Analysis of Symbol Message Passing LDPC Decoder for the Poisson PPM Channel

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***Abstract*—A simple decoding algorithm, dubbed symbol message passing decoder, is studied for q-ary low-density parity-check codes over the q-ary Poisson pulse-position modulation channel. The messages in the decoder are symbols from the finite field $\mathbb{F}_q$. To improve performance, a second decoder with an extended message set $\{\mathsf{E} \cup \mathbb{F}_q\}$ is also investigated, where E denotes an erasure. Thresholds within 1.3 dB from the Shannon limit are obtained for low field orders.**

I. Introduction

The direct detection photon-counting channel has been well-studied in the literature [1] and finds applications for optical space links with photon counting receivers. The number of received photons is often assumed to follow a Poisson distribution. Restricting the modulation to pulse position modulation (PPM) yields rates close to on-off keying (OOK) capacity for sufficiently large PPM orders which makes PPM an appealing modulation technique, e.g., for power limited deep space links.

Several channel coding techniques have been studied for the Poisson channel with PPM. Among them are Reed-Solomon (RS) codes (where the field order is matched with the PPM order) [2], convolutional codes [3], the serial concatenation of an outer convolutional code, an inner non-binary bit accumulator, and PPM named serially-concatenated pulse position modulation (SCPPM) [4], binary low-density parity-check (LDPC) codes in a bit-interleaved coded modulation (BICM) setup [5], as well as non-binary LDPC codes [6]–[8]. The proposed coded modulation schemes in [4], [8] perform close to the theoretical limits, but have the drawback of high decoding complexity. Various works in the literature target lowering the complexity of non-binary LDPC decoders [9]–[12]. Recently, a very simple symbol message passing (SMP) decoder for non-binary LDPC codes over the q-ary symmetric channel (QSC) was proposed [13]. Messages in the decoder consist of the most reliable symbol only, but variable node (VN) processors are able to exploit soft information. In [14] additive white Gaussian noise (AWGN) channels with orthogonal modulations are considered, where the field size q matches the modulation order. Also a complexity discussion is provided in [14] showing favorable decoding complexity compared to binary LDPC codes with non-binary modulations.

In this work, we extend the results of [14]. First, we adapt the SMP decoder to Poisson channels with orthogonal (PPM) modulations where the field size q and the modulation order are equal. For SMP, the exchanged messages are symbols from $\mathbb{F}_q$. Motivated by the gain achieved in [15] if erasures are allowed in the decoding algorithm, we extend SMP by including erasures in the message alphabet. We refer to the algorithm as symbol and erasure message passing (SEMP) which is related to the simplified message passing decoder introduced for the QSC in [16]. However, unlike in [16] we introduce a parameter Δ such that whenever log-likelihoods of two candidate symbols are within Δ we declare an erasure. We develop a density evolution (DE) analysis for the two different decoders which allows to design code ensembles with optimized iterative decoding thresholds.

II. Preliminaries

A. System Model

Consider LDPC codes over $\mathbb{F}_q = \{0, 1, \alpha, \dots, \alpha^{q-2}\}$ with $q = 2^m$, m a positive integer and α a primitive element of $\mathbb{F}_q$. We denote a length-N codeword as $\boldsymbol{c} = (c_1, c_2, \dots, c_N)$, with $c_i \in \mathbb{F}_q$. In this work, we match the finite field order q to the PPM order, yielding a one-to-one mapping between codeword symbols c_i and PPM symbols $\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,q})$. A PPM symbol spans over q time slots out of which one contains a pulse and the remaining $q-1$ slots are blank. We denote by P_u a PPM symbol for which the u-th time slot contains a pulse. The value $a \in \mathbb{F}_q$ of a code symbol specifies a PPM slot index $u \in \{1, 2, \dots, q\}$ through a one-to-one mapping $u = f(a)$. Hence, with slight abuse of notation we may also write the slot index u as an element of a finite field. We consider transmission over an optical channel with direct detection at the receiver. Let $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_n)$ be the received sequence. Let a received modulation symbol be $\boldsymbol{y}_i = (y_{i,0}, y_{i,1}, \dots, y_{i,\alpha^{q-2}})$, where $y_{i,u}$ is the number of received photons in the u-th slot of symbol i. Let n_s be the average number of received signal photons per pulsed slot and let n_b be the average number of received background noise photons per slot. Considering the u-th slot of modulation symbol i, the channel transition probabilities follow a Poisson distribution, i.e., for all $y \in \mathbb{N}_0$ we have

$$P_{Y_{i,u}|\boldsymbol{X}_i}(y|\mathsf{P}_{u'}) = \begin{cases} \frac{e^{-(n_\mathsf{b}+n_\mathsf{s})}(n_\mathsf{b}+n_\mathsf{s})^y}{y!}, & u' = u \\ \frac{e^{-n_\mathsf{b}} n_\mathsf{b}^y}{y!}, & \text{else.} \end{cases}$$

From now on we drop the index i whenever possible. For $a \in \mathbb{F}_q$, we have the likelihood

$$P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\mathsf{P}_a) = \prod_{u\in\mathbb{F}_q} P_{Y_u|\boldsymbol{X}}(y_u|\mathsf{P}_a) = \left(1+\frac{n_\mathsf{s}}{n_\mathsf{b}}\right)^{y_a} e^{-(n_\mathsf{s}+qn_\mathsf{b})} \prod_{u\in\mathbb{F}_q} \frac{n_\mathsf{b}^{y_u}}{y_u!}. \tag{1}$$

Let $\gamma = \frac{n_\mathsf{s}}{q}$ be the average number of received signal photons per slot.

