VIRTUAL MOSAIC KNOTS

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by

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ABSTRACT

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The study of knots and links is a main focus of the mathematical branch of topology. Classical knot theory studies knots embedded in 3-dimensional real space and has been a primary field of study since the 1 960's. Virtual knot theory, first introduced by Kauffman in 1999, studies knots embedded in thickened surfaces. Lomanoco and Kauffman introduced mosaic diagrams in order to build a quantum knot system in 2008. In 2009, Garduño extended these mosaic diagrams to include virtual knots. In order to represent knots on surfaces, Ganzell and Henrich introduced virtual mosaic knot theory in 2020 by placing knots onto $n \times n$ polygonal representations of surfaces. We extend the idea of virtual mosaic knot theory to include *virtual rectangular mosaics*, a placement of virtual knots onto $n \times m$ polygonal representations of surfaces, as well as row mosaics, a placement of virtual knots onto $1 \times m$ polygonal representations of surfaces. In this thesis, we introduce virtual rectangular mosaics and give two rectangular mosaic invariants called the tile number and row number. Included as an appendix, we give a complete row mosaic tabulation of knots with 8 or fewer crossings and virtual knots up to 4 crossings.

KEY WORDS: Topology; Knot Theory; Virtual knots; Mosaic knots

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1. Classical Knots

Knot theory is thought to have been first introduced in the 18th century by a French mathematician named Alexandre-Theophile Vandermonde. However, it was not until the 19th century that Carl Friedrich Gauss made the first steps towards the mathematical theory that we know today.

A *knot* is a simple closed curve in three-dimensional space which can be described as an injective function, $K := S^1 \hookrightarrow S^3$. The image has no loose ends, no starting or ending points, and no self-intersections. Typically, we view a knot as a projection onto $\mathbb{R}^2$. This projection is called a *knot diagram*. Examples of knot diagrams are shown in Figure 1. We study knots up to isotopy, which allows for twisting and stretching but not breaking the knots. The first and second knots in Figure 1 are both the *unknot*, also referred to as the trivial knot. The unknot is the only knot that bounds an embedded disk in 3-space.

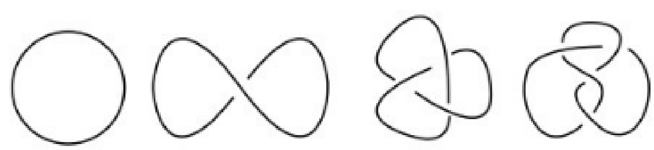


FIGURE 1. Example of knot diagrams.

Knots are composed of a single connected *component*, meaning a knot is one closed loop that passes under and over itself. Choosing a point anywhere on the knot and tracing the knot, we will always return to the starting point. A *link* is a finite union of pairwise nonintersecting knots, $L = K_1 \cup \cdots \cup K_n$. In this way, a knot can be considered a 1-component link. Two examples of simple links are given in Figure 2.

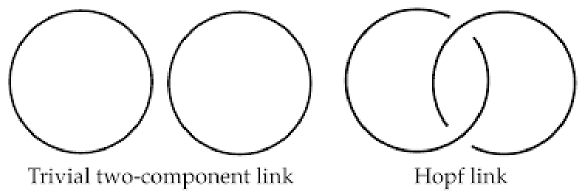


FIGURE 2. Example of links.

Two knot diagrams are called *equivalent* or *isotopic* if they can be physically manipulated (rotated, bent, twisted, stretched, etc.), without splitting and reconnecting the component, so that one looks identical to the other. *Reidemeister moves* are local modifications, which means we apply them on a smaller part of the knot diagram, leaving the rest of the diagram unchanged. There are three such moves, shown in Figure 3. Two knot diagrams represent equivalent knots if and only if they can be related by a finite sequence of Reidemeister moves.

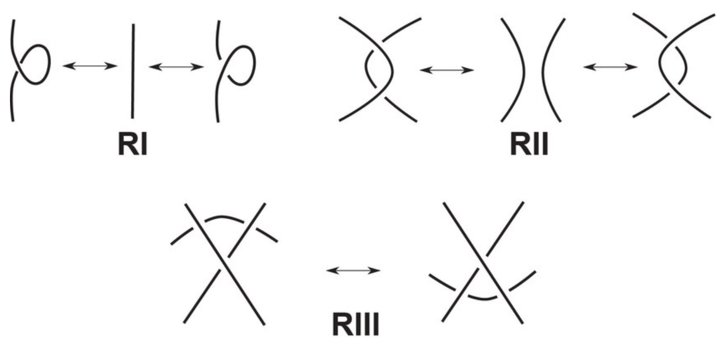


FIGURE 3. Reidemeister moves

Figure 4 shows an example of performing knot isotopy via the Reidemeister moves. What seems like a complicated knot is actually equivalent to the unknot. Each diagram in the figure differs from the next by one Reidemeister move. The *crossing number* of a knot, denoted $c(K)$, is the number of crossings the knot has.

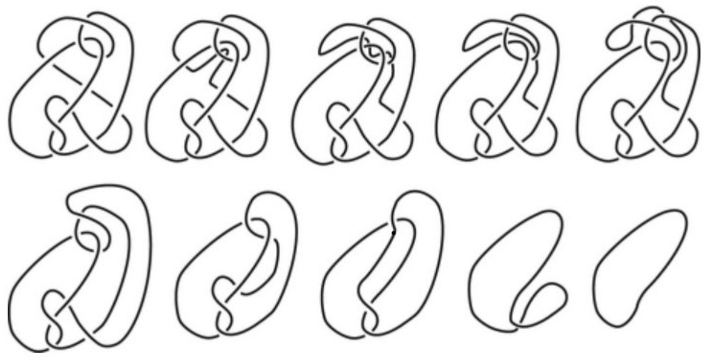


FIGURE 4. Example using Reidemeister moves.

A link can be given an *orientation* by assigning a direction of travel around each loop [6]. An example of an oriented knot is shown in Figure 5.

FIGURE 5. Example of an oriented knot.

Given an oriented knot we can define the *sign of a crossing, c*, denoted $sign(c)$ to be either positive or negative according to Figure 6. Note that the overstrand in a positive crossing has positive slope.

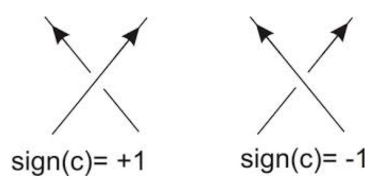


FIGURE 6. Sign of a crossing.

A knot is considered *alternating* if there is a pattern of under, over, under, over, etc. when following the orientation and tracing the knot.

1.2. Surfaces

In topology, a *surface* Σ is a two-dimensional manifold - a space that locally looks like the real plane. Every surface has a *genus* which can be thought of as the number of holes. Figure 7 has some examples of surfaces and their corresponding genus. There are non-orientable surfaces, such as the Mobius band or the Klein bottle, which means the surface has 1 side, and there are orientable surfaces which have two distinct sides. A classical result in topology is that surfaces can be classified according to their orientability and g enus. However, we will only consider orientable surfaces.

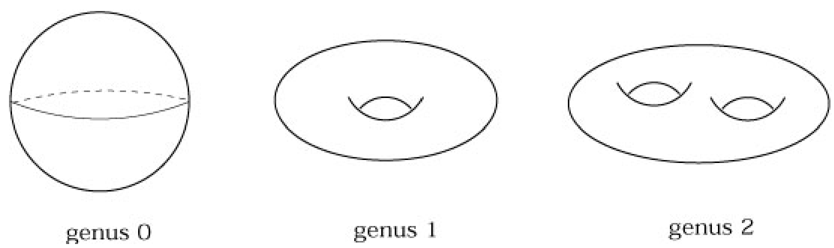


FIGURE 7. Examples of surfaces.

Some surfaces can be represented by polygons. Starting with a $2n$-sided polygon, label the edges $\{a_1, \ldots, a_n\}$ where each edge label appears exactly twice. In the form of an arrow, each edge label is given a direction. Using the edge labelings and matching the arrows, we can "glue" paired edges. This quotient space is a *closed surface*. This is defined in [7]. An example of this is shown in Figure 8. In this example we would read off the edge labeling, starting with the a on the left, as a *boundary word*, $aba^{-1}b^{-1}$, where the inverse notation describes a backwards arrow on the edge. The surface in Figure 8 is called the *torus* and it is an orientable surface of genus 1.

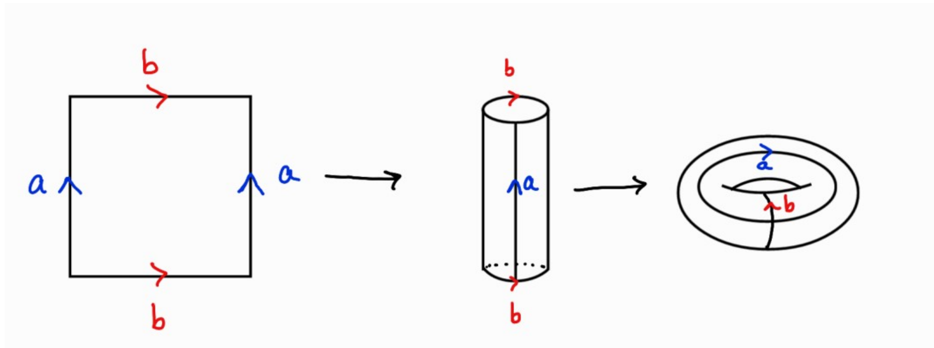


FIGURE 8. Example of the torus represented as a polygon.

The polygon represents an orientable surface if and only if the second occurrence of each edge labeling has the arrow pointed in the opposite direction of the first label. Reading off the boundary word, we would have each edge a_i appear as a_i once and a_i^{-1} once [7]. Furthermore, a polygon represents a surface with positive genus if and only if its boundary word contains a pattern of the form $a \ldots b \ldots a^{-1} \ldots b^{-1}$ [[7]Theorem 6.14].

1.3. Virtual Knots

We usually study knots as objects in the 3-sphere, S^3. However, knots can be embedded in any 3-dimensional space. *Virtual knots* and *virtual links* are embedded on thickened surfaces, $\Sigma \times [0,1]$. These knots can contain crossings that are not formed by a part of the component passing over and under itself. An example of a knot and link embedded on a thickened surface along with their corresponding virtual diagrams are illustrated in Figure 9.

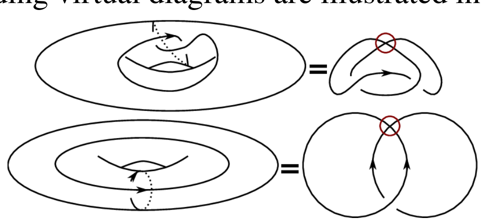


FIGURE 9. A knot and link embedded on a thickened torus.

The dotted segments represent where the components wrap around the *meridian* resulting in two parts of the components lying on different sides of the surface, which can only happen on a surface with positive genus. This creates a crossing in where a dotted segment and a solid segment intersect. These crossings are called *virtual crossings*. The corresponding knot diagram for a virtual knot with a virtual crossing will have a crossing that does not have under or overstrand, and a circle around the crossing. Figure 10 is an example of a virtual knot on a genus 2 surface.

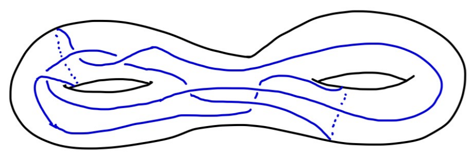


FIGURE 10. Knot on genus 2 surface.

As with classical knots, we can determine which virtual knots are equivalent up to isotopy using virtual Reidemeister moves. These moves are analogous to the classical ones, along with a fourth move as pictured in Figure 11.

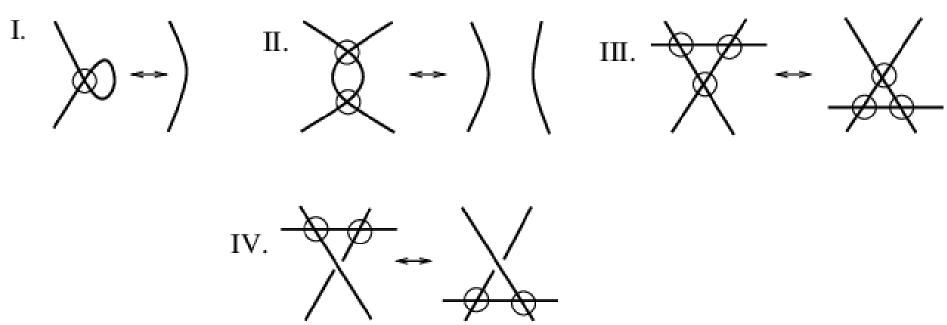


FIGURE 11. The four virtual Reidemeister moves.

Figure 12 shows the two "forbidden moves"; these moves pass an arc behind or in front of a virtual crossing. However, forbidden moves do not preserve the virtual knot, and so are forbidden.

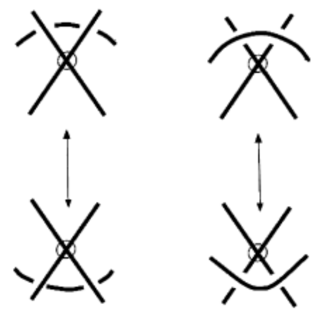


FIGURE 12. The forbidden moves.

1.4. Gauss Codes and Gauss Diagrams

Gauss diagrams provide a way of encoding knot information. Given an oriented knot diagram, we can create a Gauss diagram and, given a Gauss diagram, we can recover the corresponding knot. Beginning with an oriented knot:

1. Choose a base point on the knot, label each crossing from 1 to n in order they are passed.

2. Label each crossing with the sign of that crossing.

3. Label the outside of a circle, in a counterclockwise direction, the order of crossings in which they are passed through when starting, and ending at the base point of the knot.

4. Then insert arrows which start at one end of the circle to the other. The arrow points away from the label when going over the crossing, and towards the label when going under.

5. Each arrow is then labeled, at the tail, with the sign of the crossing.

An example of going from an oriented knot to a Gauss diagram is shown in Figure 13.

