
\begin{equation*}
\widehat{V}_{l, l+1}[\mu]=-\frac{(2 l+3)}{(2 l)!!} \mu \cos \vartheta \mathcal{V}_{l+1}[\mu] . \tag{2.2.11}
\end{equation*}
$$

It follows from 2.2.8) that $\hat{V}_{l, m}[\mu]=0$ for $m \geq l+2$ since $P_{l}^{m}=0$ for $m>l$.

## 80 2. SOLUTIONS OF LAPLACE'S EQUATION IN SPHEROIDAL COORDINATES

To avoid the difficulties usually associated with manipulations such as those of the formulas (2.2.7) and (2.2.8), it will be convenient to notice elementary recurrence relations for the internal and external harmonics. The following will be the key in the proof of Theorem 3.1.10, and it is based on the results of [239].

Proposition 2.2.5. Let $l \geq 0$ and $0 \leq m \leq l$. The functions $V_{l, m}[\mu]$ satisfy the recurrence relation

$$
\begin{equation*}
\frac{1}{l+1+m} V_{l, m}[\mu]=U_{l, m}[\mu]+\frac{\mu^{2}(l+m)}{4 l^{2}-1} V_{l-2, m}[\mu] . \tag{2.2.12}
\end{equation*}
$$

This uses the convention $V_{l-2, m}[\mu]=0$ when $m>l$.
Proof. Using (2.2.7) and (1.4.12) direct computations show that

$$
\begin{aligned}
V_{l+1, m}[\mu]= & (l+1+m) U_{l, m}[\mu] \\
& +\frac{\alpha_{l, m} \mu^{l}(l+1+m)(l+m)}{\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)(2 l+1)}\left[\cos \vartheta P_{l-1}^{m}(\cos \vartheta) P_{l}^{m}(\cosh \eta)\right. \\
& \left.-\cosh \eta P_{l}^{m}(\cos \vartheta) P_{l-1}^{m}(\cosh \eta)\right],
\end{aligned}
$$

with

$$
\alpha_{l, m}=\frac{2 l+1}{l+1-m} \alpha_{l+1, m} .
$$

Using again (1.4.13), we obtain

$$
\begin{aligned}
V_{l+1, m}[\mu]= & (l+1+m) U_{l, m}[\mu] \\
& +\frac{\alpha_{l-1, m} \mu^{l}(l+1+m)(l+m)(l+m-1)}{\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)(2 l+1)(2 l-1)} \\
& \times\left[\cosh \eta P_{l-2}^{m}(\cos \vartheta) P_{l-1}^{m}(\cosh \eta)\right. \\
& \left.-\cos \vartheta P_{l-1}^{m}(\cos \vartheta) P_{l-2}^{m}(\cosh \eta)\right] .
\end{aligned}
$$

The result now follows.

Since the basic harmonics, $U_{l, m}^{ \pm}[\mu]$ of [132] are polynomials of degree $l$ it is clear that the operations of rescaling by $1 / \mu$ or $i / \mu$ and multiplying by $\mu^{l}$ implied in (2.1.6) assure that the proper harmonics $V_{l, m}^{ \pm}[\mu]$ defined by (2.1.21) are polynomials of degree $l$ in $\mu^{2}$. From the preceding discussion, it is clear that $-\mu$ produces the same results as $\mu$, so the only powers of $\mu$ that appear are even.

From (2.2.12), we note that for the internal solid spherical harmonics (2.1.10), there holds a formula analogous to Appell differentiation of monomials,

$$
\begin{equation*}
\frac{\partial}{\partial x_{0}} U_{l+1, m}^{ \pm}[0](\mathbf{x})=(l+1+m) U_{l, m}^{ \pm}[0](\mathbf{x}) \tag{2.2.13}
\end{equation*}
$$

In contrast, $V_{l, m}^{ \pm}[\mu]$ is not so related merely to $U_{l, m}^{ \pm}[\mu]$ for $\mu \neq 0$, as we proceed to show.

Accordingly, the preceding recurrence formula for $V_{l, m}[\mu]$ combined with (2.2.7) gives a representation for the functions $V_{l, m}^{ \pm}[\mu]$ in terms of the basic harmonics $U_{l, m}^{ \pm}[\mu]$, which is surprisingly simple and reasonably efficient. The theorem is as follows:

Theorem 2.2.6. Let $l \geq 0$ and $0 \leq m \leq l$. The coefficients $v_{l, m, k}$ in the relation

$$
\begin{equation*}
V_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l-m}{2}\right]} v_{l, m, k} \mu^{2 k} U_{l-2 k, m}^{ \pm}[\mu] \tag{2.2.14}
\end{equation*}
$$

are given by

$$
\begin{equation*}
v_{l, m, k}=\frac{(l+1+m)!(2 l+1-4 k)!!}{(l+m-2 k)!(2 l+1)!!} . \tag{2.2.15}
\end{equation*}
$$

Proof. Suppose inductively that the formula holds when $l$ is replaced by $l^{\prime}<l$. Then

$$
\begin{aligned}
V_{l, m}^{ \pm}[\mu]= & (l+1+m) U_{l, m}^{ \pm}[\mu] \\
& +\frac{(l+1+m)(l+m)}{(2 l+1)(2 l-1)} \sum_{k=0}^{\left[\frac{l-2-m}{2}\right]} v_{l-2, m, k} \mu^{2(k+1)} U_{l-2(k+1), m}^{ \pm}[\mu] .
\end{aligned}
$$

Since, by 2.2.15,

$$
v_{l, m, 0}=l+1+m, \quad v_{l, m, k+1}=\frac{(l+1+m)(l+m)}{4 l^{2}-1} v_{l-2, m, k},
$$

we find that the stated formula holds, completing the proof.
The above theorem is the generalization of the result given in [239] to spheroids of arbitrary eccentricity.

Corollary 2.2.3, in conjunction with (2.2.13), immediately yield the following:

Corollary 2.2.7. Let $l \geq 0$ and $0 \leq m \leq l$. Then

$$
\begin{align*}
V_{l, m}^{ \pm}[\mu] & =\sum_{k=0}^{\left[\frac{l-m}{2}\right]} \alpha_{l+1, m, k}(l+1+m-2 k) \mu^{2 k} U_{l-2 k, m}^{ \pm}[0],  \tag{2.2.16}\\
U_{l, m}^{ \pm}[0] & =\sum_{k=0}^{\left[\frac{l-m}{2}\right]} \frac{\widetilde{\alpha}_{l+1, m, k}}{l+1+m} \mu^{2 k} V_{l-2 k, m}^{ \pm}[\mu], \tag{2.2.17}
\end{align*}
$$

where the constants $\alpha_{l, m, k}$ and $\widetilde{\alpha}_{l, m, k}$ are given by (2.2.1) and (2.2.2).
Having thus proved that internal harmonics of the form (2.2.7) which satisfy (2.2.12) likewise satisfy (2.2.14), it will be interesting to know whether the external functions will fulfill similar recurrence relations employing the equation $(2.2 .8)$. In the first place, it is evident from the form of the external functions (2.2.8), by (2.2.11), that a direct substitution of our arguments is not applicable.

Proceeding in a manner analogous to that for the internal spheroidal harmonics (2.2.7), it turns out that the functions $\widehat{V}_{l, m}^{ \pm}[\mu]$ can be computed directly using a recurrence formula as in [250]:

Proposition 2.2.8. Let $l \geq 0$ and $0 \leq m \leq l+1$. The functions $\hat{V}_{l, m}[\mu]$ satisfy the recurrence relation

$$
\begin{equation*}
\frac{\mu^{2}(l+1-m)(l-m)}{(2 l+1)(2 l+3)} \widehat{V}_{l, m}[\mu]=(l-m) \widehat{U}_{l, m}[\mu]+\widehat{V}_{l-2, m}[\mu] . \tag{2.2.18}
\end{equation*}
$$

This uses the convention $\hat{V}_{l-2, m}[\mu]=0$ when $m>l+1$.
Proof. The proof is similar to the one presented for Proposition 2.2.5, and it uses (1.4.21) and (1.4.22).

The result, which follows from (2.2.18), is in analogy to (2.2.13), involving the derivatives of the external solid spherical harmonics $(2.1 .11)$ with respect to $x_{0}$ :

$$
\begin{equation*}
\frac{\partial}{\partial x_{0}} \widehat{U}_{l+1, m}^{ \pm}[0](\mathbf{x})=-(l+2-m) \widehat{U}_{l+2, m}^{ \pm}[0](\mathbf{x}) . \tag{2.2.19}
\end{equation*}
$$

In virtue of Proposition 2.2.8, we are thus led to the following representation of the proper functions $\widehat{V}_{l, m}^{ \pm}[\mu]$ in terms of the external harmonics $\widehat{U}_{l, m}^{ \pm}[\mu]$, which have a similar proof to that of Theorem 2.2.6.

Theorem 2.2.9. Let $l \geq 0$ and $0 \leq m \leq l+1$. The coefficients $\widehat{v} l, m, k$ in the relation

$$
\begin{align*}
\widehat{V}_{l, m}^{ \pm}[\mu]= & \sum_{k=0}^{\left[\frac{l-m}{2}\right]-1} \widehat{v}_{l, m, k} \frac{1}{\mu^{2(k+1)}} \widehat{U}_{l-2 k, m}^{ \pm}[\mu] \\
& + \begin{cases}\frac{(2 l+3)!!}{(l+1-m)!(2 m+3)!!} \frac{1}{\mu^{l-m}} \widehat{V}_{m, m}^{ \pm}[\mu] & \text { if } l-m \text { is even, } \\
\frac{2(2 l+3)!!}{(l+1-m)!(2 m+5)!!} \frac{1}{\mu^{l-m-1}} \widehat{V}_{m+1, m}^{ \pm}[\mu] & \text { if } l-m \text { is odd }\end{cases} \tag{2.2.20}
\end{align*}
$$

are given by

$$
\begin{equation*}
\widehat{v}_{l, m, k}=\frac{(l-m-2 k)!(2 l+3)!!}{(l+1-m)!(2 l-1-4 k)!!} . \tag{2.2.21}
\end{equation*}
$$

The proper harmonics $V_{l, m}^{ \pm}[\mu]$ and $\widehat{V}_{l, m}^{ \pm}[\mu]$ will play a crucial role in studying the internal and external spheroidal monogenics in Chapter 3 .

### 2.3 Orthogonal Families of Proper Internal and External Spheroidal Harmonics

Since the internal harmonics (2.1.21), except for the constant factors $\alpha_{l, m}$ and the rescaling of the $\mathbf{x}$ variable, are the functions defined in [132], an essential result of that paper regarding the orthogonality and completeness of these functions in the $L_{2}$-Hilbert space can be restated as follows. To make this work self-contained, we prove the orthogonality in the following calculation of the norms.

Theorem 2.3.1. For fixed $\mu$, the set $\left\{V_{l, m}^{ \pm}[\mu]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ forms an orthogonal basis of $\operatorname{Har}_{2}\left(\Omega_{\mu}\right)$ with the norms

$$
\begin{equation*}
\left\|V_{l, m}^{ \pm}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}=2 \pi\left(1+\delta_{0, m}\right) \mu^{2 l+3} \gamma_{l, m} I_{l, m}(\mu) \tag{2.3.1}
\end{equation*}
$$

where $I_{l, m}(\mu)$ is defined by (2.1.18), and

$$
\begin{equation*}
\gamma_{l, m}=\frac{(l+1+m)(l+2-m)!(l+1+m)!}{(2 l+1)!!(2 l+3)!!} . \tag{2.3.2}
\end{equation*}
$$

For the limiting case, $\mu=0$,

$$
\begin{equation*}
\left\|V_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(\Omega_{0}\right)}^{2}=\frac{2 \pi\left(1+\delta_{0, m}\right)(l+1+m)(l+1+m)!}{(2 l+1)(2 l+3)(l-m)!} \tag{2.3.3}
\end{equation*}
$$

Proof. For simplicity, we consider the prolate case, where $0<\eta \leq \eta_{\mu}$, i.e., with $\cosh \eta_{\mu}=1 / \mu$. When $m_{1} \neq m_{2}$, we have

$$
\begin{aligned}
& \left\langle V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\langle V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\langle V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\langle V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0 .
\end{aligned}
$$

By definition of the integral (2.1.15) and using (2.2.7) for $m_{1}=m_{2}=m$, a direct computation shows that

$$
\begin{aligned}
& \left\langle V_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), h\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)} \\
& =\alpha_{l_{1}+1, m}(l+1+m) \mu^{l_{1}+3} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} \int_{0}^{2 \pi} h \Phi_{m}^{+}(\varphi) \\
& \quad \times\left[\cosh \eta P_{l_{1}}^{m}(\cos \vartheta) P_{l_{1}+1}^{m}(\cosh \eta)\right. \\
& \left.\quad-\cos \vartheta P_{l_{1}+1}^{m}(\cos \vartheta) P_{l_{1}}^{m}(\cosh \eta)\right] \sinh \eta \sin \vartheta d \varphi d \vartheta d \eta .
\end{aligned}
$$

The last integral vanishes when $h$ is a harmonic polynomial of the form

$$
P_{l_{2}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cosh \eta) \Phi_{m}^{+}(\varphi)
$$

of degree $l_{2}<l_{1}$, since

$$
\int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta d \vartheta=0
$$

and

$$
\int_{0}^{\pi} P_{l_{1}+1}^{m}(\cos \vartheta) \cos \vartheta P_{l_{2}}^{m}(\cos \vartheta) \sin \vartheta d \vartheta=0 .
$$

Hence, for $l_{1} \neq l_{2}$,

$$
\left\langle V_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), V_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0,
$$

and also

$$
\left\langle V_{l_{1}, m}[\mu] \Phi_{m}^{-}(\varphi), V_{l_{2}, m}[\mu] \Phi_{m}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)}=0 .
$$

For $l_{1}=l_{2}=l$, by 2.2.12, we find

$$
\begin{aligned}
& \left\langle V_{l, m}[\mu] \Phi_{m}^{+}(\varphi), V_{l, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)} \\
& =\alpha_{l+1, m} \alpha_{l, m}(l+1+m)^{2} \mu^{2 l+3} \int_{0}^{2 \pi}\left(\Phi_{m}^{+}(\varphi)\right)^{2} d \varphi \\
& \quad \times \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} P_{l}^{m}(\cos \vartheta) P_{l}^{m}(\cosh \eta)\left[\cosh \eta P_{l}^{m}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta)\right. \\
& \left.\quad-\cos \vartheta P_{l+1}^{m}(\cos \vartheta) P_{l}^{m}(\cosh \eta)\right] \sinh \eta \sin \vartheta d \vartheta d \eta .
\end{aligned}
$$

Now, using 1.4.13) it follows that

$$
\begin{aligned}
& t P_{l}^{m}(t) P_{l+1}^{m}(t) \\
& =\frac{1}{2 l+3}\left[(l+2-m) P_{l+2}^{m}(t)+(l+1+m) P_{l}^{m}(t)\right] P_{l}^{m}(t)
\end{aligned}
$$

with $t=\cos \vartheta$ or $\cosh \eta$.
Therefore

$$
\begin{aligned}
& \left\langle V_{l, m}[\mu] \Phi_{m}^{+}(\varphi), V_{l, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\Omega_{\mu}\right)} \\
& =\alpha_{l+1, m} \alpha_{l, m} \frac{(l+2-m)(l+1+m)^{2}}{2 l+3} \mu^{2 l+3}\left(1+\delta_{0, m}\right) \pi \\
& \quad \times\left[\int_{0}^{\eta_{\mu}} P_{l+2}^{m}(\cosh \eta) P_{l}^{m}(\cosh \eta) \sinh \eta d \eta\right. \\
& \quad \times \int_{0}^{\pi}\left(P_{l}^{m}(\cos \vartheta)\right)^{2} \sin \vartheta d \vartheta \\
& \quad-\int_{0}^{\eta_{\mu}}\left(P_{l}^{m}(\cosh \eta)\right)^{2} \sinh \eta d \eta \\
& \left.\quad \times \int_{0}^{\pi} P_{l+2}^{m}(\cos \vartheta) P_{l}^{m}(\cos \vartheta) \sin \vartheta d \vartheta\right] \\
& = \\
& \left(\alpha_{l, m}\right)^{2}(l+1+m)^{2} \frac{(l+2-m)(l+m)!}{(2 l+1)(2 l+3)(l-m)!} \mu^{2 l+3}\left(1+\delta_{0, m}\right) 2 \pi \\
& \quad \times \int_{0}^{\eta_{\mu}} P_{l+2}^{m}(\cosh \eta) P_{l}^{m}(\cosh \eta) \sinh \eta d \eta .
\end{aligned}
$$

We obtain the same value when $\Phi_{m}^{+}(\varphi)$ is replaced by $\Phi_{m}^{-}(\varphi)$ throughout, $m>0$. The limiting case, $\mu=0$, follows with the use of Proposition 2.2.5 and (2.1.19).

It will be a convenient opportunity to deliver the orthogonality of the corresponding proper spheroidal polynomials over the surface of the prescribed spheroids with respect to a suitable weight function. The following theorem generalizes a similar result in [132] to spheroids of arbitrary eccentricity.

Theorem 2.3.2. For fixed $\mu$, the set $\left\{V_{l, m}^{ \pm}[\mu]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ forms an orthogonal family over the surface of the spheroid $\Omega_{\mu}$ in the sense of the scalar product

$$
\begin{equation*}
\{f, g\}_{0, L_{2}\left(\partial \Omega_{\mu}\right)}=\int_{\partial \Omega_{\mu}} f(\mathbf{x}) g(\mathbf{x})|\zeta(\mu, \mathbf{x})|^{1 / 2} d \sigma \tag{2.3.4}
\end{equation*}
$$

where $|\zeta(\mu, \mathbf{x})|$ is defined by (2.1.9).

Proof. For the proof, we use similar ideas as in Theorem 2.3.1 and consider the prolate case again. When $m_{1} \neq m_{2}$, we have

$$
\begin{aligned}
& \left\{V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\}_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\{V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\}_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\{V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\}_{0, L_{2}\left(\Omega_{\mu}\right)}=0, \\
& \left\{V_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), V_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\}_{0, L_{2}\left(\Omega_{\mu}\right)}=0 .
\end{aligned}
$$

Let $P^{*}=\left(\eta_{\mu}, \vartheta, \varphi\right)$, i.e., with $\cosh \eta_{\mu}=1 / \mu$. By a similar argument to that used in [132], a direct computation shows that

$$
\begin{aligned}
& \left\{V_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), V_{l_{2, m}}[\mu] \Phi_{m}^{+}(\varphi)\right\}_{0, L_{2}\left(\Omega_{\mu}\right)} \\
& =|\mu|^{3} \sinh \eta_{\mu} \int_{0}^{2 \pi}\left(\Phi_{m}^{+}(\varphi)\right)^{2} d \varphi \\
& \quad \times \int_{0}^{\pi} V_{l_{1}, m}[\mu]\left(P^{*}\right) V_{l_{1, m}}[\mu]\left(P^{*}\right) \sin \vartheta\left|\sin \left(\vartheta-i \eta_{\mu}\right)\right|^{2} d \vartheta
\end{aligned}
$$

so that

$$
\left|\sin \left(\vartheta-i \eta_{\mu}\right)\right|^{2}=\cosh ^{2} \eta_{\mu}-\cos ^{2} \vartheta
$$

where we have used that

$$
\begin{aligned}
\left|\zeta\left(\mu, P^{*}\right)\right|^{1 / 2} & =|\mu|\left|1-\left(\cos \vartheta \cosh \eta_{\mu}+i \sin \vartheta \sinh \eta_{\mu}\right)^{2}\right|^{1 / 2} \\
& =|\mu|\left|\sin \left(\vartheta-i \eta_{\mu}\right)\right|
\end{aligned}
$$

By the usual argument of assuming that $l_{1}>l_{2}$ in association with the facts

$$
\begin{aligned}
& \int_{0}^{\pi} P_{l_{1}}^{m}(\cos \vartheta) P_{l_{2}-2 k}^{m}(\cos \vartheta) \sin \vartheta d \vartheta=0, \\
& \int_{0}^{\pi} P_{l_{1}+1}^{m}(\cos \vartheta) \cos \vartheta P_{l_{2}-2 k}^{m}(\cos \vartheta) \sin \vartheta d \vartheta=0,
\end{aligned}
$$

we find, by (2.2.7) and (2.2.14), that

$$
\begin{aligned}
& \left\{V_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), V_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right\}_{0, L_{2}\left(\Omega_{\mu}\right)} \\
= & \frac{2 \pi\left(1+\delta_{0, m}\right)|\mu|^{2 l_{1}+3}\left(l_{1}+1+m\right)\left(l_{1}+1-m\right)\left(l_{1}+1+m\right)!}{\left(2 l_{1}+1\right)^{2}\left(l_{1}-m\right)!}\left(\alpha_{l_{1}+1, m}\right)^{2} \\
& \times \frac{\sqrt{1-\mu^{2}}}{\mu} P_{l_{1}}^{m}\left(\frac{1}{\mu}\right)\left[\frac{1}{\mu} P_{l_{1}+1}^{m}\left(\frac{1}{\mu}\right)-\frac{l_{1}+1+m}{2 l_{1}+3} P_{l_{1}}^{m}\left(\frac{1}{\mu}\right)\right] \delta_{l_{1}, l_{2}},
\end{aligned}
$$

with the same formula when $\Phi_{m}^{+}(\varphi)$ is replaced by $\Phi_{m}^{-}(\varphi), m>0$.

Using the expressions of change of basis between different systems of spherical and spheroidal harmonics calculated in the previous section, we can now verify the following general conversion formula, which relates the proper internal spheroidal harmonics associated with $\Omega_{\mu}$ to those associated with any other $\Omega_{\tilde{\mu}}$. Here the coefficients in the underlying series must depend on $\mu, \widetilde{\mu}$ in a nonpolynomial way.

Theorem 2.3.3. Let $l \geq 0,0 \leq m \leq l$ and let $\mu, \tilde{\mu} \in[0,1) \cup i \mathbb{R}^{+}$such that $\mu \neq 0$. The coefficients $w_{l, m, k}[\widetilde{\mu}, \mu]$ in the relation

$$
\begin{equation*}
V_{l, m}^{ \pm}[\widetilde{\mu}]=\sum_{k=0}^{\left[\frac{l-m}{2}\right]} w_{l, m, k}[\widetilde{\mu}, \mu] V_{l-2 k, m}^{ \pm}[\mu] \tag{2.3.5}
\end{equation*}
$$

are given by

$$
\begin{equation*}
w_{l, m, k}[\widetilde{\mu}, \mu]={ }_{2} F_{1}\left(-k,-l+k-3 / 2 ;-l-1 / 2 ;(\widetilde{\mu} / \mu)^{2}\right) \widetilde{w}_{l, m, k} \mu^{2 k}, \tag{2.3.6}
\end{equation*}
$$

where

$$
\widetilde{w}_{l, m, k}=\frac{(l+1+m)!(2 l+3-4 k)!!}{2^{k}(l+1+m-2 k)!k!(2 l+3-2 k)!!} .
$$

Proof. We begin by replacing $\mu$ by $\widetilde{\mu}$ in (2.2.17) and replacing the terms on the right-hand side according to (2.2.16). By linear independence of the harmonic basis elements, it follows that

$$
\begin{equation*}
w_{l, m, k}[\widetilde{\mu}, \mu]=\mu^{2 k} \sum_{n=0}^{k} \alpha_{l+1, m, n} \widetilde{\alpha}_{l-2 n+1, m, k-n}(\widetilde{\mu} / \mu)^{2 n} \tag{2.3.7}
\end{equation*}
$$

in which we note that all terms are real-valued. Direct computations show that

$$
\alpha_{l+1, m, n} \widetilde{\alpha}_{l-2 n+1, m, k-n}=\widetilde{w}_{l, m, k} c_{l, k, n},
$$

where

$$
c_{l, k, n}=(-1)^{n} \frac{l!k!(2 l+1-2 n)!(2 l+3-2 k)!}{n!(l-n)!(2 l+1)!(k-n)!(2 l+3-2 n-2 k)!}
$$

is the coefficient in the polynomial

$$
{ }_{2} F_{1}\left(-k,-l+k-3 / 2 ;-l-1 / 2 ;(\widetilde{\mu} / \mu)^{2}\right)=\sum_{n=0}^{k} c_{l, k, n}(\widetilde{\mu} / \mu)^{2 n} .
$$

The result now follows.

The use of the particular coefficients $w_{l, m, k}[\widetilde{\mu}, \mu]$ in 2.3 .5 is for the following.

Corollary 2.3.4. For each $l \geq 0$ and $0 \leq m \leq l$, the limits

$$
\lim _{\widetilde{\mu} \rightarrow 0} w_{l, m, k}[\widetilde{\mu}, \mu], \quad \lim _{\mu \rightarrow 0} w_{l, m, k}[\widetilde{\mu}, \mu]
$$

exist and are given, respectively, by

$$
w_{l, m, k}[0, \mu]=\widetilde{\alpha}_{l+1, m, k} \mu^{2 k}, \quad w_{l, m, k}[\widetilde{\mu}, 0]=\alpha_{l+1, m, k} \widetilde{\mu}^{2 k},
$$

where the constants $\alpha_{l, m, k}$ and $\widetilde{\alpha}_{l, m, k}$ are given by (2.2.1) and (2.2.2).
Proof. By (2.3.7), it follows that

$$
\begin{aligned}
w_{l, m, k}[\widetilde{\mu}, \mu]= & \sum_{n=1}^{k-1} \alpha_{l+1, m, n} \widetilde{\alpha}_{l-2 n+1, m, k-n} \widetilde{\mu}^{2 n} \mu^{2(k-n)} \\
& +\alpha_{l+1, m, k} \widetilde{\alpha}_{l-2 k+1, m, 0} \widetilde{\mu}^{2 k}+\alpha_{l+1, m, 0} \widetilde{\alpha}_{l+1, m, k} \mu^{2 k} .
\end{aligned}
$$

By taking $\mu=0$ or $\widetilde{\mu}=0$, we obtain the desired limits.
Corollary 2.3 .4 extends the definition of the coefficients $w_{l, m, k}[\widetilde{\mu}, \mu]$ for the case $\mu=0$, and we thus can drop the restriction $\mu \neq 0$ stated in Theorem 2.3.3

From the considerations adduced in the previous section, it becomes clear that the orthogonality of the proper external functions may be derived using the evident connections with the proper internal ones. It will be worked out thoroughly below. The main difference between the internal and external proper harmonics lies in the additional functions (2.1.6) and (2.1.7). As we could see, $V_{l, m}[\mu]$ decomposes into summands of the form

$$
P_{l-2 k}^{m}(\cos \vartheta) P_{l-2 k}^{m}(\cosh \eta) .
$$

However, $\widehat{V}_{l, m}[\mu]$ includes not only summands of the form

$$
P_{l-2 k}^{m}(\cos \vartheta) Q_{l-2 k}^{m}(\cosh \eta)
$$

but also an additional term, $\cosh \eta \mathcal{V}_{l}[\mu]$ or $\cos \vartheta \mathcal{V}_{l}[\mu]$, as the case may be, where $\mathcal{V}_{l}[\mu]$ is given by $(2.2 .10)$. Thus, it is necessary to improve our techniques of proving the orthogonality of the external functions $\widehat{V}_{l, m}^{ \pm}[\mu]$ with respect to different inner products.

At first, we formulate the following technical proposition, a result which will be useful hereafter. It expresses an orthogonal property for the ansatz functions $\hat{V}_{l, m}[\mu]$. We here borrow from the techniques used in the earlier work [250] and extend those results.

Proposition 2.3.5. Let $\widehat{V}_{l, m}[\mu]$ be defined as in (2.2.8) and let $\mu$ be fixed. The following orthogonality relations hold for all $m=0,1, \ldots$ and each pair $(l, k)$ such that $l, k \in\{m, m+1\}$,

$$
\begin{equation*}
\int_{0}^{\pi} \widehat{V}_{l, m}[\mu] P_{k}^{m}(\cos \vartheta) \sin \vartheta d \vartheta=0 \tag{2.3.8}
\end{equation*}
$$

Proof. Fix a value of $\mu$. We denote the left-hand sides of (2.3.8) by $\mathrm{C}_{\varepsilon_{1}, \varepsilon_{2}}^{m}(\mu)$ with $l=m+\varepsilon_{1}$ and $k=m+\varepsilon_{2}$. We only use pairs in the set $\{(0,0),(0,1)$, $(1,0),(1,1)\}$.

We have from 2.2.8 that

$$
\begin{aligned}
\mathrm{C}_{(1,0)}^{m}(\mu)= & \int_{0}^{\pi} \widehat{V}_{m+1, m}[\mu] P_{m}^{m}(\cos \vartheta) \sin \vartheta d \vartheta \\
= & \frac{\beta_{m+2, m}(m+1)}{\mu^{m+4}} \int_{0}^{\pi}\left[(2 m+3) P_{m+1}^{m}(\cos \vartheta) Q_{m+1}^{m}(\cosh \eta)\right. \\
& \left.-2 m \frac{P_{m+1}^{m}(\cos \vartheta) Q_{m-1}^{m}(\cosh \eta)}{\cosh ^{2} \eta-\cos ^{2} \vartheta}\right] P_{m}^{m}(\cos \vartheta) \sin \vartheta d \vartheta .
\end{aligned}
$$

We may observe that the first term gives a zero-integral because of the orthogonality (1.4.24) of the associated Legendre functions of the first kind. The second term also has a vanishing integral because the underlying function is odd with respect to the variable $t=\cos \vartheta$. Similarly, it can be proved that $\mathrm{C}_{(0,1)}^{m}(\mu)=0$.

We now consider the two remaining integrals $\mathrm{C}_{(0,0)}^{m}(\mu)$ and $\mathrm{C}_{(1,1)}^{m}(\mu)$. For simplicity, we only sketch the proof for $\mathrm{C}_{(0,0)}^{m}(\mu)$. The other can be derived straightforwardly. We proceed with the proof using induction. Let us begin by computing $\mathrm{C}_{(0,0)}^{m}(\mu)$ for the initial values $m=0,1$. As a consequence of Neumann's formula 1.4.6), we find that

$$
Q_{0}(\cosh \eta)=\frac{1}{2} \int_{0}^{\pi} \frac{\cosh \eta \sin \vartheta}{\cosh ^{2} \eta-\cos ^{2} \vartheta} d \vartheta
$$

Using the explicit representation (2.2.9) for the $\hat{V}_{0,0}[\mu]$ we obtain, as before,

$$
\begin{aligned}
\mathrm{C}_{(0,0)}^{0}(\mu) & =\int_{0}^{\pi} \widehat{V}_{0,0}[\mu] P_{0}(\cos \vartheta) \sin \vartheta d \vartheta \\
& =\frac{3}{\mu^{3}} \log \left(\frac{\cosh \eta+1}{\cosh \eta-1}\right)-\frac{6}{\mu^{3}} Q_{0}(\cosh \eta) \\
& =0 .
\end{aligned}
$$

## 90 2. SOLUTIONS OF LAPLACE'S EQUATION IN SPHEROIDAL COORDINATES

We compute

$$
\begin{aligned}
\mathrm{C}_{(0,0)}^{1}(\mu)= & \int_{0}^{\pi} \widehat{V}_{1,1}[\mu] P_{1}^{1}(\cos \vartheta) \sin \vartheta d \vartheta \\
= & \frac{15}{2 \mu^{4}} \int_{0}^{\pi}\left[3 P_{1}^{1}(\cos \vartheta) Q_{1}^{1}(\cosh \eta)\right. \\
& \left.-2 \frac{\cosh \eta P_{1}^{1}(\cos \vartheta) Q_{0}^{1}(\cosh \eta)}{\cosh ^{2} \eta-\cos ^{2} \vartheta}\right] P_{1}^{1}(\cos \vartheta) \sin \vartheta d \vartheta \\
= & \frac{15}{2 \mu^{4}}\left[4 Q_{1}^{1}(\cosh \eta)-2 \cosh \eta Q_{0}^{1}(\cosh \eta) \int_{0}^{\pi} \frac{\sin ^{3} \vartheta}{\cosh ^{2} \eta-\cos ^{2} \vartheta} d \vartheta\right] \\
= & 0,
\end{aligned}
$$

where Neumann's formula (1.4.6) and (2.2.9) were used again to obtain

$$
\int_{0}^{\pi} \frac{\sin ^{3} \vartheta}{\cosh ^{2} \eta-\cos ^{2} \vartheta} d \vartheta=2\left[\frac{Q_{0}(\cosh \eta)}{\cosh \eta}-Q_{1}(\cosh \eta)\right] .
$$

Having thus shown that $\mathrm{C}_{(0,0)}^{0}(\mu)=\mathrm{C}_{(0,0)}^{1}(\mu)=0$, it now remains to deduce the values of $\mathrm{C}_{(0,0)}^{m}(\mu)$ from the values of the corresponding integrals $\mathrm{C}_{(0,0)}^{m-1}(\mu)$ using a recurrence relation, as we proceed to show.

For an arbitrary $m>1$, we have from (2.2.8) that

$$
\begin{aligned}
\mathrm{C}_{(0,0)}^{m}(\mu)= & \int_{0}^{\pi} \widehat{V}_{m, m}[\mu] P_{m}^{m}(\cos \vartheta) \sin \vartheta d \vartheta \\
= & \frac{\beta_{m+1, m}(2 m+1)}{\mu^{m+3}} \int_{0}^{\pi}\left[(2 m+1) P_{m}^{m}(\cos \vartheta) Q_{m}^{m}(\cosh \eta)\right. \\
& \left.-2 m \frac{\cosh \eta P_{m}^{m}(\cos \vartheta) Q_{m-1}^{m}(\cosh \eta)}{\cosh ^{2} \eta-\cos ^{2} \vartheta}\right] P_{m}^{m}(\cos \vartheta) \sin \vartheta d \vartheta \\
= & \frac{\beta_{m+1, m}(2 m+1)}{\mu^{m+3}}\left[2(2 m)!Q_{m}^{m}(\cosh \eta)\right. \\
& \left.-2 m \cosh \eta Q_{m-1}^{m}(\cosh \eta) \int_{0}^{\pi} \frac{\left[P_{m}^{m}(\cos \vartheta)\right]^{2}}{\cosh ^{2} \eta-\cos ^{2} \vartheta} \sin \vartheta d \vartheta\right] .
\end{aligned}
$$

Now, we rewrite the integral part in the above formula as follows:

$$
\begin{aligned}
& \int_{0}^{\pi} \frac{\left[P_{m}^{m}(\cos \vartheta)\right]^{2}}{\cosh ^{2} \eta-\cos ^{2} \vartheta} \sin \vartheta d \vartheta \\
= & (2 m-1)^{2} \int_{0}^{\pi} \frac{\left[P_{m-1}^{m-1}(\cos \vartheta)\right]^{2}}{\cosh ^{2} \eta-\cos ^{2} \vartheta} \sin ^{3} \vartheta d \vartheta \\
= & 2(2 m-1)!-(2 m-1)^{2} \sinh ^{2} \eta \int_{0}^{\pi} \frac{\left[P_{m-1}^{m-1}(\cos \vartheta)\right]^{2}}{\cosh ^{2} \eta-\cos ^{2} \vartheta} \sin \vartheta d \vartheta .
\end{aligned}
$$

Substituting all these computations into $\mathrm{C}_{(0,0)}^{m}$, we find

$$
\begin{aligned}
& \frac{\mu^{m+3}}{\beta_{m+1, m}(2 m+1)} \mathrm{C}_{(0,0)}^{m}(\mu) \\
= & 2(2 m)!\left[Q_{m}^{m}(\cosh \eta)-\cosh \eta Q_{m-1}^{m}(\cosh \eta)\right. \\
& \left.+(2 m-1) \sinh \eta Q_{m-1}^{m-1}(\cosh \eta)\right] \\
& -\frac{\mu^{m+2}}{\beta_{m, m-1}} 2 m(2 m-1) \sinh \eta \mathrm{C}_{(0,0)}^{m-1}(\mu),
\end{aligned}
$$

with

$$
\frac{\beta_{m+1, m}}{\beta_{m, m-1}}=\frac{2 m+3}{2 m(2 m+1)} .
$$

We shall proceed as follows. According to (1.4.22) and 1.4.23, we obtain

$$
\begin{aligned}
& Q_{m}^{m}(\cosh \eta)-\cosh \eta Q_{m-1}^{m}(\cosh \eta) \\
& +(2 m-1) \sinh \eta Q_{m-1}^{m-1}(\cosh \eta)=0
\end{aligned}
$$

Therefore

$$
\mathrm{C}_{(0,0)}^{m}(\mu)=-\frac{(2 m+1)(2 m+3)}{\mu} \sinh \eta \mathrm{C}_{(0,0)}^{m-1}(\mu) .
$$

This is an inductive formula associated with the initial values $\mathrm{C}_{(0,0)}^{0}(\mu)=$ $\mathrm{C}_{(0,0)}^{1}(\mu)=0$. It yields that $\mathrm{C}_{(0,0)}^{m}(\mu)=0$ for all $m=0,1, \ldots$ This completes the proof.

According to the previous result, we can formulate two main theorems about the orthogonality of the external functions $\widehat{V}_{l, m}^{ \pm}[\mu]$ over the exterior and surface of the prescribed spheroids. The theorems are as follows:

Theorem 2.3.6. For fixed $\mu$, the set $\left\{\widehat{V}_{l, m}^{ \pm}[\mu]: m=0, \ldots, l+1 ; l=0,1, \ldots\right\}$ forms an orthogonal family of $\operatorname{Har}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)$ with the norms

$$
\begin{equation*}
\left\|\widehat{V}_{l, m}^{ \pm}[\mu]\right\|_{\left.L_{2}\left(\mathbb{R}^{3}\right) \bar{\Omega}_{\mu}\right)}^{2}=\frac{2 \pi\left(1+\delta_{0, m}\right)}{\mu^{2 l+3}} \widehat{\gamma}_{l, m} \widehat{I}_{l, m}(\mu) \tag{2.3.9}
\end{equation*}
$$

where $\widehat{I}_{l, m}(\mu)$ is defined by 2.1.20, and

$$
\begin{equation*}
\widehat{\gamma}_{l, m}=\frac{(2 l+1)!!(2 l+3)!!(l+2-m)}{(l+m)!(l+1-m)!} . \tag{2.3.10}
\end{equation*}
$$

Proof. We will assume that $\nu<0$ because the case $\nu>0$ is similar. When $m_{1} \neq m_{2}$, we have

$$
\begin{aligned}
& \left\langle\widehat{V}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), \widehat{V}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}=0, \\
& \left\langle\widehat{V}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), \widehat{V}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}=0, \\
& \left\langle\widehat{V}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{+}(\varphi), \widehat{V}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{-}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}=0, \\
& \left\langle\widehat{V}_{l_{1}, m_{1}}[\mu] \Phi_{m_{1}}^{-}(\varphi), \widehat{V}_{l_{2}, m_{2}}[\mu] \Phi_{m_{2}}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}=0 .
\end{aligned}
$$

Using (2.1.15), we obtain

$$
\begin{aligned}
& \left\langle\widehat{V}_{l_{1}, m}[\mu] \Phi_{m}^{+}(\varphi), \widehat{V}_{l_{2}, m}[\mu] \Phi_{m}^{+}(\varphi)\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)} \\
= & \left(1+\delta_{0, m}\right) \pi \int_{0}^{\pi} \int_{\eta_{\mu}}^{\infty} \widehat{V}_{l_{1}, m}[\mu] \widehat{V}_{l_{2}, m}[\mu] d R,
\end{aligned}
$$

where $d R=\mu^{3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right) \sinh \eta \sin \vartheta d \eta d \vartheta$.
Thus we study integrals of the form

$$
\int_{0}^{\pi} \widehat{V}_{l_{1}, m}[\mu] \widehat{V}_{l_{2}, m}[\mu]\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right) \sin \vartheta d \vartheta
$$

Without loss of generality, we assume in the first instance that $l_{1}>l_{2}$ and proceed to set $\widehat{V}_{l_{1}, m}[\mu]$ as (2.2.8) and $\widehat{V}_{l_{2}, m}[\mu]$ as (2.2.20). Thus, following Theorem 2.2.9, it follows that the remaining nonvanishing integrals are, respectively,

$$
\int_{0}^{\pi} \widehat{V}_{l_{1, m}}[\mu] P_{m}^{m}(\cos \vartheta) \sin \vartheta d \vartheta
$$

according to $l_{2}-m$ being even, or

$$
\int_{0}^{\pi} \widehat{V}_{l_{1}, m}[\mu] P_{m+1}^{m}(\cos \vartheta) \sin \vartheta d \vartheta
$$

according to $l_{2}-m$ being odd.
Using representation 2.2 .20 for $\widehat{V}_{l_{1}, m}[\mu]$ again, we are led towards the integrals of the form, as stated in Proposition 2.3.5. Therefore, the remaining integrals vanish, so the orthogonality property follows. The rest of the proof is similar to that for Theorem 2.3.1.

Theorem 2.3.7. For fixed $\mu$, the set $\left\{\widehat{V}_{l, m}^{ \pm}[\mu]: m=0, \ldots, l+1 ; l=0,1, \ldots\right\}$ forms an orthogonal family over the surface of the spheroid $\Omega_{\mu}$ in the sense of the scalar product (2.3.4).

Proof. The proof goes along the same lines as those of Theorems 2.3.2 and 2.3.6, and hence, we omit the details.

## 3

## Families of Monogenic and Contragenic Functions on Spheroidal Domains

In the first part of this chapter, two distinct single one-parameter families of internal and external spheroidal monogenics are calculated. New explicit formulas for their nonscalar parts are obtained in terms of the proper spheroidal harmonics. Consequently, two orthogonal bases of monogenic functions are constructed for the interior and exterior of a spheroid. It is further shown that the orthogonality of the spheroidal monogenics in question does not depend on the eccentricity of the spheroids. Conversion formulas are obtained that relate different spheroidal monogenic systems using expressions of change of basis calculated in the previous chapter. An application of the theoretical material is made to determine the Bergman kernel function for the space of monogenic and square-integrable functions defined in a spheroid. Also, we provide plot simulations that demonstrate the effectiveness of this approach.

The second part of the chapter is focused on constructing a basis for the space of functions obtained by summing a monogenic function with an antimonogenic function. Also, we give an explicit construction of a graded basis for the space of square-integrable contragenic functions. Then we investigate the relations between the contragenic function systems for spheroids of different eccentricities. This produces the notion of "universal spheroidal contragenic function." For simplicity's sake, we have confined our discussion in this part of the chapter to the case of the region inside a spheroid. Still, it is worth mentioning that the results can be extended to the region outside a spheroid.

### 3.1 Orthogonal Bases for Monogenics in Distinct Spheroids

We have hitherto discussed those solutions of the Laplace equation, or in other words, those solutions of the equivalent equation in prolate spheroidal coordinates. In the present section, we propose to consider families of internal and external spheroidal monogenics, for which new explicit formulas for their nonscalar parts are obtained in terms of the proper spheroidal harmonics (2.1.21) and (2.1.22). We also relate the families of monogenic functions associated with a spheroid $\Omega_{\mu}$ to those defined in another spheroid via computational formulas. Moreover, we prove that these families are orthogonal with respect to the scalar $L_{2}$-inner product over the interior and exterior of a spheroidal domain. Additionally, we show the corresponding orthogonality of these families over the surface of the prescribed spheroids with respect to a suitable weight function. Although the spheroidal monogenics are nonhomogeneous, this will not affect the completeness of the obtained systems.

### 3.1.1 Internal Monogenic Spheroidal Polynomials

A basis of polynomials spanning the square-integrable solutions of $\bar{\partial} \mathbf{f}=0$ was given in [239, 242] for prolate spheroids and another in [256] (cf. [248]) for oblate spheroids, via explicit formulas. Note that the preceding prolate and oblate spheroidal monogenics can be obtained as a particular case of the present theory by appropriate interpretation. In the following, we consider the prolate and oblate cases of spheroids simultaneously.

Definition 3.1.1. Let $l \geq 0$ and $0 \leq m \leq l+1$. The basic internal monogenic spheroidal polynomials of degree $l$ and order $m$ are

$$
\begin{equation*}
\mathbf{X}_{l, m}^{ \pm}[\mu]=\partial\left(U_{l+1, m}^{ \pm}[\mu]\right), \tag{3.1.1}
\end{equation*}
$$

where the $U_{l, m}^{ \pm}[\mu]$ are defined by 2.1.6).
The prescribed polynomials $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are indeed monogenic since $U_{l+1, m}^{ \pm}[\mu]$ are harmonic, given the factorization (1.3.7) of the Laplacian. As derivatives of polynomials, they are also polynomials in $x_{0}, x_{1}, x_{2}$. It can be further shown that $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are (up to rescaling) the same polynomials defined in [239]; we will not need this fact here.

As shown in [239], the following expression will be essential in constructing the basic spheroidal monogenics. It allows us to define the zero-order monogenic polynomials.

Lemma 3.1.2. For each $l \geq 0$,

$$
\begin{equation*}
V_{l,-1}[\mu]=-\frac{1}{(l+1)(l+2)} V_{l, 1}[\mu] . \tag{3.1.2}
\end{equation*}
$$

Proof. To prove this, we use representation (2.2.7). According to (1.4.7), we find that

$$
\begin{aligned}
V_{l,-1}[\mu] & =-\frac{\alpha_{l+1,-1}}{(l+1)^{2}(l+2)^{2} \alpha_{l+1,1}} V_{l, 1}[\mu] \\
& =-\frac{1}{(l+1)(l+2)} V_{l, 1}[\mu] .
\end{aligned}
$$

The result follows.

By the lemma just proved, and by (3.1.1), we obtain the further result:

## Lemma 3.1.3.

$$
\mathbf{X}_{l,-1}^{ \pm}[\mu]= \begin{cases}\mp \frac{1}{(l+1)(l+2)} \mathbf{X}_{l, 1}^{ \pm}[\mu] & \text { if } l=1,2, \ldots  \tag{3.1.3}\\ 0 & \text { if } l=0\end{cases}
$$

Now we will work out explicit expressions for the basic spheroidal monogenics in terms of the orthogonal basis of proper harmonic polynomials, first published in [239]. It was noted in [240] that $\operatorname{Sc}\left(\mathbf{X}_{l, m}^{ \pm}[0]\right)$ is equal to $V_{l, m}^{ \pm}[0]=(l+1+m) U_{l, m}^{ \pm}[0]$, and an explicit expression for the vector part of $\mathbf{X}_{l, m}^{ \pm}[0]$ was written out. In this way, the following theorem may be considered as a generalization of the corresponding result of [240] to spheroidal domains of arbitrary eccentricity.

Theorem 3.1.4. For each $l \geq 0$ and $0 \leq m \leq l+1$, the basic internal monogenic spheroidal polynomials (3.1.1) are equal to

$$
\begin{align*}
\mathbf{X}_{l, m}^{ \pm}[\mu]= & V_{l, m}^{ \pm}[\mu]+\frac{\mathbf{i}}{2}\left[(l+1+m) V_{l, m-1}^{ \pm}[\mu]-\frac{1}{l+2+m} V_{l, m+1}^{ \pm}[\mu]\right] \\
& \mp \frac{\mathbf{j}}{2}\left[(l+1+m) V_{l, m-1}^{\mp}[\mu]+\frac{1}{l+2+m} V_{l, m+1}^{\mp}[\mu]\right] \tag{3.1.4}
\end{align*}
$$

where the $V_{l, m}^{ \pm}[\mu]$ are defined by (2.2.7). Further, the $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are polynomials in $\mu^{2}$.

Proof. The operator (1.3.2) in spheroidal coordinates 2.1.2) is

$$
\begin{aligned}
\partial= & \frac{1}{\mu\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}\left(\cos \vartheta \sinh \eta \frac{\partial}{\partial \eta}-\sin \vartheta \cosh \eta \frac{\partial}{\partial \vartheta}\right) \\
& -\frac{1}{\mu\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi)\left(\sin \vartheta \cosh \eta \frac{\partial}{\partial \eta}+\cos \vartheta \sinh \eta \frac{\partial}{\partial \vartheta}\right) \\
& -\frac{1}{\mu \sin \vartheta \sinh \eta}(-\mathbf{i} \sin \varphi+\mathbf{j} \cos \varphi) \frac{\partial}{\partial \varphi} .
\end{aligned}
$$

The first line of this expression applied to $U_{l+1, m}^{ \pm}[\mu]$ produces the scalar part of $\mathbf{X}_{l, m}^{ \pm}[\mu]$ in (3.1.4). For the nonscalar part, we use the relations (1.4.14), (1.4.16), 1.4.18) and 1.4.20) to obtain

$$
\begin{aligned}
& \frac{2}{\mu^{l+1} \alpha_{l+1, m} \Phi_{m}^{ \pm}}\left(\cos \vartheta \sinh \eta \frac{\partial}{\partial \vartheta}+\sin \vartheta \cosh \eta \frac{\partial}{\partial \eta}\right) U_{l+1, m}^{ \pm}[\mu] \\
= & (l+1+m)(l+2-m)\left[\sin \vartheta \cosh \eta P_{l+1}^{m}(\cos \vartheta) P_{l+1}^{m-1}(\cosh \eta)\right. \\
& \left.-\cos \vartheta \sinh \eta P_{l+1}^{m-1}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta)\right] \\
& +\sin \vartheta \cosh \eta P_{l+1}^{m}(\cos \vartheta) P_{l+1}^{m+1}(\cosh \eta) \\
& +\cos \vartheta \sinh \eta P_{l+1}^{m+1}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta) .
\end{aligned}
$$

Next, we use the relation (1.4.17) (valid for $|t|<1$, and replacing $1-t^{2}$ with $t^{2}-1$ for $\left.|t|>1\right)$ producing

$$
\begin{aligned}
-\frac{\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}{\mu^{l} \alpha_{l+1, m-1}} V_{l, m-1}[\mu]= & \sin \vartheta \cosh \eta P_{l+1}^{m}(\cos \vartheta) P_{l+1}^{m-1}(\cosh \eta) \\
& -\cos \vartheta \sinh \eta P_{l+1}^{m-1}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta) .
\end{aligned}
$$

Furthermore, using the expression 1.4.15 and its counterpart 1.4.19) for $|t|>1$, and then applying (1.4.13), we arrive at

$$
\begin{aligned}
& \cosh \eta \sin \vartheta P_{l+1}^{m}(\cos \vartheta) P_{l+1}^{m+1}(\cosh \eta) \\
& +\sinh \eta \cos \vartheta P_{l+1}^{m+1}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta) \\
= & \frac{\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}{(l+1-m)(l+2+m) \mu^{l} \alpha_{l+1, m+1}} V_{l, m+1}[\mu] .
\end{aligned}
$$

With these calculations at hand, we have

$$
\begin{aligned}
& -\frac{1}{\mu\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)}\left(\sin \vartheta \cosh \eta \frac{\partial}{\partial \eta}+\cos \vartheta \sinh \eta \frac{\partial}{\partial \vartheta}\right) U_{l+1, m}^{ \pm}[\mu] \\
= & \frac{(l+1+m)}{2} V_{l, m-1}[\mu] \Phi_{m}^{ \pm}-\frac{1}{2(l+2+m)} V_{l, m+1}[\mu] \Phi_{m}^{ \pm} .
\end{aligned}
$$

Similarly, one can prove that

$$
\begin{aligned}
& \frac{1}{\sin \vartheta \sinh \eta} \frac{\partial}{\partial \varphi} U_{l+1, m}^{ \pm}[\mu] \\
= & \mp \frac{m \mu^{l+1} \alpha_{l+1, m}}{\cosh ^{2} \eta-\cos ^{2} \vartheta} \Phi_{m}^{\mp} \\
& \times\left[\frac{\sinh \eta P_{l+1}^{m+1}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta)}{\sin \vartheta}+\frac{\sin \vartheta P_{l+1}^{m+1}(\cos \vartheta) P_{l+1}^{m}(\cosh \eta)}{\sinh \eta}\right] \\
= & \pm \frac{\mu}{2}\left[\frac{1}{l+2+m} V_{l, m+1}[\mu]+(l+1+m) V_{l, m-1}[\mu]\right] \Phi_{m}^{\mp} .
\end{aligned}
$$

Combining these three formulas, one straightforward obtains the desired expressions for $\left(\partial / \partial x_{1}\right) U_{l+1, m}^{ \pm}[\mu]$ and $\left(\partial / \partial x_{2}\right) U_{l+1, m}^{ \pm}[\mu]$. It turns out by the uniqueness of the representation of expression (2.2.16) that $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are polynomials in $\mu^{2}$.

Some examples of (3.1.1) in low degree provided by (3.1.4) are exhibited in Tables 3.1 and 3.2 .

It is evident that, by Proposition 2.2.5 and Theorem 3.1.4, we may find explicit expressions for the internal solid spherical monogenics $\mathbf{X}_{l, m}^{ \pm}[0]$ in terms of the basic solid spherical harmonics $U_{l, m}^{ \pm}[0]$ [240]. Applications of these functions are detailed in Chapter 5 .

Corollary 3.1.5. For all $\mathbf{x} \in \mathbb{R}^{3}$, the limit $\lim _{\mu \rightarrow 0} \mathbf{X}_{l, m}^{ \pm}[\mu](\mathbf{x})$ exists and is given by

$$
\begin{align*}
\mathbf{X}_{l, m}^{ \pm}[0](\mathbf{x})= & (l+1+m) U_{l, m}^{ \pm}[0](\mathbf{x}) \\
& +\frac{\mathbf{i}}{2}\left[(l+m)(l+1+m) U_{l, m-1}^{ \pm}[0](\mathbf{x})-U_{l, m+1}^{ \pm}[0](\mathbf{x})\right] \\
& \mp \frac{\mathbf{j}}{2}\left[(l+m)(l+1+m) U_{l, m-1}^{\mp}[0](\mathbf{x})+U_{l, m+1}^{\mp}[0](\mathbf{x})\right], \tag{3.1.5}
\end{align*}
$$

where the $U_{l, m}^{ \pm}[0]$ are defined by (2.1.10).
For general orientation, the reader is urged to read some of the existing works where the internal spherical monogenics emerged [63, [64, 65]. It is worth mentioning that at the time of the publications [64, 65, 67], closed-form representations corresponding to $\mathbf{X}_{l, m}^{ \pm}[0]$ in terms of the basic internal solid spherical harmonics (2.1.10), initially stated in [240], were not at disposal for the investigation of some basic properties of these functions [240]. They played a fundamental role in [149, 150, 151, 236, 240, 241] in

| $l$ | $m$ | $\mathbf{X}_{l, m}^{ \pm}[\mu]$ |
| :---: | :---: | :---: |
| 0 | 0 | $\mathbf{X}_{0,0}^{+}=1$ |
|  | 1 | $\begin{aligned} & \mathbf{X}_{0,1}^{+}=\mathbf{i} \\ & \mathbf{X}_{0,1}^{-}=\mathbf{j} \end{aligned}$ |
| 1 | 0 | $\mathbf{X}_{1,0}^{+}=2 x_{0}+\mathbf{i} x_{1}+\mathbf{j} x_{2}$ |
|  | 1 | $\begin{aligned} & \mathbf{X}_{1,1}^{+}=-3 x_{1}+3 \mathbf{i} x_{0} \\ & \mathbf{X}_{1,1}^{-}=-3 x_{2}+3 \mathbf{j} x_{0} \end{aligned}$ |
|  | 2 | $\begin{aligned} & \mathbf{X}_{1,2}^{+}=-6 \mathbf{i} x_{1}+6 \mathbf{j} x_{2} \\ & \mathbf{X}_{1,2}^{-}=-6 \mathbf{i} x_{2}-6 \mathbf{j} x_{1} \end{aligned}$ |
| 2 | 0 | $\mathbf{X}_{2,0}^{+}=3\left(x_{0}^{2}-\frac{x_{1}^{2}}{2}-\frac{x_{2}^{2}}{2}-\frac{\mu^{2}}{5}\right)+3 \mathbf{i} x_{0} x_{1}+3 \mathbf{j} x_{0} x_{2}$ |
|  | 1 | $\begin{aligned} & \mathbf{X}_{2,1}^{+}=-12 x_{0} x_{1}+3 \mathbf{i}\left(2 x_{0}^{2}-\frac{3 x_{1}^{2}}{2}-\frac{x_{2}^{2}}{2}-\frac{2 \mu^{2}}{5}\right)-3 \mathbf{j} x_{1} x_{2} \\ & \mathbf{X}_{2,1}^{-}=-12 x_{0} x_{2}-3 \mathbf{i} x_{1} x_{2}+3 \mathbf{j}\left(2 x_{0}^{2}-\frac{x_{1}^{2}}{2}-\frac{3 x_{2}^{2}}{2}-\frac{2 \mu^{2}}{5}\right) \end{aligned}$ |
|  | 2 | $\begin{aligned} & \mathbf{X}_{2,2}^{+}=15\left(x_{1}^{2}-x_{2}^{2}\right)-30 \mathbf{i} x_{0} x_{1}+30 \mathbf{j} x_{0} x_{2} \\ & \mathbf{X}_{2,2}^{-}=30 x_{1} x_{2}-30 \mathbf{i} x_{0} x_{2}-30 \mathbf{j} x_{0} x_{1} \end{aligned}$ |
|  | 3 | $\begin{aligned} & \mathbf{X}_{2,3}^{+}=45 \mathbf{i}\left(x_{1}^{2}-x_{2}^{2}\right)-90 \mathbf{j} x_{1} x_{2} \\ & \mathbf{X}_{2,3}^{-}=90 \mathbf{i} x_{1} x_{2}+45 \mathbf{j}\left(x_{1}^{2}-x_{2}^{2}\right) \end{aligned}$ |

Table 3.1: Basic spheroidal monogenic polynomials of degree $l=0,1,2$, parametrized by the eccentricity $\mu$.
the study of quaternionic counterparts of the well-known Bohr Theorem, Borel-Carathéodory's Theorem, and Hadamard's Real-Part Theorems on the majorant of a Taylor series, as well as Bloch's Theorem, where they were investigated in detail.

In a different context, orthogonal Appell bases (with respect to the hypercomplex derivative) of monogenic polynomials were constructed in [38, 39] (cf. [40, 41] and [238]) using systems of $\mathbb{H}$-valued internal solid spherical monogenics, which are orthogonal with respect to the quaternionic inner product (1.2.1). These bases were rediscovered in [203] and [42, 204] using a different algebraic approach based on Gelfand-Tsetlin schemes. We shall not enter into a discussion of the properties of the Gelfand-Tsetlin bases in dimension 3. They will be found very completely treated by Bock et al. [42]. We will, however, as explained later in this chapter, show that the prescribed bases can also be generated using the previous functions (3.1.5) (see Theorem 3.1.20 below).

| $l$ | $m$ | $\mathbf{X}_{l, m}^{ \pm}[\mu]$ |
| :---: | :---: | :---: |
| 3 | 0 | $\begin{aligned} \mathbf{X}_{3,0}^{+}= & 2 x_{0}\left(2 x_{0}^{2}-3 x_{1}^{2}-3 x_{2}^{2}-\frac{6 \mu^{2}}{7}\right) \\ & +3 \mathbf{i} x_{1}\left(2 x_{0}^{2}-\frac{x_{1}^{2}}{2}-\frac{x_{2}^{2}}{2}-\frac{2 \mu^{2}}{7}\right) \\ & +3 \mathbf{j} x_{2}\left(2 x_{0}^{2}-\frac{x_{1}^{2}}{2}-\frac{x_{2}^{2}}{2}-\frac{2 \mu^{2}}{7}\right) \end{aligned}$ |
|  | 1 | $\begin{aligned} \mathbf{X}_{3,1}^{+}= & 15 x_{1}\left(-2 x_{0}^{2}+\frac{x_{1}^{2}}{2}+\frac{x_{2}^{2}}{2}+\frac{2 \mu^{2}}{7}\right) \\ & +5 \mathbf{i} x_{0}\left(2 x_{0}^{2}-\frac{9 x_{1}^{2}}{2}-\frac{3 x_{2}^{2}}{2}-\frac{6 \mu^{2}}{7}\right)-15 \mathbf{j} x_{0} x_{1} x_{2} \\ \mathbf{X}_{3,1}^{-}= & 15 x_{2}\left(-2 x_{0}^{2}+\frac{x_{1}^{2}}{2}+\frac{x_{2}^{2}}{2}+\frac{2 \mu^{2}}{7}\right) \\ & -15 \mathbf{i} x_{0} x_{1} x_{2}+5 \mathbf{j} x_{0}\left(2 x_{0}^{2}-\frac{3 x_{1}^{2}}{2}-\frac{9 x_{2}^{2}}{2}-\frac{6 \mu^{2}}{7}\right) \end{aligned}$ |
|  | 2 | $\begin{aligned} \mathbf{X}_{3,2}^{+}= & 90 x_{0}\left(x_{1}^{2}-x_{2}^{2}\right)+30 \mathbf{i} x_{1}\left(-3 x_{0}^{2}+x_{1}^{2}+\frac{3 \mu^{2}}{7}\right) \\ & +30 \mathbf{j} x_{2}\left(3 x_{0}^{2}-x_{2}^{2}-\frac{3 \mu^{2}}{7}\right) \\ \mathbf{X}_{3,2}^{-}= & 180 x_{0} x_{1} x_{2}+15 \mathbf{i} x_{2}\left(-6 x_{0}^{2}+3 x_{1}^{2}+x_{2}^{2}+\frac{6 \mu^{2}}{7}\right) \\ & +15 \mathbf{j} x_{1}\left(-6 x_{0}^{2}+x_{1}^{2}+3 x_{2}^{2}+\frac{6 \mu^{2}}{7}\right) \end{aligned}$ |
|  | 3 | $\begin{aligned} & \mathbf{X}_{3,3}^{+}=105 x_{1}\left(-x_{1}^{2}+3 x_{2}^{2}\right)+315 \mathbf{i} x_{0}\left(x_{1}^{2}-x_{2}^{2}\right)-630 \mathbf{j} x_{0} x_{1} x_{2} \\ & \mathbf{X}_{3,3}^{-}=105 x_{2}\left(-3 x_{1}^{2}+x_{2}^{2}\right)+630 \mathbf{i} x_{0} x_{1} x_{2}+315 \mathbf{j} x_{0}\left(x_{1}^{2}-x_{2}^{2}\right) \end{aligned}$ |
|  | 4 | $\begin{aligned} & \mathbf{X}_{3,4}^{+}=420 \mathbf{i} x_{1}\left(-x_{1}^{2}+3 x_{2}^{2}\right)+420 \mathbf{j} x_{2}\left(3 x_{1}^{2}-x_{2}^{2}\right) \\ & \mathbf{X}_{3,4}^{-}=420 \mathbf{i} x_{2}\left(-3 x_{1}^{2}+x_{2}^{2}\right)+420 \mathbf{j} x_{1}\left(-x_{1}^{2}+3 x_{2}^{2}\right) \end{aligned}$ |

Table 3.2: Basic spheroidal monogenic polynomials of degree $l=3$, parametrized by the eccentricity $\mu$.

To motivate the relevance of the expressions stated above, explicit recurrence rules between the basic spheroidal monogenic polynomials (3.1.4) are discussed in the following.

Proposition 3.1.6. For each $l \geq 0$ and $0 \leq m \leq l+1$, the basic polynomials (3.1.1) satisfy the recursive formula

$$
\begin{align*}
\mathbf{X}_{l, m}^{ \pm}[\mu]= & \frac{(l+1+m)}{2}\left(\mathbf{X}_{l, m-1}^{ \pm}[\mu] \mathbf{i} \mp \mathbf{X}_{l, m-1}^{\mp}[\mu] \mathbf{j}\right) \\
& -\frac{1}{2(l+2+m)}\left(\mathbf{X}_{l, m+1}^{ \pm}[\mu] \mathbf{i} \pm \mathbf{X}_{l, m+1}^{\mp}[\mu] \mathbf{j}\right) . \tag{3.1.6}
\end{align*}
$$

Proof. The proof is an immediate consequence of Theorem 3.1.4 by direct inspection of the relations between the quaternionic components of the basic polynomials.

Eq. (3.1.6) requires knowledge of polynomials of a fixed degree $l$ with orders $m-1$ and $m+1$. These polynomials in their turn depend, respectively, upon polynomials of degree $l$, with orders $m-2$ and $m$. Thus, only a recursion over $m$ from smallest to largest is needed. Under these conditions, a recursive scheme based on this equation should find the backward and the basic forward polynomials for a fixed degree with various orders up to $m+1$. In particular, it is easy to find the values of $\mathbf{X}_{0,1}^{+}[\mu]$ and $\mathbf{X}_{0,1}^{-}[\mu]$, directly from the initial value $\mathbf{X}_{0,0}^{+}[\mu]$, expressed as

$$
\begin{equation*}
\mathbf{X}_{0,0}^{+}[\mu]=1=-\mathbf{X}_{0,1}^{+}[\mu] \mathbf{i}=-\mathbf{X}_{0,1}^{-}[\mu] \mathbf{j} . \tag{3.1.7}
\end{equation*}
$$

From this we easily deduce the further result:
Corollary 3.1.7. For each $l \geq 0$ and $0 \leq m \leq l+1$, the basic polynomials (3.1.1) satisfy the recurrence formula

$$
(l+1+m)\left(\mathbf{X}_{l, m-1}^{+}[\mu]-\mathbf{X}_{l, m-1}^{-}[\mu] \mathbf{k}\right)+\mathbf{X}_{l, m}^{+}[\mu] \mathbf{i}+\mathbf{X}_{l, m}^{-}[\mu] \mathbf{j}=0,
$$

with the starting value (3.1.7).
However, the computation of representations (3.1.4) and (3.1.5) would be somewhat laborious; we proceed, therefore, to investigate more convenient expressions. Given Theorem 2.3.3, it is natural to ask whether it would be possible to express the basic spheroidal monogenics associated with $\Omega_{\mu}$ to those associated with $\Omega_{\widetilde{\mu}}$. The following conversion formula can be obtained as a direct consequence of the fact that by Theorem 2.3.3, the matrix $\left(w_{l, m, k}[\mu, \widetilde{\mu}]\right)_{l, k}$ is essentially the product of $\left(\alpha_{l+1, m, k} \widetilde{\mu}^{2 k}\right)_{l, k}$ and the inverse of $\left(\widetilde{\alpha}_{l+1, m, k} \mu^{2 k}\right)_{l, k}$.
Theorem 3.1.8. Let $l \geq 0,0 \leq m \leq l+1$ and let $\mu, \widetilde{\mu} \in[0,1) \cup i \mathbb{R}^{+}$. Then

$$
\begin{equation*}
\mathbf{X}_{l, m}^{ \pm}[\widetilde{\mu}]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} w_{l, m, k}[\widetilde{\mu}, \mu] \mathbf{X}_{l-2 k, m}^{ \pm}[\mu] \tag{3.1.8}
\end{equation*}
$$

where the $w_{l, m, k}[\widetilde{\mu}, \mu]$ are given by (2.3.6.
It is of interest to remark at this point that the basic antimonogenic polynomials satisfy the same relation,

$$
\overline{\mathbf{X}}_{l, m}^{ \pm}[\tilde{\mu}]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} w_{l, m, k}[\tilde{\mu}, \mu] \overline{\mathbf{X}}_{l-2 k, m}^{ \pm}[\mu] .
$$

As a consequence of the above theorem and Corollary 2.3.4, we have the following result:

Corollary 3.1.9. Let $l \geq 0$ and $0 \leq m \leq l+1$. Then

$$
\begin{align*}
& \mathbf{X}_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} \alpha_{l+1, m, k} \mu^{2 k} \mathbf{X}_{l-2 k, m}^{ \pm}[0],  \tag{3.1.9}\\
& \mathbf{X}_{l, m}^{ \pm}[0]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} \widetilde{\alpha}_{l+1, m, k} \mu^{2 k} \mathbf{X}_{l-2 k, m}^{ \pm}[\mu],
\end{align*}
$$

where the constants $\alpha_{l, m, k}$ and $\widetilde{\alpha}_{l, m, k}$ are given by (2.2.1) and (2.2.2).
It readily follows from expression 3.1.9 that for fixed $\mu \neq 0$, the polynomials $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are generally not homogeneous.