B. Extrinsic Channels

For the DE analysis that follows we use the concept of extrinsic channels [17]. Consider a QSC with error probability ϵ and input and output alphabet $\mathcal{A} = \mathcal{B} = \mathbb{F}_q$. The transition probabilities of this QSC are

$$P(b|a) = \begin{cases} 1-\epsilon & \text{if } b = a \\ \epsilon/(q-1) & \text{otherwise.} \end{cases} \tag{2}$$

Further, consider a q-ary error and erasure channel (QEEC) with error probability ϵ, erasure probability θ, input alphabet $\mathcal{A} = \mathbb{F}_q$ and output alphabet $\mathcal{B} = \{\mathsf{E} \cup \mathbb{F}_q\}$, where E is an erasure. The transition probabilities of the QEEC are given by

$$P(b|a) = \begin{cases} 1-\epsilon-\theta & \text{if } b = a \\ \theta & \text{if } b = \mathsf{E} \\ \epsilon/(q-1) & \text{otherwise.} \end{cases} \tag{3}$$

C. Log-Likelihood Vector

For a discrete memoryless channel (DMC) with input $u \in \mathbb{F}_q$ and output $\boldsymbol{y}$, we introduce the normalized log-likelihood vector ($\boldsymbol{L}$-vector)

$$\boldsymbol{L}(\boldsymbol{y}) = [L_0(\boldsymbol{y}), L_1(\boldsymbol{y}), \ldots, L_{\alpha^{q-2}}(\boldsymbol{y})] \tag{4}$$

with elements (dubbed L-values)

$$L_u(\boldsymbol{y}) = \log\left(P(\boldsymbol{y}|u)\right) \quad \forall u \in \mathbb{F}_q. \tag{5}$$

D. Non-binary LDPC Codes

Non-binary LDPC codes can be defined by an $M \times N$ sparse parity-check matrix $\boldsymbol{H} = [h_{i,j}]$ with elements in $\mathbb{F}_q$. The parity-check matrix can be represented by a Tanner graph with N VNs corresponding to codeword symbols and M check nodes (CNs) corresponding to parity checks. Each edge connecting VN $\mathtt{v}$ and CN $\mathtt{c}$ is labeled by a non-zero element $h_{\mathtt{v},\mathtt{c}}$ of $\boldsymbol{H}$. The sets $\mathcal{N}(\mathtt{v})$ and $\mathcal{N}(\mathtt{c})$ denote the neighbors of VN $\mathtt{v}$ and CN $\mathtt{c}$, respectively. The degree of a VN $\mathtt{v}$ is the cardinality of the set $\mathcal{N}(\mathtt{v})$. Similarly, the degree of a CN $\mathtt{c}$ is the cardinality of the set $\mathcal{N}(\mathtt{c})$. The VN (CN) edge-oriented degree distribution polynomial is $\lambda(x) = \sum_i \lambda_i x^{i-1}$ ($\rho(x) = \sum_i \rho_i x^{i-1}$) where λ_i (ρ_i) is the fraction of edges incident to VNs (CNs) with degree i. An unstructured irregular LDPC code ensemble $\mathscr{C}^{q,N}_{\lambda,\rho}$ is the set of all q-ary LDPC codes with block length N and degree distribution polynomial pair $\lambda(x)$ and $\rho(x)$.

III. Decoding Algorithms

In this section, we describe two decoding algorithms for the Poisson PPM channel, namely SMP and SEMP. We denote by $m^{(\ell)}_{\mathtt{c}\to\mathtt{v}}$ the message sent from CN $\mathtt{c}$ to its neighboring VN $\mathtt{v}$. Similarly, $m^{(\ell)}_{\mathtt{v}\to\mathtt{c}}$ is the message sent from VN $\mathtt{v}$ to CN $\mathtt{c}$ at the ℓ-th iteration. The alphabet of the exchanged messages between the CNs and VNs is $\mathcal{M}_{\text{SMP}} = \mathbb{F}_q$ for SMP and $\mathcal{M}_{\text{SEMP}} = \{\mathsf{E} \cup \mathbb{F}_q\}$ for SEMP.