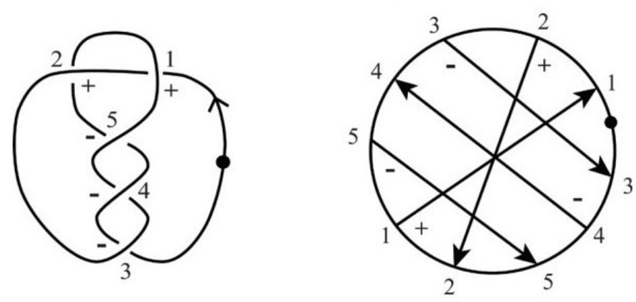


FIGURE 13. A knot and its corresponding Gauss diagram.

Using the Gauss diagram or the labeled knot diagram we can obtain the *Gauss code*. Start at the first crossing and list out the order of the crossings starting and ending at the base point of the knot diagram, or listing the order of the crossings from the Gauss diagram in a counterclockwise order. Thus, in this example, the order is as follows:

1 2 3 4 5 1 2 5 4 3.

Each crossing appears exactly twice in the Gauss code - once for going over and once for going under. To complete the Gauss code, each time the crossing appears in the sequence add in the under (U) and overstrand (O) information. Therefore, the complete Gauss code for this example is

U1 O2 O3 U4 O5 O1 U2 U5 O4 U3.

Another way this could be written is

-1 2 3 -4 5 1 -2 4 -3,

where each U from the previous notation is a negative sign. Given a Gauss diagram, a knot is alternating if consecutive arrow heads are facing the opposite way when moving around the circle, and no two consecutive arrows point in the same direction. If a Gauss diagram has all non intersecting arrows, then the Gauss diagram is of the unknot.

1.5. Intersection Index Polynomial

Knot invariants are equivalence relations which assign a given knot a characteristic that is then used to distinguish it from other knots. For any knot invariant α and knots K and K', if $\alpha(K) \neq \alpha(K')$, then K and K' are not isotopic. Such an invariant is the intersection index polynomial. We describe an algorithm for computing the intersection index polynomial from a knot diagram as defined in [6]. Given a knot diagram K, and a crossing d we *smooth* crossing d as in Figure 14.

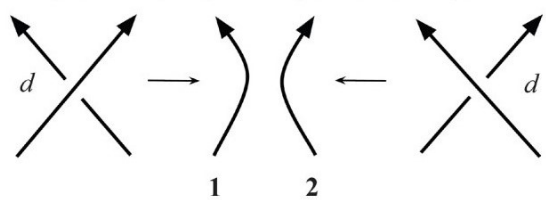


FIGURE 14. Smoothing a crossing.

Note that smoothing a crossing changes the knot; in particular, smoothing a crossing in a knot diagram produces a 2-component link. When following the orientation of a link, the smoothing is labeled with a 1 on the left side and a 2 on the right side. Let C_d be the set of classical crossings in the virtual link diagram that involve both components after crossing d is smoothed. Then each crossing in C_d is assigned a ± 1 in Figure 15. Component 2 on the left results in a $+1$ and component 1 on the left results in a -1.

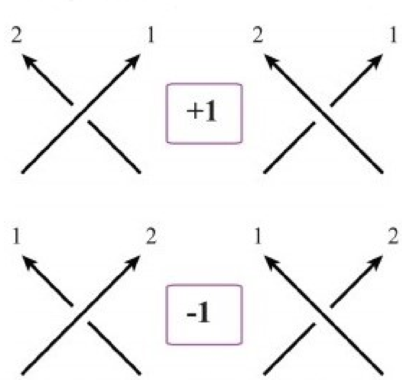


FIGURE 15. Crossings that involve two components.

For a virtual knot diagram K, and a crossing d, *the intersection index* $i(d)$ is the sum of the values $\alpha(x)$ for all classical crossings $x \in C_d$, where $\alpha(x)$ is the ± 1 that is dependent on the crossings involving both components. Therefore, $i(d) = \sum_{x \in C_d} \alpha(x)$. Figure 16 is an example of computing the intersection index of a virtual knot with 4 classical crossings labeled $\{a, b, c, d\}$.

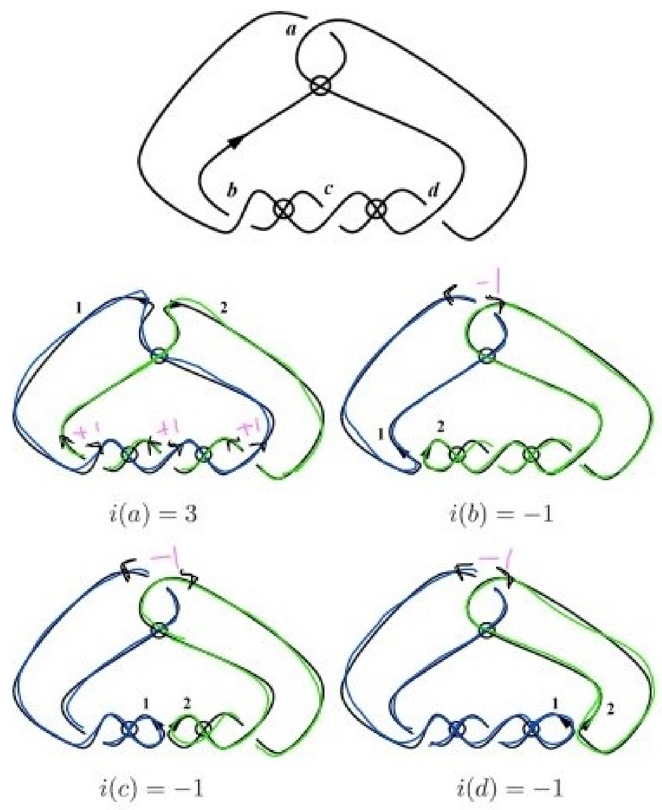


Figure 16. Example of calculating the intersection index.

The *intersection index polynomial*, $p_t(K)$, for a virtual knot K with diagram D is the sum over all classical crossings d in D of the polynomial $sign(d)(t^{|i(d)|} - 1)$. Therefore, we have the following:

$$p_t(K) = \sum_{d \in D} sign(d)(t^{|i(d)|} - 1).$$

Recall $sign(d)$ is the sign of the crossing defined in section 1.1. Therefore, using the computed intersection index of each crossing, the knot in the previous example has the following intersection index polynomial:

$$p_t(K) = (1)(t^3 - 1) + (1)(t^{|-1|} - 1) + (1)(t^{|-1|} - 1) + (1)(t^{|-1|} - 1)$$

$$= t^3 - 1 + t - 1 + t - 1 + t - 1$$

$$= t^3 + 3t - 4.$$

This polynomial is invariant of the virtual Reidemeister moves and thus a true invariant [6].

CHAPTER II

MOSAIC KNOTS

2.1. Classical Mosaic Knots

Mosaic knots, introduced by Lomanoco and Kauffman in [9], place a knot diagram onto an $n \times n$ grid of *suitably connected* tiles, where the tiles form a knot diagram and all the arcs on the tiles connect. Mosaic knots are formed from the 11 tiles illustrated in Figure 17. Figure 18 gives examples of mosaic knots.

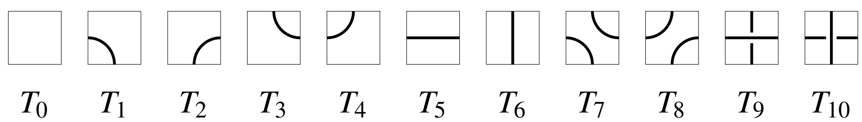


FIGURE 17. Mosaic tiles.

FIGURE 18. Examples of mosaic knots.

The *mosaic number* of a knot, K, denoted $m(K)$, is the smallest integer n for which K can be represented by an $n \times n$ mosaic.

2.2. Virtual Mosaics

Ganzell and Henrich extended the definition of mosaic knots to include virtual knots. A *virtual mosaic*, defined in [5], is an $n \times n$ array of standard mosaic tiles, together with an identification of the $4n$ edges of the array boundary, that forms a knot or link diagram on a closed, orientable surface. The genus of the surface will also be the genus of the mosaic. Figure 19 shows the virtual trefoil represented on a 2×2 virtual mosaic. Virtual crossings are created by intersections coming from the outer edge identification. Furthermore, a classical knot can be realized as a virtual mosaic in which the edge identifications do not intersect when connected.

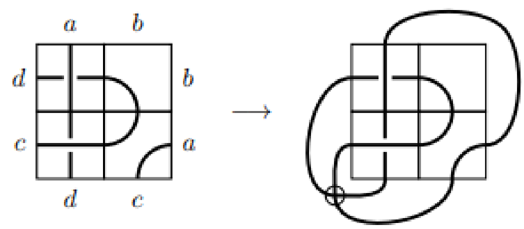


FIGURE 19. Virtual trefoil on virtual mosaic

The *virtual mosaic number* of a knot, K, denoted $m_v(K)$, is the smallest integer *n* for which *K* can be represented by an $n \times n$ virtual mosaic. An efficient way to place a knot on a mosaic is to use the Gauss Code. In placing the knot on the mosaic, it is helpful to keep in mind the crossings that precede or follow a particular crossing in the subsequence, as well as the maximum subsequence in which no crossings are repeated. However, different projections of the same knot have different Gauss codes resulting in different lengths of subsequences that have no crossings repeated.

Proposition 1. [[5]] *The figure-8 knot does not fit on a* 2×2 *mosaic.*

Proof. First, we assign an orientation to the knot and a base point, Figure 20.

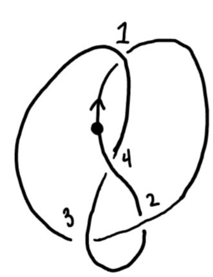


FIGURE 20. Oriented and labeled figure-8 knot.

Tracing the knot we obtain the following Gauss code:

-1 2 -3 1 -4 3 -2 4.

This is an alternating knot and moreover every Gauss code representation must have a sequence of four consecutive crossings. In this case 2 -3 1 -4 is one of the times there are four distinct crossings in a row with no repetition. Therefore, at some point in the sequence, we pass through all four crossings before coming to a crossing already accounted for. Using this information, we will prove by contradiction that the figure-8 knot, which has a minimum of four crossings, has virtual mosaic number 3. Figure 21 will illustrate this process.

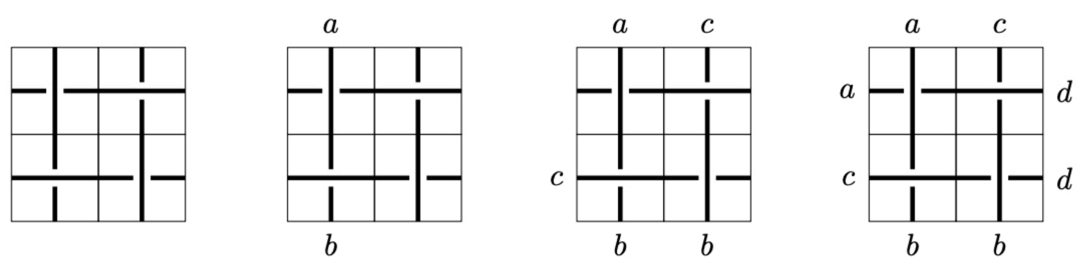


FIGURE 21. Finding virtual mosaic number for the figure 8 knot.

Suppose the figure-8 knot did fit on a 2×2 mosaic. Then since this knot is alternating we would need an alternating pattern of T_9 and T_{10} tiles as in Figure 21. Without loss of generality, we can choose an edge and label it a (in this example we started with the top left hand corner). To avoid creating a set of links label the edge directly below a with a new label, namely b, as shown in Figure 21. Since this knot is alternating, and the first b is placed at an understrand, we need the other b at an edge where the strand is going over. However, to prevent virtual crossings there must be no edges between a given pair or an even amount of edges between them. Therefore, the second b can only be placed directly to the right of the first as shown in Figure 21. Consequently, the top right label must then be a new label, c, or we would create a set of links. We have also now created the sequence of four consecutive crossings. There are only two open spots to put the second c, and these are the top right tile next to the first or the bottom left next to b. Putting c in the top right allows for a Reidemeister one move. Hence, the second c must be placed in the bottom left corner. Lastly, in Figure 21 we have both d's on the right side of the mosaic and put in the last a.

However, connecting the outer edges results in a Reidemeister one move in the top left hand corner where the a labels connect. Therefore, we can reduce the diagram to a 3 crossing diagram. Hence, the virtual mosaic number of the figure-8 knot is 3 and this is illustrated in Figure 22. □

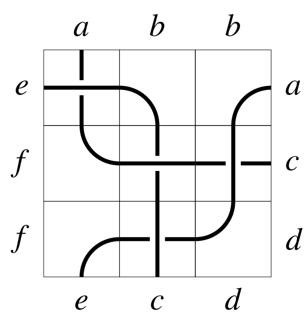


FIGURE 22. Figure-8 knot on a 3×3 mosaic.

The following is a result from [5] which provides a relationship between the classical mosaic number and the virtual mosaic number of a given knot.

Proposition 2. [[5], Proposition 1] *For a link, L, let $m(L)$ be the classical mosaic number and $m_v(L)$ the virtual mosaic number. For a link or a nontrivial knot, $m_v(L) \leq m(L) - 2$.*

Proof. Given a classical mosaic, since there are no T_9 or T_{10} tiles in the first or last rows and columns, we can delete those and identify the edges to represent the original connections. Then label edges that do not have connections with matched pairs. □

Figure 23 provides an example of this proposition using the 7_2 knot. However, this inequality is not sharp since $m(7_2) = 6$ and $m_v(7_2) = 3$, as illustrated in Figure 24.