In Subsection 3.2.3, we extend the above formulas to include the contragenic functions, which are those harmonic functions orthogonal to the monogenic functions and the antimonogenic functions in the domain under consideration.

The following theorem, more general than that of [239], includes the latter as a particular case. It addresses the orthogonality of the basic monogenic polynomials $\mathbf{X}_{l, m}^{ \pm}[\mu]$, which is the central theme of the present section. The proof here given may be taken as an alternative to that of [239].

Theorem 3.1.10. For fixed $\mu$, the set $\left\{\mathbf{X}_{l, m}^{ \pm}[\mu]: m=0, \ldots, l+1 ; l=\right.$ $0,1, \ldots\}$ forms an orthogonal family over the spheroid $\Omega_{\mu}$ in the sense of the scalar inner product 1.2.2. Their norms are given by

$$
\begin{aligned}
& \left\|\mathbf{X}_{l, m}^{ \pm}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}=\frac{\pi \mu^{2 l+3}}{(l+2)(l+2+m)(2 l+1)!!(2 l+3)!!} \\
& \\
& {\left[(l+2)(l+m)(l+1+m)(l+3-m)!(l+2+m)!I_{l, m-1}(\mu)\right.} \\
& \quad+2 \delta_{0, m}(l+2+m)(l+1)!(l+2)!I_{l, 1}(\mu) \\
& \quad+(l+2)(l+1-m)!(l+2+m)!\left(I_{l, m+1}(\mu)\right. \\
& \left.\left.\quad+2(l+2-m)(l+1+m)\left(1+\delta_{0, m}\right) I_{l, m}(\mu)\right)\right],
\end{aligned}
$$

where $I_{l, m}(\mu)$ is defined by 2.1.18). For the limiting case, $\mu=0$,

$$
\begin{equation*}
\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(\Omega_{0}\right)}^{2}=\frac{2 \pi\left(1+\delta_{0, m}\right)(l+1)(l+1+m)!}{(2 l+3)(l+1-m)!} \tag{3.1.10}
\end{equation*}
$$

Proof. By definition of the integral (1.2.2), it follows that

$$
\begin{aligned}
& \left\langle\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu], \mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{0}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{0}+\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{1}\right. \\
& \left.+\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{2}\right) d \mathbf{x} .
\end{aligned}
$$

By Theorems 3.1.4 and 2.3.1, we have

$$
\begin{equation*}
\int_{\Omega_{\mu}}\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{0}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{0} d \mathbf{x}=\left\|V_{l_{1}, m_{1}}^{ \pm}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2} \delta_{l_{1}, l_{2}} \delta_{m_{1}, m_{2}} \tag{3.1.11}
\end{equation*}
$$

Thus, to verify the orthogonality of the $\mathbf{X}_{l, m}^{ \pm}[\mu]$, it suffices to show that the vector parts of the polynomials $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are orthogonal.

Expanding the integrands and applying the trigonometric identities

$$
\begin{aligned}
\Phi_{m_{1}-1}^{ \pm} \Phi_{m_{2}-1}^{ \pm}+\Phi_{m_{1}-1}^{\mp} \Phi_{m_{2}-1}^{\mp} & =\Phi_{m_{1}-m_{2}}^{+} \\
\Phi_{m_{1}+1}^{ \pm} \Phi_{m_{2}+1}^{ \pm}+\Phi_{m_{1}+1}^{\mp} \Phi_{m_{2}+1}^{\mp} & =\Phi_{m_{1}-m_{2}}^{+}, \\
-\Phi_{m_{1}-1}^{ \pm} \Phi_{m_{2}+1}^{ \pm}+\Phi_{m_{1}-1}^{\mp} \Phi_{m_{2}+1}^{\mp} & =\mp \Phi_{m_{1}+m_{2}}^{+} \\
-\Phi_{m_{1}+1}^{ \pm} \Phi_{m_{2}-1}^{ \pm}+\Phi_{m_{1}+1}^{\mp} \Phi_{m_{2}-1}^{\mp} & =\mp \Phi_{m_{1}+m_{2}}^{+}
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
& \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{1}+\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{2}\right) d \mathbf{x} \\
= & \frac{1}{4}\left(p_{1} p_{2} \int_{\Omega_{\mu}} V_{l_{1}, m_{1}-1}[\mu] V_{l_{2}, m_{2}-1}[\mu] \Phi_{m_{1}-m_{2}}^{+} d \mathbf{x}\right. \\
& \mp \frac{p_{1}}{p_{2}+1} \int_{\Omega_{\mu}} V_{l_{1}, m_{1}-1}[\mu] V_{l_{2}, m_{2}+1}[\mu] \Phi_{m_{1}+m_{2}}^{+} d \mathbf{x} \\
& \mp \frac{p_{2}}{p_{1}+1} \int_{\Omega_{\mu}} V_{l_{1}, m_{1}+1}[\mu] V_{l_{2}, m_{2}-1}[\mu] \Phi_{m_{1}+m_{2}}^{+} d \mathbf{x} \\
& \left.+\frac{1}{\left(p_{1}+1\right)\left(p_{2}+1\right)} \int_{\Omega_{\mu}} V_{l_{1}, m_{1}+1}[\mu] V_{l_{2}, m_{2}+1}[\mu] \Phi_{m_{1}-m_{2}}^{+} d \mathbf{x}\right),
\end{aligned}
$$

where $p_{i}=l_{i}+1+m_{i}(i=1,2)$.
We continue the calculations only for the prolate case. The following identities

$$
\int_{0}^{2 \pi} \Phi_{m_{1} \pm m_{2}}^{ \pm}(\varphi) d \varphi=2 \pi \delta_{m_{1}, m_{2}}
$$

for $m_{1}, m_{2}>0$ imply that

$$
\begin{aligned}
& \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{1}+\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{2}\right) d \mathbf{x} \\
= & \frac{\pi p_{1}\left(l_{2}+1+m_{1}\right)}{2} \delta_{m_{1}, m_{2}} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1, m_{1}-1}}[\mu] V_{l_{2}, m_{1}-1}[\mu] d R \\
& \mp \pi \frac{l_{2}+1}{2\left(l_{1}+2\right)} \delta_{m_{1}, 0} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1}, 1}[\mu] V_{l_{2},-1}[\mu] d R \\
& \mp \pi \frac{l_{1}+1}{2\left(l_{2}+2\right)} \delta_{m_{1}, 0} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1},-1}[\mu] V_{l_{2}, 1}[\mu] d R \\
& +\frac{\pi}{2\left(p_{1}+1\right)\left(l_{2}+1+m_{1}\right)} \delta_{m_{1}, m_{2}} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1}, m_{1}+1}[\mu] V_{l_{2}, m_{1}+1}[\mu] d R
\end{aligned}
$$

where $d R=\mu^{3}\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right) \sin \vartheta \sinh \eta d \vartheta d \eta$.
In consequence, using (3.1.3), we find

$$
\begin{aligned}
& \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{1}+\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{2}\right) d \mathbf{x} \\
= & \frac{\pi p_{1}\left(l_{2}+1+m_{1}\right) \delta_{m_{1}, m_{2}}}{2} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1}, m_{1}-1}[\mu] V_{l_{2}, m_{1}-1}[\mu] d R \\
& \pm \frac{\pi}{\left(l_{1}+2\right)\left(l_{2}+2\right)} \delta_{m_{1}, 0} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1}, 1}[\mu] V_{l_{2}, 1}[\mu] d R \\
& +\frac{\pi}{2 p_{1}\left(l_{2}+1+m_{1}\right)} \delta_{m_{1}, m_{2}} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi} V_{l_{1, m_{1}+1}}[\mu] V_{l_{2}, m_{1}+1}[\mu] d R .
\end{aligned}
$$

Now, using Proposition 2.2.5 and applying again the orthogonality of Theorem 2.3.1, we are left with

$$
\begin{align*}
& \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{1}+\left[\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{ \pm}[\mu]\right]_{2}\right) d \mathbf{x} \\
= & \frac{\pi \mu^{2 l_{1}+3}}{\left(l_{1}+2\right)\left(2 l_{1}+1\right)!!\left(2 l_{1}+3\right)!!} \\
& \times\left[\left(l_{1}+2\right)\left(l_{1}+1+m_{1}\right)!\right. \\
& \left(\left(l_{1}+m_{1}\right)\left(l_{1}+1+m_{1}\right)\left(l_{1}+3-m_{1}\right)!I_{l_{1}, m_{1}-1}(\mu)\right. \\
& \left.\left.+\left(l_{1}-m_{1}+1\right)!I_{l_{1}, m_{1}+1}(\mu)\right)+2\left(l_{1}+1\right)!\left(l_{1}+2\right)!I_{l_{1}, 1}(\mu) \delta_{0, m_{1}}\right] \delta_{l_{1}, l_{2}} \delta_{m_{1}, m_{2}}, \tag{3.1.12}
\end{align*}
$$

with $I_{l, m}(\mu)$ defined in (2.1.18). Combining (3.1.11) and (3.1.12), we conclude that $\left\langle\mathbf{X}_{l_{1}, m_{1}}^{+}[\mu], \mathbf{X}_{l_{2}, m_{2}}^{+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$ when $l_{1} \neq l_{2}$ or $m_{1} \neq m_{2}$. Similarly, $\left\langle\mathbf{X}_{l_{1}, m_{1}}^{-}[\mu], \mathbf{X}_{l_{2}, m_{2}}^{-}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$ when $l_{1} \neq l_{2}$ or $m_{1} \neq m_{2}$.

Using the orthogonality of the system $\left\{\Phi_{m}^{ \pm}\right\}$on $[0,2 \pi]$ again, we conclude that $\left\langle\mathbf{X}_{l_{1}, m_{1}}^{ \pm}[\mu], \mathbf{X}_{l_{2}, m_{2}}^{\mp}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$ when the indices do not coincide. The calculation of the norms comes from taking $l_{1}=l_{2}$ and $m_{1}=m_{2}$ in (3.1.12) and adding expression (2.3.9). By the symmetric form of the $\mathbf{X}_{l, m}^{ \pm}[\mu]$ in (3.1.4), it follows that $\left\|\mathbf{X}_{l, m}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}=\left\|\mathbf{X}_{l, m}^{-}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}$ when $m \neq 0$. The limiting case, $\mu=0$, follows with the use of Corollary 3.1.5 and 2.3.3). The proof is now completed.

The corresponding orthogonality of the basic spheroidal monogenic polynomials over the surface of the prescribed spheroids follows immediately from Theorems 2.3.2 and 3.1.10.

Theorem 3.1.11. For fixed $\mu$, the set $\left\{\mathbf{X}_{l, m}^{ \pm}[\mu]: m=0, \ldots, l+1 ; l=\right.$ $0,1, \ldots\}$ forms an orthogonal family over the surface of the spheroid $\Omega_{\mu}$ in the sense of the scalar product

$$
\begin{equation*}
\{\boldsymbol{f}, \boldsymbol{g}\}_{0, L_{2}\left(\partial \Omega_{\mu}, \mathcal{A}\right)}=\operatorname{Sc} \int_{\partial \Omega_{\mu}} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{g}}(\mathbf{x})|\zeta(\mu, \mathbf{x})|^{1 / 2} d \sigma \tag{3.1.13}
\end{equation*}
$$

where $|\zeta(\mu, \mathbf{x})|$ is defined by (2.1.9).
In Subsection 3.2.1, we detail how the monogenic polynomials fit in the space of harmonic polynomials. In [208], it was shown that the dimension of the space $\mathcal{M}_{l}^{+}(\Omega, \mathcal{A})$ of homogeneous monogenic polynomials with values in $\mathcal{A}$ in the variables $x_{0}, x_{1}, x_{2}$ of degree $l$ is $2 l+3$ (this does not depend on the domain $\Omega$ ). Since the polynomials we are working with are generally not homogeneous (when $\mu \neq 0$ ), we consider the space

$$
\mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathcal{A}\right)=\bigcup_{0 \leq k \leq l} \mathcal{M}_{k}^{+}(\Omega, \mathcal{A})
$$

of monogenic polynomials of degree no greater than $l$ with values in $\mathcal{A}$. This class is not altered by adding monogenic polynomials of lower degree.

Thus

## Proposition 3.1.12.

$$
\begin{equation*}
\operatorname{dim} \mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathcal{A}\right)=\sum_{k=0}^{l}(2 k+3)=(l+1)(l+3) . \tag{3.1.14}
\end{equation*}
$$

Consider the collection of $2 k+3$ polynomials

$$
B_{k}[\mu]:=\left\{\mathbf{X}_{k, m}^{+}[\mu]: 0 \leq m \leq k+1\right\} \cup\left\{\mathbf{X}_{k, m}^{-}[\mu]: 1 \leq m \leq k+1\right\}
$$

By (3.1.14) and Theorem 3.1.10, the union

$$
\begin{equation*}
\bigcup_{0 \leq k \leq l} B_{k}[\mu] \tag{3.1.15}
\end{equation*}
$$

is an orthogonal basis for $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathcal{A}\right)$. In addition, $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathcal{A}\right)$ is dense in $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$. Therefore the following result, which will be of use in further discussion, can now be established:

Theorem 3.1.13. For fixed $\mu$, the set (3.1.15) forms an orthogonal basis of $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$.

Thus, as a consequence of the above theorem, we can define the Fourier expansion of a square-integrable monogenic function defined in a spheroid of arbitrary eccentricity. Using the $L_{2}$-norms stated in Theorem 3.1.10, we can normalize the proposed basic monogenic polynomials, and so the definition is as follows:

Definition 3.1.14. Suppose $\boldsymbol{f} \in \mathcal{M}_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$ and $\mu$ is fixed. The Fourier series of $\boldsymbol{f}$ with respect to the basis 3.1.15 is

$$
\begin{equation*}
\sum_{l=0}^{\infty} \sum_{m=0}^{l+1}\left(a_{l, m}^{+}[\mu] \frac{\mathbf{X}_{l, m}^{+}[\mu]}{\left\|\mathbf{X}_{l, m}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}}+a_{l, m}^{-}[\mu] \frac{\mathbf{X}_{l, m}^{-}[\mu]}{\left\|\mathbf{X}_{l, m}^{-}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}}\right) \tag{3.1.16}
\end{equation*}
$$

where the associated coefficients are uniquely defined by

$$
a_{l, m}^{ \pm}[\mu]=\frac{1}{\left\|\mathbf{X}_{l, m}^{ \pm}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}}\left\langle\boldsymbol{f}, \mathbf{X}_{l, m}^{ \pm}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} .
$$

As will be hereafter seen, the series expansion 3.1.16) plays an essential role in Chapter 5.

Obviously, according to Definition 3.1.14, $\boldsymbol{f}$ may be characterized by its coefficients using Parseval's identity:

Corollary 3.1.15. For fixed $\mu, \boldsymbol{f} \in \mathcal{M}_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$ is equivalent to

$$
\sum_{l=0}^{\infty} \sum_{m=0}^{l+1}\left[\left(a_{l, m}^{+}[\mu]\right)^{2}+\left(a_{l, m}^{-}[\mu]\right)^{2}\right]<\infty .
$$

Furthermore, it would be useful if the preceding basis (3.1.15) also has the Appell property (with respect to the hypercomplex derivative). It was shown in [256] that there does not exist an orthogonal Appell basis in the case of spaces of internal oblate spheroidal monogenics.

We shall proceed in such a manner that we compute the hypercomplex derivative of a basic monogenic spheroidal polynomial of degree $l$ and show, as expected, that the obtained polynomial is not a member of the family with degree $l-1$ like in the cases of Appell bases [39, 63, 65, 67]. We find that the hypercomplex derivative of a basic spheroidal monogenic is a combination of $[(l-m) / 2]+1$ spheroidal monogenics of lower degrees. It can be represented by all basic polynomials of degree at most $l-1$.

Theorem 3.1.16. For fixed $\mu$, the hypercomplex derivatives of the $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are equal to

$$
\begin{equation*}
\left(\frac{1}{2} \partial\right) \mathbf{X}_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l-m}{2}\right]} v_{l, m, k} \mu^{2 k} \mathbf{X}_{l-1-2 k, m}^{ \pm}[\mu], \tag{3.1.17}
\end{equation*}
$$

where the constants $v_{l, m, k}$ are given by (2.2.15).
Proof. Since $\partial / \partial x_{0}$ is a linear operator, we find, by Theorem 2.2.6, the relation

$$
\frac{\partial}{\partial x_{0}} V_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l-m}{2}\right]} v_{l, m, k} \mu^{2 k} V_{l-1-2 k, m}^{ \pm}[\mu] .
$$

The rest of the proof follows from the same principle as Theorem 3.1.9. Hence, we omit it.

Accordingly, from what has just been proved, we show that there are two spheroidal monogenic constants among the elements of the canonical basis (3.1.15), i.e., functions whose hypercomplex derivative is identically zero. Later, by dimension considerations, we will see that these generate all monogenic constants.

Proposition 3.1.17. For fixed $\mu$, the polynomials $\mathbf{X}_{l, l+1}^{ \pm}[\mu]$ are monogenic constants. Further, they do not depend on $\mu^{2}$.

Proof. The proof is a consequence of Theorem 3.1.16. It can be further proved, by Corollary 3.1.9, that $\mathbf{X}_{l, l+1}^{ \pm}[\mu]=\mathbf{X}_{l, l+1}^{ \pm}[0]$. Hence, the polynomials $\mathbf{X}_{l, l+1}^{ \pm}\left(x_{1}, x_{2}\right)$ do not depend on $\mu^{2}$.

The hypercomplex derivatives of the prescribed monogenic polynomials in its extended signification being thus computed, no difficulties can arise in restricting them to a particular limiting case. When $\mu=0$, we have readily from (3.1.17) that [63, 65]:

$$
\begin{equation*}
\left(\frac{1}{2} \partial\right) \mathbf{X}_{l, m}^{ \pm}[0]=(l+1+m) \mathbf{X}_{l-1, m}^{ \pm}[0] . \tag{3.1.18}
\end{equation*}
$$

The reader might find that without any additional work, using (3.1.18) and setting for each $l \geq 0$ and $0 \leq m \leq l+1$,

$$
\begin{equation*}
\mathbf{Y}_{l, m}^{ \pm}:=\frac{l!(1+m)!}{(l+1+m)!} \mathbf{X}_{l, m}^{ \pm}[0] \tag{3.1.19}
\end{equation*}
$$

that

$$
\begin{equation*}
\left(\frac{1}{2} \partial\right) \mathbf{Y}_{l, m}^{ \pm}=l \mathbf{Y}_{l-1, m}^{ \pm} \tag{3.1.20}
\end{equation*}
$$

Thus, the hypercomplex derivative of $\mathbf{Y}_{l, m}^{ \pm}$results again in a real multiple of the similar polynomial one degree lower [238]. The particular normalization (3.1.20) is called Appell property, which was already generalized in 1880 by Appell [21] to more general polynomial systems, nowadays called Appell systems. In [67], it was proved that the internal solid spherical monogenics (3.1.19) form, indeed, an orthogonal Appell basis for $\mathcal{M}_{2}\left(\Omega_{0}, \mathcal{A}\right)$. In [38] and [41], fundamental recursion formulas were obtained for the elements of the prescribed Appell basis.

It was further observed in [67] that for fixed $l$, the set $\left\{\mathbf{Y}_{l, 0}^{+}: l=0,1, \ldots\right\}$ coincides in dimension 3 with the family of Appell homogeneous monogenic polynomials studied by Malonek et al. in [112, 113, 114]. It is of interest to remark that these generalized Appell polynomials were extensively applied to the study of several elementary functions within hypercomplex analysis [5], [68, 69, [72, [83, 113, 219, 220], the computation of combinatorial identities [14, [71, 73], and the study of generalized Joukowski transformations in Euclidean spaces of arbitrary higher dimension [18, 82]. We call attention to the fact that the special monogenic polynomials given by Abul-Ez and Constales in [7, 8, 9, 12, 342] are deliberately similar, up to a rescaling factor, to those exploited in [113]. However, at the time of publication [7], the concept of hypercomplex differentiability or the corresponding use of the hypercomplex derivative was not reflected in the investigation of Appell sets of monogenic polynomials. These generalized polynomials were used to prove a counterpart of Hadamard's three-hyperballs Theorem within hypercomplex analysis [13].

We now turn to the discussion of our results. We show that the Appell property holds for a part of the basic internal spheroidal monogenics (providing the prescribed normalization (3.1.19) for $\mathbf{X}_{l, m}^{ \pm}[\mu]$ ).

Corollary 3.1.18. Let $\mu$ be fixed. For $l-m=0,1$, the hypercomplex derivatives of the $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are equal to

$$
\left(\frac{1}{2} \partial\right) \mathbf{X}_{l, m}^{ \pm}[\mu]=(l+1+m) \mathbf{X}_{l-1, m}^{ \pm}[\mu] .
$$

Proof. It is an immediate consequence of Theorem 3.1.16.

Having established this result, we proceed to compute the primitives of the basic monogenic polynomials according to Definition 1.3.14. This will be done by direct inversion of the formula of their hypercomplex derivatives (3.1.17).

The result is that:

Theorem 3.1.19. For fixed $\mu$, the monogenic primitives of the $\mathbf{X}_{l, m}^{ \pm}[\mu]$ are equal to

$$
\begin{equation*}
\mathcal{P}\left(\mathbf{X}_{l, m}^{ \pm}[\mu]\right)=\frac{1}{l+2+m} \mathbf{X}_{l+1, m}^{ \pm}[\mu]-\frac{\mu^{2}(l+1+m)}{(2 l+1)(2 l+3)} \mathbf{X}_{l-1, m}^{ \pm}[\mu] . \tag{3.1.21}
\end{equation*}
$$

Proof. Using (2.2.15), it is easy to verify that

$$
v_{l+1, m, k+1}=\frac{(l+2+m)(l+1+m)}{(2 l+1)(2 l+3)} v_{l-1, m, k} .
$$

We rely on formula (3.1.17) to obtain

$$
\begin{aligned}
& \frac{1}{2} \partial\left(\mathbf{X}_{l+1, m}^{ \pm}[\mu]-\mu^{2} \frac{(l+2+m)(l+1+m)}{(2 l+1)(2 l+3)} \mathbf{X}_{l-1, m}^{ \pm}[\mu]\right) \\
= & (l+2+m) \mathbf{X}_{l, m}^{ \pm}[\mu] .
\end{aligned}
$$

This leads to the theorem.

A direct examination of 3.1.21 shows that in the limiting case, $\mu=0$, we readily have that [63, 66]:

$$
\begin{equation*}
\mathcal{P}\left(\mathbf{X}_{l, m}^{ \pm}[0]\right)=\frac{1}{l+2+m} \mathbf{X}_{l+1, m}^{ \pm}[0] . \tag{3.1.22}
\end{equation*}
$$

The advantages of identities such as (3.1.17) and (3.1.21) are that they furnish concise expressions for the hypercomplex derivatives and primitives of the basic monogenic spheroidal polynomials for which many of their properties may be investigated. These are explored in more detail in Chapter 5

One of our significant results is that the three-dimensional solid spherical monogenics considered, e.g., in [39, 43, 65, [67, [236], are embedded in the prescribed one-parameter family of internal spheroidal monogenics. Hence, the latter can be seen as an extension of the former functions to arbitrarily spheroidal domains.

### 3.1.2 The Monogenic Bergman Kernel on Spheroids

In the first instance, this section is concerned with constructing an orthogonal basis of $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$ formed by monogenic spheroidal polynomials with values in $\mathbb{H}$. The importance of building this basis stems from the role it plays in the calculation of the monogenic Bergman kernel function in spheroidal domains of arbitrary eccentricity. In particular, it is proved that the constructed mapping is a mapping in $\mathbb{R}^{3}$, and some examples that illustrate the effectiveness of the approach are given.

With slight adaptation, all the results established in Subsection 3.1.1 apply to the construction of $\mathbb{H}$-valued (left) monogenic polynomials satisfying the Moisil-Teodorescu system (1.3.3). Cação gave the first example of such a development in $\Omega_{0}$, in [63, 67] (cf. [64, [65, 66]), followed by Bock [38, [39, 40, 41] (cf. [238]).

By the representation given in (3.1.4), the following theorem can be established:

Theorem 3.1.20. Let $l \geq 0$ and $0 \leq m \leq l$. For fixed $\mu$, the monogenic spheroidal polynomials defined by

$$
\begin{equation*}
\mathbf{X}_{l, m}^{Q}[\mu]=\mathbf{X}_{l, m+1}^{+}[\mu] \mathbf{i}+\mathbf{X}_{l, m+1}^{-}[\mu] \mathbf{j} \tag{3.1.23}
\end{equation*}
$$

or, more explicitly,

$$
\begin{align*}
\mathbf{X}_{l, m}^{Q}[\mu]= & -(l+2+m)\left(V_{l, m}^{+}[\mu]-\mathbf{k} V_{l, m}^{-}[\mu]\right) \\
& +\mathbf{i} V_{l, m+1}^{+}[\mu]+\mathbf{j} V_{l, m+1}^{-}[\mu] \tag{3.1.24}
\end{align*}
$$

are orthogonal over the spheroid $\Omega_{\mu}$ in the sense of the quaternionic inner product (1.2.1). Their norms are given by

$$
\left\|\mathbf{X}_{l, m}^{Q}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}=\frac{4 \pi \mu^{2 l+3}(l+2+m)(l+1-m)!(l+2+m)!}{(2 l+1)!!(2 l+3)!!}, \quad\left[(l+2-m)(l+1+m)\left(1+\delta_{0, m}\right) I_{l, m}(\mu)+I_{l, m+1}(\mu)\right], .
$$

where $I_{l, m}(\mu)$ is defined by 2.1.18). For the limiting case, $\mu=0$,

$$
\begin{equation*}
\left\|\mathbf{X}_{l, m}^{Q}[0]\right\|_{L_{2}\left(\Omega_{0}\right)}^{2}=\frac{4 \pi(l+2+m)(l+2+m)!}{(2 l+3)(l-m)!} \tag{3.1.25}
\end{equation*}
$$

Proof. By definition of the quaternionic inner product 1.2.1,

$$
\left\langle\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu], \mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right\rangle_{L_{2}\left(\Omega_{\mu}, \mathbb{H}\right)}=(\mathrm{I})+\mathbf{i}(\mathrm{II})+\mathbf{j}(\mathrm{III})+\mathbf{k}(\mathrm{IV}),
$$

where

$$
\begin{aligned}
(\mathrm{I})= & \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{0}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{0}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{1}\right. \\
& \left.+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{2}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{3}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{3}\right) d \mathbf{x}, \\
(\mathrm{II})= & \int_{\Omega_{\mu}}\left(-\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{0}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{1}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{0}\right. \\
& \left.-\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{3}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{3}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{2}\right) d \mathbf{x}, \\
(\mathrm{III})= & \int_{\Omega_{\mu}}\left(-\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{0}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{2}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{0}\right. \\
& \left.+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{3}-\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{3}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{1}\right) d \mathbf{x}, \\
(\mathrm{IV})= & \int_{\Omega_{\mu}}\left(-\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{0}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{3}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{3}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{0}\right. \\
& \left.-\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{1}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{2}+\left[\mathbf{X}_{l_{1}, m_{1}}^{Q}[\mu]\right]_{2}\left[\mathbf{X}_{l_{2}, m_{2}}^{Q}[\mu]\right]_{1}\right) d \mathbf{x} .
\end{aligned}
$$

The proof follows similar lines of Theorems 3.1.4 and 3.1.10 and is therefore omitted.

The theorem that has now been established will play an essential role in calculating the monogenic kernel function in spheroidal domains of differing eccentricity.

Further, it is seen that the chosen combination for the basic spheroidal monogenics $\mathbf{X}_{l, m+1}^{ \pm}[\mu]$ stated in Theorem 3.1 .20 follows by the underlying symmetry of the polynomials (see Proposition 3.1.6 above); other possible combinations of these polynomials could also be considered.

The corresponding orthogonality over the surface of the prescribed spheroids holds for the polynomials (3.1.23).

Theorem 3.1.21. For fixed $\mu$, the set $\left\{\mathbf{X}_{l, m}^{Q}[\mu]: m=0, \ldots, l ; l=0,1, \ldots\right\}$ forms an orthogonal family over the surface of the spheroid $\Omega_{\mu}$ in the sense of the quaternionic inner product

$$
\{\boldsymbol{f}, \boldsymbol{g}\}_{L_{2}\left(\partial \Omega_{\mu}, \mathbb{H}\right)}=\int_{\partial \Omega_{\mu}} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{g}}(\mathbf{x})|\zeta(\mu, \mathbf{x})|^{1 / 2} d \sigma
$$

where $|\zeta(\mu, \mathbf{x})|$ is defined by (2.1.9).
Some examples of (3.1.23) in low degree provided by (3.1.23) are exhibited in Table 3.3.
$\left.\begin{array}{|c|c|l|}\hline l & m & \mathbf{X}_{l, m}^{Q}[\mu] \\ \hline 0 & 0 & \mathbf{X}_{0,0}^{Q}=-2\end{array}\right]$

Table 3.3: Spheroidal monogenic basis polynomials of degree $l=0,1,2,3$, parametrized by the eccentricity $\mu$.

In [318], it was shown that the dimension of the space $\mathcal{M}_{l}^{+}(\Omega, \mathbb{H})$ consisting of homogeneous monogenic polynomials with values in $\mathbb{H}$ in the variables $x_{0}, x_{1}, x_{2}$ of degree $l$ is $l+1$, and this does not depend on the domain $\Omega$. Since the polynomials $\mathbf{X}_{l, m}^{Q}[\mu]$ are generally not homogeneous (when $\mu \neq 0$ ), we proceed to the consideration of the space of $\mathbb{H}$-valued monogenic polynomials of degree no greater than $l$ defined by

$$
\mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathbb{H}\right)=\bigcup_{0 \leq k \leq l} \mathcal{M}_{k}^{+}(\Omega, \mathbb{H}) .
$$

Thus

## Proposition 3.1.22.

$$
\begin{equation*}
\operatorname{dim} \mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathbb{H}\right)=\sum_{k=0}^{l}(k+1)=\frac{(l+1)(l+2)}{2} . \tag{3.1.26}
\end{equation*}
$$

Now, consider the collection of $k+1$ polynomials

$$
C_{k}[\mu]:=\left\{\mathbf{X}_{k, m}^{Q}[\mu]: 0 \leq m \leq k\right\} .
$$

Hence, the union

$$
\begin{equation*}
\bigcup_{0 \leq k \leq l} C_{k}[\mu] \tag{3.1.27}
\end{equation*}
$$

is an orthogonal basis for the space $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathbb{H}\right)$, and this is by (3.1.26) and Theorem 3.1.20. Moreover, $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}, \mathbb{H}\right)$ is dense in $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$.

It has thus been shown that:
Theorem 3.1.23. For fixed $\mu$, the set (3.1.27) forms an orthogonal basis of $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$.

Accordingly, from what has just been proved, we can now find an explicit representation of the monogenic Bergman kernel function in spheroidal domains of arbitrary eccentricity, which generalizes the corresponding result in [243]. We see that $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$ is a subspace of $L_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$. Thus, by the Quaternion Riesz Representation Theorem 1.2 .23 , to each $\mathbf{x}, \widetilde{\mathbf{x}} \in \Omega_{\mu}$, there exists a unique reproducing kernel $\boldsymbol{B}[\mu](\mathbf{x}, \widetilde{\mathbf{x}})$ in $\mathcal{M}_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$ such that

$$
\begin{equation*}
\boldsymbol{f}(\widetilde{\mathbf{x}})=\langle\boldsymbol{f}, \boldsymbol{B}[\mu](\cdot, \widetilde{\mathbf{x}})\rangle_{L_{2}\left(\Omega_{\mu}, \mathrm{H}\right)}, \tag{3.1.28}
\end{equation*}
$$

or equivalently,

$$
\boldsymbol{f}(\widetilde{\mathbf{x}})=\int_{\Omega_{\mu}} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{B}}[\mu](\mathbf{x}, \widetilde{\mathbf{x}}) d \mathbf{x}
$$

for any $\boldsymbol{f} \in \mathcal{M}_{2}\left(\Omega_{\mu}, \mathbb{H}\right)$.
On combining this with what has been proved above, we have further the following theorem:

Theorem 3.1.24. For fixed $\mu$, the function

$$
\boldsymbol{B}[\mu](\mathbf{x}, \widetilde{\mathbf{x}})=\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{\overline{\mathbf{X}}_{l, m}^{Q}[\mu](\widetilde{\mathbf{x}}) \mathbf{X}_{l, m}^{Q}[\mu](\mathbf{x})}{\left\|\mathbf{X}_{l, m}^{Q}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}}
$$

is the monogenic Bergman kernel function of the spheroid $\Omega_{\mu}$.

We now proceed to consider the particular case $\widetilde{\mathbf{x}}=\mathbf{0}$. By (2.2.16), it then follows that

$$
\begin{aligned}
\boldsymbol{B}[\mu](\mathbf{x}, \mathbf{0}) & =\sum_{l=0}^{\infty} \frac{\overline{\mathbf{X}}_{l, 0}^{Q}[\mu](\mathbf{0}) \mathbf{X}_{l, 0}^{Q}[\mu](\mathbf{x})}{\left\|\mathbf{X}_{l, 0}^{Q}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}} \\
& =\sum_{l=0}^{\infty} \frac{(-1)^{l+1}(4 l+3)!!\mathbf{X}_{2 l, 0}^{Q}[\mu](\mathbf{x})}{(2 \mu)^{2 l+3} l!(l+1)!\left[4(l+1)(2 l+1) I_{2 l, 0}(\mu)+I_{2 l, 1}(\mu)\right]}
\end{aligned}
$$

where $I_{l, m}(\mu)$ is defined by (2.1.18).
In accordance with Theorem 3.1.20, a straightforward observation shows that $\mathbf{X}_{2 l, 0}^{Q}[\mu]=-2(l+1) \mathbf{X}_{2 l, 0}^{+}[\mu]$ for all $l \geq 0$, and thus $\boldsymbol{B}[\mu](\mathbf{x}, \mathbf{0})$ defines a mapping in $\mathbb{R}^{3}$. It can further be seen that $|\boldsymbol{B}[\mu](\mathbf{x}, \mathbf{0})|$ does not depend on the azimuthal angle $\varphi$. Hence the image of a spheroidal domain under this mapping will be symmetric with respect to the $x_{0}$-axis.

Figures 3.1-3.6 visualize approximations of different degrees for the image of a prolate spheroid centered at the origin of eccentricity $\mu=\sqrt{3} / 2$, under the mapping $\boldsymbol{B}[\mu](\mathbf{x}, \mathbf{0})$.


Figure 3.1: $l=1$


Figure 3.2: $l=3$

### 3.1.3 External Spheroidal Monogenic Functions

An orthogonal basis for the monogenic $L_{2}$-space consisting of external spheroidal functions is of prime importance to the analysis. Much of this subsection is devoted to studying the analytical properties of the elements that constitute such a basis. The construction of the basic external spheroidal monogenics becomes much more complicated than the one for internal functions since they contain logarithmic functions. Thus a simple substitution in


Figure 3.3: $l=5$


Figure 3.5: $l=13$


Figure 3.4: $l=10$


Figure 3.6: $l=15$
our arguments is not enough. We refer to [250] for a list of the known results concerning the external prolate spheroidal monogenics before the present investigation.

In [18, 82], the authors defined the hypercomplex Joukowski transformations employing the connection between the Kelvin transform [148] applied to certain monogenic polynomials and the hypercomplex derivative of the fundamental solution (1.3.5). In [237], Morais et al. used the same strategy and proposed an orthogonal basis of external solid hyperspherical monogenics in dimension 4. In [40], the author also used a proper Kelvin transform to construct $\mathbb{H}$-valued external spherical monogenics from internal spherical monogenics. Additionally, a generalized Laurent series expansion for the spherical shell was also considered in [40]. A theoretical advantage of these procedures is that using a Kelvin transform keeps specific desirable properties of the functions, such as orthogonality, invariant.

We point out that for $\mathcal{A}$-valued functions in a spheroid, the Kelvin transform method is not directly applicable.

Moreover, ordinary methods based on the decomposition of an exterior function space into subspaces of homogeneous functions often fail to prove the completeness of a function system because of the appearance of logarithmic functions. All these remarkable observations suggest using a different approach to verify the completeness of the underlying external monogenic function system. The technique we use is based on the harmonic extension of a function defined on the boundary of the spheroid $\Omega_{\mu}$ to the external domain $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$.

In an earlier paper [250], we treated the analogous problem of constructing an orthogonal basis of elements spanning the square-integrable solutions of $\bar{\partial} \boldsymbol{f}=0$ for the space exterior of a prolate spheroid via explicit formulas. We here borrow some of these techniques and fit many of those results to the present case. In particular, we can consider the prolate and oblate examples of spheroids simultaneously. It is left to show that explicit formulas for the external functions can readily be carried through in detail.

By following the procedure discussed in Subsection 3.1.1, we define the required spheroidal monogenics to be employed for the space exterior of the prescribed spheroids as follows.

Definition 3.1.25. Let $l \geq-1$ and $0 \leq m \leq l+1$. The basic external monogenic spheroidal functions of degree $l$ and order $m$ are

$$
\begin{equation*}
\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]=\partial \widehat{U}_{l+1, m}^{ \pm}[\mu], \tag{3.1.29}
\end{equation*}
$$

where the $\widehat{U}_{l, m}^{ \pm}[\mu]$ are defined by (2.1.7).
Morais et al. [250] gave the first definition of the kind just indicated, thus defining what, by using Legendre functions of the second type, are known as the external prolate spheroidal monogenics.

We shall emphasize the apparent fact that the spheroidal monogenics (3.1.29) are not polynomials but rather algebraic functions, which are homogeneous of degree $-(l+3)$.

By similar reasoning to that by which the internal spheroidal monogenics were established, it can be shown that the explicit expressions of the external spheroidal monogenics satisfy the same type of symmetry as the internal spheroidal monogenics. However, when deducing equations such as (2.2.8) and (2.2.20), $l$ is supposed to be greater than or equal to -1 , and $m$ is a positive integer, including zero, not greater than $l+1$. The reader will also notice that, given the factorization of the Laplacian (1.3.7), the functions $\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]$ are indeed monogenic. It is left to the reader to check that $\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]$ are (up to rescaling) the same functions defined in [250].

We can now explicitly express the quaternionic components of the external spheroidal monogenics in terms of the proper external harmonic functions (2.1.22). From (1.4.8) and (2.2.8), we deduce the following preliminary result, whose proof is similar to Lemma 3.1.2,

Lemma 3.1.26. For each $l \geq 0$,

$$
\begin{equation*}
\hat{V}_{l,-1}[\mu]=-\frac{1}{(l+1)(l+2)} \hat{V}_{l, 1}[\mu] . \tag{3.1.30}
\end{equation*}
$$

The functions (3.1.30) are involved in the representation (3.1.31) for zeroorder monogenic functions. We have then the theorem:

Theorem 3.1.27. For each $l \geq 0$ and $0 \leq m \leq l$, the basic external spheroidal monogenic functions (3.1.29) are equal to

$$
\begin{align*}
\widehat{\mathbf{X}}_{-1,0}^{+}[\mu]= & \frac{-\sinh \eta \cos \vartheta+(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) \cosh \eta \sin \vartheta}{\mu^{2} \sinh \eta\left(\cosh ^{2} \eta-\cos ^{2} \vartheta\right)} \\
\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]= & \widehat{V}_{l, m}^{ \pm}[\mu]+\frac{\mathbf{i}}{2}\left[(l+2-m) \widehat{V}_{l, m-1}^{ \pm}[\mu]-\frac{1}{l+1-m} \widehat{V}_{l, m+1}^{ \pm}[\mu]\right] \\
& \mp \frac{\mathbf{j}}{2}\left[(l+2-m) \widehat{V}_{l, m-1}^{\mp}[\mu]+\frac{1}{l+1-m} \widehat{V}_{l, m+1}^{\mp}[\mu]\right],  \tag{3.1.31}\\
\widehat{\mathbf{X}}_{l, l+1}^{ \pm}[\mu]= & \widehat{V}_{l, l+1}^{ \pm}[\mu]+\frac{\mathbf{i}}{2}\left[\widehat{V}_{l, l}^{ \pm}[\mu]-\frac{\mu \cosh \eta}{2(2 l+5) \cos \theta} \widehat{V}_{l+1, l+2}^{ \pm}[\mu]\right] \\
& \mp \frac{\mathbf{j}}{2}\left[\widehat{V}_{l, l}^{\mp}[\mu]+\frac{\mu \cosh \eta}{2(2 l+5) \cos \theta} \widehat{V}_{l+1, l+2}^{\mp}[\mu]\right],
\end{align*}
$$

where the $\widehat{V}_{l, m}^{ \pm}[\mu]$ are defined by (2.2.8).
Proof. The proof follows the same lines as the proof of Theorem 3.1.4 and will be omitted.

It might be interesting to trace the connections between the internal spheroidal monogenics and the external functions just considered. In the first place, we observe that $\widehat{\mathbf{X}}_{l, 0}^{-}[\mu]$ also vanish identically. However, on account of $\widehat{V}_{l, l+1}^{ \pm}[\mu] \neq 0$, there are apparent identifiable differences between the internal and external functions. Further observation shows that the scalar parts of $\widehat{\mathbf{X}}_{l, l+1}^{ \pm}[\mu]$ do not vanish.

As a direct consequence of Theorem 3.1.27, and by Propositions 2.1.6 and 2.2.8, we may obtain an explicit representation for the external solid spherical monogenics (3.1.29) employing the external solid spherical harmonic functions [250].

Corollary 3.1.28. For all $\mathbf{x} \in \mathbb{R}^{3},|\mathbf{x}| \neq 0$, the limits $\lim _{\mu \rightarrow 0} \widehat{\mathbf{X}}_{-1,0}^{+}[\mu](\mathbf{x})$ and $\lim _{\mu \rightarrow 0} \widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu](\mathbf{x})$ exist and are given, respectively, by

$$
\begin{aligned}
\widehat{\mathbf{X}}_{-1,0}^{+}[0](\mathbf{x})= & -\frac{\overline{\mathbf{x}}}{|\mathbf{x}|^{3}} \\
\widehat{\mathbf{X}}_{l, m}^{ \pm}[0](\mathbf{x})= & -(l+2-m) \widehat{U}_{l+2, m}^{ \pm}[0](\mathbf{x}) \\
& -\frac{\mathbf{i}}{2}\left[(l+2-m)(l+3-m) \widehat{U}_{l+2, m-1}^{ \pm}[0](\mathbf{x})-\widehat{U}_{l+2, m+1}^{ \pm}[0](\mathbf{x})\right] \\
& \pm \frac{\mathbf{j}}{2}\left[(l+2-m)(l+3-m) \widehat{U}_{l+2, m-1}^{\mp}[0](\mathbf{x})+\widehat{U}_{l+2, m+1}^{\mp}[0](\mathbf{x})\right]
\end{aligned}
$$

where the $\hat{U}_{l, m}^{ \pm}[0]$ are defined by (2.1.11).
From this result, we see that the function $\widehat{\mathbf{X}}_{-1,0}^{+}[\mu]$ leads to the CauchyFueter kernel (1.3.5), except for the normalization factor $-1 / 4 \pi$ when $\mu \rightarrow$ 0 . This observation is fundamental to ensure that the external spheroidal functions are well-defined on the outer domain of the prescribed spheroid $\Omega_{\mu}$ and gives evidence of the completeness of the underlying basic external spheroidal monogenics.

Given Corollary 2.2.4, it is natural to find the direct and inverse transformation formulas that permit passing from spherical to spheroidal monogenics.

Theorem 3.1.29. Let $l \geq 0$ and $0 \leq m \leq l+1$. Then

$$
\begin{align*}
& \widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\infty} \beta_{l+1, m, k} \mu^{2 k} \widehat{\mathbf{X}}_{l+2 k, m}^{ \pm}[0],  \tag{3.1.32}\\
& \widehat{\mathbf{X}}_{l, m}^{ \pm}[0]=\sum_{k=0}^{\infty} \widehat{\beta}_{l+1, m, k} \mu^{2 k} \widehat{\mathbf{X}}_{l+2 k, m}^{ \pm}[\mu], \tag{3.1.33}
\end{align*}
$$

where the constants $\beta_{l, m, k}$ and $\widehat{\beta}_{l, m, k}$ are given by (2.2.3) and (2.2.4).
Proof. For simplicity, we only prove the direct transformation formula (3.1.32). The inverse formula (3.1.33) can be proved similarly. We fix $m, l$, $\mu$, and the choice of sign $\pm$. According to (3.1.31), we want to show that $\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]$ is equal to

$$
A=\sum_{k=0}^{\infty} \mu^{2 k}\left(A_{0, k}+\frac{\mathbf{i}}{2} A_{1, k} \mp \frac{\mathbf{j}}{2} A_{2, k}\right),
$$

where the quaternionic components are given by

$$
\begin{aligned}
A_{0, k}= & \beta_{l+1, m, k} \widehat{V}_{l+2 k, m}^{ \pm}[0], \\
A_{1, k}= & (l+2-m+2 k) \beta_{l+1, m, k} \widehat{V}_{l+2 k, m-1}^{ \pm}[0] \\
& -\frac{1}{l+1-m+2 k} \beta_{l+1, m, k} \widehat{V}_{l+2 k, m+1}^{ \pm}[0], \\
A_{2, k}= & (l+2-m+2 k) \beta_{l+1, m, k} \widehat{V}_{l+2 k, m-1}^{\mp}[0] \\
+ & \frac{1}{l+1-m+2 k} \beta_{l+1, m, k} \widehat{V}_{l+2 k, m+1}^{\mp}[0] .
\end{aligned}
$$

By (2.2.3),

$$
\begin{aligned}
A_{1, k}= & (l+2-m) \beta_{l+1, m-1, k} \widehat{V}_{l+2 k, m-1}^{ \pm}[0] \\
& -\frac{1}{l+1-m} \beta_{l+1, m+1, k} \widehat{V}_{l+2 k, m+1}^{ \pm}[0], \\
A_{2, k}= & (l+2-m) \beta_{l+1, m-1, k} \widehat{V}_{l+2 k, m-1}^{\mp}[0] \\
& +\frac{1}{l+1-m} \beta_{l+1, m+1, k} \widehat{V}_{l+2 k, m+1}^{\mp}[0],
\end{aligned}
$$

and then by Corollary 2.2.4,

$$
\begin{aligned}
& \sum_{k=0}^{\infty} A_{0, k}=\widehat{V}_{l, m}^{ \pm}[\mu], \\
& \sum_{k=0}^{\infty} A_{1, k}=(l+2-m) \widehat{V}_{l, m-1}^{ \pm}[\mu]+\frac{1}{l+1-m} \widehat{V}_{l, m+1}^{ \pm}[\mu], \\
& \sum_{k=0}^{\infty} A_{2, k}=(l+2-m) \widehat{V}_{l, m-1}^{\mp}[\mu]-\frac{1}{l+1-m} \widehat{V}_{l, m+1}^{\mp}[\mu],
\end{aligned}
$$

which justifies the assertion $\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]=A$ in view of (3.1.31).
The consideration of questions that arise about the external spheroidal monogenic functions will be postponed. For the present, it is sufficient to justify that the spherical functions defined in Corollary 3.1 .28 form an orthogonal basis of the $L_{2}$-space of $\mathcal{A}$-valued monogenic functions for the space exterior of a unit ball in $\mathbb{R}^{3}$. In the first place, it should be noted that the operator $\partial$ establishes an isomorphism between the corresponding spaces $\operatorname{Har}_{l+1}^{-}\left(\Omega_{0}\right)$ and $\mathcal{M}_{l}^{-}\left(\Omega_{0}, \mathcal{A}\right)$. Moreover, bearing in mind the Hilbert space orthogonal decomposition

$$
\mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{0}, \mathcal{A}\right)=\bigoplus_{l=0}^{\infty} \mathcal{M}_{l}^{-}\left(\Omega_{0}, \mathcal{A}\right)
$$

it follows that the collection

$$
\begin{equation*}
\left\{\widehat{\mathbf{X}}_{-1,0}^{+}[0], \widehat{\mathbf{X}}_{l, m}^{ \pm}[0]: m=0, \ldots, l+1 ; l=0,1, \ldots\right\} \tag{3.1.34}
\end{equation*}
$$

forms an orthogonal basis of $\mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{0}, \mathcal{A}\right)$.
With a view to the extension of the general theorem of [250], it will now be shown that the basic external spheroidal functions (3.1.29) are indeed orthogonal over the exterior of the prescribed spheroids.

Theorem 3.1.30. For fixed $\mu$, the set (3.1.34) forms an orthogonal family over the exterior of the spheroid $\Omega_{\mu}$ in the sense of the scalar inner product (1.2.2).

Proof. Since the external functions (3.1.29) share the same structure as the internal ones (3.1.1), the orthogonality for different degrees $l_{1} \neq l_{2}$ can be done similarly to Theorem 3.1.10. We then prove the orthogonality of the functions in cases of the same degree $l$. Looking back to the form of the external functions, we find that $\widehat{\mathbf{X}}_{l, m}^{+}[\mu]$ and $\widehat{\mathbf{X}}_{l, m}^{-}[\mu]$ are orthogonal by the orthogonality of the following pairs on $[0,2 \pi]:\left\{\Phi_{m}^{-}, \Phi_{m}^{+}\right\},\left\{\Phi_{m+1}^{-}, \Phi_{m-1}^{+}\right\}$, $\left\{\Phi_{m-1}^{-}, \Phi_{m+1}^{+}\right\}$. It suffices to check the orthogonality inside each subset of $\left\{\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]\right\}$, whose proof follows the same lines as Theorem 2.3.6.

We proceed to investigate a result about the approximation of a function $\boldsymbol{f} \in \mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)$ expanded as a linear combination of external spheroidal monogenics.

Theorem 3.1.31. Suppose $\boldsymbol{f} \in \mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right) \cap C^{1}\left(\partial \Omega_{\mu}\right)$ and let $\mu$ be fixed. The Fourier series expansion given by the expression

$$
\begin{align*}
& \widehat{a}_{-1,0}^{+}[\mu] \frac{\widehat{\mathbf{X}}_{-1,0}^{ \pm}[\mu]}{\left\|\widehat{\mathbf{X}}_{-1,0}^{ \pm}[\mu]\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}} \\
& +\sum_{l=0}^{\infty} \sum_{m=0}^{l+1}\left(\widehat{a}_{l, m}^{+}[\mu] \frac{\widehat{\mathbf{X}}_{l, m}^{+}[\mu]}{\left\|\widehat{\mathbf{X}}_{l, m}^{+}[\mu]\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}}+\widehat{a}_{l, m}^{-}[\mu] \frac{\widehat{\mathbf{X}}_{l, m}^{-}[\mu]}{\left\|\widehat{\mathbf{X}}_{l, m}^{-}[\mu]\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}}\right) \tag{3.1.35}
\end{align*}
$$

where

$$
\widehat{a}_{-1,0}^{+}[\mu]=\frac{\left\langle\boldsymbol{f}, \widehat{\mathbf{X}}_{-1,0}^{+}[\mu]\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)}}{\left\|\widehat{\mathbf{X}}_{-1,0}^{+}[\mu]\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}}, \quad \widehat{a}_{l, m}^{ \pm}[\mu]=\frac{\left\langle\boldsymbol{f}, \widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]\right\rangle_{0, L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)}}{\left\|\widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}}
$$

converges to $\boldsymbol{f}$ in the $L_{2}$-sense.

Proof. The proof is similar to that in [250], but it is necessary to employ the definitions 2.1.7 and 3.1.29. Suppose that $f \in \mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)$. Hence, there exists a real-valued harmonic function $h$ in $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ such that $(1 / 2) \partial h=\boldsymbol{f}$. Moreover, the restriction of $h$ on $\partial \Omega_{\mu}$ is a twice continuously differentiable function. Now, let $g(\vartheta, \varphi)$ be a function defined on the unit sphere, which is related to the value of $\operatorname{Tr}_{\partial \Omega_{\mu}} h$ on the prescribed spheroid by $g(\vartheta, \varphi)=$ $\operatorname{Tr}_{\partial \Omega_{\mu}} h(\vartheta, \varphi)=h\left(\eta_{\mu}, \vartheta, \varphi\right)$. The trace operator $\operatorname{Tr}_{\partial \Omega_{\mu}}$ describes just the restriction onto the boundary $\partial \Omega_{\mu}$. Since $g(\vartheta, \varphi)$ is a twice continuously differentiable function on the unit sphere, it can be expressed employing a series of surface spherical harmonics,

$$
\begin{align*}
g(\vartheta, \varphi) & =\sum_{l=0}^{\infty} \sum_{m=0}^{l} P_{l}^{m}(\cos \vartheta)\left[\alpha_{l, m}^{+} \Phi_{m}^{+}(\varphi)+\alpha_{l, m}^{-} \Phi_{m}^{-}(\varphi)\right]  \tag{3.1.36}\\
& =\operatorname{Tr}_{\partial \Omega_{\mu}} h(\vartheta, \varphi)
\end{align*}
$$

It was shown in [179, 274] that the above expansion is uniformly absolutely convergent with respect to $(\vartheta, \varphi) \in \partial \Omega_{0}$.

For simplicity, we now assume that $\nu<0$, where $\eta_{\mu}<\eta<\infty$, i.e., with $\cosh \eta_{\mu}=1 / \mu$. Extending the series expansion (3.1.36) to $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ leads to a series expansion of $h$ in terms of the external spheroidal harmonics 2.1.7):

$$
\begin{equation*}
h=\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{\beta_{l, m}}{\mu^{l+1}} \frac{Q_{l}^{m}(\cosh \eta)}{Q_{l}^{m}(1 / \mu)} P_{l}^{m}(\cos \vartheta)\left[\alpha_{l, m}^{+} \Phi_{m}^{+}(\varphi)+\alpha_{l, m}^{-} \Phi_{m}^{-}(\varphi)\right] \tag{3.1.37}
\end{equation*}
$$

Using the results of [170, pp.417-421], we find $\left|Q_{l}^{m}(\cosh \eta) / Q_{l}^{m}(1 / \mu)\right|<1$ for all $m=0, \ldots, l(l=0,1, \ldots)$. Under the previous circumstances, it then follows that the series (3.1.37) converges uniformly and absolutely to $h$ in $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$.

Moreover, since $h$ is harmonic in $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ and twice continuously differentiable on the boundary $\partial \Omega_{\mu}$, it yields the absolute uniform convergence of its first derivatives in $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu} \cup \partial \Omega_{\mu}$. In particular, the corresponding series expansion for the derivatives, namely

$$
\begin{equation*}
\sum_{l=0}^{\infty} \sum_{m=0}^{l}\left(\frac{\alpha_{l, m}^{+}[\mu]}{Q_{l}^{m}(1 / \mu)} \widehat{\mathbf{X}}_{l-1, m}^{+}[\mu]+\frac{\alpha_{l, m}^{-}[\mu]}{Q_{l}^{m}(1 / \mu)} \widehat{\mathbf{X}}_{l-1, m}^{-}[\mu]\right) \tag{3.1.38}
\end{equation*}
$$

converges uniformly and absolutely to $\boldsymbol{f}=(1 / 2) \partial h$ in $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu} \cup \partial \Omega_{\mu}$. This further implies the $L_{2}$-convergence in every subset $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu} \cap B_{r}$, where $B_{r}$ is a ball with some radius $r>0$, which contains $\bar{\Omega}_{\mu}$. More precisely, denote by $\mathbf{S}_{N}$ the finite sum of the first $N$-summands in the series (3.1.38). For any $\epsilon>0$, there exists a natural number $N(\epsilon)$ such that

$$
\sup _{\mathbf{x} \in \mathbb{R}^{3} \backslash \bar{\Omega}_{\mu} \cup \partial \Omega_{\mu}}\left|\mathbf{S}_{N(\epsilon)}(\mathbf{x})-\boldsymbol{f}(\mathbf{x})\right|<\epsilon .
$$

We have then $\left\|\mathbf{S}_{N(\epsilon)}-\boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu} \cap B_{r}\right)}^{2}<\epsilon^{2} \operatorname{vol}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu} \cap B_{r}\right)$. For the exterior domain of $B_{r}$, we use another estimation. Let $0<r_{1}<r$ such that $\Omega_{\mu} \subset B_{r_{1}}$. By the Cauchy integral formula (1.3.6), for all $\mathbf{z} \in \mathbb{R}^{3} \backslash B_{r}$, one finds

$$
\begin{aligned}
\left|\mathbf{S}_{N(\epsilon)}(\mathbf{z})-\boldsymbol{f}(\mathbf{z})\right| & \leq \frac{1}{4 \pi} \int_{\partial B_{r_{1}}}|\boldsymbol{q}(\mathbf{z}-\boldsymbol{\zeta})|\left|\mathbf{S}_{N(\epsilon)}(\boldsymbol{\zeta})-\boldsymbol{f}(\boldsymbol{\zeta})\right| d \sigma(\boldsymbol{\zeta}) \\
& <\frac{\epsilon}{4 \pi} \int_{\partial B_{r_{1}}} \frac{1}{|\mathbf{z}|^{2}-|\boldsymbol{\zeta}|^{2}} d \sigma(\boldsymbol{\zeta}) \\
& <\epsilon \frac{r_{1}^{2}}{|\mathbf{z}|^{2}-r_{1}^{2}} .
\end{aligned}
$$

Now, the $L_{2}$-norm of the difference between $\mathbf{S}_{N}$ and $\boldsymbol{f}$ can be approximated by

$$
\begin{aligned}
\left\|\mathbf{S}_{N(\epsilon)}-\boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash B_{r}\right)}^{2} & =\int_{\mathbb{R}^{3} \backslash B_{r}}\left|\mathbf{S}_{N(\epsilon)}-\boldsymbol{f}\right|^{2} d \boldsymbol{\omega} \\
& <\left(r_{1}^{2} \epsilon\right)^{2} \int_{r}^{\infty} \int_{0}^{\pi} \int_{0}^{2 \pi} \frac{\rho^{2} \sin \theta}{\left(\rho^{2}-r_{1}^{2}\right)^{2}} d \varphi d \theta d \rho \\
& <4 \pi\left(r_{1}^{2} \epsilon\right)^{2} \int_{r}^{\infty} \frac{\rho^{2}}{\left(\rho^{2}-r_{1}^{2}\right)^{2}} d \rho \\
& <4 \pi\left(r_{1}^{2} \epsilon\right)^{2}\left[\frac{r}{2\left(r^{2}-r_{1}^{2}\right)}-\frac{1}{4 r_{1}} \log \frac{r-r_{1}}{r+r_{1}}\right] .
\end{aligned}
$$

To sum up, for an arbitrary small $\epsilon>0$, we can find a natural number $N(\epsilon)$ such that $\left\|\mathbf{S}_{N(\epsilon)}-\boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}\right)}<\epsilon$. Thus, the series expansion (3.1.38) converges to $\boldsymbol{f}$ in the whole domain $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ in the sense of the $L_{2}$-norm. The theorem follows.

From this theorem, we easily deduce the further results:
Corollary 3.1.32. Suppose $\boldsymbol{f} \in \mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)$ and let $\mu$ be fixed. Then the restriction of $\boldsymbol{f}$ in $\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}$ can be represented by its Fourier series expansion with respect to the set

$$
\begin{equation*}
\left\{\widehat{\mathbf{X}}_{-1,0}^{+}[\mu], \widehat{\mathbf{X}}_{l, m}^{ \pm}[\mu]: m=0, \ldots, l+1 ; l=0,1, \ldots\right\} \tag{3.1.39}
\end{equation*}
$$

Further, this series expansion converges to $\boldsymbol{f}$ in the $L_{2}$-sense.
Corollary 3.1.33. Any function in the set (3.1.34) can be represented by its Fourier series expansion with respect to the set (3.1.39).

To conclude, the general result is that:

Theorem 3.1.34. For fixed $\mu$, the set (3.1.39) forms an orthogonal basis of $\mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)$.

Proof. This result may be proved by first approximating $\boldsymbol{f}$ in $\mathcal{M}_{2}\left(\mathbb{R}^{3} \backslash \bar{\Omega}_{\mu}, \mathcal{A}\right)$ by external solid spherical monogenics (3.1.34), and then by basic external spheroidal monogenics (3.1.39).

This theorem is the generalization of that of [250], which corresponds to the case of prolate spheroids.

### 3.2 Contragenics on Spheroidal Domains

This section builds polynomial bases for square-integrable harmonic function spaces. These functions are called contragenic and are orthogonal to the monogenic and antimonogenic $\mathcal{A}$-valued functions defined in a prolate or oblate spheroid in the sense of $L_{2}$. We further give computational formulas relating to orthogonal bases of harmonic and contragenic functions defined in spheroids of differing eccentricity. As an application, we show that there are common nontrivial contragenic functions to all spheroids of all eccentricities, thus presenting the concept of "spheroidal universally contragenic function." For simplicity, we have confined our discussion only to the case of the region inside a spheroid. It might be worth noting that all our results can also be extended to the region outside a spheroid. The results that appear in this section are published in [133] and [134.

### 3.2.1 Ambigenic Spheroidal Polynomials

It is well-known that every complex-valued harmonic function in a simply connected domain in the complex plane can be expressed as the sum of a holomorphic function and an antiholomorphic function, where these two elements are unique up to a constant summand. This fact was generalized for monogenic functions on quaternions [318] and Clifford algebras [48], and there is a similar result for monogenic functions from $\mathbb{R}^{3}$ to $\mathbb{H}$ published in [63]. Notwithstanding these achievements, it was revealed in [17] that there is no corresponding statement for $\mathcal{A}$-valued monogenic functions because the multiplication in $\mathcal{A}$ is not a closed operation in $\mathbb{H}$. This means that there are harmonic functions that cannot be expressed as the sum of a monogenic and an antimonogenic function.

Given the facts discussed above, in this section, we write out a basis for the space of internal functions obtained by summing a monogenic function and an antimonogenic function. All these orthogonal bases are composed
of elements parametrized by the shape of the corresponding spheroid. In general, aspects of antimonogenic functions are slight modifications of monogenic function facts retrieved by assuming the conjugate. But to discuss contragenic functions in the following, it is necessary to consider the subspace of the $\mathcal{A}$-valued harmonic functions generated by both the monogenic and antimonogenic functions. In [17], elements of this space were termed ambigenic functions.

As discussed in Subsection 1.3.1, an $\mathcal{A}$-valued function $\boldsymbol{f}$ is antimonogenic, if and only if $\overline{\boldsymbol{f}}$ is monogenic. Since the decomposition of an ambigenic function as a sum of a monogenic and an antimonogenic function is not unique, the set $\mathcal{M}\left(\Omega_{\mu}\right) \cap \overline{\mathcal{M}}\left(\Omega_{\mu}\right)$ of monogenic constants in the domain $\Omega_{\mu}$ must be taken into account. Going back to the representation (3.1.4), we observe that for $0 \leq k \leq l$

$$
\mathbf{X}_{k, k+1}^{ \pm}[\mu]=(k+1)\left(\mathbf{i} V_{k, k}^{ \pm}[\mu] \mp \mathbf{j} V_{k, k}^{\mp}[\mu]\right),
$$

where

$$
V_{k, k}[\mu]=(-1)^{k}(2 k+1)!!\left(x_{1}^{2}+x_{2}^{2}\right)^{k / 2} .
$$

From Proposition 3.1.17, it follows that $\mathbf{X}_{k, k+1}^{ \pm}[\mu]$ are spheroidal monogenic constants with vanishing scalar part; that is, they are the negatives of their conjugates. This observation, along with dimension considerations, makes it possible to give a canonical basis for the ambigenic polynomials defined in spheroidal domains of arbitrary eccentricity.

There are natural projections of $\mathcal{M}\left(\Omega_{\mu}\right)$ onto the subspaces

$$
\operatorname{Sc} \mathcal{M}\left(\Omega_{\mu}\right)=\left\{\operatorname{Sc}(\boldsymbol{f}): \boldsymbol{f} \in \mathcal{M}\left(\Omega_{\mu}\right)\right\} \subseteq \operatorname{Har}_{\mathbb{R}}\left(\Omega_{\mu}\right)
$$

and

$$
\operatorname{Vec} \mathcal{M}\left(\Omega_{\mu}\right)=\left\{\operatorname{Vec}(\boldsymbol{f}): \boldsymbol{f} \in \mathcal{M}\left(\Omega_{\mu}\right)\right\} \subseteq \operatorname{Har}_{\{0\} \oplus \mathbb{R}^{2}}\left(\Omega_{\mu}\right)
$$

Hence, $\operatorname{Sc} \mathcal{M}\left(\Omega_{\mu}\right)=\operatorname{Sc} \overline{\mathcal{M}}\left(\Omega_{\mu}\right)$ and $\operatorname{Vec} \mathcal{M}\left(\Omega_{\mu}\right)=-\operatorname{Vec} \overline{\mathcal{M}}\left(\Omega_{\mu}\right)$. In [236], it was further shown that the property of $\Omega_{\mu}$ of being simply-connected guarantees that $\operatorname{Sc} \mathcal{M}\left(\Omega_{\mu}\right)=\operatorname{Har}\left(\Omega_{\mu}\right)$ (that is, every harmonic function is the scalar part of a monogenic function). The corresponding vector part is unique up to the addition of a monogenic constant. The discussion of two constructive approaches for generating monogenic functions in $\Omega_{0}$ via harmonic conjugates is detailed in Chapter 5 .

