NONDEFINABILITY RESULTS FOR RESTRICTIONS OF THE EXPONENTIAL MAPS OF ABELIAN VARIETIES

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This thesis is concerned with definability questions for structures given by expanding the ordered real field by various classical functions. Initial results in this area concern the structure given by expanding the ordered real field by the real exponential function. The rich model theory of this structure immediately shows that the sine function is not definable in this structure. Bianconi showed that no restriction of the sine function is definable in this structure.

In this work we consider similar definability questions for a function that is similar to the exponential function, which is known as the Weierstrass $\wp$-function. The proofs of these results rely on a version of a functional transcendence result known as Ax's Theorem for the Weierstrass $\wp$-function. A corresponding theorem for the modular j-function is due to Pila and Tsimermann and by using this theorem we also obtain a definability result for the j-function.

Definability questions for expansions of the real field by several $\wp$-functions were considered and answered in work of Jones, Kirby and Servi. The $\wp$-function arises in the exponential map of elliptic curves, which are the abelian varieties of dimension 1. In this thesis we give a corresponding result for the exponential maps of all abelian varieties. This is joint work with Jones and Kirby.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

This thesis is concerned with definability questions for expansions of the ordered real field by restrictions of certain real analytic functions. Although these questions and results are model theoretic in nature, the proofs of these results explore the relationship between model theory and functional transcendence as well as differential algebra. This thesis can be considered in two parts. The first part of this thesis, Chapters 3,4,5 and 6, explores definability questions for expansions of the ordered real field by the restriction of a single function. A preliminary version of the work in Chapter 3 can be seen in the preprint [28]. The second part of this thesis consists of Chapter 7 and explores interdefinability questions for the expansion of the ordered real field by restrictions of several functions. The work in this chapter is in collaboration with my supervisor Gareth Jones and Jonathan Kirby.

The main area in which the work in this thesis lies is model theory, which is a branch of Mathematical Logic. One of the main objects of study in model theory are first order structures $\mathcal{M}$. These first order structures consist of a set M together with a collection of constant symbols, function symbols and relation symbols. A subset of M^n for some integer $n \geq 1$ is said to be *definable* in the structure $\mathcal{M}$ if it is the solution set of a formula in $\mathcal{M}$. A function f is said to be *definable* if its graph is a definable subset. A common problem in model theory is to consider a first order structure $\mathcal{M}$ and ask what its definable sets and definable functions are. Throughout this thesis definability means definability with parameters.

An important part of model theory that has been considered for several decades is the theory of o-minimal structures, which were first considered and developed by Van den Dries in [11], Pillay and Steinhorn in [32] as well as Knight, Pillay and Steinhorn in [23]. A detailed exposition of the basic theory of these structures is given by van den Dries in [13]. A structure $\mathcal{M} = (M; <)$, where $<$ is a dense linear order, is said to be *o-minimal* if every definable subset of M is a finite union of intervals and points. An initial example of an o-minimal structure is the ordered real field $\overline{\mathbb{R}} = (\mathbb{R}; +, \times, 0, 1, <)$. Here $+$ and $\times$ are the usual addition and multiplication and $<$ is the usual ordering. The o-minimality of this structure follows from the work of Tarski in [37]. As mentioned above throughout this thesis we are concerned with expansions of $\overline{\mathbb{R}}$ by restrictions of certain analytic functions. An initial example of such a structure is $\mathbb{R}_{\text{exp}} := (\overline{\mathbb{R}}, \exp : \mathbb{R} \to \mathbb{R})$, which was shown to be model complete by Wilkie in [39]. This combined with results of Khovanski in [18] gives that $\mathbb{R}_{\text{exp}}$ is o-minimal. Consider a real analytic function $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^m$ is an open subset that contains $[-1, 1]^m$. Then the function

$$\tilde{f}(x) := \begin{cases} f(x) & x \in [-1,1]^m \\ 0 & x \notin [-1,1]^m \end{cases}$$

is called the *restriction* of f. The expansion of $\overline{\mathbb{R}}$ by the restrictions of all real analytic functions is denoted $\mathbb{R}_{\text{an}}$. This structure is also o-minimal, which was shown by van den Dries in [12]. Another important example of an o-minimal structure is $\mathbb{R}_{\text{an,exp}} := (\mathbb{R}_{\text{an}}, \exp : \mathbb{R} \to \mathbb{R})$, the o-minimality of which was shown by van den Dries and Miller in [14]. Other results in this area include that the theory of $\mathbb{R}_{\text{exp}}$ is decidable under the assumption that Schanuel's Conjecture, an important conjecture in transcendental number theory, is true. This is due to Macintyre and Wilkie in [26].

The o-minimality of $\mathbb{R}_{\text{exp}}$ immediately implies that the sine function is not definable in $\mathbb{R}_{\text{exp}}$. For if it was then the zero set of the sine function would also be definable in $\mathbb{R}_{\text{exp}}$, which is an infinite discrete subset of the real numbers, a contradiction. A natural question at this stage is whether some restriction of sine to a bounded real interval is definable in $\mathbb{R}_{\text{exp}}$. This question was considered by Bianconi, who showed the following theorem in [5].

Theorem 1.0.1. *No non-trivial restriction of sine to a bounded interval $I \subseteq \mathbb{R}$ is definable in $\mathbb{R}_{\text{exp}}$.*

This theorem can be restated to say that no restriction of the complex exponential function to a disc in $\mathbb{C}$ is definable in $\mathbb{R}_{\exp}$. In fact one can go further. The following theorem is also due to Bianconi in [6].

Theorem 1.0.2. *Let $D \subseteq \mathbb{R}^{2n}$ be a definable open polydisc and $u, v : D \to \mathbb{R}$ two definable functions in $\mathbb{R}_{\exp}$. Suppose that $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic in D. Then u and v are already definable in $\overline{\mathbb{R}}$.*

Here the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$ is made. To begin this thesis we consider this formulation of the theorem for other transcendental functions. There certainly are transcendental functions f such that the restriction of f to a non-empty disc $D \subseteq \mathbb{C}$ is definable in the structure $(\overline{\mathbb{R}}, f|_I)$. In fact it turns out that there are examples of such functions that are not dissimilar to the exponential function. A *complex lattice* $\Omega \subseteq \mathbb{C}$ is defined to be a discrete subgroup of $\mathbb{C}$ of rank 2. To each such lattice Weierstrass associated the following function known as the Weierstrass $\wp$-function,

$$\wp(z) = \wp_\Omega(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

The $\wp$-function is a periodic function that also has an addition formula and a differential equation, which shall be recalled in the background chapter, Chapter 2. These are properties that are also satisfied by the exponential function and it is this together with the rich model theory of $\mathbb{R}_{\exp}$ that motivates the study of the model theory of these Weierstrass $\wp$-functions. Another property that both $\wp$ and the exponential function share is that they both arise in the exponential map of a commutative algebraic group. The model theory of Weierstrass $\wp$-functions has been considered by various people including Bianconi in [4] and Macintyre in [25]. In the course of his investigation into the model theory of these Weierstrass $\wp$-functions in [25], Macintyre observed the following.

Lemma 1.0.3. *Let $\Omega = \mathbb{Z} + i\mathbb{Z}$ and let $\wp = \wp_\Omega$ be its $\wp$-function. Let $D \subseteq \mathbb{C}$ be a disc that does not contain any lattice points. Then $\wp|_D$ is definable in $(\overline{\mathbb{R}}, \wp|_{(1/8,3/8)})$.*

Here the interval $(1/8, 3/8)$ is chosen for convenience as it avoids the poles of $\wp$. The lattice $\mathbb{Z} + i\mathbb{Z}$ is rather special. For example it can easily be seen that $\wp(iz) = -\wp(z)$. This observation is essentially all that is required to prove the previous lemma. The reason that the lattice $\mathbb{Z} + i\mathbb{Z}$ is so special is that it is a *real lattice* which has *complex multiplication*. A lattice Ω is a *real lattice* if it is closed under complex conjugation and has *complex multiplication* if there is a non-integer complex number α such that $\alpha\Omega \subseteq \Omega$. An initial result in this thesis is to extend the above lemma to all real lattices with complex multiplication. The assumption that Ω is real gives that the restriction of $\wp$ to a bounded real interval that does not intersect the lattice Ω is a real valued function. In fact Macintyre's observation can be extended to all lattices Ω with complex multiplication. In this thesis we show that one can go further. Namely one can show that the restriction of $\wp$ to a disc $D \subseteq \mathbb{C}$ is definable in the structure given by expanding $\overline{\mathbb{R}}$ by the restriction of $\wp_\Omega$ to an interval if and only if the lattice Ω has complex multiplication.

Theorem 1.0.4. *Let Ω be a complex lattice and let $\wp = \wp_\Omega$ be its $\wp$-function. Let I be a bounded real open interval such that the closure of I does not contain any lattice points. Then there is a non-empty disc $D \subseteq \mathbb{C}$ such that $\wp|_D$ is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$ if and only if the lattice Ω has complex multiplication.*

The proof of this theorem is spread across two chapters, Chapter 3 and Chapter 4. In Chapter 3 the theorem is proved for the case where the lattice Ω is a real lattice and in Chapter 4 we give the non-real lattice case. One direction of this theorem is the aforementioned extension of Macintyre's lemma. The proof of the converse in both chapters adapts a method of Bianconi used to prove Theorem 1.0.1. This method uses an implicit definition that is due to Wilkie in [39] and was proved more generally by Jones and Wilkie in [20]. This method of Bianconi also uses a functional transcendence result due to Ax in [1]. In Chapters 3,4 and 5 we use a version of this result for the Weierstrass $\wp$-function due to Ax in [2] and Brownawell and Kubota in [7].

An outline of the proof of this converse is the following. We assume that a restriction of $\wp$ to a disc $D \subseteq \mathbb{C}$ is definable in our structure. By the implicit definition this restriction is defined by a non-singular system of polynomials involving $\wp$ and $\wp'$. This gives an upper bound on the transcendence degree of some finitely generated extension of $\mathbb{C}$, which comes from the number of variables in our system of equations minus the number of equations. Then an application of the aforementioned functional transcendence result gives a lower bound which is contradictory. The reason for splitting the proof in to these two cases is the following. The implicit definition requires that the functions in our structure are real analytic functions. When we are in the real lattice case this is immediate from a fact about the $\wp$-function and we may produce a system of equations with $\wp$ and $\wp'$ on $\mathbb{R}$, which is done in Chapter 3. However once the condition that Ω is a real lattice is removed we must consider restrictions of the real and imaginary parts of $\wp$ and $\wp'$, which gives a system of equations in more variables. Therefore a straightforward application of Bianconi's method fails. However after making various adaptations involving the identities for the real and imaginary parts of a complex function and identities for the Weierstrass $\wp$-function we may give a proof of the non-real lattice case and finish the proof of Theorem 1.0.4, which is done in Chapter 4.

Now our attention turns to the question of which other $\wp$-functions are definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$. An *elliptic function* with respect to the lattice Ω is a meromorphic function that is periodic with respect to Ω. It is a general fact that given a complex lattice Ω any elliptic function with respect to Ω can be written as a rational function in $\wp_\Omega$ and $\wp'_\Omega$. This can be seen in [35]. Let Ω' be a complex lattice such that $\Omega \subseteq \Omega'$. Then $\wp_{\Omega'}$ is an elliptic function with respect to Ω and if Ω has complex multiplication then a restriction of $\wp_{\Omega'}$ to a disc $D \subseteq \mathbb{C}$ is definable in $(\overline{\mathbb{R}}, \wp_\Omega|_I)$. In Chapter 5 we show that this does not hold in the case when the lattice Ω is a real lattice that does not have complex multiplication. In fact this converse goes further and also shows that we do not obtain any new complex functions and can be thought of as a $\wp$-function analogue of Theorem 1.0.2.

Theorem 1.0.5. *Let $D \subseteq \mathbb{R}^{2N}$ be a definable open polydisc and $u, v : D \to \mathbb{R}$ be two functions that are both definable, in the structure $(\overline{\mathbb{R}}, \wp|_I)$, where Ω is a real lattice without complex multiplication and I is some bounded real interval that does not intersect the lattice Ω. If $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic in D, then u and v are definable in $\overline{\mathbb{R}}$.*

The proof of this theorem adapts that of Theorem 1.0.2, which uses a method similar to that used to prove Theorem 1.0.4. Here we use a different implicit definition, which arises from a theorem of Gabrielov, namely Theorem 1 in [17]. The main difference between this method and the one seen in earlier chapters is that two extra equations must be added to the system. These equations arise from the Cauchy-Riemann equations for the functions u and v.