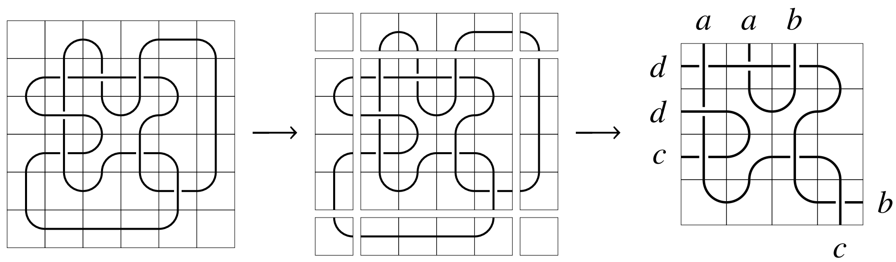


FIGURE 23. Going from a 6×6 classical mosaic to 4×4 virtual mosaic.

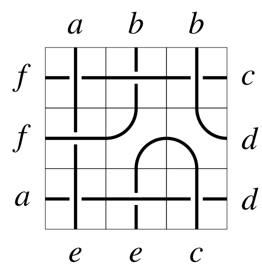


FIGURE 24. The 7_2 knot on a 3×3 mosaic.

2.3. Classical and Virtual Mosaic Moves

In the introduction, we introduced Reidemeister moves, which are used to relate equivalent knot diagrams. Since we study knots up to isotopy, it is necessary to have the same ability on a mosaic. Therefore, in this section, we give a collection of mosaic moves introduced in [5], that preserve knot type. Note in the following illustrations the light gray arcs are arcs that may or may not be on the mosaic. The dotted arcs represent a different tile that could also be there instead of the solid black line, so that each move represents several possible configurations. Each of these moves can be rotated or reflected on a different position of the mo sa ic. If there are no edge identifications on the mosaic move then the move can be performed on the edge or in the middle of the mosaic. Moves with labeled edges must be done on the edge. The first two sets of moves pertain to the interior of the mosaic and the last set are for virtual mosaics. The last set is the only set which involves the edge identifications.

The *planar isotopy moves* are shown in Figure 25. These moves are performed by replacing the tiles and replicate the local modifications done with classical knots, i.e. bending, stretching, shifting etc. For example, the P_1 move slides the segment from the top left tile to the bottom right tile.

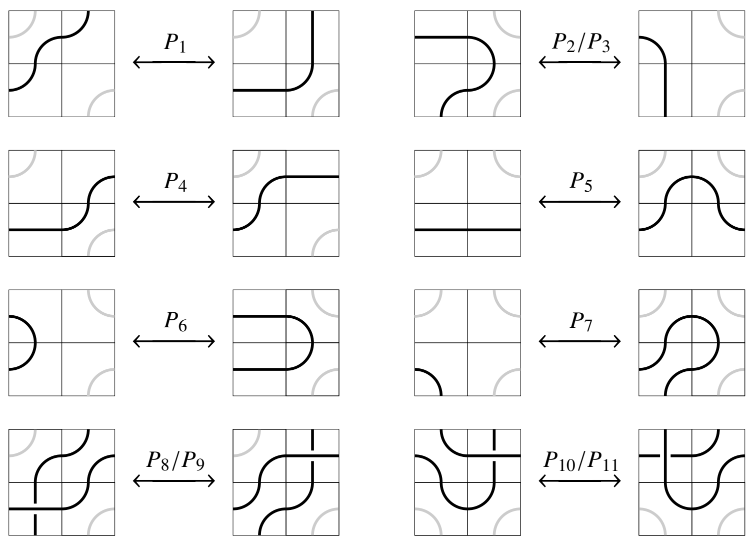


FIGURE 25. List of planar isotopy moves.

The next set of moves are tile replacements that imitate the Reidemeister moves, these are shown in Figure 26.

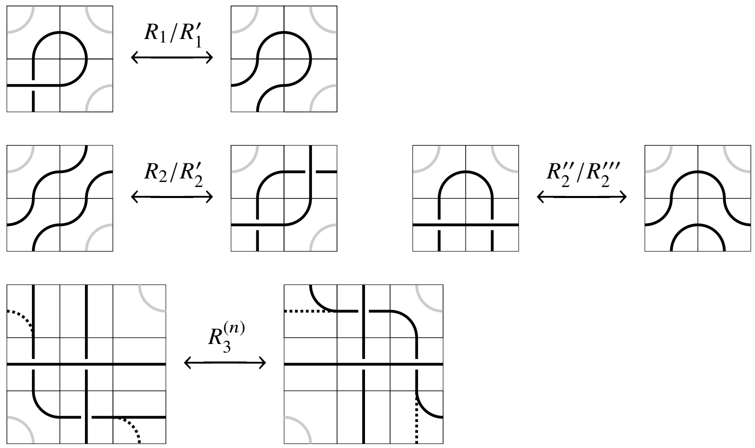


FIGURE 26. Reidemeister moves on a mosaic.

Surface isotopy moves, shown in Figure 27, allow for local modifications that involve the switching of outer edge identifications along with the strands on associated tiles. In the following diagrams, when one edge identification is switched with another edge identification, we must preserve the connections. To do so, we replace the tiles in a way that moves the strand and places it with the corresponding edge identification. For example, in the SI_3 move the edge identifications a and x are switched. Accordingly, the strand in which a is the connection must follow.

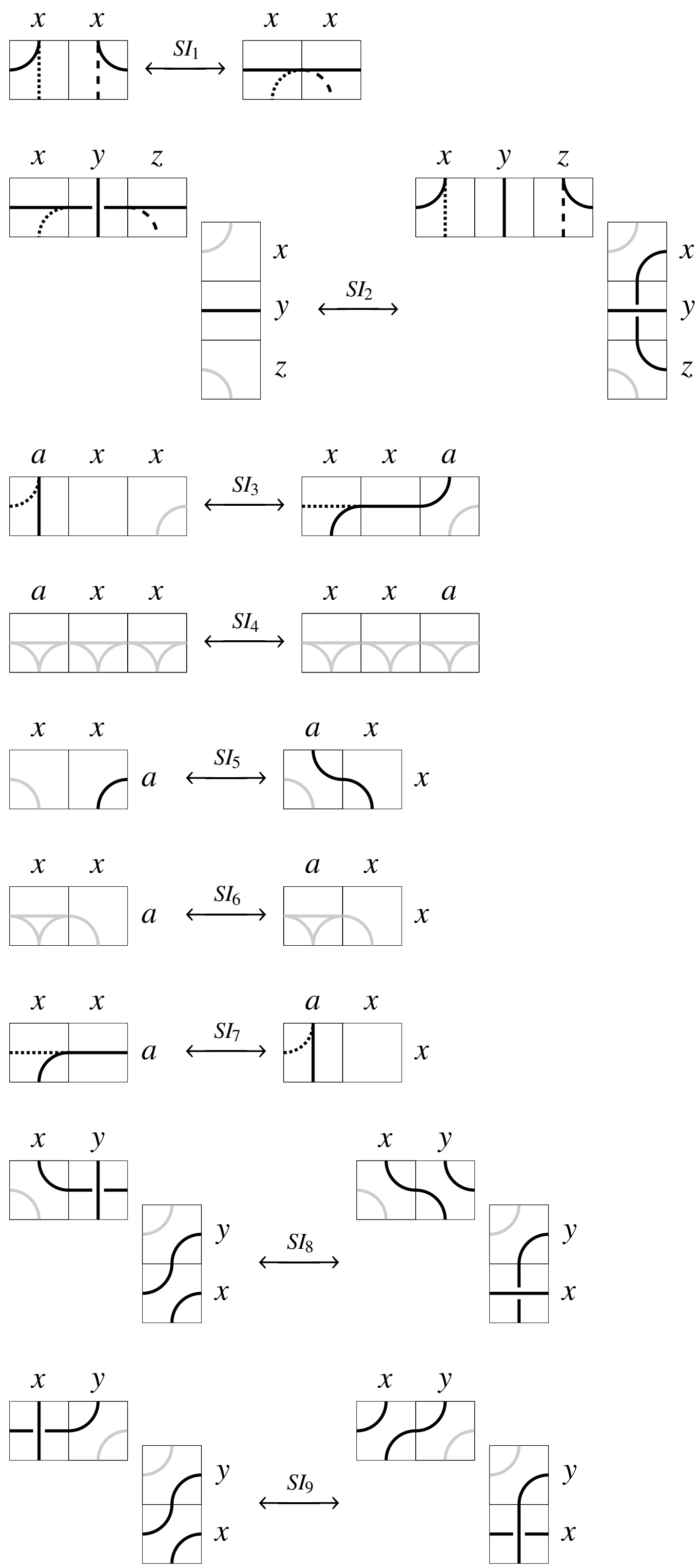


FIGURE 27. List of surface isotopy moves.

2.4. Stabilizations & Destablizations

Virtual knots and links viewed as knot diagrams on surfaces have moves that allow us to represent them on surfaces of different g e nera. T here a re a lso m oves w e c an p erform on virtual mosaics that achieve the same goal. We call them stabilization and destabilization moves. These are shown in Figure 28.

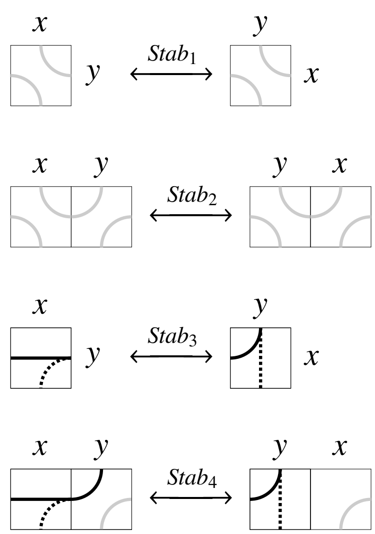


FIGURE 28. Stabilization and destabilization moves.

2.5. Mosaic Injection & Ejection

The following moves, introduced in [5], enlarge or shrink a mosaic without changing the knot type or genus. Let $\mathbb{V}^{(n)}$ denote the set of virtual n-mosaics. If $V^{(n)} \in \mathbb{V}^{(n)}$, then the ij-entry (row i, column j) of $V^{(n)}$ is denoted by $V^{(n)}_{ij}$. In [5], Ganzell and Henrich define an injection with the following function.

The *standard virtual mosaic injection*

$$\iota \colon \mathbb{V}^{(n)} \longrightarrow \mathbb{V}^{(n+2)}$$

$$V^{(n)} \mapsto V^{(n+2)}$$

will be defined as

$$V_{ij}^{(n+2)} = \begin{cases} V_{ij}^{(n)} & \text{if } 0 \leq i, j < n \\ T_5 & \text{if } i < n,\ j \geq n, \text{ and } V_{i,n-1}^{(n)} \in \{T_2, T_3, T_5, T_7, T_8, T_9, T_{10}\} \\ T_6 & \text{if } i \geq n,\ j < n, \text{ and } V_{n-1,j}^{(n)} \in \{T_1, T_2, T_6, T_7, T_8, T_9, T_{10}\} \\ T_0 & \text{otherwise,} \end{cases}$$

where the new boundary edges are labeled in adjacent pairs. The reverse process is called an *ejection*.

This function, however, contains errors, not all tiles get mapped correctly depending on the type of injection. In order to have one function that works for all injections we modified the function given above.

Definition 3. *Define a mosaic injection as*

$$V_{ij}^{(n+2)} = \begin{cases} V_{ij}^{(n)} & \textit{if } i \notin \alpha, j \notin \beta \\ T_5 & \textit{if } j \in \beta \textit{ and } V_{i,j^*-1}^{(n)} \textit{ or } V_{i,j^*}^{(n)} \in \{T_2, T_3, T_5, T_7, T_8, T_9, T_{10}\} \\ T_6 & \textit{if } i \in \alpha, \textit{ and } V_{i^*-1,j}^{(n)} \textit{ or } V_{i^*,j}^{(n)} \in \{T_1, T_2, T_6, T_7, T_8, T_9, T_{10}\} \\ T_0 & \textit{otherwise.} \end{cases}$$

Where the new rows created in the injection α_1, α_2 and the new columns β_1, β_2. Further, i^* and j^* are the values of the injection ($\iota_{i^* j^*}$).

Figure 29 illustrates an example of an injection and how to label the columns and rows.

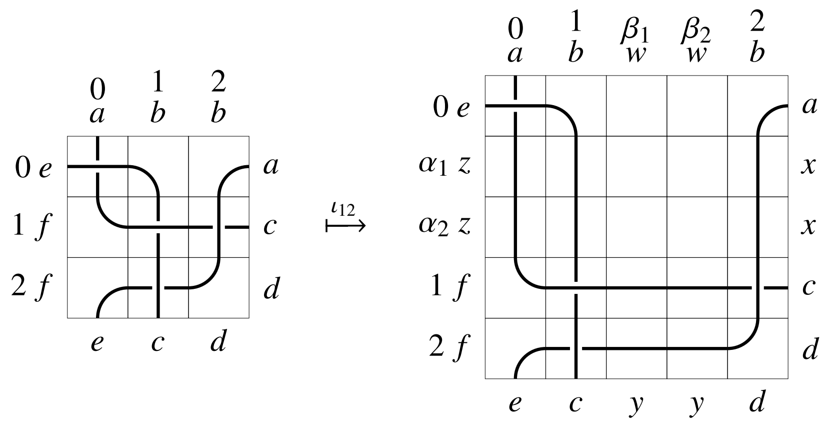


FIGURE 29. An ι_{12}-injection

It is possible to have an injection where i^* and/or j^* are 0. In this case the new rows or columns are placed above row 0 or to the left of column 0. Note this function also gives redundant values for such injections along with injections where i^* or j^* is equal n. For example let $j^* = 0$. Then, looking at the T_5 tile conditions we have $V^{(n)}_{i,j^*-1} = V^{(n)}_{i,-1}$. However, $V^n_{i,-1}$ is not a tile from the original mosaic since j should only take on values from 0 to $n-1$. Since the new function provides two different tiles in which to refer back to, we are still able to use the function, and correctly map the tiles.