A. SMP

1) Initialization: At the beginning VN $\mathtt{v}$ computes the $\boldsymbol{L}$-vector (4) and sends the symbol which has the maximum L-value to all its neighbors. From (1) and (5) the channel $\boldsymbol{L}$-vector is

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{y}) =& [L_0(\boldsymbol{y}), L_1(\boldsymbol{y}), \ldots, L_{\alpha^{q-2}}(\boldsymbol{y})] \\ L_a(\boldsymbol{y}) =& K y_a - q n_\mathsf{b} - n_\mathsf{s} \\ &+ \sum_{u\in\mathbb{F}_q} \left(y_u \log(n_\mathsf{b}) - \log(y_u!)\right) \quad \forall a \in \mathbb{F}_q \end{aligned} \tag{6}$$

where $K = \log\left(1+\frac{n_\mathsf{s}}{n_\mathsf{b}}\right)$. The outgoing VN message is computed as

$$m^{(0)}_{\mathtt{v}\to\mathtt{c}} = \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, L_a(\boldsymbol{y}) = \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, y_a. \tag{7}$$

2) CN update: The message from CN $\mathtt{c}$ to a neighboring VN $\mathtt{v}$ is obtained by determining the symbol that satisfies the parity-check equation given the incoming messages from all other neighbors. The outgoing CN message at the ℓ-th iteration is

$$m^{(\ell)}_{\mathtt{c}\to\mathtt{v}} = -h^{-1}_{\mathtt{v},\mathtt{c}} \sum_{\mathtt{v}'\in\mathcal{N}(\mathtt{c})\setminus\mathtt{v}} h_{\mathtt{v}',\mathtt{c}} m^{(\ell-1)}_{\mathtt{v}'\to\mathtt{c}}$$

where the multiplication and the sum are performed over $\mathbb{F}_q$, $h_{\mathtt{v},\mathtt{c}}$ is a parity-check matrix element and $h^{-1}_{\mathtt{v},\mathtt{c}}$ its inverse.

3) VN update: Each VN computes

$$\begin{aligned} \boldsymbol{L}^{(\ell)}_{\mathsf{ex}} &= [L^{(\ell)}_{\mathsf{ex},0}, L^{(\ell)}_{\mathsf{ex},1}, \ldots, L^{(\ell)}_{\mathsf{ex},\alpha^{q-2}}] \\ &= \boldsymbol{L}(\boldsymbol{y}) + \sum_{\mathtt{c}'\in\mathcal{N}(\mathtt{v})\setminus\mathtt{c}} \boldsymbol{L}(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}}) \end{aligned} \tag{8}$$

where $\boldsymbol{L}(\boldsymbol{y})$ is calculated according to (6). Further, we model each CN to VN message as an observation of the symbol X (associated to $\mathtt{v}$) at the output of an *extrinsic* QSC whose crossover probability is obtained via DE analysis (see Section IV). The crossover probability is used to obtain $\boldsymbol{L}(m^{(\ell)}_{\mathtt{c}'\to\mathtt{v}})$ from (2), (4) and (5). A VN passes the symbol that maximizes $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$ to its neighboring CNs, i.e.,

$$m^{(\ell)}_{\mathtt{v}\to\mathtt{c}} = \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, L^{(\ell)}_{\mathsf{ex},a}. \tag{9}$$

4) Decision: Each VN estimates the value of the respective code word symbol as

$$\begin{aligned} \hat{m}^{(\ell)}_{\mathtt{v}} &= \underset{a\in\mathbb{F}_q}{\operatorname{argmax}}\, L^{(\ell)}_{\mathsf{app},a} \\ \boldsymbol{L}^{(\ell)}_{\mathsf{app}} &= [L^{(\ell)}_{\mathsf{app},0}, L^{(\ell)}_{\mathsf{app},1}, \ldots, L^{(\ell)}_{\mathsf{app},\alpha^{q-2}}] \end{aligned} \tag{10}$$

$$= \boldsymbol{L}(\boldsymbol{y}) + \sum_{\mathsf{c}' \in \mathcal{N}(\mathsf{v})} \boldsymbol{L}(m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}}). \quad (11)$$

We remark that in (7), (9) and (10), whenever multiple maximizing arguments exist, we choose one of them uniformly at random.

B. SEMP

For the SEMP, we introduce a real-valued parameter Δ, which is chosen to maximize the iterative decoding threshold. In this work, we keep Δ constant over all iterations (but in principle one could allow Δ to vary over iterations).

1) Initialization: At the beginning, the message from VN v to a neighboring CN c is

$$m^{(0)}_{\mathsf{v} \to \mathsf{c}} = \begin{cases} a & \text{if } \exists a \in \mathbb{F}_q \text{ with } y_a > y_u + \frac{\Delta}{K} \forall u \in \mathbb{F}_q \setminus \{a\} \\ \mathsf{E} & \text{otherwise.} \end{cases}$$

2) CN update: At the ℓ-th iteration, CN c sends to a neighboring VN v the message

$$m^{(\ell)}_{\mathsf{c} \to \mathsf{v}} = \begin{cases} -h^{-1}_{\mathsf{v},\mathsf{c}} \sum\limits_{\mathsf{v}' \in \mathcal{N}(\mathsf{c}) \setminus \mathsf{v}} h_{\mathsf{v}',\mathsf{c}} m^{(\ell-1)}_{\mathsf{v}' \to \mathsf{c}} & \text{if } m^{(\ell-1)}_{\mathsf{v}' \to \mathsf{c}} \neq \mathsf{E} \\ & \forall \mathsf{v}' \in \mathcal{N}(\mathsf{c}) \\ \mathsf{E} & \text{otherwise.} \end{cases}$$