One of the key components needed in the proof of Theorem 1.0.4 and Theorem 1.0.5 is the existence of a version of Ax's theorem for the Weierstrass $\wp$-function. This allows us to obtain large lower bounds on transcendence degree. In general obtaining such bounds without an Ax statement is a difficult problem. Therefore once we have another transcendental function for which a version of Ax's theorem holds, a natural problem to consider is whether we can answer similar definability questions in expansions of $\overline{\mathbb{R}}$ by restrictions of this function. The modular j-function is a transcendental function which is defined on the upper half plane and which satisfies a version of Ax's theorem, due to Pila and Tsimermann in [30]. Also, the j-function is a real valued function when restricted to an interval on the imaginary axis. This partially motivates the following theorem, which is proved in Chapter 6.

Theorem 1.0.6. *Let $I \subseteq \mathbb{R}^{>0}$ be an interval and let $D \subseteq \mathbb{H}$ be an non-empty disc. Then the restriction of j to the disc D is not definable in the structure $(\overline{\mathbb{R}}, j|_{iI})$.*

The proof of this theorem adapts the original method of Bianconi and also returns to the original implicit definition. The form of the Ax result that is required for the proof of this theorem is very slightly different to the form stated in [30]. The form that is required is stated and proved in the background chapter, Chapter 2, and follows immediately from the form in [30].

In the final chapter of this thesis we turn our attention to a different definability question. Definability questions for expansions of $\overline{\mathbb{R}}$ by multiple $\wp$-functions were considered and answered by Jones, Kirby and Servi in [21]. As in previous cases we consider a restriction of these functions. Firstly we make the definition of this restriction more precise. Let $\mathcal{F}$ be a set of maps $f : U \to \mathbb{R}$ each defined on an open subset U of $\mathbb{R}^n$ for some n. A function $f \in \mathcal{F}$ is said to be *locally definable* in some expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$ if for each $a \in U$ there is a neighbourhood U_a of a such that the restriction of f to this neighbourhood is definable in $\mathcal{R}$. A *proper restriction* of f is a restriction of f to an open box in U with rational corners. The smallest expansion of $\overline{\mathbb{R}}$ in which all of these maps are locally definable is the expansion of $\overline{\mathbb{R}}$ by all the proper restrictions of the maps in $\mathcal{F}$, denoted $\mathbb{R}_{PR(\mathcal{F})}$. Two complex lattices Ω_1 and Ω_2 are *isogenous* if there is a non-zero complex number such that $\alpha\Omega_2 \subseteq \Omega_1$. If $f : D \to \mathbb{C}$ is a holomorphic function for a disc $D \subseteq \mathbb{C}$ whose centre is in $\mathbb{R}$ then the Schwarz reflection of f is denoted f_{SR} and is given by $f_{\mathrm{SR}}(z) = \overline{f(\bar{z})}$. The definability question we are concerned with is the case when $\mathcal{F}$ is a finite set of $\wp$-functions. This was the question considered and answered by Jones, Kirby and Servi in [21] who showed the following.

Theorem 1.0.7. *Let $\mathcal{F}_1$ consist of complex exponentiation and some Weierstrass $\wp$-functions and let $\mathcal{F}_2$ consist of Weierstrass $\wp$-functions. Suppose that none of the functions in $\mathcal{F}_2$ is isogenous to any $\wp$-function from $\mathcal{F}_1$ or isogenous to the Schwarz reflection of a $\wp$-function in $\mathcal{F}_1$. Then any set in both $\mathbb{R}_{PR(\mathcal{F}_1)}$ and in $\mathbb{R}_{PR(\mathcal{F}_2)}$ is definable in $\overline{\mathbb{R}}$.*

Over the complex field an elliptic curve $E(\mathbb{C}) = E_\Omega(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})$ is given by the equation

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3.$$

The map $\exp_E : \mathbb{C} \to E(\mathbb{C}), z \mapsto [\wp(z) : \wp'(z) : 1]$ is called the *exponential map* of E. Here the complex numbers g_2 and g_3 depend on the lattice Ω associated to $\wp$ and are known as the invariants of $\wp$. These are defined explicitly in the background chapter, Chapter 2. In fact elliptic curves are the abelian varieties of dimension 1. An abelian variety is a complete algebraic group. It is therefore natural to ask whether the previous theorem can be extended to the exponential maps of general abelian varieties. In order to formulate this question more precisely we must choose the set $\mathcal{F}$ more carefully.

Let G be an abelian variety and let $\mathcal{F}_G$ be the set consisting of: the exponential map of G, the exponential maps of all abelian subvarieties of G, the exponential maps of all abelian varieties isogenous to an abelian subvariety of G, the exponential maps of the products of these abelian varieties and the exponential maps of all abelian varieties isogenous to an abelian subvariety of these products as well as the Schwarz reflections of all these exponential maps. Then we have the following theorem.

Theorem 1.0.8. *Let G and H be abelian varieties and let $\mathcal{F}_G$ and $\mathcal{F}_H$ be their associated sets of exponential maps. Suppose that $\mathcal{F}_G \cap \mathcal{F}_H = \emptyset$.*

Then any set definable in both $\mathbb{R}_{PR(\mathcal{F}_G)}$ and $\mathbb{R}_{PR(\mathcal{F}_H)}$ is semialgebraic. (In other words, it is definable in $\overline{\mathbb{R}}$).

The general strategy used in the proof of this theorem is an adaptation of the method used in [21] to prove Theorem 1.0.7. This uses the method of predimensions due to Hrushovski in [19]. In Chapter 7 the predimension used is different to that in [21]. In [21] the group rank in the definition of this predimension arises from the dimension of the graph of the exponential map of the elliptic curves associated to each $\wp$-function. Here we introduce the notion of $G^{\max}$, an abelian variety of maximal dimension with respect to a certain property and which is defined up to isogeny. The group rank is then defined to be the dimension of the vector space of Kähler differentials that is spanned by the differentials associated to the exponential map of this $G^{\max}$ evaluated at a particular point. The definition of this $G^{\max}$ requires an Ax-type result, which follows from a result of Kirby, namely Proposition 3.7 in [22]. The material in this chapter is in collaboration with my supervisor Gareth Jones and Jonathan Kirby.

In summary, the format of this thesis is the following. The next chapter will consist of background material on the Weierstrass $\wp$-function and the modular j-function as well as all the versions of Ax's theorem that shall be needed. Also included in this background chapter are the statements of both of the implicit definitions that are required as well as an explanation on how we may obtain upper bounds on transcendence degree from the non-singular systems given by these implicit definitions. In Chapter 3 we give the proof of Theorem 1.0.4 in the case where the lattice Ω is real and in Chapter 4 we give the case where the lattice Ω is not isogenous to a real lattice. In this latter chapter the proof splits into two subcases, namely when the lattice Ω is isogenous to its complex conjugate and when it is not. In Chapters 5 and 6 the proofs of Theorems 1.0.5 and 1.0.6 are given, respectively. Finally in Chapter 7 we give the proof of Theorem 1.0.8 after some background on abelian varieties and differential forms. To finish this thesis we give a conclusion which comments on each of the results proved in this work and contains some discussion on the various ways in which these results could be extended in future research.

Chapter 2

Background

In this chapter we shall give the background material needed throughout this thesis. This will begin with an overview of the Weierstrass $\wp$-function and the modular j-function. Throughout this work we shall require an important result from functional transcendence known as Ax's theorem. In fact various versions of this theorem shall be used for both the $\wp$-function and the j-function and therefore in Section 2.3 we shall list all of them. These functional transcendence results give large lower bounds on the transcendence degree of certain finitely generated extensions of $\mathbb{C}$ and some of the methods used throughout this thesis will involve opposing them with upper bounds that become contradictory if the theorems fail. These upper bounds shall arise from various non-singular systems of equations given by implicit definitions. Therefore we conclude this background chapter by stating these implicit definitions and explaining how these upper bounds follow from the system of equations. There are two implicit definitions used throughout this thesis the first of which is due to Wilkie in [39] and which Bianconi refers to in [5] as the 'Desingularisation Theorem'. The second of these implicit definitions is due to a result of Gabrielov. Although this result is well known this is as far as I am aware the first use of this result to obtain an implicit definition of this kind. The systems of equations that we shall use are systems of algebraic functions and so the proof of the upper bound is given here. For systems of polynomials it is a standard argument.

2.1 The Weierstrass ℘-function

This first background section contains material on the Weierstrass ℘-function, a complex meromorphic function associated to a complex lattice. Firstly we give the definition of a complex lattice as well as various types of complex lattices that will be central to the results at the beginning of this thesis.

Definition 2.1.1. Let $\Omega \subseteq \mathbb{C}$. Then Ω is said to be a *complex lattice* if there exist complex numbers ω_1 and ω_2 such that $\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}, \quad \Im(\omega_2/\omega_1) > 0\}$. In other words Ω is a discrete subgroup of rank 2. The set $\{\omega_1, \omega_2\}$ is referred to as an oriented basis for the lattice Ω.

Definition 2.1.2. Let Ω_1 and Ω_2 be complex lattices. If there is a non-zero complex number α such that $\alpha\Omega_1 \subseteq \Omega_2$ then the lattices Ω_1 and Ω_2 are *isogenous*. This is denoted $\Omega_1 \sim \Omega_2$.

Definition 2.1.3. Let Ω be a complex lattice. If there is a non-integer complex number α such that $\alpha\Omega \subseteq \Omega$ then the lattice Ω has *complex multiplication*.

Definition 2.1.4. Let Ω be a complex lattice. If Ω is closed under complex conjugation, in other words if $\overline{\Omega} = \Omega$, then Ω is a *real lattice*.

The following definition is in Sections 19-20 of [38].

Definition 2.1.5. Let Ω be a real lattice with an oriented basis $\{\omega_1, \omega_2\}$. Then,

1. If ω_1 is real and ω_2 is purely imaginary then Ω is known as a *rectangular* lattice.

2. If $\omega_1 = \overline{\omega_2}$ so that $\omega_3 := \omega_1 + \omega_2$ is real and $\omega_4 := \omega_2 - \omega_1$ is purely imaginary then Ω is known as a *rhombic* lattice.

Remark 2.1.6. In Sections 19 and 20 of [38] it can be seen that the real lattices are precisely the rectangular and rhombic lattices.

To each complex lattice Ω Weierstrass associated a function, the Weierstrass ℘-function. The rest of this subsection will consist of the definition of the Weierstrass ℘-function as well as some important properties of this function. This may be seen in various books such as [9], [38] as well as Chapter 6 in [35].

Definition 2.1.7. Let $\Omega \subseteq \mathbb{C}$ be a complex lattice. Then for all $z \in \mathbb{C}$,

$$\wp(z) = \wp_\Omega(z) = \frac{1}{z^2} + \sum_{z \in \Omega^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where $\Omega^* = \Omega \setminus \{0\}$.

The following lemma consists of several basic facts about the Weierstrass $\wp$-function and is therefore stated without proof. These facts can be seen in Chapter 3 of [9].

Lemma 2.1.8. *Let Ω be a complex lattice and $\wp = \wp_\Omega$ be its associated Weierstrass $\wp$-function. Then,*

1. *$\wp$ is a meromorphic function with double poles at precisely the points in the lattice Ω (and analytic elsewhere).*

2. *$\wp$ is periodic with respect to Ω.*

Definition 2.1.9. For a complex lattice Ω the *invariants* of $\wp$ are defined to be the complex numbers

$$g_2 = g_2(\Omega) = 60 \sum_{\omega \in \Omega^*} \frac{1}{\omega^4} \qquad \text{and} \qquad g_3 = g_3(\Omega) = 140 \sum_{\omega \in \Omega^*} \frac{1}{\omega^6}. \tag{2.1}$$

The content of the following proposition can be seen in Section 18 of [38].

Proposition 2.1.10. *Let Ω be a real lattice and $\wp$ its Weierstrass $\wp$-function. Then the restriction of $\wp$ to an interval I that does not contain any poles and is on either the real or imaginary axis is a real valued function and the associated invariants g_2 and g_3 are real numbers.*

The Weierstrass $\wp$-function satisfies a differential equation, the proof of which can be seen for example in Theorem 3 in Chapter 3 of [9].

Theorem 2.1.11. *For all $z \in \mathbb{C} \setminus \Omega$ we have that,*

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3. \tag{2.2}$$

Therefore the functions $\wp$ and $\wp'$ are algebraically dependent. Differentiating both sides of this differential equation gives that

$$\wp''(z) = 6\wp^2(z) - \frac{g_2}{2}. \tag{2.3}$$

In particular for any $n \geq 2$ the derivative $\wp^{(n)}$ may be written as a polynomial in $\wp$ and $\wp'$. Another crucial property of $\wp$ is its addition formula. This can be seen in Theorem 6 in Chapter 3 of [9].