As was shown in [17], the dimension of the space $\mathcal{M}_{l}^{+}\left(\Omega_{0}\right)+\overline{\mathcal{M}}_{l}^{+}\left(\Omega_{0}\right)$ of homogeneous ambigenic polynomials is $4 l+4$ when $l \geq 1$. As discussed in the previous section, the polynomial basis for spheroidal harmonics are generally not homogenous (when $\mu \neq 0$ ). Thus we continue to work with all degrees up to $l$. The dimension of the space $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}\right)+\overline{\mathcal{M}}_{l}^{*}\left(\Omega_{\mu}\right)$ (not a direct sum) of ambigenic polynomials of degree at most $l$ is

## Proposition 3.2.1.

$$
\begin{aligned}
\operatorname{dim}\left(\mathcal{M}_{l}^{*}\left(\Omega_{\mu}\right)+\overline{\mathcal{M}}_{l}^{*}\left(\Omega_{\mu}\right)\right) & =\sum_{k=0}^{l} \operatorname{dim}\left(\mathcal{M}_{k}^{+}\left(\Omega_{0}\right)+\overline{\mathcal{M}}_{k}^{+}\left(\Omega_{0}\right)\right) \\
& =3+\sum_{k=1}^{l}(4 k+4) \\
& =2 l(l+3)+3 .
\end{aligned}
$$

Before we state the main result of the present section, the following elementary lemma will be required:

Lemma 3.2.2. Let $\mu$ be fixed. For $m \neq 0$,

$$
\left\langle\mathbf{X}_{k, m}^{+}[\mu], \overline{\mathbf{X}}_{k, m}^{+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=\left\langle\mathbf{X}_{k, m}^{-}[\mu], \overline{\mathbf{X}}_{k, m}^{-}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} .
$$

Proof. By definition of the integral (1.2.2) it follows that

$$
\begin{aligned}
& \left\langle\mathbf{X}_{k, m}^{+}[\mu], \overline{\mathbf{X}}_{k, m}^{+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & \int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{k, m}^{+}[\mu]\right]_{0}^{2}-\left[\mathbf{X}_{k, m}^{+}[\mu]\right]_{1}^{2}-\left[\mathbf{X}_{k, m}^{+}[\mu]\right]_{2}^{2}\right) d \mathbf{x} \\
= & \int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left(V_{k, m}[\mu]\right)^{2} d \eta d \vartheta \int_{0}^{2 \pi} \cos ^{2}(m \varphi) d \varphi \\
& -\frac{1}{4} \int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left[(k+1+m) V_{k, m-1}[\mu]-\frac{1}{k+2+m} V_{k, m+1}[\mu]\right]^{2} d \eta d \vartheta \\
& \times \int_{0}^{2 \pi} \cos ^{2}(m \varphi) d \varphi \\
& -\frac{1}{4} \int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left[(k+1+m) V_{k, m-1}[\mu]-\frac{1}{k+2+m} V_{k, m+1}[\mu]\right]^{2} d \eta d \vartheta \\
& \times \int_{0}^{2 \pi} \sin ^{2}(m \varphi) d \varphi .
\end{aligned}
$$

Since $m \neq 0$, the two values $\int_{0}^{2 \pi}\left[\Phi_{m}^{ \pm}(\varphi)\right]^{2} d \varphi$ are equal, and therefore

$$
\begin{aligned}
\left\langle\mathbf{X}_{k, m}^{+}[\mu], \overline{\mathbf{X}}_{k, m}^{+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} & =\int_{\Omega_{\mu}}\left(\left[\mathbf{X}_{k, m}^{-}[\mu]\right]_{0}^{2}-\left[\mathbf{X}_{k, m}^{-}[\mu]\right]_{1}^{2}-\left[\mathbf{X}_{k, m}^{-}[\mu]\right]_{2}^{2}\right) d \mathbf{x} \\
& =\left\langle\mathbf{X}_{k, m}^{-}[\mu], \overline{\mathbf{X}}_{k, m}^{-}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}
\end{aligned}
$$

Unfortunately, it is not clear how to extract an orthogonal basis of ambigenic functions from the list $\left\{\mathbf{X}_{l, m}^{ \pm}[\mu], \overline{\mathbf{X}}_{l, m}^{ \pm}[\mu]\right\}$ when $\mu \neq 0$. It will now be shown that this can be solved by making a slight modification of the ideas given in [17]. Bearing in mind the Gram-Schmidt orthogonalization procedure, a definition of the ambigenic spheroidal functions for the interior of the prescribed spheroids is as follows.

Definition 3.2.3. Let $l \geq 0$. The basic internal ambigenic spheroidal polynomials of degree $l$ and order $m$ are

$$
\begin{aligned}
\mathbf{Y}_{l, m}^{++}[\mu] & =\mathbf{X}_{l, m}^{+}[\mu] \text { for } m=0, \ldots, l+1, \\
\mathbf{Y}_{l, m}^{-+}[\mu] & =\mathbf{X}_{l, m}^{-}[\mu] \text { for } m=1, \ldots, l, \\
\mathbf{Y}_{l, m}^{+-}[\mu] & =\overline{\mathbf{X}}_{l, m}^{+}[\mu]-\gamma_{l, m}[\mu] \mathbf{X}_{l, m}^{+}[\mu] \text { for } m=0, \ldots, l, \\
\mathbf{Y}_{l, m}^{--}[\mu] & =\overline{\mathbf{X}}_{l, m}^{-}[\mu]-\gamma_{l, m}[\mu] \mathbf{X}_{l, m}^{-}[\mu] \text { for } m=1, \ldots, l+1,
\end{aligned}
$$

where

$$
\gamma_{l, m}[\mu]=\left\{\begin{array}{cc}
\frac{\left\langle\mathbf{X}_{l, m}^{+}[\mu], \overline{\mathbf{X}}_{l, m}^{+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}}{\left\|\mathbf{X}_{l, m}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}} & \text { if } 0 \leq m \leq l  \tag{3.2.1}\\
0 & \text { if } m=l+1
\end{array}\right.
$$

The result of [17], which gives the construction of an orthogonal basis for the space of ambigenic functions defined in $\Omega_{0}$, is a particular case of the following more general result:

Proposition 3.2.4. For fixed $\mu$, the collection of $2 l(l+3)+3$ polynomials

$$
\begin{aligned}
& \left\{\mathbf{Y}_{k, m}^{++}: 0 \leq m \leq k+1\right\} \cup\left\{\mathbf{Y}_{k, m}^{-+}: 1 \leq m \leq k\right\} \\
& \\
& \cup\left\{\mathbf{Y}_{k, m}^{+-}: 0 \leq m \leq k\right\} \cup\left\{\mathbf{Y}_{k, m}^{--}: 1 \leq m \leq k+1\right\}
\end{aligned}
$$

where $0 \leq k \leq l$, forms an orthogonal basis in $L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$ for the subspace of ambigenic polynomials of degree at most $l$.

Proof. Bearing in mind that $\mu$ is fixed, we write $\mathbf{X}_{k, m}^{ \pm}, \mathbf{Y}_{k, m}^{ \pm \pm}, \gamma_{k, m}$ for $\mathbf{X}_{k, m}^{ \pm}[\mu]$, $\mathbf{Y}_{k, m}^{ \pm, \pm}[\mu], \gamma_{k, m}[\mu]$. Since there are $2 l(l+3)+3$ polynomials in the given list, it suffices to prove the orthogonality to conclude that they generate the ambigenic polynomials. Now, because the collection $\left\{\mathbf{X}_{k, m}^{ \pm}: m=0, \ldots, k+\right.$ $1 ; k=0, \ldots, l\}$ is an orthogonal basis of $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}\right)$, it then follows that

$$
\begin{aligned}
& \left\langle\mathbf{Y}_{k, m}^{++}, \overline{\mathbf{Y}}_{k, m}^{-+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=\left\langle\mathbf{Y}_{k, m}^{++}, \overline{\mathbf{Y}}_{k, m}^{--}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& =\left\langle\mathbf{Y}_{k, m}^{+-}, \overline{\mathbf{Y}}_{k, m}^{-+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=\left\langle\mathbf{Y}_{k, m}^{+-}, \overline{\mathbf{Y}}_{k, m}^{--}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0 .
\end{aligned}
$$

Since

$$
\begin{aligned}
& \left\langle\mathbf{Y}_{k_{1}, m_{1}}^{+-}, \mathbf{Y}_{k_{2}, m_{2}}^{+-}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& =\left\langle\overline{\mathbf{X}}_{k_{1}, m_{1}}^{+}-\gamma_{k_{1}, m_{1}} \mathbf{X}_{k_{1}, m_{1}}^{+}, \overline{\mathbf{X}}_{k_{2}, m_{2}}^{+}-\gamma_{k_{2}, m_{2}} \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& = \\
& =\left\langle\overline{\mathbf{X}}_{k_{1}, m_{1}}^{+}, \overline{\mathbf{X}}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}-\gamma_{k_{2}, m_{2}}\left\langle\overline{\mathbf{X}}_{k_{1}, m_{1}}^{+}, \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& \quad-\gamma_{k_{1}, m_{1}}\left\langle\mathbf{X}_{k_{1}, m_{1}}^{+}, \overline{\mathbf{X}}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}+\gamma_{k_{1}, m_{1}} \gamma_{k_{2}, m_{2}}\left\langle\mathbf{X}_{k_{1}, m_{1}}^{+}, \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)},
\end{aligned}
$$

it will be enough to study $\left\langle\overline{\mathbf{X}}_{k_{1}, m_{1}}^{+}, \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}$ and $\left\langle\mathbf{X}_{k_{1}, m_{1}}^{+}, \overline{\mathbf{X}}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}$ :

$$
\begin{aligned}
\left\langle\overline{\mathbf{X}}_{k_{1}, m_{1}}^{+}, \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}= & \int_{\Omega_{\mu}}\left[\left[\mathbf{X}_{k_{1}, m_{1}}^{+}\right]_{0}\left[\mathbf{X}_{k_{2}, m_{2}}^{+}\right]_{0}\right. \\
& \left.-\left(\left[\mathbf{X}_{k_{1}, m_{1}}^{+}\right]_{1}\left[\mathbf{X}_{k_{2}, m_{2}}^{+}\right]_{1}+\left[\mathbf{X}_{k_{1}, m_{1}}^{+}\right]_{2}\left[\mathbf{X}_{k_{2}, m_{2}}^{+}\right]_{2}\right)\right] d \mathbf{x}
\end{aligned}
$$

but from the proof of Theorem 3.1.10, we have

$$
\begin{aligned}
& \left\langle\overline{\mathbf{X}}_{k_{1}, m_{1}}^{+}, \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& =\left(\left\|\operatorname{Sc}\left(\mathbf{X}_{k_{1}, m_{1}}^{+}\right)\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}-\left\|\operatorname{Vec}\left(\mathbf{X}_{k_{1}, m_{1}}^{+}\right)\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}\right) \delta_{k_{1}, k_{2}} \delta_{m_{1}, m_{2}}
\end{aligned}
$$

Now, we find that

$$
\left\langle\mathbf{Y}_{k_{1}, m_{1}}^{++}, \mathbf{Y}_{k_{2}, m_{2}}^{+-}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=\left\langle\mathbf{X}_{k_{1}, m_{1}}^{+}, \overline{\mathbf{X}}_{k_{2}, m_{2}}^{+}-\gamma_{k_{2}, m_{2}} \mathbf{X}_{k_{2}, m_{2}}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} .
$$

By the above observations, we see that these polynomials are orthogonal when $k_{1} \neq k_{2}$ or $m_{1} \neq m_{2}$, and when the indices coincide,

$$
\begin{aligned}
\left\langle\mathbf{Y}_{k, m}^{++}, \mathbf{Y}_{k, m}^{+-}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}= & \left\langle\mathbf{X}_{k, m}^{+}, \overline{\mathbf{X}}_{k, m}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& -\frac{\left\langle\mathbf{X}_{k, m}^{+}, \overline{\left.\mathbf{X}_{k, m}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}}\left\|\mathbf{X}_{k, m}^{+}\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{+}\right.}{\|}\left\|_{k, m}^{2}\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2} \\
= & 0
\end{aligned}
$$

Moreover, by the orthogonality of the system $\left\{\Phi_{k}^{+}, \Phi_{l}^{-}: k \geq 0, l>0\right\}$ in $[0,2 \pi]$, it is clear that

$$
\left\langle\mathbf{Y}_{k_{1}, m_{1}}^{++}, \mathbf{Y}_{k_{2}, m_{2}}^{--}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0
$$

and further

$$
\left\langle\mathbf{Y}_{k, m}^{++}, \mathbf{Y}_{k, m}^{--}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0
$$

Similarly, by Lemma 3.2.2, we have

$$
\begin{aligned}
\left\langle\mathbf{Y}_{k, m}^{-+}, \mathbf{Y}_{k, m}^{--}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}= & \left\langle\mathbf{X}_{k, m}^{-}, \overline{\mathbf{X}}_{k, m}^{-}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& -\frac{\left\langle\mathbf{X}_{k, m}^{+}, \overline{\mathbf{X}}_{k, m}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}}{\left\|\mathbf{X}_{k, m}^{+}\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}}\left\|\mathbf{X}_{k, m}^{-}\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2} \\
= & 0
\end{aligned}
$$

which completes the proof.

### 3.2.2 Contragenic Spheroidal Polynomials

Contragenic functions were unknown until [17]. Their very existence was not contemplated until that study was published. But to be able to consider the "monogenic part" of a given harmonic function, it is of the utmost importance that contragenic functions are understood. Unlike harmonicity and monogenicity, contragenicity is not a local property as it depends on the domain under consideration. Therefore, it cannot be defined by the direct application of any differential operator.

The dimensions over $\mathbb{R}$ of the relevant polynomial spaces are summarized in Table 3.4. The subscript $*$ refers to polynomials of degree at most $l$.

| Space of polynomials | $\operatorname{dim}_{\mathbb{R}}$ |
| :---: | :---: |
| $\operatorname{Har}_{l}^{*}(\mathbb{R})$ | $(l+1)^{2}$ |
| $\operatorname{Har}_{l}^{*}\left(\mathbb{R}^{3}\right)$ | $3(l+1)^{2}$ |
| $\mathcal{M}_{l}^{*}, \overline{\mathcal{M}}_{l}^{*}$ | $(l+1)(l+3)$ |
| $\mathcal{M}_{l}^{*} \cap \overline{\mathcal{M}}_{l}^{*}$ | $2 l+3$ |
| $\mathcal{M}_{l}^{*}+\overline{\mathcal{M}}_{l}^{*}$ | $2 l(l+3)+3$ |

Table 3.4: Dimensions of spaces of polynomials $(l \geq 0)$.

Table 3.4 refers to polynomials defined in the whole $\mathbb{R}^{3}$, or likewise as their restrictions to any domain. For compact domains such as the spheroids $\Omega_{\mu}$, the functions are automatically square-integrable.

Orthogonal complements can be used to quantify how a harmonic function is not ambigenic. Since, by definition, real-valued functions are orthogonal in $L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$ to functions taking values in $\mathbf{i} \mathbb{R}+\mathbf{j} \mathbb{R}$, there is a natural orthogonal direct sum decomposition of the space of square-integrable ambigenic functions [17], namely

## Proposition 3.2.5.

$$
\begin{align*}
& \mathcal{M}_{2}\left(\Omega_{\mu}\right)+\overline{\mathcal{M}}_{2}\left(\Omega_{\mu}\right)=\operatorname{Sc} \mathcal{M}_{2}\left(\Omega_{\mu}\right) \oplus \operatorname{Vec} \mathcal{M}_{2}\left(\Omega_{\mu}\right),  \tag{3.2.2}\\
& \mathcal{M}_{2}\left(\Omega_{\mu}\right) \cap \overline{\mathcal{M}}_{2}\left(\Omega_{\mu}\right) \subseteq \operatorname{Vec} \mathcal{M}_{2}\left(\Omega_{\mu}\right)
\end{align*}
$$

Using the previous result, we are positioned to formulate the following general definition of a contragenic function.

Definition 3.2.6. In any domain $\Omega \subseteq \mathbb{R}^{3}$, a function $\boldsymbol{h} \in \operatorname{Har}(\Omega) \cap L_{2}(\Omega, \mathcal{A})$ is called $\Omega$-contragenic when it is orthogonal in $L_{2}(\Omega, \mathcal{A})$ to all square-integrable ambigenic functions, that is, if it lies in

$$
\mathcal{N}(\Omega)=\left(\mathcal{M}_{2}(\Omega)+\overline{\mathcal{M}}_{2}(\Omega)\right)^{\perp}
$$

where the orthogonal complement is taken in $\operatorname{Har}(\Omega) \cap L_{2}(\Omega, \mathcal{A})$.
Unlike the spaces of harmonic, monogenic, antimonogenic, and ambigenic functions, the above definition of $\mathcal{N}(\Omega)$ involves the $L_{2}$-inner product. Thus, it depends on domain $\Omega$, which cannot be omitted from the notation without ambiguity.

Our principal concern is a basis for the contragenic spheroidal functions, which will enable us to express an arbitrary harmonic function predictably as a sum of an ambigenic function and a contragenic function in $\Omega_{\mu}$.

The following discussion focuses on spaces of polynomials of degree no greater than $l$. As regards Definition 3.2.6, let $\mathcal{N}_{l}\left(\Omega_{\mu}\right) \subset \mathcal{N}\left(\Omega_{\mu}\right)$ denote the subspace of contragenic polynomials of degree $l$, and let $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)=$ $\bigcup_{k=0}^{l} \mathcal{N}_{k}\left(\Omega_{\mu}\right)$ be the subspace of contragenic polynomials of degree no greater than $l$. Nonzero constant harmonic functions are never contragenic so that we will have no use for $\mathcal{N}_{0}^{*}\left(\Omega_{\mu}\right)=\{\mathbf{0}\}$.

Thus we have the successive orthogonal complements

$$
\mathcal{N}_{l}\left(\Omega_{\mu}\right)=\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right) \ominus \mathcal{N}_{l-1}^{*}\left(\Omega_{\mu}\right),
$$

which are composed of polynomials of degree precisely $l$. It can further be seen that

$$
\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)=\bigoplus_{k=1}^{l} \mathcal{N}_{k}\left(\Omega_{\mu}\right),
$$

and there is a Hilbert space orthogonal decomposition

$$
\mathcal{N}^{*}\left(\Omega_{\mu}\right)=\bigoplus_{k=1}^{\infty} \mathcal{N}_{k}\left(\Omega_{\mu}\right)
$$

of the full collection of contragenic functions in $L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$.
It was shown in [17] that the homogeneous polynomials of degree $l$, which are contragenic on $\Omega_{0}$, form a space of dimension $2 l-1$ with $l \geq 1$ (this dimension count is simply the difference of $3 \operatorname{dim} \operatorname{Har}_{l}^{+}\left(\Omega_{0}\right)=3(2 l+1)$ and $\left.\operatorname{dim}\left(\mathcal{M}_{l}^{+}\left(\Omega_{0}\right)+\overline{\mathcal{M}}_{l}^{+}\left(\Omega_{0}\right)\right)=4 l+4\right)$. The same study also observed that if a real-valued harmonic homogeneous polynomial is completed as the scalar part of a monogenic function (unique up to adding a monogenic constant), then the vector part can also be seen as a homogeneous polynomial of the
same degree. Since the spheroidal harmonics and monogenics are generally not homogeneous (when $\mu \neq 0$ ), it is preferable to combine the dimensions up to $l$; we have $\operatorname{dim} \mathcal{N}_{l}^{*}\left(\Omega_{0}\right)=l^{2}$ for the unit ball. Now, because the dimension of an orthogonal complement within a fixed vector space does not depend upon the choice of the $L_{2}$-inner product, and the harmonic and the ambigenic polynomials of degree less than or equal to $l$ do not rely on the domain under consideration, it is clear that, in general, we have $\operatorname{dim} \mathcal{N}_{l}\left(\Omega_{\mu}\right)=2 l-1$ also for $l \geq 1$.

Accordingly, we have the following result:

## Proposition 3.2.7.

$$
\operatorname{dim} \mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)=\sum_{k=0}^{l} \operatorname{dim} \mathcal{N}_{k}\left(\Omega_{\mu}\right)=l^{2} .
$$

By using the vector parts of the basic monogenic spheroidal polynomials (3.1.1) as building blocks, an orthogonal basis of $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$ for $\mu \neq 0$ can be found, such as is required. In this connection, we define the contragenic spheroidal functions to be employed for the space interior of the prescribed spheroids as follows.

Definition 3.2.8. Let $l \geq 1$ and $0 \leq m \leq l-1$. The basic internal contragenic spheroidal polynomials of degree $l$ and order $m$ are

$$
\begin{align*}
\mathbf{Z}_{l, m}^{ \pm}[\mu]= & \frac{a_{l, m}[\mu]}{l+1+m}\left[\operatorname{Vec}\left(\mathbf{X}_{l, m}^{\mp}[\mu]\right) \mp \operatorname{Vec}\left(\mathbf{X}_{l, m}^{ \pm}[\mu]\right) \mathbf{k}\right] \\
& -\left[\operatorname{Vec}\left(\mathbf{X}_{l, m}^{\mp}[\mu]\right) \pm \operatorname{Vec}\left(\mathbf{X}_{l, m}^{ \pm}[\mu]\right) \mathbf{k}\right], \tag{3.2.3}
\end{align*}
$$

where

$$
\begin{equation*}
a_{l, m}[\mu]=\frac{1}{l+1+m}\left(\frac{\left\|V_{l, m+1}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}}{(l+2+m)\left\|V_{l, m-1}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}}\right)^{2} . \tag{3.2.4}
\end{equation*}
$$

By Lemma 3.1.2, it follows that $a_{l, 0}[\mu]=l+1$. Thus,

$$
\mathbf{Z}_{l, 0}^{+}[\mu]=-2 \operatorname{Vec}\left(\mathbf{X}_{l, 0}^{+}[\mu]\right) \mathbf{k} .
$$

We have not yet proved that the proposed polynomials are indeed contragenic. That it is so, will be shown hereafter. Given Theorem 3.1.4, we shall now express the basic polynomials (3.2.3) in terms of the $\mathbf{i}, \mathbf{j}$ components of the basic monogenic polynomials (3.1.1) in a completed developed form.

By Definition 3.2.8, we set

$$
\begin{equation*}
b_{l, m}^{ \pm}[\mu]:=\frac{a_{l, m}[\mu] \pm(l+1+m)}{l+1+m} \tag{3.2.5}
\end{equation*}
$$

It then follows that

$$
\begin{aligned}
\mathbf{Z}_{l, m}^{ \pm}[\mu]= & \mathbf{i}\left(b_{l, m}^{-}[\mu]\left[\mathbf{X}_{l, m}^{\mp}[\mu]\right]_{1} \mp b_{l, m}^{+}[\mu]\left[\mathbf{X}_{l, m}^{ \pm}[\mu]\right]_{2}\right) \\
& +\mathbf{j}\left(b_{l, m}^{-}[\mu]\left[\mathbf{X}_{l, m}^{\mp}[\mu]\right]_{2} \pm b_{l, m}^{+}[\mu]\left[\mathbf{X}_{l, m}^{ \pm}[\mu]\right]_{1}\right) .
\end{aligned}
$$

Moreover, from $\mathbf{X}_{l, m}^{ \pm}[\mu] \mathbf{k}=\mathbf{i}\left[\mathbf{X}_{l, m}^{ \pm}[\mu]\right]_{2}-\mathbf{j}\left[\mathbf{X}_{l, m}^{ \pm}[\mu]\right]_{1}+\mathbf{k}\left[\mathbf{X}_{l, m}^{ \pm}[\mu]\right]_{0}$, the above expression becomes

$$
\begin{align*}
\mathbf{Z}_{l, m}^{ \pm}[\mu]= & \mathbf{i}\left(a_{l, m}[\mu] V_{l, m-1}^{\mp}[\mu]+\frac{1}{l+2+m} V_{l, m+1}^{\mp}[\mu]\right) \\
& \pm \mathbf{j}\left(a_{l, m}[\mu] V_{l, m-1}^{ \pm}[\mu]-\frac{1}{l+2+m} V_{l, m+1}^{ \pm}[\mu]\right) \tag{3.2.6}
\end{align*}
$$

for $1 \leq m \leq l-1$, while for $m=0$,

$$
\mathbf{Z}_{l, 0}^{+}[\mu]=\frac{1}{l+2}\left(\mathbf{i} V_{l, 1}^{-}[\mu]-\mathbf{j} V_{l, 1}^{+}[\mu]\right),
$$

where the proper functions $V_{l, m}^{ \pm}[\mu]$ are defined by (2.2.7).
Some examples of $\mathbf{Z}_{l, m}^{ \pm}[\mu]$ in low degree provided by (3.2.6) are given in Table 3.5

| $l$ | $m$ | $\mathbf{Z}_{l, m}^{ \pm}[\mu]$ |
| :---: | :---: | :---: |
| 1 | 0 | $\mathbf{Z}_{1,0}^{+}=-\mathbf{i} x_{2}+\mathbf{j} x_{1}$ |
| 2 | 0 | $\mathbf{Z}_{2,0}^{+}=-3 \mathbf{i} x_{0} x_{2}+3 \mathbf{j} x_{0} x_{1}$ |
|  | 1 | $\begin{aligned} \mathbf{Z}_{2,1}^{+}= & 6 \mathbf{i} x_{1} x_{2}+\frac{3 \mathbf{j}}{30-20 \mu^{2}+6 \mu^{4}}\left[25 x_{2}^{2}-2 \mu^{2}-10 x_{2}^{2} \mu^{2}\right. \\ & +4 \mu^{4}+x_{2}^{2} \mu^{4}-2 \mu^{6}+10 x_{0}^{2}\left(-1+\mu^{2}\right)^{2} \\ & \left.+x_{1}^{2}\left(-35+30 \mu^{2}-11 \mu^{4}\right)\right] \\ \mathbf{Z}_{2,1}^{-}= & \frac{3 \mathbf{i}}{30-20 \mu^{2}+6 \mu^{4}}\left[-35 x_{2}^{2}-2 \mu^{2}+30 x_{2}^{2} \mu^{2}+4 \mu^{4}\right. \\ & -11 x_{2}^{2} \mu^{4}-2 \mu^{6}+x_{1}^{2}\left(-5+\mu^{2}\right)^{2} \\ & \left.+10 x_{0}^{2}\left(-1+\mu^{2}\right)^{2}\right]+6 \mathbf{j} x_{1} x_{2} \end{aligned}$ |
| 3 | 0 | $\begin{aligned} \mathbf{Z}_{3,0}^{+}= & \frac{3}{14} \mathbf{i} x_{2}\left(-28 x_{0}^{2}+7 x_{1}^{2}+7 x_{2}^{2}+4 \mu^{2}\right) \\ & -\frac{3}{14} \mathbf{j} x_{1}\left(-28 x_{0}^{2}+7 x_{1}^{2}+7 x_{2}^{2}+4 \mu^{2}\right) \end{aligned}$ |
|  | 1 | $\begin{aligned} \mathbf{Z}_{3,1}^{+}= & 30 \mathbf{i} x_{0} x_{1} x_{2}+\frac{15 \mathbf{j} x_{0}}{70-84 \mu^{2}+30 \mu^{4}}\left[49 x_{2}^{2}\right. \\ & -6 \mu^{2}-42 x_{2}^{2} \mu^{2}+12 \mu^{4}+9 x_{2}^{2} \mu^{4}-6 \mu^{6}+14 x_{0}^{2}\left(-1+\mu^{2}\right)^{2} \\ & \left.+x_{1}^{2}\left(-91+126 \mu^{2}-51 \mu^{4}\right)\right] \\ \mathbf{Z}_{3,1}^{-}= & \frac{15 \mathbf{i} x_{0}}{70-84 \mu^{2}+30 \mu^{4}}\left[-91 x_{2}^{2}-6 \mu^{2}\right. \\ & +126 x_{2}^{2} \mu^{2}+12 \mu^{4}-51 x_{2}^{2} \mu^{4}-6 \mu^{6}+x_{1}^{2}\left(7-3 \mu^{2}\right)^{2} \\ & \left.+14 x_{0}^{2}\left(-1+\mu^{2}\right)^{2}\right]+30 \mathbf{j} x_{0} x_{1} x_{2} \end{aligned}$ |
|  | 2 | $\begin{aligned} \mathbf{Z}_{3,2}^{+}= & -\frac{30 \mathbf{i} x_{2}}{35-14 \mu^{2}+3 \mu^{4}}\left[-21 x_{2}^{2}-2 \mu^{2}+14 x_{2}^{2} \mu^{2}+4 \mu^{4}\right. \\ & \left.-5 x_{2}^{2} \mu^{4}-2 \mu^{6}+x_{1}^{2}\left(-7+\mu^{2}\right)^{2}+14 x_{0}^{2}\left(-1+\mu^{2}\right)^{2}\right] \\ & -\frac{30 \mathbf{j} x_{1}}{35-14 \mu^{2}+3 \mu^{4}}\left[49 x_{2}^{2}-2 \mu^{2}-14 x_{2}^{2} \mu^{2}+4 \mu^{4}+x_{2}^{2} \mu^{4}\right. \\ & \left.-2 \mu^{6}+14 x_{0}^{2}\left(-1+\mu^{2}\right)^{2}+x_{1}^{2}\left(-21+14 \mu^{2}-5 \mu^{4}\right)\right] \\ \mathbf{Z}_{3,2}^{-}= & \frac{60 \mathbf{i} x_{1}}{35-14 \mu^{2}+3 \mu^{4}}\left[28 x_{2}^{2}+\mu^{2}-14 x_{2}^{2} \mu^{2}-2 \mu^{4}+4 x_{2}^{2} \mu^{4}\right. \\ & \left.+\mu^{6}-7 x_{0}^{2}\left(-1+\mu^{2}\right)^{2}+x_{1}^{2}\left(-7+\mu^{4}\right)\right] \\ & -\frac{60 \mathbf{j} x_{2}}{35-14 \mu^{2}+3 \mu^{4}}\left[-7 x_{2}^{2}+\mu^{2}-2 \mu^{4}+x_{2}^{2} \mu^{4}\right. \\ & \left.+\mu^{6}-7 x_{0}^{2}\left(-1+\mu^{2}\right)^{2}+2 x_{1}^{2}\left(14-7 \mu^{2}+2 \mu^{4}\right)\right] \end{aligned}$ |

Table 3.5: Spheroidal contragenic polynomials of low degree, parametrized by the eccentricity $\mu$.

The general theorem is the following:
Theorem 3.2.9. The $l^{2}$ polynomials $\mathbf{Z}_{k, m}^{ \pm}[\mu]$ (with $1 \leq k \leq l, 0 \leq m \leq k-1$ ) are contragenic. Further, they form an orthogonal basis for $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$ when $\mu$ is fixed.
Proof. For simplicity, we will continue to write $\mathbf{X}_{l, m}^{ \pm}, \mathbf{Z}_{l, m}^{ \pm}$, in place of $\mathbf{X}_{l, m}^{ \pm}[\mu]$, $\mathbf{Z}_{l, m}^{ \pm}[\mu]$ considering that $\mu$ is fixed. First, we prove that $\mathbf{Z}_{k, m}^{ \pm}$are contragenic. As they have no scalar parts, it suffices to show that they are orthogonal to $\operatorname{Vec} \mathcal{M}_{l}^{*}\left(\Omega_{\mu}\right)$. To do this, we use the basis obtained by dropping the scalar parts of the basis for $\mathcal{M}_{l}^{*}\left(\Omega_{\mu}\right)$ given in Theorem 3.1.10. Since

$$
\left\{\Phi_{m_{1}}^{+}, \Phi_{m_{2}}^{-}: m_{1} \geq 0, m_{2} \geq 1\right\}
$$

is a system of orthogonal functions in $[0,2 \pi]$, then when $1 \leq m_{1} \leq k_{1}$ and $1 \leq m_{2} \leq k_{2}$, we readily see that

$$
\left\langle\mathbf{Z}_{k_{1}, m_{1}}^{+}, \operatorname{Vec}\left(\mathbf{X}_{k_{2}, m_{2}}^{+}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=\left\langle\mathbf{Z}_{k_{1}, m_{1}}^{-}, \operatorname{Vec}\left(\mathbf{X}_{k_{2}, m_{2}}^{-}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0
$$

On the other hand, when $m_{1}>0$ and $m_{2} \geq 0$, we have that

$$
\begin{aligned}
\left\langle\mathbf{Z}_{k_{1}, m_{1}}^{ \pm}, \operatorname{Vec}\left(\mathbf{X}_{k_{2}, m_{2}}^{\mp}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}= & b_{k_{1}, m_{1}}^{-} \int_{\Omega_{\mu}}\left[\mathbf{X}_{k_{1}, m_{1}}^{\mp}\right]_{1}\left[\mathbf{X}_{k_{2}, m_{2}}^{\mp}\right]_{1} d \mathbf{x} \\
& \mp b_{k_{1}, m_{1}}^{+} \int_{\Omega_{\mu}}\left[\mathbf{X}_{k_{1}, m_{1}}^{ \pm}\right]_{2}\left[\mathbf{X}_{k_{2}, m_{2}}^{\mp}\right]_{1} d \mathbf{x} \\
& +b_{k_{1}, m_{1}}^{-} \int_{\Omega_{\mu}}\left[\mathbf{X}_{k_{1}, m_{1}}^{\mp}\right]_{2}\left[\mathbf{X}_{k_{2}, m_{2}}^{\mp}\right]_{2} d \mathbf{x} \\
& \pm b_{k_{2}, m_{2}}^{+} \int_{\Omega_{\mu}}\left[\mathbf{X}_{k_{1}, m_{1}}^{ \pm}\right]_{1}\left[\mathbf{X}_{k_{2}, m_{2}}^{\mp}\right]_{2} d \mathbf{x}
\end{aligned}
$$

where the coefficients $b_{k, m}^{ \pm}$are defined by (3.2.5).
Since the system

$$
\left\{\operatorname{Vec}\left(\mathbf{X}_{k, m}^{+}\right), \operatorname{Vec}\left(\mathbf{X}_{j, l}^{-}\right): 0 \leq k \leq l, 0 \leq m \leq k, 1 \leq j \leq l, 1 \leq l \leq j\right\}
$$

is orthogonal, the previous integrals may be written as

$$
\begin{align*}
& \left\langle\mathbf{Z}_{k_{1}, m_{1}}^{ \pm}, \operatorname{Vec}\left(\mathbf{X}_{k_{2}, m_{2}}^{\mp}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}^{\eta_{\mu}} \\
= & \frac{\pi}{2}\left[2 a_{k_{1}, m_{1}}\left(k_{1}+1+m_{1}\right) \int_{0}^{\eta_{\mu}} \int_{0}^{\pi}\left(V_{k_{1}, m_{1}-1}[\mu]\right)^{2} d \vartheta d \eta\right. \\
& -\frac{2}{\left(k_{1}+2+m_{1}\right)^{2}} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi}\left(V_{k_{1}, m_{1}+1}[\mu]\right)^{2} d \vartheta d \eta \\
& \left.\mp \frac{2 \delta_{0, m_{1}}}{\left(k_{1}+2\right)^{2}} \int_{0}^{\eta_{\mu}} \int_{0}^{\pi}\left(V_{k_{1}, 1}[\mu]\right)^{2} d \vartheta d \eta\right] \delta_{k_{1}, k_{2}} \delta_{m_{1}, m_{2}} . \tag{3.2.7}
\end{align*}
$$

Furthermore, using the expression (3.2.6) and recalling that

$$
\begin{aligned}
2 \operatorname{Vec}\left(\mathbf{X}_{k, m}^{-}\right)= & \frac{\mathbf{i}}{2}\left[(k+1+m) V_{k, m-1}^{-}[\mu]-\frac{1}{k+2+m} V_{k, m+1}^{-}[\mu]\right] \\
& +\frac{\mathbf{j}}{2}\left[(k+1+m) V_{k, m-1}^{+}[\mu]+\frac{1}{k+2+m} V_{k, m+1}^{+}[\mu]\right]
\end{aligned}
$$

when $m>0$, by (3.2.4), it hence follows that

$$
\begin{aligned}
& \left\langle\mathbf{Z}_{k, m}^{+}, \operatorname{Vec}\left(\mathbf{X}_{k, m}^{-}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & \frac{1}{2}\left[\int_{\Omega_{\mu}}\left(a_{k, m} V_{k, m-1}^{-}[\mu]+\frac{1}{k+2+m} V_{k, m+1}^{-}[\mu]\right)\right. \\
& \times\left((k+1+m) V_{k, m-1}^{-}[\mu]-\frac{1}{k+2+m} V_{k, m+1}^{-}[\mu]\right) d \mathbf{x} \\
& +\int_{\Omega_{\mu}}\left(a_{k, m} V_{k, m-1}^{+}[\mu]-\frac{1}{k+2+m} V_{k, m+1}^{+}[\mu]\right) \\
& \left.\times\left((k+1+m) V_{k, m-1}^{+}[\mu]+\frac{1}{k+2+m} V_{k, m+1}^{+}[\mu]\right) d \mathbf{x}\right] \\
= & a_{k, m}\left\|V_{k, m-1}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2}-\frac{1}{(k+2+m)^{2}}\left\|V_{k, m+1}^{+}[\mu]\right\|_{L_{2}\left(\Omega_{\mu}\right)}^{2} \\
= & 0 .
\end{aligned}
$$

Similarly, the orthogonality of $\left\{\Phi_{m}^{+}, \Phi_{l}^{-}\right\}$gives

$$
\left\langle\mathbf{Z}_{k, m}^{-}, \operatorname{Vec}\left(\mathbf{X}_{k, m}^{+}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0
$$

Next, we compute

$$
\begin{aligned}
& \left\langle\mathbf{Z}_{k_{1}, 0}^{+}, \operatorname{Vec}\left(\mathbf{X}_{k_{2}, m}^{ \pm}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & \frac{1}{2\left(k_{1}+2\right)}\left[\left(k_{2}+1+m\right) \int_{\Omega_{\mu}} V_{k_{1}, 1}^{-}[\mu] V_{k_{2}, m-1}^{ \pm}[\mu] d \mathbf{x}\right. \\
& -\frac{1}{k_{2}+2+m} \int_{\Omega_{\mu}} V_{k_{1}, 1}^{-}[\mu] V_{k_{2}, m+1}^{ \pm}[\mu] d \mathbf{x} \\
& \pm\left(\left(k_{2}+1+m\right) \int_{\Omega_{\mu}} V_{k_{1}, 1}^{+}[\mu] V_{k_{2}, m-1}^{\mp}[\mu] d \mathbf{x}\right. \\
& \left.\left.+\frac{1}{k_{2}+2+m} \int_{\Omega_{\mu}} V_{k_{1}, 1}^{+}[\mu] V_{k_{2}, m+1}^{\mp}[\mu] d \mathbf{x}\right)\right] \\
= & 0
\end{aligned}
$$

which follows by the orthogonality of $\left\{\Phi_{m}^{+}, \Phi_{l}^{-}\right\}$.

For $k_{1} \neq k_{2}$, by the orthogonality of the system $\left\{V_{k_{1}, m_{1}}^{+}, V_{k_{2}, m_{2}}^{-}: 0 \leq k_{1} \leq\right.$ $\left.l_{1}, 0 \leq k_{2} \leq l_{2}, 0 \leq m_{1} \leq k_{1}, 1 \leq m_{2} \leq k_{2}, l_{1}, l_{2} \geq 0\right\}$ it remains to check that

$$
\begin{aligned}
& \left\langle\mathbf{Z}_{k, 0}^{+}, \operatorname{Vec}\left(\mathbf{X}_{k, 2}^{-}\right)\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& =\frac{(k+1+m)}{2(k+2)}\left(\int_{\Omega_{\mu}}\left(V_{k, 1}^{-}[\mu]\right)^{2} d \mathbf{x}-\int_{\Omega_{\mu}}\left(V_{k, 1}^{+}[\mu]\right)^{2} d \mathbf{x}\right) \\
& =0
\end{aligned}
$$

where the last equality is a consequence of

$$
\begin{aligned}
\int_{\Omega_{\mu}}\left(V_{k, 1}^{-}[\mu]\right)^{2} d \mathbf{x} & =\int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left(V_{k, 1}[\mu]\right)^{2} d \eta d \vartheta \int_{0}^{2 \pi} \sin ^{2} \varphi d \varphi \\
& =\int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left(V_{k, 1}[\mu]\right)^{2} d \eta d \vartheta \int_{0}^{2 \pi} \cos ^{2} \varphi d \varphi \\
& =\int_{\Omega_{\mu}}\left(V_{k, 1}^{+}[\mu]\right)^{2} d \mathbf{x}
\end{aligned}
$$

We have then verified that the functions $\mathbf{Z}_{k, m}^{ \pm}$are contragenic. It remains to prove the orthogonality of the system

$$
\left\{\mathbf{Z}_{k, m}^{ \pm}: k \geq 1,0 \leq m \leq k-1\right\}
$$

Using the expression (3.2.6), when $m_{1}, m_{2} \geq 1$ we then get

$$
\begin{aligned}
& \left\langle\mathbf{Z}_{k_{1}, m_{1}}^{ \pm}, \mathbf{Z}_{k_{2}, m_{2}}^{ \pm}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & a_{k_{1}, m_{1}} a_{k_{2}, m_{2}} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}-1}^{\mp}[\mu] V_{k_{2}, m_{2}-1}^{\mp}[\mu] d \mathbf{x} \\
& +\frac{a_{k_{1}, m_{1}}}{k_{2}+2+m_{2}} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}-1}^{\mp}[\mu] V_{k_{2}, m_{2}+1}^{\mp}[\mu] d \mathbf{x} \\
& +\frac{a_{k_{2}, m_{2}}}{k_{1}+2+m_{1}} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}+1}^{\mp}[\mu] V_{k_{2}, m_{2}-1}^{\mp}[\mu] d \mathbf{x} \\
& +\frac{1}{\left(k_{1}+2+m_{1}\right)\left(k_{2}+2+m_{2}\right)} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}+1}^{\mp}[\mu] V_{k_{2}, m_{2}+1}^{\mp}[\mu] d \mathbf{x} \\
& +a_{k_{1}, m_{1}} a_{k_{2}, m_{2}} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}-1}^{ \pm}[\mu] V_{k_{2}, m_{2}-1}^{ \pm}[\mu] d \mathbf{x} \\
& -\frac{a_{k_{1}, m_{1}}}{k_{2}+2+m_{2}} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}-1}^{ \pm}[\mu] V_{k_{2}, m_{2}+1}^{ \pm}[\mu] d \mathbf{x} \\
& -\frac{a_{k_{2}, m_{2}}}{k_{1}+2+m_{1}} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}+1}^{ \pm}[\mu] V_{k_{2}, m_{2}-1}^{ \pm}[\mu] d \mathbf{x} \\
& +\frac{1}{\left(k_{1}+2+m_{1}\right)\left(k_{2}+2+m_{2}\right)} \int_{\Omega_{\mu}} V_{k_{1}, m_{1}+1}^{ \pm}[\mu] V_{k_{2}, m_{2}+1}^{ \pm}[\mu] d \mathbf{x} .
\end{aligned}
$$

Thus, by similar reasoning to that used in the proof of Theorem 3.1.10, we obtain

$$
\begin{aligned}
& \left\langle\mathbf{Z}_{k_{1}, m_{1}}^{ \pm}, \mathbf{Z}_{k_{2}, m_{2}}^{ \pm}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & 2 \pi \delta_{m_{1}, m_{2}} \delta_{k_{1}, k_{2}}\left[\left(a_{k_{1}, m_{1}}\right)^{2} \int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left(V_{k_{1}, m_{1}-1}[\mu]\right)^{2} d \eta d \vartheta\right. \\
& \left.+\frac{1}{\left(k_{1}+2+m_{1}\right)^{2}} \int_{0}^{\pi} \int_{0}^{\eta_{\mu}}\left(V_{k_{1}, m_{1}+1}[\mu]\right)^{2} d \eta d \vartheta\right] .
\end{aligned}
$$

On the other hand, when $1 \leq m \leq k_{1}$, we have that

$$
\begin{aligned}
& \left\langle\mathbf{Z}_{k_{1}, 0}^{+}, \mathbf{Z}_{k_{2}, m}^{ \pm}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & \frac{1}{k_{1}+2}\left(\frac{1}{k_{2}+2+m} \int_{\Omega_{\mu}} V_{k_{1}, 1}^{-}[\mu] V_{k_{2}, m+1}^{\mp}[\mu] d \mathbf{x}\right. \\
& +a_{k_{2}, m} \int_{\Omega_{\mu}} V_{k_{1}, 1}^{-}[\mu] V_{k_{2}, m-1}^{\mp}[\mu] d \mathbf{x} \mp a_{k_{2}, m} \int_{\Omega_{\mu}} V_{k_{1}, 1}^{+}[\mu] V_{k_{2}, m-1}^{ \pm}[\mu] d \mathbf{x} \\
& \left. \pm \frac{1}{k_{2}+2+m} \int_{\Omega_{\mu}} V_{k_{1}, 1}^{+}[\mu] V_{k_{2}, m+1}^{ \pm}[\mu] d \mathbf{x}\right) .
\end{aligned}
$$

Whence, it is clear that $\left\langle\mathbf{Z}_{k_{1}, 0}^{+}, \mathbf{Z}_{k_{2}, m}^{-}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$. It remains to check that also $\left\langle\mathbf{Z}_{k_{1}, 0}^{+}, \mathbf{Z}_{k_{2}, m}^{+}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$. But this follows again from the formula for the cosine of a sum of angles and $\int_{0}^{2 \pi} \Phi_{m}^{+} d \varphi=0$. Finally, by the orthogonality of the system $\left\{\Phi_{m}^{ \pm}\right\}$, it then follows that $\left\langle\mathbf{Z}_{k_{1}, m_{1}}^{ \pm}, \mathbf{Z}_{k_{2}, m_{2}}^{\mp}\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$. This furnishes the complete proof.

Theorem 3.2 .9 is the generalization of that of [17], which corresponds to the case of the unit ball. It has, of course, essential consequences. The object here will be to discuss an orthogonal decomposition for the Hilbert space of square-integrable $\mathcal{A}$-valued harmonic functions in $\Omega_{\mu}$. We shall observe that subspaces analogous to the homogeneous polynomials are obtained by defining $\hat{\operatorname{Har}}_{l}\left(\Omega_{\mu}\right)$ as the orthogonal component of $\operatorname{Har}_{l-1}^{*}\left(\Omega_{\mu}\right)$ in $\operatorname{Har}_{l}^{*}\left(\Omega_{\mu}\right)$. This gives an orthogonal Hilbert space decomposition

$$
\operatorname{Har}\left(\Omega_{\mu}\right) \cap L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)=\bigoplus_{l=0}^{\infty} \hat{\operatorname{Har}}_{l}\left(\Omega_{\mu}\right)
$$

For $\mu=0$, this is, in fact, the well-known decomposition by solid spherical harmonics [25, p. 81]. Similarly, let $\widehat{\mathcal{N}}_{l}\left(\Omega_{\mu}\right)$ be the orthogonal component of $\mathcal{N}_{l-1}^{*}\left(\Omega_{\mu}\right)$ in $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$; whence, $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)=\bigoplus_{k=1}^{l} \widehat{\mathcal{N}}_{k}\left(\Omega_{\mu}\right)$. Thus

$$
\begin{equation*}
\widehat{\operatorname{Har}}_{l}\left(\Omega_{\mu}\right)=\widehat{\mathcal{M}}_{l}\left(\Omega_{\mu}\right) \oplus \widehat{\widehat{\mathcal{M}}_{l}\left(\Omega_{\mu}\right)} \oplus \widehat{\mathcal{N}}_{l}\left(\Omega_{\mu}\right), \tag{3.2.8}
\end{equation*}
$$

where the monogenic part $\widehat{\mathcal{M}}_{l}\left(\Omega_{\mu}\right)$ is defined analogously.
From the observations just made, we will now infer a result, which expresses for contragenics the analogy of the well-known denseness of the harmonic polynomials and the monogenic polynomials in the corresponding Hilbert spaces of harmonic and monogenic functions.

Theorem 3.2.10. The functions $\mathbf{Z}_{k, m}^{ \pm}[\mu]$ span a dense set in $\mathcal{N}\left(\Omega_{\mu}\right)$. Therefore the functions $\mathbf{Y}_{k, m}^{ \pm \pm}[\mu], \mathbf{Z}_{k, m}^{ \pm}[\mu]$ form an orthogonal basis for the Hilbert space $\operatorname{Har}\left(\Omega_{\mu}\right) \cap L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)$.

Proof. Let $\mathbf{Z} \in \mathcal{N}\left(\Omega_{\mu}\right)$ be an arbitrary contragenic function. Write $\mathbf{Z}=$ $\sum_{0}^{\infty} \mathbf{U}_{k}$, where $\mathbf{U}_{k} \in \operatorname{Har}_{k}\left(\Omega_{\mu}\right)$, and let $\mathbf{U}_{k}=\mathbf{Y}_{k}+\mathbf{Z}_{k}$ be the decomposition of $\mathbf{U}_{k}$ into ambigenic and contragenic polynomials. Thus $\mathbf{Z}=\mathbf{Y}+\sum_{1}^{\infty} \mathbf{Z}_{k}$, where $\mathbf{Y}=\sum_{0}^{\infty} \mathbf{Y}_{k}$ is both ambigenic and contragenic, $\mathbf{Y}=\mathbf{0}$, and thus $\mathbf{Z}=\sum_{1}^{\infty} \mathbf{Z}_{k}$. Hence, by Proposition 3.2 .4 and Theorem 3.2.9, it then follows that $\mathbf{Z} \in \oplus \widehat{\mathcal{N}}_{k}\left(\Omega_{\mu}\right)$, as required.

To sum up, the orthogonal decomposition of square-integrable harmonic functions in $\Omega_{\mu}$ as $\operatorname{Har}\left(\Omega_{\mu}\right)=\left(\mathcal{M}_{2}\left(\Omega_{\mu}\right)+\overline{\mathcal{M}}_{2}\left(\Omega_{\mu}\right)\right) \oplus \mathcal{N}\left(\Omega_{\mu}\right)$ justifies the idea of referring to the "ambigenic part" or the "monogenic part" of any harmonic function $\Omega_{\mu} \rightarrow \mathcal{A}$ (the latter being determined up to an additive monogenic constant). Theorem 3.2.10 provides a method of calculating this part by obtaining the Fourier coefficients as in any Hilbert space and then discarding the contragenic and antimonogenic terms.

In closing this section, we call attention to the fact that the notion of contragenicity depends on the domain under consideration, which implies that it is not a local property. In this way, contragenicity differs from harmonicity and monogenicity since those are both local properties. For example, the restriction of a contragenic function to a subdomain does not need to be contragenic. In particular, it is impossible to seek a condition on the derivatives of a harmonic function that can detect if it is contragenic or not. It is still unknown whether such a condition exists when associated with a fixed domain, such as a sphere or a spheroid.

### 3.2.3 Relations among Contragenic Functions

In this section, we investigate functions that are contragenic for spheroids of differing eccentricity. Most of our attention is related to different systems of harmonic functions in $\mathcal{N}\left(\Omega_{\mu}\right)$ to those in $\mathcal{N}\left(\Omega_{\widetilde{\mu}}\right)$, where $\mu \neq \widetilde{\mu}$. While manipulating the underlying formulas is essentially algebraic, it should be borne in mind that we are dealing with continuously varying families of function spaces, which are determined by integration over varying domains.

When it is asserted that the notion of orthogonality is different for different spheroids, the assertion must be taken to mean that the definition of a contragenic function does not imply that an $L_{2}$-function belonging to the space $\mathcal{N}_{l}^{*}\left(\Omega_{\widetilde{\mu}}\right)$ should also be in $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$, when $\widetilde{\mu} \neq \mu$. In other words, we may not expect a general formula like

$$
" \mathbf{Z}_{l, m}^{ \pm}[\widetilde{\mu}]=\sum z_{n, m, k}[\widetilde{\mu}, \mu] \mathbf{Z}_{l-2 k, m}^{ \pm}[\mu] "
$$

analogous to the results of Theorems 2.3.3 and 3.1.8 for harmonic and monogenic functions. It will here be shown that the intersection of all of the $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$ is nontrivial, giving what may be called universal contragenic functions in the context of spheroids.

These observations being made, we consider the particular ambigenic spheroidal polynomials

$$
\begin{equation*}
\mathbf{A}_{l, m}^{ \pm}[\mu]:=2 \operatorname{Vec}\left(\mathbf{X}_{l, m}^{ \pm}[\mu]\right)=\mathbf{X}_{l, m}^{ \pm}[\mu]-\overline{\mathbf{X}}_{l, m}^{ \pm}[\mu], \tag{3.2.9}
\end{equation*}
$$

where $l \geq 1$ and $0 \leq m \leq l+1$. In accordance with Definition 3.2.3, we further observe that

$$
\mathbf{A}_{l, m}^{ \pm}[\mu]=-\mathbf{Y}_{l, m}^{ \pm,-}[\mu]+\left(1-\gamma_{l, m}[\mu]\right) \mathbf{Y}_{l, m}^{ \pm,+}[\mu],
$$

where the coefficients $\gamma_{l, m}[\mu]$ are given by (3.2.1).
The following two preliminary lemmas are required to prove a general formula relating systems of harmonic and contragenic functions associated with spheroids of differing eccentricity:

Lemma 3.2.11. For fixed $\mu$, the set $\left\{\mathbf{A}_{k, m}^{ \pm}[\mu]: 1 \leq k \leq l, 0 \leq m \leq k+1\right\}$ forms an orthogonal family over the interior of the spheroid $\Omega_{\mu}$ in the sense of the scalar inner product (1.2.2).

Proof. For simplicity of notation, we denote by $\sigma_{k, m}=1-\gamma_{k, m}[\mu]$. A direct computation shows that

$$
\begin{aligned}
& \left\langle\mathbf{A}_{k_{1}, m_{1}}^{ \pm}[\mu], \mathbf{A}_{k_{2}, m_{2}}^{ \pm}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
= & \left\langle\mathbf{Y}_{k_{1}, m_{1}}^{ \pm,}[\mu], \mathbf{Y}_{k_{2}, m_{2}}^{ \pm,}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}-\sigma_{k_{2}, m_{2}}\left\langle\mathbf{Y}_{k_{1}, m_{1}}^{ \pm,-}[\mu], \mathbf{Y}_{k_{2}, m_{2}}^{ \pm,+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& -\sigma_{k_{1}, m_{1}}\left\langle\mathbf{Y}_{k_{1}, m_{1}}^{ \pm,+}[\mu], \mathbf{Y}_{k_{2},-m_{2}}^{ \pm,}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)} \\
& +\sigma_{k_{1}, m_{1}} \sigma_{k_{2}, m_{2}}\left\langle\mathbf{Y}_{k_{1}, m_{1}}^{ \pm+}[\mu], \mathbf{Y}_{k_{2}, m_{2}}^{ \pm,+}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right) .} .
\end{aligned}
$$

By Theorem 3.2.4 the system $\left\{\mathbf{Y}_{k, m}^{ \pm, \pm}[\mu]: k \geq 1,0 \leq m \leq k+1\right\}$ is orthogonal. It then follows that $\left\langle\mathbf{A}_{k_{1}, m_{1}}^{ \pm}[\mu], \mathbf{A}_{k_{2}, m_{2}}^{ \pm}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$, when $k_{1} \neq k_{2}$ or $m_{1} \neq m_{2}$. In a similar manner as above, we can show that $\left\langle\mathbf{A}_{k_{1}, m_{1}}^{ \pm}[\mu], \mathbf{A}_{k_{2}, m_{2}}^{\mp}[\mu]\right\rangle_{0, L_{2}\left(\Omega_{\mu}, \mathcal{A}\right)}=0$.

Lemma 3.2.12. Let $l \geq 1$ and $1 \leq m \leq l-1$. The basic contragenic polynomials (3.2.3) are equal to

$$
\begin{align*}
\mathbf{Z}_{l, 0}^{+}[\mu]= & \frac{2}{l+2} V_{l, 1}[\mu] \boldsymbol{\Psi}_{+, 1}^{-},  \tag{3.2.10}\\
\mathbf{Z}_{l, m}^{ \pm}[\mu]= & (l+1+m) a_{l, m}[\mu] V_{l, m-1}[\mu] \boldsymbol{\Psi}_{-, m-1}^{\mp} \\
& +\frac{1}{l+2+m} V_{l, m+1}[\mu] \Psi_{+, m+1}^{\mp}, \tag{3.2.11}
\end{align*}
$$

where the $V_{l, m}[\mu]$ are defined by (2.2.7), $\boldsymbol{\Psi}_{+, m}^{ \pm}=\mathbf{i} \Phi_{m}^{ \pm}(\varphi) \pm \mathbf{j} \Phi_{m}^{\mp}(\varphi)$ and $\Psi_{-, m}^{ \pm}=\mathbf{i} \Phi_{m}^{ \pm}(\varphi) \mp \mathbf{j} \Phi_{m}^{\mp}(\varphi)$.

Proof. It is a simple matter to verify that

$$
\begin{aligned}
& \mathbf{\Psi}_{+, m}^{ \pm} \mathbf{k}= \pm \boldsymbol{\Psi}_{+, m}^{\mp}, \\
& \boldsymbol{\Psi}_{-, m}^{ \pm} \mathbf{k}=\mp \boldsymbol{\Psi}_{-, m}^{\mp}, \\
& \mathbf{i} V_{l, m}^{ \pm}[\mu]+\mathbf{j} V_{l, m}^{\mp}[\mu]=V_{l, m}[\mu] \mathbf{\Psi}_{ \pm, m}^{ \pm}, \\
& \mathbf{i} V_{n, m}^{ \pm}[\mu]-\mathbf{j} V_{l, m}^{\mp}[\mu]=V_{l, m}[\mu] \mathbf{\Psi}_{\mp, m}^{ \pm} .
\end{aligned}
$$

Now, by (3.2.9), we find

$$
\begin{align*}
\mathbf{A}_{l, m}^{ \pm}[\mu]= & (l+1+m) V_{l, m-1}[\mu] \boldsymbol{\Psi}_{-, m-1}^{ \pm} \\
& -\frac{1}{l+2+m} V_{l, m+1}[\mu] \Psi_{+, m+1}^{ \pm} . \tag{3.2.12}
\end{align*}
$$

The rest of the proof follows from Definition 3.2 .8 by considering (3.2.3).
With regard to Lemma 3.2.12, it will be convenient to decompose the harmonic polynomials just considered, $V_{l, m-1}[\mu] \Psi_{-, m-1}^{\mp}$ and $V_{l, m+1}[\mu] \Psi_{+, m+1}^{\mp}$, as the sum of basic contragenic and ambigenic polynomials.

Proposition 3.2.13. Let $l \geq 1$ and $0 \leq m \leq l+1$. Then

$$
\begin{aligned}
V_{l, m-1}[\mu] \Psi_{-, m-1}^{ \pm} & =\frac{1}{(l+1+m)\left(a_{l, m}[\mu]+1\right)}\left(\mathbf{Z}_{l, m}^{\mp}[\mu]+\mathbf{A}_{l, m}^{ \pm}[\mu]\right), \\
V_{l, m+1}[\mu] \Psi_{+, m+1}^{ \pm} & =\frac{l+2+m}{a_{l, m}[\mu]+1}\left(\mathbf{Z}_{l, m}^{\mp}[\mu]-a_{l, m}[\mu] \mathbf{A}_{l, m}^{ \pm}[\mu]\right)
\end{aligned}
$$

where the $a_{l, m}[\mu]$ are given by (3.2.4).
Proof. The proof immediately follows from Lemma 3.2 .12 by adding and subtracting instances of (3.2.10), (3.2.11), and (3.2.12).

We then proceed by stating and proving the main result of the section, which provides many examples for which $\mathbf{Z}_{l, m}^{ \pm}[\widetilde{\mu}] \notin \mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$ for $m \geq 1$. It will be further observed that there are common nontrivial contragenic functions to all spheroids of all eccentricities.

Additionally, we introduce the following notations:

$$
\begin{align*}
& z_{l, 0, k}^{\mathrm{C}}[\widetilde{\mu}, \mu]=\frac{l+2-2 k}{l+2} w_{l, 1, k}[\widetilde{\mu}, \mu], \\
& z_{l, m, k}^{\mathrm{C}}[\widetilde{\mu}, \mu]= \begin{cases}\frac{a_{l, m}[\widetilde{\mu}]+1}{a_{l-2 k, m}[\mu]+1} w_{l, m, k}[\widetilde{\mu}, \mu], & k=0, \ldots,\left[\frac{l-1-m}{2}\right], \\
\frac{a_{l, m}[\widetilde{\mu}]}{a_{l-2 k, m}[\mu]+1} w_{l, m, k}[\widetilde{\mu}, \mu], & k=\left[\frac{l-m}{2}\right], \ldots,\left[\frac{l+1-m}{2}\right],\end{cases} \\
& z_{l, m, k}^{\mathrm{A}}[\widetilde{\mu}, \mu]= \begin{cases}\frac{a_{l, m}[\widetilde{\mu}]-a_{l, m}[\mu]}{a_{l-2 k, m}[\mu]+1} w_{l, m, k}[\widetilde{\mu}, \mu], & k=0, \ldots,\left[\frac{l-1-m}{2}\right], \\
\frac{a_{l, m}[\widetilde{\mu}]}{a_{l-2 k, m}[\mu]+1} w_{l, m, k}[\widetilde{\mu}, \mu], & k=\left[\frac{l-m}{2}\right], \ldots,\left[\frac{l+1-m}{2}\right],\end{cases} \tag{3.2.13}
\end{align*}
$$

where $1 \leq m \leq l-1$, and the $w_{l, m, k}[\widetilde{\mu}, \mu]$ are given by (2.3.6).
We are now ready to express the decomposition of basic contragenics for one spheroid in terms of the basic contragenics $(3.2 .3)$ and the ambigenics (3.2.9) of any other.

Proposition 3.2.14. Let $l \geq 1$. Then

$$
\begin{aligned}
& \mathbf{Z}_{l, 0}^{+}[\tilde{\mu}]=\sum_{k=0}^{\left[\frac{l-1}{2}\right]} z_{l, 0, k}^{\mathrm{C}}[\widetilde{\mu}, \mu] \mathbf{Z}_{l-2 k, 0}^{+}[\mu], \\
& \mathbf{Z}_{l, m}^{ \pm}[\widetilde{\mu}]=\sum_{k=0}^{\left[\frac{[+1-m}{}=1\right.}\left(z_{l, m, k}^{\mathrm{C}}[\widetilde{\mu}, \mu] \mathbf{Z}_{l-2 k, m}^{ \pm}[\mu]+z_{l, m, k}^{\mathrm{A}}[\widetilde{\mu}, \mu] \mathbf{A}_{l-2 k, m}^{ \pm}[\mu]\right)
\end{aligned}
$$

for $1 \leq m \leq l-1$.
Proof. We first prove the case $m=0$. By applying Theorem 2.3.3 to (3.2.10) with $\widetilde{\mu}$ in place of $\mu$, we obtain

$$
\mathbf{Z}_{l, 0}^{+}[\widetilde{\mu}]=\frac{2}{l+2} \sum_{k=0}^{\left[\frac{l-1}{2}\right]} w_{l, 1, k}[\widetilde{\mu}, \mu] V_{l-2 k, 1}[\mu] \Psi_{+, 1}^{-},
$$

which is reduced to the first statement after another application of (3.2.10). Similarly, for $m \geq 1$,

$$
\begin{align*}
& \mathbf{Z}_{l, m}^{ \pm}[\widetilde{\mu}]=(l+1+m) a_{l, m}[\widetilde{\mu}]\left[\frac{l+1-m}{2}\right] \\
& \sum_{k=0} w_{l, m-1, k}[\widetilde{\mu}, \mu] V_{l-2 k, m-1}[\mu] \mathbf{\Psi}_{-, m-1}^{ \pm}  \tag{3.2.14}\\
&+\frac{1}{l+2+m} \sum_{k=0}^{\left[\frac{l-1-m}{2}\right]} w_{l, m+1, k}[\widetilde{\mu}, \mu] V_{l-2 k, m+1}[\mu] \boldsymbol{\Psi}_{+, m+1}^{ \pm} .
\end{align*}
$$

We observe from the definitions leading to Corollary 2.2.7 that

$$
\alpha_{l+1, m-1, n} \widetilde{\alpha}_{l+1-2 n, m-1, k-n}=\frac{l+1+m-2 k}{l+1+m} \alpha_{l+1, m, n} \widetilde{\alpha}_{l+1-2 n, m, k-n}
$$

and, by (2.3.7), we find

$$
\begin{aligned}
\frac{l+1+m}{l+1+m-2 k} w_{l, m-1, k}[\widetilde{\mu}, \mu] & =w_{l, m, k}[\widetilde{\mu}, \mu] \\
& =\frac{l+2+m-2 k}{l+2+m} w_{l, m+1, k}[\widetilde{\mu}, \mu] .
\end{aligned}
$$

With these calculations at hand, we further apply Proposition 3.2.13 to show that

$$
\begin{aligned}
& (l+1+m) w_{l, m-1, k}[\widetilde{\mu}, \mu] V_{l-2 k, m-1}[\mu] \mathbf{\Psi}_{-, m-1}^{ \pm} \\
& =\frac{1}{a_{l-2 k, m}[\mu]+1} w_{l, m, k}[\widetilde{\mu}, \mu]\left(\mathbf{Z}_{l-2 k, m}^{\mp}[\mu]+\mathbf{A}_{l-2 k, m}^{ \pm}[\mu]\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{l+2+m} w_{l, m+1, k}[\tilde{\mu}, \mu] V_{l-2 k, m+1}[\mu] \Psi_{+, m+1}^{ \pm} \\
& =\frac{1}{a_{l-2 k, m}[\mu]+1} w_{l, m, k}[\widetilde{\mu}, \mu]\left(\mathbf{Z}_{l-2 k, m}^{\mp}[\mu]-a_{l-2 k, m}[\mu] \mathbf{A}_{l-2 k, m}^{ \pm}[\mu]\right)
\end{aligned}
$$

Inserting these two relations into the respective sums of (3.2.14) gives the desired result.

The chief interest of this proposition arises from the fact that it provides information about the intersection of the spaces of contragenic functions up to degree $l$. The general theorem may now be stated:

Theorem 3.2.15. Let $l \geq 1$. The following statements hold:
(i) $\mathbf{Z}_{l, 0}^{+}[\mu] \in \mathcal{N}_{l}^{*}\left(\Omega_{0}\right)$ for all $\mu$;
(ii) $\mathbf{Z}_{l, m}^{ \pm}[\mu] \notin \mathcal{N}_{l}^{*}\left(\Omega_{0}\right)$ when $\mu \neq 0$ and $1 \leq m \leq l-1$.

Proof. The first statement is an immediate consequence of the first formula of Proposition 3.2.14. Now, consider a basic element $\mathbf{Z}_{l, m}^{ \pm}[\mu]$ of $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$, with $\mu \neq 0$ and $1 \leq m \leq l-1$. We have then, by the second formula of Proposition 3.2.14,

$$
\mathbf{Z}_{l, m}^{ \pm}[\mu]=\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]}\left(z_{l, m, k}^{\mathrm{C}}[\mu, 0] \mathbf{Z}_{l-2 k, m}^{ \pm}[0]+z_{l, m, k}^{\mathrm{A}}[\mu, 0] \mathbf{A}_{l-2 k, m}^{ \pm}[0]\right)
$$

Suppose that $\mathbf{Z}_{l, m}^{ \pm}[\mu] \in \mathcal{N}_{l}^{*}\left(\Omega_{0}\right)$. Since the right-hand side of the above expression is orthogonal to all $\Omega_{0}$-ambigenics, we obtain

$$
\sum_{k=0}^{\left[\frac{l+1-m}{2}\right]} z_{l, m, k}^{\mathrm{A}}[\mu, 0] \mathbf{A}_{l-2 k, m}^{ \pm}[0]=0
$$

So by the linear independence of the ambigenic polynomials (3.2.9), it then follows that $z_{l, m, k}^{\mathrm{A}}[\mu, 0]=0$ for all $k$. The case in (3.2.13) where $k$ is $[(l-m) / 2]$ or $[(l+1-m) / 2]$ tells us that $a_{l, m}[\mu]=0$, which is manifestly false by (3.2.4). Consequently, $\mathbf{Z}_{l, m}^{ \pm}[\mu] \notin \mathcal{N}_{l}^{*}\left(\Omega_{0}\right)$ as claimed, and accordingly, the theorem has been established.