CHAPTER III

VIRTUAL RECTANGULAR MOSAICS

3.1. Virtual Rectangular Mosaics

In this section we extend the definition of virtual mosaic knots to include those that are not square. This minimizes the amount of tiles necessary and provides a more efficient way of representing knots.

Definition 4. *A* virtual rectangular mosiac *is an $m \times n$ array of standard mosaic tiles, together with an identification of the $2m + 2n$ edges of the array boundary that forms a knot or link diagram on a closed, orientable surface.*

The virtual trefoil has virtual mosaic number 2 and must have four tiles to be represented on a square mosaic. However, we can achieve a representation of the virtual trefoil using only two tiles. This is illustrated in Figure 30. As a consequence of using rectangular mosaics, a new invariant other than the virtual mosaic number is needed.

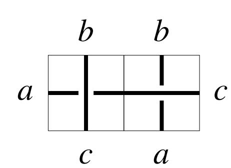


FIGURE 30. Virtual trefoil on rectangular mosaic.

Definition 5. *The* tile number *of a virtual mosaic, denoted $T_v(K)$, is the number of tiles necessary to represent a knot on a mosaic.*

Therefore, the virtual trefoil has tile number 2 or $T_v(K) = 2$. A lower bound for tile number is the classical crossing number since classical crossings are the only crossings on the mosaic grid. We show there are families of knots whose tile number is equal to their crossing number. This requires defining the following special type of rectangular mosaic.

Definition 6. *A* row mosaic *is a* $1 \times n$ *mosaic with an identification of the* $2n+2$ *edges of the array boundary that forms a knot or link diagram on a closed, orientable surface.*

An example of a row mosaic representing the classical knot 7_1 is shown in Figure 31. As seen in the example, all the crossings are in a single horizontal row.

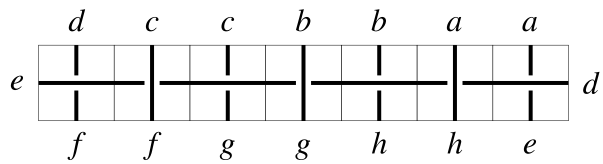


FIGURE 31. A row mosaic representation of the 7_1 knot.

We define a new invariant specific to row mosaics.

Definition 7. *The* row number *of a knot K, denoted* $\rho(K)$*, is the minimum number of tiles necessary to represent a knot on a row mosaic.*

We show that every classical knot can be presented on a row mosaic. In 2008, the following theorem was introduced by Adams, Shinjo, and Tanka [3].

Theorem 8. [[3], Theorem 1.2] *Every knot has a projection that can be decomposed into two sub-arcs such that each sub-arc never crosses itself.*

An example of a knot decomposed into two sub-arcs is shown in Figure 32.

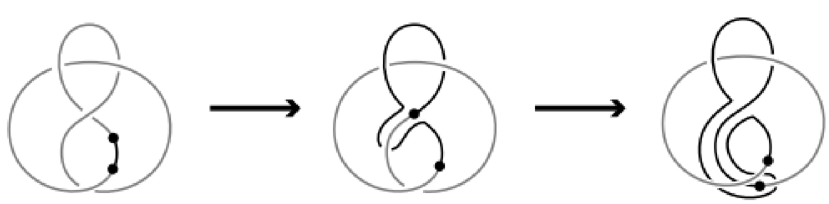


FIGURE 32. The Figure 8 knot decomposed into two sub-arcs that do not cross themselves.

In 2018, Owad expanded this result in [10].

Theorem 9. [[10], Theorem 2.5] *Every knot has a diagram that is composed of a single straight strand and collection of semicircles, all with their centers on the straight strand.*

This theorem provides a direct connection between a specific type of knot diagram and row mosaics. Moreover, this connection leads to several results including families of knots that can be represented by a row mosaic where the row number is equal to the crossing number. Figure 33 shows one of the sub-arcs can be straightened out while the other is composed of semicircles where their centers meet along the straightened sub-arc. Therefore, we have a knot diagram where all the crossings are in a straight line.

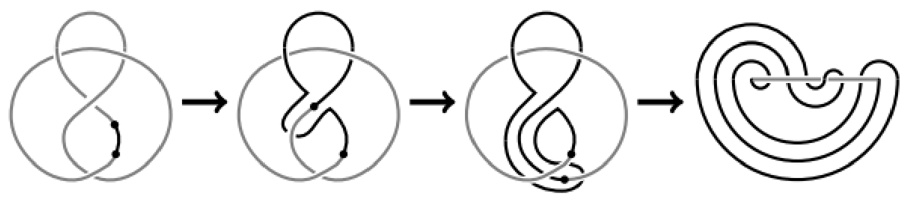


FIGURE 33. The Figure 8 knot with the grey arc bent to be straight.

These type of knot diagrams are called *straight*, or the knot is said to be in straight position. If the knot in straight position realizes its crossing number, then the knot is called *perfectly straight*. Some knots, however, have straight diagrams where the number of crossings is greater than the number of crossings in the reduced knot diagram. An example is shown in Figure 34. The number of crossings it takes for a knot to be in the straight position is called the *straight number*, denoted $str(K)$. An algorithm to put a knot into straight position is shown in [11]. Figure 34 is also an example of using the algorithm which takes the gray strand and lifts it up under the horizontal line and then back under putting the knot into straight position.

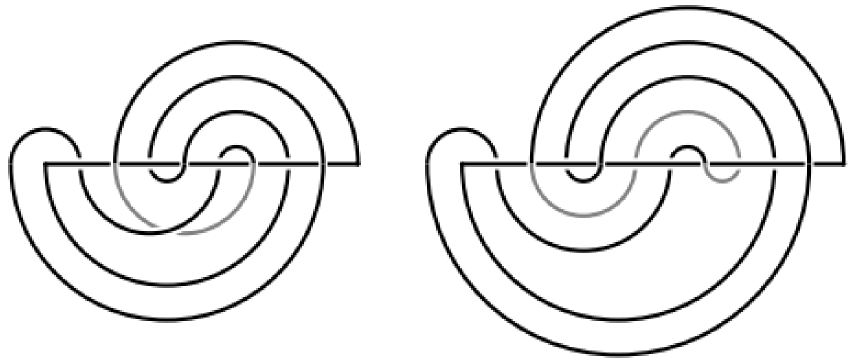


FIGURE 34. Knot 9_{32} on the left with 9 crossings reduced and not in straight position. On the right the 9_{32} with 10 crossings in straight position.

Proposition 10. *Every classical knot can be put on a row mosaic. For a knot K,* $str(K) = \rho(K)$.

Proof. Since every knot can be put into straight position, we can represent each crossing with a T_9 or T_{10} crossing tile. Then using edge identifications, preserve the connections of the semicircles. Therefore, every classical knot K can be put on a row mosaic and $str(K) = \rho(K)$. □

Proposition 11. *Given the minimal row mosaic representing a classical knot, the genus of the mosaic is 0.*

Proof. Suppose we have a minimal row mosaic of a classical knot. Then this knot is also in straight position. By Theorem 7, the outer edge identifications produce a collection of semicircles. Furthermore, by Theorem 6, these semicircles do not cross one another. A general mosaic with n crossings and $2n+2$ edge identifications is shown in Figure 35.

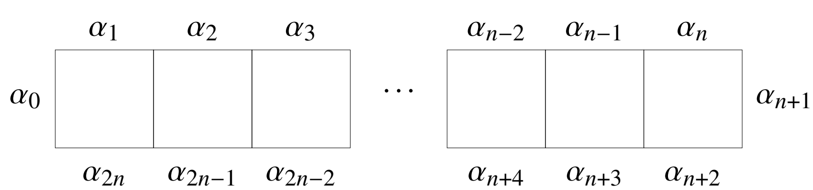
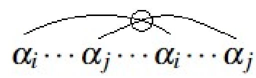


FIGURE 35. An n-crossing $1 \times n$ mosaic.

A surface will only have positive genus if two pairs split one another in the sequence, i.e. $\alpha_i \cdots \alpha_j \cdots \alpha_i \cdots \alpha_j$. However, if the outer edges have such pairs we have the following



which produces a virtual crossing. Thus, there are no pairs that interlock and hence, the genus of the mosaic is zero. □

We can combinatorially tabulate row mosaics for alternating knots with a fixed row number by using an alternating pattern of T_9 and T_{10} tiles, and then considering all possible edge labelings. We are able to manually produce a knot table for all classical knots with 8 crossings or fewer. This table is included in the appendix.

To find all the different combinations of the outer edge identifications we use a python program that gives every balanced combination of n sets of parenthesis. This means before closing a pair of parenthesis, all parenthesis inside the pair must also be closed. For example, the string (()(())) would translate to $abbcddca$. This ensures the edge identifications, when connected, do not produce crossings off the mosaic. Then we can narrow down the list by taking out the combinations where the edge identifications produce a link. An exhaustive search of all edge pairings was used to create the row mosaic table.

Example 12. Consider in cases, a 6-crossing knot on a 2×3 mosaic and note all knots with 6-crossings are alternating. First, there are 10 edge identifications. Therefore, to find all the possible edge identifications, produce all the combinations of 5 sets of balanced parenthesis. In Figure 36, looking at the outer edge identifications, if 1 is identified to 2, 2 identified to 7, 3 identified to 6, 10 identified to 4 or 9 identified to 5, we get at least a two component link. Thus, the combinations where those positions in the string are equal can be thrown out. Another way links can be produced are with 2 side by side edge identifications being equal along with the ones directly across from them being equal. An occurrence of this appears in Figure 36, when 1 is identified to 2 and 8 identified to 7. The last types of strings we can rule out are those that produce a Reidemeister 1 move, such as 1 and 10 being equal. After narrowing down the list of combinations, we can check the remaining candidates to see which ones produce a knot and if so which one.

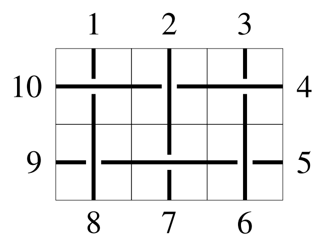


FIGURE 36. A 2×3 mosaic with no identifications.

Using this tabulation, we observe the following:

Proposition 13. *If a knot K has 7 crossings or fewer the row number is equal to the crossing number,* $\rho(K) = c(K)$.

Proof. Using the method above, we were able to find a row mosaic representation for each knot and provide a table of these knots in the appendix. The proposition can be checked manually. □

Proposition 14. *All virtual knots that have up to and including 4 crossings have a row number equal to the crossing number and the genus of the mosaic is the same as the knot.*

Proof. Looking at a knot table of every virtual knot with four crossings or less, each one has a sequence of n consecutive crossings where n is the crossing number. Therefore, these knots can be represented by a row mosaic with $\rho(K) = n$. After identifying a row mosaic for a given virtual knot, we can identify the surface obtained by following the boundary identifications, as in [7]. In each case, there exists a row mosaic representation where the genus of the row mosaic is equal to the genus of the knot, as given in [2]. There is a table of these virtual knots included in the appendix. □

Proposition 15. *Other families of knots where $\rho(K) = c(K)$:*

1. *All torus knots $T_{p,q}$.*

2. *All n-pretzel knots.*

3. *All 2-Bridge knots $K_{p/q}$ where the continued fraction of p/q has length less than 6.*

Proof. These knots are perfectly straight proven in [10], hence $\rho(K) = c(K)$. □

Definition 16. *Let m be the maximum number of crossings in a knot diagram, or the longest subsequence, in a Gauss code representing a knot diagram, in which no crossing is repeated.*

Proposition 17. *Given a knot diagram or Gauss, code the $\rho(K) \leq 2^r(m+1)-1$ where r is the remaining number of crossings not included in the longest subsequence with no repeated crossings.*

Proof. This is a direct result from Lemma 3.1 in [10], which states $str(K) \leq 2^r(m+1)-1$. Since $str(K) = \rho(K)$, we have the same result for row number. □

Note that equivalent knot projections may not have the same value for m. This means two projections for equivalent knots with the same amount of crossings may not both be able to be represented by a row mosaic. The following proposition is a result of Proposition 15.

Proposition 18. *Given a classical knot, K, if $m = c(K)$ then $\rho(K) = c(K)$.*

Proof. By Proposition 15, $\rho(K) \leq 2^r(m+1)-1$. Assuming $m = c(K)$ implies $r = 0$. Hence,

$$\begin{aligned}\rho(K) &\leq 2^0(c(K)+1)-1 \\ &\leq c(K)+1-1 \\ &\leq c(K).\end{aligned}$$

Since K is a classical knot all crossings must be represented by a tile, thus $\rho(K) = c(K)$. □

Adams et al.[1] introduced diagrams representing knots where their crossings are stacked. These diagrams are called *petal diagrams*. An example of such diagram is shown in Figure 37. The number of petals it takes to represent a given knot is called the *petal number*.

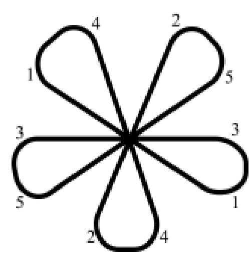


FIGURE 37. Petal diagram of a trefoil knot.

A relationship between straight number and petal number is given in [10]. We can use the to establish the following.

Proposition 19. *For a knot with $\rho(K) = n$, $p(K) \leq 3n$.*

Proof. Let $p(K)$ denote the petal number, then given a knot where $str(K) = n$, $p(K) \leq 3n$. Since $\rho(K) = str(K)$, it follows that $p(K) \leq 3n$. □

Even though every knot has a row mosaic, a row mosaic does not always minimize the tile number.

Proposition 20. *There exist knots such that $\rho(K) > T_v(K)$.*

Proof. The 10_{88} knot are one of these knots. The row number for this knot is 12, however the tile number is 10 since this knot fits on a 2×5 virtual mosaic. Both representations are shown in Figure 38.