3) VN update: The message from VN v to CN c is obtained by first computing $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$ defined in (8). $\boldsymbol{L}(\boldsymbol{y})$ is calculated according to (6) and, for SEMP, the extrinsic channel is a QEEC whose error and erasure probabilities can be estimated via DE analysis (see Section IV). The error and erasure probabilities are used to obtain $\boldsymbol{L}(m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}})$ from (3), (4) and (5). Second, for the outgoing message we pick

$$m^{(\ell)}_{\mathsf{v} \to \mathsf{c}} = \begin{cases} a & \text{if } \exists a \in \mathbb{F}_q \text{ with } L^{(\ell)}_{\mathsf{ex},a} > L^{(\ell)}_{\mathsf{ex},u} + \Delta \\ & \forall u \in \mathbb{F}_q \setminus \{a\} \\ \mathsf{E} & \text{otherwise.} \end{cases}$$

4) Decision: Each VN computes $\boldsymbol{L}^{(\ell)}_{\mathsf{app}}$ defined in (11) by using the error and erasure probabilities of the extrinsic QEEC. The final decision is

$$\hat{m}^{(\ell)}_{\mathsf{v}} = \operatorname*{argmax}_{a \in \mathbb{F}_q} L^{(\ell)}_{\mathsf{app},a}.$$

IV. Density Evolution Analysis

We provide a DE analysis for the two decoding algorithms SMP and SEMP. In particular, we are interested in the iterative decoding threshold of non-binary irregular LDPC code ensembles. For the analysis we make use of the all-zero codeword assumption since both channel and decoder fulfill the symmetry conditions [18], [19]

$$\begin{aligned} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\mathsf{P}_a) =& P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}^{+a}|\mathsf{P}_0) \\ \boldsymbol{y}^{+a} =& (y_a, y_{a+1}, \ldots, y_{\alpha^{q-2}+a}) \\ P(m|a) =& P(m+a|0) \end{aligned}$$

where the sum is over $\mathbb{F}_q$. We denote by $\ell_{\max}$ the maximum number of decoding iterations.

A. SMP

We partition the message alphabet $\mathcal{M}_{\text{SMP}}$ into 2 disjoint sets $\mathcal{I}_0 = \{0\}, \mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$ where $|\mathcal{I}_0| = 1$, $|\mathcal{I}_1| = q-1$. Due to symmetry, the messages in the same set have the same probability. Let $p^{(\ell)}_{\mathcal{I}_k}$ be the probability that a VN to CN message belongs to the set $\mathcal{I}_k$ at the ℓ-th iteration and $s^{(\ell)}_{\mathcal{I}_k}$ the probability that a CN to VN message belongs to the set $\mathcal{I}_k$, where $k \in \{0,1\}$. The ensemble iterative decoding threshold $\gamma^\star$ is defined as the minimum γ for which $p^{(\ell)}_{\mathcal{I}_0} \to 1$ as $\ell \to \infty$. DE proceeds as follows.

1) Initialization: Under the all-zero codeword assumption, the elements of $\boldsymbol{Y}$ are Poisson distributed with expectation

$$\mathbb{E}[Y_u] = \begin{cases} n_{\mathsf{s}} + n_{\mathsf{b}} & u = 0 \\ n_{\mathsf{b}} & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} p^{(0)}_{\mathcal{I}_0} =& e^{-(n_{\mathsf{s}}+qn_{\mathsf{b}})} \sum_{y=0}^{\infty} \frac{(n_{\mathsf{s}}+n_{\mathsf{b}})^y}{y!} \sum_{t=0}^{q-1} \binom{q-1}{t} \times \\ & \frac{1}{t+1} \Big(\frac{n_{\mathsf{b}}^y}{y!} \Big)^t \Big(\sum_{i=0}^{y-1} \frac{n_{\mathsf{b}}^i}{i!} \Big)^{q-1-t} \\ p^{(0)}_{\mathcal{I}_1} =& 1 - p^{(0)}_{\mathcal{I}_0}. \end{aligned}$$

2) CN update: For $\ell = 1, 2, \ldots, \ell_{\max}$ we have

$$\begin{aligned} s^{(\ell)}_{\mathcal{I}_0} =& \frac{1}{q} \Big[1 + (q-1) \rho \Big(\frac{q \cdot p^{(\ell-1)}_{\mathcal{I}_0} - 1}{q-1} \Big) \Big] \\ s^{(\ell)}_{\mathcal{I}_1} =& 1 - s^{(\ell)}_{\mathcal{I}_0}. \end{aligned}$$

The extrinsic channel is a QSC with error probability $s^{(\ell)}_{\mathcal{I}_1}$.