It is significant to note in this connection that Theorem 3.2.15 does not assert that $\mathbf{Z}_{l, 0}^{+}[\mu]$ lies in the top-level slice $\mathcal{N}_{l}\left(\Omega_{0}\right)$ of $\mathcal{N}_{l}^{*}\left(\Omega_{0}\right)$. We now have the result that
Corollary 3.2.16. Let $l \geq 1$. Then

$$
\operatorname{dim} \bigcap_{\mu \in[0,1) \cup i \mathbb{R}^{+}} \mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right) \geq l .
$$

Proof. The result is an immediate consequence of the fact that Theorem 3.2.15 applies to arbitrary $\mu$, so the intersection contains a fixed $l$-dimensional subspace of $\mathcal{N}_{l}^{*}\left(\Omega_{0}\right)$.

Furthermore, it follows from Theorem 3.2.15 that the common intersection $\cap \mathcal{N}^{*}\left(\Omega_{\mu}\right)$ of the full spaces of contragenic functions on spheroids is infinite-dimensional, containing all the contragenic polynomials $\mathbf{Z}_{l, m}^{+}[\mu]$ for which $m=0$. Although this phenomenon is not fully understood, it seems likely that these contragenic polynomials have unique characteristics, given their more straightforward structure 3.2 .10 . Further questions regarding the exact relations among the spaces $\mathcal{N}_{l}^{*}\left(\Omega_{\mu}\right)$ remain open. If the method employed in the proof of Theorem 3.2.15 is applied to linear combinations of the $\mathbf{Z}_{l, m}^{ \pm}[\mu]$ instead of just to these generators individually, transcendental equations related to $(3.2 .4$ appear. These equations may be a subject for future work.

142 3. MONOGENICS AND CONTRAGENICS ON SPHEROIDAL DOMAINS

## 4

## Prolate Spheroidal Wave Functions associated with the QFT

The first part of this chapter begins with a discussion of the existence of a theory of functions with quaternionic values and of three real variables, which is determined by a Moisil-Teodorescu type operator with quaternionic variable coefficients and it is intimately related to the theory of PSWFs. We proceed to study the energy concentration problem of band-limited quaternionic signals under the QFT defined by (1.6.4) and prove a quaternionic version of Donoho-Stark's uncertainty principle. Keys to the analysis are certain $c$-Quaternionic Prolate Spheroidal Wave Functions (from now on abbreviated as $c$-QPSWFs), which possess several unique properties that make them most useful for the study of band-limited functions. We prove that the $c$-QPSWFs are maximally concentrated in both the spatial and frequency domains among band-limited quaternionic functions of a given energy. The $c$-QPSWFs are orthogonal and complete over two different bounded domains along the Euclidean space $\mathbb{R}^{3}$, under certain symmetry assumptions: the space of square-integrable quaternionic functions and the reproducing kernel Hilbert space of band-limited quaternionic signals.

In the second part of the chapter, the $c$-QPSWFs are used to investigate the band-limited extrapolation problem. The mean-square convergence of the quaternionic Slepian series for band-limited functions is also discussed. More importantly, the maximization problem of the $c$-QPSWFs is analyzed. It is shown how to use the $c$-QPSWFs to examine the energy concentration of a signal in the spatial and frequency domains. In the application part, we study the $c$-QPSWFs restricted in the spatial domain to the unit ball $\Omega_{0}$ and frequency domain to the ball $c \Omega_{0}$ of radius $c>0$ and establish some of their
fundamental properties.

### 4.1 The Connection between the PSWFs and the Notion of $c$-hyperholomorphicity

In the present section, we show that the theory of PSWFs may also be determined by a Moisil-Teodorescu type operator with quaternionic variable coefficients [103]. It will follow from the fact that every metaharmonic function can be decomposed into a direct sum of two functions from the conjugate classes of $\mathcal{D}_{c}$-hyperholomorphy [189]. As a result, we explain the connections between the PSWFs, on the one hand, and the quaternionic $c$ hyperholomorphic and $c$-antihyperholomorphic functions, $c=\mu k, \mu \in(0,1)$, $k \in \mathbb{R}^{+}$, on the other.

By (1.5.1), the following factorization holds for the Helmholtz operator in three-dimensional Cartesian coordinates:

$$
\begin{aligned}
\Delta_{3}+k^{2} & =\left(k+\mathbf{i} \frac{\partial}{\partial x_{0}}+\mathbf{j} \frac{\partial}{\partial x_{1}}+\mathbf{k} \frac{\partial}{\partial x_{2}}\right)\left(k-\mathbf{i} \frac{\partial}{\partial x_{0}}-\mathbf{j} \frac{\partial}{\partial x_{1}}-\mathbf{k} \frac{\partial}{\partial x_{2}}\right) \\
& =: D_{c} \bar{D}_{c} .
\end{aligned}
$$

By using notations already employed in Subsection 1.5.1, the Helmholtz operator can then be written as

$$
\begin{aligned}
& W_{\boldsymbol{\Phi}}\left(\Delta_{3}+k^{2}\right) W_{\boldsymbol{\Psi}} \\
& =W_{\boldsymbol{\Phi}}\left(D_{c}\right) W_{\boldsymbol{\Psi}} W_{\boldsymbol{\Phi}}\left(\bar{D}_{c}\right) W_{\boldsymbol{\Psi}} \\
& =W_{\boldsymbol{\Phi}}\left(k+\mathbf{i} \frac{\partial}{\partial x_{0}}+\mathbf{j} \frac{\partial}{\partial x_{1}}+\mathbf{k} \frac{\partial}{\partial x_{2}}\right) W_{\boldsymbol{\Psi}} W_{\boldsymbol{\Phi}}\left(k-\mathbf{i} \frac{\partial}{\partial x_{0}}-\mathbf{j} \frac{\partial}{\partial x_{1}}-\mathbf{k} \frac{\partial}{\partial x_{2}}\right) W_{\boldsymbol{\Psi}} .
\end{aligned}
$$

Straightforward computations show that

$$
W_{\Phi} \frac{\partial}{\partial x_{0}} W_{\Psi}=\frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}\left(\xi \frac{\partial}{\partial \xi}-t \frac{\partial}{\partial t}\right) \cos \varphi-\frac{\sin \varphi}{h_{2}(\xi, t)} \frac{\partial}{\partial \varphi},
$$

where $h_{1}(\xi, t)$ and $h_{2}(\xi, t)$ are defined in (1.5.5).
Similarly,

$$
W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{1}} W_{\boldsymbol{\Psi}}=\frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}\left(\xi \frac{\partial}{\partial \xi}-t \frac{\partial}{\partial t}\right) \sin \varphi+\frac{\cos \varphi}{h_{2}(\xi, t)} \frac{\partial}{\partial \varphi},
$$

and

$$
W_{\boldsymbol{\Phi}} \frac{\partial}{\partial x_{2}} W_{\boldsymbol{\Psi}}=\frac{\mu}{h_{1}^{2}(\xi, t)}\left[\left(\xi^{2}-1\right) t \frac{\partial}{\partial \xi}+\left(1-t^{2}\right) \xi \frac{\partial}{\partial t}\right] .
$$

In this manner, it follows that

$$
\begin{align*}
\mathcal{D}_{c}:= & W_{\boldsymbol{\Phi}} D_{c} W_{\boldsymbol{\Psi}}  \tag{4.1.1}\\
= & W_{\boldsymbol{\Phi}}\left(k+\mathbf{i} \frac{\partial}{\partial x_{0}}+\mathbf{j} \frac{\partial}{\partial x_{1}}+\mathbf{k} \frac{\partial}{\partial x_{2}}\right) W_{\boldsymbol{\Psi}} \\
= & k+\left[\frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) \xi+\frac{\mu \mathbf{k}}{h_{1}^{2}(\xi, t)}\left(\xi^{2}-1\right) t\right] \frac{\partial}{\partial \xi} \\
& +\left[-\frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) t+\frac{\mu \mathbf{k}}{h_{1}^{2}(\xi, t)}\left(1-t^{2}\right) \xi\right] \frac{\partial}{\partial t} \\
& +\left[\frac{1}{h_{2}(\xi, t)}(-\mathbf{i} \sin \varphi+\mathbf{j} \cos \varphi)\right] \frac{\partial}{\partial \varphi} .
\end{align*}
$$

Analogously,

$$
\begin{align*}
\overline{\mathcal{D}}_{c}:= & W_{\boldsymbol{\Phi}} \bar{D}_{c} W_{\mathbf{\Psi}}  \tag{4.1.2}\\
= & k-\left[\frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) \xi+\frac{\mu \mathbf{k}}{h_{1}^{2}(\xi, t)}\left(\xi^{2}-1\right) t\right] \frac{\partial}{\partial \xi} \\
& -\left[-\frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) \eta+\frac{\mu \mathbf{k}}{h_{1}^{2}(\xi, t)}\left(1-t^{2}\right) \xi\right] \frac{\partial}{\partial t} \\
& -\left[\frac{1}{h_{2}(\xi, t)}(-\mathbf{i} \sin \varphi+\mathbf{j} \cos \varphi)\right] \frac{\partial}{\partial \varphi} .
\end{align*}
$$

The operators $\mathcal{D}_{c}$ and $\overline{\mathcal{D}}_{c}$ are well-defined on

$$
\mathbb{R}^{3} \backslash\left\{(\xi, t, \varphi) \in \mathbb{R}^{3}: h_{1}(\xi, t) \neq 0, h_{2}(\xi, t) \neq 0\right\},
$$

but to unify the notations and simplify the calculations further, we choose them acting on $\Omega_{\xi, t, \varphi}$, defined as (1.5.6).

We introduce a particular class of $\mathbb{H}$-valued functions analogous to holomorphic complex-valued functions.

Definition 4.1.1. Given a real number $c>0$, a function $\boldsymbol{f} \in C^{1}\left(\Omega_{\xi, t, \varphi}, \mathbb{H}\right)$ is called
(i) c-hyperholomorphic in $\Omega_{\xi, t, \varphi}$ if $\mathcal{D}_{c} \boldsymbol{f}=\mathbf{0}$ identically in $\Omega_{\xi, t, \varphi}$;
(ii) c-antihyperholomorphic in $\Omega_{\xi, t, \varphi}$ if $\overline{\mathcal{D}}_{c} \boldsymbol{f}=\mathbf{0}$ identically in $\Omega_{\xi, t, \varphi}$.

It is understood that $\mathcal{D}_{c}, \overline{\mathcal{D}}_{c}$ are applied to $\boldsymbol{f}$ after the change of variables $\left(x_{0}, x_{1}, x_{2}\right) \leftrightarrow(\xi, t, \varphi)$, which itself depends on the choice of $\mu$. The next lemma follows immediately from the definition.

Lemma 4.1.2. The operators (4.1.1) and (4.1.2) factor the differential operator induced by equation (1.5.3); that is,

$$
\begin{equation*}
\mathcal{D}_{c} \overline{\mathcal{D}}_{c}=\overline{\mathcal{D}}_{c} \mathcal{D}_{c}=\frac{1}{h_{1}^{2}(\xi, t)} \mathcal{W}, \tag{4.1.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{W}=\mathcal{W}_{\xi, t, \varphi}+c^{2}\left(\xi^{2}-t^{2}\right) \tag{4.1.4}
\end{equation*}
$$

and $\mathcal{W}_{\xi, t, \varphi}$ is defined by 1.5.4.
The factorization property (4.1.3) suggests that nullsolutions of the $\mathcal{D}_{c^{-}}$ operator play the same role for the $\mathcal{W}$-operator as the usual holomorphic functions in one complex variable or monogenic functions of quaternionic analysis play for the corresponding Laplace operators. Furthermore, they are related to researched $c$-hyperholomorphic functions for the Helmholtz operator [189, 190]. However, note that there is an essential difference as the operators (4.1.1) and (4.1.2) have nonconstant coefficients, and the theories of such operators are much more sophisticated.

We proceed to show how to relate the solutions of the angular and radial prolate spheroidal equations (1.5.8) and (1.5.9) directly to the notion of $c$ hyperholomorphicity of Definition 4.1.1. A discussion of the relationship between a quaternionic function theory and the theory of the PSWFs (of order zero) was considered in [103]. However, in treatments such as [103] and [248], the PSWFs degenerate to a segment as the eccentricity of the spheroid decreases. The present approach does not suffer this drawback as it is based on explicit formulas, whose elements are parametrized by the shape of the spheroids. We start by considering a real-valued solution to the operator $\left(1 / h_{1}^{2}(\xi, t)\right) \mathcal{W}$, say $\Phi_{n, 0}(\xi, t)$, given by 1.5.7). Bearing in mind the factorization (4.1.3), we may then apply either the operator 4.1.1) or its conjugate operator (4.1.2) to $\Phi_{n, 0}(\xi, t), n=0,1, \ldots$ More precisely, we set $\boldsymbol{\phi}_{n, 0}:=(1 / 2 k) \overline{\mathcal{D}}_{c}\left[\Phi_{n, 0}\right]$ and $\bar{\phi}_{n, 0}:=(1 / 2 k) \mathcal{D}_{c}\left[\Phi_{n, 0}\right]$. It is clear from the context that we have suppressed that the family of conjugate functions $\left\{\phi_{n, 0}, \bar{\phi}_{n, 0}\right\}$ depend on the parameter $c$.

Following what has been conducted above, direct observations show that

$$
\mathcal{D}_{c} \boldsymbol{\phi}_{n, 0}=\mathcal{D}_{c}\left(\frac{1}{2 k} \overline{\mathcal{D}}_{c} \Phi_{n, 0}\right)=\frac{1}{2 k} \mathcal{D}_{c} \overline{\mathcal{D}}_{c} \Phi_{n, 0}=\frac{1}{2 k} \frac{1}{h_{1}^{2}(\xi, t)} \mathcal{W} \Phi_{n, 0}=\mathbf{0}
$$

and, similarly, $\overline{\mathcal{D}}_{c} \overline{\boldsymbol{\phi}}_{n, 0}=\mathbf{0}$. It then follows that each pair of solutions of the corresponding equations (1.5.8) and (1.5.9) generate a $c$-hyperholomorphic function and a $c$-antihyperholomorphic function. Furthermore, the decomposition $\boldsymbol{\phi}_{n, 0}+\bar{\phi}_{n, 0}=\Phi_{n, 0}$ holds for all $n=0,1, \ldots$ This is related to the
fact that if $\Phi_{n, 0}$ is a metaharmonic function (i.e., $\Delta_{3} \Phi_{n, 0}+k^{2} \Phi_{n, 0}=0$ ), then there exist (uniquely) two functions $\phi_{n, 0}^{(1)}$ and $\boldsymbol{\phi}_{n, 0}^{(2)}$ from the conjugate classes of $\mathcal{D}_{c}$-hyperholomorphy such that $\Phi_{n, 0}=\boldsymbol{\phi}_{n, 0}^{(1)}+\boldsymbol{\phi}_{n, 0}^{(2)}$ [189].

Now we will work out explicit expressions of the functions $\phi_{n, 0}$, which follow by straightforward calculations.

Proposition 4.1.3. Given a real number $c>0$, the $\boldsymbol{\phi}_{n, 0}(n=0,1, \ldots)$ are equal to

$$
\begin{align*}
& \boldsymbol{\phi}_{n, 0}(\xi, t, \varphi)=\frac{1}{2} S_{n, 0}(c, t) R_{n, 0}(c, \xi) \\
& +(\mathbf{i} \cos \varphi+\mathbf{j} \sin \varphi) \frac{h_{2}(\xi, t)}{h_{1}^{2}(\xi, t)}\left[t R_{n, 0}(c, \xi) \frac{d S_{0, n}(c, t)}{d t}-\xi \frac{d R_{n, 0}(c, \xi)}{d \xi} S_{n, 0}(c, t)\right] \\
& -\frac{\mu \mathbf{k}}{h_{1}^{2}(\xi, t)}\left[\left(1-t^{2}\right) \xi R_{n, 0}(c, \xi) \frac{d S_{0, n}(c, t)}{d t}+\left(\xi^{2}-1\right) t \frac{d R_{n, 0}(c, \xi)}{d \xi} S_{n, 0}(c, t)\right], \tag{4.1.5}
\end{align*}
$$

where $S_{n, 0}(c, t)$ is a solution of 1.5.8) and $R_{n, 0}(c, \xi)$ is a solution of (1.5.9).
The suggested $\phi_{n, 0}$ are illustrated below. It can further be seen that for each $n$, the functions $\left|\phi_{n, 0}\right|^{2}$ do not depend on $\varphi$, by Proposition 4.1.3. Hence the plots of these functions will be symmetric with respect to the $x_{0}$-axis. The following figures show the function $\left|\phi_{0,0}(\xi, t)\right|^{2}$ in the spatial domain $(\xi, t) \in(1,1 / \mu] \times(-1,1)$ provided by (4.1.5). The parameters are $c=0.1,2,3,4$ and $\mu=c$.


Figure 4.1: $c=0.1$


Figure 4.2: $c=2$


Figure 4.3: $c=3$


Figure 4.4: $c=4$

We now plot the functions $\left|\phi_{n, 0}(\xi, t)\right|^{2}(n \in\{0, \ldots, 5\})$ for $c \in[0.1,17]$.


Figure 4.5: $\left|\phi_{0,0}\right|^{2}$


Figure 4.7: $\left|\phi_{2,0}\right|^{2}$


Figure 4.9: $\left|\phi_{4,0}\right|^{2}$


Figure 4.6: $\left|\phi_{1,0}\right|^{2}$


Figure 4.8: $\left|\phi_{3,0}\right|^{2}$


Figure 4.10: $\left|\phi_{5,0}\right|^{2}$

### 4.2 The $c$-QPSWFs

We propose, in the present section, to introduce the $c$-QPSWFs. We shall study the $c$-QPSWFs in detail and present some of their applications to represent band-limited quaternionic functions. The property that we shall be most concerned with is the orthogonality of the $c$-QPSWFs over two different three-dimensional spaces, under the assumption of a certain kind of symmetry: the space of square-integrable quaternionic functions on a cube and the reproducing kernel Hilbert space of band-limited quaternionic signals. The treatment given here is a generalization provided by Slepian and Pollak in [299] and Landau and Pollak in [198].

It is quite surprising that by slightly modifying the standard methods described in Subsection 1.5.2, one can obtain a nearly complete theory, as in the case of the multi-dimensional PSWFs [300]. There will appear significant algebraic complications because of the underlying noncommutativity of quaternions; nevertheless, the essential nature of the arguments will remain unchanged. For instance, see [251, 350] for a list of the known results concerning the $c$-QPSWFs before the present investigation.

### 4.2.1 Space-Limited and Band-Limited Quaternionic Signals

Let $\mathbf{T}$ and $\mathbf{W}$ be two cubes in $\mathbb{R}^{3}$ centered at the origin with edges parallel to the coordinate axes, of volumes $|\mathbf{T}|=8 T^{3}$ and $|\mathbf{W}|=8 W^{3}$ respectively, where $T$ and $W$ are fixed positive real numbers. We call $\mathbf{T}$ the spatial-domain, and $\mathbf{W}$ the frequency-domain. In the interest of simplicity, we henceforth choose $\mathbf{W}$ to be a scaled version of $\mathbf{T}$. We write $\mathbf{W}=c \mathbf{T}$, meaning that $\mathbf{x} \in \mathbf{T}$ if and only if $c \mathbf{x} \in \mathbf{W}$, with $c$ a positive constant. We henceforth assume $\mathbf{T}$ and $\mathbf{W}$ so chosen.

To begin with, we study the relationship between two closed subspaces of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ : the subspace $\mathcal{D}(\mathbf{T})$ of all $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ supported in $\mathbf{T}$ and the subspace $\mathcal{B}(\mathbf{W})$ of all $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ whose QFTs are supported in $\mathbf{W}$. We will show that several questions about $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$ can be answered in terms of the eigenvalues of the operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$, where $D_{\mathbf{T}}$ and $B_{\mathbf{W}}$ are the projections onto $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$, respectively. As will later be seen, the operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ may be written as a finite convolution. The behavior of its eigenvalues is worth highlighting because it differs markedly from that established in [350] for the class of finite convolutions whose QFTs are defined by (1.6.1). Apart from this application, interpretable as describing how the energy of a function in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ can be distributed over space and frequency, we prove a quaternionic counterpart of the Donoho-Stark uncertainty
principle associated with the QFT defined by (1.6.4). Similar questions for the case where more general measurable spaces replace the domains $\mathbf{T}$ and $\mathbf{W}$ are not considered in the present discussion.

We have adapted the following definitions of $\mathbb{H}$-valued functions within our context: $\varepsilon$-concentrated (in energy) in the spatial and frequency domains, from [99].
Definition 4.2.1. Let $\varepsilon_{\mathbf{T}}, \varepsilon_{\mathbf{W}} \geq 0$. We say that a function $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ is $\varepsilon_{\mathbf{T}}$-concentrated on $\mathbf{T}$ if there is a function $\boldsymbol{g}(\mathbf{x})$ vanishing outside $\mathbf{T}$ such that

$$
\|\boldsymbol{f}-\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)} \leq \varepsilon_{\mathbf{T}}\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)} .
$$

Similarly, we say that $\boldsymbol{\mathcal { F }}(\boldsymbol{f})$ is $\varepsilon_{\mathbf{W}}$-concentrated on $\mathbf{W}$ if there is a function $\boldsymbol{h}(\boldsymbol{\omega})$ vanishing outside $\mathbf{W}$ with

$$
\|\mathcal{F}(\boldsymbol{f})-\boldsymbol{h}\|_{L_{2}\left(\mathbb{R}^{3}\right)} \leq \varepsilon_{\mathbf{W}}\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)} .
$$

We consider two types of $\mathbb{H}$-valued square-integrable functions and their corresponding space-limited and band-limited spaces. The definitions of a space-limited and a band-limited quaternionic signal are more general than those of Zou et al., presented in [350]. A quaternionic function theory of generalized two-dimensional PSWFs was developed in [350] using definition (1.6.2) but not employing (1.6.4) as considered in this text.

Definition 4.2.2. We say that an $\mathbb{H}$-valued function $\boldsymbol{f}$ with finite energy is space-limited if it vanishes for all $\mathbf{x} \in \mathbb{R}^{3} \backslash \mathbf{T}$.

Definition 4.2.3. We say that an $\mathbb{H}$-valued function $\boldsymbol{f}$ with finite energy is band-limited with band $\mathbf{W}$ if $\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})$ vanishes for all $\boldsymbol{\omega} \in \mathbb{R}^{3} \backslash \mathbf{W}$.

From any $\boldsymbol{f}$ in $L_{2}\left(\mathbb{R}^{2}, \mathbb{H}\right)$, we define two restriction operators on $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ for each of which $\boldsymbol{f}$ becomes either a space-limited or a band-limited function; we will call them, respectively, space-limiting and band-limiting operators.
Definition 4.2.4. Let $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. We define the space-limiting operator by

$$
\begin{equation*}
\left(D_{\mathbf{T}} \boldsymbol{f}\right)(\mathbf{x})=\chi_{\mathbf{T}}(\mathbf{x}) \boldsymbol{f}(\mathbf{x}), \tag{4.2.1}
\end{equation*}
$$

where $\chi_{\mathbf{T}}$ is the characteristic function of $\mathbf{T}$. The subspace of quaternionic signals that are space-limited to $\mathbf{T}$, namely all functions $\boldsymbol{f}$ in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ such that $D_{\mathbf{T}} \boldsymbol{f}=\boldsymbol{f}$, is denoted by $\mathcal{D}(\mathbf{T})$.

We regard $D_{\mathbf{T}}$ as an operator whose effect on a function $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ is to produce its space-limited version. According to Definition 4.2.1 we say that $\boldsymbol{f}$ is $\varepsilon_{\mathbf{T}}$-concentrated on $\mathbf{T}$, if and only if

$$
\begin{equation*}
\left\|\boldsymbol{f}-D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \leq \varepsilon_{\mathbf{T}}\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)} . \tag{4.2.2}
\end{equation*}
$$

Similarly, we define an operator whose effect on a quaternionic function is to produce its band-limited version.
Definition 4.2.5. Let $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. We define the band-limiting operator by

$$
\begin{align*}
\left(B_{\mathbf{W}} \boldsymbol{f}\right)(\mathbf{x}) & =\mathcal{F}^{-1}\left[\chi_{\mathbf{W}} \mathcal{F}(\boldsymbol{f})\right](\mathbf{x}) \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \tag{4.2.3}
\end{align*}
$$

We denote by

$$
\begin{equation*}
\mathcal{B}(\mathbf{W})=\left\{\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right): \mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}) \equiv \mathbf{0}, \boldsymbol{\omega} \in \mathbb{R}^{3} \backslash \mathbf{W}\right\} \tag{4.2.4}
\end{equation*}
$$

the space of quaternionic signals that are band-limited to $\mathbf{W}$, namely all functions in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ such that $B_{\mathbf{W}} \boldsymbol{f}=\boldsymbol{f}$.

The expression 4.2 .3 for $B_{\mathbf{W}}$ defines a function whose QFT agrees with $\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})$ in $\boldsymbol{\omega} \in \mathrm{W}$ and vanishes elsewhere; thus, the operation $B_{\mathrm{W}} \boldsymbol{f}$ is entirely analogous to $D_{\mathrm{W}} \mathcal{F}(\boldsymbol{f})$.

If $\boldsymbol{g}:=B_{\mathrm{W}} \boldsymbol{f}$, then $\boldsymbol{\mathcal { F }}(\boldsymbol{g})$ vanishes outside W. With Definition 4.2.1 in place, it is clear that $\boldsymbol{g}$ is the closest function to $\boldsymbol{f}$ with the following property: $\mathcal{F}(\boldsymbol{f})$ is $\varepsilon_{\mathbf{W}}$-concentrated on $\mathbf{W}$, if and only if

$$
\begin{align*}
\left\|\mathcal{F}(\boldsymbol{f})-\mathcal{F}\left(B_{\mathbf{W}} \boldsymbol{f}\right)\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} & =(2 \pi)^{3 / 2}\left\|\boldsymbol{f}-B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \\
& \leq \varepsilon_{\mathbf{W}}\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)} . \tag{4.2.5}
\end{align*}
$$

In accordance with Definition 1.2 .24 , we will now show that the space $\mathcal{B}(\mathbf{W})$ is a RKQHS. The product of three sinc functions gives the corresponding three-dimensional reproducing kernel.
Theorem 4.2.6. Let $\mathbf{x}, \mathbf{y}$ be points in $\mathbb{R}^{3}$. The space $\mathcal{B}(\mathbf{W})$ is a RKQHS on $\mathbb{R}^{3}$. The unique reproducing kernel $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is given by

$$
\begin{equation*}
K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})=\prod_{j=0}^{2}\left[\left(\frac{W}{\pi}\right) K\left(\frac{W\left(x_{j}-y_{j}\right)}{\pi}\right)\right] \tag{4.2.6}
\end{equation*}
$$

where $K$ denotes the sinc function defined by 1.5.17).
Proof. Let $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$. Applying Schwarz's inequality (1.2.4) and Parseval's identity (1.6.11) to the representation 4.2.3), we obtain

$$
\begin{aligned}
|\boldsymbol{f}(\mathbf{x})| & \leq \frac{1}{(2 \pi)^{3}}\left(\int_{\mathbf{W}}|\boldsymbol{F}(\boldsymbol{f})(\boldsymbol{\omega})|^{2} d \boldsymbol{\omega}\right)^{1 / 2}\left(\int_{\mathbf{W}}|\overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x})|^{2} d \boldsymbol{\omega}\right)^{1 / 2} \\
& \leq \frac{\sqrt{|\mathbf{W}|}}{(2 \pi)^{3 / 2}}\|\boldsymbol{f}(\mathbf{x})\|_{L_{2}\left(\mathbb{R}^{3}\right)}
\end{aligned}
$$

for all $\mathbf{x}$. Hence, it follows that $\mathcal{B}(\mathbf{W})$ is a RKQHS. In this case, the reproducing kernel is obtained from the inverse QFT (1.6.9) of the function $\chi_{\mathbf{w}}(\boldsymbol{\omega}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y})$ :

$$
\begin{align*}
K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) & =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega}  \tag{4.2.7}\\
& =\frac{1}{(2 \pi)^{3}} \prod_{j=0}^{2}\left(\int_{-W}^{W} e^{\mathbf{u} \omega_{j}\left(x_{j}-y_{j}\right)} d \omega_{j}\right) .
\end{align*}
$$

To transform this expression further, we use Lemma 1.1 .3 to find

$$
\begin{aligned}
\frac{1}{2 \pi} \int_{-W}^{W} e^{\mathbf{u} \omega_{j}\left(x_{j}-y_{j}\right)} d \omega_{i} & =\frac{(-\mathbf{u})}{2 \pi x_{j}} \int_{-W}^{W} \mathbf{u}\left(x_{j}-y_{j}\right) e^{\mathbf{u} \omega_{j}\left(x_{j}-y_{j}\right)} d \omega_{j} \\
& =\frac{(-\mathbf{u})}{\pi x_{j}}\left(\frac{e^{\mathbf{u}\left(x_{j}-y_{j}\right) W}-e^{-\mathbf{u}\left(x_{j}-y_{j}\right) W}}{2}\right) \\
& =\frac{(-\mathbf{u})}{\pi x_{j}} \operatorname{Vec}\left(e^{\mathbf{u}\left(x_{j}-y_{j}\right) W}\right) \\
& =\frac{\sin \left[W\left(x_{j}-y_{j}\right)\right]}{\pi\left(x_{j}-y_{j}\right)},
\end{aligned}
$$

and hence the result follows.
It is an immediate and essential consequence from the above result that norm convergence in $\mathcal{B}(\mathbf{W})$ does indeed imply uniform convergence in the whole $\mathbb{R}^{3}$.

Lemma 4.2.7. Let $\left\{\boldsymbol{f}_{n}\right\}$ be a sequence in $\mathcal{B}(\mathbf{W})$. If $\boldsymbol{f}_{n}$ converges to $\boldsymbol{f}$ in the RKQHS-norm, then $\lim _{n \rightarrow \infty}\left|\boldsymbol{f}_{n}(\mathbf{x})-\boldsymbol{f}(\mathbf{x})\right|=0$ for all $\mathbf{x} \in \mathbb{R}^{3}$.

An essential matter for investigation is the determination of specific properties of the spaces $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$. In light of the considerations above adduced, it appears certain that a band-limited function $\boldsymbol{f}(\mathbf{x})$, which vanishes for $\mathbf{x} \in \mathbf{T}$, must vanish identically; in other words, $\mathcal{D}(\mathbf{T}) \cap \mathcal{B}(\mathbf{W})=\{\mathbf{0}\}$. It will be asserted later that there is a nonzero minimum angle between the spaces $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$. It will be shown that the eigenvalues of the convolution operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ measure the angles formed by the two subspaces. Simultaneously, its eigenfunctions are a convenient basis for the study of questions regarding the relationship between quaternionic functions and their QFTs. These spaces will be discussed in detail in Subsection 4.3.

The remarks made regarding space-limiting and band-limiting a quaternionic function show that $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$ are linear subspaces of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. We proceed with the following lemma:

Lemma 4.2.8. The spaces $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$ are complete and orthogonal to their complements $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right) \backslash \mathcal{D}(\mathbf{T})$ and $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right) \backslash \mathcal{B}(\mathbf{W})$.

Proof. The range of the operator $D_{\mathbf{T}}$ is a subspace of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ that is isometrically isomorphic to $L_{2}(\mathbf{T}, \mathbb{H})$. By abuse of notation, we shall continue to denote it by $\mathcal{D}(\mathbf{T})$. It then follows that $\mathcal{D}(\mathbf{T})$ is complete. Moreover, it is clear that

$$
\begin{equation*}
\left\langle D_{\mathbf{T}} \boldsymbol{f}, \boldsymbol{f}-D_{\mathbf{T}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}=\mathbf{0} \tag{4.2.8}
\end{equation*}
$$

for all $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, since each one vanishes where the other does not. We proceed to show that $\mathcal{B}(\mathbf{W})$ is a closed subspace of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. By Lemma 4.2 .7 , we prove this assertion in the following manner. Suppose a sequence $\left\{\boldsymbol{f}_{n}\right\}$ in $\mathcal{B}(\mathbf{W})$ converges in the mean-square to $\boldsymbol{f}$. Clearly $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. By Plancherel's identity 1.6.10), the corresponding QFT converges likewise,

$$
\int_{\mathbb{R}^{3}}\left|\boldsymbol{f}_{n}(\mathrm{x})-\boldsymbol{f}(\mathrm{x})\right|^{2} d \mathrm{x}=\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}}\left|\boldsymbol{\mathcal { F }}\left(\boldsymbol{f}_{n}\right)(\boldsymbol{\omega})-\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})\right|^{2} d \boldsymbol{\omega}
$$

Thus $\boldsymbol{\mathcal { F }}(\boldsymbol{f})(\boldsymbol{\omega})$ must vanish for $\boldsymbol{\omega} \in \mathbb{R}^{3} \backslash \mathbf{W}$ and $\boldsymbol{f}(\mathbf{x})$ has band $\mathbf{W}$. Now, employing again 1.6.10 we have then,

$$
\begin{equation*}
\left\langle B_{\mathbf{W}} \boldsymbol{f}, \boldsymbol{f}-B_{\mathrm{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}=\mathbf{0} \tag{4.2.9}
\end{equation*}
$$

for all $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$; the argument then proceeds as before.
In the following sections, it will be shown that there are certain functions $D_{\mathbf{T}} \boldsymbol{\psi}_{n}$ and $\boldsymbol{\psi}_{n}$ which span, respectively, the spaces $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$. Let us now consider the space $\mathcal{D}(\mathbf{T}) \cup \mathcal{B}(\mathbf{W})$ and its complement $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right) \backslash$ $(\mathcal{D}(\mathbf{T}) \cup \mathcal{B}(\mathbf{W}))=(\mathcal{D}(\mathbf{T}))^{\perp} \cap(\mathcal{B}(\mathbf{W}))^{\perp}$. The space $(\mathcal{D}(\mathbf{T}) \cup \mathcal{B}(\mathbf{W}))^{\perp}$ contains all such $\boldsymbol{f}$ for which $D_{\mathrm{T}} \boldsymbol{f}=\boldsymbol{f}$ and $B_{\mathrm{W}} \boldsymbol{f}=\boldsymbol{f}$, namely all such functions that are space-limited to $\mathbf{T}$ and band-limited to $\mathbf{W}$. There do not seem to be any known quaternionic function that possesses this property. However, this function must exist since $\mathcal{D}(\mathbf{T}) \cup \mathcal{B}(\mathbf{W}) \neq L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, and hence $(\mathcal{D}(\mathbf{T}) \cup \mathcal{B}(\mathbf{W}))^{\perp}$ is not empty. Whether a construction scheme for the missing functions can be developed is a matter of great concern [198], but it is an issue that goes beyond the scope of the present study.

Having made these observations, we proceed to consider a further aspect of the spaces $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$, which is substantially the one given by Landau et al. [198]. Because these results are proved in a manner precisely similar to that in which the corresponding ones in the one-dimensional case were established in [198], we state the lemmas without proof.

Lemma 4.2.9. The space $\mathcal{D}(\mathbf{T})+\mathcal{B}(\mathbf{W})$ is closed.

Lemma 4.2.10. There are infinitely many functions in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, which are orthogonal to $\mathcal{D}(\mathbf{T})+\mathcal{B}(\mathbf{W})$.

In preparation for the following section, it will now be shown that the operators defined by (4.2.1) and (4.2.3) satisfy the usual requirement of projection operators on a complex linear space and will be referred to as orthogonal projection operators of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ onto $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$, respectively.

Lemma 4.2.11. The operator $D_{\mathbf{T}}$ is an orthogonal projection of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ onto $\mathcal{D}(\mathbf{T})$, and $B_{\mathbf{W}}$ is an orthogonal projection of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ onto $\mathcal{B}(\mathbf{W})$.

Proof. According to Definition 4.2.4, it is clear that $D_{\mathbf{T}}$ is left-linear. Moreover, it is easily seen that $D_{\mathbf{T}}$ is idempotent (that is, $D_{\mathbf{T}}^{2}=D_{\mathbf{T}}$ ), and the fact that $D_{\mathbf{T}}$ is self-adjoint follows from

$$
\begin{aligned}
\left\langle D_{\mathbf{T}} \boldsymbol{f}, \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} & =\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \chi_{\mathbf{T}}(\mathbf{x}) \overline{\boldsymbol{g}}(\mathbf{x}) d \mathbf{x} \\
& =\left\langle\boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} .
\end{aligned}
$$

Similarly, by Definition 4.2.5, the range of $B_{\mathbf{W}}$ consists of all functions in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ with spectrum in $\mathbf{W}$, namely those $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ such that $\operatorname{supp} \boldsymbol{\mathcal { F }}(\boldsymbol{f}) \subset \mathbf{W}$. We may proceed in this manner to show that $B_{\mathbf{W}}$ is a left-linear operator. We use Definition 4.2.5 and Theorem4.2.6 to prove that each of the quantities $\mathcal{F}^{-1}\left[\chi_{\mathbf{w}} \mathcal{F}\left([\boldsymbol{f}]_{i}\right)\right](\mathbf{x})(i=0,1,2,3)$ is a real-valued function.

Accordingly, by the Tonelli-Hobson Theorem, we get

$$
\begin{aligned}
B_{\mathbf{W}}[\boldsymbol{f}]_{i}(\mathbf{x}) & =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \mathcal{F}\left([\boldsymbol{f}]_{i}\right)(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}}\left[\int_{\mathbb{R}^{3}}[\boldsymbol{f}(\mathbf{y})]_{i} \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \mathbf{y}\right] d \boldsymbol{\omega} \\
& =\int_{\mathbb{R}^{3}}[\boldsymbol{f}(\mathbf{y})]_{i} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) d \mathbf{y},
\end{aligned}
$$

where the last step follows from changing the order of integration (justified by the absolute convergence of the integrals involved). It is then clear that $B_{\mathbf{W}}$ can be further decomposed as follows:

$$
\begin{aligned}
B_{\mathbf{W}} \boldsymbol{f} & =\mathcal{F}^{-\mathbf{1}}\left\{\chi_{\mathbf{W}}\left[\mathcal{F}\left([\boldsymbol{f}]_{0}\right)+\mathbf{i} \mathcal{F}\left([\boldsymbol{f}]_{1}\right)+\mathbf{j} \mathcal{F}\left([\boldsymbol{f}]_{2}\right)+\mathbf{k} \mathcal{F}\left([\boldsymbol{f}]_{3}\right)\right]\right\} \\
& =B_{\mathbf{W}}[\boldsymbol{f}]_{0}(\mathbf{x})+\mathbf{i} B_{\mathbf{W}}[\boldsymbol{f}]_{1}(\mathbf{x})+\mathbf{j} B_{\mathbf{W}}[\boldsymbol{f}]_{2}(\mathbf{x})+\mathbf{k} B_{\mathbf{W}}[\boldsymbol{f}]_{3}(\mathbf{x})
\end{aligned}
$$

where $\mathcal{F}\left([\boldsymbol{f}]_{i}\right)(i=0,1,2,3)$ are $\mathbb{H}$-valued functions. Because of the (left-) linearity property (1.6.7) of the QFT, it follows that the operator $B_{\mathbf{W}}$ is also left-linear. It can be further seen that $B_{\mathbf{W}}$ is idempotent.

Now, we use Plancherel's identity 1.6 .10 to show that $B_{\mathbf{W}}$ is self-adjoint:

$$
\begin{aligned}
\left\langle B_{\mathbf{W}} \boldsymbol{f}, \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} & =\left\langle\mathcal{F}^{-1}\left[\chi_{\mathbf{w}} \mathcal{F}(\boldsymbol{f})\right], \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\frac{1}{(2 \pi)^{3}}\left\langle\chi_{\mathbf{w}} \mathcal{F}(\boldsymbol{f}), \mathcal{F}(\boldsymbol{g})\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\frac{1}{(2 \pi)^{3}}\left\langle\boldsymbol{\mathcal { F }}(\boldsymbol{f}), \chi_{\mathbf{W}} \mathcal{F}(\boldsymbol{g})\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\left\langle\boldsymbol{f}, \mathcal{F}^{-1}\left[\chi_{\mathbf{w}} \mathcal{F}(\boldsymbol{g})\right]\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\left\langle\boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} .
\end{aligned}
$$

The lemma is now thoroughly established.
We may then describe how the energy of a function in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ can be distributed over both space and frequency. Let $\boldsymbol{f}(\mathbf{x}) \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ have total energy $E=\|f\|_{L_{2}\left(\mathbb{R}^{3}\right)}$. The space-limited version of $\boldsymbol{f}(\mathbf{x})$ has total energy $E_{\mathbf{T}}=\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\|\boldsymbol{f}\|_{L_{2}(\mathbf{T})} \leq E$. Since $D_{\mathbf{T}} \boldsymbol{f}$ cannot be bandlimited, its QFT has nonvanishing energy in $\boldsymbol{\omega} \in \mathbb{R}^{3} \backslash \mathbf{W}$. The band-limited version of $D_{\mathbf{T}} \boldsymbol{f}$, namely $B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}$, will have total energy $E_{\mathrm{WT}}<E_{\mathbf{T}} \leq E$. The operation $B_{\mathbf{W}} D_{\mathbf{T}}$ transforms a member of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ into a member of $\mathcal{B}(\mathbf{W})$ with smaller total energy. The question is then to identify which members of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ lose the smallest fraction of their energy under such a transformation. That is, for which $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right) \backslash\{\mathbf{0}\}$ is

$$
\frac{\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}}
$$

a maximum? This can be answered in terms of the eigenvalues of the operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ (see Subsection 4.3 for details).

For any $\boldsymbol{f}$ in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, using Fubini's Theorem and Eq. 4.2.7) we may write the composite operator $B_{\mathbf{W}} D_{\mathbf{T}}$, which first space-limits and then bandlimits, explicitly as

$$
\begin{align*}
\left(B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right)(\mathbf{x}) & =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \mathcal{F}\left(D_{\mathbf{T}} \boldsymbol{f}\right)(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}}\left(\int_{\mathbf{T}} \boldsymbol{f}(\mathbf{y}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) d \mathbf{y}\right) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{T}} \boldsymbol{f}(\mathbf{y})\left(\int_{\mathbf{W}} \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega}\right) d \mathbf{y} \\
& =\int_{\mathbb{R}^{3}} \chi_{\mathbf{T}}(\mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{y}) d \mathbf{y} \tag{4.2.10}
\end{align*}
$$

where $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is defined by (4.2.6). By the Tonelli-Hobson Theorem, the operator $D_{\mathbf{T}} B_{\mathbf{W}}$ may be defined in an analogous fashion:

$$
\begin{aligned}
\left(D_{\mathbf{T}} B_{\mathbf{W}} \boldsymbol{f}\right)(\mathbf{x}) & =\chi_{\mathbf{T}}(\mathbf{x}) \frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\chi_{\mathbf{T}}(\mathbf{x}) \frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}}\left(\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{y}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) d \mathbf{y}\right) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\chi_{\mathbf{T}}(\mathbf{x}) \int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{y})\left(\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega}\right) d \mathbf{y} \\
& =\chi_{\mathbf{T}}(\mathbf{x}) \int_{\mathbb{R}^{3}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{y}) d \mathbf{y}
\end{aligned}
$$

From what has been shown above, it is clear that the two operators, $D_{\mathbf{T}}$ and $B_{\mathbf{W}}$, do not commute.

However, it is unnecessary to proceed with a detailed analysis of $D_{\mathbf{T}} B_{\mathbf{W}}$ because the two operators, $D_{\mathbf{T}} B_{\mathbf{W}}$ and $B_{\mathbf{W}} D_{\mathbf{T}}$, have the same spectrum as shown below. We shall hereafter take $B_{\mathbf{W}} D_{\mathbf{T}}$ as our operator of fundamental concern.

What is essential here is contained in the following lemma:
Lemma 4.2.12. The operator $B_{\mathbf{W}} D_{\mathbf{T}}$ is compact on $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$.
Proof. In the first place, it is easily seen that $B_{\mathbf{W}} D_{\mathbf{T}}$ is a (left-)linear operator. Now, we set $g_{\mathbf{y}}(\mathbf{x}):=K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$. Noting that $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ defined by (4.2.6) is an even function, we find from the space-shift property (1.6.8) of the QFT that

$$
\begin{equation*}
\mathcal{F}\left(g_{\mathbf{y}}\right)(\boldsymbol{\omega})=\chi_{\mathbf{w}}(\boldsymbol{\omega}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) \tag{4.2.11}
\end{equation*}
$$

Parseval's identity (1.6.11), together with Lemma 1.1.3), gives further

$$
\begin{aligned}
\int_{\mathbb{R}^{3}}\left|g_{\mathbf{y}}(\mathbf{x})\right|^{2} d \mathbf{x} & =\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}}\left|\mathcal{F}\left(g_{\mathbf{y}}\right)(\boldsymbol{\omega})\right|^{2} d \boldsymbol{\omega} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}}\left|\chi_{\mathbf{w}}(\boldsymbol{\omega})\right|^{2} d \boldsymbol{\omega} .
\end{aligned}
$$

Proceeding in this manner, we find that $B_{\mathbf{W}} D_{\mathbf{T}}$ defined by 4.2.10 is an integral operator whose kernel is square-integrable:

$$
\begin{align*}
& \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}}\left|\chi_{\mathbf{T}}(\mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})\right|^{2} d \mathbf{x} d \mathbf{y} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}}\left|\chi_{\mathbf{T}}(\mathbf{y})\right|^{2} d \mathbf{y} \int_{\mathbb{R}^{3}}\left|\chi_{\mathbf{w}}(\boldsymbol{\omega})\right|^{2} d \boldsymbol{\omega} \\
& =\frac{1}{(2 \pi)^{3}}|\mathbf{T}||\mathbf{W}| . \tag{4.2.12}
\end{align*}
$$

Let $\left\{\boldsymbol{f}_{n}\right\}$ be a bounded sequence in $L_{2}(\mathbf{T}, \mathbb{H})$. Then, there exists a subsequence $\left\{\boldsymbol{f}_{n_{k}}\right\}$, which converges weakly to some $\boldsymbol{f} \in L_{2}(\mathbf{T}, \mathbb{H})$. By (4.2.12), it follows that $\int_{\mathbb{R}^{3}}\left|\chi_{\mathbf{T}}(\mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})\right|^{2} d \mathbf{y}<\infty$ for almost every $\mathbf{x} \in \mathbb{R}^{3}$. For any such $\mathbf{x} \in \mathbb{R}^{3}$,

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}_{n_{k}}(\mathbf{y}) d \mathbf{y} & =\lim _{k \rightarrow \infty}\left\langle\boldsymbol{f}_{n_{k}}, K_{\mathbf{W}}(\mathbf{x}, \cdot)\right\rangle_{L_{2}(\mathbf{T}, \mathbb{H})} \\
& =\left\langle\boldsymbol{f}, K_{\mathbf{W}}(\mathbf{x}, \cdot)\right\rangle_{L_{2}(\mathbf{T}, \boldsymbol{H})} \\
& =\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{y}) d \mathbf{y},
\end{aligned}
$$

which shows that $\left(B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}_{n_{k}}\right)(\mathbf{x})$ converges pointwise to $\left(B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right)(\mathbf{x})$ for almost every $\mathbf{x}$.

Schwarz's inequality (1.2.4 yields

$$
\left|\left(B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}_{n_{k}}\right)(\mathbf{x})\right| \leq \sup _{k}\left\|\boldsymbol{f}_{n_{k}}\right\|_{L_{2}(\mathbf{T})}\left(\int_{\mathbf{T}}\left|K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})\right|^{2} d \mathbf{y}\right)^{1 / 2}
$$

for all $k$, and thus, by the Lebesgue Dominated Convergence Theorem,

$$
\begin{aligned}
& \lim _{k \rightarrow \infty}\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}_{n_{k}}-B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} \\
& =\lim _{k \rightarrow \infty} \int_{\mathbb{R}^{3}}\left|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}_{n_{k}}(\mathbf{x})-B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}(\mathbf{x})\right|^{2} d \mathbf{x} \\
& =0
\end{aligned}
$$

This concludes the proof.
We may further observe that, since the separate operators $B_{\mathbf{W}}$ and $D_{\mathbf{T}}$ are both self-adjoint and do not commute, the operator $B_{\mathbf{W}} D_{\mathbf{T}}$ cannot be self-adjoint. Nevertheless, the eigenvalues of the operator $B_{\mathbf{W}} D_{\mathbf{T}}$ are all positive real numbers, an important fact whose proof we shall postpone for the moment. By the above observation, Proposition 4.2 .14 below justifies considering combinations of the separated operators $D_{\mathbf{T}}$ and $B_{\mathbf{W}}$, such as $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$. It will be further shown that the spectra of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$ are identical.

Hence, we may proceed as follows. In accordance with Definition 4.2.4 and Eq. (4.2.10) the quaternionic Hilbert-Schmidt operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$, which takes $D_{\mathbf{T}}$ into itself, may be written explicitly as

$$
\begin{equation*}
\left(D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right)(\mathbf{x})=\int_{\mathbb{R}^{3}} \chi_{\mathbf{T}}(\mathbf{x}) \chi_{\mathbf{T}}(\mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{y}) d \mathbf{y} \tag{4.2.13}
\end{equation*}
$$

where $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is the kernel defined in 4.2.6).

It will be convenient to study certain properties of the kernel in (4.2.13). The properties in question are as follows:

Proposition 4.2.13. The kernel $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ is real, symmetric, and positive definite.

Proof. From Theorem 4.2.6, we may first observe that the kernel given in (4.2.6) is real and symmetric; that is, $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})=K_{\mathbf{W}}(\mathbf{y}, \mathbf{x})$. We note further that

$$
\begin{align*}
& \int_{\mathbf{T}}\left[\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{f}}(\mathbf{y}) d \mathbf{x}\right] d \mathbf{y} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{T}} \int_{\mathbf{T}} \boldsymbol{f}(\mathbf{x})\left[\int_{\mathbf{W}} \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{y}) d \boldsymbol{\omega}\right] \overline{\boldsymbol{f}}(\mathbf{y}) d \mathbf{x} d \mathbf{y} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}}\left|\int_{\mathbf{T}} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \mathbf{x}\right|^{2} d \boldsymbol{\omega}>0, \tag{4.2.14}
\end{align*}
$$

whenever

$$
\int_{\mathbf{T}}|\boldsymbol{f}(\mathrm{x})|^{2} d \mathrm{x}>0
$$

It follows from the fact that, if it were equal to zero for some nonzero signal $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$, then $\boldsymbol{f}$ would be identically zero on $\mathbf{T}$. This cannot happen since nontrivial $\mathbb{H}$-valued functions in $\mathbb{R}^{3}$ compose $\mathcal{B}(\mathbf{W})$.

In the following proposition, we prove some elementary properties of the operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$.

Proposition 4.2.14. (i) The operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ acting on $\mathbf{T}$ is bounded by 1, self-adjoint, positive, and compact.
(ii) The distinct nonzero eigenvalues of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$ are the same, with the same respective multiplicities.
(iii) Given two eigenfunctions $\boldsymbol{\psi}_{i}$ with $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{i}=\lambda_{i} \boldsymbol{\psi}_{i}(i=1,2)$ for the eigenvalues $\lambda_{1} \neq \lambda_{2}$, we deduce $\left\langle D_{\mathbf{T}} \boldsymbol{\psi}_{1}, D_{\mathbf{T}} \boldsymbol{\psi}_{2}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}=\mathbf{0}$. Also, the eigenfunctions of $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$ corresponding to different eigenvalues are automatically orthogonal on $\mathbf{T}$.

Proof. First, since projections are bounded by 1, self-adjoint and idempotent,

$$
\begin{aligned}
\left\langle D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}, \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} & =\left\langle B_{\mathbf{W}}^{2} D_{\mathbf{T}} \boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} \\
& <\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2},
\end{aligned}
$$

therefore, $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ is bounded by 1 , self-adjoint, and positive. Moreover, by Lemma 4.2.12 the operator $B_{\mathbf{W}} D_{\mathbf{T}}$ is compact, so are $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$, and this by Lemma 1.2 .17 . Next, write $A \sim B$ if the compact operators $A$ and $B$ have the same nonzero eigenvalues, including multiplicities. We may observe that, if $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}=\lambda \boldsymbol{\psi}$ and $\lambda \neq 0$, then $\boldsymbol{\psi}=D_{\mathbf{T}} \boldsymbol{\psi}$ since $D_{\mathbf{T}}$ is a projection and $\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \neq 0$. An application of the projection $B_{\mathbf{W}}$ to the equation now yields $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}\left(B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}\right)=\lambda B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}$, so that $\lambda$ is likewise an eigenvalue of $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$. Similar reasoning applies to the other direction. Moreover, since the QFT is invertible, we have then

$$
D_{\mathbf{T}} B_{\mathbf{W}} B_{\mathbf{W}} D_{\mathbf{T}}=\left(D_{\mathbf{T}} B_{\mathbf{W}}\right)\left(B_{\mathbf{W}} D_{\mathbf{T}} D_{\mathbf{T}} B_{\mathbf{W}}\right)\left(D_{\mathbf{T}} B_{\mathbf{W}}\right)^{-1}
$$

so $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$ are similar, and $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \sim B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$. Now, according to Theorem 1.2 .29 and using the fact that $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ (resp. $\left.B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}\right)$ is self-adjoint, it follows that eigenfunctions of these operators corresponding to different eigenvalues are orthogonal on $\mathbf{T}$. This concludes the proof.

We can fall back in this manner on the theory of quaternionic compact, self-adjoint operators discussed in Subsection 1.2.3, to obtain essential properties of the eigenfunctions of the operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and their respective eigenvalues; that is, the $\lambda$ 's for which the characteristic quaternion equation of the form $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}=\lambda \boldsymbol{\psi}$ has a nontrivial solution, $\boldsymbol{\psi} \in \mathcal{B}(\mathbf{W})$. It shall be remarked that, in the one-dimensional case, all eigenvalues of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ have multiplicity one. The proof of this property uses the differential equation (1.5.8) of the PSWFs [299]. Although this is also true for symmetric regions in higher dimensions [301], we can only conclude that each eigenvalue has finite multiplicity, in general.

From Proposition 4.2.14, there is no assurance that each eigenfunction belongs to an eigenvalue different from all the rest. Of course, if we have two degenerate eigenfunctions $\boldsymbol{\psi}_{1}, \boldsymbol{\psi}_{2}$ of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ (i.e., with a common eigenvalue), we can always construct orthogonal eigenfunctions. Since $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ is left-linear, it is clear that every linear combination of $\boldsymbol{\psi}_{1}$ and $\boldsymbol{\psi}_{2}$ is also an eigenfunction of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$. Consider $\boldsymbol{\psi}^{(1)}:=\boldsymbol{\alpha}_{1} \boldsymbol{\psi}_{1}+\boldsymbol{\beta}_{1} \boldsymbol{\psi}_{2}$, and $\boldsymbol{\psi}^{(2)}:=$ $\boldsymbol{\alpha}_{2} \boldsymbol{\psi}_{1}+\boldsymbol{\beta}_{2} \boldsymbol{\psi}_{2}$ for quaternionic constants $\boldsymbol{\alpha}_{i}, \boldsymbol{\beta}_{i}$. The functions $\boldsymbol{\psi}^{(1)}$ and $\boldsymbol{\psi}^{(2)}$ are orthogonal on $\mathbf{T}$, provided

$$
\begin{aligned}
\mathbf{0}= & \left\langle D_{\mathbf{T}} \boldsymbol{\psi}^{(1)}, D_{\mathbf{T}} \boldsymbol{\psi}^{(2)}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
= & \boldsymbol{\alpha}_{1} \overline{\boldsymbol{\alpha}_{2}}\left\|D_{\mathbf{T}} \boldsymbol{\psi}_{1}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}+\boldsymbol{\alpha}_{1}\left\langle D_{\mathbf{T}} \boldsymbol{\psi}_{1}, D_{\mathbf{T}} \boldsymbol{\psi}_{2}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \overline{\boldsymbol{\beta}_{2}} \\
& +\boldsymbol{\beta}_{1}\left\langle D_{\mathbf{T}} \boldsymbol{\psi}_{2}, D_{\mathbf{T}} \boldsymbol{\psi}_{1}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \overline{\boldsymbol{\alpha}_{2}}+\boldsymbol{\beta}_{1} \overline{\boldsymbol{\beta}_{2}}\left\|D_{\mathbf{T}} \boldsymbol{\psi}_{2}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} .
\end{aligned}
$$

Although this equation can be satisfied by choosing the adequate underlying constants $\boldsymbol{\alpha}_{i}, \boldsymbol{\beta}_{i} \in \mathbb{H}$, such development is beyond the scope of the present work.

By the Spectral Theorem 1.2.30for compact and self-adjoint operators, we can then find a countably infinite set of band-limited quaternionic functions $\boldsymbol{\psi}_{0}(\mathbf{x}), \boldsymbol{\psi}_{1}(\mathrm{x}), \boldsymbol{\psi}_{2}(\mathbf{x}), \ldots$, and a set of real positive numbers $1>\lambda_{0} \geq \lambda_{1} \geq$ $\lambda_{2} \geq \cdots$ such that as $n \rightarrow \infty, \lim \lambda_{n}=0$, which we will henceforth assume done. The notation used above hides the fact that both the $\boldsymbol{\psi}$ 's and the $\lambda$ 's depend upon the space-bandwidth product $|\mathbf{T}||\mathbf{W}| / 8$, which now plays the role of the time-bandwidth product $2 T W$ in the one-dimensional case. In Subsection 4.2.2, we show that the eigenfunctions $\boldsymbol{\psi}_{n}$ enjoy a remarkable double orthogonality property: not only are they orthogonal over the threedimensional Euclidean space $\mathbb{R}^{3}$, but their restrictions to $\mathbf{T}$ are also mutually orthogonal and, when normalized, form a basis for both $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$ (see Theorem 4.2 .24 below). This double orthogonality makes the sequence $\left\{\boldsymbol{\psi}_{n}\right\}$ an ideal basis for considering the many problems in which information about a band-limited quaternionic function is given on a domain. We will illustrate it with the question of extrapolation, already mentioned (see Subsection 4.2.3 below).

Although $B_{\mathbf{W}} D_{\mathbf{T}}$ is not self-adjoint, we proceed to show that all of its eigenvalues are real and, in fact, equal to those of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$, respectively, of $B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}}$.

Proposition 4.2.15. The spectrum of the operator $B_{\mathbf{W}} D_{\mathbf{T}}$ consists of positive eigenvalues, bounded by 1 and accumulating at zero.

Proof. We assume that $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}=\lambda \boldsymbol{\psi}$. An application by $B_{\mathbf{W}} D_{\mathbf{T}}$ to the equation leads to $\left(B_{\mathbf{W}} D_{\mathbf{T}}\right)^{2} \boldsymbol{\psi}=\lambda B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}$, which shows that $\lambda$ is an eigenvalue of $B_{\mathbf{W}} D_{\mathbf{T}}$ (it corresponds to the eigenfunction $B_{\mathbf{W}} D_{\mathbf{T}} \psi$ ). Hence, every eigenvalue of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ is also an eigenvalue of $B_{\mathbf{W}} D_{\mathbf{T}}$. Conversely, assume that $B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}=\lambda \boldsymbol{\psi}$. Applying $D_{\mathbf{T}}$ to the equation leads to $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}=\lambda D_{\mathbf{T}} \boldsymbol{\psi}$, which is equivalent to $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} D_{\mathbf{T}} \boldsymbol{\psi}=\lambda D_{\mathbf{T}} \boldsymbol{\psi}$. This means that $\lambda$ is also an eigenvalue of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$, and thus every eigenvalue of $B_{\mathbf{W}} D_{\mathbf{T}}$ is also an eigenvalue of $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$. It follows then that $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$ and $B_{\mathbf{W}} D_{\mathbf{T}}$ (as well as $D_{\mathbf{T}} B_{\mathbf{W}}$ ) have the same set of eigenvalues. The rest of the proof follows from Proposition 4.2.14.

The following theorem, which will be used in further discussion, generalizes the uncertainty principle due to Donoho and Stark in [99] within our context. In short, this result asserts that a signal and its FT cannot both be well-concentrated around their respective means: narrowing one broadens necessarily the other. The classical version of this theorem is particularly
crucial as it can be applied to signal recovery. Many variations and related information about this result can be found in [123, 143]. Other versions of this result were given, for locally compact abelian groups, by Smith [305], Özaydin and Przebinda [263], for operators on Banach spaces, by Goha and Goodman [141], and for the Dunkl Transform by Soltani [308]. A multidimensional generalization of this theorem in which the QFT is defined by (1.6.1) and the sets are measurable was first proved, as in [99], by Chen et al. in [76].

According to Definition 4.2.1, we extend this result within our context as follows:

Theorem 4.2.16 (Uncertainty principle of Donoho and Stark). Suppose that $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, not identically zero, is $\varepsilon_{\mathbf{T}}$-concentrated on $\mathbf{T}$ and $\mathcal{F}(\boldsymbol{f})$ is $\varepsilon_{\mathbf{W}}$-concentrated on $\mathbf{W}$. Then

$$
|\mathbf{T}||\mathbf{W}| \geq(2 \pi)^{3}\left(1-\varepsilon_{\mathbf{T}}-\frac{1}{(2 \pi)^{3}} \varepsilon_{\mathbf{W}}\right)^{2}
$$

Proof. This is proved in a manner similar to [99]. We rely on the norm of the operator $B_{\mathbf{W}} D_{\mathbf{T}}$ :

$$
\left\|B_{\mathbf{W}} D_{\mathbf{T}}\right\|=\sup _{f \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \frac{\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}}
$$

and, in accordance with 4.2.10 , define the Hilbert-Schmidt norm of $B_{\mathbf{W}} D_{\mathbf{T}}$ to be

$$
\left\|B_{\mathbf{W}} D_{\mathbf{T}}\right\|_{H S}:=\left(\int_{\mathbb{R}^{3}} \int_{\mathbf{T}}\left|K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})\right|^{2} d \mathbf{y} d \mathbf{x}\right)^{1 / 2}
$$

With these definitions in place and using Cauchy-Schwarz inequality (1.2.4), it turns out that $\left\|B_{\mathbf{W}} D_{\mathbf{T}}\right\|_{H S} \geq\left\|B_{\mathbf{W}} D_{\mathbf{T}}\right\|$. Moreover, since

$$
\left|K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})\right|^{2}=\left|K_{\mathbf{W}}(\mathbf{y}, \mathbf{x})\right|^{2}
$$

by (4.2.12), we have

$$
\begin{equation*}
\left\|B_{\mathbf{W}} D_{\mathbf{T}}\right\|_{H S}=\frac{1}{(2 \pi)^{3 / 2}}(|\mathbf{T}||\mathbf{W}|)^{1 / 2} \tag{4.2.15}
\end{equation*}
$$

By Definitions 4.2.4 and 4.2 .5 and the fact that $\left\|B_{\mathbf{W}}\right\|=1$, if $\boldsymbol{f}$ is $\varepsilon_{\mathbf{T}^{-}}$ concentrated on $\mathbf{T}$ and $\mathcal{F}(\boldsymbol{f})$ is $\varepsilon_{\mathbf{W}}$-concentrated on $\mathbf{W}$, then it follows from (4.2.2) and 4.2.5) that

$$
\begin{aligned}
& \left\|\boldsymbol{f}-B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \\
& \leq\left\|\boldsymbol{f}-B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}+\left\|B_{\mathbf{W}}\left(\boldsymbol{f}-D_{\mathbf{T}} \boldsymbol{f}\right)\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \\
& \leq\left(\varepsilon_{\mathbf{T}}+\frac{1}{(2 \pi)^{3 / 2}} \varepsilon_{\mathbf{W}}\right)\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)} .
\end{aligned}
$$

Proceeding in this manner, we obtain

$$
\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \geq\left(1-\varepsilon_{\mathbf{T}}-\frac{1}{(2 \pi)^{3 / 2}} \varepsilon_{\mathbf{W}}\right)\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}
$$

or, equivalently, that

$$
\left\|B_{\mathbf{W}} D_{\mathbf{T}}\right\| \geq 1-\varepsilon_{\mathbf{T}}-\frac{1}{(2 \pi)^{3 / 2}} \varepsilon_{\mathbf{W}}
$$

This inequality, together with 4.2.15, implies the theorem.
It is not in the scope of the present work to thoroughly discuss the applications of Theorem 4.2.16. But to help the reader understand the importance of this result in a quaternionic context, set $\varepsilon_{\mathbf{T}}=\varepsilon_{\mathbf{W}}=0$ in Theorem 4.2.16 and observe that $\boldsymbol{f}$ is concentrated on $\mathbf{T}$ if and only if $\operatorname{supp} \boldsymbol{f} \subset \mathbf{T}$ and $\mathcal{F}(\boldsymbol{f})$ is concentrated on $\mathbf{W}$ if and only if $\operatorname{supp} \boldsymbol{\mathcal { F }}(\boldsymbol{f}) \subset \mathbf{W}$. Hence, $|\mathbf{T}||\mathbf{W}| \geq(2 \pi)^{3}$. This means that a nonzero function and its QFT cannot both be highly concentrated, independently of the concentration sets $\mathbf{T}$ and $\mathbf{W}$ chose.

The proof of Theorem 4.2.16 is more prosperous than the stated conclusion. We note that since $\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \leq\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}$, the norm $\left\|D_{\mathbf{T}} B_{\mathbf{W}}\right\|$ satisfies the identity

$$
\begin{aligned}
\left\|D_{\mathbf{T}} B_{\mathbf{W}}\right\| & =\sup _{\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \frac{\left\|D_{\mathbf{T}} B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}} \\
& =\sup _{\boldsymbol{g} \in \mathcal{B}(\mathbf{W})} \frac{\left\|D_{\mathbf{T}} \boldsymbol{g}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} .
\end{aligned}
$$

Thus the quantity $\left\|D_{\mathbf{T}} B_{\mathbf{W}}\right\|$ measures how nearly concentrated on $\mathbf{T}$ a function $\boldsymbol{g} \in \mathcal{B}(\mathbf{W})$ can be. It will be discussed in detail in Section 4.3.

### 4.2.2 Definition and Properties of the $c$-QPSWFs

The previous section presented a preliminary account of specific properties of the spectrum of the convolution integral operator $D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}}$. It was shown that the introduction of such an operator was suggested by the operation that corresponds to space-limiting and band-limiting a quaternionic signal. The fundamental discovery of Slepian and his collaborators Landau and Pollak in the early 60s [198, 199, 299], which states that the PSWFs of order zero are maximally concentrated within a given time interval, suggests the development of a general theory of eigenfunctions and eigenvalues of a finite version of the QFT. The procedure we adopt consists of identifying under
what circumstances a square-integrable quaternionic function and its QFT are simultaneously concentrated. The account of the theory given in the present section extends the results from [300] and [350] to quaternionic signals using the QFT defined by (1.6.4).

We begin by introducing the $c$-QPSWFs in the finite-QFT setting. This, we shall do in the following manner. We have already seen that the classical PSWFs are solutions of the integral equation (1.5.14), which involves an exponential factor containing the parameter $c=T W$. In the new definition, we use the same strategy to construct the underlying integral equation and first replace (1.6.5) in 1.6 .4 with the quaternionic Fourier kernel $\overline{\boldsymbol{E}}(\mathbf{y}, c \mathbf{x})$, where again we write $\mathbf{W}=c \mathbf{T}$. We will keep referring to the scale factor $c$ as the Slepian frequency. Second, by combining the noncommutativity of the underlying multiplication and the left-linearity property (1.6.7) of the QFT, it makes sense to multiply the eigenvalues from the left of the corresponding quaternionic signals. Carrying this further, we are thus led to the following statement, containing a definition of the $c$-QPSWFs.