Theorem 2.1.12. *For complex numbers z and w such that $z - w \notin \Omega$ we have that,*

$$\wp(z+w) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2 - \wp(z) - \wp(w). \tag{2.4}$$

From this addition formula the duplication formula for $\wp$ can be deduced, namely,

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z). \tag{2.5}$$

The function $\wp'$ also has an addition formula. However this is less well known and may be deduced from the identity

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(w) & \wp'(w) & 1 \\ \wp(z+w) & -\wp'(z+w) & 1 \end{vmatrix} = 0, \tag{2.6}$$

which can be seen in page 363 of [10]. From this identity we have for all complex numbers z and w such that $z - w \notin \Omega$,

$$\wp'(z+w) = \frac{\wp(w)\wp'(z) - \wp'(w)\wp(z) - \wp(z+w)(\wp'(z) - \wp'(w))}{\wp(z) - \wp(w)}. \tag{2.7}$$

Remark 2.1.13. By repeatedly applying the addition and duplication formulas for $\wp$, one can obtain a formula for $\wp(nz)$ as a rational function in $\wp(z)$ and $\wp'(z)$ for any natural number n. Similarly we can write $\wp(z)$ in terms of $\wp(z/n)$ and $\wp'(z/n)$. Rearranging this and using the differential equation for $\wp$ gives that we can write $\wp(z/n)$ as a function in $\wp(z)$. Due to the introduction of square roots this function is algebraic. The square roots introduce the problem of branches.

In our applications of this formula we will always be able to assume that the domain of this algebraic function can be chosen so that it is also analytic.

The final definition and theorem of this section can be seen in Sections 2 and 3 of Chapter 6 in [35].

Definition 2.1.14. An *elliptic function* with respect to a lattice Ω is a meromorphic function f on $\mathbb{C}$ that satisfies

$$f(z+\omega) = f(z)$$

for all $z \in \mathbb{C}$ and $\omega \in \Omega$. The field of elliptic functions with respect to a lattice Ω is denoted $\mathbb{C}(\Omega)$.

The functions $\wp_\Omega(z)$ and $\wp'_\Omega(z)$ are elliptic functions with respect to the lattice Ω. The following theorem is Theorem 3.2 in Chapter 6 of [35].

Theorem 2.1.15. *Let Ω be a complex lattice. Then,*

$$\mathbb{C}(\Omega) = \mathbb{C}(\wp_\Omega(z), \wp'_\Omega(z)).$$

2.2 The modular j-function

This section concerns the modular j-function but firstly we give some further background on complex lattices.

Definition 2.2.1. Let Ω be a complex lattice generated by ω_1 and ω_2 such that $\Im(\omega_2/\omega_1) > 0$. Then the quotient $\tau = \omega_2/\omega_1 \in \mathbb{H}$ is known as the *period ratio* of Ω. The lattice generated by 1 and τ is denoted $\Omega_\tau = \langle 1, \tau \rangle$.

This next definition can be seen in Section 4 of Chapter 6 of [35].

Definition 2.2.2. Let Ω_1 and Ω_2 be complex lattices. Then Ω_1 and Ω_2 are said to be *homothetic* if there is a non-zero complex number α such that $\alpha\Omega_1 = \Omega_2$.

The following lemma is Lemma 1.2 in Chapter 1 of [34].

Lemma 2.2.3. *1. Let Ω be a complex lattice and let $\{\omega_1, \omega_2\}$ and $\{\omega'_1, \omega'_2\}$ be two oriented bases for Ω such that ω_2/ω_1 and $\omega'_2/\omega'_1 \in \mathbb{H}$. Then there are integers a, b, c, d with $ad - bc = 1$ such that*

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2.$$

2. *Let $\tau_1, \tau_2 \in \mathbb{H}$. Then $\Omega_{\tau_1} = \langle 1, \tau_1 \rangle$ is homothetic to $\Omega_{\tau_2} = \langle 1, \tau_2 \rangle$ if and only if there are integers a, b, c, d with $ad - bc = 1$ such that*

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}.$$

3. *Let Ω be a complex lattice. Then there is a $\tau \in \mathbb{H}$ such that Ω is homothetic to Ω_τ.*

This next definition can be seen in Section 4 of Chapter 1 of [34]. Recall from the introduction that every elliptic curve defined over $\mathbb{C}$ is associated to a complex lattice Ω.

Definition 2.2.4. Let E_τ be the elliptic curve associated to the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Then the holomorphic function $j : \mathbb{H} \to \mathbb{C}$ is defined by,

$$j(\tau) = 1728 \frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)},$$

where the complex numbers g_2 and g_3 are the invariants of the complex lattice Ω with period ratio τ as seen in Definition 2.1.9.

It turns out that the modular j-function may be written rather differently, namely it has a q-expansion with (positive) integer coefficients. This may be seen in Proposition 7.4 in Chapter 1 of [34] and the explicit coefficients are in Example 6.2.2 of Chapter 2 of [34].

Proposition 2.2.5. *Let $q = e^{2\pi i z}$. Then,*

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots.$$

Remark 2.2.6. From the q-expansion it is clear that the restriction of j to $\mathbb{H} \cap i\mathbb{R}$ is a real valued function.

By Theorem 4.1 in [34] the j-function is a modular function of weight zero. That is, for all $z, w \in \mathbb{C}$ we have that $j(z) = j(w)$ if and only if there is some matrix $\gamma \in SL_2(\mathbb{Z})$ such that

$$w = \frac{az+b}{cz+d}, \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If γ is a matrix in $GL_2^+(\mathbb{Q})$, the group of 2×2 matrices with rational entries and positive determinant, then there is a unique positive integer M such that $M\gamma \in GL_2(\mathbb{Z})$ and the entries of $M\gamma$ are relatively prime. By Proposition 23 in [41] we have that for each positive integer M there is a polynomial $\Phi_M \in \mathbb{Z}[X, Y]$ such that $\Phi_M(j(z), j(w)) = 0$ if and only if there is a matrix $\gamma \in GL_2^+(\mathbb{Q})$ such that $z = \gamma w$ and $\det(M\gamma) = M$. Finally we note as in [30] that j satisfies a nonlinear third order differential equation, namely

$$\frac{j'''}{j'} - \frac{3}{2}\left(\frac{j''}{j'}\right)^2 + \left(\frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}\right)(j')^2 = 0. \tag{2.8}$$

2.3 Variations on Ax's theorem

Throughout this thesis various functional transcendence results shall be used. These are all versions of Ax's theorem, which shall not be needed here but is stated for completeness and can be seen in [1].

Theorem 2.3.1. *Let $z_1, \ldots, z_n$ be complex power series with no constant term that are linearly independent over $\mathbb{Q}$. Then,*

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)] \geq n + 1.$$

The first functional transcendence result that we shall need is the $\wp$-function version of Ax's theorem. This is due to Ax in [2] and is also due to Brownawell and Kubota in [7]. Throughout these statements we assume that the complex lattice Ω does not have complex multiplication. (The theorem is known for the case where the lattice Ω has complex multiplication but as this is not needed in this thesis we do not state this version of the theorem. The only difference in the complex multiplication case is that the linear independence hypotheses are required over the field $\mathbb{Q}(\tau)$).

Theorem 2.3.2. *Let $z_1, \ldots, z_n$ be complex power series with no constant term that are linearly independent over $\mathbb{Q}$. Then,*

$$\operatorname{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \wp(z_1), \ldots, \wp(z_n)] \geq n + 1.$$

This theorem may be extended to multiple $\wp$-functions $\wp_1, \ldots, \wp_m$ provided that they satisfy a certain independence condition. Let Ω be a complex lattice with an oriented basis $\{\omega_1, \omega_2\}$. Recall from Section 2.2 that the *period ratio* of Ω is defined to be $\tau := \omega_2/\omega_1$. Now let $\Omega_1, \ldots, \Omega_m$ be complex lattices each of which has a fixed oriented basis. The period ratios of $\Omega_1, \ldots, \Omega_m$ are denoted by $\tau_1, \ldots, \tau_m$ respectively. The independence condition is that for all $i, j = 1, \ldots, m$ with $i \neq j$ there do not exist integers a, b, c, d with $ad - bc \neq 0$ such that

$$\tau_j = \frac{a\tau_i + b}{c\tau_i + d}.$$

Theorem 2.3.3 (Brownawell & Kubota)**.** *Suppose $\Omega_1, \ldots, \Omega_m$ are complex lattices each of which does not have complex multiplication and whose period ratios satisfy the above independence condition. Let $\wp_1, \ldots, \wp_m$ be their corresponding $\wp$-functions. Let $z_1, \ldots, z_n$ be complex power series with no constant term that are linearly independent over $\mathbb{Q}$. Then we have that*

$$\operatorname{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \wp_1(z_1), \ldots, \wp_1(z_n), \ldots, \wp_m(z_1), \ldots, \wp_m(z_m)] \geq nm + 1.$$

Remark 2.3.4. In practice we shall use slightly different versions of the previous two theorems. The version we shall use is where $z_1, \ldots, z_n$ are analytic functions defined on a disc $D \subseteq \mathbb{C}$ centred at α. The linear independence assumption becomes that $z_1 - z_1(\alpha), \ldots z_n - z_n(\alpha)$ are linearly independent over $\mathbb{Q}$. These versions of the above theorems can be easily deduced from the original statements. Here we present this deduction as a series of statements whose proofs follow from the original theorem and the statements in the series. This deduction is presented for Theorem 2.3.2. The corresponding version of Theorem 2.3.3 shall also be stated.

Corollary 2.3.5. *Suppose $z_1, \ldots, z_n$ are analytic functions on a disc D centred at zero. Also suppose that $z_1, \ldots, z_n$ vanish at zero and are linearly independent over $\mathbb{Q}$. Then*

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \wp(z_1), \ldots, \wp(z_n)] \geq n + 1.$$

Proof. As $z_i(0) = 0$ for $i = 1, \ldots, n$ we have that the power series associated to z_i has no constant term. Now apply Theorem 2.3.2. □

Corollary 2.3.6. *Let $z_1, \ldots, z_n$ be analytic functions on a disc D centred at zero and suppose that $z_1 - z_1(0), \ldots, z_n - z_n(0)$ are linearly independent over $\mathbb{Q}$. Then*

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \wp(z_1), \ldots, \wp(z_n)] \geq n + 1.$$

Proof. Apply Corollary 2.3.5 to the functions $f_i(x) = z_i(x) - z_i(0)$ and use the addition formula for $\wp$ seen in (2.4) and the differential equation seen in (2.2) to obtain the desired result for the functions $z_1, \ldots, z_n$. □

Finally we have the version of Theorem 2.3.2 that we shall use throughout this thesis.

Theorem 2.3.7. *Let $z_1, \ldots, z_n$ be analytic functions on a disc D centred at $\alpha \in \mathbb{C}$ and suppose that $z_1 - z_1(\alpha), \ldots, z_n - z_n(\alpha)$ are linearly independent over $\mathbb{Q}$. Then*

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \wp(z_1), \ldots, \wp(z_n)] \geq n + 1.$$

Proof. Let $f_i(w) = z_i(w + \alpha) - z_i(\alpha)$. The functions f_i are analytic and defined on a disc centred at zero. Also we have that $f_i(w) - f_i(0) = z_i(w + \alpha) - z_i(\alpha)$ are linearly independent as the $z_i(z) - z_i(\alpha)$ are linearly independent. Now apply Theorem 2.3.6 to $f_1, \ldots, f_n$ and once again use the addition formula for $\wp$ seen in (2.4) and the differential equation seen in (2.2) to obtain the desired result for the functions $z_1, \ldots, z_n$. □

For completeness we state the corresponding version of Theorem 2.3.3.

Theorem 2.3.8. *Suppose $\Omega_1, \ldots, \Omega_m$ are complex lattices each of which does not have complex multiplication and whose period ratios satisfy the above condition. Let $\wp_1, \ldots, \wp_m$ be their corresponding $\wp$-functions. Let $z_1, \ldots, z_n$ be analytic functions on a disc D centred at $\alpha \in \mathbb{C}$ and suppose that $z_1 - z_1(\alpha), \ldots, z_n - z_n(\alpha)$ are linearly independent over $\mathbb{Q}$. Then we have that*

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[z_1, \ldots, z_n, \wp_1(z_1), \ldots, \wp_1(z_n), \ldots, \wp_m(z_1), \ldots, \wp_m(z_n)] \geq nm + 1.$$

This next functional transcendence result is a version of Ax's theorem for the modular j-function and is due to Pila and Tsimerman in [30].