3) VN update: We introduce the random vector $\boldsymbol{F}^{(\ell)} = (F^{(\ell)}_0, \ldots, F^{(\ell)}_{\alpha^{q-2}})$, where $F^{(\ell)}_a$ denotes the random variable (RV) associated to the number of incoming CN messages to a degree d VN that take value $a \in \mathcal{M}_{\text{SMP}}$ at the ℓ-th iteration, and $f^{(\ell)}_a$ is its realization. The entries of $\boldsymbol{L}(m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}})$ in (8) are

$$\begin{aligned} L_u\big(m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}}\big) =& \log\big(P(m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}}|u)\big) \\ P(m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}}|u) =& \begin{cases} s^{(\ell)}_{\mathcal{I}_0} & \text{if } m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}} = u \\ s^{(\ell)}_{\mathcal{I}_1}/(q-1) & \text{if } m^{(\ell)}_{\mathsf{c}' \to \mathsf{v}} \neq u. \end{cases} \end{aligned}$$

The elements of $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$ in (9) are

$$\begin{aligned} L^{(\ell)}_{\mathsf{ex},a} =& K y_a + \mathsf{D}^{(\ell)} f^{(\ell)}_a + w_1 \\ \mathsf{D}^{(\ell)} =& \log(s^{(\ell)}_{\mathcal{I}_0}) - \log(s^{(\ell)}_{\mathcal{I}_1}/(q-1)) \quad (12) \\ w_1 =& \sum_{u \in \mathbb{F}_q} (y_u \log(n_{\mathsf{b}}) - \log(y_u!)) - q n_{\mathsf{b}} - n_{\mathsf{s}} \\ & + (d-1) \log(s^{(\ell)}_{\mathcal{I}_1}/(q-1)). \quad (13) \end{aligned}$$

Note that w_1 in (13) is independent of a. It can be thus ignored when computing $\boldsymbol{L}^{(\ell)}_{\mathsf{ex}}$. We obtain $p^{(\ell)}_{\mathcal{I}_0}$ and $p^{(\ell)}_{\mathcal{I}_1}$ in (14) and (15), where $\mathcal{S}_t$ is a subset of $\mathbb{F}_q \setminus \{0\}$ of size t and $\forall a \in \mathbb{F}_q \setminus \{0\}$

$$\Pr\{Y_a = y_a\} = \begin{cases} e^{-n_{\mathsf{b}}} \frac{n_{\mathsf{b}}^{y_a}}{y_a!} & y_a \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

$$p^{(\ell)}_{\mathcal{I}_0} = \sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}\} \sum_{y=0}^{\infty} \frac{e^{-(n_\mathsf{s}+n_\mathsf{b})}(n_\mathsf{s}+n_\mathsf{b})^y}{y!} \sum_{t=0}^{q-1} \frac{1}{t+1} \sum_{\mathcal{S}_t} \Big(\prod_{a\in\mathcal{S}_t} \Pr\Big\{Y_a = y + \mathsf{D}^{(\ell)} \frac{f_0^{(\ell)} - f_a^{(\ell)}}{K}\Big\}\Big) \times \Big(\prod_{a\in\mathbb{F}_q\setminus\{0,\mathcal{S}_t\}} \Pr\Big\{Y_a < y + \mathsf{D}^{(\ell)} \frac{f_0^{(\ell)} - f_a^{(\ell)}}{K}\Big\}\Big) \tag{14}$$

$$p^{(\ell)}_{\mathcal{I}_1} = 1 - p^{(\ell)}_{\mathcal{I}_0} \tag{15}$$

Further, the second sum is over integer vectors $\boldsymbol{f}^{(\ell)}$ for which we have $0 \le f_u^{(\ell)} \le d-1\,\forall u\in\mathbb{F}_q$, $\sum_{u\in\mathbb{F}_q} f_u^{(\ell)} = d-1$,

$$\Pr\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}\} = \binom{d-1}{f_0^{(\ell)},\dots,f_{\alpha^{q-2}}^{(\ell)}} (s^{(\ell)}_{\mathcal{I}_0})^{f_0^{(\ell)}} \Big(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\Big)^{d-1-f_0^{(\ell)}}.$$

B. SEMP

We partition the message alphabet $\mathcal{M}_{\text{SEMP}}$, due to symmetry, into 3 disjoint sets such that the messages in the same set have the same probability. We have $\mathcal{I}_0 = \{0\}, \mathcal{I}_1 = \{a : a \in \mathbb{F}_q \setminus \{0\}\}$ and $\mathcal{I}_2 = \{\mathsf{E}\}$.

1) Initialization: We have

$$p^{(0)}_{\mathcal{I}_0} = \sum_{y=0}^{\infty} \Pr\{Y_0 = y\} \prod_{u\in\mathbb{F}_q\setminus\{0\}} \Pr\{Y_u < y - \Delta/K\}$$

$$p^{(0)}_{\mathcal{I}_1} = \sum_{a\in\mathbb{F}_q\setminus\{0\}} \sum_{y=0}^{\infty} \Pr\{Y_a = y\} \prod_{u\in\mathbb{F}_q\setminus\{a\}} \Pr\{Y_u < y - \Delta/K\}$$

$$p^{(0)}_{\mathcal{I}_2} = 1 - p^{(0)}_{\mathcal{I}_0} - p^{(0)}_{\mathcal{I}_1}$$

where for $y \in \mathbb{N}_0$ and $a \in \mathbb{F}_q$

$$\Pr\{Y_a = y\} = \begin{cases} e^{-(n_\mathsf{s}+n_\mathsf{b})} \frac{(n_\mathsf{s}+n_\mathsf{b})^y}{y!} & a = 0 \\ e^{-n_\mathsf{b}} \frac{n_\mathsf{b}^y}{y!} & a \in \mathbb{F}_q \setminus \{0\} \end{cases}$$