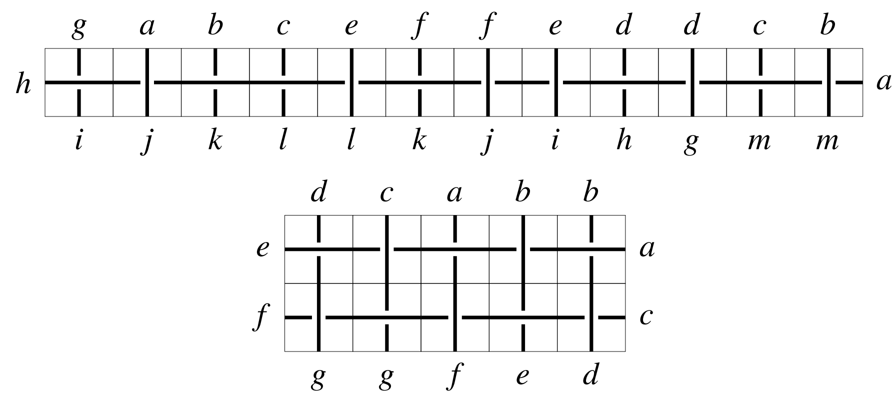


FIGURE 38. The 10_{88} knot on a row and 2×5 mosaic.

□

There are also knots whose tile number can be realized on both a row mosaic, and a virtual rectangular mosaic. The 6_3 knot is an example of such and both representations are shown in Figure 39. The only other alternating knot with 8 crossings or less where the tile number can be realized on a row mosaic and a virtual rectangular mosaic is the 8_{15} knot.

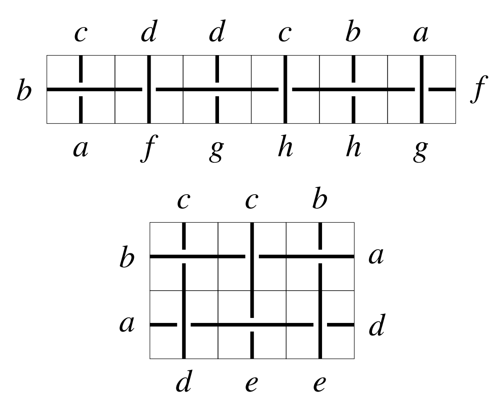


FIGURE 39. The 6_3 knot represented on a row mosaic and rectangular mosaic.

This leads to the following question. Given a knot what is the smallest $m \times n$ virtual rectangular mosaic that can realize the knot if m, n are both greater than 1? Using the exhaustive method explained in Example 10, we are able to provide a partial answer to this question. The only classical knots that fit on a 2×3 virtual mosaic are $3_1, 4_1$, and 5_2. The only alternating knots that fit on a 2×4 mosaic are $5_1, 6_1, 6_2, 7_4, 7_5, 7_6, 7_7$ and 8_{15}. These are included in a table in the appendix.

The following was an open question proposed in [5]. By Proposition 1, $m_v(K) \leq m(K) - 2$ for nontrivial K, but sometimes this inequality is strict. If equality does not always hold, is there a fixed integer r for which $m_v(K) + r \geq m(K) - 2$? Here r measures how far the inequality in the proposition is off by, and we provide a partial answer to this question by giving values of r for knots with 10 or fewer crossings.

Proposition 21. *For knots with more than 3 crossings up to and including knots with 8 or 10,* $r = 1$.

Proof. The lowest $m_v(K)$ in the set of knots with $3 < c(K) \leq 8$ is 3 ([5]), and the highest $m(K)$ is 6 ([8]). Therefore, subbing in 2 for r in the inequality we get $3 + 1 \geq 6 - 2$. For knots with 10 crossings, $m(K) \in \{6, 7\}$ ([4]). Therefore, $m_v(K) \in \{4, 5\}$. Again using the lowest $m_v(K)$ and the highest $m(K)$ and plugging those values into the inequality we get $4 + 1 \geq 7 - 2$. □

Proposition 22. *For knots with 9 crossings,* $r = 2$.

Proof. The lowest $m_v(K)$ of a 9 crossing knot is 3 ([4]) and the highest $m(K)$ is 7. Therefore, $3 + 2 \geq 7 - 2$. □

From this we can conclude that the set of all knots with 10 or fewer crossings have $r = 2$.

3.2. Rectangular Mosaic Injections

For square mosaics, an injection has to insert two rows and two columns to preserve squareness. However, this is not the case for rectangular mosaics. Rectangular mosaics allow for the insertion of only two rows or only two columns. Therefore, we modify the injection function to define a row injection, and a column injection. For the following definitions x is used to denote there is no injection in a row or column.

Definition 23. *A row injection, V_{ix}, injects two rows after the i^{th} row, $i = 0, \ldots, m$ and is defined as*

$$V_{ix}^{((m+2)\times n)} = \begin{cases} V_{ij}^{(m\times n)} & \text{if } i \notin \alpha \\ T_6 & \text{if } i \in \alpha \text{ and } V_{i^*-1,j} \text{ or } V_{i^*,j} \in \{T_1, T_2, T_6, T_7, T_8, T_9, T_{10}\} \\ T_0 & \text{otherwise.} \end{cases}$$

Figure 40 gives an example of a V_{1x} row injection.

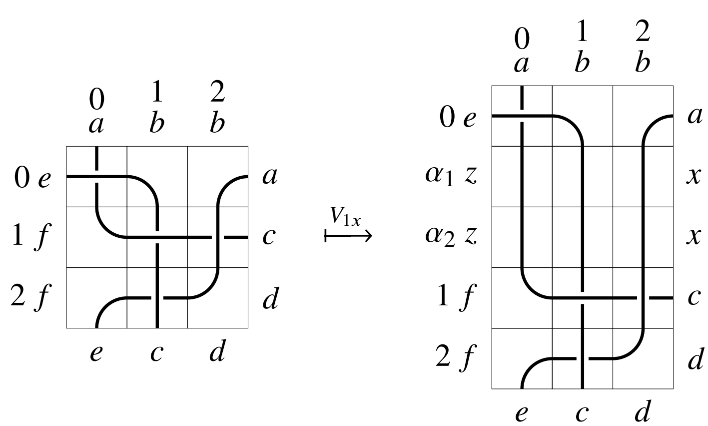


FIGURE 40. A V_{1x}-injection

Definition 24. *A column injection, V_{xj}, inserts two columns after the j^{th} column, $j = 0, \ldots, n$ is defined as*

$$V_{xj}^{(m\times(n+2))} = \begin{cases} V_{ij}^{(m\times n)} & \textit{if } j \notin \beta \\ T_5 & \textit{if } j \in \beta \textit{ and } V_{i,j^*-1} \textit{ or } V_{i,j^*} \in \{T_1, T_2, T_6, T_7, T_8, T_9, T_{10}\} \\ T_0 & \textit{otherwise.} \end{cases}$$

An example of a column injection is shown in Figure 41.

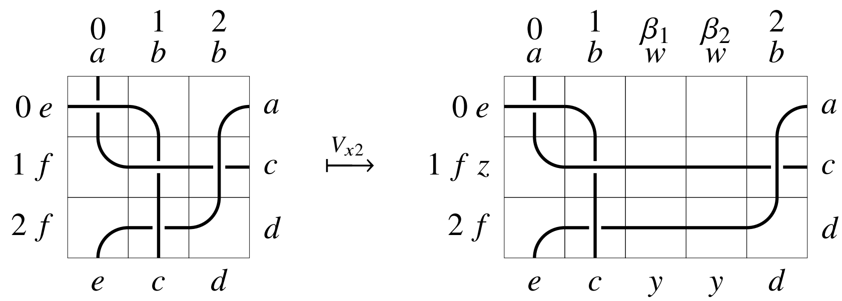
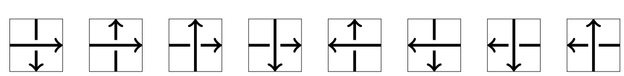


FIGURE 41. A V_{x2}-injection

3.3. Intersection Index Polynomial of Virtual Knot Mosaics

In this section we extend the computation of virtual knot invariants to virtual knot mosaics by intersection index polynomial can be computed on a mosaic. Recall this invariant involves smoothing each classical crossing according to orientation. To represent this computation on a mosaic we use oriented T_7, T_8, T_9, and T_{10} tiles. Each T_9 or T_{10} representing a crossing c will be positive or negative depending on $sign(c)$. These are shown in Figure 42.



T_9^- T_9^+ T_{10}^- T_{10}^+ T_9^- T_9^+ T_{10}^- T_{10}^+

FIGURE 42. Oriented crossing tiles.

Note using the first four tiles and rotating them, we can can obtain the last four. Smoothing these tiles requires oriented T_7 and T_8 tiles, and this is shown in Figure 43. Recall a smoothing produces two components. Therefore, following orientation, the component on the left is 1 and the component on the right is 2.

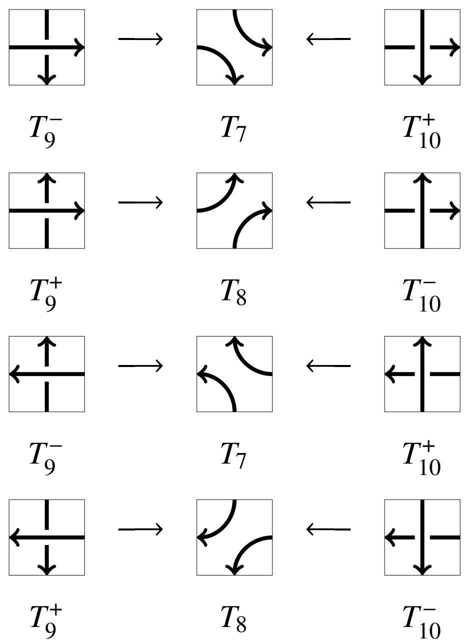
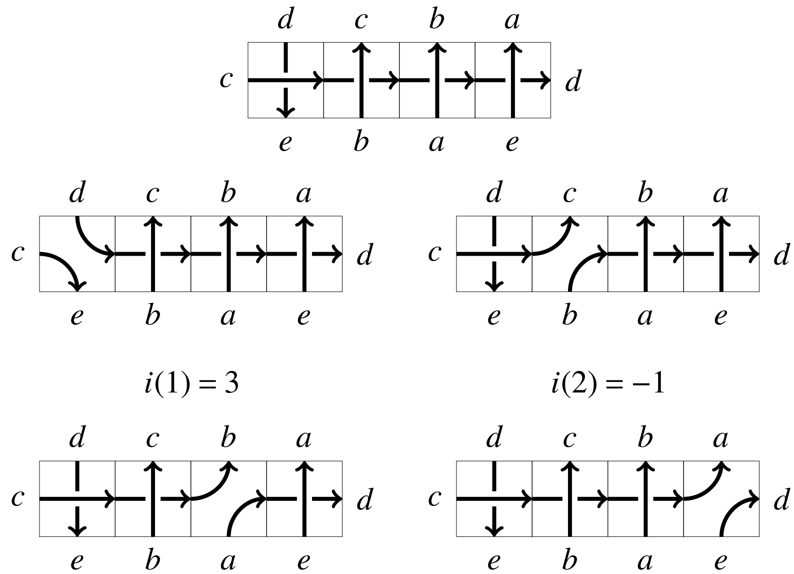


FIGURE 43. Smoothing oriented crossing tiles.

Now that we have shown an analogous algorithm for smoothing each crossing, we can realize the intersection index polynomial of a virtual knot on a virtual mosaic. First, assign an orientation to the mosaic. Then for each crossing tile, take an oriented T_9 or T_{10} tile and replace it with the corresponding oriented T_7 or T_8 tile. Figure 44 provides an example of computing the intersection index polynomial on a row mosaic.

Example 25. This knot is oriented from left to right and uses the edge identifications to travel around the knot. Using the computed weights of the crossings above we get

$$p_t(K) = -t^3 - 3t + 4$$



$i(1) = 3$ $\quad$ $i(2) = -1$

$i(3) = -1$ $\quad$ $i(4) = -1$

Figure 44. Example of computing the intersection index polynomial on a virtual mosaic.

Therefore, in a way, computing invariants of a virtual knot represented by a virtual mosaic is more efficient. Especially for an invariant such as the intersection index polynomial, where the only crossings necessary for the computation are classical crossings, which happen to be the only crossings on the virtual mosaic.

We can also use mosaic knots to help find families with similar characteristics. For example nontrivial knots containing some or all crossings with weight zero. These knots have $p_t(K) = 0$. Figure 45 shows two knots with similar patterns and all crossings have weight zero. These knots are also not the unknot.

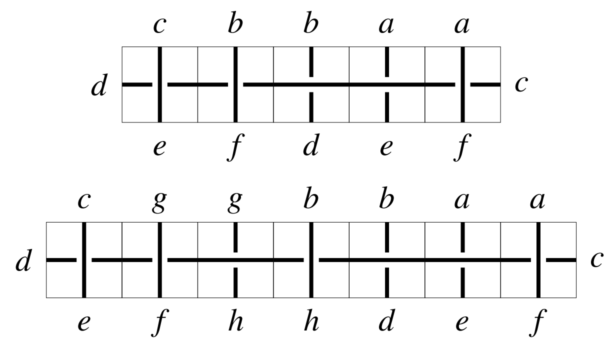


FIGURE 45. Knots whose crossings are all weight zero and are not the unknot.

CHAPTER IV

OPEN PROBLEMS

Question 1. The non-alternating knots with 8 crossings do not fit on a 2×4 virtual mosaic that has alternating tiles since there there can be no crossings off the mosaic. Can these three knots be put on a 2×4 mosaic? There would have to be two T_9 or T_{10} tiles next to one another.

Question 2. We showed how to compute one invariant on a mosaic. However, are there other invariants in which a mosaic would make it easier to compute by hand or easier to code?

Question 3. Would computing invariants using mosaic knots help find knots with similar characteristics? For example nontrivial knots containing some or all crossings with weight zero?

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APPENDIX

The following tables are included:

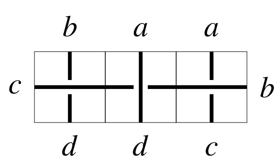
1. Row mosaics for all classical knots up to 8 crossings.