Theorem 2.3.9. *Let K be a characteristic zero differential field with m commuting derivations D_i. Let $C = \cap_i \ker(D_i)$ be the constant field of K. Let $z_i, j(z_i), j'(z_i), j''(z_i), j'''(z_i) \in K^\times$ for all $i = 1, \ldots, n$ such that*

$$D_k j_i = j_i' D_k z_i, \quad D_k j_i' = j_i'' D_k z_i \qquad \textit{and} \qquad D_k j_i'' = j_i''' D_k z_i$$

for all i, k. Suppose that $\Phi_M(j(z_i), j(z_j)) \neq 0$ for all positive integers M and for all $i, j = 1, \ldots, n$ where $i \neq j$ and also suppose that $j(z_i) \notin C$ for all i. Then we have that,

$$\begin{aligned}\text{tr.deg}_C C(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n),\\ j'(z_1), \ldots, j'(z_n), j''(z_1), \ldots, j''(z_n)) \geq 3n + \text{rank}(D_k z_i)_{i,k}.\end{aligned}$$

In this thesis we shall require a version of this result for analytic functions as in the $\wp$-function case. Now we state and prove this version of the theorem, the proof of which follows immediately from Theorem 2.3.9 using a method which was also used by Ax in [1] to prove Theorem 1 from Theorem 3 in [1].

Theorem 2.3.10. *Let $z_1, \ldots, z_n$ be analytic functions defined on a disc $D \subseteq \mathbb{C}$, which take values in the upper half plane, such that $j(z_1), \ldots, j(z_n)$ are non-constant. Suppose that $\Phi_M(j(z_i), j(z_j)) \neq 0$ for all positive integers M and for all $i, j = 1, \ldots, n$ where $i \neq j$. Then,*

$$\text{tr.deg}_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n, j(z_1), \ldots, j(z_n), j'(z_1), \ldots, j'(z_n), j''(z_1), \ldots, j''(z_n)] \geq 3n + 1.$$

Proof. Apply Theorem 2.3.9 where K is the quotient field of the ring of analytic functions on D, the constant field $C = \mathbb{C}$ and the set of derivations $\Delta = \{d/dz\}$. As $j(z_1), \ldots, j(z_n)$ are non-constant we have that $j(z_i) \notin C$ for $i = 1, \ldots, n$. By the hypothesis in the statement the condition on modular polynomials holds and we may apply Theorem 2.3.9.

□

2.4 Implicit definitions

In this section we shall give the implicit definitions that are used throughout this thesis and then conclude this background chapter by illustrating how such an implicit definition gives rise to an upper bound on the transcendence degree of certain finitely generated extensions of $\mathbb{C}$. Firstly we give a precise definition of a property that is used in the statement of these implicit definitions.

Definition 2.4.1. Let $\mathcal{F}$ be a countable collection of real analytic functions defined on a bounded interval I in $\mathbb{R}$. Let $f \in \mathcal{F}$. If the derivatives of f may be written as a polynomial with coefficients in $\mathbb{C}$ in terms of a finite number of the functions in $\mathcal{F}$ then we say that the set $\mathcal{F}$ is *closed under differentiation.*

Consider the structure $(\overline{\mathbb{R}}, \mathcal{F})$ with $\mathcal{F}$ as above. Then if all the derivatives of the functions defined by terms are also defined by terms we say that the structure $(\overline{\mathbb{R}}, \mathcal{F})$ has a *ring of terms that is closed under differentiation.*

2.4.1 Desingularisation

The first implicit definition comes from ideas of Wilkie in [39] and is referred to by Bianconi in [5] as the Desingularisation Theorem. A more general form of this implicit definition was proved by Jones and Wilkie in [20]. Let $\tilde{\mathbb{R}} = (\overline{\mathbb{R}}, \mathcal{F})$ be an expansion of $\overline{\mathbb{R}}$ by a set $\mathcal{F}$ of total analytic functions in one variable, closed under differentiation. We also assume that $\tilde{\mathbb{R}}$ has a model complete theory and as $\mathcal{F}$ is closed under differentiation the ring of terms of $\tilde{\mathbb{R}}$ is closed under differentiation. Before stating the first implicit definition that is used in this thesis we give a definition.

Definition 2.4.2. Let $f_1 : I \to \mathbb{R}$ be a function definable in the structure $\tilde{\mathbb{R}} = (\overline{\mathbb{R}}, \mathcal{F})$. Then we say that f_1 is *implicitly $\mathcal{F}$-defined* if there are some integers $n, l \geq 1$, polynomials $P_1, \ldots, P_n$ in $\mathbb{R}[y_1, \ldots, y_{(l+1)(n+1)}]$ and functions $f_2, \ldots, f_n : I \to \mathbb{R}$ such that for all $z \in I$,

$$F_1(z, f_1(z), \ldots, f_n(z)) = 0$$
$$\vdots$$
$$F_n(z, f_1(z), \ldots, f_n(z)) = 0$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}} (z, f_1(z), \ldots, f_n(z)) \neq 0,$$

where

$$
\begin{aligned}
F_i(z, f_1(z), \ldots, f_n(z)) = P_i(z, f_1(z), \ldots, f_n(z),\\
g_1(z), g_1(f_1(z)), \ldots, g_1(f_n(z)), \ldots,\\
g_l(z), g_l(f_1(z)), \ldots, g_l(f_n(z)))
\end{aligned}
$$

for $g_1, \ldots, g_l \in \mathcal{F}$.

Theorem 2.4.3 (Jones & Wilkie)**.** *Let $f : I \to \mathbb{R}$, for some open interval $I \subseteq \mathbb{R}$, be a definable function in $\tilde{\mathbb{R}}$. Then there are subintervals $I_1, \ldots, I_m \subseteq I$ such that $I \setminus (\cup_{k=1}^m I_k)$ is a finite set and f is implicitly $\mathcal{F}$-defined on each of these subintervals.*

If $f, g : I \to \mathbb{R}$ are both definable in $\tilde{\mathbb{R}}$ then it is clear that there is a subinterval $I' \subseteq I$ on which they are both implicitly $\mathcal{F}$-defined. In order to apply this desingularisation theorem we must consider total functions. However in practice we consider expansions of $\overline{\mathbb{R}}$ by restricted analytic functions, defined on some bounded real open interval denoted $I = (a, b)$ say. In order to make these functions total they are composed with a bijection from $\mathbb{R}$ to I. This is a standard trick and here we describe this function and give some detail on its derivative and other formulae that we shall need. Throughout this thesis the interval I will change but the notation for the bijection will remain consistent throughout. Firstly define $A : (a, b) \to \mathbb{R}$ by

$$A(t) = \frac{t - \frac{b+a}{2}}{\left(\frac{b-a}{2}\right)^2 - \left(t - \frac{b+a}{2}\right)^2}, \tag{2.9}$$

which is a bijection. Differentiating gives that

$$A'(t) = \frac{\left(\frac{b-a}{2}\right)^2 + \left(t - \frac{b+a}{2}\right)^2}{\left(\left(\frac{b-a}{2}\right)^2 - \left(t - \frac{b+a}{2}\right)^2\right)^2}, \tag{2.10}$$

which does not vanish. The compositional inverse, $B = A^{-1}$ is also differentiable and

$$B'(t) = \frac{\left(\left(\frac{b-a}{2}\right)^2 - \left(B(t) - \frac{b+a}{2}\right)^2\right)^2}{\left(\frac{b-a}{2}\right)^2 + \left(B(t) - \frac{b+a}{2}\right)^2} \tag{2.11}$$

also does not vanish. Finally we define,

$$B_1(t) = \frac{1}{\left(\frac{b-a}{2}\right)^2 + \left(B(t) - \frac{b+a}{2}\right)^2}. \tag{2.12}$$

From (2.10) we can observe that

$$\begin{aligned}\frac{1}{A'(B(t))} &= \frac{\left(\left(\frac{b-a}{2}\right)^2 - \left(B(t) - \frac{b+a}{2}\right)^2\right)^2}{\left(\frac{b-a}{2}\right)^2 + \left(B(t) - \frac{b+a}{2}\right)^2} \\ &= B'(t). \end{aligned} \tag{2.13}$$

This observation does not depend on the choice of interval I, it is true for all such functions $A : (a, b) \to \mathbb{R}$. Let $f_1, \ldots, f_l$ be restricted real analytic functions defined on some bounded open real interval I say. Then the structure $(\overline{\mathbb{R}}, f_1 \circ B, \ldots, f_l \circ B, B, B_1)$ is an expansion of $\overline{\mathbb{R}}$ by total analytic functions. The structures $(\overline{\mathbb{R}}, f_1, \ldots, f_l)$ and $(\overline{\mathbb{R}}, f_1 \circ B, \ldots, f_l \circ B, B, B_1)$ are equivalent in the sense that they have the same definable sets. Also if the structure $(\overline{\mathbb{R}}, f_1, \ldots, f_l)$ has a ring of terms with parameters from $\mathbb{R}$ that is closed under differentiation then so does the structure $(\overline{\mathbb{R}}, f_1 \circ B, \ldots, f_l \circ B, B, B_1)$.

2.4.2 Consequences of a result of Gabrielov

Here we give a similar implicit definition, which does not require total functions. This implicit definition is obtained from a model completeness result of Gabrielov in [17]. Although the theorem of Gabrielov is well known, as far as I am aware this is the first application of this theorem in order to obtain an implicit definition of this kind. Firstly we state Gabrielov's theorem and give some background terminology from [17]. Then we state and prove the implicit definition.

Definition 2.4.4. Let $\Phi = \{\phi_j\}$ be a set of real analytic functions ϕ_j defined and analytic on a neighbourhood of the closed unit cube $[0, 1]^{n_j} \subseteq \mathbb{R}^{n_j}$. For every $n \geq 0$, we define $A_n = A_n(\Phi)$ as the minimal set of functions with the following properties:

1. The constants 0 and 1 and a coordinate function x_1 on $\mathbb{R}$ belong to A_1.

2. $\phi_j \in A_{n_j}$ for each j.

3. If $\phi, \psi \in A_n$ then $\phi \pm \psi$ and $\phi \cdot \psi \in A_n$.

4. If $\phi(x_1, \ldots, x_n) \in A_n$ then $\phi(x_{i(1)}, \ldots, x_{i(n)}) \in A_{n+m}$, for any mapping $i : \{1, \ldots, n\} \to \{1, \ldots, n+m\}$.

5. If $\phi(x) \in A_n$ then $\partial\phi(x)/\partial x_v \in A_n$ for $v = 1, \ldots, n$.

Definition 2.4.5. A subset $X \subseteq [0,1]^n$ is called Φ-*semianalytic* if it is a finite union of sets of the form

$$\{x \in [0,1]^n : f_i(x) = 0, \text{ for } i = 1, \ldots, M; g_j(x) > 0, \text{ for } j = 1, \ldots, N\}, \quad (2.14)$$

where f_i, g_j are analytic functions from $A_n(\Phi)$. A subset $Y \subseteq [0,1]^n$ is called Φ-*subanalytic* if it is an image of the projection to $\mathbb{R}^n$ of a Φ-semianalytic subset $X \subseteq [0,1]^{m+n}$.

Definition 2.4.6. For a set $X \subseteq [0,1]^n$, let $\overline{X}$ be the closure and $\tilde{X} = [0,1]^n \setminus X$ its complement in $[0,1]^n$ and $\partial X = \overline{X} \setminus X$ its frontier. A semianalytic set $X \subseteq R^n$ is *non-singular* of dimension k at a point $x_0 \in X$ if there exist real analytic functions $h_1(x), \ldots, h_{n-k}(x)$ defined in an open set U containing x_0 such that $dh_1 \wedge \cdots \wedge dh_{n-k} \neq 0$ at x_0 and $X \cap U = \{x \in U : h_1(x) = \cdots = h_{n-k}(x) = 0\}$. A semianalytic set is *effectively non-singular* if the functions $h_1, \ldots, h_{n-k}$ can be chosen from the f_i when X is of the form (2.14). The dimension of a set X is defined as the maximum of its dimensions at non-singular points.

Theorem 2.4.7 (Gabrielov)**.** *Let Y be a Φ-subanalytic subset of $[0,1]^n$. Then $\tilde{Y} = [0,1]^n \setminus Y$ is Φ-subanalytic.*

Consider a set of restricted real analytic functions Φ and a subanalytic set Y defined from the functions in Φ. Then by the previous theorem the complement of Y is defined by functions in the algebra generated by the functions in Φ, their partial derivatives, the constants 0 and 1 and the coordinate functions. In particular we have the following corollary.