$$\Pr\{Y_a < w\} = \sum_{j=0}^{\lceil w\rceil - 1} \Pr\{Y_a = j\}.$$

2) CN update: For $\ell = 1, 2, \dots, \ell_{\max}$ we have

$$s^{(\ell)}_{\mathcal{I}_0} = \frac{1}{q}\Big[\rho(1 - p^{(\ell-1)}_{\mathcal{I}_2}) + (q-1)\rho\Big(\frac{q\cdot p^{(\ell-1)}_{\mathcal{I}_0} - 1 + p^{(\ell-1)}_{\mathcal{I}_2}}{q-1}\Big)\Big]$$

$$s^{(\ell)}_{\mathcal{I}_2} = 1 - \rho(1 - p^{(\ell-1)}_{\mathcal{I}_2})$$
$$s^{(\ell)}_{\mathcal{I}_1} = 1 - s^{(\ell)}_{\mathcal{I}_0} - s^{(\ell)}_{\mathcal{I}_2}.$$

The extrinsic channel is a QEEC with error probability $s^{(\ell)}_{\mathcal{I}_1}$ and erasure probability $s^{(\ell)}_{\mathcal{I}_2}$.

3) VN update: We extend the random vector $\boldsymbol{F}^{(\ell)}$ to $\boldsymbol{F}^{(\ell)} = (F_0^{(\ell)}, \dots, F_{\alpha^{q-2}}^{(\ell)}, F_\mathsf{E}^{(\ell)})$, where $F_a^{(\ell)}$ denotes the RV associated to the number of incoming CN messages to a degree d VN that take value $a \in \mathcal{M}_{\text{SEMP}}$ at the ℓ-th iteration, and $f_a^{(\ell)}$ is its realization. The entries of $\boldsymbol{L}(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}})$ in (8) are

$$L_u(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}) = \log\Big(P(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}|u)\Big)$$

$$P(m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}}|u) = \begin{cases} s^{(\ell)}_{\mathcal{I}_0} & \text{if } m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} = u \\ s^{(\ell)}_{\mathcal{I}_1}/(q-1) & \text{if } m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} \in \mathbb{F}_q\setminus\{u\} \\ s^{(\ell)}_{\mathcal{I}_2} & \text{if } m^{(\ell)}_{\mathsf{c}'\to\mathsf{v}} = \mathsf{E}. \end{cases}$$

The elements of $\boldsymbol{L}^{(\ell)}_{\text{ex}}$ in (9) are

$$\begin{aligned} L^{(\ell)}_{\text{ex},a} &= K y_a + \mathsf{D}^{(\ell)} f_a^{(\ell)} + w_2 \\ w_2 &= \sum_{u\in\mathbb{F}_q} (y_u \log(n_\mathsf{b}) - \log(y_u!)) + f_\mathsf{E}^{(\ell)} \log(s^{(\ell)}_{\mathcal{I}_2}) \\ &\quad + (d-1-f_\mathsf{E}^{(\ell)}) \log\Big(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\Big) - q n_\mathsf{b} - n_\mathsf{s}. \end{aligned} \tag{16}$$

where $\mathsf{D}^{(\ell)}$ is defined in (12) Note that w_2 in (16) is independent of a. It can be thus ignored when computing $\boldsymbol{L}^{(\ell)}_{\text{ex}}$. We obtain

$$p^{(\ell)}_{\mathcal{I}_0} = \sum_d \lambda_d \sum_{\boldsymbol{f}^{(\ell)}} \Pr\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}\} \sum_{y=0}^{\infty} \Pr\{Y_0 = y\} \prod_{u\in\mathbb{F}_q\setminus\{0\}} \Pr\{Y_u < y + \frac{\mathsf{D}^{(\ell)}(f_0^{(\ell)} - f_u^{(\ell)}) - \Delta}{K}\}$$

$$p^{(\ell)}_{\mathcal{I}_1} = \sum_d \lambda_d \sum_{\substack{a\in\mathbb{F}_q\setminus\{0\} \\ \boldsymbol{f}^{(\ell)}}} \Pr\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}\} \sum_{y=0}^{\infty} \Pr\{Y_a = y\} \prod_{u\in\mathbb{F}_q\setminus\{a\}} \Pr\{Y_u < y + \frac{\mathsf{D}^{(\ell)}(f_a^{(\ell)} - f_u^{(\ell)}) - \Delta}{K}\}$$

$$p^{(\ell)}_{\mathcal{I}_2} = 1 - p^{(\ell)}_{\mathcal{I}_0} - p^{(\ell)}_{\mathcal{I}_1}$$

where the second sum is over integer vectors $\boldsymbol{f}^{(\ell)}$ for which $0 \le f_u^{(\ell)} \le d-1$ for all $u \in \mathcal{M}_{\text{SEMP}}$ and $\sum_{u\in\mathcal{M}_{\text{SEMP}}} f_u^{(\ell)} = d-1$ and

$$\Pr\{\boldsymbol{F}^{(\ell)} = \boldsymbol{f}^{(\ell)}\} = \binom{d-1}{f_0^{(\ell)},\dots,f_\mathsf{E}^{(\ell)}} (s^{(\ell)}_{\mathcal{I}_0})^{f_0^{(\ell)}} \times (s^{(\ell)}_{\mathcal{I}_2})^{f_\mathsf{E}^{(\ell)}} \Big(\frac{s^{(\ell)}_{\mathcal{I}_1}}{q-1}\Big)^{d-1-f_0^{(\ell)}-f_\mathsf{E}^{(\ell)}}.$$