2. Row mosaics for all virtual knots up to and including 4 crossings.

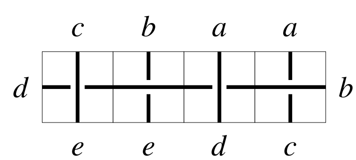
3. All classical alternating knots with 8 crossings or fewer that fit on a 2×3 or 2×4 virtual mosaic.

All of the knots in this appendix were checked using Miller's programs knotfolio (`https://kmill.github.io/knotfolio/`) and virtual knotfolio (`http://tmp.esoteri.casa/virtual-knotfolio/`).

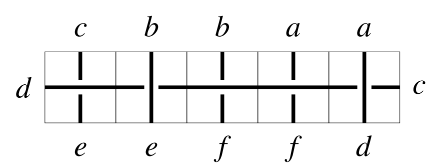
.1 Row Mosaics of Knots 3_1 to 8_{21}



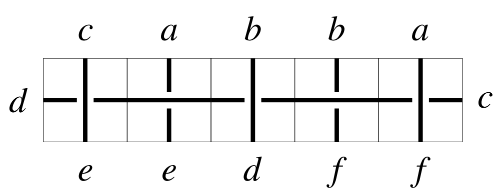
3_1



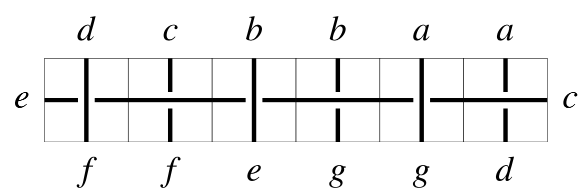
4_1



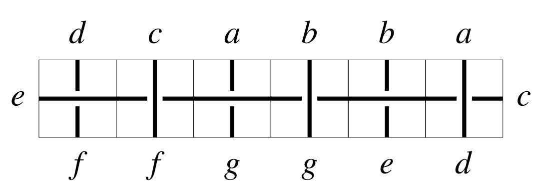
5_1



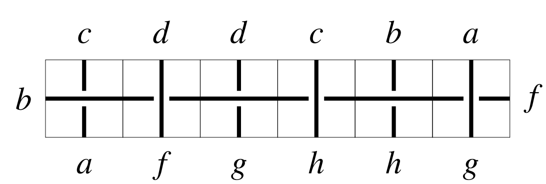
5_2



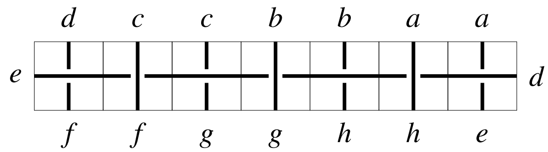
6_1



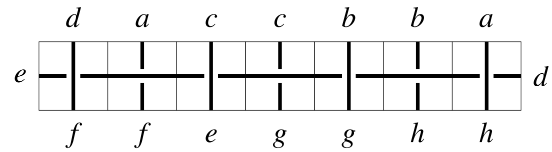
6_2



6_3



7_1



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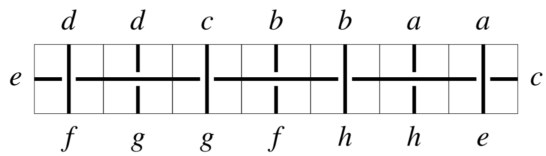
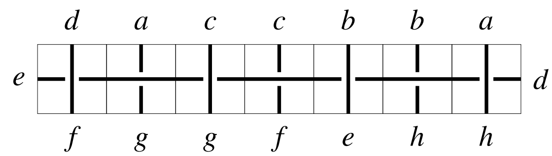
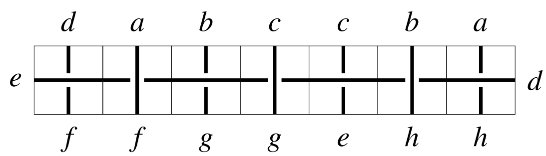


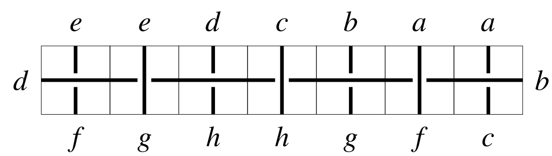
7_3



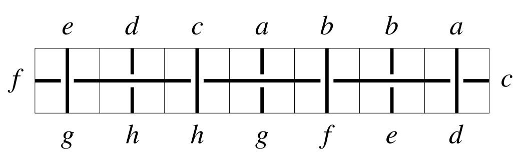
7_4



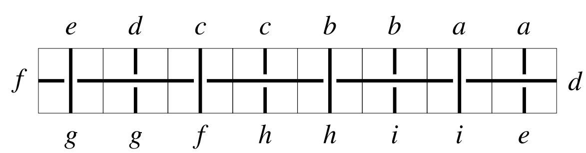
7_5



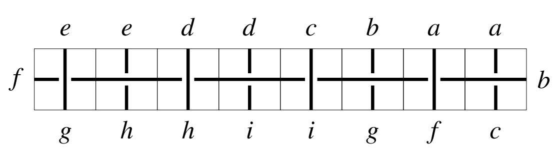
7_6



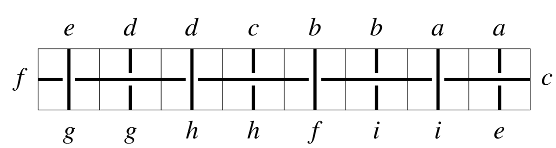
7_7



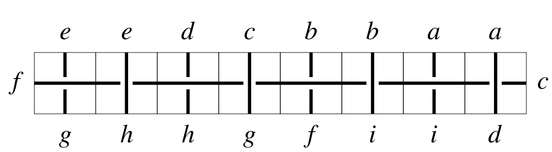
8_1



8_2



8_3



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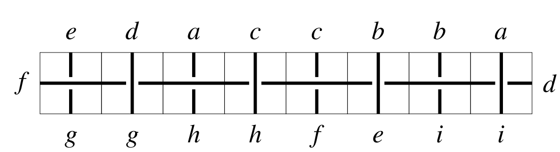

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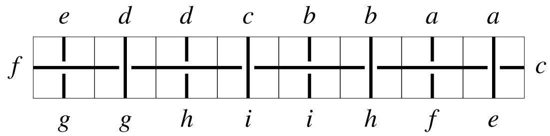

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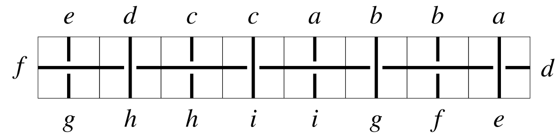

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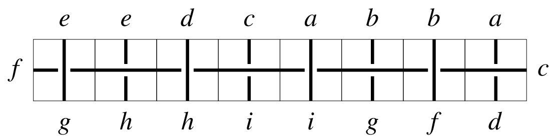
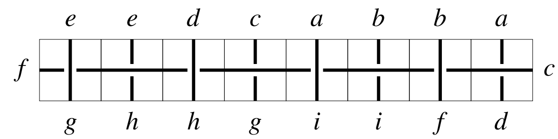
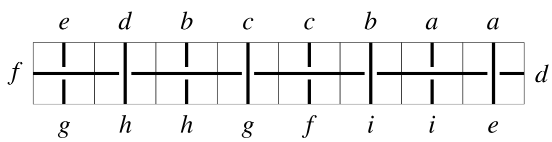


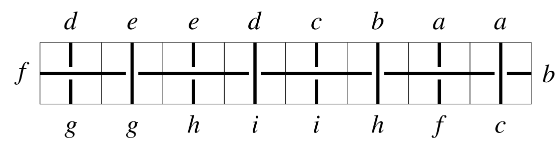
8_8



8_9



8_{10}



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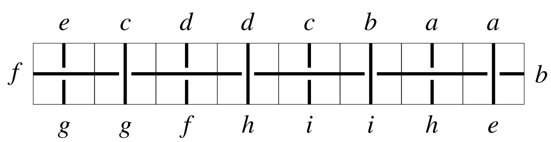

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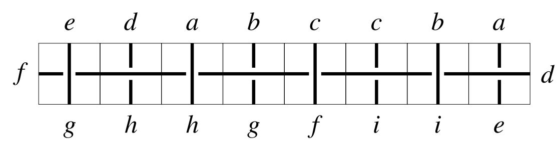

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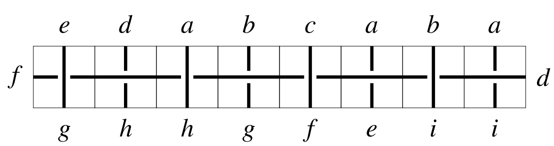

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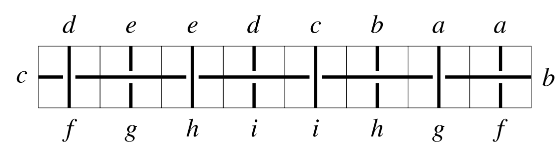

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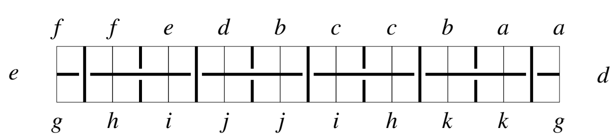

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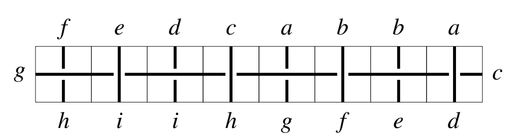

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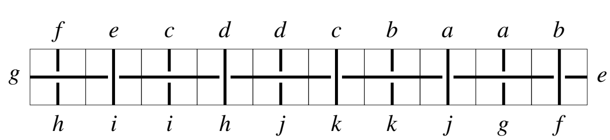
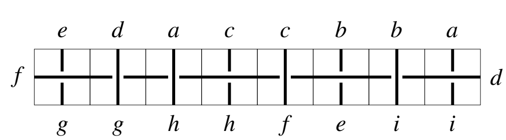


8_{18}



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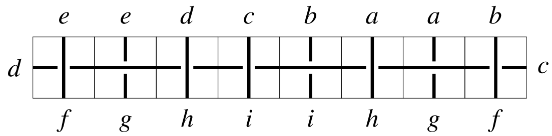

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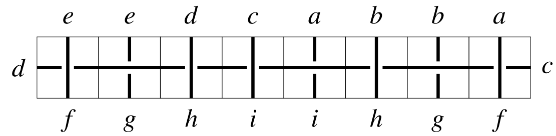
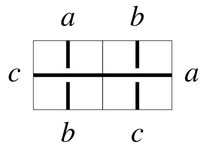
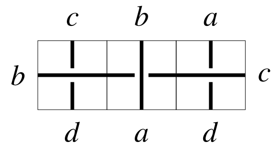


8_{21}

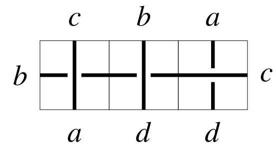
.2 Row Mosaics of Virtual knots up to 4 crossings



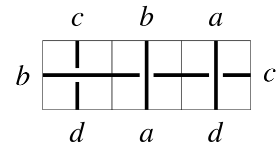
$2.1, g = 1$



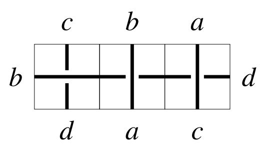
$3.1, g = 2$



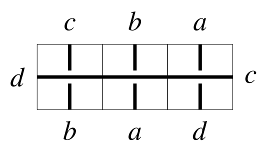
$3.2, g = 1$



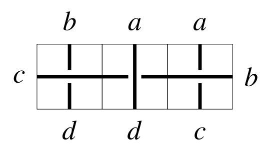
$3.3, g = 2$



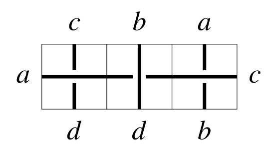
$3.4, g = 2$



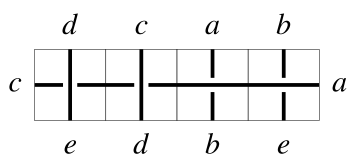
$3.5, g = 1$



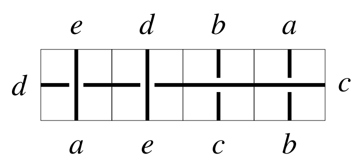
$3.6, g = 0$



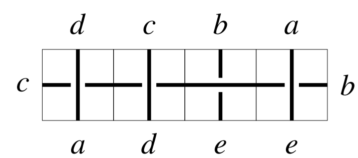
$3.7, g = 1$



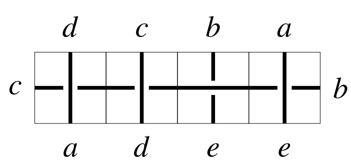
$4.1, g = 2$



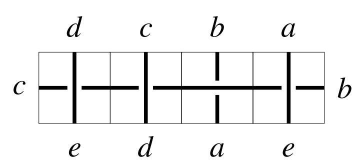
$4.2, g = 2$



$4.3, g = 2$



$4.4, g = 2$



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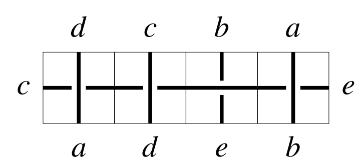

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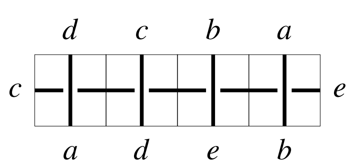
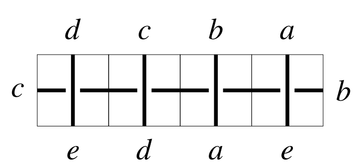
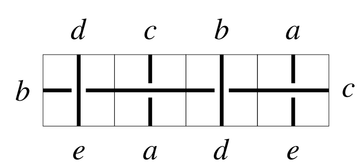