Corollary 2.4.8. *Let $\mathcal{F}$ be an infinite collection of real analytic functions that are defined on a bounded closed interval in $\mathbb{R}$ that is closed under differentiation. Then the structure $(\overline{\mathbb{R}}, \mathcal{F})$ is model complete.*

The following lemma is Lemma 3 in [17] and is required for the proof of the implicit definition.

Lemma 2.4.9. *Let X be a Φ-semianalytic set in $[0,1]^{m+n}$, and let $Y = \pi X \subseteq [0,1]^n, d = \dim Y$. Then there exist finitely many Φ-semianalytic subsets X'_v and a Φ-subanalytic subset V of X such that $Y = (\pi V) \cup \bigcup_v \pi X'_v$ and*

1. *X'_v is effectively non-singular, $\dim X'_v = d$ and $\pi : X'_v \to Y$ has rank d at every point of X'_v for each v.*
2. $\dim \pi V < d$
3. *$X'_u \cap X'_v = \emptyset$, for $u \neq v$.*

Now we shall state and prove the implicit definition that arises from Gabrielov's theorem.

Proposition 2.4.10. *Let $\mathcal{F}$ be a set of real analytic functions defined on a neighbourhood in $[0,1]$ that contains a closed interval I, suppose that $\mathcal{F}$ is closed under differentiation and consider the structure $(\overline{\mathbb{R}}, \mathcal{F}|_I)$, where $\mathcal{F}|_I \coloneqq \{g|_I : g \in \mathcal{F}\}$. Let $f : U \to I^k$ where $U \subseteq I^m$ for some $m, k \geq 1$ be a function definable in $(\overline{\mathbb{R}}, \mathcal{F})$ and let $f_1, \ldots, f_k : U \to I$ denote its coordinate functions.*

Then there exist integers $n, l \geq 1$, polynomials $P_1, \ldots, P_n$ in $\mathbb{R}[y_1, \ldots, y_{(l+1)(m+n)}]$, functions $f_{k+1}, \ldots, f_n : B \to I$ for an open box $B \subseteq U$ and $g_1, \ldots, g_l \in \mathcal{F}$ such that for all $\bar{z} = (z_1, \ldots, z_m) \in B$,

$$F_1(\bar{z}, f_1(\bar{z}), \ldots, f_n(\bar{z})) = 0$$
$$\vdots$$
$$F_n(\bar{z}, f_1(\bar{z}), \ldots, f_n(\bar{z})) = 0$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\ldots,n \\ j=m+1,\ldots,m+n}} (\bar{z}, f_1(\bar{z}), \ldots, f_n(\bar{z})) \neq 0,$$

where

$$\begin{aligned} F_i(\bar{z}, f_1(\bar{z}), \ldots, f_n(\bar{z})) = P_i(\bar{z}, f_1(\bar{z}), \ldots, f_n(\bar{z}), & \\ g_1(z_1), \ldots, g_1(z_m), g_1(f_1(\bar{z})), \ldots, g_1(f_n(\bar{z})), \ldots, & \\ g_l(z_1), \ldots, g_l(z_m), g_l(f_1(\bar{z})), \ldots, g_l(f_n(\bar{z}))). & \end{aligned}$$

Proof. Here the functions in $\mathcal{F}$ are defined on a neighbourhood in $[0,1]$ rather than a neighbourhood containing $[0,1]$. This has a slight impact on the definitions and results of Gabrielov that we wish to apply, namely that the interval $I \subseteq [0,1]$ takes the place of $[0,1]$ in the above statements. Let $Y = \Gamma(f) \subseteq \mathbb{R}^{m+1}$ be the graph of f. Clearly $\dim Y = m$. Then Y is a definable set in the structure $(\overline{\mathbb{R}}, \mathcal{F})$ and by the corollary Y is a $\mathcal{F}$-subanalytic set of dimension m. So Y has an existential definition. By definition $Y = \pi X$ where X is a $\mathcal{F}$-semianalytic subset of $\mathbb{R}^{m+n}$ for some n. By Lemma 2.4.9 we have that $Y = (\pi V) \cup \bigcup \pi X'_v$ where X'_v are effectively non-singular $\mathcal{F}$-semianalytic sets of dimension m and πV is small. It is enough to prove the result for $Y = \pi X'_v$ for a single effectively non-singular set X'_v. By the definition of an effectively non-singular set and the rank condition the function f may be defined by a non-singular system of $m+n-m$ equations as described in the statement. □

2.4.3 Upper bounds on transcendence degree

The non-singular systems given by these two implicit definitions lead to upper bounds on the transcendence degree of certain finitely generated extensions of $\mathbb{C}$. In order to obtain this well-known upper bound we shall state and prove a lemma, the proof of which uses the following lemma, which can be seen in 3.4.30 in [27].

Lemma 2.4.11. *Let F be a real closed field and let $X \subseteq F^n$ be semialgebraic. In particular, let $\phi(v,w)$ be a formula and let $a \in F^m$ be such that $X = \{x \in F^n : \phi(x,a)\}$. If $K \supseteq F$ is a real closed field, define* $\dim_K(X)$*, the algebraic dimension of X in K to be the maximum transcendence degree of $F(c_1, \ldots, c_n)$ over F, where $c \in K^n$ and $K \models \phi(c,a)$. Define* $\dim(X)$*, the algebraic dimension of X to be the maximum value of* $\dim_K(X)$ *as K ranges over all real closed extensions of F.*

Then every k-cell has algebraic dimension k.

Lemma 2.4.12. *Let K be the field of germs of meromorphic functions at zero. Suppose $F_1, \ldots, F_n : U \to \mathbb{C}$ are algebraic and analytic on open $U \subseteq \mathbb{C}^{m+n}$ with $\phi_1, \ldots, \phi_{m+n} \in K$ such that $(\phi_1(0), \ldots, \phi_{m+n}(0)) \in U \cap \mathbb{R}^{m+n}$ and $F_1, \ldots, F_n$ are real valued on $U \cap \mathbb{R}^{m+n}$ and that,*

$$F_1(\phi_1(t),\ldots,\phi_{m+n}(t)) = 0$$
$$\vdots$$
$$F_n(\phi_1(t),\ldots,\phi_{m+n}(t)) = 0$$

and

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,m+n}} (\phi_1(t),\ldots,\phi_{m+n}(t)) = n$$

for all small real t. Then,

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}(\phi_1,\ldots,\phi_{m+n}) \leq m.$$

Proof of Lemma 2.4.12. Let $a = (\phi_1(0),\ldots,\phi_{m+n}(0)) \in \mathbb{R}^{m+n}$ and assume that

$$\det\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,n\\ j=m+1,\ldots,m+n}} (a) \neq 0.$$

Apply the implicit function theorem to get that there is a neighbourhood $B_1 \times B_2$ of a where $B_1 \subseteq \mathbb{R}^m$ and $B_2 \subseteq \mathbb{R}^n$ and functions $g_1,\ldots,g_n : B_1 \to B_2$ definable in $\overline{\mathbb{R}}$ and analytic such that for $(x,y) \in B_1 \times B_2$ we have that

$$F_1(x,y) = 0$$
$$\vdots$$
$$F_n(x,y) = 0$$

if and only if $y = (g_1(x),\ldots,g_n(x))$. Write $g = (g_1,\ldots,g_n)$.

Claim 2.4.13. *There is an elementary extension $\mathcal{R}$ of $\overline{\mathbb{R}}$ such that $\mathcal{R}$ contains the germs $\phi_1,\ldots,\phi_{m+n}$ (restricted to real t).*

Proof of Claim. Consider the structure $\mathbb{R}_{\text{an}}$ and the germs of all the functions in $\mathbb{R}_{\text{an}}$ on the interval $(0,\epsilon)$. By o-minimality these form a Hardy field of $\mathbb{R}_{\text{an}}$, which is a model of the theory of $\mathbb{R}_{\text{an}}$ denoted $\mathcal{H}$. This is in fact an elementary extension of $\mathbb{R}_{\text{an}}$.

Now consider $\mathcal{H}$ with just its field structure denoted $\mathcal{R}$. This is an elementary extension of $\overline{\mathbb{R}}$, containing the germs $\phi_1,\ldots,\phi_{m+n}$, as required. □

Consider $\Gamma(g)$ as an m-cell. Therefore by the above lemma $\Gamma(g)$ has algebraic dimension m. Let $\Phi(x,y)$ be such that $\Gamma(g) = \{x \in \mathbb{R}^{n+m} : \overline{\mathbb{R}} \models \Phi(x,b)\}$. Let $c \in \mathcal{R}^{m+n}$ be such that $\mathcal{R} \models \Phi(c,b)$. Then by definition,

$$\text{tr.deg}_{\mathbb{R}}\mathbb{R}(c_1, \ldots, c_{n+m}) \leq m$$

and therefore

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}(c_1, \ldots, c_{n+m}) \leq m.$$

The germs $\phi_1, \ldots, \phi_{m+n}$ are such that the restriction of $(\phi_1, \ldots, \phi_{m+n})$ to U lies in the graph of g and so

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}(\phi_1, \ldots, \phi_{m+n}) \leq m$$

as required.

□

Chapter 3

Nondefinability for the Weierstrass $\wp$-function: The real lattice case

In this chapter we state and prove the first main result of this thesis, which is Theorem 1.0.4 for the case when the associated complex lattice Ω is a real lattice. Firstly we give a precise statement of the theorem that shall be proved in this chapter. A preliminary version of this theorem can be seen in a preprint of my own on arxiv, namely [28].

Theorem 3.0.1. *Let Ω be a real lattice and let $\wp = \wp_\Omega$ be its $\wp$-function. Let I be a bounded real open interval such that $\overline{I} \cap \Omega = \emptyset$. Then there is a non-empty disc $D \subseteq \mathbb{C}$ such that the restriction $\wp|_D$ is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$ if and only if the lattice Ω has complex multiplication.*

From Remark 2.1.6 we know that the real lattices are either rectangular or rhombic. The proof of this theorem is similar for each of these cases but both are given here. As noted in the introduction to this thesis one direction of this result is an extension of Macintyre's lemma, (Lemma 1.0.3).

3.1 Proof of Theorem 3.0.1

3.1.1 Macintyre's lemma for real lattices with complex multiplication

The first half of the proof of Theorem 3.0.1 is an extension of Macintyre's lemma to the case of all real lattices with complex multiplication. The proof of this lemma follows that of Macintyre for the case where the lattice Ω is $\mathbb{Z} + i\mathbb{Z}$. Recall that a lattice Ω has *complex multiplication* if there is a non-integer complex number α such that $\alpha\Omega \subseteq \Omega$.

Lemma 3.1.1. *Let Ω be a real lattice with complex multiplication and let $\wp = \wp_\Omega$ be its $\wp$-function. Let I be a bounded real open interval that does not contain a lattice point and whose endpoints are not lattice points. Then the restriction of $\wp$ to any complex disc that does not contain a lattice point is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$.*

Proof. As Ω has complex multiplication there is a non-integer complex number α such that $\alpha\Omega \subseteq \Omega$. Firstly we show that the function $\wp|_{\alpha I}$ is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$. Let $z \in \mathbb{C}$ and define $f(z) = \wp(\alpha z)$. Then for any $\omega \in \Omega$,

$$f(z + \omega) = \wp(\alpha z + \alpha\omega) = \wp(\alpha z) = f(z)$$

as $\alpha\Omega \subseteq \Omega$. Therefore f is a meromorphic function that is periodic with respect to the lattice Ω and so f is an elliptic function with respect to Ω. By Theorem 2.1.15 the function f may be written as a rational function R in $\wp(z)$ and $\wp'(z)$. Similarly the function $g(z) = \wp'(\alpha z)$ may be written as a rational function S in $\wp(z)$ and $\wp'(z)$. Therefore the functions $\wp$ and $\wp'$ restricted to αI are definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$.

Now consider some disc D contained in $I \times \alpha I$ that does not contain a lattice point. For $z \in D$ it is clear that we may write $z = x + \alpha y$ for $x, y \in I$. We can assume that $x - \alpha y \notin \Omega$. Then by the addition formula for $\wp$, namely (2.4), we have that

$$\wp(z) = \wp(x + \alpha y) = \frac{1}{4}\left(\frac{\wp'(x) - S(\wp(y), \wp'(y))}{\wp(x) - R(\wp(y), \wp'(y))}\right)^2 - \wp(x) - R(\wp(y), \wp'(y)).$$

As every function in this expression is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$ the function $\wp(z)$ is definable in this structure for all $z \in D$. Hence the function $\wp|_D$ is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$. Using the addition and duplication formulae gives us that $\wp$ restricted to any disc in $\mathbb{C}$ that does not contain a lattice point is definable in $(\overline{\mathbb{R}}, \wp|_I)$ as required.