Definition 4.2.17. Given a real number $c>0$, the $c-Q P S W F s \boldsymbol{\psi}_{n}(n=$ $0,1, \ldots)$ are the solutions of the integral equation

$$
\begin{equation*}
\boldsymbol{\mu}_{n} \boldsymbol{\psi}_{n}(\mathbf{x})=\int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) \overline{\boldsymbol{E}}(\mathbf{y}, c \mathbf{x}) d \mathbf{y} \tag{4.2.16}
\end{equation*}
$$

for any $\mathbf{x} \in \mathbf{T}$, where $\boldsymbol{\mu}_{n} \in \mathbb{H}$ are called the eigenvalues corresponding to the eigenfunctions $\boldsymbol{\psi}_{n}$.

We will call this definition the finite-QFT form of the $c$-QPSWFs and was first given, in a form substantially identical with the above, by Morais et al. in [251]. The notation in this definition conceals that both the $\boldsymbol{\mu}$ 's and the $\boldsymbol{\psi}$ 's are functions of the parameter $c$. When it is necessary to make this dependence explicit, we write $\boldsymbol{\mu}_{n}=\boldsymbol{\mu}_{n}(c)$ and $\boldsymbol{\psi}_{n}(\mathbf{x})=\boldsymbol{\psi}_{n}(c, \mathbf{x}), n=$ $0,1,2, \ldots$.

Later in this section, we show that a solution of equation 4.2.16) is entirely equivalent to a solution of a more straightforward quaternionic integral equation with kernel arising from the sinc functions (respectively, of the form (4.2.13) when the symmetries discussed in the previous subsection are maintained. We shall accordingly hereafter take (4.2.16) as our equation of fundamental concern.

Naturally, a considerable simplification occurs when for each $n$, the function $\boldsymbol{\psi}_{n}(\mathbf{x})$ is even or odd as a function of a quaternionic variable. Due to the symmetry of the domain $\mathbf{T}$ about the origin (that is, $\mathbf{x} \in \mathbf{T}$ implies $-\mathbf{x} \in \mathbf{T}$ ), if we change variables in (4.2.16) by replacing $\mathbf{x}$ and $\mathbf{y}$ with their negatives, then the quaternionic kernel remains unchanged as well as the domain of
integration. Therefore $\boldsymbol{\psi}_{n}(\mathbf{x})$ and $\boldsymbol{\psi}_{n}(-\mathbf{x})$ are both solutions to the integral equation (4.2.16) corresponding to the same eigenvalue $\boldsymbol{\mu}_{n}$. If $\boldsymbol{\psi}_{n}(\mathbf{x})$ is not even or odd, there will be two independent solutions, one even and one odd, corresponding to that eigenvalue, namely $\boldsymbol{\psi}_{\mathbf{e}}(\mathbf{x}):=(1 / 2)\left[\boldsymbol{\psi}_{n}(\mathbf{x})+\boldsymbol{\psi}_{n}(-\mathbf{x})\right]$ and $\boldsymbol{\psi}_{\mathbf{o}}(\mathbf{x}):=(1 / 2)\left[\boldsymbol{\psi}_{n}(\mathbf{x})-\boldsymbol{\psi}_{n}(-\mathbf{x})\right]$. The eigenfunctions of (4.2.16) can be chosen to be either even or odd functions of $\mathbf{x}$.

The quaternionic conjugate of (4.2.16) is given by

$$
\begin{equation*}
\overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) \overline{\boldsymbol{\mu}_{n}}=\int_{\mathbf{T}} \boldsymbol{E}(\mathbf{y}, c \mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{y}) d \mathbf{y} \tag{4.2.17}
\end{equation*}
$$

Multiply 4.2.16) by $\overline{\boldsymbol{\psi}_{n}}(\mathbf{x})$ from the right-hand side and integrate over $\mathbf{T}$, and multiply (4.2.17) by $\boldsymbol{\psi}_{n}(\mathbf{x})$ from the left-hand side and integrate over T. Combining these equations, one finds by using the symmetry of $\mathbf{T}$ and Fubini's Theorem that

$$
\begin{aligned}
& \left(\boldsymbol{\mu}_{n} \pm \overline{\boldsymbol{\mu}_{n}}\right) \int_{\mathbf{T}}\left|\boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x} \\
& =\int_{\mathbf{T}}\left(\int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) \overline{\boldsymbol{E}}(\mathbf{y}, c \mathbf{x}) d \mathbf{y}\right) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x} \\
& \quad \pm \int_{\mathbf{T}}\left(\int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{E}}(-\mathbf{x}, c \mathbf{y}) d \mathbf{x}\right) \overline{\boldsymbol{\psi}_{n}}(\mathbf{y}) d \mathbf{y} \\
& =\int_{\mathbf{T}}\left(\int_{\mathbf{T}}\left[\boldsymbol{\psi}_{n}(\mathbf{y}) \pm \boldsymbol{\psi}_{n}(-\mathbf{y})\right] \overline{\boldsymbol{E}}(\mathbf{y}, c \mathbf{x}) d \mathbf{y}\right) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x} .
\end{aligned}
$$

If $\boldsymbol{\psi}_{n}(\mathbf{x})$ is even, by choosing the negative sign in this equation, then one obtains $\boldsymbol{\mu}_{n}-\overline{\boldsymbol{\mu}_{n}}=\mathbf{0}$, whereas if $\boldsymbol{\psi}_{n}(\mathbf{x})$ is odd, by choosing the plus sign, one finds $\boldsymbol{\mu}_{n}+\overline{\boldsymbol{\mu}_{n}}=\mathbf{0}$.

We have thus shown that:
Proposition 4.2.18. The eigenvalues of (4.2.16) associated with even eigenfunctions are real, and the eigenvalues of (4.2.16) associated with odd eigenfunctions are purely quaternionic.

It should be observed that the solutions of (4.2.16) can be chosen either even (in which case the eigenvalue $\mu$ is real) or odd (in which case $\boldsymbol{\mu}$ is purely quaternionic), if and only if both $\mathbf{T}$ and $\mathbf{W}$ are symmetric about the origin. If one or both domains $\mathbf{T}$ and $\mathbf{W}$ is asymmetric, then no eigenfunctions of fixed parity exist.

Our present concern is with the completeness properties of the $c$-QPSWFs in two subspaces of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. The following theorem gives these remarkable properties.

Theorem 4.2.19. Given a real number $c>0$, the $c-Q P S W F s \boldsymbol{\psi}_{n}(n=$ $0,1, \ldots$ ) are band-limited functions to $\mathbf{W}$ and their QFTs are given by

$$
\begin{equation*}
\mathcal{F}\left(\boldsymbol{\psi}_{n}\right)(\boldsymbol{\omega})=\left(\frac{2 \pi}{c}\right)^{3}\left(\boldsymbol{\mu}_{n}\right)^{-1} \boldsymbol{\psi}_{n}\left(\frac{\boldsymbol{\omega}}{c}\right) \chi_{\mathbf{w}}(\boldsymbol{\omega}) \tag{4.2.18}
\end{equation*}
$$

Further, they are complete in $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$.
Proof. From Proposition 4.2 .18 and representation 1.6.6, it follows that equation (4.2.16) is equivalent to the pair of equations

$$
\begin{align*}
\mu_{e} \boldsymbol{\psi}_{e}(\mathbf{x}) & =\int_{\mathbf{T}} \cos (c\langle\mathbf{x}, \mathbf{y}\rangle) \boldsymbol{\psi}_{e}(\mathbf{y}) d \mathbf{y}  \tag{4.2.19}\\
-\boldsymbol{\mu}_{o} \mathbf{u} \boldsymbol{\psi}_{o}(\mathbf{x}) & =\int_{\mathbf{T}} \sin (c\langle\mathbf{x}, \mathbf{y}\rangle) \boldsymbol{\psi}_{o}(\mathbf{y}) d \mathbf{y} \tag{4.2.20}
\end{align*}
$$

in which $\mu_{e}$ and $\boldsymbol{\mu}_{o}$ are, respectively, real and purely quaternionic constants. The eigenfunctions of (4.2.19) must be even, and $\mu_{e}=0$ cannot be an eigenvalue of this equation, so the only even square-integrable quaternionic function in $\mathbf{T}$ for which

$$
\int_{\mathbf{T}} \cos (c\langle\mathbf{x}, \mathbf{y}\rangle) \boldsymbol{\psi}(\mathbf{y}) d \mathbf{y}=\mathbf{0}, \quad \mathbf{x} \in \mathbf{T}
$$

is $\boldsymbol{\psi}(\mathbf{y}) \equiv \mathbf{0}$. It then follows from [282, p.234] that the eigenfunctions of (4.2.19) are complete in the class of even quaternionic functions squareintegrable in T. A similar argument shows that the solutions of 4.2.20) are complete in the class of odd quaternionic functions square-integrable in T. Solutions of 4.2.19) are automatically orthogonal to solutions of (4.2.20) by symmetry. We have thus shown that the solutions of (4.2.16) are complete in $\mathcal{D}(\mathbf{T})$.

Now, a change of variables converts 4.2.16 into

$$
\left(\frac{c}{2 \pi}\right)^{3} \boldsymbol{\mu}_{n} \boldsymbol{\psi}_{n}(\mathbf{x})=\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}} \boldsymbol{\psi}_{n}\left(\frac{\boldsymbol{\omega}}{c}\right) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega}
$$

which shows that $\boldsymbol{\psi}_{n} \in \mathcal{B}(\mathbf{W})$. Indeed, since the functions $\boldsymbol{\psi}_{n}(\boldsymbol{\omega} / c)$ are complete in $\boldsymbol{\omega} \in \mathbf{W}$, Parseval's identity shows that the $\boldsymbol{\psi}_{n}(\mathbf{x})$ are complete in $\mathcal{B}(\mathbf{W})$. The result follows.

Eq. 4.2.18) states the exciting fact that the finite-QFT of $\boldsymbol{\psi}_{n}(\mathbf{x})$ restricted to $\mathbf{W}$ has the same form as the $\boldsymbol{\psi}_{n}$ except for a scale change. There does not seem to be any known quaternionic function that possesses a similar property.

It is possible to find an interesting relation between the $c$-QPSWFs at different scales by applying Eq. 4.2.16). We proceed to show that the product of a quaternionic exponential with a $c$-QPSWF of Slepian frequency $c>0$ is a band-limited function with frequency $2 c$. This lemma contains a quaternionic counterpart of a result due to Shkolnisky [296], who established the lemma for the case of complex exponentials of form $e^{i c(\mathbf{x}, \boldsymbol{\omega}\rangle}$, where $\mathbf{x}, \boldsymbol{\omega} \in$ $\Omega_{0}$.

Proposition 4.2.20. Given a real number $c>0, \boldsymbol{\omega} \in \mathbf{T}$, there exists a band-limited function $\mathbf{\Psi}$ on $\mathbf{T}$ such that

$$
\boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, c \mathbf{x})=\int_{\mathbf{T}} \boldsymbol{\Psi}(\boldsymbol{\sigma}) \overline{\boldsymbol{E}}(\boldsymbol{\sigma}, 2 c \mathbf{x}) d \boldsymbol{\sigma}
$$

for any $\mathbf{x} \in \mathbf{T}$, where $\boldsymbol{\psi}_{n}$ is a solution of (4.2.16) and $\boldsymbol{\Psi}$ satisfies

$$
\begin{equation*}
\int_{\mathbf{T}}|\boldsymbol{\Psi}(\boldsymbol{\sigma})|^{2} d \boldsymbol{\sigma}=\frac{4}{\left|\boldsymbol{\mu}_{n}\right|^{2}}\left\|D_{\mathbf{T}} \boldsymbol{\psi}_{n}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} \tag{4.2.21}
\end{equation*}
$$

Proof. Multiplying both sides of 4.2.16) from the right-hand side by $\overline{\boldsymbol{E}}(\boldsymbol{\omega}, c \mathbf{x})$, it follows that

$$
\begin{equation*}
\boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, c \mathbf{x})=\left(\boldsymbol{\mu}_{n}\right)^{-1} \int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) \overline{\boldsymbol{E}}(\mathbf{y}+\boldsymbol{\omega}, \mathbf{c x}) d \mathbf{y} \tag{4.2.22}
\end{equation*}
$$

Applying the change of variables $\boldsymbol{\sigma}=(\mathbf{y}+\boldsymbol{\omega}) / 2$ to 4.2.22), we obtain

$$
\begin{equation*}
\boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, c \mathbf{x})=\left(\boldsymbol{\mu}_{n}\right)^{-1} \int_{\mathbf{T}_{\boldsymbol{\omega}}^{*}} 2 \boldsymbol{\psi}_{n}(2 \boldsymbol{\sigma}-\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\sigma}, 2 c \mathbf{x}) d \boldsymbol{\sigma} \tag{4.2.23}
\end{equation*}
$$

where $\mathbf{T}_{\omega}^{*}$ is a cube obtained by shifting the domain $\mathbf{T}$ by the amount of $\boldsymbol{\omega} / 2$; that is if $\mathbf{x} \in \mathbf{T}_{\boldsymbol{\omega}}^{*}$ then $\mathbf{x}-\boldsymbol{\omega} / 2 \in \mathbf{T}$. Therefore, we can write 4.2.23) as

$$
\boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, c \mathbf{x})=\int_{\mathbf{T}} \boldsymbol{\Psi}(\boldsymbol{\sigma}) \overline{\boldsymbol{E}}(\boldsymbol{\sigma}, 2 c \mathbf{x}) d \boldsymbol{\sigma}
$$

where

$$
\begin{equation*}
\boldsymbol{\Psi}(\boldsymbol{\sigma}):=\chi_{\mathbf{T}_{\omega}^{*}}(\boldsymbol{\sigma})\left(\boldsymbol{\mu}_{n}\right)^{-1} 2 \boldsymbol{\psi}_{n}(2 \boldsymbol{\sigma}-\boldsymbol{\omega}) \tag{4.2.24}
\end{equation*}
$$

Equality (4.2.21) follows straightforwardly from 4.2.24).
The proposition now established may be of considerable importance since it can be applied to construct an approximation scheme for quaternionic exponentials. As a development of a numerical technique for representing a band-limited quaternionic function as an expansion in c-QPSWFs goes beyond the scope of the present work, reference is here made to [296], where details on the subject can be found.

The $c$-QPSWFs possess several unique properties that make them useful in the study of band-limited quaternionic functions. They are also the eigenfunctions of an integral equation with kernel arising from the sinc functions (see Eq. (4.2.25) below), called the low-pass filtering form of the $c$-QPSWFs, which is a quaternionic analog of 1.5.16). Theorem 4.2.21 asserts that to solve (4.2.16), it suffices to solve (4.2.25) and vice-versa. Most of the algebraic properties of the $c$-QPSWFs and the corresponding eigenvalues are deduced from (4.2.25).

Theorem 4.2.21. Given a real number $c>0$, the $c-Q P S W F s \boldsymbol{\psi}_{n}(n=$ $0,1, \ldots$ ) are solutions of the integral equation

$$
\begin{equation*}
\lambda_{n} \boldsymbol{\psi}_{n}(\mathbf{x})=D_{\mathbf{T}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})\left(=\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\psi}_{n}(\mathbf{y}) d \mathbf{y}\right) \tag{4.2.25}
\end{equation*}
$$

for any $\mathbf{x} \in \mathbf{T}$, where the $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is defined by (4.2.6). The parameters $\lambda_{n}:=(c /(2 \pi))^{3}\left|\boldsymbol{\mu}_{n}\right|^{2}, n=0,1, \ldots$ are the eigenvalues corresponding to the eigenfunctions $\boldsymbol{\psi}_{n}$.

Proof. Bearing in mind that the kernel $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is real so that it can commute with any quaternionic number. Now, consider the relation

$$
\begin{equation*}
\int_{\mathrm{W}} \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathrm{x}-\mathbf{y}) d \boldsymbol{\omega}=\int_{\mathrm{W}} \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathrm{y}) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} . \tag{4.2.26}
\end{equation*}
$$

From above and Fubini's Theorem, the right-hand side of equation 4.2.25) gives

$$
\begin{aligned}
& \int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) d \mathbf{y} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbf{W}}\left(\int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{y}) d \mathbf{y}\right) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega}
\end{aligned}
$$

Using the property $\boldsymbol{E}(\mathbf{x}, c \boldsymbol{\omega})=\overline{\boldsymbol{E}}(-\mathbf{x}, c \boldsymbol{\omega})$, and the definition 4.2.16) of the $c$-QPSWFs, the above integral reads now as follows:

$$
\begin{aligned}
\int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) d \mathbf{y} & =\frac{1}{(2 \pi)^{3}} \boldsymbol{\mu}_{n} \int_{\mathbf{W}} \boldsymbol{\psi}_{n}\left(\frac{\boldsymbol{\omega}}{c}\right) \boldsymbol{E}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\left(\frac{c}{2 \pi}\right)^{3} \boldsymbol{\mu}_{n} \int_{\mathbf{W} / c} \boldsymbol{\psi}_{n}(\boldsymbol{\omega}) \boldsymbol{E}(c \boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\left(\frac{c}{2 \pi}\right)^{3} \boldsymbol{\mu}_{n} \int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(-\mathbf{x}, c \boldsymbol{\omega}) d \boldsymbol{\omega} \\
& =\left(\frac{c}{2 \pi}\right)^{3}\left(\boldsymbol{\mu}_{n}\right)^{2} \boldsymbol{\psi}_{n}(-\mathbf{x}) .
\end{aligned}
$$

From the last equality and the fact that the $\boldsymbol{\psi}_{n}$ is even or odd with $n$ (see Proposition 4.2.18 above), we find

$$
\begin{equation*}
\lambda_{n}=\left(\frac{c}{2 \pi}\right)^{3}\left|\boldsymbol{\mu}_{n}\right|^{2} \tag{4.2.27}
\end{equation*}
$$

which gives the left-hand side of (4.2.25).

It should be emphasized that the reduction of Eq. (4.2.16) to the more straightforward integral equation (4.2.25) is only possible because both $\mathbf{T}$ and $\mathbf{W}$ are symmetric about the origin and $\mathbf{W}=c \mathbf{T}$ with $c>0$. Bearing in mind the noncommutativity of products of $\mathbb{H}$-valued signals with the QFT kernel (4.2.16), the low-pass filtering form (4.2.25), which connects the $\mathbb{H}$ valued signals with a real-valued kernel, provides an easy way to study the $c$-QPSWFs.

It is clear from the context that we have suppressed that the eigenvalues $\lambda_{n}$ depend on the parameter $c$ as well. It should be observed that the completeness of the functions $\boldsymbol{\psi}_{n}$ in $\mathcal{D}(\mathbf{T})$ assures us that the quantities (4.2.27) are the only eigenvalues of (4.2.25) and that if these quantities are distinct, the $\boldsymbol{\psi}_{n}$ are (apart from multiplicative constants) the unique $\mathcal{D}(\mathbf{T})$ solutions of 4.2 .25 ). If several of the quantities 4.2.27) are equal for different values of $n$, then linear combinations of the corresponding $\boldsymbol{\psi}_{n}$ will also satisfy (4.2.25). Within the sense of this degeneracy, the $\boldsymbol{\psi}_{n}$ are the unique solutions of (4.2.25). Moreover, by Theorem 4.2.19, the solutions of (4.2.16) are complete, and thus it follows that they are also a complete set of solutions of 4.2.25).

In Section 4.3, it will be shown that the largest eigenvalue of 4.2.25) measures the least angle between $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$, and its associated eigenfunction $\boldsymbol{\psi}_{0}$ is the best-concentrated function in $\mathbf{T}$. This result will play a crucial role in the study of specific questions regarding the relationship between the energy concentration of a quaternionic signal in the spatial and frequency spaces.

In the integral equations (4.2.16) and (4.2.25) of the $c$-QPSWFs, it is assumed that the spatial and frequency domains of definition are symmetric about the origin. However, the question is whether these properties can be extended to domains that are not necessarily symmetric about the origin but some other points. In such a case, the following modified results, which are more general than those of Definition 4.2 .17 and Theorem 4.2.21, hold similar proof to Theorem 4.2.21.

Proposition 4.2.22. Given a real number $c>0, \mathbf{P} \in \mathbb{R}^{3}$, if the $c-Q P S W F s$
satisfy the following variant of the Fourier property

$$
\boldsymbol{\psi}_{n}(\mathbf{x}):=\left(\boldsymbol{\mu}_{n}\right)^{-1}\left[\int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) \overline{\boldsymbol{E}}(\mathbf{y}+\mathbf{P}, c \mathbf{x}) d \mathbf{y}\right]
$$

then they satisfy the corresponding integral eigenvalue equation

$$
\boldsymbol{\psi}_{n}(\mathbf{x})=\frac{1}{\lambda_{n}}\left[\int_{\mathbf{T}} D_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\psi}_{n}(\mathbf{y})\right] \boldsymbol{E}(\mathbf{P}, c \mathbf{x})
$$

where the parameters $\lambda_{n}$ are given by (4.2.27), and the kernel is

$$
D_{\mathbf{W}}(\mathbf{x}, \mathbf{y})=\prod_{j=0}^{2}\left[\left(\frac{W}{\pi}\right) K\left(\frac{W\left(x_{j}-y_{j}-P j\right)}{\pi}\right)\right] .
$$

The problem that led to Eq. 4.2.25 only requires that equation to hold for $\mathbf{x} \in \mathbf{T}$. The following proposition, which we call the all-pass filtering form of the $c$-QPSWFs, extends the spatial-domain of the $c$-QPSWFs from $\mathbf{T}$ to $\mathbb{R}^{3}$.

Proposition 4.2.23. Given a real number $c>0$, the $c-Q P S W F s \boldsymbol{\psi}_{n}(n=$ $0,1, \ldots$ ) satisfy the integral equation

$$
\begin{equation*}
\boldsymbol{\psi}_{n}(\mathbf{x})=\left(\boldsymbol{\psi}_{n} * K_{\mathbf{W}}\right)(\mathbf{x})\left(=\int_{\mathbb{R}^{3}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\psi}_{n}(\mathbf{y}) d \mathbf{y}\right) \tag{4.2.28}
\end{equation*}
$$

where the $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is defined by (4.2.6) and $*$ denotes the convolution operation.

Proof. The proof follows from the reproducing property of the kernel $K_{\mathbf{W}}$. We set $g_{\mathbf{x}}(\mathbf{y}):=K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$. Using 4.2.11) and Plancherel's Theorem (1.6.10), a direct computation shows that

$$
\begin{aligned}
\int_{\mathbb{R}^{3}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\psi}_{n}(\mathbf{y}) d \mathbf{y} & =\int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n}(\mathbf{y}) g_{\mathbf{x}}(\mathbf{y}) d \mathbf{y} \\
& =\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}} \mathcal{F}\left(\boldsymbol{\psi}_{n}\right)(\boldsymbol{\omega}) \chi_{\mathbf{W}}(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\left(B_{\mathbf{W}} \boldsymbol{\psi}_{n}\right)(\mathbf{x}) \\
& =\boldsymbol{\psi}_{n}(\mathbf{x}) .
\end{aligned}
$$

Using the fact that the eigenfunctions $\boldsymbol{\psi}_{n}(\mathbf{x})$ are now defined for all $\mathbf{x}$, we proceed to show that the $\boldsymbol{\psi}_{n}(\mathbf{x})$ are doubly orthogonal in the sense that they are orthogonal over a given $\mathbf{T}$ as well as over $\mathbb{R}^{3}$. It will follow that the
functions $\boldsymbol{\psi}_{0}, \boldsymbol{\psi}_{1}, \boldsymbol{\psi}_{2}, \ldots$ that solve Eq. (4.2.25) are normalized to unity over the whole Euclidean space $\mathbb{R}^{3}$, that is, $\left\|\boldsymbol{\psi}_{n}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}=1, n=0,1, \ldots$. This surprising observation emphasizes the underlying uniqueness of the properties associated with the $c$-QPSWFs. This double orthogonality property was first recognized for the PSWFs (of order zero) by Slepian and Pollak in [299]. A later investigation by Rhodes [281] revealed that this property is possessed by all of the PSWFs of arbitrary order.

The following general result may be now deduced.
Theorem 4.2.24. There are a countably infinite set of band-limited quaternionic functions $\boldsymbol{\psi}_{0}(\mathbf{x}), \boldsymbol{\psi}_{1}(\mathbf{x}), \boldsymbol{\psi}_{2}(\mathbf{x}), \ldots$ and a set of real positive numbers $\lambda_{0} \geq \lambda_{1} \geq \lambda_{2} \geq \cdots$, bounded by 1 and accumulating at zero, with the following properties (for all nonnegative integers $n_{1}$ and $n_{2}$ ):
(i) The functions $D_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{x})$ are orthogonal and complete in $\mathcal{D}(\mathbf{T})$ :

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} D_{\mathbf{T}} \psi_{n_{1}}(\mathbf{x}) \overline{D_{\mathbf{T}} \psi_{n_{2}}}(\mathbf{x}) d \mathbf{x}=\lambda_{n_{1}} \delta_{n_{1}, n_{2}} \tag{4.2.29}
\end{equation*}
$$

(ii) The $\boldsymbol{\psi}_{n_{1}}(\mathbf{x})$ are orthonormal in $\mathbb{R}^{3}$ and complete in $\mathcal{B}(\mathbf{W})$ :

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n_{1}}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n_{2}}}(\mathbf{x}) d \mathbf{x}=\delta_{n_{1}, n_{2}} \tag{4.2.30}
\end{equation*}
$$

Proof. Since $\boldsymbol{\psi}_{n} \in \mathcal{B}(\mathbf{W})$, from Lemma 4.2.11 and Proposition 4.2.14, it then follows that

$$
\begin{aligned}
\left\langle D_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}, D_{\mathbf{T}} \boldsymbol{\psi}_{n_{2}}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} & =\left\langle D_{\mathbf{T}} B_{\mathbf{W}} \boldsymbol{\psi}_{n_{1}}, D_{\mathbf{T}} B_{\mathbf{W}} \boldsymbol{\psi}_{n_{2}}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\left\langle B_{\mathbf{W}} D_{\mathbf{T}}^{2} B_{\mathbf{W}} \boldsymbol{\psi}_{n_{1}}, \boldsymbol{\psi}_{n_{2}}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\left\langle B_{\mathbf{W}} D_{\mathbf{T}} B_{\mathbf{W}} \boldsymbol{\psi}_{n_{1}}, \boldsymbol{\psi}_{n_{2}}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\lambda_{n_{1}} \delta_{n_{1}, n_{2}} .
\end{aligned}
$$

For the proof of Statement (ii), we make use of the low-pass filtering form 4.2 .25 of the $c$-QPSWFs and extend the domain of definition of $\boldsymbol{\psi}_{n}$ from $\mathbf{T}$ to $\mathbb{R}^{3}$ in the sense that

$$
\begin{equation*}
\boldsymbol{\psi}_{n}(\mathbf{x})=\frac{1}{\lambda_{n}} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\psi}_{n}(\mathbf{y}) d \mathbf{y} \tag{4.2.31}
\end{equation*}
$$

for all $\mathbf{x} \in \mathbb{R}^{3}$. The quaternionic conjugate of (4.2.31) is

$$
\overline{\boldsymbol{\psi}_{n}}(\mathbf{x})=\frac{1}{\lambda_{n}} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \overline{\psi_{n}}(\mathbf{y}) d \mathbf{y} .
$$

By the symmetry of $\mathbf{T}$ and Tonelli-Hobson Theorem, after interchanging the order of integration, we have

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n_{1}}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n_{2}}}(\mathbf{x}) d \mathbf{x} \\
& =\frac{1}{\lambda_{n_{1}} \lambda_{n_{2}}} \int_{\mathbf{T}} \int_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{y}) \overline{\boldsymbol{\psi}_{n_{2}}}(\mathbf{t})\left(\int_{\mathbb{R}^{3}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{t}) d \mathbf{x}\right) d \mathbf{y} d \mathbf{t}
\end{aligned}
$$

We note further that $K_{\mathbf{W}}$ is an even function and that from 4.2.6, (4.2.11), and Plancherel's Theorem (1.6.10), we obtain the reproducing property

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} K_{\mathbf{W}}(\mathbf{y}, \mathbf{x}) K_{\mathbf{W}}(\mathbf{x}, \mathbf{t}) d \mathbf{x}=K_{\mathbf{W}}(\mathbf{y}, \mathbf{t}) \tag{4.2.32}
\end{equation*}
$$

By proceeding in this manner, one finds

$$
\begin{aligned}
\int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n_{1}}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n_{2}}}(\mathbf{x}) d \mathbf{x} & =\frac{1}{\lambda_{n_{1}} \lambda_{n_{2}}} \int_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{y})\left(\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{y}, \mathbf{t}) \overline{\boldsymbol{\psi}_{n_{2}}}(\mathbf{t}) d \mathbf{t}\right) d \mathbf{y} \\
& =\frac{1}{\lambda_{n_{1}}} \int_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{y}) \overline{\boldsymbol{\psi}_{n_{2}}}(\mathbf{y}) d \mathbf{y} \\
& =\frac{1}{\lambda_{n_{1}}} \lambda_{n_{1}} \delta_{n_{1} n_{2}} \\
& =\delta_{n_{1} n_{2}}
\end{aligned}
$$

For the last equality, we have used the orthogonality 4.2 .29 of the $\boldsymbol{\psi}_{n}$ in $\mathbf{T}$. Thus, the orthogonality of the $\boldsymbol{\psi}_{n}$ over $\mathbf{T}$ implies orthogonality over the whole $\mathbb{R}^{3}$ and vice-versa.

The functions $\boldsymbol{\psi}_{n}$ constitute the three-dimensional $c$-QPSWFs associated with the sets $\mathbf{T}$ and $\mathbf{W}$.

We shall observe that (4.2.25) determines $\boldsymbol{\psi}_{n}$ and $\lambda_{n}$ up to a multiplicative constant, and normalization in 4.2.30 defines this constant. Throughout this chapter, the symbols $\lambda_{n}$ and $\psi_{n}$ will always bear the meaning of Theorem 4.2 .24 Since $\left\|\boldsymbol{\psi}_{n}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}=1$ and $\left\|D_{\mathbf{T}} \boldsymbol{\psi}_{n}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}=\lambda_{n}$, a small value of $\lambda_{n}$ implies that $\boldsymbol{\psi}_{n}$ has most of its energy outside $\mathbf{T}$, while a value of $\lambda_{n}$ near one means that $\boldsymbol{\psi}_{n}$ is mostly concentrated in $\mathbf{T}$.

Under the normalization referred to Theorem 4.2.24, the kernel $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ defined by 4.2.6 has an expansion in terms of the eigenfunctions $\boldsymbol{\psi}_{n}$, given by

$$
\begin{equation*}
K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})=\sum_{n=0}^{\infty} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{y}) \tag{4.2.33}
\end{equation*}
$$

This relation, a form of Mercer's Theorem [229, 282] may be easily verified by substituting the right-hand side of (4.2.33) into 4.2.25 and then using the orthogonality (4.2.29).

Another essential feature of the c-QPSWFs is the generalized Shannon number, which we define in the same way as in the original Slepian design, i.e., as the sum over all eigenvalues $\lambda_{n}$. Using properties (4.2.29) and (4.2.30) of the eigenfunctions $\boldsymbol{\psi}_{n}$, it may be shown that this number, $N_{\text {Shannon }}$, only depends on the space-bandwidth product, namely

$$
\begin{equation*}
N_{\text {Shannon }}=\sum_{n=0}^{\infty} \lambda_{n}=\int_{\mathbf{T}} \lim _{\mathbf{y} \rightarrow \mathbf{x}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) d \mathbf{x}=\frac{1}{(2 \pi)^{3}}|\mathbf{T}||\mathbf{W}| . \tag{4.2.34}
\end{equation*}
$$

The quantity (4.2.34) approximately represents the number of eigenfunctions that are well-concentrated within the selected region of interest. More precisely, it measures the dimension of the subspace spanned by the band-limited signals that are well-localized [199].

### 4.2.3 Extrapolation of a Band-limited Quaternionic Function by the $c$-QPSWFs

Suppose one seeks to extrapolate a band-limited function known only on the spatial-domain $\mathbf{T}$ to values outside this domain using the $c$-QPSWFs. In principle, this extrapolation can be done. For example, one could calculate successive derivatives of $\boldsymbol{f}$ at some point in $\mathbf{T}$ and form a Taylor series representation that would converge anywhere. However, such a Taylor series would necessarily be truncated. The resultant approximation to $\boldsymbol{f}(\mathbf{x})$ would be a polynomial, which for sufficiently large values of $|\mathbf{x}|$, would result in a poor approximation of $\boldsymbol{f}$. This approximation is not band-limited. The $c$-QPSWFs provide an alternative approach in specific least-squares approximation problems. We propose an extrapolation routine that computes the values of $\boldsymbol{f}(\mathbf{x})$, for any $\mathrm{x} \in \mathbb{R}^{3}$, relying only on the knowledge of this function restricted to the spatial-domain $\mathbf{T}$. In the following, we do not specify the dependence of the notation on the parameter $c>0$ whenever it is clear from the context. Property (ii) of Theorem 4.2 .24 leads us to the following definition.

Definition 4.2.25. Suppose $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$. For any $\mathrm{x} \in \mathbb{R}^{3}$, the left-sided quaternionic Slepian series of $\boldsymbol{f}$ is

$$
\begin{equation*}
\boldsymbol{f}(\mathbf{x})=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathbf{x}) \tag{4.2.35}
\end{equation*}
$$

where $\boldsymbol{a}_{n}=\int_{\mathbb{R}^{3}} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x}$ are called the Slepian coefficients.
The following result studies the mean-square convergence of (4.2.35), which will be useful in Section 4.3 .

Lemma 4.2.26. Suppose $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$. Then

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \int_{\mathbb{R}^{3}}\left|\boldsymbol{f}(\mathbf{x})-\sum_{n=0}^{N} \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}=0 \tag{4.2.36}
\end{equation*}
$$

where the coefficients $\boldsymbol{a}_{n}$ are quaternionic constants, which can be determined from values of $\boldsymbol{f}$ in $\mathbf{T}$ :

$$
\begin{equation*}
\boldsymbol{a}_{n}=\frac{1}{\lambda_{n}} \int_{\mathbf{T}} \boldsymbol{f}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x} \tag{4.2.37}
\end{equation*}
$$

Proof. According to Definition 4.2.25 and Parseval's identity,

$$
\begin{equation*}
\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2}=\int_{\mathbb{R}^{3}}|\boldsymbol{f}(\mathbf{x})|^{2} d \mathbf{x} \tag{4.2.38}
\end{equation*}
$$

$\boldsymbol{f}$ may be characterized by its coefficients, and the convergence in 4.2.35) is in the mean-square sense:

$$
\lim _{N \rightarrow \infty} \int_{\mathbb{R}^{3}}\left|\boldsymbol{f}(\mathrm{x})-\sum_{n=0}^{N} \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathrm{x})\right|^{2} d \mathbf{x}=0
$$

Multiplying (4.2.35) by $\overline{\boldsymbol{\psi}_{n}}(\mathbf{x})$ from the right-hand side, integrating over $\mathbf{T}$, and using 4.2.29), we obtain (4.2.37).

The above result suggests approximating $\boldsymbol{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{3}$ by the $N$-th partial sum, namely

$$
\begin{equation*}
\boldsymbol{f}_{N}(\mathrm{x})=\sum_{n=0}^{N} \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathrm{x}) \tag{4.2.39}
\end{equation*}
$$

where the coefficients $\boldsymbol{a}_{n}$ are given by (4.2.37). The approximation 4.2.39) is itself band-limited, and the mean-square error is

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}\left|\boldsymbol{f}(\mathbf{x})-\boldsymbol{f}_{N}(\mathbf{x})\right|^{2} d \mathbf{x}=\sum_{n=N+1}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \tag{4.2.40}
\end{equation*}
$$

According to (4.2.38), this error can be made as small as desired by making $N$ sufficiently large. In the sense of (4.2.40), the extrapolation remains good for all $\mathrm{x} \in \mathbb{R}^{3}$. The error in fitting $\boldsymbol{f}_{N}$ to $\boldsymbol{f}$ in $\mathbf{T}$ is given by

$$
\begin{equation*}
\int_{\mathbf{T}}\left|\boldsymbol{f}(\mathbf{x})-\boldsymbol{f}_{N}(\mathbf{x})\right|^{2} d \mathbf{x}=\sum_{n=N+1}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \lambda_{n} \tag{4.2.41}
\end{equation*}
$$

As the $\lambda_{n}$ approach zero rapidly for sufficiently large $n$, it may happen that (4.2.41) is small for values of $N$ for which (4.2.40) is still large.

Suppose now $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ is known in the spatial-domain $\mathbf{T}$, but $\boldsymbol{f}$ is not necessarily in $\mathcal{B}(\mathbf{W})$. Applying the proof of Theorem 4.2.24, it follows that $\boldsymbol{f}$ may still be represented by (4.2.35) with the coefficients given by 4.2.37). Nevertheless, this representation is valid now only for $\mathbf{x} \in \mathbf{T}$. If $\boldsymbol{f} \notin \mathcal{B}(\mathbf{W})$, then the series 4.2.35 does not converge in the mean-square sense over the whole three-dimensional Euclidean space $\mathbb{R}^{3}$.

The previous investigations provide, by Definition 4.2.1 and Lemma 4.2.26, an answer to the question which arises as regards the quality of approximating space-limited and band-limited functions at a level $\varepsilon$ for some $\mathbf{T}$ and $\mathbf{W}$ by the $c$-QPSWFs.

Proposition 4.2.27. Suppose $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, not identically zero, is $\varepsilon_{\mathbf{T}^{-}}$ concentrated on $\mathbf{T}$ and $\mathcal{F}(\boldsymbol{f})$ is $\varepsilon_{\mathbf{W}}$-concentrated on $\mathbf{W}$. Then, for any positive integer $N$, we have

$$
\left\|\boldsymbol{f}-D_{\mathbf{T}} \boldsymbol{f}_{N}\right\|_{L_{2}(\mathbf{T})} \leq\left(\varepsilon_{\mathbf{T}}+\frac{1}{(2 \pi)^{3 / 2}} \varepsilon_{\mathbf{W}}+\sqrt{\lambda_{N+1}}\right)\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)} .
$$

Proof. It is similar to that of Theorem 4.2.16, but it is necessary to employ Lemma 4.2.26. The rest of the proof follows from (4.2.41) and the fact that the $\lambda_{n}$ are monotonically decreasing in the interval $(0,1)$.

To sum up the above observations, and bearing in mind the efficiency and applicability of the present approach, we proceed to present a practical routine for calculating a quaternionic signal $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$ known only in the spatial domain. The unknown information for $\boldsymbol{f}$ outside $\mathbf{T}$ will be filled in step-by-step. When viewed in this manner, the technique is spoken of as the band-limited quaternionic signal extrapolation. It is based on the well-known Papoulis-Gerchberg algorithm [265], as discussed in [350].

According to Proposition 4.2.23, we define the $i$ th step of the algorithm in the following manner:

$$
\mathcal{F}\left(\boldsymbol{f}_{i}\right)(\boldsymbol{\omega})=\mathcal{F}\left(\boldsymbol{g}_{i-1}\right)(\boldsymbol{\omega}) \chi_{\mathbf{W}}(\boldsymbol{\omega}), \quad i=1,2, \ldots
$$

or, equivalently,

$$
\begin{equation*}
\boldsymbol{f}_{i}(\mathbf{x})=\boldsymbol{g}_{i-1}(\mathbf{x}) * K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \tag{4.2.42}
\end{equation*}
$$

where $K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})$ is defined by 4.2.6), and

$$
\begin{aligned}
\boldsymbol{g}_{i-1}(\mathrm{x}) & =D_{\mathbf{T}} \boldsymbol{f}(\mathrm{x})+\left[1-\chi_{\mathbf{T}}(\mathrm{x})\right] \boldsymbol{f}_{i-1}(\mathrm{x}) \\
& = \begin{cases}D_{\mathbf{T}} \boldsymbol{f}(\mathrm{x}) & \text { if } \mathrm{x} \in \mathbf{T}, \\
\boldsymbol{f}_{i-1}(\mathrm{x}) & \text { if } \mathrm{x} \notin \mathbf{T},\end{cases}
\end{aligned}
$$

with $\boldsymbol{f}_{0}(\mathrm{x})=\mathbf{0}$.
By the assumption, it is quite natural to ask how to construct the $i$ th iteration $\boldsymbol{f}_{i}(\mathbf{x})$ using the $c$-QPSWFs and their corresponding eigenvalues and, consequently, to check if the proposed method is effective. This discussion introduces the following lemma.
Lemma 4.2.28. Suppose $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$. For any $\mathbf{x} \in \mathbb{R}^{3}$, the function of the ith iteration 4.2.42 is given by

$$
\boldsymbol{f}_{i}(\mathrm{x})=\boldsymbol{f}(\mathrm{x})-\sum_{n=0}^{\infty} \boldsymbol{a}_{n}\left(1-\lambda_{n}\right)^{i} \boldsymbol{\psi}_{n}(\mathrm{x}),
$$

where $\lambda_{n}$ are the eigenvalues of (4.2.25), and $\boldsymbol{\psi}_{n}$ the corresponding eigenfunctions. Further, $\boldsymbol{f}_{i}$ converges (in the mean-square sense) to $\boldsymbol{f}$ when $i$ approaches infinity.
Proof. This lemma is proved by induction. Let $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$. By Definition 4.2.25. for $\mathbf{x} \in \mathbb{R}^{3}, \boldsymbol{f}(\mathbf{x})=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathbf{x})$. For simplicity of description, we assume that $\boldsymbol{f}(\mathbf{x})=\boldsymbol{\psi}_{m}(\mathbf{x})$ for a fixed $m$, and that there exists a constant $C_{i}$ such that

$$
\begin{equation*}
\boldsymbol{f}_{i}(\mathbf{x})=\left[1-\left(1-\lambda_{m}\right)^{i}\right] \boldsymbol{\psi}_{m}(\mathbf{x})=: C_{i} \boldsymbol{\psi}_{m}(\mathbf{x}) \tag{4.2.43}
\end{equation*}
$$

for all $i \in \mathbb{N}$. If $i=1$, then $\boldsymbol{f}_{1}(\mathbf{x})=\left[1-\left(1-\lambda_{m}\right)^{1}\right] \boldsymbol{\psi}_{m}(\mathbf{x})$ is true. Now, suppose that 4.2.43 holds for $i=k$. We need to show that 4.2.43) also holds for $i=k+1$. By 4.2.42), it follows that $\boldsymbol{g}_{k}(\mathbf{x})=\bar{D}_{\mathbf{T}} \boldsymbol{\psi}_{m}(\mathbf{x})+$ $C_{k}\left[1-\chi_{\mathbf{T}}(\mathbf{x})\right] \boldsymbol{\psi}_{m}(\mathbf{x})$. Whence

$$
\boldsymbol{f}_{k+1}(\mathbf{x})=\left[C_{k} \boldsymbol{\psi}_{m}(\mathbf{x})+\left(1-C_{k}\right) D_{\mathbf{T}} \boldsymbol{\psi}_{m}(\mathbf{x})\right] * K_{\mathbf{W}}(\mathbf{x}, \mathbf{y})
$$

The low-pass filtering form (4.2.25) and the all-pass filtering form (4.2.28) of the $c$-QPSWFs, together with Proposition 4.2.15, lead to

$$
\boldsymbol{f}_{k+1}(\mathbf{x})=C_{k} \boldsymbol{\psi}_{m}(\mathbf{x})+\left(1-C_{k}\right) \lambda_{m} \boldsymbol{\psi}_{m}(\mathbf{x})=C_{k+1} \boldsymbol{\psi}_{m}(\mathbf{x})
$$

Hence, we obtain an iteration equation involving constants $C_{k}$ and $C_{k+1}$, namely $C_{k+1}=C_{k}+\left(1-C_{k}\right) \lambda_{m}$. It was proved that $C_{1}=\lambda_{m}$. A direct computation shows that $C_{k+1}=1-\left(1-\lambda_{m}\right)^{k+1}$. It follows then that

$$
\boldsymbol{f}_{i}(\mathbf{x})=C_{i} \boldsymbol{\psi}_{m}(\mathbf{x})=\left[1-\left(1-\lambda_{m}\right)^{i}\right] \boldsymbol{\psi}_{m}(\mathbf{x})
$$

for $\boldsymbol{f}(\mathbf{x})=\boldsymbol{\psi}_{m}(\mathbf{x})$. Applying these results to $\boldsymbol{f}(\mathbf{x})=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathbf{x})$, we conclude that

$$
\begin{aligned}
\boldsymbol{f}_{i}(\mathbf{x}) & =\sum_{n=0}^{\infty}\left[1-\left(1-\lambda_{n}\right)^{i}\right] \boldsymbol{a}_{n} \boldsymbol{\psi}_{n}(\mathbf{x}) \\
& =\boldsymbol{f}(\mathbf{x})-\sum_{n=0}^{\infty} \boldsymbol{a}_{n}\left(1-\lambda_{n}\right)^{i} \boldsymbol{\psi}_{n}(\mathbf{x})
\end{aligned}
$$

which proves the assertion.
We proceed to show that the error in the fit of $\boldsymbol{f}_{i}$ to $\boldsymbol{f}$, say $\boldsymbol{e}_{i}(\mathbf{x})$, tends to zero as $i$ approaches infinity. By (1.6.9) and (1.2.4), we find

$$
\begin{aligned}
\left|\boldsymbol{e}_{i}(\mathbf{x})\right|^{2} & =\left|\boldsymbol{f}(\mathbf{x})-\boldsymbol{f}_{i}(\mathbf{x})\right|^{2} \\
& \leq \frac{1}{(2 \pi)^{6}} \int_{\mathbf{W}}\left|\mathcal{F}(\boldsymbol{f})(\boldsymbol{\omega})-\mathcal{F}\left(\boldsymbol{f}_{i}\right)(\boldsymbol{\omega})\right|^{2} d \boldsymbol{\omega} \int_{\mathbf{W}}|\overline{\boldsymbol{E}}(\boldsymbol{\omega}, \mathbf{x})|^{2} d \boldsymbol{\omega} \\
& =\frac{1}{(2 \pi)^{3}}|\mathbf{W}| E_{i},
\end{aligned}
$$

where the mean-square error $E_{i}:=\left\|\boldsymbol{e}_{i}(\mathbf{x})\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}$ denotes the energy of error $\boldsymbol{e}_{i}$. Now, using 4.2.40 and 4.2.30), we find

$$
E_{i}=\int_{\mathbb{R}^{3}}\left|\sum_{n=0}^{\infty} \boldsymbol{a}_{n}\left(1-\lambda_{n}\right)^{i} \boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}=\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2}\left(1-\lambda_{n}\right)^{2 i}
$$

By 4.2.38), $E=\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2}<\infty$. Hence, for any $\varepsilon>0$, there exists an $N$ such that $\sum_{n>N}\left|\boldsymbol{a}_{n}\right|^{2}<\varepsilon$. Moreover, since the $\lambda_{n}$ are monotonically decreasing in the interval $(0,1)$, it follows at once that $1-\lambda_{n} \leq 1-\lambda_{N}$, for $n \leq N$. We then have

$$
\begin{aligned}
E_{i} & \leq\left(1-\lambda_{N}\right)^{2 i} \sum_{n=0}^{N}\left|\boldsymbol{a}_{n}\right|^{2}+\sum_{n=N+1}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \\
& \leq\left(1-\lambda_{N}\right)^{2 i} E+\varepsilon .
\end{aligned}
$$

Now, since $0<1-\lambda_{N}<1$, it follows that $E_{i} \rightarrow 0$ as $i \rightarrow \infty$. Thus $\left|\boldsymbol{e}_{i}(\mathbf{x})\right| \leq \sqrt{\left(|\mathbf{W}| E_{i}\right) /(2 \pi)^{3}} \rightarrow 0$ as $i \rightarrow \infty$. This completes the proof.

### 4.3 The $c$-QPSWFs vs. the Energy Extremal Problem

In the present section, we bring back the question posed in Subsection 4.2.1 about a quaternionic counterpart of Slepian's spatial-frequency concentration problem: Under what conditions the energy conservation ratio

$$
\begin{equation*}
\frac{\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}} \tag{4.3.1}
\end{equation*}
$$

for $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right) \backslash\{\mathbf{0}\}$ is a maximum? As we already pointed out, to answer this question, we proceed on and apply the space-limiting operator
(4.2.1) to a given nonzero quaternionic signal in the spatial domain and employ the band-limiting operator (4.2.3) in the frequency domain. From the definition in (4.2.10), it is clear that the underlying signal is different from the original one, as its energy is reduced by at least one of the operators. In the latter part of this section, it will be found the possible proportions of the energy of a quaternionic signal generated by this double truncation in a given spatial-domain $\mathbf{T}$ and a given frequency-domain $\mathbf{W}$, as well as the signals which best simultaneously maximize the spatial-frequency concentration, by (4.3.1). The techniques used in the proofs of the following results are mainly due to Slepian and Pollak [299] and Landau and Pollak [198]. Even though the essential nature of the quaternionic counterparts of the corresponding results in the one-dimensional case is nearly unchanged, we should note that we consider spaces of $\mathbb{H}$-valued functions, thus yielding a much richer theory. In this sense, these results extend the classical results of [198, 299] to a broader space.

The following generalization of the result of [299] answers the above question by finding the quaternionic band-limited signals that are maximally concentrated in a given spatial domain.

Theorem 4.3.1. Let $\boldsymbol{f}$ be a nonzero signal in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$. The maximum value of (4.3.1) can be obtained if $\boldsymbol{f}$ is a multiple of the eigenfunction of (4.2.25) belonging to the largest eigenvalue $\lambda_{0}$.

Proof. By definition (4.2.10) of the operator $B_{\mathbf{W}} D_{\mathbf{T}}$ and 4.2.32), it follows from the Tonelli-Hobson Theorem that

$$
\begin{aligned}
& \left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}(\mathbf{t})\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} \\
& =\int_{\mathbb{R}^{3}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}(\mathbf{t}) \overline{B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}}(\mathbf{t}) d \mathbf{t} \\
& =\int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} D_{\mathbf{T}} \boldsymbol{f}(\mathbf{y}) \overline{D_{\mathbf{T}} \boldsymbol{f}}(\mathbf{x})\left(\int_{\mathbb{R}^{3}} K_{\mathbf{W}}(\mathbf{y}, \mathbf{t}) K_{\mathbf{W}}(\mathbf{t}, \mathbf{x}) d \mathbf{t}\right) d \mathbf{y} d \mathbf{x} . \\
& =\int_{\mathbf{T}} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{y}) \overline{\boldsymbol{f}}(\mathbf{x}) d \mathbf{y} d \mathbf{x} .
\end{aligned}
$$

The above reasoning shows that the quantity $\left\|B_{\mathrm{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}$ depends only on values of $\boldsymbol{f}$ in $\mathbf{T}$. Consequently, the ratio (4.3.1) is equal to the maximum of

$$
\frac{\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}}{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}}=\frac{\int_{\mathbf{T}} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{y}, \mathbf{x}) \boldsymbol{f}(\mathbf{y}) \overline{\boldsymbol{f}}(\mathbf{x}) d \mathbf{y} d \mathbf{x}}{\int_{\mathbf{T}}|\boldsymbol{f}(\mathbf{x})|^{2} d \mathbf{x}}
$$

over all $f \in \mathcal{D}(\mathbf{T})$.
Now, by Definition 4.2.25 and Lemma 4.2.26, for $\mathbf{x} \in \mathbf{T}$, it follows that $D_{\mathbf{T}} \boldsymbol{f}(\mathrm{x})=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})$. Moreover, since $K_{\mathbf{W}}$ is real and using (4.2.25)
in Theorem 4.2.21, we find

$$
\begin{aligned}
& \int_{\mathbf{T}} \int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{f}(\mathbf{y}) \overline{\boldsymbol{f}}(\mathbf{x}) d \mathbf{y} d \mathbf{x} \\
& =\sum_{n_{1}=0}^{\infty} \sum_{n_{2}=0}^{\infty} \boldsymbol{a}_{n_{1}} \int_{\mathbf{T}}\left(\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) D_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{y}) d \mathbf{y}\right) \overline{D_{\mathbf{T}} \boldsymbol{\psi}_{n_{2}}}(\mathbf{x}) \overline{\boldsymbol{a}_{n_{2}}} d \mathbf{x} \\
& =\sum_{n_{1}=0}^{\infty} \sum_{n_{2}=0}^{\infty} \boldsymbol{a}_{n_{1}} \lambda_{n_{1}}\left(\int_{\mathbf{T}} D_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{x}) \overline{D_{\mathbf{T}} \boldsymbol{\psi}_{n_{2}}}(\mathbf{x}) d \mathbf{x}\right) \overline{\boldsymbol{a}_{n_{2}}} .
\end{aligned}
$$

Now, using 4.2.29) of Theorem 4.2.24 it follows that

$$
\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}=\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \lambda_{n}^{2}
$$

Since the $\lambda_{n}$ are real, we apply (4.2.29) again and find

$$
\begin{align*}
\int_{\mathbf{T}}|\boldsymbol{f}(\mathbf{x})|^{2} d \mathbf{x} & =\sum_{n_{1}=0}^{\infty} \sum_{n_{2}=0}^{\infty} \boldsymbol{a}_{n_{1}}\left(\int_{\mathbf{T}} D_{\mathbf{T}} \boldsymbol{\psi}_{n_{1}}(\mathbf{x}) \overline{D_{\mathbf{T}} \boldsymbol{\psi}_{n_{2}}}(\mathbf{x}) d \mathbf{x}\right) \overline{\boldsymbol{a}_{n_{2}}} \\
& =\sum_{n_{1}=0}^{\infty}\left|\boldsymbol{a}_{n_{1}}\right|^{2} \lambda_{n_{1}} \tag{4.3.2}
\end{align*}
$$

Thus the energy preservation ratio (4.3.1) is given by

$$
\frac{\left\|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}}{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}}=\frac{\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \lambda_{n}^{2}}{\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \lambda_{n}}
$$

Since $\lambda_{0} \geq \lambda_{n}$, if $n \geq 1$, to make the energy ratio maximal, $\boldsymbol{f}(\mathbf{x})$ should be a multiple of $D_{\mathbf{T}} \boldsymbol{\psi}_{0}(\mathbf{x})$, where $D_{\mathbf{T}} \boldsymbol{\psi}_{0}(\mathbf{x})$ denotes the eigenfunction of (4.2.25) belonging to the largest eigenvalue $\lambda_{0}$. Thus $\boldsymbol{f}(\mathbf{x})=\boldsymbol{a}_{0} D_{\mathbf{T}} \boldsymbol{\psi}_{0}(\mathbf{x})$, where $a_{0} \in \mathbb{H}$.

Before proceeding with the discussion of the energy extremal properties between the spatial and frequency domains involving the $c$-QPSWFs, we first investigate the existence of a nonzero least angle between the two subspaces $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$ of $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ and discuss its properties. The following reasoning extends the one presented in [198]. This result will allow us to infer the possible proportions of a signal's energy in the spatial and frequency domains.

According to Definition 1.2.12, we define the canonical angle between $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$ as follows:

Definition 4.3.2. Let $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$ and $\boldsymbol{g} \in \mathcal{D}(\mathbf{T})$. The number $\arg (\boldsymbol{f}, \boldsymbol{g})$ is called the canonical angle between $\mathcal{B}(\mathbf{W})$ and $\mathcal{D}(\mathbf{T})$ if it satisfies

$$
\begin{equation*}
\cos \arg (\boldsymbol{f}, \boldsymbol{g})=\frac{\operatorname{Sc}\left(\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right)}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} \tag{4.3.3}
\end{equation*}
$$

The discussion below concerns the extremal values of the angle between space-limited and band-limited quaternionic signals. The question naturally arises: What are the extremal values of $\arg (\boldsymbol{f}, \boldsymbol{g})$ between $\boldsymbol{g} \in \mathcal{D}(\mathbf{T})$ and $f \in \mathcal{B}(\mathbf{W})$ under the QFT defined by (1.6.4)? The following two results give this extremal value. Despite the difficulties arising from the fact that the underlying multiplication is not commutative, it will be shown that the essential nature of the arguments for quaternionic signals remains unchanged.

Lemma 4.3.3. If $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$ is fixed, then the $\arg (\boldsymbol{f}, \boldsymbol{g})$ between $\boldsymbol{f}$ and any $\boldsymbol{g} \in \mathcal{D}(\mathbf{T})$ satisfies $\inf _{\boldsymbol{g} \in \mathcal{D}(\mathbf{T})} \arg (\boldsymbol{f}, \boldsymbol{g})>0$. This infimum equals

$$
\arccos \frac{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}}
$$

and is assumed by $\boldsymbol{g}=k D_{\mathbf{T}} \boldsymbol{f}$ for any constant $k>0$.
Proof. By the same reasoning as in [198], if $\boldsymbol{g}$ is any function in $\mathcal{D}(\mathbf{T})$, it is clear that

$$
\operatorname{Sc}\left(\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right) \leq\left|\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right|=\left|\left\langle D_{\mathbf{T}} \boldsymbol{f}, \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right|
$$

since $\boldsymbol{f}=\boldsymbol{f}-D_{\mathrm{T}} \boldsymbol{f}+D_{\mathrm{T}} \boldsymbol{f}$ and by 4.2.8), $\left\langle\boldsymbol{f}-D_{\mathrm{T}} \boldsymbol{f}, \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}=\mathbf{0}$.
Furthermore, we rely on

$$
\left|\left\langle D_{\mathbf{T}} \boldsymbol{f}, \boldsymbol{g}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right| \leq\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}
$$

to show that

$$
\begin{aligned}
\frac{\operatorname{Sc}\left(\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}\left(\mathbb{R}^{3}, H\right)}\right)}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} & \leq \frac{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} \\
& =\frac{\operatorname{Sc}\left(\left\langle\boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right)}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}} .
\end{aligned}
$$

Since $\cos (\arg )$ is monotone decreasing in the interval $(0, \pi)$, it follows that $\arg (\boldsymbol{f}, \boldsymbol{g}) \geq \arg \left(\boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right)$ for any $\boldsymbol{g} \in \mathcal{D}(\mathbf{T})$. This inequality becomes an equality when $\boldsymbol{g}$ and $D_{\mathbf{T}} \boldsymbol{f}$ are proportional, which proves the lemma.

Building on the previous lemma, we proceed to find the least $\arg (\boldsymbol{f}, \boldsymbol{g})$ of arbitrary $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$ and $\boldsymbol{g} \in \mathcal{D}(\mathbf{T})$. We show that

$$
\begin{equation*}
\inf _{\boldsymbol{f} \in \mathcal{B}(\mathbf{W}), \boldsymbol{g} \in \mathcal{D}(\mathbf{T})} \arg (\boldsymbol{f}, \boldsymbol{g}) \tag{4.3.4}
\end{equation*}
$$

is assumed by specific functions so that the spaces $\mathcal{B}(\mathbf{W})$ and $\mathcal{D}(\mathbf{T})$ form indeed the least angle. The result may be stated as follows:

Theorem 4.3.4. Given $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$ and $\boldsymbol{g} \in \mathcal{D}(\mathbf{T})$, there exists a least angle between $\mathcal{B}(\mathbf{W})$ and $\mathcal{D}(\mathbf{T})$ that satisfies

$$
\inf _{\boldsymbol{f} \in \mathcal{B}(\mathbf{W}), \boldsymbol{g} \in \mathcal{D}(\mathbf{T})} \arg (\boldsymbol{f}, \boldsymbol{g})=\arccos \sqrt{\lambda_{0}}
$$

if and only if $\boldsymbol{f}=\boldsymbol{\psi}_{0}$ and $\boldsymbol{g}=D_{\mathbf{T}} \boldsymbol{\psi}_{0}$, where $\lambda_{0}$ is the largest eigenvalue of (4.2.25), and $\boldsymbol{\psi}_{0}$ the corresponding eigenfunction.

Proof. By the preceding lemma,

$$
\min _{\boldsymbol{g} \in \mathcal{D}(\mathbf{T})} \arg (\boldsymbol{f}, \boldsymbol{g})=\arccos \frac{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}}
$$

and therefore, (4.3.4) yields

$$
\begin{equation*}
\inf _{f \in \mathcal{B}(\mathbf{W}), \boldsymbol{g} \in \mathcal{D}(\mathbf{T})} \arg (\boldsymbol{f}, \boldsymbol{g})=\inf _{\boldsymbol{f} \in \mathcal{B}(\mathbf{W})} \arccos \frac{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} \tag{4.3.5}
\end{equation*}
$$

and the infimum on the left of (4.3.5) will be assumed if the infimum on the right is attained. According to Definition 4.2.25 and Lemma 4.2.26, for $\mathrm{x} \in \mathbf{T}, D_{\mathbf{T}} \boldsymbol{f}(\mathrm{x})=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})$. Moreover, by (4.2.38) and 4.3.2), we find

$$
\begin{equation*}
\arccos \frac{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}}=\arccos \left(\frac{\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2} \lambda_{n}}{\sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2}}\right)^{1 / 2} \tag{4.3.6}
\end{equation*}
$$

Since $\lambda_{0} \geq \lambda_{n}$, if $n \geq 1$, the minimum possible value of (4.3.6), namely $\arccos \sqrt{\lambda_{0}}$, is assumed if $\boldsymbol{f}=\boldsymbol{\psi}_{0}$ and $\boldsymbol{g}=D_{\mathbf{T}} \boldsymbol{\psi}_{0}$.

We have thus found that $\mathcal{B}(\mathbf{W})$ and $\mathcal{D}(\mathbf{T})$ have a least angle between them. Consequently, a space-limited quaternionic function and a bandlimited quaternionic function cannot be very close together unless they are very small "of unit norm."

Following the notation already employed, we now introduce the fraction of energy of a nonzero quaternionic signal $\boldsymbol{f}$ in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ that lies in a given spatial domain as

$$
\begin{equation*}
\alpha(\mathbf{T})[\boldsymbol{f}]=\alpha(\mathbf{T}):=\frac{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} . \tag{4.3.7}
\end{equation*}
$$

Similarly, we define the fraction of the signal energy that lies in a given frequency domain as

$$
\begin{equation*}
\beta(\mathbf{W})[\boldsymbol{f}]=\beta(\mathbf{W}):=\frac{\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}{\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} . \tag{4.3.8}
\end{equation*}
$$

We can think of $\alpha(\mathbf{T})$ as the fraction of the energy of $\boldsymbol{f}$ in the space of receiver functions, and $\beta(\mathbf{W})$ is the fraction of the energy of $\boldsymbol{f}$ in the space of transmitter functions. A first observation shows that if the signal is bandlimited to $\mathbf{W}$, then $\beta(\mathbf{W})=1$. Analogously, if the signal is space-limited to $\mathbf{T}$, then $\alpha(\mathbf{T})=1$. As $\boldsymbol{f}(\mathbf{x})$ ranges overall functions in $L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$, the quantities $\alpha(\mathbf{T})$ in (4.3.7) and $\beta(\mathbf{T})$ in (4.3.8) will take values over the interval $[0,1]$.

We proceed to prove an uncertainty principle that constrains the possible range of values that $\alpha(\mathbf{T})$ and $\beta(\mathbf{W})$ can take, provided the subspaces $\mathcal{B}(\mathbf{W})$ and $\mathcal{D}(\mathbf{T})$ form a nonzero least angle.

Proposition 4.3.5. Suppose the subspaces $\mathcal{B}(\mathbf{W})$ and $\mathcal{D}(\mathbf{T})$ form a nonzero least angle $\vartheta$. Then $\arccos \alpha(\mathbf{T})+\arccos \beta(\mathbf{W}) \geq \vartheta$.

Proof. By the same reasoning as used in Lemma 4.3.3, it can be easily seen that

$$
\begin{aligned}
\cos \arg \left(\boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right) & =\frac{\operatorname{Sc}\left(\left\langle D_{\mathbf{T}} \boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right)}{\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}} \\
& =\alpha(\mathbf{T})
\end{aligned}
$$

Similarly, $\beta(\mathbf{W})=\cos \arg \left(\boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right)$. Lemma 1.2 .13 shows further that

$$
\begin{aligned}
\arccos \alpha(\mathbf{T})+\arccos \beta(\mathbf{W}) & =\arg \left(\boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right)+\arg \left(\boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right) \\
& \geq \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right) \\
& \geq \vartheta
\end{aligned}
$$

where the last step follows since $D_{\mathbf{T}} \boldsymbol{f} \in \mathcal{D}(\mathbf{T})$ and $B_{\mathbf{W}} \boldsymbol{f} \in \mathcal{B}(\mathbf{W})$, and these two subspaces form the least angle $\vartheta$.

The above proposition has an unusual physical interpretation. Suppose the space of all the functions that a transmitter can generate and the space of all the functions a receiver can receive form a nonzero least angle. Then there exist no functions that can have arbitrarily large fractions of energy in those two spaces of functions.

The main results of this section, which consist of finding the signals that reach the extremal values of $(\alpha(\mathbf{T}), \beta(\mathbf{W}))$, are summarized in the following
four theorems. We show that the quaternionic counterparts of the corresponding results in the one-dimensional case are nearly unchanged. It will be convenient first to discuss the possible values that the pair $(\alpha(\mathbf{T}), \beta(\mathbf{W}))$ can take if the signal is band-limited.

Theorem 4.3.6. For any nonzero function $\boldsymbol{f} \in \mathcal{B}(\mathbf{W})$ such that $\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}=$ $1,\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\alpha(\mathbf{T})$ and $\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\beta(\mathbf{W})$, the quantity $\alpha(\mathbf{T})$ is less or equal than $\sqrt{\lambda_{0}}$, where $\lambda_{0}$ is the largest eigenvalue of 4.2.25).

Proof. According to Definition 4.2.25, $D_{\mathbf{T}} \boldsymbol{f}(\mathbf{x})=\sum_{n=0}^{\infty} \boldsymbol{a}_{n} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})$ with $a_{n} \in \mathbb{H}$. Clearly, by 4.3.2), it follows that

$$
\alpha^{2}(\mathbf{T})=\int_{\mathbb{R}^{3}}\left|D_{\mathbf{T}} \boldsymbol{f}(\mathbf{x})\right|^{2} d \mathbf{x}=\sum_{n=0}^{\infty} \lambda_{n}\left|\boldsymbol{a}_{n}\right|^{2} \leq \lambda_{0} \sum_{n=0}^{\infty}\left|\boldsymbol{a}_{n}\right|^{2},
$$

since $\lambda_{0} \geq \lambda_{n}$, if $n \geq 1$. Hence, the extremal condition is $\boldsymbol{g}^{*}(\mathbf{x})=D_{\mathbf{T}} \boldsymbol{\psi}_{0}(\mathbf{x})$. It then follows that $\alpha^{2}(\mathbf{T})=\left\|\boldsymbol{g}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}=\lambda_{0}$. For any linear combination of $\boldsymbol{\psi}_{n}$, the quantity $\alpha(\mathbf{T})$ is less than $\sqrt{\lambda_{0}}$.

We proceed to discuss the range of values of $\beta(\mathbf{W})$ when $\alpha(\mathbf{T})$ is fixed, starting by considering the case when $\alpha(\mathbf{T})=0$.