C. Surrogate Erasure Channel

For $n_\mathsf{b} = 0$, the Poisson PPM channel can be modeled as a q-ary erasure channel (QEC) with erasure probability $\epsilon = \exp(-n_\mathsf{s})$ [2], [3]. Thus, for low n_b we may rely on a simplified DE analysis on a surrogate QEC to find optimized ensembles under SMP and SEMP decoding for the Poisson PPM channel. The derivation of DE for SMP and SEMP on the QEC is omitted due to space limitations, but it follows the steps of [13], [15] for the QSC: the transition probabilities or $\boldsymbol{L}$-vector of the QSC are replaced by those of the QEC.

TABLE I

THRESHOLD $\gamma^{\star}$ OF $R = 1/2$ LDPC ENSEMBLES UNDER SMP/SEMP FOR $n_{\mathsf{b}} = 0.1$. ENSEMBLES OPTIMIZED FOR THE SURROGATE QEC ARE MARKED WITH $\bullet$. AS REFERENCES: SHANNON LIMIT γ_{Sh} AND THRESHOLD $\gamma^{\star}_{\mathsf{lit}}$ OF NON-BINARY LDPC ENSEMBLE UNDER BP FROM [6, EXAMPLE 1].

	Decoder	q	$\lambda(x)$	$\rho(x)$	$\gamma^{\star}$[dB]	γ_{Sh}[dB]	$\gamma^{\star}_{\mathsf{lit}}$[dB]
	SMP	4	$0.2486x^2 + 0.4556x^3 + 0.2958x^{11}$	$0.9633x^8 + 0.0367x^9$	-3.48	-5.6	-5.42
	SEMP		$0.7328x^2 + 0.012x^3 + 0.2552x^{11}$	$0.5188x^6 + 0.4812x^7$	-4.14		
$\bullet$	SMP		$0.1979x^2 + 0.7769x^3 + 0.0252x^{11}$	$0.3441x^6 + 0.6559x^7$	-3.42		
$\bullet$	SEMP		$0.7555x^2 + 0.0018x^3 + 0.2427x^{11}$	$0.6301x^6 + 0.3699x^7$	-4.13		
	SMP	8	$0.3344x^2 + 0.3334x^3 + 0.3322x^{11}$	$0.0103x^7 + 0.9897x^8$	-6.2	-8.35	-8.07
	SEMP		$0.595x^2 + 0.0029x^3 + 0.4021x^{11}$	$0.3721x^7 + 0.6279x^8$	-6.53		
	SMP	16	$0.3691x^2 + 0.2812x^3 + 0.3497x^{11}$	$0.0089x^7 + 0.9911x^8$	-8.88	-11.02	-10.73
	SEMP	16	$0.60421x^2 + 0.0093x^3 + 0.3865x^{11}$	$0.4938x^7 + 0.5062x^8$	-8.99		
	SMP	32	$0.4711x^2 + 0.1276x^3 + 0.4013x^{11}$	$0.0128x^7 + 0.9779x^8 + 0.0093x^9$	-11.56	-13.59	-13.37
	SEMP	32	$0.7068x^2 + 0.0044x^3 + 0.2888x^{11}$	$0.3014x^6 + 0.6986x^7$	-11.69		

TABLE II

THRESHOLD $\gamma^{\star}$ OF $R = 1/2$ LDPC CODE ENSEMBLES UNDER SMP/SEMP FOR $n_{\mathsf{b}} = 0.002$. ENSEMBLES OPTIMIZED FOR THE SURROGATE QEC ARE MARKED WITH $\bullet$. AS REFERENCES: SHANNON LIMIT γ_{Sh} AND THRESHOLD $\gamma^{\star}_{\mathsf{lit}}$ OF NON-BINARY LDPC ENSEMBLE UNDER BP FROM [6, EXAMPLE 1].