□

3.1.2 The converse of Theorem 3.0.1

Let Ω be a complex lattice which does not have complex multiplication and let $\wp = \wp_\Omega$ be its $\wp$-function. Recall that the interval I is a bounded real open interval that does not contain a lattice point and whose endpoints are not lattice points. As the structure $(\overline{\mathbb{R}}, \wp|_I)$ is o-minimal the derivative $\wp'|_I$ is definable in this structure and so the structures $(\overline{\mathbb{R}}, \wp|_I)$ and $(\overline{\mathbb{R}}, \wp|_I, \wp'|_I)$ are equivalent in the sense of having the same definable sets. From the formula for the second derivative of $\wp$, (2.3), and the differential equation for $\wp$, (2.2), it is clear that the nth derivative of $\wp$ may be written as a polynomial in $\wp$ and $\wp'$ for all $n \geq 2$. Therefore the set $\{\wp|_I, \wp'|_I\}$ is closed under differentiation and so by Gabrielov's theorem the structure $(\overline{\mathbb{R}}, \wp|_I, \wp'|_I)$ is model complete. Bianconi has model completeness results for $\wp$ in [4] but they do not appear to be applicable here. Recall from Section 2.4.1 the semialgebraic function $B : \mathbb{R} \to I$. The structures $(\overline{\mathbb{R}}, \wp|_I, \wp'|_I)$ and $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$ are also equivalent in the sense of having the same definable sets and so it suffices to prove Theorem 3.0.1 in the structure $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$. They also have the same existentially and universally definable sets. Let X be a universally definable subset in $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$, then it is also universally definable in $(\overline{\mathbb{R}}, \wp|_I, \wp'|_I)$ and by model completeness it is an existentially definable subset in $(\overline{\mathbb{R}}, \wp|_I, \wp'|_I)$, which is an existentially definable subset in $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$. Therefore the structure $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$ is also model complete. It also has a ring of terms that is closed under differentiation. This follows from the fact that $\{\wp|_I, \wp'|_I\}$ is closed under differentiation and the definitions of B' and B_1. From the background chapter we know that the real lattices may be separated into two types, the rectangular and rhombic lattices. The proof here is given for both the rectangular and rhombic lattices. The main difference between the two proofs is the choice of the interval I, which is given explicitly, beginning with the rectangular lattice case.

For the rectangular case, Section 19 of [38] gives that it is possible to choose generators ω_1 and ω_2 for the lattice Ω so that ω_1 is real and ω_2 is purely imaginary. The interval I is chosen so that iI does not contain any lattice points. There are two sub cases to consider namely $|\omega_1| \leq |\omega_2|$ and $|\omega_2| \leq |\omega_1|$. In the first of these sub cases the interval is $I = (\omega_1/8, 3\omega_1/8)$ and in the second the interval is $(\omega_2/8i, 3\omega_2/8i)$. For each of these intervals the appropriate functions A and $B = A^{-1}$ may be defined and we assume that we are in the first of these sub cases.

Now we assume for a contradiction that there is a non-empty disc D such that the restriction $\wp|_D$ is definable in the structure $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$. By translating and scaling we may suppose that D contains the interval iI, which does not contain any lattice points. Let $f_1 : I \to \mathbb{R}$ be defined as $f_1(t) = \wp(it)$, a real valued function. This is a definable function in the structure $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$ and now we apply Theorem 2.4.3 to the function f_1 in order to give an implicit definition. Therefore for some integer $n \geq 1$ and some subinterval $I' \subseteq I$ there are polynomials $P_1^*, \dots, P_n^* \in \mathbb{R}[y_1, \dots, y_{5n+5}]$ certain functions $f_2, \dots, f_n : I' \to \mathbb{R}$ such that for all $t \in I'$,

$$\begin{aligned} F_1(t, f_1(t), \dots, f_n(t)) &= 0 \\ &\vdots \\ F_n(t, f_1(t), \dots, f_n(t)) &= 0 \end{aligned}$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (t, f_1(t), \dots, f_n(t)) \neq 0,$$

where for $i = 1, \dots, n$ we have that

$$\begin{aligned} F_i(t, f_1(t), \dots, f_n(t)) = P_i^*(t, f_1(t), \dots f_n(t), \\ &\wp(B(t)), \wp(B(f_1(t))), \dots, \wp(B(f_n(t))), \\ &\wp'(B(t)), \wp'(B(f_1(t))), \dots, \wp'(B(f_n(t))), \\ &B(t), B(f_1(t)), \dots, B(f_n(t)), \\ &B_1(t), B_1(f_1(t)), \dots, B_1(f_n(t))). \end{aligned}$$

As $\wp$ and $\wp'$ are algebraically dependent and B and B_1 are algebraic functions we have that the functions $F_1, \dots, F_n$ may be written as algebraic functions in $t, f_1(t), \ldots, f_n(t)$ and $\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))$. In defining these algebraic functions square roots are introduced from the differential equation for $\wp$ and the definition of B, which may affect the analyticity of these algebraic functions. For example we wish to avoid points where $\wp'$ vanishes, in other words where $B(t), B(f_1(t)), \ldots, B(f_n(t))$ are zeros of $\wp'$. However as the image of B is in I then by shrinking and shifting the interval I if necessary we may avoid these zeros. These algebraic functions are denoted $P_1, \ldots, P_n$ and their domain is a small open set in $\mathbb{R}^{2n+2}$ which, perhaps after first shrinking the interval I', contains the set

$$\{[t, f_1(t), \ldots, f_n(t), \wp(B(t)), \wp(B(f_1(t))), \ldots \wp(B(f_n(t)))] : t \in I'\}$$

and $P_1, \ldots, P_n$ are analytic on this domain. Therefore for $i = 1, \ldots, n$ we have that

$$F_i(x_1, \ldots, x_{n+1}) = P_i(x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})))$$

and in particular for all $t \in I'$,

$$F_i(t, f_1(t), \ldots, f_n(t)) = P_i[t, f_1(t), \ldots, f_n(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))] = 0$$

for algebraic functions $P_1, \ldots, P_n$. Now we take n to be minimal such that there is some interval I' and algebraic functions $P_1, \ldots, P_n$ in $2(n+1)$ variables and $F_i(x_1, \ldots, x_{n+1}) = P_i(x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})))$ and there are also functions $f_2, \ldots, f_n$ whose domains are I' such that $F_i(t, f_1(t), \ldots, f_n(t)) = 0$ and $\det(\partial F_i/\partial x_j)(t, f_1(t), \ldots, f_n(t)) \neq 0$ for all $t \in I'$ and $P_1, \ldots, P_n$ are analytic on their respective domains. Note that the subinterval I' and the functions $f_2, \ldots, f_n$ as well as the algebraic functions $P_1, \ldots, P_n$ may not be the same as those given here. For $i = 1, \ldots, n$ and $j = 2, \ldots, n+1$,

$$\frac{\partial F_i}{\partial x_j}(x_1, \ldots, x_{n+1}) = \frac{\partial P_i}{\partial y_j}(\bar{y}) + B'(x_j)\wp'(B(x_j))\frac{\partial P_i}{\partial y_{j+n+1}}(\bar{y}) \tag{3.1}$$

where

$$\bar{y} = (x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1}))).$$

The functions $B \circ f_1, \ldots, B \circ f_n$ are restricted real analytic functions defined on the real interval I' and can therefore be continued to analytic functions on a disc $D' \subseteq \mathbb{C}$ centred at β in I'. In order to obtain a lower bound on transcendence degree we apply Theorem 2.3.7. Firstly we prove a linear independence claim so that this theorem may be applied.

Claim 3.1.2. *Let $f_0(t) = t$. The functions $B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly independent over $\mathbb{Q}$ on the disc D'.*

Proof of Claim. It suffices to prove this claim for the restriction of these functions to the interval I'. Suppose that $B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly dependent over $\mathbb{Q}$. Then we have that for all $t \in I'$

$$a_0(B(t) - B(\beta)) + a_1(B(f_1(t)) - B(f_1(\beta))) + \cdots + a_n(B(f_n(t)) - B(f_n(\beta))) = 0 \tag{3.2}$$

for $a_0, \ldots, a_n \in \mathbb{Q}$ not all zero. Now suppose that for some rational a we have that for all $t \in I'$,

$$a(B(t) - B(\beta)) = B(f_1(t)) - B(f_1(\beta)).$$

Then as $f_1(t) = \wp(iB(t))$ and B is an algebraic function we have that $\wp$ is definable in $\overline{\mathbb{R}}$, a contradiction.

Therefore $a_i \neq 0$ for some $i = 2, \ldots, n$. We take this to be a_n and upon dividing both sides of (3.2) by a_n and rewriting the rationals $a_0, \ldots, a_{n-1}$ we may write,

$$\begin{aligned} B(f_n(t)) =& B(f_n(\beta)) + a_0(B(t) - B(\beta)) + a_1(B(f_1(t)) - B(f_1(\beta))) \\ &+ \cdots + a_{n-1}(B(f_{n-1}(t)) - B(f_{n-1}(\beta))) \end{aligned}$$

for rationals $a_0, \ldots, a_{n-1}$ not all zero and so

$$f_n(t) = A[B(f_n(\beta)) + a_0(B(t) - B(\beta)) + a_1(B(f_1(t)) - B(f_1(\beta))) + \cdots + a_{n-1}(B(f_{n-1}(t)) - B(f_{n-1}(\beta)))].$$

Define the functions $\tilde{A}, \tilde{B} : \mathbb{R}^n \to \mathbb{R}$ as

$$\tilde{A}(t_1, \ldots, t_n) = A[B(f_n(\beta)) + a_0(t_1 - B(\beta)) + \cdots + a_{n-1}(t_n - B(f_{n-1}(\beta)))] \tag{3.3}$$

and

$$\tilde{B}(t_1, \ldots, t_n) = B(f_n(\beta)) + a_0(t_1 - B(\beta)) + \cdots + a_{n-1}(t_n - B(f_{n-1}(\beta))). \tag{3.4}$$

So that

$$\tilde{A}(B(t), B(f_1(t)), \ldots, B(f_{n-1}(t))) = f_n(t)$$

and

$$\tilde{B}(B(t), B(f_1(t)), \ldots, B(f_{n-1}(t))) = B(f_n(t))$$

for all $t \in I'$. Recall from Remark 2.1.13 that $\wp(Nz)$ may be written as a rational function in $\wp(z)$ and $\wp'(z)$ where N is an integer. By rearranging this formula $\wp(z/N)$ can be written in terms of $\wp(z)$ and $\wp'(z)$. This rearrangement introduces square roots and therefore $\wp(z/N)$ is an algebraic function in $\wp(z)$ and $\wp'(z)$. By shifting and shrinking the interval I' if necessary the domain of this algebraic function may be chosen so that it is analytic. Once again by introducing roots and altering the domain if necessary $\wp(Nz)$ and $\wp(z/N)$ may both be written as algebraic functions in $\wp(z)$. Let $\mathcal{V}$ be the algebraic function in the variables $v_1, \ldots, v_n$ such that,

$$\mathcal{V}(\wp(B(z_1)), \ldots, \wp(B(z_n))) = \wp(\tilde{B}(B(z_1), \ldots, B(z_n))). \tag{3.5}$$

Differentiating (3.5) with respect to z_j for $j = 2, \ldots, n$ and evaluating at

$(t, f_1(t), \ldots, f_{n-1}(t))$ and using the expression (3.4) gives that

$$B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\frac{\partial \mathcal{V}}{\partial v_j}(\tilde{v}(t)) = a_{j-1}B'(f_{j-1}(t))\wp'(B(f_n(t))). \tag{3.6}$$

where

$$\tilde{v} = \tilde{v}(t) = [\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_{n-1}(t)))]$$

for all $t \in I'$. Now for $i = 1, \ldots, n$ define

$$\begin{aligned} Q_i(w_1, \ldots, w_{2n}) = P_i(w_1, \ldots, w_n, \tilde{A}(B(w_1), \ldots, B(w_n)), \\ w_{n+1}, \ldots, w_{2n}, \mathcal{V}(w_{n+1}, \ldots, w_{2n})) \end{aligned}$$

and also define

$$G_i(u_1, \ldots, u_n) = Q_i(u_1, \ldots, u_n, \wp(B(u_1)), \ldots, \wp(B(u_n)))$$

for all $i = 1, \ldots, n$. Therefore for all $t \in I'$ we have that,

$$G_i(t, f_1(t), \ldots, f_{n-1}(t)) = 0$$

for all $i = 1, \ldots, n$. Therefore we have a system of algebraic functions $Q_1, \ldots, Q_n$ in fewer variables. These algebraic functions have a domain which is an open set in $\mathbb{R}^{2n}$ that contains the set

$$\{[t, f_1(t), \ldots, f_{n-1}(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_{n-1}(t)))] : t \in I'\}.$$