Theorem 4.3.7. There is a nonzero function $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ such that $\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}=1,\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\alpha(\mathbf{T})$ and $\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\beta(\mathbf{W})$, under the following condition: if $\alpha(\mathbf{T})$ equals 0 on $\mathbf{T}$, then $\beta(\mathbf{W})$ is greater than or equal to 0 and less than 1 on $\mathbf{W}$.

Proof. This is proved similarly to [198]. We first observe that if the quantity $\alpha(\mathbf{T})$ equals 0 , then $\beta(\mathbf{W}) \neq 1$. Although $\beta(\mathbf{W})$ cannot attain the value 1 , we can still find an underlying quaternionic signal for which $\beta(\mathbf{W})$ is arbitrarily close to 1 .

Let

$$
\begin{equation*}
\mathcal{G}:=\left\{\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right):\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}=1,\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\alpha(\mathbf{T})\right\} \tag{4.3.9}
\end{equation*}
$$

be a given function class. For $\alpha(\mathbf{T})=0$, we construct a signal $\boldsymbol{f}^{*}(\mathbf{x})$ in $\mathcal{G}$ as follows:

$$
\begin{equation*}
\boldsymbol{f}^{*}(\mathrm{x}):=\frac{\boldsymbol{\psi}_{n}(\mathrm{x})-D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathrm{x})}{\sqrt{1-\lambda_{n}}} \tag{4.3.10}
\end{equation*}
$$

where $\lambda_{n}$ and $\boldsymbol{\psi}_{n}$ are, respectively, an eigenvalue and a corresponding eigenfunction of (4.2.25). In this way, it is easy to see that

$$
\begin{aligned}
\left\|\boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}= & \frac{1}{1-\lambda_{n}} \int_{\mathbb{R}^{3}}\left[\boldsymbol{\psi}_{n}(\mathbf{x})-D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})\right]\left[\overline{\boldsymbol{\psi}_{n}}(\mathbf{x})-\overline{D_{\mathbf{T}} \boldsymbol{\psi}_{n}}(\mathbf{x})\right] d \mathbf{x} \\
= & \frac{1}{1-\lambda_{n}}\left[\int_{\mathbb{R}^{3}}\left|\boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}-\int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{D_{\mathbf{T}} \boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x}\right. \\
& \left.-\int_{\mathbb{R}^{3}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x}+\int_{\mathbb{R}^{3}}\left|D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}\right] \\
= & \frac{1-2 \lambda_{n}+\lambda_{n}}{1-\lambda_{n}} \\
= & 1
\end{aligned}
$$

and $\alpha(\mathbf{T})=\left\|D_{\mathbf{T}} \boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=0$. Hence, $\boldsymbol{f}^{*} \in \mathcal{G}$. We proceed to compute $\left\|B_{\mathbf{W}} \boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}$. Because $\boldsymbol{\psi}_{n}(\mathbf{x}) \in \mathcal{B}(\mathbf{W})$, we find

$$
B_{\mathbf{W}} \boldsymbol{f}^{*}(\mathbf{x})=\frac{\boldsymbol{\psi}_{n}(\mathbf{x})-B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})}{\sqrt{1-\lambda_{n}}}
$$

Hence, we obtain

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}}\left|B_{\mathbf{W}} \boldsymbol{f}^{*}(\mathbf{x})\right|^{2} d \mathbf{x} \\
= & \frac{1}{1-\lambda_{n}} \int_{\mathbb{R}^{3}}\left[\boldsymbol{\psi}_{n}(\mathbf{x})-B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})\right]\left[\overline{\boldsymbol{\psi}_{n}}(\mathbf{x})-\overline{B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}}(\mathbf{x})\right] d \mathbf{x} \\
= & \frac{1}{1-\lambda_{n}}\left[\int_{\mathbb{R}^{3}}\left|\boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}-\int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x}\right. \\
& \left.-\int_{\mathbb{R}^{3}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x}+\int_{\mathbb{R}^{3}}\left|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}\right] .
\end{aligned}
$$

The first integral on the right-hand side follows from the orthogonality (4.2.30) of the $c$-QPSWFs.

Straightforward computations of the second integral further show that

$$
\begin{aligned}
\int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{B_{\mathbf{W}} D_{\mathbf{T}} \psi_{n}}(\mathbf{x}) d \mathbf{x} & =\left\langle B_{\mathbf{W}} \boldsymbol{\psi}_{n}, D_{\mathbf{T}} \boldsymbol{\psi}_{n}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\left\langle\boldsymbol{\psi}_{n}, D_{\mathbf{T}} \boldsymbol{\psi}_{n}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =\lambda_{n} \\
& =\int_{\mathbb{R}^{3}} \overline{\boldsymbol{\psi}_{n}(\mathbf{x}) \overline{B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})} d \mathbf{x}} \\
& =\int_{\mathbb{R}^{3}} B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x}
\end{aligned}
$$

For the last integral, we use the definition 4.2.10 of the operator $B_{\mathbf{W}} D_{\mathbf{T}}$, and 4.2 .32 and 4.2 .25 . It then follows from the Tonelli-Hobson Theorem that

$$
\begin{aligned}
& \int_{\mathbb{R}^{3}}\left|B_{\mathbf{W}} D_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x} \\
& =\int_{\mathbb{R}^{3}}\left(\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{y}) \boldsymbol{\psi}_{n}(\mathbf{y}) d \mathbf{y}\right)\left(\int_{\mathbf{T}} K_{\mathbf{W}}(\mathbf{x}, \mathbf{z}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{z}) d \mathbf{z}\right) d \mathbf{x} \\
& =\int_{\mathbf{T}} \int_{\mathbf{T}} \boldsymbol{\psi}_{n}(\mathbf{y}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{z}) K_{\mathbf{W}}(\mathbf{y}, \mathbf{z}) d \mathbf{y} d \mathbf{z} \\
& =\lambda_{n} \int_{\mathbf{T}}\left|\boldsymbol{\psi}_{n}(\mathbf{z})\right|^{2} d \mathbf{z} \\
& =\lambda_{n}^{2}
\end{aligned}
$$

With these calculations at hand, we finally obtain that $\beta(\mathbf{W})=\sqrt{1-\lambda_{n}}$. Since the sequence of eigenvalues $\left\{\lambda_{n}\right\}_{n=0}^{\infty}$ is monotone decreasing in the interval $(0,1)$, the $\lambda_{n}$ 's can be made arbitrarily close to 0 . Thus, there exist functions in $\mathcal{G}$ with values of $\beta(\mathbf{W})$ arbitrarily close to 1 . On the other hand, it is clear that when both energy ratios $\alpha(\mathbf{T})$ and $\beta(\mathbf{W})$ are equal to 0 , the underlying quaternionic signal must be the identically zero signal. Nevertheless, we shall also show that there exists a signal that is not identically zero when $\beta(\mathbf{W})$ is arbitrarily close to 0 .

We proceed to find a function $\boldsymbol{g}(\mathbf{x})$ such that $\mathcal{F}(\boldsymbol{g})(\boldsymbol{\omega})=\mathcal{F}\left(\boldsymbol{f}^{*}\right)(\boldsymbol{\omega}-\boldsymbol{\sigma})$ for any $\boldsymbol{\sigma} \in \mathbb{R}^{3}$, where $\boldsymbol{f}^{*}$ is defined above as 4.3.10). If such a $\boldsymbol{g}(\mathbf{x})$ exists, then

$$
\begin{aligned}
\beta(\mathbf{W}) & =\frac{1}{(2 \pi)^{3 / 2}}\left\|\mathcal{F}\left(B_{\mathbf{W}} \boldsymbol{g}\right)\right\|_{L_{2}\left(\mathbb{R}^{3}\right)} \\
& =\frac{1}{(2 \pi)^{3 / 2}}\left(\int_{\mathbf{W}}\left|\mathcal{F}\left(\boldsymbol{f}^{*}\right)(\boldsymbol{\omega}-\boldsymbol{\sigma})\right|^{2} d \boldsymbol{\omega}\right)^{1 / 2} \\
& =\frac{1}{(2 \pi)^{3 / 2}}\left(\int_{\mathbf{W}_{\boldsymbol{\sigma}}}\left|\mathcal{F}\left(\boldsymbol{f}^{*}\right)(\boldsymbol{\omega})\right|^{2} d \boldsymbol{\omega}\right)^{1 / 2},
\end{aligned}
$$

where $\mathbf{W}_{\boldsymbol{\sigma}}$ is a cube with edges of length $2 W$ and translated by the amount of $\boldsymbol{\sigma}$. We shall observe that the quantity $\beta(\mathbf{W})$ is continuous in $\boldsymbol{\sigma}$ for a fixed $\mathbf{W}$, and by Theorem 1.6 .2 property (vi), it approaches zero as $|\boldsymbol{\sigma}| \rightarrow \infty$. Thus, $\beta(\mathbf{W})$ can be arbitrarily close to 0 .

It remains to construct $\boldsymbol{f}(\mathbf{x})$ and then check whether it belongs to the class $\mathcal{G}$. Since $\boldsymbol{g}(\mathbf{x})$ satisfies $\mathcal{F}(\boldsymbol{g})(\boldsymbol{\omega})=\mathcal{F}\left(\boldsymbol{f}^{*}\right)(\boldsymbol{\omega}-\boldsymbol{\sigma})$, by definition (1.6.9), it follows that

$$
\begin{aligned}
\boldsymbol{g}(\mathrm{x}) & =\frac{1}{(2 \pi)^{3}} \int_{\mathbb{R}^{3}} \mathcal{F}\left(\boldsymbol{f}^{*}\right)(\boldsymbol{\omega}) \overline{\boldsymbol{E}}(\boldsymbol{\omega}+\boldsymbol{\sigma}, \mathbf{x}) d \boldsymbol{\omega} \\
& =\boldsymbol{f}^{*}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\sigma}, \mathbf{x}) .
\end{aligned}
$$

It can further be shown that

$$
\begin{aligned}
\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} & =\int_{\mathbb{R}^{3}}\left|\boldsymbol{f}^{*}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\sigma}, \mathbf{x})\right|^{2} d \mathbf{x} \\
& =\left\|\boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} \\
& =1
\end{aligned}
$$

and

$$
\alpha(\mathbf{T})=\int_{\mathbf{T}}\left|\boldsymbol{f}^{*}(\mathbf{x}) \overline{\boldsymbol{E}}(\boldsymbol{\sigma}, \mathbf{x})\right|^{2} d \mathbf{x}=0
$$

Hence, $\boldsymbol{g} \in \mathcal{G}$ and $\beta(\mathbf{W})$ can be arbitrarily close to 0 when $|\boldsymbol{\sigma}| \rightarrow \infty$; that is, $0 \leq \beta(\mathbf{W})<1$. This completes the proof.

From the properties of the QFT, we may conclude that the extremal properties for band-limited functions proved so far have their corresponding space-limited counterparts. In this manner, if $\alpha(\mathbf{T})=0$, then it follows that $0 \leq \beta(\mathbf{W})<1$. Also, if $\beta(\mathbf{W})=0$, then it follows that $0 \leq \alpha(\mathbf{T})<1$. One can further conclude a similar result as Theorem 4.3.6 for the extremal value $\alpha(\mathbf{T})=1$. For any nonzero quaternionic signal $\boldsymbol{f} \in \mathcal{D}(\mathbf{T})$, i.e., for which $\alpha(\mathbf{T})=1$, we may find that $\beta(\mathbf{W}) \leq \sqrt{\lambda_{0}}$. If $\beta(\mathbf{W})=\sqrt{\lambda_{0}}$, then $\boldsymbol{f}^{*}(\mathbf{x})=\left(D_{\mathbf{T}} \boldsymbol{\psi}_{0}(\mathbf{x})\right) / \sqrt{\lambda_{0}}$.

It is now left to prove that for all quaternionic signals for which $0<$ $\alpha(\mathbf{T})<\sqrt{\lambda_{0}}$ holds, the quantity $\beta(\mathbf{W})$ is not limited.

Theorem 4.3.8. There is a nonzero function $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ such that $\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}=1,\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\alpha(\mathbf{T})$ and $\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\beta(\mathbf{W})$, under the following condition: if $0<\alpha(\mathbf{T})<\sqrt{\lambda_{0}}$, then the quantity $\beta(\mathbf{W})$ can take on any value in the interval $[0,1]$.

Proof. Let $0<\alpha(\mathbf{T})<\sqrt{\lambda_{0}}$. Since the sequence $\left\{\lambda_{n}\right\}_{n=1}^{\infty}$ is monotone decreasing in the interval $(0,1)$, and $\lambda_{n} \rightarrow 0$ when $n$ approaches infinity, we can find an eigenvalue $\lambda_{n}$ such that $\lambda_{n}<\alpha(\mathbf{T})$. Let $\boldsymbol{\psi}_{n}$ be the eigenfunction corresponding to the eigenvalue $\lambda_{n}$. Now, consider the signal

$$
\boldsymbol{f}^{*}(\mathbf{x})=\frac{\sqrt{\alpha^{2}(\mathbf{T})-\lambda_{n}} \boldsymbol{\psi}_{0}(\mathbf{x})+\sqrt{\lambda_{0}-\alpha^{2}(\mathbf{T})} \boldsymbol{\psi}_{n}(\mathbf{x})}{\sqrt{\lambda_{0}-\lambda_{n}}}
$$

It is thus seen that $\boldsymbol{f}^{*} \in \mathcal{B}(\mathbf{W})$ since $\boldsymbol{\psi}_{0}, \boldsymbol{\psi}_{n} \in \mathcal{B}(\mathbf{W})$.

From the orthogonality 4.2.30 of the $c$-QPSWFs, it follows that

$$
\begin{aligned}
\left\|\boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}= & \frac{1}{\lambda_{0}-\lambda_{n}}\left[\left(\alpha^{2}(\mathbf{T})-\lambda_{n}\right) \int_{\mathbb{R}^{3}}\left|\boldsymbol{\psi}_{0}(\mathbf{x})\right|^{2} d \mathbf{x}\right. \\
& -\sqrt{\left(\alpha^{2}(\mathbf{T})-\lambda_{n}\right)\left(\lambda_{0}-\alpha^{2}(\mathbf{T})\right)} \int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{0}(\mathbf{x}) \overline{\boldsymbol{\psi}_{n}}(\mathbf{x}) d \mathbf{x} \\
& -\sqrt{\left(\alpha^{2}(\mathbf{T})-\lambda_{n}\right)\left(\lambda_{0}-\alpha^{2}(\mathbf{T})\right)} \int_{\mathbb{R}^{3}} \boldsymbol{\psi}_{n}(\mathbf{x}) \overline{\boldsymbol{\psi}_{0}}(\mathbf{x}) d \mathbf{x} \\
& \left.+\left(\lambda_{0}-\alpha^{2}(\mathbf{T})\right) \int_{\mathbb{R}^{3}}\left|\boldsymbol{\psi}_{n}(\mathbf{x})\right|^{2} d \mathbf{x}\right] \\
= & \frac{1}{\lambda_{0}-\lambda_{n}}\left[\left(\alpha^{2}(\mathbf{T})-\lambda_{n}\right)+\left(\lambda_{0}-\alpha^{2}(\mathbf{T})\right)\right] \\
= & 1 .
\end{aligned}
$$

Similarly, using (4.2.29), we find that

$$
\begin{aligned}
\left\|D_{\mathbf{T}} \boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} & =\frac{1}{\lambda_{0}-\lambda_{n}}\left[\left(\alpha^{2}(\mathbf{T})-\lambda_{n}\right) \lambda_{0}+\left(\lambda_{0}-\alpha^{2}(\mathbf{T})\right) \lambda_{n}\right] \\
& =\alpha^{2}(\mathbf{T}),
\end{aligned}
$$

which implies that $\boldsymbol{f}^{*} \in \mathcal{G}$, where $\mathcal{G}$ was defined in (4.3.9). Likewise, we find that $\left\|B_{\mathbf{W}} \boldsymbol{f}^{*}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\beta(\mathbf{W})=1$. Consequently, if $0<\alpha(\mathbf{T})<\sqrt{\lambda_{0}}$, then there exists a quaternionic signal such that $\beta(\mathbf{W})=1$. The verification of $0 \leq \beta(\mathbf{W})<1$ is similar to Theorem 4.3.7.

We conclude this section by studying the range of possible values of $\beta(\mathbf{W})$ for which $\sqrt{\lambda_{0}} \leq \alpha(\mathbf{T})<1$.

Theorem 4.3.9. There is a nonzero function $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ such that $\|\boldsymbol{f}\|_{L_{2}\left(\mathbb{R}^{3}\right)}=1,\left\|D_{\mathbf{T}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\alpha(\mathbf{T})$ and $\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}=\beta(\mathbf{W})$, under the following condition: the maximum of $\beta(\mathbf{W})$ is assumed by

$$
\begin{equation*}
\arccos \alpha(\mathbf{T})+\arccos \beta(\mathbf{W}) \geq \arccos \sqrt{\lambda_{0}} \tag{4.3.11}
\end{equation*}
$$

as such $\sqrt{\lambda_{0}} \leq \alpha(\mathbf{T})<1$, where $\lambda_{0}$ is the largest eigenvalue of 4.2.25).
Proof. Let $\sqrt{\lambda_{0}} \leq \alpha(\mathbf{T})<1$. For a function $\boldsymbol{f} \in \mathcal{G}$, where $\mathcal{G}$ was defined in 4.3.9), we find its projections onto $\mathcal{D}(\mathbf{T})$ and $\mathcal{B}(\mathbf{W})$. By Lemmas 4.2.9 and 4.2.10, we can then decompose $\boldsymbol{f}$ as follows:

$$
\begin{equation*}
\boldsymbol{f}=\boldsymbol{\lambda} D_{\mathrm{T}} \boldsymbol{f}+\boldsymbol{\mu} B_{\mathrm{W}} \boldsymbol{f}+\boldsymbol{g} \tag{4.3.12}
\end{equation*}
$$

where $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{H},\left\langle\boldsymbol{g}, D_{\mathbf{T}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}=\mathbf{0}$, and $\left\langle\boldsymbol{g}, B_{\mathbf{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}=\mathbf{0}$.

Now, taking the quaternionic inner product of the decomposition (4.3.12), respectively, with $\boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}, B_{\mathrm{W}} \boldsymbol{f}$, and $\boldsymbol{g}$ over the whole $\mathbb{R}^{3}$, and using the fact that $\boldsymbol{f} \in \mathcal{G}$, we obtain

$$
\begin{aligned}
1 & =\alpha^{2}(\mathbf{T}) \overline{\boldsymbol{\lambda}}+\beta^{2}(\mathbf{W}) \overline{\boldsymbol{\mu}}+\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}, \\
\alpha^{2}(\mathbf{T}) & =\alpha^{2}(\mathbf{T}) \overline{\boldsymbol{\lambda}}+\left\langle D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \overline{\boldsymbol{\mu}}, \\
\beta^{2}(\mathbf{W}) & =\left\langle B_{\mathbf{W}} \boldsymbol{f}, D_{\mathbf{T}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \overline{\boldsymbol{\lambda}}+\beta^{2}(\mathbf{W}) \overline{\boldsymbol{\mu}}, \\
\langle\boldsymbol{g}, \boldsymbol{f}\rangle & =\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2} .
\end{aligned}
$$

By eliminating the terms $\langle\boldsymbol{g}, \boldsymbol{f}\rangle, \overline{\boldsymbol{\lambda}}$, and $\overline{\boldsymbol{\mu}}$ from the above equations, we find that, for $\alpha(\mathbf{T}) \beta(\mathbf{W}) \neq 0$,

$$
\begin{aligned}
& \beta^{2}(\mathbf{W})-2 \operatorname{Sc}\left\langle D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)} \\
& =-\alpha^{2}(\mathbf{T})+\left(1-\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}\right)\left(1-\frac{\left|\left\langle D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}\right|^{2}}{\alpha^{2}(\mathbf{T}) \beta^{2}(\mathbf{W})}\right) .
\end{aligned}
$$

By considering

$$
\frac{\operatorname{Sc}\left\langle D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}}{\left\|D_{\mathbf{T}} \boldsymbol{g}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}\left\|B_{\mathbf{W}} \boldsymbol{f}\right\|_{L_{2}\left(\mathbb{R}^{3}\right)}}=\cos \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right),
$$

where $\arg \left(D_{\mathbf{T}} \boldsymbol{g}, B_{\mathbf{W}} \boldsymbol{g}\right)$ is the angle formed between $D_{\mathbf{T}} \boldsymbol{f} \in \mathcal{D}(\mathbf{T})$ and $B_{\mathbf{W}} \boldsymbol{f} \in$ $\mathcal{B}(\mathbf{W})$, by Theorem 4.3.4 $\arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right) \geq \arccos \sqrt{\lambda_{0}}$. Further computations show that

$$
\begin{aligned}
& \beta^{2}(\mathbf{W})-2 \alpha(\mathbf{T}) \beta(\mathbf{W}) \cos \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right) \\
& =-\alpha^{2}(\mathbf{T})+\left(1-\|\boldsymbol{g}\|_{L_{2}\left(\mathbb{R}^{3}\right)}^{2}\right)\left(1-\cos ^{2} \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right)\right) \\
& \leq-\alpha^{2}(\mathbf{T})+\sin ^{2} \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right) .
\end{aligned}
$$

Simplifying the above inequality, we obtain

$$
\begin{aligned}
& \left(\beta(\mathbf{W})-\alpha(\mathbf{T}) \cos \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right)\right)^{2} \\
& \quad \leq\left(1-\alpha^{2}(\mathbf{T})\right) \sin ^{2} \arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right)
\end{aligned}
$$

with equality, if and only if $\boldsymbol{g}=\mathbf{0}$ and $\left\langle D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right\rangle_{L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)}$ is real.
We then find

$$
\beta(\mathbf{W}) \leq \cos \left(\arg \left(D_{\mathbf{T}} \boldsymbol{f}, B_{\mathbf{W}} \boldsymbol{f}\right)-\arccos \alpha(\mathbf{T})\right)
$$

We conclude that $\arccos \alpha(\mathbf{T})+\arccos \beta(\mathbf{W}) \geq \arccos \sqrt{\lambda_{0}}$. Equality is attained by setting $\boldsymbol{g}(\mathbf{x})=p \boldsymbol{\psi}_{0}(\mathbf{x})+q D_{\mathbf{T}} \boldsymbol{\psi}_{0}(\mathbf{x})$, where

$$
p=\sqrt{\frac{1-\alpha^{2}(\mathbf{T})}{1-\lambda_{0}}}, \text { and } q=\frac{\alpha(\mathbf{T})}{\sqrt{\lambda_{0}}}-\sqrt{\frac{1-\alpha^{2}(\mathbf{T})}{1-\lambda_{0}}} .
$$

The proof is completed.

Theorem 4.3.9 constrains the possible values of the pair $(\alpha(\mathbf{T}), \beta(\mathbf{W}))$ when $\mathbf{T}$ and $\mathbf{W}$ are specified because the property $0<\lambda_{0}<1$ always holds. Therefore, the fractions of the energy of a nonzero signal $\boldsymbol{f} \in L_{2}\left(\mathbb{R}^{3}, \mathbb{H}\right)$ in two random domains cannot be arbitrarily large simultaneously, no matter what sets of concentration we choose. That is to say, if the fraction of energy of a nonzero signal in a given spatial domain is specified, then the fraction of its energy on a given frequency domain must remain below a certain maximum. It is a generalization of the classical uncertainty principle proved by Landau et al. in [198, which states that any nonzero signal cannot have arbitrarily large proportions of energy in both a finite time-domain and a finite frequency-domain.

In the following figure, we describe the set $(\alpha(\mathbf{T}), \beta(\mathbf{W}))$ as the region of the square $0 \leq \alpha(\mathbf{T}) \leq 1,0 \leq \beta(\mathbf{T}) \leq 1$ lying above the ellipse (4.3.11). It exhibits the impossibility of the simultaneous confinement of a given quaternionic signal and its amplitude spectrum.


Figure 4.11: The relationship between $\alpha(\mathbf{T})$ and $\beta(\mathbf{W})$ for $c=1$.

### 4.4 Constructing $c$-QPSWFs on the Ball

Subsection 4.2.2 shows that when $\mathbf{T}$ is a cube centered at the origin and $\mathbf{W}=$ $c \mathbf{T}$, then solutions of Eq. 4.2.16) are also solutions to a more straightforward integral equation. In the present section, we show that if $\mathbf{T}$ and $\mathbf{W}$ are spherical, then the angular part of the solutions of (4.2.16) can be found explicitly. The separation of variables in spherical coordinates leads to a set of one-dimensional radial integral equations. The treatment given here is a generalization of that provided by Slepian in [300].

Before we proceed, we give a version of the Funk-Hecke formula for the surface spherical monogenics of the form (3.1.24), which we formulate in a preliminary lemma.

Lemma 4.4.1. Let $\boldsymbol{f}$ be an $\mathbb{H}$-valued integrable function over $[-1,1]$. For $\boldsymbol{\omega}^{\prime} \in \partial \Omega_{0}$ and nonnegative integers $l$, $m$ with $0 \leq m \leq l$, the Funk-Hecke formula holds for the surface spherical monogenics of the form (3.1.24):

$$
\begin{equation*}
\int_{\boldsymbol{\omega} \in \partial \Omega_{0}} \boldsymbol{f}\left(<\boldsymbol{\omega}, \boldsymbol{\omega}^{\prime}>\right) \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\omega}) d \sigma(\boldsymbol{\omega})=2 \pi \boldsymbol{\lambda}_{l} \mathbf{X}_{l, m}^{Q}[0]\left(\boldsymbol{\omega}^{\prime}\right) \tag{4.4.1}
\end{equation*}
$$

with $\boldsymbol{\lambda}_{l}=\int_{-1}^{1} P_{l}(t) \boldsymbol{f}(t) d t$, where $P_{l}(t)$ denotes the Legendre polynomial of degree $l$.

Proof. The result follows straightforwardly from the well-known Funk-Hecke formula for surface spherical harmonics [286, Thm. A.34], but it is necessary to employ the definition (3.1.24).

By Definition 4.2.17 and the above lemma, the following theorem will now be established:

Theorem 4.4.2. Let $\mathbf{u}$ be any pure quaternion such that $\mathbf{u}^{2}=-1$. Given a real number $c>0$, the eigenvalues and eigenfunctions of the finite-QFT supported on $\mathbf{T}=\Omega_{0}$, namely

$$
\begin{equation*}
\boldsymbol{\mu}_{l, m, n} \boldsymbol{\psi}_{l, m, n}(\mathbf{x})=\int_{\Omega_{0}} \boldsymbol{\psi}_{l, m, n}(\mathbf{y}) \overline{\boldsymbol{E}}(\mathbf{y}, c \mathbf{x}) d \mathbf{y} \tag{4.4.2}
\end{equation*}
$$

for any $\mathbf{x} \in \Omega_{0}$ are, respectively, $\boldsymbol{\mu}_{l, m, n}=(2 \pi)^{3 / 2} \mathbf{u}^{l} \beta_{l, n}(n, l=0,1,2, \ldots)$ and $\boldsymbol{\psi}_{l, m, n}(\mathbf{x})=\left[(2 l+3) C_{l, m}\right]^{-1 / 2} R_{l, n}(|\mathbf{x}|) \mathbf{X}_{l, m}^{Q}[0](\mathbf{x} /|\mathbf{x}|)(m=0, \ldots, l)$ with $C_{l, m}$ given by (3.1.25), where $R_{l, n}(|\mathbf{x}|)$ are radial eigenfunctions and $\beta_{l, n}$ the corresponding eigenvalues of the integral equation

$$
\begin{equation*}
\beta_{l, n} R_{l, n}(|\mathbf{x}|)=\int_{0}^{1} \frac{J_{l+1 / 2}(c \rho|\mathbf{x}|)}{(c \rho|\mathbf{x}|)^{1 / 2}} R_{l, n}(\rho) \rho^{2} d \rho . \tag{4.4.3}
\end{equation*}
$$

Further, the eigenfunctions $\boldsymbol{\psi}_{l, m, n}$ are orthogonal and complete in $\mathcal{D}\left(\Omega_{0}\right)$ and $\mathcal{B}\left(c \Omega_{0}\right)$, and orthogonal on $\mathbb{R}^{3}$.

Proof. Since by Lemma 1.6.2, the QFT is spherically symmetric, the problem of finding the eigenfunctions and eigenvalues of (4.4.2) can be solved by the method of separation of variables. In the first place, by expanding $\boldsymbol{\psi}$ in surface spherical monogenics, $\boldsymbol{\psi}=\sum_{l} \sum_{m}\left[(2 l+3) C_{l, m}\right]^{-1 / 2} R_{l, m}(\rho) \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\eta})$,
where $\mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\eta})$ are of the form (3.1.24), and $R_{l, m}(\rho)$ is a real-valued function defined on the interval $[0,1]$, we obtain

$$
\begin{align*}
& \boldsymbol{\mu}_{l, m} \boldsymbol{\psi}_{l, m}(\mathbf{x}) \\
& =\sum_{l=0}^{\infty} \sum_{m=0}^{l} \int_{0}^{1} \iint_{\boldsymbol{\eta} \in \partial \Omega_{0}}\left[(2 l+3) C_{l, m}\right]^{-1 / 2} R_{l, m}(\rho) \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\eta}) \overline{\boldsymbol{E}}(\rho \boldsymbol{\eta}, c \mathbf{x}) \rho^{2} d \boldsymbol{\eta} d \rho . \tag{4.4.4}
\end{align*}
$$

Next, write $\mathbf{x}=r \boldsymbol{\xi}$ with $r \geq 0, \boldsymbol{\xi} \in \partial \Omega_{0}$ so that $\overline{\boldsymbol{E}}(\rho \boldsymbol{\eta}, \operatorname{cr} \boldsymbol{\xi})=\exp (\mathbf{u s c}\langle\boldsymbol{\eta}, \boldsymbol{\xi}\rangle)$, where $s=\rho r$. Moreover, we have by the Funk-Hecke formula (4.4.1) and Lemma 1.4.3

$$
\begin{align*}
& \iint_{\boldsymbol{\eta} \in \partial \Omega_{0}} \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\eta}) \overline{\boldsymbol{E}}(\rho \boldsymbol{\eta}, c r \boldsymbol{\xi}) d \boldsymbol{\eta} \\
& =2 \pi\left(\int_{-1}^{1} P_{l}(t) \exp (\mathbf{u} s c t) d t\right) \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\xi}) \\
& =\mathbf{u}^{l}(2 \pi)^{3 / 2} \frac{J_{l+1 / 2}(c s)}{(c s)^{1 / 2}} \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\xi}) . \tag{4.4.5}
\end{align*}
$$

We emphasize that in the above equation, the function $J_{l+1 / 2}(c s)$ is independent of the index $m$. As a matter of fact, by substituting (4.4.5) into (4.4.4), it is thus seen that both $R_{l, m}(\rho)$ and $\boldsymbol{\mu}_{l, m}$ are independent of $m$, and so (4.4.4) becomes

$$
\begin{align*}
& \beta_{l, n} \boldsymbol{\psi}_{l, m, n}(r, \boldsymbol{\xi}) \\
& =\left[(2 l+3) C_{l, m}\right]^{-1 / 2} \mathbf{X}_{l, m}^{Q}[0](\boldsymbol{\xi}) \int_{0}^{1} \frac{J_{l+1 / 2}(c \rho r)}{(c \rho r)^{1 / 2}} R_{l, n}(\rho) \rho^{2} d \rho \tag{4.4.6}
\end{align*}
$$

for all $r \in[0,1]$, where

$$
\begin{equation*}
\boldsymbol{\mu}_{l, m, n}=\boldsymbol{\mu}_{l, n}=(2 \pi)^{3 / 2} \mathbf{u}^{l} \beta_{l, n} \tag{4.4.7}
\end{equation*}
$$

with $n, l=0,1, \ldots$. A simple argument shows that 4.4.6 reduces to the integral equation of the form (4.4.3). Now, by similar reasoning to that by which the general properties of the $c$-QPSWFs were established, it follows from Proposition 4.2.18 that the eigenfunctions of 4.4 .2 are even or odd, and the corresponding eigenvalues are real or pure quaternionic according to the parity of these eigenfunctions. The domain on which the eigenfunctions are defined can be extended from $\Omega_{0}$ to $\mathbb{R}^{3}$ by requiring that 4.4.2) holds for all $\mathbf{x} \in \mathbb{R}^{3}$. Theorems 4.2.19 and 4.2.24 further ensure that the eigenfunctions are orthogonal and complete in $\mathcal{D}\left(\Omega_{0}\right)$ and $\mathcal{B}\left(c \Omega_{0}\right)$, and orthogonal on $\mathbb{R}^{3}$. Thus, the result follows.

The notation above hides the fact that both the $\boldsymbol{\mu}_{l, m, n}$ 's and the radial part $R_{l, n}$ of the $\boldsymbol{\psi}_{l, m, n}$ 's are functions of the parameter $c$. When it is necessary to make this dependence explicit, we write $\boldsymbol{\mu}_{l, m, n}=\boldsymbol{\mu}_{l, m, n}(c), R_{l, n}(|\mathbf{x}|)=$ $R_{l, n}(c,|\mathbf{x}|)$, and $\boldsymbol{\psi}_{l, m, n}(\mathbf{x})=\boldsymbol{\psi}_{l, m, n}(c, \mathbf{x})$.

It follows from the preceding theorem that the radial function $R_{l, n}$ coincides with the corresponding radial factor of the even and odd PSWFs of order zero treated by Slepian et al. [299]. We also observe that the onedimensional integral equation (4.4.3) is independent of the index $m$, and thus, by (4.4.7), the expected degeneracy of eigenvalues occurs due to the spherical symmetry.

The next proposition shows the relations between the $c$-QPSWFs and the PSWFs, described in [300].

Proposition 4.4.3. Let $n, l \geq 0$ and $0 \leq m \leq l$. Given a real number $c>0$, the $c$-QPSWFs supported on $\Omega_{0}$ are equal to

$$
\begin{aligned}
\boldsymbol{\psi}_{l, m, n}(\mathbf{x})= & \frac{1}{2(2 l+1)(l+2+m)}\left[-\left(1+\delta_{0, m}\right)\left(\psi_{l, m, n}^{+}(\mathbf{x})-\mathbf{k} \psi_{l, m, n}^{-}(\mathbf{x})\right)\right. \\
& \left.+(l-m)\left(\mathbf{i} \psi_{l, m+1, n}^{+}(\mathbf{x})+\mathbf{j} \psi_{l, m+1, n}^{-}(\mathbf{x})\right)\right]
\end{aligned}
$$

where

$$
\psi_{l, m, n}^{ \pm}(\mathbf{x})=\frac{(2 l+1)(l-m)!}{2 \pi\left(1+\delta_{0, m}\right)(l+m)!} R_{l, n}(|\mathbf{x}|) U_{l, m}^{ \pm}[0](\mathbf{x} /|\mathbf{x}|)
$$

Proof. The proof follows in a rather straightforward way from the representation (3.1.24) and Theorem 4.4.2.

By adding these results together, and making the substitution $\varphi_{l, n}(r)=$ $r R_{l, n}(r)$ with $r=|\mathbf{x}|$, we shall note that 4.4.3) can also be written as a finite-Hankel transform [180]:

$$
\begin{equation*}
\gamma_{l, n} \varphi_{l, n}(r)=\int_{0}^{1} J_{l+1 / 2}(c \rho r) \sqrt{c \rho r} \varphi_{l, n}(\rho) d \rho, \quad 0 \leq r \leq 1 \tag{4.4.8}
\end{equation*}
$$

where $\gamma_{l, n}=c \beta_{l, n}, n, l=0,1, \ldots$. We observe that $\varphi_{l, n}(0)=0$. The even and odd PSWFs (of order zero) correspond to the eigenfunctions of (4.4.8) for $l=-1$ and 0 ; in these two cases, the kernels become $(2 / \pi)^{1 / 2} \cos (c \rho r)$ and $(2 / \pi)^{1 / 2} \sin (c \rho r)$, respectively. Proceeding in this way, we now consider the operator $M_{c}: L_{2}([0,1]) \rightarrow L_{2}([0,1])$ defined by

$$
M_{c} \varphi_{l, n}(r)=\int_{0}^{1} J_{l+1 / 2}(c \rho r) \sqrt{c \rho r} \varphi_{l, n}(\rho) d \rho .
$$

In the first place, it is known from [300] that the operator, $M_{c}$, is compact and self-adjoint on $L_{2}([0,1])$, and $\gamma=0$ is not an eigenvalue of this operator.

The standard spectral theory assures the existence of an orthogonal basis of eigenfunctions $\varphi_{l, n}$ of $L_{2}([0,1])$. One of the main results found by Slepian in [300] was that solutions of (4.4.8) are also solutions of the Sturm-Liouville problem:

$$
\begin{equation*}
L_{c} \varphi_{l, n}(r)=-\chi_{l, n} \varphi_{l, n}(r) \tag{4.4.9}
\end{equation*}
$$

where $L_{c}$ is the differential operator given by

$$
\begin{equation*}
L_{c} \varphi_{l, n}(r)=\left(1-r^{2}\right) \frac{d \varphi_{l, n}(r)}{d r}-2 r \frac{d \varphi_{l, n}(r)}{d r}+\left(-\frac{l(l+1)}{r^{2}}-c^{2} r^{2}\right) \varphi_{l, n}(r), \tag{4.4.10}
\end{equation*}
$$

where $0<r<1$. The operator $L_{c}$ is self-adjoint for square-integrable and twice-differentiable functions that vanish at the origin. Bounded solutions of (4.4.9) exist only for a countable set of real values of $\chi_{l, n}, n, l=0,1, \ldots$, which we label so that $\chi_{l, 0} \leq \chi_{l, 1} \leq \chi_{l, 2} \leq \ldots$. Slepian [300] called the solutions of (4.4.9) generalized prolate spheroidal functions. In [173], Hurtley called these solutions hyperspheroidal functions.

With the help of specific formulas for the eigenfunctions and eigenvalues of (4.4.9) found in [300], the following result may be established:

Proposition 4.4.4. Let $\mathbf{u}$ be any pure quaternion such that $\mathbf{u}^{2}=-1$. Given a real number $c>0$, the eigenfunctions and eigenvalues of the finite-QFT supported on $\Omega_{0}$ are equal to

$$
\begin{aligned}
\boldsymbol{\psi}_{l, m, n}(\mathbf{x}) & =\frac{(l-m)!\varphi_{l, n}(c,|\mathbf{x}|)}{4 \pi(l+2+m)(l+2+m)!|\mathbf{x}|} \mathbf{X}_{l, m}^{Q}[0](\mathbf{x} /|\mathbf{x}|), \\
\boldsymbol{\mu}_{l, n} & =(2 \pi)^{3 / 2} \mathbf{u}^{l}\left[\frac{(l+1+2 n)(l+2+2 n)}{c}+O(c)\right]
\end{aligned}
$$

as $c \rightarrow 0$, where

$$
\begin{equation*}
\varphi_{l, n}(c,|\mathbf{x}|)=\frac{(2 n)!!(2 l+1)!!}{(2 n+2 l+1)!!}|\mathbf{x}|^{l} P_{n}^{(l+1 / 2,0)}\left(1-2|\mathbf{x}|^{2}\right)+O\left(c^{2}\right) \tag{4.4.11}
\end{equation*}
$$

In 4.4.11), $P_{\lambda}^{(\alpha, \beta)}(x)$ denotes the classical Jacobi polynomial of degree $\lambda$ in $x$, and with real parameters $\alpha$ and $\beta$.

The radial eigenfunctions $\varphi_{l, n}(0,|\mathbf{x}|)$ are known as the Zernike circle polynomials. These polynomials form a basis of orthogonal polynomials that arise in expanding the optical wavefront in imaging systems with circular pupils. These polynomials were first introduced by Zernike's Nobel prize [344, 345] about eighty years ago connected with his phase contrast and
knife-edge tests and since then have been extensively discussed in the literature [1, 2, 3, 4, 28, 36, 249, 257, 260, 321. It may further be shown that the functions $\varphi_{l, n}(0,|\mathbf{x}|)$ are the same as those of a second-order homogeneous differential equation, which arises in the theory of orthogonal ball polynomials [84, Theorem 11.1.5]. The three-dimensional quaternionic Zernike spherical polynomials introduced by Morais et al. in [249] can be obtained as a particular case of the present theory by appropriate interpretation. Further generalizations were made in [70], in which several analytical and algebraical properties of the quaternionic Zernike spherical polynomials were provided.

194 4. THE C-QUATERNIONIC PROLATE SPHEROIDAL WAVE FUNCTIONS

## 5

## Applications

In the first part of the chapter, we present some applications and discuss two constructive approaches for the generation of harmonic conjugates to find monogenic functions in $\mathbb{R}^{3}$. The first algorithm is based on different systems of harmonic and monogenic functions proposed in previous chapters. In contrast, the second one is presented, employing an integral representation. We give some examples of function spaces illustrating the techniques involved. More specifically, we discuss the monogenic weighted Hardy and Bergman spaces consisting of all functions with values in $\mathcal{A}$, which are monogenic in $\Omega_{0}$ and satisfy growth restrictions other than boundedness. We end up proving the boundedness of the underlying harmonic conjugation operators in specific weighted spaces.

In the second part of the chapter, the focus is direct to the geometric mapping properties of monogenic functions. Another application of the theory developed in the previous sections is the generalization of Bloch's Theorem to monogenic mappings defined in the unit ball of the three-dimensional Euclidean space. We can explicitly compute a lower bound for the Bloch constant.

The results that appear in the first part of the chapter were published in [245]. The results of the second part appeared in [241].

### 5.1 On Riesz Systems of Harmonic Conjugates

Given a harmonic function $u$ in a domain $\Omega$ of $\mathbb{R}^{4}$, the problem of finding a harmonic conjugate $\boldsymbol{v}$, which holds $\boldsymbol{f}(\mathbf{x})=u+\boldsymbol{v}$ monogenic in $\Omega$ and generalizes the well-known case of the complex plane, was first introduced by Sudbery in [318]. The author proposed an algorithm for the calculation of
quaternion-valued monogenic functions. Later and independently, Xu considered the problem of conjugate harmonics in the framework of Clifford analysis [336]. In [337] and [338], the constructions of conjugate harmonics to the Poisson kernel in the open unit ball and the upper half-space were obtained. The extension and completeness of these results were obtained in [49] and [50]. However, no effort was made to establish the function spaces to which these conjugate harmonics and the whole monogenic function belong. In [24], this question was replied for conjugate harmonics via Sudbery's formula in the scale of Bergman spaces. Nevertheless, these results do not apply to functions with values in $\mathcal{A}$.

We begin to recall from [152] an algorithm to determine a "unique" $\boldsymbol{f} \in$ $\mathcal{M}_{2}\left(\Omega_{0}\right)$ via conjugate harmonics, which makes essential use of the orthogonal bases of solid spherical harmonics and monogenics discussed in Chapters 2 and 3, respectively. See [152, 245] for a list of the known results before the present investigation. In the literature, similar ideas can be found in the works of Moisil in [231] and Stein and Weiß in [311]. However, in [311], the authors presented an approach based on the gradient of harmonic functions in the upper half-space, which are radial in two variables. Therefore the link is not immediate and will not be developed here.

In the following, assume that $h$ is a square-integrable harmonic function defined in $\Omega_{0}$. Given Theorems 2.3.1 and 3.1.10 and representation (3.1.5), start by considering the Fourier expansion of $h$ in terms of the orthogonal basis formed by the scalar parts of the solid spherical monogenics. Then replace the scalar parts of each harmonic polynomial with the full monogenic polynomial to obtain a Fourier series expansion of the form (3.1.16). In doing so, we consider that the monogenic polynomials are not normalized, and the coefficients of the series expansion need to be corrected. It results in an additional condition on the original Fourier coefficients of $h$. We formulate this idea in detail in the following theorem [152].

Theorem 5.1.1. Let $h \in \operatorname{Har}_{2}\left(\Omega_{0}\right)$ be expressed as a Fourier series expansion:

$$
\begin{equation*}
h(\mathbf{x})=\sum_{l=0}^{\infty} \sum_{m=0}^{l}\left(a_{l, m}^{+}[0] \frac{\mathrm{Sc}\left(\mathbf{X}_{l, m}^{+}[0]\right)}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, m}^{+}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}+a_{l, m}^{-}[0] \frac{\mathrm{Sc}\left(\mathbf{X}_{l, m}^{-}[0]\right)}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, m}^{-}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}\right) . \tag{5.1.1}
\end{equation*}
$$

If, additionally, the Fourier coefficients satisfy the condition

$$
\begin{equation*}
\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(l+1)(2 l+1)}{(l+1)^{2}-m^{2}}\left[\left(a_{l, m}^{+}[0]\right)^{2}+\left(a_{l, m}^{-}[0]\right)^{2}\right]<\infty \tag{5.1.2}
\end{equation*}
$$

then the series

$$
\begin{equation*}
\sum_{l=0}^{\infty} \sum_{m=0}^{l}\left(a_{l, m}^{+}[0] \frac{\operatorname{Vec}\left(\mathbf{X}_{l, m}^{+}[0]\right)}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, m}^{+}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}+a_{l, m}^{-}[0] \frac{\operatorname{Vec}\left(\mathbf{X}_{l, m}^{-}[0]\right)}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, m}^{-}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}\right) \tag{5.1.3}
\end{equation*}
$$

converges in the mean-square sense and it defines a vector-valued function $\boldsymbol{v} \in \operatorname{Har}_{2}\left(\Omega_{0}\right)$ such that $\boldsymbol{f}(\mathbf{x})=u(\mathbf{x})+\boldsymbol{v}(\mathbf{x}) \in \mathcal{M}_{2}\left(\Omega_{0}\right)$. Further, $\boldsymbol{v}$ is unique up to the addition of a nontrivial monogenic constant.

We will call $(h, \boldsymbol{v})$ a pair of conjugate harmonic functions in $\Omega_{0}$. This construction is obtained step-by-step, where each simple step exhibits the existence and uniqueness (up to the addition of a monogenic constant) of a vector-valued harmonic function $\boldsymbol{v}$ conjugate to $h$, which will make $\boldsymbol{f}$ an $\mathcal{A}$ valued monogenic function. To see this, observe that by adding a monogenic constant $\varphi$ to $\boldsymbol{v}$, the resulting function $\widetilde{\boldsymbol{v}}=\boldsymbol{v}+\boldsymbol{\varphi}$ is also conjugate to $h$. In the remainder of this section, we study how the quality of $h$ influences the quality of $\boldsymbol{v}$ and then examine how $h$ and $\boldsymbol{v}$ together define a suitable space for $\boldsymbol{f}$. Such a result will allow the definition of a continuous operator between spaces of harmonic and monogenic functions given by the construction of harmonic conjugates.

Before developing the general theory further, let us make two observations. First, by the direct construction of the expansion (5.1.3), we obtain a total of $2 l+1$ monogenic polynomials of degree $l$. However, since $\operatorname{dim} \mathcal{M}_{l}^{+}(\Omega, \mathcal{A})=2 l+3$, adding two monogenic constants, the number of independent polynomials needed to form a basis for $\mathcal{M}_{2}\left(\Omega_{0}\right)$ can indeed be attained. Secondly, it should be remarked that the criterion described in Theorem 5.1.1 is not well-applicable in practice. It is not yet clear how to characterize which condition of the theorem describes a function space (for the functions $h$ ). If we suppose for the moment more smoothness of the given function $h$, we can count exponential decay of the Fourier coefficients. Thus, by Definition 1.3.12, we can formulate a general sufficient condition that guarantees the convergence of the series expansion 5.1.3), and hence the existence of a function $\boldsymbol{f}$ in $\mathcal{M}_{2,1}\left(\Omega_{0}\right)$. The following theorem answers this question.

Theorem 5.1.2. Let $h \in \operatorname{Har}_{2}\left(\Omega_{0}\right)$. If the absolute values of the coefficients of the series expansion (5.1.1) satisfy the condition (5.1.2) and, additionally, are less than $C /(l+1)^{1+\alpha}$ with $\alpha>1$ for some constant $C>0$, then there exists a function $\boldsymbol{f} \in \mathcal{M}_{2,1}\left(\Omega_{0}\right)$ such that $\operatorname{Sc}(\boldsymbol{f})=h$ in $\Omega_{0}$.

Proof. Let expand $h \in \operatorname{Har}_{2}\left(\Omega_{0}\right)$ as in Theorem 5.1.1. First, we replace the scalar parts of each monogenic polynomial by the full polynomial itself. On
introducing suitable correction factors, we can then rewrite the underlying series as a Fourier expansion in terms of solid spherical monogenics. Hence, we obtain a function $\boldsymbol{f}$ in $\mathcal{M}_{2}\left(\Omega_{0}\right)$, represented by the expansion:

$$
\boldsymbol{f}=\sum_{l=0}^{\infty} \sum_{m=0}^{l} \sqrt{\frac{(l+1)(2 l+1)}{(l+1)^{2}-m^{2}}}\left(a_{l, m}^{+}[0] \mathbf{X}_{l, m}^{+}[0]+a_{l, m}^{-}[0] \mathbf{X}_{l, m}^{-}[0]\right) .
$$

Since the hypercomplex derivative of a monogenic function $\boldsymbol{f}$ is again monogenic, it remains to prove that $(1 / 2) \partial \boldsymbol{f} \in L_{2}\left(\Omega_{0}, \mathcal{A}\right)$. The above series converges in the $L_{2}$-sense. So it converges uniformly to $\boldsymbol{f}$ in each compact subset of $\Omega_{0}$. Accordingly, the series of all partial derivatives converge uniformly to the corresponding partial derivatives of $\boldsymbol{f}$ in compact subsets of $\Omega_{0}$. Since the operator $\partial$ is continuous, we can thus take the hypercomplex derivative of the series term-by-term, and by Corollary 3.1.18 one finds

$$
\begin{equation*}
\left(\frac{1}{2} \partial\right) \boldsymbol{f}=\sum_{l=1}^{\infty} \sum_{m=0}^{l} \sqrt{l(2 l+3)}\left(a_{l, m}^{+}[0] \mathbf{X}_{l-1, m}^{+}[0]+a_{l, m}^{-}[0] \mathbf{X}_{l-1, m}^{-}[0]\right) . \tag{5.1.4}
\end{equation*}
$$

We now proceed to use the $L_{2}$-convergence of (5.1.4) to find that the expansion

$$
\begin{equation*}
\sum_{l=1}^{\infty} l(2 l+3) \sum_{m=1}^{l}\left[\left(a_{l, m}^{+}[0]\right)^{2}+\left(a_{l, m}^{-}[0]\right)^{2}\right] \tag{5.1.5}
\end{equation*}
$$

also converges in the mean-square sense. Using the hypothesis on the upper bounds of the Fourier coefficients, it follows from (5.1.5) that

$$
\|\boldsymbol{f}\|_{L_{2}\left(\Omega_{0}\right)}^{2}<\sum_{l=1}^{\infty} \frac{6 C^{2}}{(l+1)^{2 \alpha-1}} .
$$

Since the assumption that $\alpha>1$ leads to the convergence of the above series, it follows that (5.1.4) converges and hence $(1 / 2) \partial \boldsymbol{f} \in L_{2}\left(\Omega_{0}, \mathcal{A}\right)$. The theorem has now been completely established.

From the preceding considerations, we deduce the further result:
Lemma 5.1.3. Let $\alpha>-1$. Suppose that $\boldsymbol{f} \in \mathcal{M}_{2}\left(\Omega_{0}\right)$ has the expansion $\boldsymbol{f}(\mathbf{x})=\sum_{l=0}^{\infty} \mathbf{P}_{l}(\mathbf{x})$, where

$$
\mathbf{P}_{l}(\mathbf{x})=\sum_{m=0}^{l}\left(a_{l, m}^{+}[0] \frac{\mathbf{X}_{l, m}^{+}[0]}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, m}^{+}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}+a_{l, m}^{-}[0] \frac{\mathbf{X}_{l, m}^{-}[0]}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, m}^{-}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}\right)
$$

Then

$$
\begin{equation*}
\|\boldsymbol{f}\|_{L_{2, \alpha}\left(\Omega_{0}\right)} \approx\left(\sum_{l=0}^{\infty} \frac{1}{(l+1)^{\alpha+1}}\left\|\mathbf{P}_{l}\right\|_{L_{2}\left(\partial \Omega_{0}\right)}^{2}\right)^{1 / 2} \tag{5.1.6}
\end{equation*}
$$

(For any $A, B>0, A \approx B$ denotes the two-sided estimate $c_{1} A \leq B \leq c_{2} A$ for some constants $c_{1}, c_{2}>0$.)

Proof. In view of the homogeneity and orthogonality of the monogenic polynomials $\mathbf{P}_{l}$ for each $l=0,1, \ldots$, term-by-term integration yields

$$
\begin{equation*}
\left(M_{2}(\boldsymbol{f} ; \rho)\right)^{2}=\int_{\partial \Omega_{0}}|\boldsymbol{f}(\rho \boldsymbol{\eta})|^{2} d \sigma(\boldsymbol{\eta})=\sum_{l=0}^{\infty} \rho^{2 l}\left\|\mathbf{P}_{l}\right\|_{L_{2}\left(\partial \Omega_{0}\right)}^{2}, \tag{5.1.7}
\end{equation*}
$$

where $\rho \in[0,1)$.
In this manner, we obtain

$$
\begin{equation*}
\|\boldsymbol{f}\|_{\mathcal{H}_{2}\left(\Omega_{0}\right)}=\|\boldsymbol{f}\|_{L_{2}\left(\partial \Omega_{0}\right)}=\left(\sum_{l=0}^{\infty}\left\|\mathbf{P}_{l}\right\|_{L_{2}\left(\partial \Omega_{0}\right)}^{2}\right)^{1 / 2} \tag{5.1.8}
\end{equation*}
$$

In order to obtain an equivalent norm in the Bergman spaces, $B_{2, \alpha}\left(\Omega_{0}\right)$ or $\mathcal{B}_{2, \alpha}\left(\Omega_{0}\right)$, we integrate 5.1.7) over the interval $(0, r)$ for every $r \in(0,1)$, and obtain

$$
\begin{aligned}
& 3 \int_{0}^{r}(1-\rho)^{\alpha}\left(M_{2}(\boldsymbol{f} ; \rho)\right)^{2} \rho^{2} d \rho \\
& =3 \sum_{l=0}^{\infty}\left\|\mathbf{P}_{l}\right\|_{L_{2}\left(\partial \Omega_{0}\right)}^{2} \int_{0}^{r}(1-\rho)^{\alpha} \rho^{2 l+2} d \rho
\end{aligned}
$$

By the Stirling's formula, we have then

$$
\int_{0}^{1}(1-\rho)^{\alpha} \rho^{2 l+2} d \rho \approx \frac{\alpha!}{2^{\alpha+1}} \frac{1}{(l+1)^{\alpha+1}}
$$

as $l \rightarrow \infty$. By letting $r$ approach $1^{-}$we obtain (5.1.6).
As was shown in [35], similar results apply to the more general class of Clifford algebra-valued functions expanded into spherical harmonics. The Hardy norm 5.1.8 and the unweighted Bergman $L_{2,0}\left(\Omega_{0}\right)$-norm were obtained in this setting. For real-valued harmonic functions in the unit ball in $\mathbb{R}^{n}$, the equivalence of (5.1.6) was obtained in [270].

It will be observed that we cannot assume a priori that the existence of a function $h$ in $\operatorname{Har}\left(\Omega_{0}\right)$ or $H_{2}\left(\Omega_{0}\right)$ necessarily implies the existence of a function $\boldsymbol{f}$ in $\mathcal{H}_{2}\left(\Omega_{0}\right)$, constructed as in Theorem 5.1.1. It will thus appear that the inequality

$$
\begin{equation*}
\|\boldsymbol{f}\|_{\mathcal{H}_{2}\left(\Omega_{0}\right)} \leq C\|h\|_{H_{2}\left(\Omega_{0}\right)} \tag{5.1.9}
\end{equation*}
$$

fails for some constant $C>0$.
To see this we consider, for example, the expansion

$$
h(\mathbf{x})=\sum_{l=0}^{\infty} a_{l, l}^{+}[0] \frac{\operatorname{Sc}\left(\mathbf{X}_{l, l}^{+}[0]\right)}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, l}^{+}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}},
$$

where $a_{l, l}^{+}[0]=(l+1)^{-3 / 2}$. In the first place, we find

$$
\|h(\mathbf{x})\|_{H_{2}\left(\Omega_{0}\right)}^{2}=\sum_{l=0}^{\infty}(2 l+3)\left(a_{l, l}^{+}[0]\right)^{2}<\infty .
$$

However, in accordance with Theorem 5.1.1, a straightforward calculation shows that

$$
\begin{aligned}
\|\boldsymbol{f}\|_{\mathcal{H}_{2}\left(\Omega_{0}\right)}^{2} & =\left\|\sum_{l=0}^{\infty} a_{l, l}^{+}[0] \frac{\mathbf{X}_{l, l}^{+}[0]}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, l}^{+}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}}\right\|_{\mathcal{H}_{2}\left(\Omega_{0}\right)}^{2} \\
& =\sum_{l=0}^{\infty}(l+1)(2 l+3)\left(a_{l, l}^{+}[0]\right)^{2} \\
& =\infty
\end{aligned}
$$

which contradicts (5.1.9). Having made this observation, we have then to investigate whether, or under what conditions, concrete a priori criteria for the given harmonic function are capable of ensuring the existence of a "unique" monogenic function.

We now infer from Theorem 5.1.1, in combination with Lemma 5.1.3, an essential result of the boundedness of the harmonic conjugation operator in some given weighted spaces.
Theorem 5.1.4. Let $h \in \operatorname{Har}_{2}\left(\Omega_{0}\right)$. The harmonic conjugation operator $h \rightarrow \boldsymbol{f}$ is
(i) bounded from $H_{2}\left(\Omega_{0}\right)$ into $\mathcal{B}_{2}\left(\Omega_{0}\right)$ :

$$
\|\boldsymbol{f}\|_{L_{2}\left(\Omega_{0}\right)} \leq\|h\|_{H_{2}\left(\Omega_{0}\right)}
$$

(ii) bounded from $B_{2, \alpha}\left(\Omega_{0}\right)$ (for $\alpha>-1$ ) into $\mathcal{B}_{2, \alpha+1}\left(\Omega_{0}\right)$ :

$$
\|\boldsymbol{f}\|_{L_{2, \alpha+1}\left(\Omega_{0}\right)} \leq\|h\|_{L_{2, \alpha}\left(\Omega_{0}\right)} .
$$

The previous inequality is sharp in the sense that the exponent $\alpha+1$ on the left-hand side cannot be replaced by any smaller one, and the operator $h \rightarrow \boldsymbol{f}$ is unbounded from $B_{2, \alpha}\left(\Omega_{0}\right)$ to $\mathcal{B}_{2, \alpha+1-\epsilon}\left(\Omega_{0}\right)$ for any $\epsilon>0$.
Proof. For simplicity of presentation, we set $\boldsymbol{f}(\mathbf{x})=\sum_{l=0}^{\infty} \mathbf{P}_{l}(\mathbf{x})$, where $\mathbf{P}_{l}$ is given as in Lemma 5.1.3. A direct computation shows that

$$
\begin{aligned}
\|\boldsymbol{f}\|_{L_{2}\left(\Omega_{0}\right)}^{2} & =\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(2 l+1)(2 l+3)}{(l+1)^{2}-m^{2}}\left[\left(a_{l, m}^{+}[0]\right)^{2}+\left(a_{l, m}^{-}[0]\right)^{2}\right] \\
& \leq \sum_{l=0}^{\infty}(2 l+3) \sum_{m=0}^{l}\left[\left(a_{l, m}^{+}[0]\right)^{2}+\left(a_{l, m}^{-}[0]\right)^{2}\right] \\
& =\|h\|_{H_{2}\left(\Omega_{0}\right)}^{2},
\end{aligned}
$$

and this is the required result.
For the proof of the second part,

$$
\begin{aligned}
\|\boldsymbol{f}\|_{L_{2, \alpha+1}\left(\Omega_{0}\right)}^{2} & =\sum_{l=0}^{\infty} \frac{1}{(l+1)^{\alpha+2}} \sum_{m=0}^{l} \frac{(2 l+1)(2 l+3)}{(l+1)^{2}-m^{2}}\left[\left(a_{l, m}^{+}[0]\right)^{2}+\left(a_{l, m}^{-}[0]\right)^{2}\right] \\
& \leq \sum_{l=0}^{\infty} \frac{2 l+3}{(l+1)^{\alpha+1}} \sum_{m=0}^{l}\left[\left(a_{l, m}^{+}[0]\right)^{2}+\left(a_{l, m}^{-}[0]\right)^{2}\right] \\
& =\|h\|_{L_{2, \alpha}\left(\Omega_{0}\right) .}^{2} .
\end{aligned}
$$

We proceed to find a counterexample for the inequality

$$
\begin{equation*}
\|\boldsymbol{f}\|_{L_{2, \alpha+1-\epsilon}\left(\Omega_{0}\right)} \leq C(\alpha, \epsilon)\|h\|_{L_{2, \alpha}\left(\Omega_{0}\right)} \tag{5.1.10}
\end{equation*}
$$

for some constant $C(\alpha, \epsilon)>0$. Let $\epsilon$ be arbitrarily chosen and fixed such that $0<\epsilon<1$. Now, consider the expansion

$$
h(\mathbf{x})=\sum_{l=0}^{\infty} a_{l, l}^{+}[0] \frac{\operatorname{Sc}\left(\mathbf{X}_{l, l}^{+}[0]\right)}{\left\|\operatorname{Sc}\left(\mathbf{X}_{l, l}^{+}[0]\right)\right\|_{L_{2}\left(\Omega_{0}\right)}},
$$

where $a_{l, l}^{+}[0]=(l+1)^{(\alpha-1-\epsilon) / 2}$.
We thus have

$$
\|h\|_{L_{2, \alpha}\left(\Omega_{0}\right)}^{2} \leq 3 \sum_{l=0}^{\infty} \frac{1}{(l+1)^{\alpha}}\left(a_{l, l}^{+}[0]\right)^{2}<\infty
$$

and

$$
\|\operatorname{Sc}(\boldsymbol{f})\|_{L_{2, \alpha+1-\epsilon}\left(\Omega_{0}\right)}^{2} \geq 2 \sum_{l=0}^{\infty} \frac{1}{(l+1)^{\alpha-\epsilon}}\left(a_{l, l}^{+}[0]\right)^{2}=\infty,
$$

which contradicts 5.1.10). This completes the proof.
It is interesting to note that the Hardy space $\mathcal{H}_{2}\left(\Omega_{0}\right)$ can be regarded as the limiting case of the weighted Bergman space $\mathcal{B}_{2, \alpha}\left(\Omega_{0}\right)$ as $\alpha$ approaches $-1^{+}$. Accordingly, if we identify $\mathcal{B}_{2,-1}\left(\Omega_{0}\right)$ with $\mathcal{B}_{2}\left(\Omega_{0}\right)$, then it turns out for Theorem 5.1.4 that Property (i) can be seen as a generalization of Property (ii).

Furthermore, if we want to make the above theorem more precise, then we need a priori criteria for the given function $h$ that ensure the convergence of the constructed series expansion of $\boldsymbol{f}=h+\boldsymbol{v}$ in $\mathcal{M}_{2}\left(\Omega_{0}\right)$ or another suitable space. Of course, the additional assumption (5.1.2) of Theorem 5.1.1 is such a criterion, but it is not well-applicable in practice. It is not known at present whether there is a known function space that is defined precisely by these conditions.

We now proceed to consider an alternative algorithm to computing a "unique" pair of conjugate harmonic functions in $\mathbb{R}^{3}$ whose properties we can investigate to provide a characterization for the boundedness of harmonic conjugation operators on specific weighted function spaces. The proposed algorithm may be stated as follows:
Theorem 5.1.5. Suppose that $h \in \operatorname{Har}_{2}\left(\Omega_{0}\right)$ has continuous second derivatives. If

$$
\begin{equation*}
[\boldsymbol{v}(\mathbf{x})]_{1}:=-x_{0} \int_{0}^{1} \frac{\partial h\left(\rho x_{0}, x_{1}, x_{2}\right)}{\partial x_{1}} d \rho+w\left(x_{1}, x_{2}\right) \tag{5.1.11}
\end{equation*}
$$

where $w\left(x_{1}, x_{2}\right)$ is a function such that $\Delta_{3} w=\frac{\partial^{2} h\left(0, x_{1}, x_{2}\right)}{\partial x_{0} \partial x_{1}}$, and

$$
[\boldsymbol{v}(\mathbf{x})]_{2}:=\int_{0}^{1}\left[-\left|\begin{array}{cc}
x_{0} & x_{2}  \tag{5.1.12}\\
\frac{\partial h(t x)}{\partial x_{0}} & \frac{\partial h(t x)}{\partial x_{2}}
\end{array}\right|+\left|\begin{array}{cc}
x_{1} & x_{2} \\
\frac{\partial[\boldsymbol{v}(t x)]_{1}}{\partial x_{1}} & \frac{\partial[\boldsymbol{v}(t x)]_{1}}{\partial x_{2}}
\end{array}\right|\right] d t,
$$

then there exists a function $\boldsymbol{f} \in \mathcal{M}_{2}\left(\Omega_{0}\right)$ such that $\operatorname{Sc}(\boldsymbol{f})=h,[\boldsymbol{f}]_{1}=[\boldsymbol{v}]_{1}$ and $[\boldsymbol{f}]_{2}=[\boldsymbol{v}]_{2}$ in $\Omega_{0}$. Further, the most general monogenic function $\boldsymbol{g}$ having $h$ as its scalar part is given by $\boldsymbol{g}(\mathbf{x})=\boldsymbol{f}(\mathbf{x})+\boldsymbol{\varphi}\left(x_{1}, x_{2}\right)$, where $\boldsymbol{\varphi}$ is a nontrivial monogenic constant.