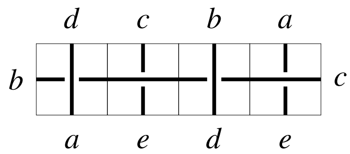
$4.7, g = 2$



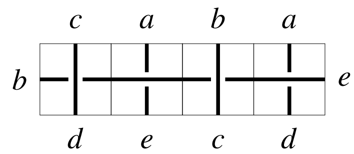
$4.8, g = 2$



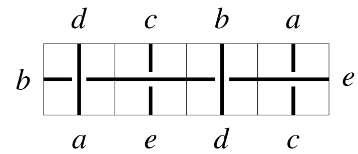
$4.9, g = 2$



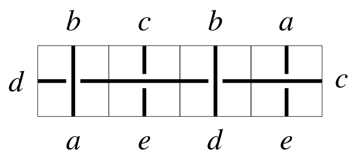
$4.10, g = 2$



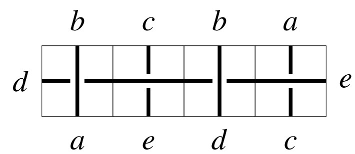
$4.11, g = 2$



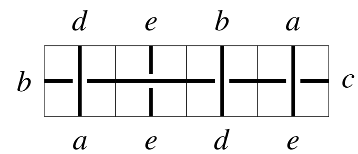
$4.12, g = 1$



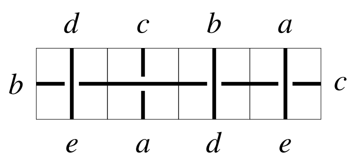
$4.13, g = 2$



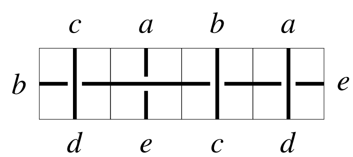
$4.14, g = 2$



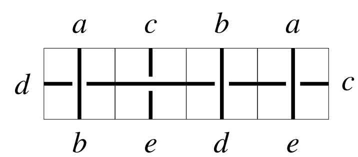
$4.15, g = 2$



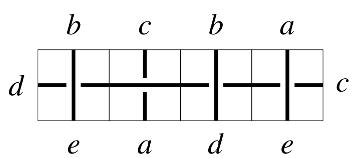
$4.16, g = 2$



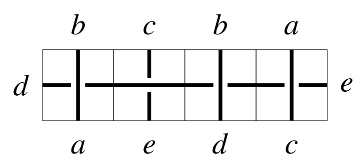
$4.17, g = 2$



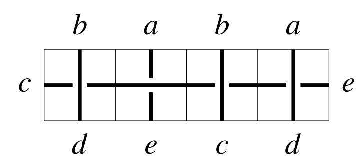
$4.18, g = 2$



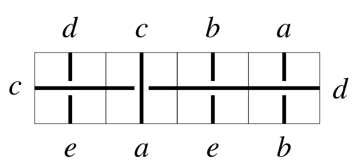
$4.19, g = 2$



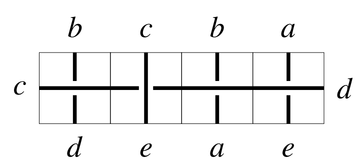
$4.20, g = 2$



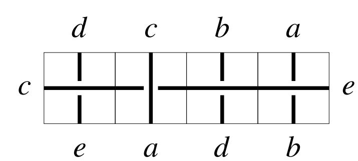
$4.21, g = 2$



$4.22, g = 2$



$4.23, g = 2$



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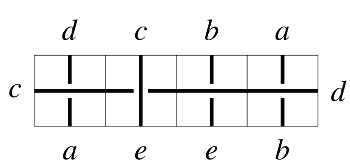

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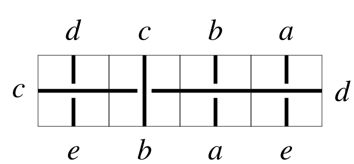

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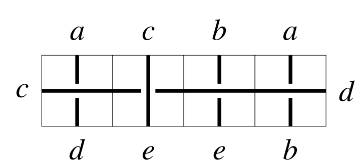
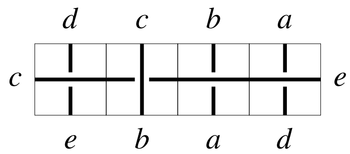
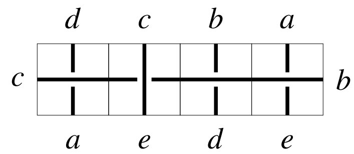


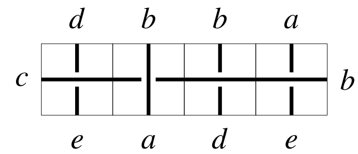
$4.27, g = 2$



4.28, $g = 2$



4.29, $g = 2$



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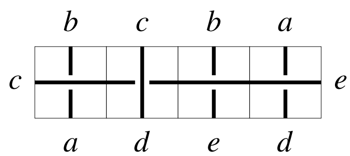

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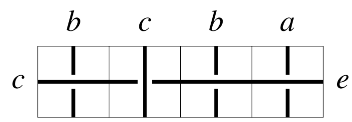

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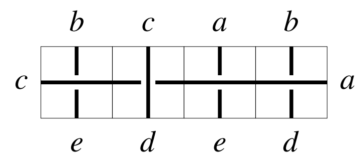

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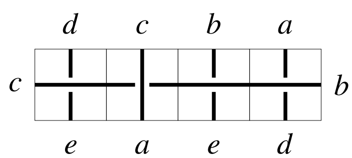

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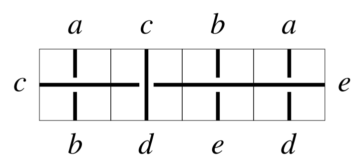

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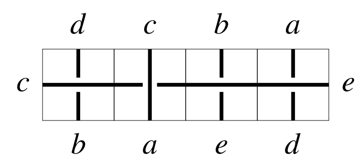

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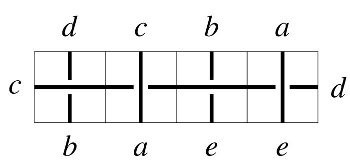

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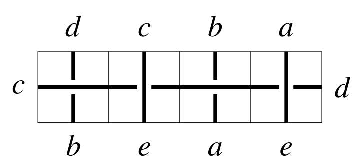

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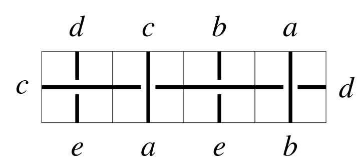

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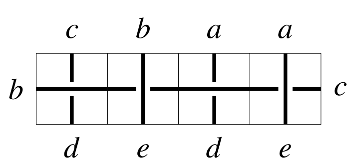

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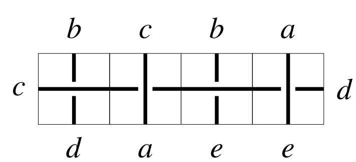

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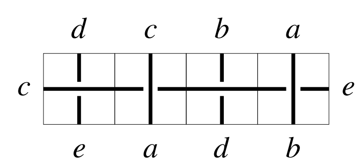

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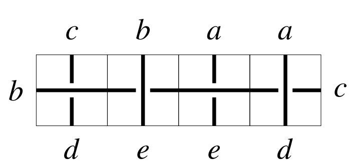

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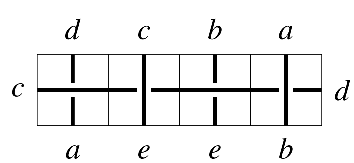

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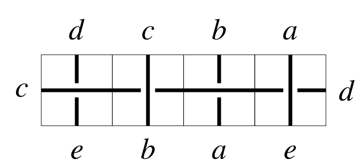

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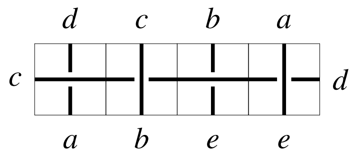

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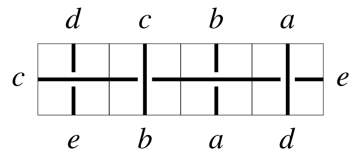

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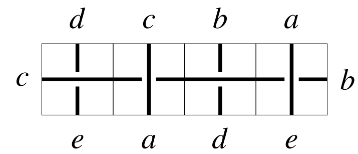

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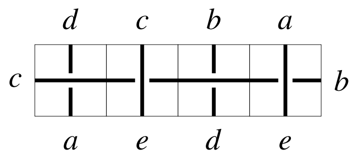

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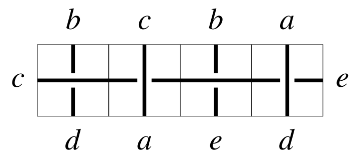

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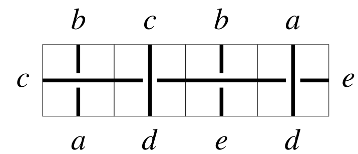

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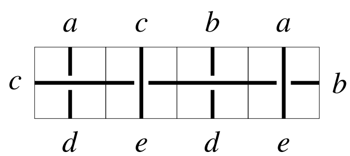

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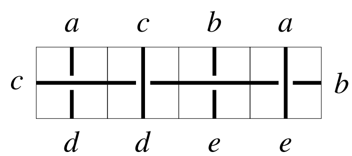

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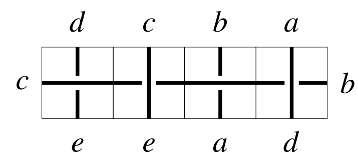

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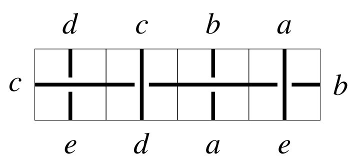

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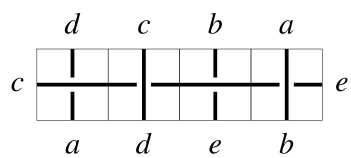

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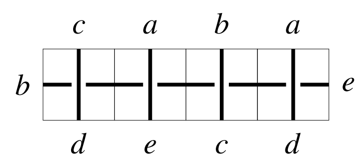

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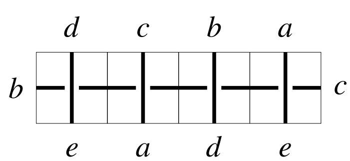

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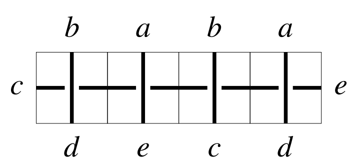

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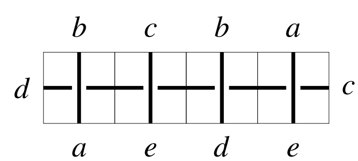

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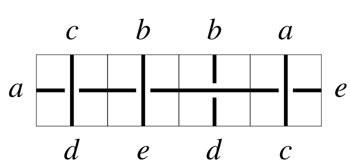

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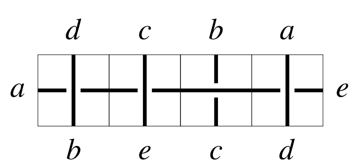

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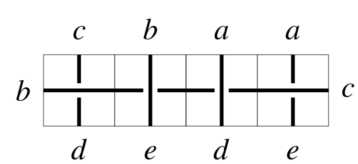

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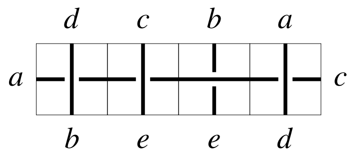

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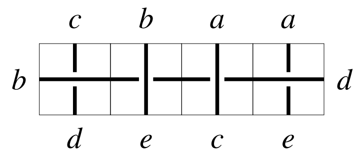

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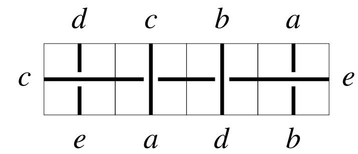

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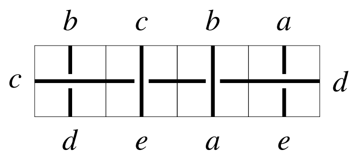

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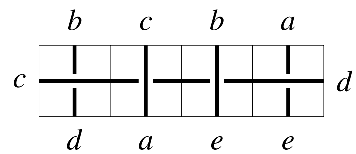

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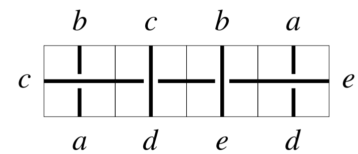

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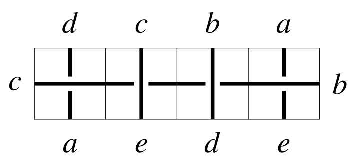

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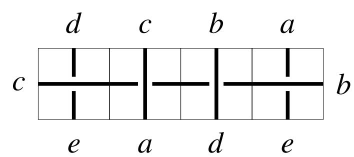

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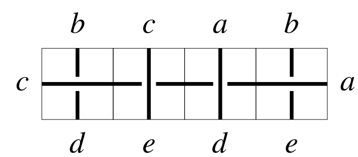

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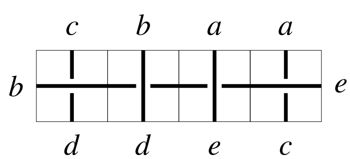

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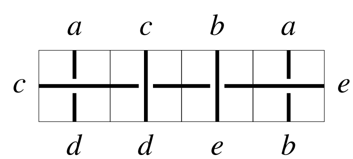

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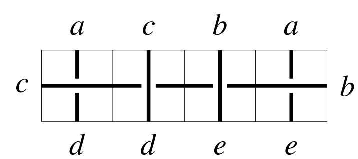

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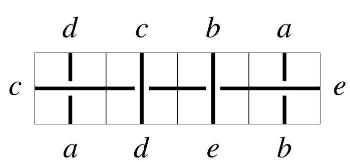

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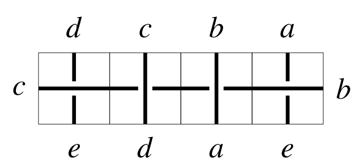