If one of the $(n-1) \times (n-1)$ minors of the matrix

$$\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n}} (t, f_1(t), \ldots, f_n(t))$$

is non-zero for some $t \in I'$ then we have a contradiction to the minimality of n. Hence we assume all these minors are zero. For $i = 1, \ldots, n$ and $j = 2, \ldots, n$ we have that

$$\frac{\partial G_i}{\partial u_j} = \frac{\partial Q_i}{\partial w_j} + B'(u_j)\wp'(B(u_j))\frac{\partial Q_i}{\partial w_{j+n}}.$$

Now differentiating Q_i with respect to w_j for $j = 2, \ldots, n$ gives,

$$\frac{\partial Q_i}{\partial w_j} = \frac{\partial P_i}{\partial y_j} + a_{j-1}B'(u_j)A'[B(f_n(\beta)) + a_0(B(u_1) - B(\beta)) + \\ \cdots + a_{n-1}(B(u_n) - B(f_{n-1}(\beta)))]\frac{\partial P_i}{\partial y_{n+1}}.$$

Also differentiating Q_i with respect to w_{j+n} for $j = 2, \ldots, n$ gives,

$$\frac{\partial Q_i}{\partial w_{j+n}} = \frac{\partial P_i}{\partial y_{j+n+1}} + \frac{\partial P_i}{\partial y_{2n+2}}\frac{\partial \mathcal{V}}{\partial v_j}.$$

Here the partial derivatives of Q_i are evaluated at

$$(u_1, \ldots, u_n, \wp(B(u_1)), \ldots, \wp(B(u_n)))$$

and the partial derivatives of P_i are evaluated at

$$[u_1, \ldots, u_n, \tilde{A}(B(u_1), \ldots, B(u_n)), \\ \wp(B(u_1)), \ldots, \wp(B(u_n)), \mathcal{V}(\wp(B(u_1)), \ldots, \wp(B(u_n)))].$$

Putting this all together and using (3.6) as well as (3.1) we can see that upon evaluating at $(t, f_1(t), \ldots, f_{n-1}(t))$ we have for all $j = 2, \ldots, n$ that

$$\begin{aligned}\frac{\partial G_i}{\partial u_j} &= \frac{\partial P_i}{\partial y_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial P_i}{\partial y_{n+1}} \\ &\quad + B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\left(\frac{\partial P_i}{\partial y_{j+n+1}} + \frac{\partial P_i}{\partial y_{2n+2}}\frac{\partial \mathcal{V}}{\partial v_j}\right) \\ &= \frac{\partial F_i}{\partial x_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial P_i}{\partial y_n} \\ &\quad + a_{j-1}B'(f_{j-1}(t))\wp'(B(f_n(t)))\frac{\partial P_i}{\partial y_{2n+2}} \\ &= \frac{\partial F_i}{\partial x_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\Big(\frac{\partial P_i}{\partial y_{n+1}} \\ &\qquad\qquad + B'(f_n(t))\wp'(B(f_n(t)))\frac{\partial P_i}{\partial y_{2n+2}}\Big),\end{aligned}$$

where the partial derivatives of F_i are evaluated at $(t, f_1(t), \ldots, f_n(t))$ for all $t \in I'$. Hence we have that

$$\frac{\partial G_i}{\partial u_j} = \frac{\partial F_i}{\partial x_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial F_i}{\partial x_n}.$$

As all the $(n-1)\times(n-1)$ minors of the matrix

$$\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,n}}(t, f_1(t), \ldots, f_n(t))$$

are zero for all $t \in I'$ we have that the determinant

$$\begin{vmatrix} F & \begin{matrix}\frac{\partial F_1}{\partial x_{n+1}} \\ \vdots \\ \frac{\partial F_n}{\partial x_{n+1}}\end{matrix}\end{vmatrix}(t, f_1(t), \ldots, f_n(t)) = 0,$$

where F is the matrix

$$F = \left(\frac{\partial F_i}{\partial x_j} + a_{j-1}B'(x_j)A'(x_{n+1})\frac{\partial F_i}{\partial x_n}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,n+1}}.$$

Fix some $t \in I'$ and consider the matrix

$$\begin{pmatrix} F & \begin{matrix}\frac{\partial F_1}{\partial x_{n+1}} \\ \vdots \\ \frac{\partial F_n}{\partial x_{n+1}}\end{matrix}\end{pmatrix}(t, f_1(t), \ldots, f_n(t)).$$

This can be obtained from the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,n+1}}(t, f_1(t), \dots, f_n(t))$$

using the column operations $c_k \to c_k + a_k B'(f_k(t))A'(B(f_n(t)))c_n$ for $k = 1, \dots, n-1$. These column operations do not change the value of the determinant and so

$$\det\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,n+1}}(t, f_1(t), \dots, f_n(t)) = 0.$$

As t was arbitrary we have that for all $t \in I'$,

$$\det\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,n+1}}(t, f_1(t), \dots, f_n(t)) = 0,$$

a contradiction.

□

Observe that the addition of $it - i\beta$ to the list $B(t) - B(\beta), \dots, B(f_n) - B(f_n(\beta))$ does not destroy the linear independence. If it did then we may write

$$\begin{aligned} i(t-\beta) = &a_0(B(t) - B(\beta)) + a_1(B(f_1(t)) - B(f_1(\beta))) \\ &+ \cdots + a_n(B(f_n(t)) - B(f_n(\beta))) \end{aligned}$$

for rational $a_0, \dots, a_n$ not all zero. For any $t \in I'$ the left hand side of this is purely imaginary and the right hand side is real, a contradiction. Applying Theorem 2.3.7 to the list of functions $it, B, B \circ f_1, \dots, B \circ f_n$ gives that

$$\begin{aligned} &\operatorname{tr.deg}_{\mathbb{C}}\mathbb{C}[it, B(t), B \circ f_1 \dots, B \circ f_n, \\ &\qquad \wp(it), \wp(B(t)), \wp(B(f_1)), \dots, \wp(B(f_n))] \geq n+3. \end{aligned}$$

Now we find a contradictory upper bound on this transcendence degree. Firstly we show that for all $t \in I'$ the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,2n+2}}(\tilde{y}(t))$$

where

$$\tilde{y} = \tilde{y}(t) = [t, f_1(t), \ldots, f_n(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))]$$

has maximal rank n. This is done by a similar argument to that of Claim 4 in the proof of Theorem 4 in [6]. Recall that for $i = 1, \ldots, n$

$$\begin{aligned} F_i(t, f_1(t), \ldots, f_n(t)) = P_i[&t, f_1(t), \ldots, f_n(t), \\ &\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))]. \end{aligned}$$

Let $\tilde{x} = \tilde{x}(t) = (t, f_1(t), \ldots, f_n(t))$. Therefore from the expression (3.1) we have that the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}} (\tilde{x})$$

is given by multiplying the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,2n+2}} (\tilde{y})$$

by a $(2n+1) \times n$ matrix M where,

$$M = \begin{pmatrix} & 0 & B'(f_1(t))\wp'(B(f_1(t))) & \ldots & 0 \\ I_{n-1} & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \ldots & B'(f_n(t))\wp'(B(f_n(t))) \end{pmatrix}^T.$$

By the nonsingularity of the system $F_1, \ldots, F_n$ it is clear that the rows of

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}} (\tilde{x})$$

are linearly independent over $\mathbb{R}$ and so the rows of

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,2n+2}} (\tilde{y})$$

are also linearly independent over $\mathbb{R}$ for all $t \in I'$. Therefore for all $t \in I'$ the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,2n+2}} (\tilde{y}(t))$$

has maximal rank n. The upper bound on transcendence degree comes from the number of variables minus the number of equations, giving a bound of $n+2$, which is smaller than our lower bound thus providing a contradiction. By applying Lemma 2.4.12 we have upon restricting the functions $f_0, f_1, \dots, f_n, \wp(B(f_0))$, $\wp(B(f_1)), \dots, \wp(B(f_n))$ to some subinterval $I'' \subseteq I'$ if necessary the following upper bound on transcendence degree. Namely,

$$\operatorname{tr.deg}_{\mathbb{C}}\mathbb{C}[t, f_1(t), \dots, f_n(t), \wp(B(t)), \wp(B(f_1)), \dots, \wp(B(f_n))] \leq n+2.$$

As $f_1(t) = \wp(it)$ and $B(t), t$ as well as $B(f_i(t)), f_i(t)$ are algebraically dependent for $i = 1, \dots, n$ we have that

$$\begin{aligned}\operatorname{tr.deg}_{\mathbb{C}}\mathbb{C}[&it, B(t), B \circ f_1, \dots, B \circ f_n, \\ &\wp(it), \wp(B(t)), \wp(B(f_1)), \dots, \wp(B(f_n))] \leq n+2.\end{aligned}$$

So we have found upper and lower bounds on the transcendence degree of some finitely generated extension of $\mathbb{C}$ that are incompatible. Hence we have a contradiction as required and the theorem is proved in the rectangular lattice case.

For the rhombic lattice case one may choose generators ω_1 and ω_2 of the lattice Ω so that $\overline{\omega}_1 = \omega_2$. Then $\omega_3 \coloneqq \omega_1 + \omega_2$ and $\omega_4 \coloneqq \omega_2 - \omega_1$ are real and purely imaginary respectively. Again there are two sub-cases to consider, namely when $|\omega_3| \leq |\omega_4|$ and $|\omega_4| \leq |\omega_3|$. The corresponding intervals are $(\omega_3/8, 3\omega_3/8)$ and $(\omega_4/8i, 3\omega_4/8i)$ respectively. We give the proof for the first of these sub cases and pass to the corresponding auxiliary structure.

For a contradiction assume that the lattice Ω does not have complex multiplication and that there is a non-empty disc D such that the function $\wp|_D$ is definable in $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$. As in the rectangular case we may assume that the function $f_1(t) = \wp(it)$ is definable in the structure $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$ and by once again applying Theorem 2.4.3 we have that $\wp(it)$ is defined by a nonsingular system of equations

$$\begin{aligned}F_i(t, f_1(t), \ldots, f_n(t)) = P_i[&t, f_1(t), \ldots, f_n(t),\\ &\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))]\end{aligned}$$

for $i = 1, \ldots, n$ where for all $t \in I'$ where I' is a bounded real open interval we have that $F_i(t, f_1(t), \ldots, f_n(t)) = 0$ and the functions P_i are algebraic functions whose domain is an open subset of $\mathbb{R}^{2n+2}$ which, after perhaps shrinking the interval I', contains the set

$$\{[t, f_1(t), \ldots, f_n(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))] : t \in I'\}$$

and which are analytic on this domain. After reproving the corresponding linear independence claim, which is simply a reproduction of the same argument in the rectangular case, we may apply Theorem 2.3.7 to get that

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[&it, B(t), B \circ f_1 \ldots, B \circ f_n,\\ &\wp(it), \wp(B(t)), \wp(B(f_1)), \ldots, \wp(B(f_n))] \geq n + 3.\end{aligned}$$

By a repetition of the earlier discussion in the rectangular lattice case we know that the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,2n+2}} [t, f_1(t), \ldots, f_n(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t)))]$$

has maximal rank n. Therefore

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[t, f_1, \ldots, f_n, \wp(B(t)), \ldots, \wp(B(f_1)), \ldots, \wp(B(f_n))] \leq n + 2$$

and once again by algebraic dependence we have that

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[it, B(t), B \circ f_1, \ldots, B \circ f_n,$$
$$\wp(it), \wp(B(t)), \wp(B(f_1)), \ldots, \wp(B(f_n))] \leq n + 2.$$

These are incompatible upper and lower bounds on the transcendence degree of some finitely generated extension of $\mathbb{C}$. Therefore we have a contradiction and so we have proven the theorem in the rhombic lattice case. As these are all of the cases of real lattices the proof is complete.

Remark 3.1.3. The restriction of the $\wp$-function to a disc not containing any points in its associated lattice is essentially a convenience. There is no obstruction to definability when the disc contains lattice points, beyond the $\wp$-function itself not being defined at such points. Let $D \subseteq \mathbb{C}$ be a disc containing a single lattice point $\omega \in \Omega$ and consider the function $f(z) = (z - \omega)^2\wp(z)$. If Ω has complex multiplication then as $(z - \omega)^2$ is definable it is clear by a repetition of the proof of Lemma 3.1.1 that $f|_D$ is definable in $(\overline{\mathbb{R}}, \wp|_I)$. Now we suppose that Ω does not have complex multiplication and assume for a contradiction that $f|_D$ is definable in $(\overline{\mathbb{R}}, \wp|_I)$. Then $f|_{D'}$ is definable in $(\overline{\mathbb{R}}, \wp|_I)$ for some disc $D' \subseteq D$ that does not contain ω. Therefore $\wp|_{D'}$ is definable in $(\overline{\mathbb{R}}, \wp|_I)$, a contradiction to Theorem 3.0.1. Therefore the statement of Theorem 3.0.1 may be extended to all complex discs.