	Decoder	q	$\lambda(x)$	$\rho(x)$	$\gamma^{\star}$[dB]	γ_{Sh}[dB]	$\gamma^{\star}_{\mathsf{lit}}$[dB]
	SMP	4	$0.2055x^2 + 0.6953x^3 + 0.0992x^{11}$	$0.0246x^6 + 0.9648x^7 + 0.0106x^8$	-4.68	-7.45	-7.23
	SEMP		$0.6871x^2 + 0.3129x^{11}$	$0.143x^6 + 0.857x^7$	-6.3		
$\bullet$	SMP		$0.1979x^2 + 0.7769x^3 + 0.0252x^{11}$	$0.3441x^6 + 0.6559x^7$	-4.68		
$\bullet$	SEMP		$0.7555x^2 + 0.0018x^3 + 0.2427x^{11}$	$0.6301x^6 + 0.3699x^7$	-6.3		
	SMP	8	$0.2152x^2 + 0.5352x^3 + 0.2496x^{11}$	$0.1481x^7 + 0.8519x^8$	-7.61	-10.38	-10.19
	SEMP		$0.7083x^2 + 0.0093x^4 + 0.2824x^{11}$	$0.3348x^6 + 0.6652x^7$	-9.1		
	SMP	16	$0.2284x^2 + 0.4866x^3 + 0.285x^{11}$	$0.969x^8 + 0.031x^9$	-10.59	-13.3	-13.1
	SEMP		$0.6285x^2 + 0.0095x^3 + 0.362x^{11}$	$0.7132x^7 + 0.2868x^8$	-11.9		
	SMP	32	$0.2456x^2 + 0.4206x^3 + 0.3338x^{11}$	$0.6673x^8 + 0.3327x^9$	-13.57	-16.24	-15.99
	SEMP		$0.5868x^2 + 0.0359x^3 + 0.3773x^{11}$	$0.4963x^7 + 0.5037x^8$	-14.6		

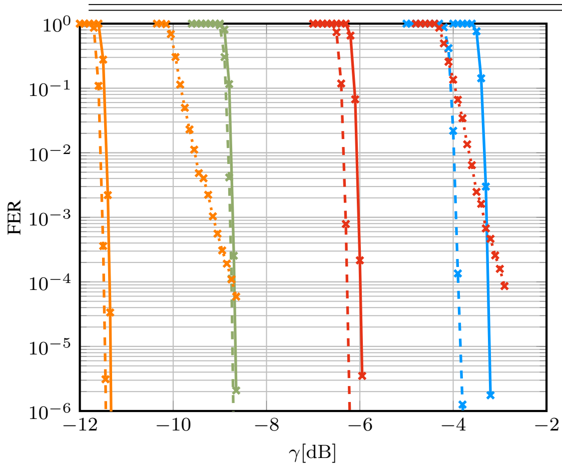

Fig. 1. FER versus γ of optimized codes via DE under SMP (, , ,) and SEMP (, , ,) for $n_{\mathsf{b}} = 0.1$ and $q = 4$ (), $q = 8$ (), $q = 16$ () and $q = 32$ (). As a reference: performance of ARJA code from [20] under SMP for $q = 8$ () and $q = 32$ ().

V. NUMERICAL RESULTS

With the help of the DE analysis presented in Section IV, we designed optimized rate $R = 1/2$ irregular LDPC code ensembles for $q \in \{4, 8, 16, 32\}$, $n_{\mathsf{b}} \in \{0.002, 0.1\}$ for both SMP and SEMP decoding. The maximum VN degree was restricted to 12 and the number of iterations to 50. The optimized degree distributions are provided in Tables I and II. SEMP shows visible gains over SMP for small values of q (e.g. > 0.6 dB for $q = 4$), while for $q = 32$ the iterative decoding thresholds nearly coincide. A comparison with the Shannon limit reveals an increasing gap for increasing q, ranging from 1.2 dB for $q = 4$ to 1.6 dB for $q = 32$ in Table II. As a comparison with the literature, we provide iterative decoding thresholds of non-binary LDPC code ensembles under belief propagation (BP) decoding [6, Example 1] which show an almost constant gap of 0.3 dB to the Shannon limit. We also observe from Tables I and II that the gap to the Shannon limit increases as n_{b} increases, e.g., from 1.3 dB for $n_{\mathsf{b}} = 0.002$ to 1.8 dB for $n_{\mathsf{b}} = 0.1$ in case of $q = 8$ and SEMP. The complexity analysis of SMP was provided in [14] showing that SMP decoding might be a good choice when low-complexity decoding is targeted. Finally, DE on a surrogate QEC yields ensembles with similar thresholds as DE on the Poisson PPM channel for $n_{\mathsf{b}} \in \{0.002, 0.1\}$, confirming the validity of a surrogate QEC code design.

For completeness, simulation results with $n_{\mathsf{b}} = 0.1$ and a maximum of 50 decoding iterations are shown in Fig. 1 in terms of frame error rate (FER) versus γ for $q \in \{4, 8, 16, 32\}$. All codes have a block length $N = 10000$ in q-ary symbols. The obtained FERs closely follow the predicted thresholds. To illustrate the need for a tailored code design, we also simulated an off-the-shelf accumulate-repeat-4-jagged-accumulate (AR4JA) code from [20] for $q \in \{8, 32\}$. The performance under SEMP is close to the one under SMP and therefore is removed from the Figure. Under SMP the codes show a significant loss compared to an optimized design.

VI. CONCLUSION

This work investigates two simple decoding algorithms, called SMP and SEMP for q-ary LDPC codes on the Poisson PPM channel. A DE analysis is developed to find optimized code ensembles which for small $q \leq 8$ under SEMP show a gap to the Shannon limit of less than 1.3 dB. For higher q the presented decoding algorithms might be of interest whenever low complexity decoding is the primary goal.

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