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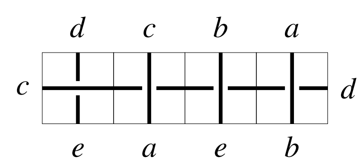

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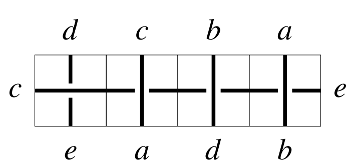

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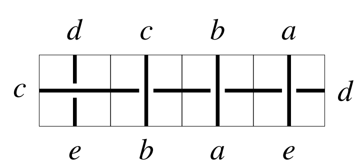

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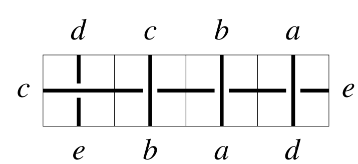

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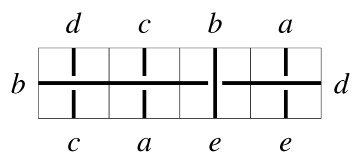

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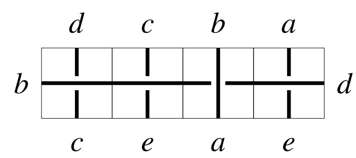

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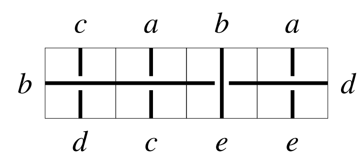

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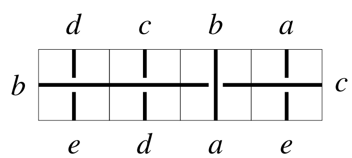

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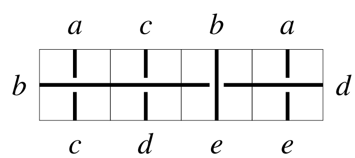

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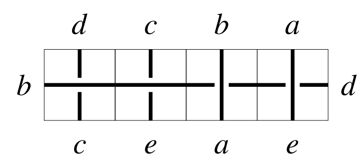

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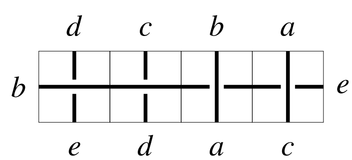

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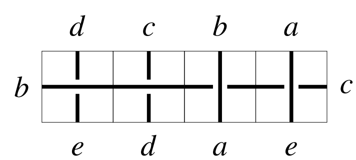

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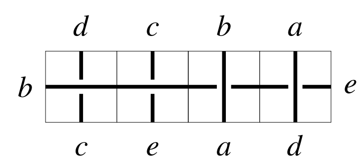

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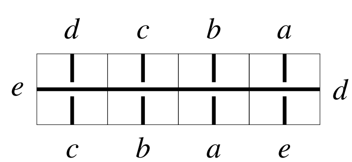

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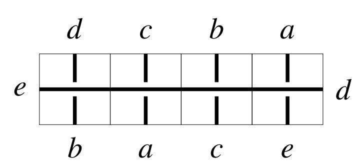

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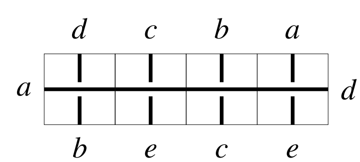

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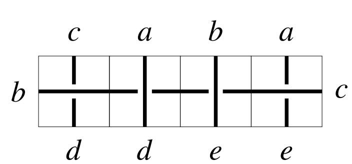

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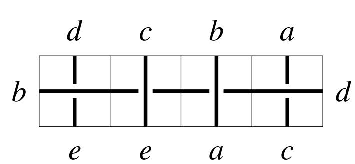

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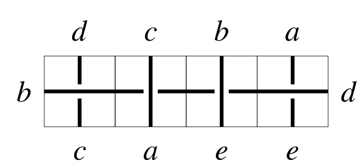

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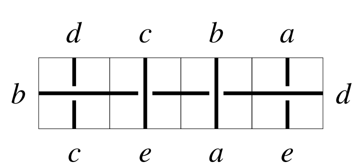

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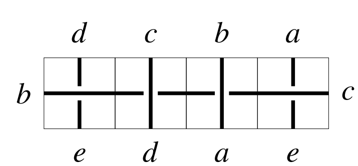
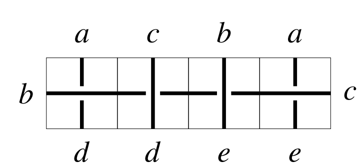
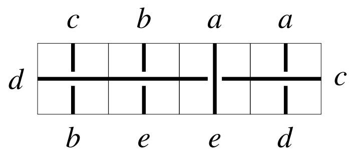


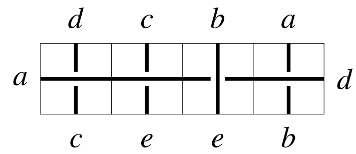
4.98, $g = 2$



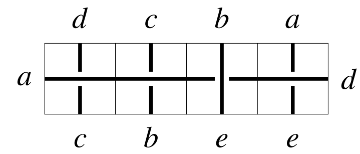
4.99, $g = 1$



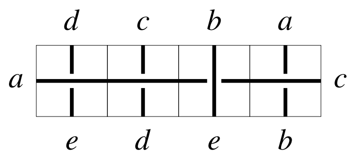
$4.100, g = 1$



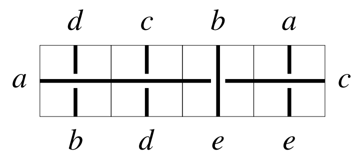
$4.101, g = 1$



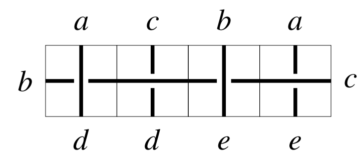
$4.102, g = 1$



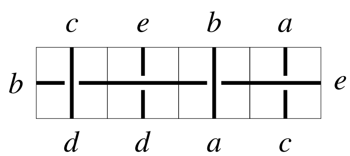
$4.103, g = 2$



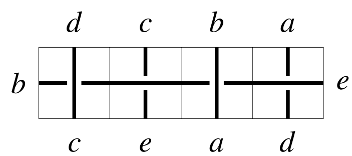
$4.104, g = 1$



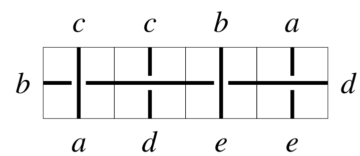
$4.105, g = 1$



$4.106, g = 1$

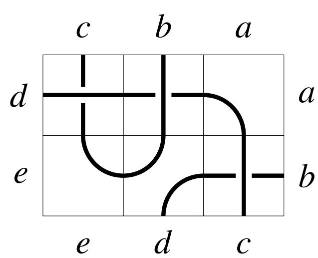


$4.107, g = 2$

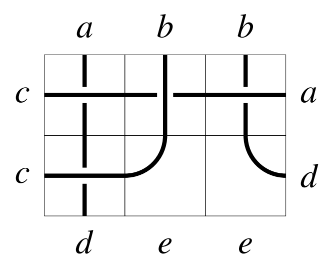


$4.108, g = 0$

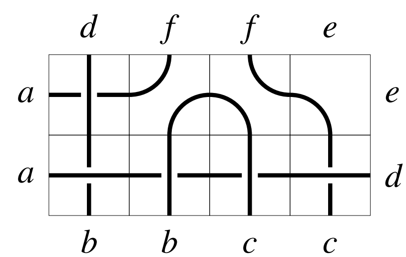
.3 Alternating Knots that fit on 2×3 and 2×4 Virtual Mosaics



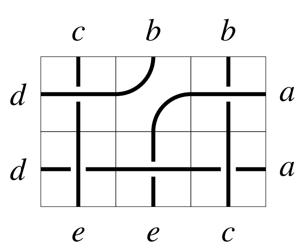
3_1



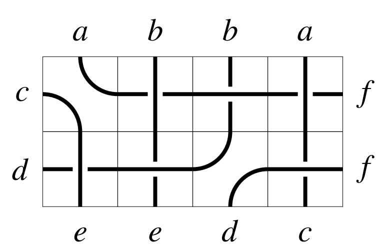
4_1



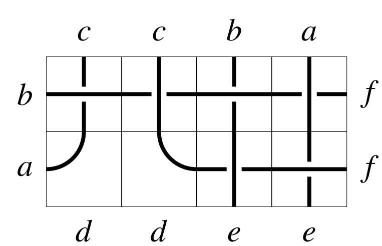
5_1



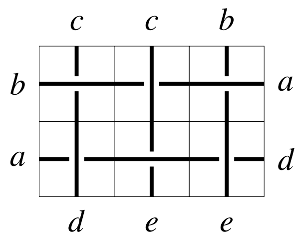
5_2



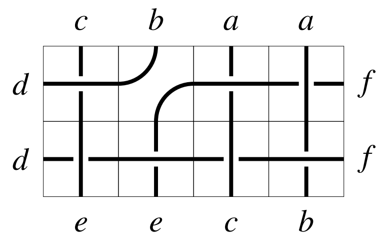
6_1



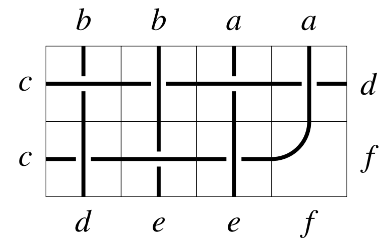
6_2



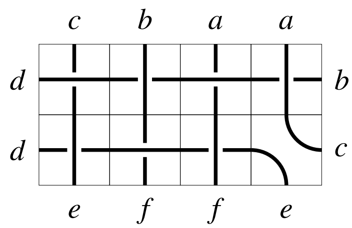
6_3



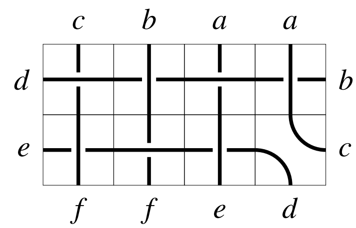
7_4



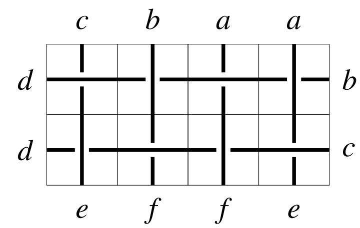
7_5



7_6



7_7



8_{15}

RACHEL MEYERS

EDUCATION

Sam Houston State University, Huntsville, TX ***January 2021 - Present***
College of Science and Engineering Technology
Master of Science in Mathematics,
Department of Mathematics and Statistics

Sam Houston State University, Huntsville, TX ***January 2018 - December 2020***
College of Science and Engineering Technology
Bachelor of Science in Mathematics,
Department of Mathematics and Statistics

Lone Star College ***Aug. 2016 - May 2018***
Associate of Arts

CAREER OBJECTIVE

To expand my knowledge in the field of mathematics while gaining career-like experience.

RESEARCH PROJECTS

Independent Study **Summer/Fall 2020**

Did an independent study on classical and virtual knot theory.

Master's Thesis **Spring 2021-Present**

Currently, working on a master's thesis. So far it will entail, but not be limited to, classical, virtual and mosaic knot theory. This paper will include original research expanding the idea of mosaics to those that are rectangular.

WORKSHOPS

Homotopical Methods in Fixed Point Theory **July 11-15, 2022**

CONFERENCE PRESENTATIONS

SIAM Fall 2021

- **Participated in a poster presentation on classical and virtual mosaic knot theory.**

TWIMS Spring 2022

- **Gave a talk on classical and virtual mosaic knot theory.**

TECHNICAL STRENGTHS

Research and Analysis	**Knot Theory**
Software & Tools	**Latex, GAP, Matlab**
Programming	**Java, Cocalc**

WORK EXPERIENCE

Teaching Assistant at SHSU **Fall 2022**

- **Currently, teaching two courses Math 1314 and Math N014**

Graduate Assistant at SHSU **Spring 2021-Spring 2022**

- **Graded, proctored, and tutored for specific professors of various math courses.**

Teaching Assistant for SHSU STEM Center **July 2019-Present**

- **Teaching assistant for STEM summer bridge program for the math section. This job entailed helping teach students the importance of how to learn math and applying those study skills. I held a 2-hour TA session everyday during this program. In fall 2020 I held calculus workshops to help students succeed in their calculus courses along with being a TA for a winter mini course in calculus.**

GUIA Fellow **Fall 2020-Spring 2022**

- **Helped plan at the end of each semester our Graduate Undergraduate Instructor Academy. Then at the beginning of the new semester would helped at the training. My job was to create the surveys for each training session then collect and analyze the data. Fall 2021 helped create the report that was sent to the NSF and in return received a scholarship.**

Math Center at SHSU Academic Success Center **August 2019-December 2020**

- **Mathematics tutor. Most common subjects include Statistics and Calculus.**

Grader for SHSU Mathematics and Statistics Department **Spring 2020**

- **Graded assignments and quizzes for a professor. Along with grading, when COVID made us move online, made answer keys for the professor's assignments, and quizzes.**

Math Tutor at Tutor's Station **April 2019 - Present**

- **Tutored a variety of math courses from junior high to college along with GRE, TSI and SAT math prep.**

ACHIEVEMENTS

Glenda J. St. Andrie Scholarship recipient
Mozelle Powell Spanish Scholarship recipient
Graduate Achievement Scholarship recipient
Dean's List at Sam Houston Spring 2020

President's List Spring 2020
GUIA Fellow Fall 2020, Spring 2021, Fall 2021, Spring 2022
President of AMS SHSU chapter Fall 2021-Spring 2021
Outstanding Graduate Student award recipient Spring 2022

PROFESSIONAL ORGANIZATIONS

American Mathematical Society (AMS) - President of SHSU Chapter Fall 2021-Spring 2022
Pi Mu Epsilon - Spring 2022

REFERENCES

References are available upon request.