Proof. In the first place, we prove that $\boldsymbol{f}=h+\boldsymbol{v}$ is monogenic, i.e. it satisfies the Riesz system (1.3.4). On account of the assumption about $[\boldsymbol{v}]_{1}$, it follows that

$$
[\boldsymbol{v}(\mathbf{x})]_{1}=-\int_{0}^{x_{0}} \frac{\partial h\left(t, x_{1}, x_{2}\right)}{\partial x_{1}} d t+w\left(x_{1}, x_{2}\right)
$$

for all $\mathbf{x} \in \Omega_{0}$.
Thus

$$
\begin{equation*}
\frac{\partial[\boldsymbol{v}(\mathbf{x})]_{1}}{\partial x_{0}}=-\frac{\partial h(\mathbf{x})}{\partial x_{1}} . \tag{5.1.13}
\end{equation*}
$$

We have now

$$
\begin{aligned}
\Delta[\boldsymbol{v}(\mathbf{x})]_{1} & =\frac{\partial^{2}[\boldsymbol{v}(\mathbf{x})]_{1}}{\partial x_{0}^{2}}+\Delta_{x_{1}, x_{2}}[\boldsymbol{v}(\mathbf{x})]_{1} \\
& =-\frac{\partial^{2} h(\mathbf{x})}{\partial x_{0} \partial x_{1}}-\int_{0}^{x_{0}} \frac{\partial}{\partial x_{1}} \Delta_{x_{1}, x_{2}} h\left(t, x_{1}, x_{2}\right) d t+\Delta_{x_{1}, x_{2}} w\left(x_{1}, x_{2}\right) \\
& =-\frac{\partial^{2} h(\mathbf{x})}{\partial x_{0} \partial x_{1}}+\int_{0}^{x_{0}} \frac{\partial^{3} h\left(t, x_{1}, x_{2}\right)}{\partial x_{0}^{2} \partial x_{1}} d t+\Delta_{x_{1}, x_{2}} w\left(x_{1}, x_{2}\right) \\
& =-\frac{\partial^{2} h(\mathbf{x})}{\partial x_{0} \partial x_{1}}+\frac{\partial^{2} h(\mathbf{x})}{\partial x_{0} \partial x_{1}}-\frac{\partial^{2} h\left(0, x_{1}, x_{2}\right)}{\partial x_{0} \partial x_{1}}+\Delta_{x_{1}, x_{2}} w\left(x_{1}, x_{2}\right) \\
& =0 .
\end{aligned}
$$

Hence $[\boldsymbol{v}(\mathbf{x})]_{1}$ is harmonic.
Next, we define the function

$$
\begin{align*}
F(\mathbf{x}):= & \int_{0}^{x_{2}}\left(\frac{\partial h(0,0, t)}{\partial x_{0}}-\frac{\partial[\boldsymbol{v}(0,0, t)]_{1}}{\partial x_{1}}\right) d t+\int_{0}^{x_{1}} \frac{\partial\left[\boldsymbol{v}\left(0, t, x_{2}\right)\right]_{1}}{\partial x_{2}} d t \\
& -\int_{0}^{x_{0}} \frac{\partial h\left(t, x_{1}, x_{2}\right)}{\partial x_{2}} d t . \tag{5.1.14}
\end{align*}
$$

We proceed to compute the partial derivatives of $F$ with respect to $x_{0}, x_{1}, x_{2}$. It follows that

$$
\begin{equation*}
\frac{\partial F(\mathbf{x})}{\partial x_{0}}=-\frac{\partial h(\mathbf{x})}{\partial x_{2}} . \tag{5.1.15}
\end{equation*}
$$

Using (5.1.13), we have then

$$
\begin{align*}
\frac{\partial F(\mathbf{x})}{\partial x_{1}} & =\frac{\partial\left[\boldsymbol{v}\left(0, x_{1}, x_{2}\right)\right]_{1}}{\partial x_{2}}-\int_{0}^{x_{0}} \frac{\partial^{2} h\left(t, x_{1}, x_{2}\right)}{\partial x_{1} \partial x_{2}} d t \\
& =\frac{\partial\left[\boldsymbol{v}\left(0, x_{1}, x_{2}\right)\right]_{1}}{\partial x_{2}}+\int_{0}^{x_{0}} \frac{\partial^{2}\left[\boldsymbol{v}\left(t, x_{1}, x_{2}\right)\right]_{1}}{\partial x_{0} \partial x_{2}} d t \\
& =\frac{\partial\left[\boldsymbol{v}\left(0, x_{1}, x_{2}\right)\right]_{1}}{\partial x_{2}}+\left.\frac{\partial\left[\boldsymbol{v}\left(t, x_{1}, x_{2}\right)\right]_{1}}{\partial x_{2}}\right|_{0} ^{x_{0}} \\
& =\frac{\partial[\boldsymbol{v}(\mathbf{x})]_{1}}{\partial x_{2}} . \tag{5.1.16}
\end{align*}
$$

In view of the harmonicity of $h$ and $[\boldsymbol{v}]_{1}$, and using (5.1.13), it thus follows that

$$
\begin{align*}
\frac{\partial F(\mathbf{x})}{\partial x_{2}}= & \frac{\partial h\left(0,0, x_{2}\right)}{\partial x_{0}}-\frac{\partial\left[\boldsymbol{v}\left(0,0, x_{2}\right)\right]_{1}}{\partial x_{1}}+\int_{0}^{x_{1}} \frac{\partial^{2}\left[\boldsymbol{v}\left(0, t, x_{2}\right)\right]_{1}}{\partial x_{2}^{2}} d t \\
& -\int_{0}^{x_{0}} \frac{\partial^{2} h\left(t, x_{1}, x_{2}\right)}{\partial x_{2}^{2}} d t \\
= & \frac{\partial h\left(0,0, x_{2}\right)}{\partial x_{0}}-\int_{0}^{x_{1}}\left(\frac{\partial^{2}\left[\boldsymbol{v}\left(0, t, x_{2}\right)\right]_{1}}{\partial x_{0}^{2}}+\frac{\partial^{2}\left[\boldsymbol{v}\left(0, t, x_{2}\right)\right]_{1}}{\partial x_{1}^{2}}\right) d t \\
& -\frac{\partial\left[\boldsymbol{v}\left(0,0, x_{2}\right)\right]_{1}}{\partial x_{1}}+\int_{0}^{x_{0}}\left(\frac{\partial^{2} h\left(t, x_{1}, x_{2}\right)}{\partial x_{0}^{2}}+\frac{\partial^{2} h\left(t, x_{1}, x_{2}\right)}{\partial x_{1}^{2}}\right) d t \\
= & \frac{\partial h\left(0,0, x_{2}\right)}{\partial x_{0}}-\int_{0}^{x_{1}}\left(-\frac{\partial^{2} h\left(0, t, x_{2}\right)}{\partial x_{0} \partial x_{1}}+\frac{\partial^{2}\left[\boldsymbol{v}\left(0, t, x_{2}\right)\right]_{1}}{\partial x_{1}^{2}}\right) d t \\
& -\frac{\partial\left[\boldsymbol{v}\left(0,0, x_{2}\right)\right]_{1}}{\partial x_{1}}+\int_{0}^{x_{0}}\left(\frac{\partial^{2} h\left(t, x_{1}, x_{2}\right)}{\partial x_{0}^{2}}-\frac{\partial^{2}\left[\boldsymbol{v}\left(t, x_{1}, x_{2}\right)\right]_{1}}{\partial x_{0} \partial x_{1}}\right) d t \\
= & \frac{\partial h(\mathbf{x})}{\partial x_{0}}-\frac{\partial[\boldsymbol{v}(\mathbf{x})]_{1}}{\partial x_{1}} . \tag{5.1.17}
\end{align*}
$$

We proceed to show that $[\boldsymbol{v}(\mathbf{x})]_{2}=F(\mathbf{x})$ for all $\mathbf{x} \in \Omega_{0}$. In accordance with (5.1.14), we write $F$ as a curve-line integral:

$$
\begin{align*}
& F(\mathbf{x}) \\
= & \int_{(0,0,0)}^{\left(x_{0}, x_{1}, x_{2}\right)} \tag{5.1.18}
\end{align*}\left[-\frac{\partial h(\mathbf{y})}{\partial x_{2}} d y_{0}+\frac{\partial[\boldsymbol{v}(\mathbf{y})]_{1}}{\partial x_{2}} d y_{1}+\left(\frac{\partial h(\mathbf{y})}{\partial x_{0}}-\frac{\partial[\boldsymbol{v}(\mathbf{y})]_{1}}{\partial x_{1}}\right) d y_{2}\right] .
$$

Thus, the above integral is path independent in view of the conditions 5.1.15, (5.1.16) and 5.1.17), namely

$$
\left\{\begin{array}{ccc}
\nabla F & = & \left(-\frac{\partial h}{\partial x_{2}}, \frac{\partial[\boldsymbol{v}]_{1}}{\partial x_{2}}, \frac{\partial h}{\partial x_{0}}-\frac{\partial[\boldsymbol{v}]_{1}}{\partial x_{1}}\right), \\
\operatorname{curl} \nabla F & = & 0 .
\end{array}\right.
$$

It is convenient to choose a path of integration in (5.1.18) whose segments are parallel to the coordinate axes. A suitable change of variables in (5.1.18) leads to

$$
\begin{equation*}
F(\mathbf{x})=\int_{0}^{1}\left[-x_{0} \frac{\partial h(t \mathbf{x})}{\partial x_{2}}+x_{2} \frac{\partial h(t \mathbf{x})}{\partial x_{0}}+x_{1} \frac{\partial[\boldsymbol{v}(t \mathbf{x})]_{1}}{\partial x_{2}}-x_{2} \frac{\partial[\boldsymbol{v}(t \mathbf{x})]_{1}}{\partial x_{1}}\right] d t, \tag{5.1.19}
\end{equation*}
$$

which corresponds to the general form of 5.1.12). Thus, $F(\mathbf{x})=[\boldsymbol{v}(\mathbf{x})]_{2}$ for all $\mathbf{x} \in \Omega_{0}$. This establishes that $\boldsymbol{f}=h+\mathbf{i}[\boldsymbol{v}]_{1}+\mathbf{j}[\boldsymbol{v}]_{2}$ is a solution of the Riesz system (1.3.4). Now, let $\boldsymbol{g}$ be the most general monogenic function such that $\mathrm{Sc}(\boldsymbol{g})=h$. On account of the assumption about $\boldsymbol{f}$, it follows that $2 h(\mathbf{x})=$ $\boldsymbol{g}(\mathrm{x})+\overline{\boldsymbol{g}}(\mathrm{x})=\boldsymbol{f}(\mathrm{x})+\overline{\boldsymbol{f}}(\mathrm{x})$. Therefore $\boldsymbol{f}(\mathrm{x})-\boldsymbol{g}(\mathrm{x})+(\overline{\boldsymbol{f}(\mathrm{x})-\boldsymbol{g}(\mathrm{x})})=0$, and so $[\boldsymbol{f}(\mathbf{x})-\boldsymbol{g}(\mathbf{x})]_{0}=0$ for all $\mathbf{x} \in \Omega_{0}$. Since $\boldsymbol{f}(\mathbf{x})-\boldsymbol{g}(\mathbf{x})$ is monogenic in $\Omega_{0}$, it is then clear that $\boldsymbol{f}(\mathbf{x})-\boldsymbol{g}(\mathbf{x})$ reduces to a monogenic constant $\varphi$ with $\operatorname{Sc}(\boldsymbol{\varphi})=0$. We now see that $\boldsymbol{g}(\mathbf{x})=\boldsymbol{f}(\mathbf{x})+\boldsymbol{\varphi}\left(x_{1}, x_{2}\right)$ for all $\mathbf{x} \in \Omega_{0}$. This completes the proof of the theorem.

The above theorem shows that there are as many $\mathcal{A}$-valued monogenic functions as there are harmonic functions in $\mathbb{R}^{3}$. In the following two sections, we shall illustrate how these techniques can be applied to problems that arise as to the discussion of the monogenic Hardy and weighted Bergman spaces in $\Omega_{0}$.

### 5.1.1 Harmonic conjugates in monogenic weighted Hardy spaces

In this section, we rely on various techniques used in Theorem 5.1.5to further develop results in the general setting of the monogenic weighted Hardy spaces and to discuss some applications.

We now briefly recall some basic facts about the Poisson kernel, which will be used to estimate the size of certain integrals.

Lemma 5.1.6 (see [188]). Let $\Omega$ be a bounded domain in $\mathbb{R}^{3}$ and $\partial \Omega \in C^{2}$, and let $P_{\Omega}(\mathbf{x}, \boldsymbol{\eta})$ be the Poisson kernel for $\Omega$. Then

$$
P_{\Omega}(\mathbf{x}, \boldsymbol{\eta}) \approx \frac{\operatorname{dist}(\mathbf{x}, \partial \Omega)}{|\mathbf{x}-\boldsymbol{\eta}|^{3}}
$$

for all $\mathbf{x} \in \Omega, \boldsymbol{\eta} \in \partial \Omega$.
For any fixed $\rho, r \in(0,1)$, we consider the inner domain of the following oblate spheroid:

$$
\begin{equation*}
\Omega_{\rho, r}=\left\{\mathbf{x} \in \mathbb{R}^{3}: \frac{x_{0}^{2}}{\rho^{2} r^{2}}+\frac{x_{1}^{2}}{r^{2}}+\frac{x_{2}^{2}}{r^{2}}<1\right\} . \tag{5.1.20}
\end{equation*}
$$

Now we estimate the size of the Poisson kernel for $\Omega_{\rho, r}$, which will be of use in the further discussion.

Lemma 5.1.7. Let $P_{\Omega_{\rho, r}}(\mathbf{x}, \boldsymbol{\eta})$ be the Poisson kernel for $\Omega_{\rho, r}$. Then

$$
P_{\Omega_{\rho, r}}(\mathbf{x}, \boldsymbol{\eta}) \approx \frac{\operatorname{dist}\left(\mathbf{x}, \partial \Omega_{\rho, r}\right)}{|\mathbf{x}-\boldsymbol{\eta}|^{3}}
$$

for all $\mathbf{x} \in \Omega_{\rho, r}, \boldsymbol{\eta} \in \partial \Omega_{\rho, r}$. In particular,

$$
P_{\Omega_{\rho, r}}(\mathbf{0}, \boldsymbol{\eta}) \approx \frac{\rho r}{|\boldsymbol{\eta}|^{3}}
$$

for all $\boldsymbol{\eta} \in \partial \Omega_{\rho, r}$.
Before we proceed any further, we need the following preliminary lemmas and some notation.

Lemma 5.1.8. For any $\alpha>0$ and $\beta>1$, there holds

$$
\int_{0}^{1} \frac{t^{\alpha-1}}{(1-\rho t)^{\beta}} d t \sim \frac{1}{(\beta-1)(1-\rho)^{\beta-1}}
$$

as $\rho \rightarrow 1^{-}$.
Proof. The proof is straightforward and will be omitted.
Lemma 5.1.9 (see [347]). Let $h=h\left(x_{1}, x_{2}\right)$ be a nonnegative superharmonic function in the unit disk $\mathcal{D}=\left\{x_{1}^{2}+x_{2}^{2}<1\right\}$, and let $\gamma>-1$ and $0<p<2+\gamma$. Then there is a constant $C(p, \gamma, a)>0$ such that $\|h\|_{L_{p, \gamma}(\mathcal{D})} \leq C(p, \gamma, a) h(a)$ for any point $a \in \mathcal{D}$.

Lemma 5.1.10 (see [122]). Let $h(\mathbf{x})$ be a nonnegative subharmonic function in $\Omega_{0}$. If $M_{1}(h ; \rho)=\int_{\partial \Omega_{0}} h(\rho \boldsymbol{\eta}) d \sigma(\boldsymbol{\eta})$ is bounded on $\rho \in[0,1)$, then $h(\mathbf{x})$ has a harmonic majorant $u(\mathbf{x}) \in H_{1}\left(\Omega_{0}\right)$ such that $h(\mathbf{x}) \leq u(\mathbf{x})$ for all $\mathbf{x} \in \Omega_{0}$, and $\|u\|_{H_{1}\left(\Omega_{0}\right)} \leq C \sup _{\rho \in(0,1)} M_{1}(h ; \rho)$ for some constant $C>0$.

By means of this result, the following lemma may be deduced:
Lemma 5.1.11. Let $1 \leq p<\infty, \beta>0, \alpha>-1, \boldsymbol{\lambda}=\left(\lambda_{0}, \lambda_{1}, \lambda_{2}\right)$ a multiindex of nonnegative integers such that $|\boldsymbol{\lambda}|=\lambda_{0}+\lambda_{1}+\lambda_{2}$, and let $m$ be a positive integer. Then

$$
\begin{align*}
&\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)} \approx \sum_{|\lambda|<m}\left|\partial^{\lambda} h(\mathbf{0})\right|+\sum_{|\lambda|=m}\left\|\partial^{\lambda} h\right\|_{H_{p, \beta+m}\left(\Omega_{0}\right)},  \tag{5.1.21}\\
&\|h\|_{L_{\alpha, p}\left(\Omega_{0}\right)} \approx \sum_{|\lambda|<m}\left|\partial^{\lambda} h(\mathbf{0})\right|+\sum_{|\lambda|=m}\left\|\partial^{\lambda} h\right\|_{L_{p, \alpha+p m}\left(\Omega_{0}\right)} \tag{5.1.22}
\end{align*}
$$

for all $h \in \operatorname{Har}\left(\Omega_{0}\right)$. Further,

$$
\begin{align*}
\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)} & \approx|h(\mathbf{0})|+\|\nabla h\|_{H_{p, \beta+1}\left(\Omega_{0}\right)}  \tag{5.1.23}\\
\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)} & \approx|h(\mathbf{0})|+\|\nabla h\|_{L_{p, \alpha+p}\left(\Omega_{0}\right)} \tag{5.1.24}
\end{align*}
$$

The constants involved in the equivalence of the different norms depend on some or all of $p, \beta, \alpha, m$.

Proof. The proofs of (5.1.21) and (5.1.23) can be found in [24]. We omit the proofs of (5.1.22) and (5.1.24) because they follow the same lines, without essential change.

We are now in a position to establish the following general theorem, which provides criteria for the existence of an $\mathcal{A}$-valued function $\boldsymbol{f}$ in the space $\mathcal{H}_{p, \beta}\left(\Omega_{0}\right)$.

Theorem 5.1.12. Let $h \in \operatorname{Har}\left(\Omega_{0}\right)$ and let $w\left(x_{1}, x_{2}\right)$ be a solution of the equation

$$
\begin{equation*}
\Delta_{x_{1}, x_{2}} w=\frac{\partial^{2} h\left(0, x_{1}, x_{2}\right)}{\partial x_{0} \partial x_{1}} \tag{5.1.25}
\end{equation*}
$$

such that $h(a)$ is finite at some point $a=\left(a_{1}, a_{2}\right)$ with $a_{1}^{2}+a_{2}^{2}<1$. If $h \in H_{p, \beta}\left(\Omega_{0}\right)$ for some $\beta>0$ and $1<p<\infty$, then there exists a function $\boldsymbol{f}$ in $\mathcal{H}_{p, \beta}\left(\Omega_{0}\right)$ such that $\operatorname{Sc}(\boldsymbol{f})=h$ in $\Omega_{0}$, and a constant $C(p, \beta, a)>0$ such that

$$
\|\boldsymbol{f}\|_{\mathcal{H}_{p, \beta}\left(\Omega_{0}\right)} \leq C(p, \beta, a)\left(\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)}+|w(a)|\right)
$$

Proof. In the first place, we use Theorem 5.1.5 to construct a monogenic function $\boldsymbol{f}=h+\mathbf{i}[\boldsymbol{v}]_{1}+\mathbf{j}[\boldsymbol{v}]_{2}$ for a given real-valued harmonic function $h$, where the quaternionic components $[\boldsymbol{v}]_{1}$ and $[\boldsymbol{v}]_{2}$ can be found, respectively, by (5.1.11) and (5.1.12). For any point $\mathbf{x}=r \boldsymbol{\eta} \in \Omega_{0}$, by (5.1.11) we have then

$$
\begin{align*}
\left|[\boldsymbol{v}(\mathbf{x})]_{1}\right| & \leq\left|x_{0}\right| \int_{0}^{1}\left|\frac{\partial h\left(\rho x_{0}, x_{1}, x_{2}\right)}{\partial x_{1}}\right| d \rho+\left|w\left(x_{1}, x_{2}\right)\right| \\
& =: \widetilde{v_{1}}(\mathbf{x})+\left|w\left(x_{1}, x_{2}\right)\right| \tag{5.1.26}
\end{align*}
$$

Minkowski's inequality gives

$$
M_{p}\left(\widetilde{v_{1}} ; r\right) \leq \int_{0}^{1}\left(\int_{|\mathbf{x}|=r}\left|x_{0}\right|^{p}\left|\frac{\partial h\left(\rho x_{0}, x_{1}, x_{2}\right)}{\partial x_{1}}\right|^{p} d \sigma\right)^{1 / p} d \rho
$$

Denote by $u(\mathbf{y})$ the smallest harmonic majorant of the subharmonic function $\left|\frac{\partial h(\mathbf{y})}{\partial x_{1}}\right|^{p}$ in the ball $B(\sqrt{r})=\left\{\mathbf{x} \in \mathbb{R}^{3}:|\mathbf{x}|<\sqrt{r}\right\}$. The previous inequality
becomes

$$
\begin{aligned}
M_{p}\left(\widetilde{v_{1}} ; r\right) & \leq \int_{0}^{1}\left(\int_{|\mathbf{x}|=r}\left|x_{0}\right|^{p} u\left(\rho x_{0}, x_{1}, x_{2}\right) d \sigma\right)^{1 / p} d \rho \\
& \leq r \int_{0}^{1}\left(\int_{|\mathbf{x}|=r} u\left(\rho x_{0}, x_{1}, x_{2}\right) d \sigma\right)^{1 / p} d \rho \\
& =r \int_{0}^{1}\left(\int_{\partial \Omega_{\rho, r}} u(\mathbf{y}) d \sigma\right)^{1 / p} d \rho
\end{aligned}
$$

where the spheroid $\Omega_{\rho, r}$ was defined in 5.1.20).
Now, we write the Poisson integral representation of $u$ in $\Omega_{\rho, r} \subset B(\sqrt{r})$, and estimate it at the origin:

$$
u(\mathbf{x})=\int_{\partial \Omega_{\rho, r}} P_{\Omega_{\rho, r}}(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d \sigma .
$$

By Lemma 5.1.7, we have

$$
\begin{aligned}
u(\mathbf{0}) & =\int_{\partial \Omega_{\rho, r}} P_{\Omega_{\rho, r}}(\mathbf{0}, \mathbf{y}) u(\mathbf{y}) d \sigma \\
& \geq C \int_{\partial \Omega_{\rho, r}} \frac{\rho r}{|\mathbf{y}|^{3}} u(\mathbf{y}) d \sigma \\
& \geq C \frac{\rho}{r^{2}} \int_{\partial \Omega_{\rho, r}} u(\mathbf{y}) d \sigma .
\end{aligned}
$$

From this we find

$$
\begin{aligned}
M_{p}\left(\widetilde{v_{1}} ; r\right) & \leq r \int_{0}^{1}\left(\int_{\partial \Omega_{\rho, r}} u(\mathbf{y}) d \sigma\right)^{1 / p} d \rho \\
& \leq C r \int_{0}^{1}\left(\frac{r^{2}}{\rho} u(\mathbf{0})\right)^{1 / p} d \rho \\
& =C(p) r^{1+2 / p}(u(\mathbf{0}))^{1 / p} .
\end{aligned}
$$

By the mean-value property for harmonic functions and Lemma 5.1.10, it follows that

$$
\begin{align*}
M_{p}\left(\widetilde{v_{1}} ; r\right) & \leq C(p) r^{1+2 / p} \int_{0}^{1}\left(\frac{1}{\rho r} M_{1}(u ; \sqrt{\rho r})\right)^{1 / p} d \rho \\
& =C(p) r^{1+1 / p} \int_{0}^{1} \frac{1}{\rho^{1 / p}}\left(M_{1}(u ; \sqrt{\rho r})\right)^{1 / p} d \rho \\
& \leq C(p) r^{1+1 / p} \int_{0}^{1} \frac{1}{\rho^{1 / p}} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{\rho r}\right) d \rho \tag{5.1.27}
\end{align*}
$$

It follows from these considerations and, in particular from Lemma 5.1.8, that

$$
\begin{aligned}
M_{p}\left(\widetilde{v_{1}} ; r\right) & \leq C(p) r^{1+1 / p} \int_{0}^{1} \frac{(1-\sqrt{\rho r})^{\beta+1} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{\rho r}\right)}{\rho^{1 / p}(1-\sqrt{\rho r})^{\beta+1}} d \rho \\
& \leq C(p, \beta) r^{1+1 / p}\left\|\frac{\partial h}{\partial x_{1}}\right\|_{H_{p, \beta+1}\left(\Omega_{0}\right)} \int_{0}^{1} \frac{1}{\rho^{1 / p}(1-\rho r)^{\beta+1}} d \rho \\
& \leq \frac{C(p, \beta)}{(1-r)^{\beta}}\left\|\frac{\partial h}{\partial x_{1}}\right\|_{H_{p, \beta+1}\left(\Omega_{0}\right)} .
\end{aligned}
$$

We have then, in accordance with Lemma 5.1.11,

$$
\begin{align*}
(1-r)^{\beta} M_{p}\left(\widetilde{v_{1}} ; r\right) & \leq C\left\|\frac{\partial h}{\partial x_{1}}\right\|_{H_{p, \beta+1}\left(\Omega_{0}\right)} \\
& \leq C\|\nabla h\|_{H_{p, \beta+1}\left(\Omega_{0}\right)} \\
& \leq C\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)} \tag{5.1.28}
\end{align*}
$$

for all $r \in(0,1)$.
We proceed to estimate the last term of (5.1.26) by means of Lemma 5.1.9. It is known from [139] that the solution $w\left(x_{1}, x_{2}\right)$ of the Poisson equation (5.1.25) in $\mathcal{D}$ with vanishing boundary values on the unit circle is the

Green potential of $\frac{\partial^{2} h\left(0, x_{1}, x_{2}\right)}{\partial x_{0} \partial x_{1}}$. We split the function $\frac{\partial^{2} h\left(0, x_{1}, x_{2}\right)}{\partial x_{0} \partial x_{1}}$ into its positive and negative parts, namely $w=w^{+}-w^{-}$, where $w^{+}=\max \{w, 0\}$ and $w^{-}=\max \{-w, 0\}$ are nonnegative superharmonic functions in $\mathcal{D}$. By Lemma 5.1.9, it follows that $\left\|w^{ \pm}\right\|_{L_{p, p}(\mathcal{D})} \leq C(p, a) w^{ \pm}(a)$ for some constant $C(p, a)>0$.

Thus,

$$
\|w\|_{L_{p, p}(\mathcal{D})} \leq\left\|w^{+}\right\|_{L_{p, p}(\mathcal{D})}+\left\|w^{-}\right\|_{L_{p, p}(\mathcal{D})} \leq C(p, a)|w(a)| .
$$

Since the integral means of $w^{ \pm}, M_{p}\left(w^{ \pm} ; r\right)$, are decreasing functions of $r$, then

$$
\sup _{1 / 2 \leq r<1}(1-r)^{\beta} M_{p}\left(w^{ \pm} ; r\right) \leq C\left\|w^{ \pm}\right\|_{L_{p, p}(\mathcal{D})} \leq C(p, \beta, a) w^{ \pm}(a)
$$

for any $\beta>0$. Whence,

$$
\begin{equation*}
\sup _{1 / 2 \leq r<1}(1-r)^{\beta} M_{p}(w ; r) \leq C(p, \beta, a)|w(a)|, \tag{5.1.29}
\end{equation*}
$$

and this conclusion holds, not only for $w\left(x_{1}, x_{2}\right)$, but also for its extension in $\Omega_{0}$, namely $w(\mathbf{x})$. Accordingly, by (5.1.26), 5.1.28) and (5.1.29), we have

$$
\begin{align*}
\left\|[\boldsymbol{v}]_{1}\right\|_{H_{p, \beta}\left(\Omega_{0}\right)} & \leq C(p, \beta) \sup _{1 / 2 \leq r<1}(1-r)^{\beta} M_{p}\left([\boldsymbol{v}]_{1} ; r\right) \\
& \leq C(p, \beta, a)\left(\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)}+|w(a)|\right) . \tag{5.1.30}
\end{align*}
$$

Proceeding in a similar manner, we use (5.1.19) to estimate the quaternionic component $[\boldsymbol{v}]_{2}$ :

$$
\begin{aligned}
\left|[\boldsymbol{v}(\mathbf{x})]_{2}\right| \leq & \int_{0}^{1}\left(\left|x_{0}\right|\left|\frac{\partial h(t \mathbf{x})}{\partial x_{2}}\right|+\left|x_{2}\right|\left|\frac{\partial h(t \mathbf{x})}{\partial x_{0}}\right|+\left|x_{1}\right|\left|\frac{\partial[\boldsymbol{v}(t \mathbf{x})]_{1}}{\partial x_{2}}\right|\right. \\
& \left.+\left|x_{2}\right|\left|\frac{\partial[\boldsymbol{v}(t \mathbf{x})]_{1}}{\partial x_{1}}\right|\right) d t \\
\leq & \sqrt{2} \int_{0}^{1}\left(|\nabla h(t \mathbf{x})|+\left|\nabla[\boldsymbol{v}(t \mathbf{x})]_{1}\right|\right) d t .
\end{aligned}
$$

Now, we use the function $[\boldsymbol{v}]_{2}$ and Minkowski's inequality as above, and obtain

$$
\begin{equation*}
M_{p}\left([\boldsymbol{v}]_{2} ; r\right) \leq C \int_{0}^{1} M_{p}(\nabla h ; t r) d t+C \int_{0}^{1} M_{p}\left(\nabla[\boldsymbol{v}]_{1} ; t r\right) d t \tag{5.1.31}
\end{equation*}
$$

It follows immediately from these estimates that

$$
\begin{aligned}
M_{p}\left([\boldsymbol{v}]_{2} ; r\right) \leq & C \sup _{0<t<1}(1-t r)^{\beta+1} M_{p}(\nabla h ; t r) \int_{0}^{1} \frac{1}{(1-t r)^{\beta+1}} d t \\
& +C \sup _{0<t<1}(1-t r)^{\beta+1} M_{p}\left(\nabla[\boldsymbol{v}]_{1} ; t r\right) \int_{0}^{1} \frac{1}{(1-t r)^{\beta+1}} d t \\
\leq & C(1-r)^{-\beta}\left(\|\nabla h\|_{H_{p, \beta+1}\left(\Omega_{0}\right)}+\left\|\nabla[\boldsymbol{v}]_{1}\right\|_{H_{p, \beta+1}\left(\Omega_{0}\right)}\right) .
\end{aligned}
$$

We observe now that, by Lemma 5.1.11 and 5.1.30, we have

$$
\begin{aligned}
\left\|[\boldsymbol{v}]_{2}\right\|_{H_{p, \beta}\left(\Omega_{0}\right)} & \leq C\|\nabla h\|_{H_{p, \beta+1}\left(\Omega_{0}\right)}+C\left\|\nabla[\boldsymbol{v}]_{1}\right\|_{H_{p, \beta+1}\left(\Omega_{0}\right)} \\
& \leq C\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)}+C\left\|[\boldsymbol{v}]_{1}\right\|_{H_{p, \beta}\left(\Omega_{0}\right)} \\
& \leq C(p, \beta, a)\left(\|h\|_{H_{p, \beta}\left(\Omega_{0}\right)}+|w(a)|\right)
\end{aligned}
$$

and therefore the theorem has been established.
The question of whether there is a similar characterization for monogenic weighted Bergman spaces will be considered in the next section.

### 5.1.2 Harmonic conjugates in monogenic weighted Bergman spaces

From the point of view adopted in Theorem 5.1.12, we proceed to prove a similar result for the monogenic weighted Bergman spaces $\mathcal{B}_{p, \alpha}\left(\Omega_{0}\right)$ in the range $\alpha>-1$. The proof of this result is based on Theorem 5.1.5, along with some well-known inequalities.

With a view to the extension of the general theorem, the following lemma will be required:

Lemma 5.1.13 (see [121]). Let $1 \leq p<\infty, \gamma<-1<\alpha$ and let $h(\rho)>0$ for $\rho \in(0,1)$. Then there exists a constant $C(p, \alpha, \gamma)>0$ such that

$$
\int_{0}^{1}(1-\rho)^{\alpha} \rho^{\gamma}\left(\int_{0}^{\rho} h(t) d t\right)^{p} d \rho \leq C(p, \alpha, \gamma) \int_{0}^{1}(1-\rho)^{\alpha+p} \rho^{\gamma+p} h^{p}(\rho) d \rho .
$$

It now remains for us to present a weighted Bergman space estimate; and this required estimate is founded upon the following lemma:
Lemma 5.1.14. If $1 \leq p<\infty$ and $\gamma<-1<\alpha$, then there exists a constant $C(p, \alpha, \gamma)>0$ such that

$$
\left(\int_{0}^{1}(1-\rho)^{\alpha}\left(M_{p}(h ; \rho)\right)^{p} \rho^{\gamma} d \rho\right)^{1 / p} \leq C(p, \alpha, \gamma)\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)}
$$

for all $h \in \operatorname{Har}\left(\Omega_{0}\right)$.

Proof. The result follows in a straightforward way from the subharmonicity of $|h|^{p}$ and the monotonicity of the integral $M_{p}(h ; \rho)$ with respect to $\rho \in$ $[0,1)$.

The following theorem will now be established:
Theorem 5.1.15. Let $h \in \operatorname{Har}\left(\Omega_{0}\right)$ and let $w\left(x_{1}, x_{2}\right)$ be a solution of (5.1.25) such that $w(a)$ is finite at some point $a=\left(a_{1}, a_{2}\right)$ with $a_{1}^{2}+a_{2}^{2}<1$. If $h \in B_{p, \alpha}\left(\Omega_{0}\right)$ for some $\alpha>-1$ and $1<p<\infty$, then there exists a function $\boldsymbol{f}$ in $\mathcal{B}_{p, \alpha}\left(\Omega_{0}\right)$ such that $\operatorname{Sc}(\boldsymbol{f})=h$ in $\Omega_{0}$, and a constant $C(p, \alpha, a)>0$ such that

$$
\|\boldsymbol{f}\|_{L_{p, \alpha}\left(\Omega_{0}\right)} \leq C(p, \alpha, a)\left(\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)}+|w(a)|\right) .
$$

Proof. As in Theorem 5.1.12, this is proved at once by applying Theorem 5.1.5 to the construction of a monogenic function $\boldsymbol{f}=h+\mathbf{i}[\boldsymbol{v}]_{1}+\mathbf{j}[\boldsymbol{v}]_{2}$ for a given real-valued harmonic function $h$, where the quaternionic components $[\boldsymbol{v}]_{1},[\boldsymbol{v}]_{2}$ can be obtained, respectively, by (5.1.11) and (5.1.12). We proceed to estimate $\left|[\boldsymbol{v}(\mathbf{x})]_{1}\right|$ as in (5.1.26). By (5.1.27), we have then

$$
\begin{aligned}
M_{p}\left(\widetilde{v_{1}} ; r\right) & \leq C_{p} r^{1+1 / p} \int_{0}^{1} \frac{1}{\rho^{1 / p}} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{\rho r}\right) d \rho \\
& =C_{p} r^{2 / p} \int_{0}^{r} \frac{1}{t^{1 / p}} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{t}\right) d t
\end{aligned}
$$

where

$$
\widetilde{v_{1}}(\mathbf{x})=\left|x_{0}\right| \int_{0}^{1}\left|\frac{\partial h\left(\rho x_{0}, x_{1}, x_{2}\right)}{\partial x_{1}}\right| d \rho .
$$

Raising both sides of the above expression to the power of $p$, integrating, and using Lemma 5.1.14 we find

$$
\begin{aligned}
\left\|\widetilde{v_{1}}\right\|_{L_{p, \alpha}\left(\Omega_{0}\right)}^{p} & \leq C \int_{0}^{1}(1-r)^{\alpha}\left[M_{p}\left(\widetilde{v_{1}} ; r\right)\right]^{p} d r \\
& \leq C \int_{0}^{1}(1-r)^{\alpha} r^{2}\left[\int_{0}^{r} t^{-1 / p} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{t}\right) d t\right]^{p} d r \\
& \leq C \int_{0}^{1}(1-r)^{\alpha} r^{-1-\delta}\left[\int_{0}^{r} t^{-1 / p} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{t}\right) d t\right]^{p} d r
\end{aligned}
$$

where the parameter $\delta>0$ can be chosen arbitrarily. In order to apply the Hardy inequality of Lemma 5.1.13, we choose $\delta=(p-1) / 2$. Thus, by Lemma
5.1.14 it follows that

$$
\begin{aligned}
\left\|\widetilde{v_{1}}\right\|_{L_{p, \alpha}\left(\Omega_{0}\right)}^{p} & \leq C \int_{0}^{1}(1-r)^{\alpha+p} r^{p-1-\delta}\left[r^{-1 / p} M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{r}\right)\right]^{p} d r \\
& =C \int_{0}^{1}(1-r)^{\alpha+p} r^{(p-1) / 2-1}\left[M_{p}\left(\frac{\partial h}{\partial x_{1}} ; \sqrt{r}\right)\right]^{p} d r \\
& \leq C(p, \alpha)\left\|\frac{\partial h}{\partial x_{1}}\right\|_{L_{p, \alpha+p}\left(\Omega_{0}\right)}^{p}
\end{aligned}
$$

By Lemma 5.1.11, we also have

$$
\begin{equation*}
\|\widetilde{v}\|_{L_{p, \alpha}\left(\Omega_{0}\right)} \leq C(p, \alpha)\|\nabla h\|_{L_{p, \alpha+p}\left(\Omega_{0}\right)} \leq C(p, \alpha)\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)} \tag{5.1.32}
\end{equation*}
$$

Next, we estimate the term $w\left(x_{1}, x_{2}\right)$ in (5.1.26) as in Theorem 5.1.12. Making use of Lemma 5.1.9, we have $\left\|w^{ \pm}\right\|_{L_{p, p}(\mathcal{D})} \leq C(p, a) w^{ \pm}(a)$ for nonnegative superharmonic functions $w^{ \pm}$.

Since the integral means $M_{p}\left(w^{ \pm} ; r\right)$ of the superharmonic functions $w^{ \pm}$ are decreasing functions of $r$, we thus obtain

$$
\begin{aligned}
\left\|w^{ \pm}\right\|_{L_{p, p}(\mathcal{D})} & \geq\left[2 \int_{0}^{r}(1-t)^{p}\left(M_{p}\left(w^{ \pm} ; t\right)\right)^{p} t d t\right]^{1 / p} \\
& \geq C_{p} M_{p}\left(w^{ \pm} ; r\right)
\end{aligned}
$$

for all $r \in[1 / 2,1)$. For any $\alpha>-1$, it then follows that

$$
\begin{align*}
{\left[\int_{1 / 2}^{1}(1-r)^{\alpha}\left(M_{p}\left(w^{ \pm} ; r\right)\right)^{p} r d r\right]^{1 / p} } & \leq C(p, \alpha)\left\|w^{ \pm}\right\|_{L_{p, p}(\mathcal{D})} \\
& \leq C(p, \alpha, a) w^{ \pm}(a) \tag{5.1.33}
\end{align*}
$$

We note in passing that inequality (5.1.33) remains valid for the extension $w(\mathbf{x})$ of $W\left(x_{1}, x_{2}\right)$ in $\Omega_{0}$. Accordingly, by (5.1.32) and 5.1.33), we have

$$
\begin{align*}
\left\|[\boldsymbol{v}]_{1}\right\|_{L_{p, \alpha}\left(\Omega_{0}\right)} & \leq C(p, \alpha)\left[\int_{1 / 2}^{1}(1-r)^{\alpha}\left(M_{p}\left([\boldsymbol{v}]_{1} ; r\right)\right)^{p} d r\right]^{1 / p} \\
& \leq C(p, \alpha, a)\left(\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)}+|w(a)|\right) \tag{5.1.34}
\end{align*}
$$

We proceed to estimate the quaternionic component $[\boldsymbol{v}]_{2}$. By (5.1.31) and (5.1.34), and using Lemma 5.1.11, we deduce also

$$
M_{p}\left([\boldsymbol{v}]_{2} ; r\right) \leq C \int_{0}^{1} M_{p}(\nabla h ; t r) d t+C \int_{0}^{1} M_{p}\left(\nabla[\boldsymbol{v}]_{1} ; t r\right) d t
$$

Hence

$$
\begin{aligned}
\left\|[\boldsymbol{v}]_{2}\right\|_{L_{p, \alpha}\left(\Omega_{0}\right)} & \leq C\|\nabla h\|_{L_{p, \alpha+p}\left(\Omega_{0}\right)}+C\left\|\nabla[\boldsymbol{v}]_{1}\right\|_{L_{p, \alpha+p}\left(\Omega_{0}\right)} \\
& \leq C\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)}+C\left\|[\boldsymbol{v}]_{1}\right\|_{L_{p, \alpha}\left(\Omega_{0}\right)} \\
& \leq C(p, \alpha, a)\left(\|h\|_{L_{p, \alpha}\left(\Omega_{0}\right)}+|w(a)|\right),
\end{aligned}
$$

and thus the theorem is established.

### 5.2 A Bloch-type theorem for monogenic functions

Bloch's classical theorem, being a traditional object of analysis, occupies a special place in the geometric theory of holomorphic functions. It asserts that if $f$ is a holomorphic function on a region that contains the closure of the unit disk centered at the origin such that $f(0)=0$ and $\left|f^{\prime}(0)\right|=1$, then the image domain contains a disk of radius $1 / 72$. The optimal value is known as Bloch's constant, and $1 / 72$ is not the best possible. The original proof of Bloch [37] depends on the Wiman theory of the comparison of two power series involving integral functions [331]. A treatise was published independently by Landau and Valiron [197], in which Bloch's arguments were considerably simplified. There are many other proofs of Bloch's Theorem, including works by Landau [196], Carathéodory [74], Heins [162], Pommerenke [272], Estermann [111], Ahlfors [15], and Remmert [277].

In the present section, we present a generalization of Bloch's Theorem for $\mathcal{A}$-monogenic functions defined in the unit ball of the Euclidean space $\mathbb{R}^{3}$. We also give an explicit lower bound for the Bloch constant.

### 5.2.1 Estimates for monogenic functions bounded with respect to their hypercomplex derivatives

Following the results obtained in Chapter 3, relating to solid spherical monogenics, we begin by finding estimates for the Fourier coefficients of an $\mathcal{A}$ valued monogenic function expanded in series of solid spherical monogenics by the growth of the maximum of the modulus of its hypercomplex derivative. Let us introduce some notation, which we shall use in the sequel: let $\mathcal{M}(\boldsymbol{f} ; \rho):=\max _{B_{\rho}}|\boldsymbol{f}(\mathbf{x})|$ for all $0 \leq|\mathbf{x}| \leq \rho$ denote the maximum modulus of $\boldsymbol{f}$. In view of Definition 1.3.14, we shall now prove a useful upper bound estimate for $\mathcal{M}(\mathcal{P}[(1 / 2) \partial \boldsymbol{f}(\mathbf{x})-(1 / 2) \partial \boldsymbol{f}(\mathbf{0})] ; \rho)$ in terms of the $C$-norm of $(1 / 2) \partial \boldsymbol{f}(\mathbf{x})-(1 / 2) \partial \boldsymbol{f}(\mathbf{0})$.

Lemma 5.2.1. Let $\boldsymbol{f} \in \mathcal{M}_{2}\left(B_{\rho}\right)$ be such that $\boldsymbol{f}(\mathbf{0})=\mathbf{0}$. Then,

$$
\begin{aligned}
& \left|\mathcal{P}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right]\right| \\
& \leq \frac{2}{\sqrt{3}} \frac{|\mathbf{x}|^{2}\left(4|\mathbf{x}|^{2}+9 \rho^{2}-11|\mathbf{x}| \rho\right)}{(\rho-|\mathbf{x}|)^{3}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) ; \rho\right)
\end{aligned}
$$

for all $0 \leq|\mathbf{x}|<\rho$.
Proof. Bearing in mind that the orthogonality and completeness of the set $\left\{\mathbf{X}_{l, m}^{ \pm}[0]: m=0, \ldots, l+1 ; l=0,1, \ldots\right\}$ in $\mathcal{M}_{2}\left(\Omega_{0}\right)$ implies its orthogonality and completeness in $\mathcal{M}_{2}\left(B_{\rho}\right)$, so that

$$
\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}^{2}=\frac{\rho^{2 l+3}}{2 l+3}\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(\Omega_{0}\right)}^{2}
$$

for all $l=0,1, \ldots$, we can consider the Fourier expansion of a function $\boldsymbol{f}$ in $\mathcal{M}_{2}\left(B_{\rho}\right)$ :

$$
\begin{equation*}
\boldsymbol{f}=\sum_{l=0}^{\infty} \sum_{m=0}^{l+1}\left(\tilde{a}_{l, m}^{+}[0] \frac{\mathbf{X}_{l, m}^{+}[0]}{\left\|\mathbf{X}_{l, m}^{+}[0]\right\|_{L_{2}\left(B_{\rho}\right)}}+\widetilde{a}_{l, m}^{-}[0] \frac{\mathbf{X}_{l, m}^{-}[0]}{\left\|\mathbf{X}_{l, m}^{-}[0]\right\|_{L_{2}\left(B_{\rho}\right)}}\right), \tag{5.2.1}
\end{equation*}
$$

where $\widetilde{a}_{l, m}^{ \pm}[0]=\left(1 /\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}\right)\left\langle\boldsymbol{f}, \mathbf{X}_{l, m}^{ \pm}[0]\right\rangle_{0, L_{2}\left(B_{\rho}, \mathcal{A}\right)}$. Since the prescribed monogenic polynomials $\mathbf{X}_{l, m}^{ \pm}[0]$ are homogeneous, the assumption $\boldsymbol{f}(\mathbf{0})=\mathbf{0}$ yields to $\widetilde{a}_{0,0}^{+}[0]=\widetilde{a}_{0,1}^{+}[0]=\widetilde{a}_{0,1}^{-}[0]=0$. Moreover, since the series (5.2.1) converges in $L_{2}\left(B_{\rho}\right)$, it follows that it converges uniformly to $\boldsymbol{f}$ in each compact subset of $B_{\rho}$; and also, the series of all partial derivatives converges uniformly to the corresponding partial derivatives of $\boldsymbol{f}$ in compact subsets of $B_{\rho}$. We proceed to apply the hypercomplex derivative of the series (5.2.1) term-by-term, and subtract the quantity of $(1 / 2) \partial \boldsymbol{f}(\mathbf{0})$. Thus, by (3.1.18), we have

$$
\begin{align*}
& \left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) \\
& =\sum_{l=2}^{\infty} \sum_{m=0}^{l} \frac{(l+1+m)}{\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}}\left(\tilde{a}_{l, m}^{+}[0] \mathbf{X}_{l-1, m}^{+}[0]+\tilde{a}_{l, m}^{-}[0] \mathbf{X}_{l-1, m}^{-}[0]\right) . \tag{5.2.2}
\end{align*}
$$

We consequently apply the primitive operator $\mathcal{P}$ of the above series term-byterm, and by (3.1.22), we see that

$$
\mathcal{P}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right]=\sum_{l=2}^{\infty} \sum_{m=0}^{l} \frac{\left(\tilde{a}_{l, m}^{+}[0] \mathbf{X}_{l, m}^{+}[0]+\widetilde{a}_{l, m}^{-}[0] \mathbf{X}_{l, m}^{-}[0]\right)}{\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}} .
$$

### 5.2. A BLOCH-TYPE THEOREM FOR MONOGENIC FUNCTIONS 215

Now, we investigate the relationships between the Fourier coefficients $\widetilde{a}_{l, m}^{ \pm}[0]$ and the $C$-norm of " $(1 / 2) \partial \boldsymbol{f}(\mathbf{x})-(1 / 2) \partial \boldsymbol{f}(\mathbf{0})$ ". Multiply both sides of (5.2.2) by the orthogonal set $\left\{\mathbf{X}_{l-1, m}^{ \pm}[0]: m=0, \ldots, l ; l=1,2, \ldots\right\}$ and integrate over $B_{\rho}$. It then follows that

$$
\begin{aligned}
\tilde{a}_{l, m}^{ \pm}[0]= & \frac{1}{(l+1+m)} \frac{\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}}{\left\|\mathbf{X}_{l-1, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}^{2}} \\
& \times \int_{B_{\rho}}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right] \mathbf{X}_{l-1, m}^{ \pm}[0] d \mathbf{x}
\end{aligned}
$$

for each $l \geq 2$ and $m=0, \ldots, l$.
In view of the relations just made, it is clear that the Fourier coefficients in (5.2.1) can now be determined from a scalar inner product between " $(1 / 2) \partial \boldsymbol{f}(\mathbf{x})-(1 / 2) \partial \boldsymbol{f}(\mathbf{0})$ " and each element on the foregoing set, consisting of $2 l+1$ polynomials. Accordingly, using the following pointwise estimates proved in [240],

$$
\begin{equation*}
\left|\mathbf{X}_{l, m}^{ \pm}[0](\mathbf{x})\right| \leq \frac{1}{2}(l+1) \sqrt{\frac{(l+1+m)!}{(l+1-m)!}}|\mathbf{x}|^{l} \tag{5.2.3}
\end{equation*}
$$

for all $l \geq 0$ and $m=0, \ldots, l+1$, we then have

$$
\begin{aligned}
\left|\widetilde{a}_{l, m}^{ \pm}[0]\right| \leq & \sqrt{\frac{4 \pi \rho^{3}}{3}} \frac{\left\|\mathbf{X}_{l, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}}{\left\|\mathbf{X}_{l-1, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}} \\
& \times \frac{1}{(l+1+m)} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) ; \rho\right) .
\end{aligned}
$$

It thus appears, from the estimates of the Fourier coefficients obtained so far, that

$$
\begin{aligned}
\left|\mathcal{P}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right]\right| \leq & \sqrt{\frac{4 \pi \rho^{3}}{3}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) ; \rho\right) \\
& \times \sum_{l=2}^{\infty} \sum_{m=0}^{l} \frac{\left(\left|\mathbf{X}_{l, m}^{+}[0]\right|+\left|\mathbf{X}_{l, m}^{-}[0]\right|\right)}{(l+1+m)\left\|\mathbf{X}_{l-1, m}^{ \pm}[0]\right\|_{L_{2}\left(B_{\rho}\right)}} .
\end{aligned}
$$

Finally, using again (5.2.3), a straightforward computation shows that

$$
\begin{aligned}
& \left|\mathcal{P}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right]\right| \\
& \leq \sqrt{\frac{2}{3} \rho^{3}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) ; \rho\right) \sum_{l=2}^{\infty} \frac{|\mathbf{x}|^{l}}{\sqrt{\rho^{2 l+1}}}(1+2 l \sqrt{l+1}) \sqrt{1+\frac{1}{l}} \\
& \leq \frac{2 \rho}{\sqrt{3}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) ; \rho\right) \sum_{l=2}^{\infty}\left(\frac{|\mathbf{x}|}{\rho}\right)^{l}(l+1)^{2} .
\end{aligned}
$$

Thus the lemma is proved.
We proceed further to prove an estimate for $\mathcal{M}((1 / 2) \partial \boldsymbol{f}(\mathbf{x})-(1 / 2) \partial \boldsymbol{f}(\mathbf{0}) ; \rho)$ in terms of the $C$-norm of $(1 / 2) \partial \boldsymbol{f}(\mathbf{x})$. The following lemma will then be established:

Lemma 5.2.2. Let $\boldsymbol{f} \in \mathcal{M}_{2}\left(B_{\rho}\right)$. Then

$$
\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right| \leq \frac{6|\mathbf{x}| \rho}{(\rho-|\mathbf{x}|)^{2}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right)
$$

for all $0 \leq|\mathbf{x}|<\rho$.
Proof. By the Cauchy integral formula 1.3.6, it follows that

$$
\boldsymbol{f}(\mathbf{x})-\boldsymbol{f}(\mathbf{0})=\frac{1}{4 \pi} \int_{\partial B_{\rho}}[\boldsymbol{q}(\mathbf{x}-\mathbf{y})-\boldsymbol{q}(-\mathbf{y})] \mathbf{n}(\mathbf{y}) \boldsymbol{f}(\mathbf{y}) d \sigma(\mathbf{y})
$$

where the following inequality holds [145, p. 50]:

$$
|\boldsymbol{q}(\mathbf{x}-\mathbf{y})-\boldsymbol{q}(-\mathbf{y})| \leq \frac{|\mathbf{x}-\mathbf{y}|\left(|\mathbf{y}|^{2}+|\mathbf{y}||\mathbf{x}-\mathbf{y}|+2|\mathbf{x}-\mathbf{y}|^{2}\right)|\mathbf{x}|}{|\mathbf{x}-\mathbf{y}|^{3}|\mathbf{y}|^{3}}
$$

Thus, we have

$$
\begin{aligned}
& \left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right| \\
& =\frac{1}{4 \pi}\left|\int_{\partial B_{\rho}}[\boldsymbol{q}(\mathbf{x}-\mathbf{y})-\boldsymbol{q}(-\mathbf{y})] \mathbf{n}(\mathbf{y})\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{y}) d \sigma(\mathbf{y})\right| \\
& \leq \frac{1}{4 \pi} \int_{\partial B_{\rho}} \frac{|\mathbf{x}-\mathbf{y}|\left(|\mathbf{y}|^{2}+|\mathbf{y}||\mathbf{x}-\mathbf{y}|+2|\mathbf{x}-\mathbf{y}|^{2}\right)|\mathbf{x}|}{|\mathbf{x}-\mathbf{y}|^{3}|\mathbf{y}|^{3}}\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{y})\right| d \sigma(\mathbf{y}) \\
& \leq \frac{1}{4 \pi}|\mathbf{x}| \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right)\left[\int_{\partial B_{\rho}}\left(\frac{1}{\rho} \frac{1}{|\mathbf{x}-\mathbf{y}|^{2}}+\frac{1}{\rho^{2}} \frac{1}{|\mathbf{x}-\mathbf{y}|}+\frac{2}{\rho^{3}}\right)\right] d \sigma(\mathbf{y}) \\
& \leq \frac{|\mathbf{x}|}{(\rho-|\mathbf{x}|)^{2}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right)\left(2 \rho-|\mathbf{x}|+\frac{2}{\rho}(\rho-|\mathbf{x}|)^{2}\right) \\
& \leq \frac{|\mathbf{x}|}{(\rho-|\mathbf{x}|)^{2}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right)\left(4 \rho+\frac{2|\mathbf{x}|^{2}}{\rho}-5|\mathbf{x}|\right) \\
& \leq \frac{|\mathbf{x}|}{\left(\rho-|\mathbf{x}|^{2}\right.} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right)(6 \rho-5|\mathbf{x}|) \\
& \leq \frac{6|\mathbf{x}| \rho}{(\rho-|\mathbf{x}|)^{2}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right)
\end{aligned}
$$

which furnishes the required estimate.

### 5.2. A BLOCH-TYPE THEOREM FOR MONOGENIC FUNCTIONS 217

### 5.2.2 The Bloch Theorem

We shall, in this section, state and prove a quaternionic version of Bloch's Theorem. It will be asserted that if $\boldsymbol{f}$ is an $\mathcal{A}$-valued square-integrable and monogenic function on a region containing the closure of the unit ball in $\mathbb{R}^{3}$ and such that its hypercomplex derivative is normalized to 1 at the origin, then there is an open subset of the unit ball on which $\boldsymbol{f}$ maps one-to-one onto some ball of radius at least $R$. It will be observed that the conclusion of the theorem holds with $R$ equal to $1 / 120-(31096 \sqrt{3}) / 20511149>1 / 150$. It will further be seen that the value of $R$ does not depend on the choice of $\boldsymbol{f}$. Although the assumption is inherently differential geometric, the proof techniques remain purely theoretical.

The present will be a convenient opportunity for studying the monotony of the following real-valued function:

$$
\begin{equation*}
g(r)=\frac{r}{2}-8 \sqrt{3} \frac{r^{3} \rho\left(4 r^{2}+9 \rho^{2}-11 r \rho\right)}{(\rho-r)^{5}} \tag{5.2.4}
\end{equation*}
$$

for all $r \in(0, \rho)$. It can be easily proved that $g$ is an increasing function near the point $r=0$, and if $r$ approaches $\rho$ then $g$ is decreasing. It follows at once that $g^{\prime}(\rho / 30)>0$ and $g^{\prime}(\rho / 20)<0$. It can be seen further that $g^{\prime \prime}$ has only one real zero and this zero must be positive, which means that $g^{\prime \prime}(r)<0$ in $(0, \rho)$. Accordingly, $\rho / 30$ estimates from below the only real zero of $g^{\prime}(r)$ in $(0, \rho)$; and thus,

$$
g\left(\frac{\rho}{30}\right)=\left(\frac{1}{60}-\frac{62192}{20511149} \sqrt{3}\right) \rho
$$

is a lower estimate for the maximum of $g$ in $(0, \rho)$. We have thus obtained the following lemma:

Lemma 5.2.3. Let $g$ be defined by (5.2.4) for all $r \in(0, \rho)$. Then $g$ has only one maximum in $(0, \rho)$ at $r=r_{\max }$. Further, $g\left(r_{\max }\right)>\left(\frac{1}{60}-\frac{62192}{20511149} \sqrt{3}\right) \rho$ and $r_{\max }>\rho / 30$.

The following proposition follows at once from Lemmas 5.2.1, 5.2.2 and 5.2.3:

Proposition 5.2.4. Let $\boldsymbol{f} \in \mathcal{M}_{2}\left(\bar{B}_{\rho}\right)$ be nonconstant and satisfy the normalization $\mathcal{M}((1 / 2) \partial \boldsymbol{f}(\mathbf{x}), \rho) \leq 2|(1 / 2) \partial \boldsymbol{f}(\mathbf{a})|$ for $\mathbf{a} \in B_{\rho}$. Then the image domain contains a ball of radius $\left(\frac{1}{60}-\frac{62192}{20511149} \sqrt{3}\right) \rho|(1 / 2) \partial \boldsymbol{f}(\mathbf{a})|$. (We note that $1 / 60-(62192 \sqrt{3}) / 20511149>1 / 75)$.

Proof. We may assume for the moment that $\mathbf{a}=\boldsymbol{f}(\mathbf{a})=\mathbf{0}$. The result is trivial if $(1 / 2) \partial \boldsymbol{f}(\mathbf{0})=\mathbf{0}$, so it may be assumed that $(1 / 2) \partial \boldsymbol{f}(\mathbf{0}) \neq \mathbf{0}$. By (3.1.22), we thus have

$$
\begin{aligned}
\mathcal{P}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right] & =\boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) \frac{1}{2} \mathbf{X}_{1,0}^{+}[0](\mathbf{x}) \\
& =: \mathbf{A}(\mathbf{x}),
\end{aligned}
$$

where $(1 / 2) \partial \boldsymbol{f}(\mathbf{0})(1 / 2) \mathbf{X}_{1,0}^{+}[0](\mathbf{x})$ denotes the linear term from the corresponding (left-sided) Taylor series of $\boldsymbol{f}$. In accordance with Lemmas 5.2.1 and 5.2.2, a simple computation shows that

$$
\begin{align*}
|\mathbf{A}(\mathbf{x})| & =\left|\mathcal{P}\left[\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right]\right| \\
& \leq \frac{12}{\sqrt{3}} \frac{|\mathbf{x}|^{3} \rho\left(4|\mathbf{x}|^{2}+9 \rho^{2}-11|\mathbf{x}| \rho\right)}{(\rho-|\mathbf{x}|)^{5}} \mathcal{M}\left(\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x}) ; \rho\right) . \tag{5.2.5}
\end{align*}
$$

Let $r \in(0, \rho)$. Since $\mathbf{X}_{1,0}^{+}[0](\mathbf{x})=2 x_{0}+\mathbf{i} x_{1}+\mathbf{j} x_{2}$, it then follows that

$$
\left|\boldsymbol{f}(\mathbf{x})-\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0}) \frac{1}{2} \mathbf{X}_{1,0}^{+}[0](\mathbf{x})\right| \geq \frac{r}{2}\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right|-|\boldsymbol{f}(\mathbf{x})|,
$$

which holds for all $\mathbf{x}$ with $|\mathbf{x}|=r$. In accordance with (5.2.5), the assumption $\mathcal{M}((1 / 2) \partial \boldsymbol{f}(\mathbf{x}), \rho) \leq 2|(1 / 2) \partial \boldsymbol{f}(\mathbf{0})|$ yields to

$$
|\boldsymbol{f}(\mathbf{x})| \geq\left(\frac{r}{2}-8 \sqrt{3} \frac{r^{3} \rho\left(4 r^{2}+9 \rho^{2}-11 r \rho\right)}{(\rho-r)^{5}}\right)\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right|
$$

By translation of the corresponding Taylor series of $\boldsymbol{f}$ around the origin, we can extend the above arguments to an arbitrary point $\mathbf{a} \in B_{\rho}$. Thus, by Lemma 5.2.3, we have

$$
|\boldsymbol{f}(\mathrm{x})-\boldsymbol{f}(\mathbf{a})| \geq\left(\frac{1}{60}-\frac{62192}{20511149} \sqrt{3}\right) \rho\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{a})\right|
$$

for all $|\mathbf{x}|=\rho / 30$. The result follows.
The extension of Bloch's Theorem will now be established:
Theorem 5.2.5. If $\boldsymbol{f} \in \mathcal{M}_{2}\left(\bar{\Omega}_{0}\right)$ is nonconstant, then its image domain contains a ball of radius

$$
\left(\frac{1}{120}-\frac{31096}{20511149} \sqrt{3}\right) \mathcal{M}\left(\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})\right|(1-|\mathbf{x}|) ; 1\right)>\frac{1}{150}\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{0})\right| .
$$

### 5.2. A BLOCH-TYPE THEOREM FOR MONOGENIC FUNCTIONS 219

Proof. To every $\boldsymbol{f}$ in $\mathcal{M}_{2}\left(\bar{\Omega}_{0}\right)$, we assign the function $|(1 / 2) \partial \boldsymbol{f}(\mathbf{x})|(1-|\mathbf{x}|)$, which is continuous on $\bar{\Omega}_{0}$. It assumes its maximum at a point $\mathbf{y} \in \Omega_{0}$. With $t:=(1-|\mathbf{y}|) / 2$, we have

$$
\mathcal{M}\left(\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{x})\right|(1-|\mathbf{x}|), 1\right)=2 t\left|\left(\frac{1}{2} \partial\right) \boldsymbol{f}(\mathbf{y})\right|
$$

for $B_{t}(\mathbf{y}) \subset \Omega_{0}$. We have also $1-|\mathbf{x}| \geq t$ for $\mathbf{x} \in B_{t}(\mathbf{y})$. Next, from $|(1 / 2) \partial \boldsymbol{f}(\mathbf{x})|(1-|\mathbf{x}|) \leq 2 t|(1 / 2) \partial \boldsymbol{f}(\mathbf{y})|$ we then deduce that $|(1 / 2) \partial \boldsymbol{f}(\mathbf{x})| \leq$ $2|(1 / 2) \partial \boldsymbol{f}(\mathbf{y})|$ for all $\mathbf{x} \in B_{t}(\mathbf{y})$. Thus, by Proposition 5.2.4. we see that the image domain of $\boldsymbol{f}$ contains a ball about $\boldsymbol{f}(\mathbf{y})$ of radius

$$
R=\left(\frac{1}{60}-\frac{62192}{20511149} \sqrt{3}\right) t|(1 / 2) \partial \boldsymbol{f}(\mathbf{y})|
$$

The quaternionic version of Bloch's Theorem is further contained in the following:

Theorem 5.2.6. Let $\boldsymbol{f} \in \mathcal{M}_{2}\left(\bar{\Omega}_{0}\right)$ be nonconstant and satisfy the normalization $\mid(1 / 2) \partial) \boldsymbol{f}(\mathbf{0}) \mid=1$. Then the image domain contains a ball of radius $1 / 120-(31096 \sqrt{3}) / 20511149>1 / 150$.

Further, it remains to investigate the best possible value of the Bohr radius for which Bloch's Theorem still holds.

## 6

## Conclusions and Suggestions for Further Study

In the present thesis, a quaternionic function theory related to spheroidal functions was developed in two separate contexts. In the first part, distinct orthogonal bases were constructed for the spaces of harmonic, monogenic, and contragenic functions defined in spheroids, particularly prolates and oblates. For spheroidal domains of arbitrary eccentricity, formulas that relate systems of harmonic, monogenic, and contragenic functions from one domain to another were described. Correspondingly, it was shown that there are common contragenic functions for spheroids of any eccentricity. Various applications associated with the prescribed spheroidal harmonics and monogenics are possible due to their unique structure combined with explicit series representations of the hypercomplex derivative and primitive of a monogenic function. In one of these applications, two constructive approaches were discussed to generate monogenic functions via harmonic conjugates. Another application focused on the generalization of Bloch's Theorem for monogenic functions defined in the unit ball of the three-dimensional Euclidean space. In the second part of the thesis, a space-frequency theory for band-limited quaternionic functions was developed, which runs parallel to the time-frequency analysis of band-limited functions produced by Slepian, Landau, and Pollak.

As a result of this study, some interesting but unresolved questions arose and are discussed in the following. As future work, besides applying our results to concrete boundary value problems, we intend to use the constructed bases for the study of convex and starlike univalent monogenic functions and their geometrical properties. In general, it is still an open problem of describing monogenic functions via their global geometric mapping properties. It is still unknown which domains can be mapped to a ball (or spheroid)
and which domains can be the image of such simple domains. Interest in questions of this type has increased in connection with constructing a theory of monogenic mappings. A first global result was considered by Almeida et al. in [18], in which the authors studied the global behavior of a generalized Joukowski transformation in the context of Clifford analysis (cf. [82]). It is hoped that the explicit expressions we found for the internal and external spheroidal monogenics may shed light on these problems. It would also be interesting to know whether the results of Chapter 4 could be extended to more general integral transforms, such as the Quaternion Linear Canonical Transform [184, 186] and the Windowed Quaternion Fourier Transform [224, 223]. To the best of our knowledge, no one has discussed this sort of question until now, even though they seem to pave the way to promising future work.

## Bibliography

[1] R.M. Aarts and A.J.E.M. Janssen, (2009), On-axis and far field sound radiation from resilient flat and dome-shaped radiators, J. Acoust. Soc. Am. 125, pp.1444-1455.
[2] R.M. Aarts and A.J.E.M. Janssen, (2009), Sound radiation quantities arising from a resilient circular radiator, J. Acoust. Soc. Am. 126, pp.1776-1787.
[3] R.M. Aarts and A.J.E.M. Janssen, (2010), Sound radiation from a resilient cap on a rigid sphere, J. Acoust. Soc. Am. 127, pp.2262-2273.
[4] R.M. Aarts and A.J.E.M. Janssen, (2011), Spatial impulse responses from a flexible baffled circular piston, J. Acoust. Soc. Am. 129, pp.2952-2959.
[5] M. Abdalla, M. Abul-Ez and J. Morais, (2018), On the Construction of Generalized Monogenic Bessel Polynomials, Math. Methods Appl. Sci., Vol. 41, No. 18, pp.9335-9348.
[6] M. Abramowitz and I.A. Stegun (Eds.), 1972, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover.
[7] M. Abul-Ez and D. Constales, (1990), Basic sets of polynomials in Clifford analysis, Complex Var., Vol. 14, No. 1-4, pp.177-185.
[8] M. Abul-Ez and D. Constales, (1991), Linear substitution for basic sets of polynomials in Clifford analysis, Port. Math., Vol. 48, No. 2, pp.143-154.
[9] M. Abul-Ez, (1992), Inverse sets of polynomials in Clifford analysis, Arch. Math., Vol. 5, pp.561-567.
[10] M. Abul-Ez, (1998), Bessel polynomial expansions in spaces of holomorphic functions, J. Math. Anal. Appl., Vol. 221, No. 1, pp.177-190.
[11] M. Abul-Ez, (2000), Hadamard product of bases of polynomials in Clifford analysis, Complex Var., Vol. 43, No. 2, pp.109-128.
[12] M. Abul-Ez and D. Constales, (2003), The square root base of polynomials in Clifford analysis, Arch. Math., Vol. 80, No. 5, pp.486495.
[13] M. Abul-Ez, D. Constales, J. Morais and M. Zayed, (2014), Hadamard three-hyperballs type theorem and overconvergence of special monogenic simple series. J. Math. Anal. Appl., Vol. 412, pp.426-434.
[14] L. Aceto, H. Malonek and G. Tomaz, (2017), Matrix approach to hypercomplex Appell polynomials, Appl. Numer. Math., Vol. 116, pp.2-9.
[15] L.V. Ahlfors, 1973, Conformal invariants: Topics in geometric function theory, McGraw-Hill, New York.
[16] M.V. Aliev, A.M. Belov, A.V. Ershov and M.A. Chicheva, (2007), Parallel algorithms for a hypercomplex discrete Fourier transform, Pattern Recognit. Image Anal., Vol. 17, No.1, pp.1-5.
[17] C. Álvarez-Peña, R.M. Porter, (2014), Contragenic functions of three variables, Complex Anal. Oper. Theory, Vol. 8, No. 2, pp.409-427.
[18] R. Almeida and H. Malonek, (2010), A note on a generalized Joukowski transformation, Appl. Math. Lett., Vol. 23, pp.1174-1178.
[19] L. Andrews, 1998, Special Functions of Mathematics for Engineers, SPIE Optical Engineering Press, Bellingham, Oxford University Press, Oxford.
[20] V. Antonov and A. Baranov, (2002), Relation between the expansions of an external potential in spherical functions and spheroidal harmonics, Tech. Phys., Vol. 47, No. 3, pp.361-363.
[21] P. Appell, (1880), Sur une class de polynomes, Ann. Sci. Éc. Norm. Supér, Vol. 9, pp.119-144.
[22] I. Arnaoudov and G. Venkov, (2010), Relations between spheroidal and spherical harmonics, Comptes rendus de l'Académie bulgare des sciences: sciences mathématiques et naturelles, Vol. 7, pp.971-978.
[23] F.M. Arscott, 1964, Periodic Differential Equations. An introduction to Mathieu, Lamé, and allied functions, International Series of Monographs in Pure and Applied Mathematics, Oxford : New York, Paris: Pergamon Press.
[24] K. Avetisyan, K. Gürlebeck and W. Sprössig, (2009), Harmonic conjugates in weighted Bergman spaces of quaternion-valued functions, Comput. Methods Funct. Theory, Vol. 9, No. 2, pp.593-608.
[25] S. Axler, P. Bourdon and W. Ramey, 2001, Harmonic Function Theory, New York: Springer.
[26] V. Balabaev, (1974), Quaternionic analogue of the Cauchy-Riemann system in four-dimensional space and some its applications, Soviet Acad. Sci., Doklady, Vol. 214, No. 3, pp.489-491 (Russian).
[27] V. Balabaev, (1978), On a system of equations in quaternions in four-dimensional complex space, Math. Notes, Vol. 23, No. 1, pp.41-46 (Russian).
[28] S. Bará, J. Arines, J. Ares and P. Prado, (2006), Direct transformation of Zernike eye aberration coefficients between scaled, rotated, and/or displaced pupils, J. Opt. Soc. Am., Vol. 23, No. 9, pp.2061-2066.
[29] P. Bas, N.Le Bihan and J.M. Chassery, (2003), Color image water marking using quaternion Fourier transform, In: Proceedings of the IEEE International Conference on Acoustics Speech and Signal Processing, ICASSP, Hong-Kong, pp.521-524.
[30] W.N. Bailey, (1933), On the product of two Legendre polynomials, Math. Proc. Cambridge Philos. Soc., Vol. 29, No. 2, pp.173-177.
[31] E. Bayro-Corrochano, (2005), Multi-resolution image analysis using the quaternion wavelet transform, Numer. Algorithms, Vol. 39, 1-3, pp.35-55.
[32] E. Bayro-Corrochano, (2006), The theory and use of the quaternion wavelet transform, J. Math. Imaging Vis., Vol. 24, pp.19-35.
[33] E. Bayro-Corrochano, N. Trujillo and M. Naranjo, (2007), Quaternion Fourier descriptors for preprocessing and recognition of spoken words using images of spatiotemporal representations, J. Math. Imaging Vis., Vol. 28, No. 2, pp.179-190.
[34] W.W. Bell, 1968, Special Functions for Scientists and Engineers, Great Britain, Butler and Tanner Ltd, Frome and London.
[35] S. Bernstein, K. Gürlebeck, L.F. Reséndis and L.M. Tovar, (2005), Dirichlet and Hardy spaces of harmonic and monogenic functions, Z. Anal. Anwend., Vol. 24, pp.635-672.
[36] A.B. Bhatia and E. Wolf, (1954), On the circle polynomials of Zernike and related orthogonal sets, Math. Proc. Cambridge Philos. Soc., Vol. 50, pp.40-48.
[37] A. Bloch, (1925), Les théorèmes de M. Valiron sur les fonctions entières et la théorie de l'uniformisation, Annales de la Faculté des sciences de Toulouse: Mathématiques, Vol. 17, pp.1-22.
[38] S. Bоск, 2009, Über funktionentheoretische Methoden in der räumlichen Elastizitätstheorie, Ph.D. thesis, Bauhaus-University Weimar.
[39] S. Bock and K. Gürlebeck, (2010), On a generalized Appell system and monogenic power series, Math. Methods Appl. Sci., Vol. 33, No. 4, pp. 394-411.
[40] S. Bock, (2012), On a three dimensional analogue to the holomorphic z-powers: Laurent series expansions, Complex Var., Vol. 57, No. 12, pp.1271-1287.
[41] S. Bock, (2012), On a three dimensional analogue to the holomorphic z-powers: power series and recurrence formulae, Complex Var., Vol. 57, No. 12, pp.1349-1370.
[42] S. Bock, K. Gürlebeck, R. Lávička and V. Souček, (2012), The Gel'fand-Tsetlin bases for spherical monogenics in dimension 3, Rev. Mat. Iberoam., Vol. 28, No. 4, pp.1165-1192.
[43] S. Bock, (2018), On a Class of Monogenic Functions with (Logarithmic) Line Singularities, Adv. Appl. Clifford Algebr., Vol. 28, No. 6.
[44] C.J. Bouwkamp, (1947), On spheroidal wave functions of order zero, J. Math. Phys. Mass. Inst. Tech., Vol. 26, pp.79-92.
[45] G.D. Boyd and J.P. Gordon, (1961), Confocal Multimode Resonator for Millimeter Through Optical Wavelength Masers, Bell Sys. Tech. J., Vol. 40, pp.489-508.
[46] J.P. Boyd, (2003), Approximation of an analytic function on a finite real interval by a bandlimited function and conjectures on properties of prolate spheroidal functions, Appl. Comput. Harmon. Anal., No. 2, Vol. 15, pp.168-176.
[47] J.P. Boyd, (2004), Prolate spheroidal wavefunctions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudospectral algorithms, J. Comput. Phys., No. 2, Vol. 199, pp.688-716.
[48] F. Brackx, R. Delanghe and F. Sommen, 1982, Clifford analysis, Pitman Advanced Publishing Program.
[49] F. Brackx, R. Delanghe and F. Sommen, (2002), On conjugate harmonic functions in Euclidean space, Math. Methods Appl. Sci., Vol. 25, No. 16-18, pp.1553-1562.
[50] F. Brackx and R. Delanghe, (2003), On harmonic potential fields and the structure of monogenic functions, Zeitschrift für Anal. und ihre Anwendung, Vol. 22, No. 2, pp.261-273.
[51] F. Brackx, N. De Schepper and F. Sommen, (2005), The CliffordFourier transform, J. Fourier Anal. Appl., Vol. 1, No. 6, pp.669-681.
[52] F. Brackx, N. De Schepper, and F. Sommen, (2006), The twodimensional Clifford-Fourier transform, J. Math. Imaging Vis., Vol. 26, No. 1, pp.5-18.
[53] F. Brackx, H. De Schepper and F. Sommen, (2007), A theoretical framework for wavelet analysis in a Hermitean Clifford setting, Commun. Pure Appl. Anal., Vol. 6, No. 3, pp.549-567.
[54] F. Brackx, H. De Schepper, N. De Schepper and F. Sommen, (2009), Generalized Hermitean Clifford-Hermite polynomials and the associated wavelet transform, Math. Methods Appl. Sci., Vol. 32, No. 5, pp.606-630.
[55] F. Brackx, E. Hitzer and S.J. Sangwine, (2013), History of quaternion and Clifford Fourier transforms and wavelets, In: Quaternion and Clifford Fourier Transforms and Wavelets (eds. Eckhard Hitzer and Stephen Sangwine), Trends in Mathematics Series, Vol. 27, Springer Basel AG, pp.XI-XXVII.
[56] H.A. Buchdahl, N.P. Buchdahl, P.J. Stiles, (1977), On a relation between spherical and spheroidal harmonics, J. Phys. A: Math. Gen., Vol. 10, No. 11, pp.1833-1836.
[57] T. BüLow, 1999, Hypercomplex spectral signal representations for the processing and analysis of images, Ph.D. diss., Christian-Albrechts University, Kiel, Germany.
[58] T. Bülow, M. Felsberg and G. Sommer, (2001), Non-commutative hypercomplex Fourier transforms of multidimensional signals, in G. Sommer (ed.), Geom. Comp. Clifford Alg., Springer, Berlin, Heidelberg, pp.187-207.
[59] A.L.V. Buren and J.E. Boisvert, (2002), Accurate calculation of prolate spheroidal radial functions of the first kind and their first derivatives, Quart. Appl. Math., Vol. 60, No. 3, pp.589-599.
[60] A.L.V. Buren and J.E. Boisvert, (2004), Improved calculation of prolate spheroidal radial functions of the second kind and their first derivatives, Quart. Appl. Math., Vol. 62, No. 3, pp.493-507.
[61] W.E. Byerly, (1959), An Elementary Treatise on Fourier's Series: and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics, New York: Dover, pp.251-258.
[62] I. Cação, K. Gürlebeck and H. Malonek, (2001), Special monogenic polynomials and $L_{2}$-approximation, Adv. Appl. Clifford Algebr., Vol. 11, pp.47-60.
[63] I. CAÇão, 2004, Constructive Approximation by Monogenic polynomials, Ph.D. diss., University of Aveiro.
[64] I. Cação, K. Gürlebeck and S. Bock, (2005), Complete Orthonormal Systems of Spherical Monogenics - A Constructive Approach, L. H. Son et al. (ed.), Methods of Complex and Clifford Analysis, SAS International Publications, Delhi, pp.241-260.
[65] I. Cação, K. Gürlebeck and B. Bock, (2006), On Derivatives of Spherical Monogenics, Complex Var., Vol. 51, No. 8-11, pp.847-869.
[66] I. Cação and K. Gürlebeck, (2007), On monogenic primitives of monogenic functions, Complex Var., Vol. 52, No. 10-11, pp.1081-1100.
[67] I. CAÇÃO, (2010), Complete orthonormal sets of polynomial solutions of the Riesz and Moisil-Teodorescu systems in $\mathbb{R}^{3}$, Numer. Algorithms, Vol. 55, No. 2-3, pp.191-203.
[68] I. Cação, M.I. Falcão and H. Malonek, (2011), Laguerre derivative and monogenic Laguerre polynomials: an operational approach, Math. Comput. Modeling, Vol. 53, pp.1084-1094.
[69] I. Cação and H. Malonek, (2011), On an Hypercomplex Generalization of Gould-Hopper and Related Chebyshev Polynomials, In: Murgante B., Gervasi O., Iglesias A., Taniar D., Apduhan B.O. (eds) Computational Science and Its Applications - ICCSA 2011. ICCSA 2011. Lecture Notes in Computer Science, Vol. 6784. Springer, Berlin, Heidelberg, pp.316-326.
[70] I. Cação and J. Morais, (2014), An orthogonal set of weighted quaternionic Zernike spherical functions, B. Murgante et al. (Eds.): ICCSA 2014, Part I, LNCS 8579, pp.103-116.
[71] I. Cação, H. Malonek and G. Tomaz, (2017), Shifted Generalized Pascal Matrices in the Context of Clifford Algebra-Valued Polynomial Sequences, In: Gervasi O. et al. (eds) Computational Science and Its Applications - ICCSA 2017. ICCSA 2017. Lecture Notes in Computer Science, Vol. 10405. Springer, Cham, pp.409-421.
[72] I. Cação, M.I. Falcĩo and H. Malonek, (2017), Three-term recurrence relations for systems of Clifford algebra-valued orthogonal polynomials, Adv. Appl. Clifford Algebras, Vol. 27, pp.71-85.
[73] I. Cação, M.I. Falcĩo and H. Malonek, (2018), On generalized Vietoris' number sequences, Discret. Appl. Math., Vol. 269, No. 30, pp.77-85.
[74] C. Carathéodory, (1929), Über die Winkelderivierten von beschränkten analytischen Funktionen, Sitz. Ber. Preuss. Akad., Phys.-Math., IV, pp.1-18.
[75] Q.Y. Chen, D. Gottlieb and J.S. Hesthaven, (2005), Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs, SIAM J. Numer. Anal., No. 5, Vol. 43, pp.1912-1933.
[76] L. Chen, K. Kou and M. Liu, (2015), Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform, J. Math. Anal. Appl., Vol. 423, No. 1, pp.681-700.
[77] D. Cheng and K. Kou, (2018), Generalized sampling expansions associated with quaternion Fourier transform, Math. Methods Appl. Sci., Vol. 41, No. 11, pp.4021-4032.
[78] D. Cheng and K. Kou, (2019), Plancherel theorem and quaternion Fourier transform for square integrable functions, Complex Var., Vol. 64, No. 2, pp.223-242.
[79] W.K. Clifford, (1878), Applications of Grassmann's Extensive Algebra, Amer. J. Math., Vol. 1, No. 4, pp.350-358.
[80] F. Colombo, I. Sabadini, D.C. Struppa, 2011, Noncommutative Functional Calculus: Theory and Applications of SliceHyperholomorphic Functions, Progress in Mathematics series, 289, Birkhäuser, Basel.
[81] R. Courant and D. Hilbert, 1989, Methods of Mathematical Physics, Vol. 1, English edition translated from the German original, Wiley Interscience, New York.
[82] C. Cruz, M.I. Falcão and H. Malonek, (2011), 3D mappings by generalized Joukowski transformations, in: Computational Science and Its Applications - ICCSA 2011, Lecture Notes in Computer Science, Vol. 6784, Santander, pp.358-373.
[83] C. Cruz, M.I. Falcão and H. Malonek, (2014), Monogenic pseudocomplex power functions and their applications, Math. Methods Appl. Sci., Vol. 37, pp.1723-1735.
[84] F. Dai and Y. Xu, 2013, Approximation Theory and Harmonic Analysis on Spheres and Balls, Springer-Verlag.
[85] G. Darwin, (1902), Ellipsoidal harmonic analysis, Phil. Trans. Roy. Soc. A, Vol. 197, pp.461-557.
[86] G. Dassios, 2012, Ellipsoidal Harmonics, Theory and Applications, Cambridge University Press.
[87] P. Davis, 1963, Interpolation and Approximation. Blaisdell Publishing Company, New York.
[88] C.A. Deavours, (1973), The quaternion calculus, American Mathematical Monthly, Washington, DC: Mathematical Association of America, Vol. 80, No. 9, pp.995-1008.
[89] R. Delanghe, (1970), On regular-analytic functions with values in a Clifford-algebra, Math. Ann., Vol. 185, pp.91-111.
[90] R. Delanghe, (1972), On the singularities of functions with values in a Clifford algebra, Math. Ann., Vol. 196, pp.293-319.
[91] R. Delanghe, F. Sommen and V. Soucek, (1992), Clifford Agebras and Spinor-Valued Functions, Kluwer Academic Publishers Group, Dordrecht.
[92] R. Delanghe, (2001), Clifford analysis: history and perspective, Comput. Methods Funct. Theory, Vol. 1, No. 1, pp.107-153.
[93] R. Delanghe, R. S. Krausshar and H. Malonek, (2001), Differentiability of functions with values in some real associative algebras: approaches to an old problem, Bulletin de la Société Royale des Sciences de Liège, Vol. 70, No. 4-6, pp.231-249.
[94] R. Delanghe, (2007), On homogeneous polynomial solutions of the Riesz system and their harmonic potentials, Complex Var., Vol. 52, No. 10-11, pp.1047-1062.
[95] R. Delanghe, (2009), On Homogeneous Polynomial Solutions of the Moisil-Théodoresco System in $\mathbb{R}^{3}$, Comput. Methods Funct. Theory, Vol. 9, No. 1, pp.199-212.
[96] M.A. Delsuc, (1988), Spectral representation of 2D NMR spectra by hypercomplex numbers, Journal of magnetic resonance, Vol. 77, No. 1, pp.119-124.
[97] A. Dixon, (1904), On the Newtonian potential, Quart. J. Math., Vol. 35, pp.283-296.
[98] J.M. Dixon and R. Lacroix, (1974), Some useful relations using spherical harmonics and Legendre polynomials, J. Phys. A: Math. Nucl. Gen., Vol. 6, No. 8, 552.
[99] D.L. Donoho and P.B. Stark, (1989), Uncertainty principles and signal recovery, SIAM J. Appl. Math., Vol. 49, No. 3, pp.906-931.
[100] F.J. Dyson, (1962), Sattistical Theory of Energy Levels of Complex Systems, III, J. of Math. Phys., Vol. 3, pp.166-175.
[101] A.D. Dzhuraev, (1982), On the Moisil-Teodorescu system, Pitman Res. Not. in Math. Ser., Vol. 262, pp.186-203.
[102] J. Ebling and G. Scheuermann, (2005), Clifford Fourier transform on vector fields, IEEE Trans. Vis. Comp. Graph., Vol. 11, No. 4, pp.469479.
[103] M.E. Luna-Elizarrarás, J. Morais, M.A. PÉrez-de la Rosa and M. Shapiro, (2016), On a Version of Quaternionic Function Theory Related to Chebyshev Polynomials and Modified Sturm-Liouville Operators, Quart. Appl. Math., Vol. 74, pp.165-187.
[104] T.A. Ell, 1992, Hypercomplex Spectral Transformations, Ph.D. diss., University of Minnesota.
[105] T.A. Ell, (1993), Quaternion-Fourier transforms for analysis of twodimensional linear time-invariant partial differential systems, in: Proceedings of the 32nd IEEE Conference on Decision Control, pp.18301841.
[106] T.A. Ell and S.J. Sangwine, (2007), Hypercomplex Fourier transforms of color images, IEEE Trans. Image Process., Vol. 16, pp.22-35.
[107] T.A. Ell, N.Le Bihan and and S.J. Sangwine, 2014, Quaternion Fourier transforms for signal and image processing, John Wiley \& Sons, Inc.
[108] A. Erdélyi, 1953, Higher Transcendental Functions, Vol. I, McGrawHillBook Co., Inc., New York.
[109] R.R. Ernst, G. Bodenhausen and A. Wokaun, 1987, Principles of Nuclear Magnetic Resonance in One and Two Dimensions, International Series of Monographs on Chemistry. Oxford University Press.
[110] V. Erofeenko, (1988), Addition Theorems, Nauka i Tekhnika, Minsk (in Russian).
[111] T. Estermann, (1971), Notes on Landau's proof of Picard's "Great" Theorem, Sudies in Pure Mathematics presented to R. Rado, ed. L. Mirsky. Acad. Press London, New York, pp.101-106.
[112] M. Falcão, J. Cruz and H. Malonek, (2006), Remarks on the generation of monogenic functions, Proc. of the 17-th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, ISSN 1611-4086 (K. Gürlebeck and C. Könke, eds.), Bauhaus-University Weimar, 2006.
[113] M. Falcão, and H. Malonek, (2007), Generalized Exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions, Numerical Analysis and Applied Mathematics (T.E. Simos, G. Psihoyios, and Ch. Tsitouras, eds.), AIP Conference Proceedings, Vol. 936, Melville, NY: American Institute of Physics (AIP), pp.750-753.
[114] M. Falcão, and H. Malonek, (2007), Special Monogenic polynomials - Properties and Applications, Numerical Analysis and Applied Mathematics (T.E. Simos, G. Psihoyios, and Ch. Tsitouras, eds.), AIP Conference Proceedings, Vol. 936, Melville, NY: American Institute of Physics (AIP), pp.764-767.
[115] P.E. Falloon, P.C. Abbott and J.B. Wang, (2003), Theory and computation of the spheroidal wave functions, J. Physics A, Vol. 36, No. 20.
[116] A. Fedotowsky and G. Boivin, (1972), Finite Fourier selftransforms, Q. Appl. Math., Vol. 30, No. 3, pp.235-254.
[117] L. Fejér, (1909), Über die Laplacesche Reihe, Math. Ann. 67, pp.76109.
[118] N.M. Ferrers, 1897, An elementary treatise on spherical harmonics and subjects connected with them, London: MacMillan and Co.
[119] N.M. Ferrers, (1897), On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae, and elliptic rings, of variable densities, Quarterly Journal of Pure and Applied Mathematics, Vol. 14, pp.1-22.
[120] C. Flammer, 1957, Spheroidal Wave Functions. Stanford, CA: Stanford University Press.
[121] T.M. Flett, (1968), Mean values of power series, Pac. J. Math., Vol. 25, pp.463-494.
[122] T.M. Flett, (1970), Inequalities for the pth mean values of harmonic and subharmonic functions with $p \leq 1$, Proc. London Math. Soc., Vol. 20, pp.249-275.
[123] G.B. Folland and A. Sitaram, (1997), The uncertainty principle: A mathematical survey, J. Fourier Anal. Appl., Vol. 3, No. 3, pp.207238.
[124] G. Frobenius, (1878), Über lineare Substitutionen and bilineare Formen, J. Reine Angew. Math., Vol. 84, pp.1-63.
[125] W.H.J. Fuchs, (1963), On the eigenvalues of an integral equation, Notices Amer. Math. Soc., Vol. 10, 352.
[126] R. Fueter, (1932), Analytische Funktionen einer Quaternionenvariablen, Comment. Math. Helv., Vol. 4, pp.9-20.
[127] R. Fueter, (1935), Die Funktionentheorie der Differentialgleichungen $\Delta u=0$ und $\Delta \Delta u=0$ mit vier reellen Variablen, Comm. Math. Helv., Vol. 7, pp.307-330.
[128] R. Fueter, (1935), Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen, Comment. Math. Helv., Vol. 8, pp.371-378.
[129] R. Fueter, (1936), Die Singularitäten der eindeutigen regulären Funktionen einer Quaternionenvariablen I, Comment. Math. Helv., Vol. 9, pp.320-334.
[130] R. Fueter, (1940), Reguläre Funktionen einer Quaternionenvariablen, Lecture notes, Spring Semester, Math. Inst. Univ. Zürich.
[131] R. Fueter, (1949), Functions of a Hyper Complex Variable, Lecture notes written and supplemented by E. Bareiss, Math. Inst. Univ. Zürich, Fall Semester.
[132] P. Garabedian, (1953), Orthogonal harmonic polynomials, Pacific J. Math. Vol., 3, No. 3, pp.585-603.
[133] R. García A., J. Morais, R. M. Porter, (2018), Contragenic functions on spheroidal domains, Math. Methods Appl. Sci., Vol. 31, No. 7, pp.2575-2589.
[134] R. García A., J. Morais, R. M. Porter, (2020), Relations among spheroidal and spherical harmonics, App. Math. Comput., Vol. 384, 125147.
[135] M. Gaudin, (1961), Sur la loi limite de l'espacement des valeurs propres d'une matrice aléatorre, Nuclear Phys., Vol. 25, pp.447-458.
[136] G. Gentili, C. Stoppato and D.C. Struppa, 2013, Regular Functions of a Quaternionic Variable, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg.
[137] S. Georgiev and J. Morais, (2013), Bochner's Theorems in the framework of Quaternion Analysis, In: Quaternion and Clifford Fourier Transforms and Wavelets (eds. Eckhard Hitzer and Stephen Sangwine), Trends in Mathematics Series, Vol. 27, Springer Basel AG, pp.85-104.
[138] S. Georgiev, J. Morais, K.I. Kou and W. Sproessig, (2013), Bochner-Minlos Theorem and Quaternion Fourier Transform. In: Quaternion and Clifford Fourier Transforms and Wavelets (eds. Eckhard Hitzer and Stephen Sangwine), Trends in Mathematics Series, Vol. 27, Springer Basel AG, pp.105-120.
[139] D. Gilbarg and N.S. Trudinger, 1983, Elliptic Partial Differential Equations of Second Order, Springer-Verlag: Berlin, Heidelberg, N.Y.
[140] J. Gilbert and M. Murray, 1991, Clifford algebras and Dirac operators in harmonic analysis, Cambridge Univ. Press, Cambridge and New York.
[141] S.S. Goha and T.N.T. Goodman, (2006), Uncertainty principles in Banach spaces and signal recovery, J. Approx. Theory, Vol. 143, No. 1, pp.26-35.
[142] G. Green, (1835), On the determination of the exterior and interior attractions of ellipsoids of variable densities, Trans. Cambridge Philos. Soc., Vol. 5, pp.395-430.
[143] K. Gröchenig, 2001, Foundations of Time-Frequency Analysis, Birkhäuser, Boston.
[144] K. GÜRLEBECK, (1988), Interpolation and best approximation in spaces of monogenic functions, Wissenschafteliche Zeitschrift der TU Karl-Marx-Stadt, 30, pp.38-40.
[145] K. Gürlebeck and W. Sprössig, 1989, Quaternionic Analysis and Elliptic Boundary Value Problems, Akademie Verlag, Berlin.
[146] K. Gürlebeck and W. Sprössig, 1997, Quaternionic Calculus for Engineers and Physicists, John Wiley and Sons, Chichester.
[147] K. GÜRlebeck and H. Malonek, (1999), A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications, Complex Var., Vol. 39, No. 3, pp.199-228.
[148] K. Gürlebeck, K. Habetha and W. Sprössig, 2008, Holomorphic Functions in the Plane and n-dimensional Space, Birkhäuser Verlag, Basel-Boston-Berlin.
[149] K. Gürlebeck, J. Morais and P. Cerejeiras, 2009, BorelCarathéodory Type Theorem for monogenic functions, Complex Anal. Oper. Theory, Vol. 3, No. 1, pp.99-112.
[150] K. Gürlebeck and J. Morais, (2009), Bohr Type Theorems for Monogenic Power Series, Comput. Methods Funct. Theory, Vol. 9, No. 2, pp.633-651.
[151] K. Gürlebeck and J. Morais, (2010), On Bohr's phenomenon in the context of Quaternionic analysis and related problems, in: Le Hung Son (ed.); Tutschke, Wolfgang (ed.) Algebraic structures in partial
differential equations related to complex and Clifford analysis. Ho Chi Minh City University of Education Press, pp.9-24.
[152] K. Gürlebeck and J. Morais, (2011), On orthonormal polynomial solutions of the Riesz system in $\mathbb{R}^{3}$, Recent Adv. Comput. Appl. Math., pp.143-158.
[153] K. Gürlebeck, K. Habetha and W. Sprössig, 2016, Application of Holomorphic Functions in Two and Higher Dimensions, Birkhäuser Verlag, Basel-Boston-Berlin.
[154] K.E. Gustafson and D.K.M. Rao, 1997, Numerical Range, Springer, New York.
[155] R. Hamilton, 1866, Elements of Quaternions, Longmans Green, reprinted by Chelsea, New York.
[156] S. Hanish, R.V. Baier, A.L.Van Buren and B.J. King, 1970, Tables of radial spheroidal wave functions, volume 1, prolate, $m=0$.
[157] S. Hanish, R.V. Baier, A.L. Van Buren and B.J. King, 1970, Tables of radial spheroidal wave functions, volume 2, prolate, $m=1$.
[158] S. Hanish, R.V. Baier, A.L. Van Buren and B.J. King, 1970, Tables of radial spheroidal wave functions, volume 3, prolate, $m=2$.
[159] H. Heine, 1842, De aequationibus nonnullis differentialibus, Ph.D. diss., Universität Göttingen, Göttingen.
[160] E Heine, (1843), Über einige Aufgaben, welche auf partielle Differentialgleichungen führen, J. reine angew. Math. 26, pp.185-216.
[161] E Heine, 1878, Handbuch der Kugelfunktionen, Verlag G. Reimer, Berlin.
[162] M. Heins, 1962, Selected Topics in the Classical Theory of Functions of a Complex Variable, Holt, Rinehart and Winston, New York.
[163] D. Hestenes, (1968), Multivector calculus, J. Math. Anal. Appl., pp.313-325.
[164] Hilbert, 1933, Über die invarianten Eigenschaften spezieller binärer Formen, insbesondere der Kugelfunktionen, Springer, Berlin, Heidelberg.
[165] M.J.M. Hill, (1883), On functions of more than two variables analogous to tesseral harmonics, Trans. Camb. Philos. Soc., Vol. 13, pp.273299.
[166] E. Hitzer and B. Mawardi, (2006), Uncertainty principle for the Clifford geometric algebra based on Clifford Fourier transform, in Wavelet Analysis and Applications, T. Qian, M. I. Vai, and Y. Xu, Eds., Applied and Numerical Harmonic Analysis, pp.45-54, Springer, Berlin, Germany.
[167] E. Hitzer, (2007), Quaternion fourier transform on quaternion fields and generalizations, Adv. Appl. Clifford Algebras, Vol. 17, pp.497-517.
[168] E. Hitzer, (2007), Tutorial on Fourier transformations and wavelet transformations in Clifford geometric algebra, In K. Tachibana, editor, Lecture notes of the International Workshop for Computational Science with Geometric Algebra (FCSGA2007), pp.65-87.
[169] E. Hitzer and B. Mawardi, (2008), Clifford fourier transform on multivector fields and uncertainty principles for dimensions $n=2$ (mod 4) and $n=3(\bmod 4)$, Adv. Appl. Clifford Algebras, Vol. 18, pp.715-736.
[170] E. Hobson, 1931, The theory of spherical and ellipsoidal harmonics, Cambridge.
[171] H.E. Hunter, 1965, Tables of prolate spheroidal functions for $m=0$ : Volume I.
[172] H.E. Hunter, 1965, Tables of prolate spheroidal functions for $m=0$ : Volume II.
[173] J.C. Hurtley, 1969, Hyperspheroidal functions, in Proceedings of the symposium on quasioptics, Polytechnic Press of Brooklyn, New York.
[174] V. Iftimie, (1965), Fonctions hypercomplexes, Bull. Math. Soc. Sci. Math. R. S. Roumanie, Vol. 9, No. 57, pp.279-332.
[175] R.C. Jacobson, (1953), Lectures in abstract Algebra II, D. von Math. Mag., Vol. 24, pp.237-246.
[176] J.E. Jamison, 1970, Extension of some theorems of complex functional analysis to linear spaces over the quaternions and Cayley numbers, Ph.D. diss., University of Missouri-Rolla.
[177] G. Jansen, (2000), Transformation properties of spheroidal multipole moments and potentials, J. Phys. A: Math. Gen., Vol. 33, No. 7, pp.1375-1394.
[178] I. Kantor and A. Solodovnikov, 1989, Hypercomplex Numbers: An Elementary Introduction to Algebras, Springer-Verlag, New York.
[179] H. Kalf, (1995), On the Expansion of a Function in Terms of Spherical Harmonics in Arbitrary Dimensions, Bull. Belg. Math. Soc. Simon Stevin, Vol. 2, pp.361-380.
[180] A.Karoui and T.Moumni, (2009), Spectral analysis of the finite Hankel transforbv $m$ and circular prolate spheroidal wave functions, J. Comput. Appl. Math., Vol. 23, No. 2, pp.315-333.
[181] F. Klein, (1881), Über Lamé'sche Functionen, Math. Ann., Vol. 18, pp.237-246.
[182] H. Kogelnik and T. Li, (1966), Laser beams and resonators, Proc. IEEE, Vol. 54, pp.1312-1329.
[183] I.V. Komarov, L.I. Ponomarev and S.Y. Slavyanov, 1976, Spheroidal and Coulomb Spheroidal Functions, Nauka, Moscow (Russian).
[184] K. Kou, Jian-Yu Ou and J. Morais, (2013), On uncertainty principle for quaternionic linear canonical transform, Abstr. and Appl. Anal., Vol. 2013, 725952.
[185] K. Kou, J. Morais and Y. Zhang, (2013), Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis, Math. Methods Appl. Sci., Vol. 36, No. 9, pp.1028-1041.
[186] K. Kou and J. Morais, (2014), Asymptotic behaviour of the quaternion linear canonical transform and the Bochner-Minlos theorem, App. Math. Comput., Vol. 247, No. 15, pp.675-688.
[187] K. I. Kou, M. S. Liu, J. Morais, and C. Zou, (2017), Envelope detection using generalized analytic signal in 2D QLCT domains. Multidimens. Syst. Signal Process., 28, 1343-1366.
[188] S. Krantz, (2005), Calculation and estimation of the Poisson kernel, J. Math. Anal. Appl., Vol. 302, pp.143-148.
[189] V. Kravchenko and M. Shapiro, 1996, Integral Representations for Spatial Models of Mathematical Physics, Research Notes in Mathematics, Pitman Advanced Publishing Program, London.
[190] V. Kravchenko, 2003, Applied quaternionic analysis, Research and Exposition in Mathematics, Lemgo: Heldermann Verlag, Vol. 28.
[191] N. Krylov, (1947), Sur les quaternions de W.R. Hamilton et la notion de la monogenéité, Dokl. Akad. Nauk SSSR 55, pp.787-788.
[192] G. Lamé,, (1839), Mémoire sur l'équilibre des températures dans un ellipsoïde à trois axes inégaux, J. Math. Pures Appl., pp.26-163.
[193] C. Lanczos, 1919, Die Funktionentheoretischen Beziehungen Der Maxwellschen Aethergleichungen, Ph.D. diss., Budapest.
[194] B. Landa and Y. Shkolnisky,, (2017), Approximation scheme for essentially bandlimited and space-concentrated functions on a disk, Appl. Comput. Harmon. Anal., Vol. 43, No. 3, pp.381-403.
[195] B. Landa and Y. Shkolnisky,, (2017), Steerable principal components for space-frequency localized images, SIAM J. Imaging Sci., Vol. 10, No. 2, pp.508-534.
[196] E. Landau, (1927), Der Picard-Schottkysche Satz und die Blochsche Konstante, Sitz. Bel'. Preuss. Akad., Phys.-Math, Vol. 1, No. 1, pp.4650.
[197] E. Landau and G. Valiron, (1929), A Deduction from Schwarz's Lemma, London Math. Soc., Vol. s1-4, No. 3, pp.162-163.
[198] H.J. Landau and H.O. Pollak, (1961), Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty - II, Bell Sys. Tech. J., Vol. 40, pp.65-84.
[199] H.J. Landau and H.O. Pollak, (1962), Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty - III, Bell Sys. Tech. J., Vol. 41, pp.1295-1336.
[200] H.J. Landau, (1965), The Eigenvalue Behaviour of Certain Convolution Equations, Trans. Amer. Math. Soc., Vol. 115, pp.242-256.
[201] H.J. Landau and H. Widom, (1980), Eigenvalue distribution of time and frequency limiting, J. Math. Anal. Appl., Vol. 77, No. 2, pp.469-481.
[202] P.S. Laplace, 1785, Théorie des Attractions des Sphéroïdes et de la Figure des Planètes, Vol. III, Paris: Mechanique Celeste.
[203] R. LÁVIČKA, (2010), Canonical bases for sl(2, © $)$-modules of spherical monogenics in dimension 3, Arch. Math. (Brno), Vol. 46, No. 5, pp.339349.
[204] R. LÁvička, (2012), Complete Orthogonal Appell Systems for Spherical Monogenics, Complex Anal. Oper. Theory, Vol. 6, No. 2, pp.477-489.
[205] N. Lebedev, 1972, Special Functions and their Applications, Dover, New York.
[206] A-M. Legendre, (1785), Recherches sur l'attraction des sphéroïdes homogènes, Mémoires de Mathématiques et de Physique, présentés à L'Académie des Sciences (Paris), Vol. 19, pp.411-435.
[207] L. Li, M. Leong, T. Yeo, P. Kooi and K. Tan, (1998), Computations of spheroidal harmonics with complex arguments: a review with an algorithm, Phys. Rev. E, Vol. 58, No. 5, pp.6792-6806.
[208] H. Leutwiler, (2001), Quaternionic analysis in $\mathbb{R}^{3}$ versus its hyperbolic modification, en: Brackx, F., Chisholm, J.S.R., Soucek, V. (eds.) Proceedings of the NATO Advanced Research Workshop held in Prague, October 30 - November 3, 2000 25, Kluwer Academic Publishers, Dordrecht, Boston, London.
[209] C. Li, A. McIntosh and T. Qian, 1996, Clifford Algebras, Fourier Transform and Singular Convolution Operators On Lipschitz Surfaces, A. McIntosh, Clifford Alg. Four. Theory, Sing. Integrals, and Harm. Functions on Lipschitz Dom. J. Ryan (ed.), Chap. 1 in Cliff. Alg. in Anal. and Rel. Topics, CRC Press.
[210] L.W. Li, X.K. Kang and M.S. Leong, 2002, Spheroidal Wave Functions in Electromagnetic Theory, John Wiley \& Sons.
[211] F. Lindemann, (1882), Entwicklung der Functionen einer complexen Variabeln nach Laméshen Functionen und nach Zugeordneten der Kugelfunctionen, Math. Ann., Vol. 19, pp.323-386.
[212] J. Liouville, (1846), Sur diverses questions d'analyse et de Physique mathématique concernant L'ellipsoïde, J. Math. Pures Appl., 11, pp.217-236.
[213] P. Lounesto, 2001, Clifford Algebras and Spinors, Cambridge University Press, New York-Melbourne.
[214] H. Malonek, 1987, Zum Holomorphiebegriff in höheren Dimensionen, Habilitation Thesis, Halle.
[215] H. Malonek, (1990), Power series representation for monogenic functions in $\mathbb{R}^{m+1}$ based on a permutational product, Complex Var., Vol. 15, No. 3, pp.181-191.
[216] H. Malonek, (2001), Contributions to a geometric function theory in higher dimensions by Clifford analysis methods: Monogenic functions and $M$-conformal mappings, in: Clifford Analysis and its Applications ed. F. Brackx et al., Kluwer, NATO Sci. Ser. II, Math. Phys. Chem. 25, pp.213-222.
[217] H. Malonek, (2003), Quaternions in applied sciences: A historical perspective of a mathematical concept. In Proceedings of the International Conference on the Applications of Computer Science and Mathematics in Architecture and Civil Engineering. Bauhaus-Universität Weimar.
[218] H. Malonek, (2004), Selected topics in hypercomplex function theory, in: Clifford algebras and potential theory, Eriksson, S.-L.(ed.), University of Joensuu, Report Series 7, pp.111-150.
[219] H. Malonek and G. Tomaz, (2008), On generalized Euler polynomials in Clifford analysis, Int. J. of Pure and Appl. Math., Vol. 44, No. 3, pp.447-465.
[220] H. Malonek and G. Tomaz, (2009), Bernoulli polynomials and Pascal matrices in the context of Clifford analysis, Discrete Appl. Math., Vol. 157, No. 4, pp.838-847.
[221] B. Mawardi and E. Hitzer, (2006), Clifford Fourier transform and uncertainty principle for the Clifford geometric algebra $\mathcal{C} \ell_{0,3}, \mathrm{Adv}$. App. Cliff. Alg., Vol. 16, No. 1, pp.41-61.
[222] B. Mawardi, E. Hitzer, A. Hayashi and R. Ashino, (2008), An uncertainty principle for quaternion Fourier transform, Comput. Math. Appl., Vol. 56, No. 9, pp.2411-2417.
[223] B. Mawardi, (2010), Generalized Fourier transform in Clifford algebra $\mathcal{C} \ell_{0,3}$, Far East J. Math. Sci., Vol. 44, No. 2, pp.143-154.
[224] B. Mawardi, E. Hitzer, R. Ashino and R. Vaillancourt, (2010), Windowed Fourier transform of two-dimensional quaternionic signals, App. Math. Comput., Vol. 28, No. 6, pp.2366-2379.
[225] B. Mawardi, (2011), Generalized Fourier transform in real Clifford algebra $\mathcal{C} \ell_{0, n}$, Far East J. Math. Sci., Vol. 48, No. 1, pp.11-24.
[226] J. Meixner and F.W. Schäfke, 1954, Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin.
[227] J. Meixner, F.W. Schäfke and G. Wolf, 1980, Mathieu Functions and Spheroidal Functions and Their Mathematical Foundations: Further Studies, Lecture Notes in Mathematics, Vol. 837, SpringerVerlag, Berlin-New York.
[228] A. Melijhzon, (1948), Because of monogenicity of quaternions, Dokl. Akad. Nauk. USSR, Vol. 59, pp.431-434 (Russian).
[229] J. Mercer, (1909), Functions of positive and negative type, and their connection with the theory of integral equations, Philos. Trans. Roy. Soc. A, Vol. 209, pp.415-446.
[230] I. Mitelman and M. Shapiro, (1995), Differentiation of the Martinelli-Bochner integrals and the notion of hyperderivability, Math. Nachr., Vol. 172, pp.211-238.
[231] G. Moisil, (1931), Sur la généralisation des fonctions conjuguées, Rend. Acad. Naz. dei Lincei, Vol. 14, pp.401-408.
[232] G. Moisil and N. Théodoresco, (1931), Fonctions holomorphes dans l'espace, Matematica (Cluj) 5, pp.142-159.
[233] P. Moon and D. Spencer, 1961, Field Theory Handbook, Springer, Berlin.
[234] I.C. Moore and M. Cada, (2004), Prolate spheroidal wave functions, an introduction to the Slepian series and its properties, Appl. Comput. Harmon. Anal., No. 3, Vol. 16, pp.208-230.
[235] J.A. Morrison, (1965), Eigenfunctions of the finite Fourier transform operator over a hyperellipsoidal region, J. Math, and Phys. 44, pp.245-254.
[236] J. Morais, 2009, Approximation by homogeneous polynomial solutions of the Riesz system in $\mathbb{R}^{3}$, Ph.D. diss., Bauhaus-Universität Weimar.
[237] J. Morais, H.T. Le and W. Sprössig, (2010), Approximation of monogenic functions by means of monogenic polynomials in $\mathbb{R}^{4}$, ICNAAM 2010: International Conference of Numerical Analysis and Applied Mathematics 2010. AIP Conference Proceedings, Vol. 1281, pp.1488-1491.
[238] J. Morais and H.T. Le, (2011), Orthogonal Appell Systems of Monogenic Functions in the Cylinder, Math. Methods Appl. Sci., Vol. 34, No. 12, pp.1472-1486.
[239] J. Morais, (2011), A complete orthogonal system of spheroidal monogenics, J. Numer. Anal. Ind. Appl. Math., Vol. 6, No. 3-4, pp.105-119.
[240] J. Morais and K. Gürlebeck, (2012), Real-part estimates for solutions of the Riesz system in $\mathbb{R}^{3}$, Complex Var., Vol. 57, No. 5, pp.505522.
[241] J. Morais and K. Gürlebeck, (2012), Bloch's theorem in the context of quaternion analysis, Comput. Methods Funct. Theory, Vol. 12, No. 2, pp.541-558.
[242] J. Morais, (2013), An orthogonal system of monogenic polynomials over prolate spheroids in $\mathbb{R}^{3}$, Math. Comput. Model., Vol. 57, pp.425434.
[243] J. Morais, K.I. Kou and W. Sprössig, (2013), Generalized holomorphic Szegö kernel in 3D spheroids, Comput. Math. Appl., Vol. 65, pp.576-588.
[244] J. Morais, K.I. Kou and S. Georgiev, (2013), On convergence properties of 3D spheroidal monogenics, Int. J. Wavelets, Multiresolut. Inf. Process., Vol. 11, No. 3, 1350024.
[245] J. Morais, K. Avetisyan and K. Gürlebeck, (2013), On Riesz systems of harmonic conjugates in $\mathbb{R}^{3}$, Math. Methods Appl. Sci., Vol. 36, No. 12, pp.1598-1614.
[246] J. Morais, (2014), Computational aspects of the continuum quaternionic wave functions for hydrogen, Ann. Physics, Vol. 349, pp.171-188.
[247] J. Morais, S. Georgiev and W. Sprössig, 2014, Real Quaternionic Calculus Handbook, Birkhäuser, Basel.
[248] J. Morais, M.A. Pérez-de la Rosa and K.I. Kou, (2015), Computational geometric and boundary value properties of oblate spheroidal quaternionic wave functions, Wave Motion, Vol. 57, pp.112-128.
[249] J. Morais and I. Cação, (2015), Quaternion Zernike spherical polynomials, Math. Comp., Vol. 84, pp.1317-1337.
[250] J. Morais, M.H. Nguyen and K.I. Kou, (2016), On 3D orthogonal prolate spheroidal monogenics, Math. Methods Appl. Sci., Vol. 39, No. 4, pp.635-648.
[251] J. Morais and K.I. Kou, (2016), Constructing prolate spheroidal quaternion wave signals on the sphere, Math. Methods Appl. Sci., Vol. 39, No. 14, pp.3961-3978.
[252] P. Morse and H. Feshbach, 1953, Methods of Theoretical Physics, I, II, McGraw-Hill, New York.
[253] T. Moumni and A. Zayed, (2014), A generalisation of the prolate spheroidal wave functions with applications to sampling, Integral Transforms Spec. Funct., Vol. 25, No. 6, pp.433-447.
[254] C. MüLler, 1966, Spherical Harmonics, Lectures Notes in Mathematics, Vol. 17, Berlin: Springer-Verlag.
[255] F. Neumann, 1878, Beiträge zur Theorie der Kugelfunktionen, II, Leipzig.
[256] H.M. Nguyen, K. Gürlebeck, J. Morais and S. Bock, (2014), On orthogonal monogenics in oblate spheroidal domains and recurrence formulae, Integral Transforms Spec. Funct., Vol. 25, No. 7, pp.513-527.
[257] B.R.A. Nijboer, 1942, The diffraction theory of aberrations, Ph.D. diss., University of Groningen, Groningen.
[258] A. Nikiforov and V. Uvarov, 1988, Special functions of Mathematical Physics, Birkhaüser Verlag, Basel.
[259] C. Niven, 1880, On the Conduction of heat in ellipsoids of revolution, Philosophical transactions of the Royal Society of London, Vol. 171.
[260] R.J. Noll, (1976), Zernike polynomials and atmospheric turbulence, J. Opt. Soc. Am., Vol. 66, pp.207-211.
[261] F.W.J. Olver, D.W. Lozier, R. Boisvert and C.W. Clark, 2010, Handbook of Mathematical Functions, Cambridge University Press, Cambridge.
[262] A. Osipov, V. Rokhlin and H. Xiao, 2013, Prolate spheroidal wave functions of order zero, Applied Mathematical Sciences Monograph Series, Vol. 187, Springer.
[263] Ozaydin and Przebinda, (2004), An entropy-based uncertainty principle for a locally compact abelian group, Journal of Functional Analysis, Vol. 215, pp.241-252.
[264] R. Paley and N. Wiener, 1987, Fourier Transforms in the Complex Domain, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence (Reprint of the 1934 original).
[265] A. Papoulis, (1975), A new algorithm in spectral analysis and bandlimited extrapolation, IEEE Transactions on Circuits Syst., Vol. 19, pp.735-742.
[266] S. Pei, M. Yeh and T. Luo, (1999), Fractional Fourier series expansion for finite signal and dual extension to discrete-time fractional Fourier transform, IEEE Trans. on signal processing, Vol. 47, No. 10, pp.2883-2888.
[267] S. Pei, J. Ding and J. Chang, (2001), Efficient implementation of quaternion Fourier transform convolution and correlation by 2-Dcomplex FFT, IEEE Trans. Signal Process, Vol. 49, pp.2783-2797.
[268] S. Pei and J. Ding, (2005), Generalized prolate spheroidal wave functions for optical finite fractional Fourier and linear canonical transform, J. Opt. Soc. Amer. A, Vol. 22, No. 3, pp.460-474.
[269] M. Petrich, (1963), On the Number of Orthogonal Signals which can be Placed in a WT-Product, J. Soc. Ind. Appl. Math., Vol. 11, pp.936940.
[270] A.I. Petrosyan, (2008), On weighted harmonic Bergman spaces, Demonst. Math., Vol. 41, pp.73-83.
[271] A.D. Polyanin and A.V. Manzhirov, 2007, Handbook of Mathematics for Engineers and Scientists, Chapman \& Hall/CRC.
[272] Ch. Pommerenke, (1964), Über die Faberschen Polynome schlichter Funktionen, Math. Z., Vol. 85, pp.197-208.
[273] I.R. Porteous, 1995, Clifford Algebras and the Classical Groups, Cambridge University Press, Cambridge.
[274] D. Ragozin, (1972), Uniform convergence of spherical harmonic expansions, Math. Ann., Vol. 195, pp.87-94.
[275] E.D. Rainville, 1960, Special Functions, The Macmillan Company, New York.
[276] D.K. RaO, (1976), A triangle inequality for angles in a Hilbert space, Revisa Colombiana de Mathematicas, Vol. X, pp.95-97.
[277] R. Remmert, 1998, Classical Topics in Complex Function Theory, Berlin: Springer-Verlag.
[278] J.B. Reyes and R. Delanghe, (2008), On the solutions of the Moisil-Théodoresco system, Math. Methods Appl. Sci., Vol. 31, No. 12, pp.1427-1439.
[279] J.B. Reyes and M. Shapiro, (2009), Clifford analysis versus its quaternionic counterparts, Math. Methods Appl. Sci., Vol. 33, No. 9, pp.1089-1101.
[280] D.R. Rhodes, (1963), The Optimum Line Source for the Best MeanSquare Approximation to a Given Radiation Pattern, IEEE Trans. Antennas and Propagation, Vol. 11, No. 4, pp.440-446.
[281] D.R. Rhodes, (1965), On some double orthogonality properties of the spheroidal and Mathieu functions, J. Math. and Phys., Vol. 44, pp.5265.
[282] F. Riesz and B. Sz-Nagy, 1955, Functional Analysis, Ungar, New York.
[283] M. Riesz, 1958, Clifford numbers and spinors, Inst. Phys. Sci. and Techn. Lect. Ser., Vol. 38. Maryland.
[284] K.F. Riley, M. P. Hobson and S.J. Bence, 1998, Mathematical Methods for Physics and Engineering: A Comprehensive Guide, American Journal of Physics, Vol. 67.
[285] V. Rokhlin and H. Xiao, (2007), Approximate formulae for certain prolate spheroidal wave functions valid for large values of both order and band-limit, Appl. Comput. Harmon. Anal., Vol. 22, No. 1, pp.105-123.
[286] B. Rubin, 2015, Introduction to Radon transforms: with elements of Fractional calculus and Harmonic Analysis, Cambridge University Press, Cambridge.
[287] J. Ryan, (1986), Left regular polynomials in even dimensions, and tensor products of Clifford algebras, In Clifford algebras and their applications in mathematical physics, J. S. R. Chisholm and A. K. Common (eds.), pp.133-147.
[288] I. Sabadini, M. Shapiro and D. Struppa, (2000), Algebraic analysis of the Moisil-Theodorescu system, Complex Var., Vol. 40, No. 4, pp.333-357.
[289] S. Said, N.L. Bihan and S.J. Sangwine, (2008), Fast complexified quaternion Fourier transform, IEEE Trans. Signal Process., Vol. 56, pp.1522-1531.
[290] S.J. Sangwine and T.A. Ell. The discrete Fourier transform of a colour image. In J. M. Blackledge and M. J. Turner, editors, Image Processing II Mathematical Methods, Algorithms and Applications, pages 430-441, Chichester, 2000. Horwood Publishing for Institute of Mathematics and its Applications. Proceedings Second IMA Conference on Image Processing, De Montfort University, Leicester, UK, September 1998.
[291] S.J. Sangwine and T.A. Ell, (2007), Hypercomplex Fourier transforms of color images, IEEE Trans. Image Process., Vol. 16, No. 1, pp.22-35.
[292] G. Sansone, 1959, Orthogonal Functions, Pure and Applied Mathematics, Vol. IX. Interscience Publishers, New York.
[293] G. Scheffers, (1893), Verallgemeinerung der Grundlagen der gewöhnlichen complexen Zahlen, Berichte kgl. Sächs. Ges. der Wiss., Vol. 52.
[294] K. Serkh, 2015, On generalized prolate spheroidal functions, Technical report, Technical Report TR-1519, Department of Mathematics, Yale University.
[295] Y. Shkolnisky, M. Tygert and V. Rokhlin, (2006), Approximation of bandlimited functions, Appl. Comput. Harmon. Anal., Vol. 21, No. 3, pp.413-420.
[296] Y. Shkolnisky, (2007), Prolate spheroidal wave functions on a discIntegration and approximation of two-dimensional bandlimited functions, Appl. Comput. Harmon. Anal., Vol. 22, No. 2, pp.235-256.
[297] F.J. Simons, F.A. Dahlen and M.A. Wieczorek, (2006), Spatiospectral concentration on a sphere, SIAM Review, Vol. 48, No. 3, pp.504-536.
[298] D. Slepian, (1954), Estimation of Signal Parameters in the Presence of Noise, IRE Trans. PGIT-3, pp.68-89.
[299] D. Slepian and H.O. Pollak, (1961), Prolate spheroidal wave functions, fourier analysis, and uncertainty - I, Bell System Tech. J., Vol. 40, pp.43-64.
[300] D. Slepian, (1964), Prolate spheroidal wave functions, Fourier analysis and uncertainity - IV: Extensions to many dimensions; generalized prolate spheroidal functions, Bell System Tech. J., Vol. 43, pp.30093057.
[301] D. Slepian, (1965), Some asymptotic expansions for prolate spheroidal wave functions, J. Math. Phys., Vol. 44, No. 2, pp.99-140.
[302] D. Slepian, (1976), On bandwidth, Proc. IEEE, Vol. 64, No. 3, pp.292-300.
[303] D. Slepian, (1978), Prolate spheroidal wave functions, Fourier analysis, and uncertainty-V: The discrete case, Bell System Tech. J., Vol. 57, No. 5, pp.1371-1430.
[304] D. Slepian, (1983), Some comments on Fourier analysis, uncertainty, and modeling, SIAM Rev., Vol. 25, No. 3, pp.379-393.
[305] K. Smith, (1990), The uncertainty principle on groups, SIAM J. Appl. Math., Vol. 50, No. 3, pp.876-882.
[306] F. Sommen, (1982), Hypercomplex Fourier and Laplace transforms I, Illinois J. Math., Vol. 26, No. 2, pp.332-352.
[307] F. Sommen, (1983), Hypercomplex Fourier and Laplace transforms II, Complex Var., Vol. 1, No. 2-3, pp.209-238.
[308] F. Soltani, (2014), L L Donoho-Stark Uncertainty Principles for the Dunkl Transform on $\mathbb{R}^{d}$, J. Phys. Math., Vol. 5, No. 1.
[309] R. Soummer, C. Aime and P.E. Falloon, (2003), Stellar coronagraphy with prolate apodized circular apertures, Astron. Astrophys., Vol. 397, No. 3, pp.1161-1172.
[310] W. Sprössig, (1978), Räumliches analogon zum komplexen Toperator, Beiträge zur Analysis, Vol. 12, pp.127-137.
[311] E.M. Stein and G. Weiss, (1960), On the theory of harmonic functions of several variables, Part I: The theory of $H^{p}$ spaces, Acta Math., Vol. 103, No. 1-2, pp.25-62.
[312] E. Stein, 1970, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton NJ.
[313] E. Stein and G. Weiss, 1987, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton NJ.
[314] T.J. Stieltjes, (1885), Sur certains polynômes qui vérifient une équation différentielle linéaire du second ordre et sur la théorie des fonctions de Lamé, Acta Mathematica, 6, pp.321-326.
[315] J.A. Stratton, P.M. Morse, L.J. Chu and R.A. Hutner, 1941, Elliptic Cylinder and Spheroidal Wave Functions, John Wiley \& Sons, New York.
[316] J.A. Stratton, P.M. Morse, L.J. Chu, J.D.C. Little AND F.J. Corgbato, 1956, Spheroidal Wave Functions, John Wiley \& Sons, New York.
[317] M.J.O. Strutt, (1932), Lamésche - Mathieusche - und Verwandte Funktionen in Physik und Technik, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 1, Verlag Julius Springer, Berlin (In German).
[318] A. Sudbery, (1979), Quaternionic analysis, Math. Proc. Cambridge Phil. Soc., Vol. 85, pp.199-225.
[319] D. Suschowk, (1959), Formulae for 25 of the Associated Legendre Functions of the Second Kind, Mathematical Tables and Other Aids to Computation, Vol. 13, No. 68, pp.303-305.
[320] G. SzegÖ, (1935), A problem concerning orthogonal polynomials, Trans. Amer. Math. Soc., Vol. 37, pp.196-206.
[321] W.J. Tango, (1977), The circle polynomials of Zernike and their application in optics, Appl. Phys., Vol. 13, pp.327-332.
[322] O. Teichmüller, (1936), Operatoren im Wachsschen Raum, J. reine und angew. Math., Vol. 174, pp.73-124.
[323] W. Thomson and P.G. Tait Oxford, 1867, Treatise on natural philosophy, Oxford.
[324] W.J. Thompson, 1997, Atlas for Computing Mathematical Functions, John Wiley \& Sons.
[325] W.J. Thompson, 1999, Spheroidal wave functions, Computing in Science and Engineering.
[326] G. Walter and T. Soleski, (2005), A new friendly method of computing prolate spheroidal wave functions and wavelets, Appl. Comput. Harmon. Anal., Vol. 19, No. 3, pp.432-443.
[327] G. Walter, (2005), Prolate spheroidal wavelets: translation, convolution, and differentiation made easy, J. Fourier Anal. Appl., Vol. 11, No. 1, pp.73-84.
[328] G. Walter and X. Shen, (2004), Wavelets based on prolate spheroidal wave functions, J. Fourier Anal. Appl., Vol. 10, No. 1, pp.126.
[329] G. Walter and X. Shen, (2003), Sampling with prolate spheroidal wave functions, Sampl. Theory Signal Image Process., Vol. 2, No. 1, pp.25-52.
[330] E.T. Whittaker and G.N. Watson, 1927, A course in modern analysis, Cambridge, Cambridge Univ. Press.
[331] A. Wiman, (1914), Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem größten Gliede der zugehörigen Taylorschen Reihe, Acta Math., Vol. 37, pp.305-326.
[332] E. T. Whittaker and G. N. Watson, 1920, A Course of Modern Analysis, Cambridge University Press.
[333] J. Whittaker, 1949, Sur les séries de base de polynômes quelconques, Collection de monographies sur la theorie des fonctions, Paris: GauthierVillars., Vol. VI.
[334] H. Xiao, V. Rokhlin and N. Yarvin, (2001), Prolate spheroidal wavefunctions, quadrature and interpolation. Inverse Problems, Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000), Vol. 17, No. 4, pp.805-838.
[335] H. Xiao, 2001, Prolate spheroidal wave functions, quadrature, interpolation, and asymptotic formulae, Ph.D. diss., Yale University.
[336] Z. Xu, 1989, Boundary Value Problems and Function Theory for SpinInvariant Differential Operators, Ph.D. diss., Ghent University.
[337] Z. Xu, J. Chen and W. Zhang, (1990), A harmonic conjugate of the Poisson kernel and a boundary value problem for monogenic functions in the unit ball of $\mathbb{R}^{n}(n \geq 2)$, Simon Stevin, Vol. 64, pp.187-201.
[338] Z. Xu and C. Zhou, (1993), On boundary value problems of Riemann-Hilbert type for monogenic functions in a half space of $\mathbb{R}^{m}$ ( $m \geq 2$ ), Complex Variables: Theory and Appl., Vol. 22, pp.181-194.
[339] Y. Yang and K. Kou, (2016), Novel uncertainty principles associated with 2D quaternion Fourier transforms, Integral Transform Spec. Funct., Vol. 27, No. 3, pp.213-226.
[340] M.H. Yeh, (2008), Relationships among various 2-D quaternion Fourier transforms, IEEE Signal Process. Lett., Vol. 15, pp.669-672.
[341] A.I. ZAYED, (2007), A generalization of the prolate spheroidal wave functions, Proc. Amer. Math. Soc., Vol. 135, No. 7, pp.2193-2203.
[342] M. Zayed, M. Abul-Ez and J. Morais, (2012), Generalized derivative and primitive of Cliffordian bases of polynomials constructed through Appell monomials, Comput. Meth. Func. Theo., Vol.12, pp.501-515.
[343] S.M. Zemyan, 2010, The classical theory of integral equations a concise treatment, New York: Springer.
[344] F. Zernike, (1934), Diffraction theory of the knife-edge test and its improved version, the phase-contrast method, Monthly Notices of the Royal Astronomical Society, Vol. 94, No. 5, pp.377-384.
[345] F. Zernike and H.C. Brinkman, (1935), Hypersphärische Funktionen und die in sphärischen Bereichen orthogonalen Polynome, Proc. Royal Acad. Amsterdam, Vol. 38, pp.161-170.
[346] S. Zhang and J. Jin, 1996, Computation of Special Functions, Wiley, New York.
[347] S. Zhao, (1992), On the weighted Lp-integrability of nonnegative Msuperharmonic functions, Proc. Am. Math. Soc., Vol. 115, pp.677-685.
[348] H. Zhao, Q. Ran, J. Ma and L. Tan, (2010), Generalized prolate spheroidal wave functions associated with linear canonical transform, IEEE Trans. Signal Process., Vol. 58, No. 6, pp.3032-3041.
[349] H. Zhao, R. Wang, D. Song and D. Wu, (2014), Maximally concentrated sequences in both time and linear canonical transform domains, Signal Image Video Process., Vol. 8, No. 5, pp.819-829.
[350] C. Zou, K. Kou and J. Morais, (2018), Prolate spheroidal wave functions associated with the quaternionic Fourier transform, Math. Methods Appl. Sci., Vol. 41, No. 11, pp.4003-4020.

## Index

$L_{2}$-norm, 33
$\varepsilon_{\mathbf{T}^{-}}$-concentrated, 150
$\varepsilon_{\mathbf{W}}$-concentrated, 150
c-QPSWFs, 163
adjoint operation, 37
all-pass filtering form, 169
angle, 34
angular function, 65
angular prolate spheroidal equation, 51
angular prolate spheroidal functions, 51
angular-frequency variables, 58
antimonogenic function, 39
Appell differentiation, 81
associated Legendre functions of the first kind, 46
associated Legendre functions of the second kind, 46
azimuthal angle, 65
azimuthal function, 65
band-limited function, 150
band-limiting operator, 151
basic external monogenic spheroidal functions, 115
basic internal ambigenic spheroidal polynomials, 125
basic internal contragenic
spheroidal polynomials, 129
basic internal monogenic spheroidal polynomials, 94
basic spheroidal harmonics, 66
Bessel function of the first kind, 46
Bloch's Theorem, 217
Bochner-Minlos Theorem, 56
Bohr radius, 219
boundary, 30
bounded operator, 35, 158
canonical angle, 178
Cassini surfaces, 67
Cauchy's integral formula, 41
Cauchy-Bunyakovsky-Schwarz inequality, 33
Cauchy-Fueter kernel, 40
characteristic function, 150
characteristic quaternion equation, 159
Clifford algebra, 29
closed, 36
closure, 30
coaxial spheroidal domain, 64
colatitude angle, 65
compact operator, 35, 156, 158
complete orthogonal set, 35
completeness, 35
conjugate generalized

Cauchy-Riemann operator, 38
contragenic function, 128
degenerate eigenfunctions, 159
eccentricity, 49, 64
eigenfunction, 37
eigenfunctions, 158
eigenmanifold, 37
eigenvalue, 37
eigenvalues, 158
energy conservation ratio, 176
Euclidean unit ball, 64
external harmonics, 67
external solid spherical harmonics, 69
finite-Hankel transform, 191
finite-QFT form, 163
frequency-domain, 149
Fubini's Theorem, 155, 164
Funk-Hecke formula, 189
Gaussian hypergeometric function, 47
generalized Cauchy-Riemann operator, 38
generalized Shannon number, 172
harmonic Hardy space, 42
harmonic weighted Bergman spaces, 42
harmonic weighted Hardy spaces, 42
Helmholtz operator, 49
Hilbert-Schmidt norm, 161
homogeneous ambigenic polynomials, 123
hypercomplex derivative, 43
internal harmonics, 67
internal solid spherical harmonics, 69
internal solid spherical monogenics, 97
inverse quaternion Fourier kernel, 61

Kronecker delta, 29
Laplace operator, 41
Lebesgue's Dominated Convergence Theorem, 59
left-linear space, 30
Legendre function of the second kind, 45
Legendre polynomial, 45
Legendre's associated differential equation, 45
Legendre's differential equation, 45
low-pass filtering form, 167
maximum modulus, 213
mean-square error, 173
Mercer's Theorem, 171
Moisil-Teodorescu system, 39
monogenic Bergman kernel function, 112
monogenic constant, 43
monogenic function, 38
monogenic Hardy space, 42
monogenic primitive, 44
monogenic Sobolev-type space, 43
monogenic weighted Bergman spaces, 42, 210
monogenic weighted Hardy spaces, 42, 204

Neumann's formula, 45
norm, 28
oblate spheroid, 63,64
operator norm, 35
orthogonal, 33
orthogonal Appell basis, 107
orthogonal basis, 35
orthogonal projection, 37, 154
orthonormal set, 35
pair of conjugate harmonic functions, 197
Parseval's identity, 61, 165
Plancherel's Theorem, 61
Poisson kernel, 205
positive definite kernel, 158
positive operator, 37,158
prolate spheroid, 63, 64
prolate spheroidal coordinates, 49 , 65
prolate spheroidal domain, 49
prolate spheroidal harmonics, 65
proper ambigenic spheroidal polynomials, 137
proper spheroidal harmonics, 75
pure quaternion, 27
quaternion, 27
quaternion conjugate, 27
quaternion exponential function, 28
quaternion Fourier kernel, 58
Quaternion Fourier Transform, 56
Quaternion Fourier-Stieltjes Transform, 56
quaternion inverse, 28
Quaternion Riesz Representation Theorem, 36
quaternion-valued function, 30
quaternionic Hilbert space, 33
quaternionic Hilbert-Schmidt operator, 157
quaternionic inner product, 32
quaternionic Zernike spherical polynomials, 193
radial coordinate, 65
radial function, 65
radial prolate spheroidal equation, 51
radial prolate spheroidal functions, 51
reduced quaternion-valued function, 30
reduced quaternions, 30
reproducing kernel, 151
Reproducing Kernel Quaternion
Hilbert Space, 36
Riemann-Lebesgue, 59
Riesz system, 40
scalar inner product, 33
scalar part, 27
self-adjoint, 37
self-adjoint operator, 158
Shannon number, 53
Slepian frequency, 54, 163
space-bandwidth product, 160, 172
space-limited function, 150
space-limiting operator, 150
space-variables, 58
spatial-domain, 149
Spectral Theorem, 160
spherically symmetric, 60
surface spherical harmonics, 120
symmetric kernel, 158
the band-limited quaternionic signal extrapolation, 174
The Fourier expansion of a square-integrable monogenic function defined in a spheroid, 105
the left-sided quaternionic Slepian series, 172
The Prolate spheroidal wave functions, 49
the right-sided Quaternion Fourier
Transform, 58
the scalar Helmholtz equation, 49
the skew-field of quaternions, 28
The Spectral Theorem, 38
Tonelli-Hobson Theorem, 154, 177
total energy, 61
two-sided monogenic function, 40

Uncertainty principle of Donoho and Stark, 160
unit quaternion, 28
universal contragenic functions, 137
vector part, 27
Zernike circle polynomials, 192


[^0]:    ${ }^{1}$ The definition, here made of a continuous quaternionic function with compact support, will be discussed in detail in Subsection 4.2.1