Chapter 4

Nondefinability for the Weierstrass $\wp$-function: The general lattice case

In the previous chapter the assumption that Ω is a real lattice gives that the restriction of the function $\wp_\Omega$ to a bounded real interval that does not intersect Ω is a real valued function. However the proof of Macintyre's lemma in the previous chapter has no dependence on the lattice Ω being real, it merely requires the assumption that the lattice has complex multiplication. Therefore it is natural to remove this assumption and consider a restriction of the real and imaginary parts of $\wp_\Omega$.

There are complex lattices that are non-real but are isogenous to a real lattice and later in this chapter an example of such a lattice is given. Recall that two lattices Ω and Ω' are isogenous if there is some non-zero complex number α such that $\alpha\Omega \subseteq \Omega'$. From the definition of $\wp$ it can be seen that $\alpha^2\wp_{\alpha\Omega}(z) = \wp_\Omega(z/\alpha)$ and so if some restriction of $\wp_\Omega$ is definable in some expansion of $\overline{\mathbb{R}}$ then the same restriction of $\wp_{\alpha\Omega}$ is also definable in this expansion using the parameter α. Therefore if a lattice Ω is isogenous to a real lattice then we can obtain a version of the converse direction of Theorem 3.0.1 by applying the theorem for the $\wp$-function associated to this real lattice. In this chapter we prove the following theorem. In this chapter we denote the real and imaginary parts of $\wp$ by $\Re(\wp)$ and $\Im(\wp)$ respectively.

Theorem 4.0.1. *Let Ω be a complex lattice and let I be a bounded real open interval that does not intersect Ω and whose endpoints are not in Ω. Then there* *is a non-empty disc $D \subseteq \mathbb{C}$ such that $\wp|_D$ is definable in $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$ if and only if the lattice Ω has complex multiplication.*

In the case where the lattice Ω is isogenous to a real lattice then this has been shown in Theorem 3.0.1. In this chapter we give the proof of this theorem in the case when the lattice Ω is not isogenous to a real lattice. One direction of this theorem will involve extending the result of Macintyre from the real lattice case to all complex lattices that have complex multiplication. As noted above the proof of this result only relies on Ω having complex multiplication and does not depend on the lattice Ω being real. Therefore the proof of this direction is very similar to the proof of the corresponding direction for Theorem 3.0.1, namely Lemma 3.1.1.

For the converse we would like to once again use the method of Bianconi. As before we can show that a non-trivial restriction of $\wp$ to a line segment within the disc D can be defined implicitly by a system of equations that is non-singular. Then we look to find contradictory upper and lower bounds on the transcendence degree of some finitely generated extension of $\mathbb{C}$. However there is a problem in using this method here, which was outlined in the introduction. Here we give a more detailed explanation of this problem. The presence of an extra function in the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$ leads to an extra $n+1$ variables in the system of equations. This gives an upper bound of $2n+3$ whilst the lower bound remains $n+3$. These bounds are not contradictory and so this more literal minded application of Bianconi's method fails. Therefore in order to obtain the desired contradiction as in the previous chapter, the upper bound must be lowered or the lower bound raised. However it turns out that after making a minor alteration the method of Bianconi can in fact be used here. This involves using a stronger version of the theorem of Brownawell and Kubota, which involves multiple $\wp$-functions, in order to raise the lower bound. Recall from the background chapter that this theorem requires a certain independence condition on the period ratios of the $\wp$-functions $\wp_{\Omega_1}, \dots, \wp_{\Omega_m}$. The period ratios of $\Omega_1, \dots, \Omega_m$ are denoted by $\tau_1, \dots, \tau_m$ respectively. The independence condition is that for all $i, j = 1, \dots, m$ with $i \neq j$ there do not exist integers a, b, c, d with $ad - bc \neq 0$ such that

$$\tau_j = \frac{a\tau_i + b}{c\tau_i + d}.$$

Now we wish to find another $\wp$-function that arises naturally from our system of equations and show that its lattice is independent from Ω in the above sense.

The function $\overline{\wp_\Omega(\bar{z})}$ is an elliptic function with respect to the lattice $\overline{\Omega}$ with period ratio $\bar{\tau}$ and from the usual identities we know that,

$$\Re(\wp(z)) = \frac{1}{2}(\wp(z) + \overline{\wp(\bar{z})}) \qquad \text{and} \qquad \Im(\wp(z)) = \frac{1}{2i}(\wp(z) - \overline{\wp(\bar{z})}).$$

We write $\tilde{\wp}(z) = \tilde{\wp}_\Omega(z) = \wp_{\overline{\Omega}}(z) = \overline{\wp_\Omega(\bar{z})}$. It follows from the identities that an algebraic function in $\Re(\wp)$ and $\Im(\wp)$ can be rewritten as an algebraic function in $\wp$ and $\tilde{\wp}$. Now we prove a well-known lemma which relates this independence condition and the isogeny of lattices.

Lemma 4.0.2. *Let Ω and Ω' be complex lattices with oriented bases $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$ and period ratios τ and τ'. Then Ω and Ω' are isogenous if and only if there are integers a, b, c, d with $ad - bc > 0$ such that*

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Proof. Suppose that Ω and Ω' are isogenous. Then there is a non-zero complex number α such that $\alpha\Omega' \subseteq \Omega$. Therefore there are integers a, b, c, d such that $\alpha\omega_1' = d\omega_1 + c\omega_2$ and $\alpha\omega_2' = b\omega_1 + a\omega_2$ and so

$$\tau' = \frac{b\omega_1 + a\omega_2}{d\omega_1 + c\omega_2} = \frac{a\tau + b}{c\tau + d}.$$

Also,

$$\begin{aligned} \tau' &= \frac{(a\tau + b)(c\bar{\tau} + d)}{|c\tau + d|^2} \\ &= \frac{ac|\tau|^2 + ad\tau + bc\bar{\tau} + bd}{|c\tau + d|^2} \end{aligned}$$

and as $\Im(\tau') > 0$ we have that $ad - bc > 0$ as required.

For the converse suppose that there are integers a, b, c, d with $ad - bc > 0$ such that

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Then

$$\begin{aligned}\frac{\omega_1(c\tau+d)}{\omega_1'}\Omega' &= \omega_1(c\tau+d)\Omega_{\tau'}\\ &= \omega_1((c\tau+d)\mathbb{Z}+(a\tau+b)\mathbb{Z})\\ &\subseteq \omega_1\Omega_\tau\\ &= \Omega.\end{aligned}$$

□

Therefore if the lattice Ω is not isogenous to $\overline{\Omega}$ we have that there are no integers a, b, c, d with $ad - bc \neq 0$ such that $\bar{\tau} = (a\tau + b)/(c\tau + d)$ and we may therefore use Theorem 2.3.8 in the proof of Theorem 4.0.1 in this case.

However if Ω is isogenous to $\overline{\Omega}$ then there are integers a, b, c, d such that the above relation holds. Therefore if Ω is isogenous to its conjugate and is not isogenous to a real lattice then one can not apply Theorem 2.3.8 with the Weierstrass $\wp$-functions $\wp_\Omega$ and $\tilde{\wp}_\Omega$ in this case. To show that such lattices exist we give the following lemma, which is from a private correspondence with Harry Schimdt and I thank him for his contribution.

Lemma 4.0.3. *Let $L = \{\alpha \in \mathbb{C}^\times : \Omega + \alpha\overline{\Omega}$ is a lattice$\}$. Then the following hold,*

1. *Suppose that Ω is of the form $\mathbb{Z} + \tau\mathbb{Z}$. Then $\alpha \in L$ if and only if $\{\alpha, \alpha\bar{\tau}\} \subseteq \mathbb{Q} + \tau\mathbb{Q}$.*

2. *A lattice Ω is isogenous to a real lattice if and only if $L \cap S^1$ is non-empty, where S^1 denotes the unit circle.*

3. *Let $\theta \in S^1 \cap \mathbb{H}$ and let N be a natural number that is non-square. Let Ω be the lattice $\Omega = \langle 1, \sqrt{N}\theta\rangle$ and suppose that Ω does not have complex multiplication. Then Ω is isogenous to its conjugate but not to a real lattice.*

Proof. For the first part of the lemma let $\alpha \in L$. Then $\Omega + \alpha\overline{\Omega}$ is a lattice denoted Ω' say. Let $\{\omega_1', \omega_2'\}$ be an oriented basis for Ω'. Then we have that

$$1 = a\omega_1' + b\omega_2' \tag{4.1}$$
$$\tau = c\omega_1' + d\omega_2' \tag{4.2}$$
$$\alpha = e\omega_1' + f\omega_2' \tag{4.3}$$
$$\alpha\bar{\tau} = g\omega_1' + h\omega_2' \tag{4.4}$$

for integers a, b, c, d, e, f, g, h. Therefore $\omega_2' = (1 - a\omega_1')/b$ and so substituting this into (4.2) gives that $\omega_1' = p + q\tau$ for rational p and q. Now substituting this and the equation for ω_2' into (4.3) and (4.4) gives that α and $\alpha\bar{\tau}$ are in $\mathbb{Q} + \tau\mathbb{Q}$ as required.

Now suppose that $\{\alpha, \alpha\bar{\tau}\} \subseteq \mathbb{Q} + \tau\mathbb{Q}$. Then we may write $\alpha = p + q\tau$ and $\alpha\bar{\tau} = r + s\tau$ for rationals p, q, r, s. Let $\omega' \in \Omega + \alpha\overline{\Omega}$. Then ω' can be written as $a + b\tau + c(p + q\tau) + d(r + s\tau)$ for integers a, b, c, d. Multiplying by a common denominator for the rationals p, q, r, s denoted M say, which depends only on α, gives $M\omega' = m + n\tau \in \Omega$ for integers m and n. Then $M(\Omega + \alpha\overline{\Omega}) \subseteq \Omega$ and as $M\Omega \subseteq M(\Omega + \alpha\overline{\Omega})$ we have that $\Omega + \alpha\overline{\Omega}$ contains and is contained in a lattice, which are discrete groups of rank 2. Clearly $\Omega + \alpha\overline{\Omega}$ is also a discrete group and as it contains and is contained in a discrete group of rank 2 it also has rank 2. Therefore it is a lattice and so $\alpha \in L$ as required.

For part 2, suppose $\theta \in L \cap S^1$, so that $\Omega' = \Omega + \theta\overline{\Omega}$ is a lattice. We may write $\theta = \beta/\bar{\beta}$ for some $\beta \in \mathbb{C}$. Then $\bar{\beta}\Omega' = \bar{\beta}\Omega + \beta\overline{\Omega}$ is a real lattice and as $\Omega \subseteq \Omega'$ the lattice $\bar{\beta}\Omega'$ is isogenous to Ω. Now suppose that Ω' is a real lattice that is isogenous to Ω. Therefore we have that $\beta\overline{\Omega} \subseteq \Omega'$ for some $\beta \in \mathbb{C}^\times$. Let $\theta = \beta/\bar{\beta}$ and consider $\Omega + \theta\overline{\Omega}$, which contains the lattice Ω. As $\beta\overline{\Omega} \subseteq \Omega'$ and Ω' is a real lattice we have that

$$\bar{\beta}(\Omega + \theta\Omega) = \bar{\beta}\Omega + \beta\overline{\Omega} \subseteq \Omega' + \Omega' \subseteq \Omega'.$$

Then as $\Omega + \theta\overline{\Omega}$ contains and is contained in a lattice it is a discrete subgroup of rank 2 as discussed in the proof of part 1 and so it is a lattice and therefore θ is in L as required.

For the third part of the lemma note that $\bar{\theta} = 1/\theta$. Then $(\sqrt{N}/\theta)\Omega = (\sqrt{N}/\theta)\mathbb{Z} + N\mathbb{Z} \subseteq \overline{\Omega}$ and so Ω is isogenous to its conjugate. Now suppose for a contradiction that Ω is isogenous to a real lattice. By part 2 there is some $\theta' \in S^1 \cap L$ and by part 1 we have that $\{\theta', \theta'\sqrt{N}/\theta\} \subseteq \mathbb{Q} + \sqrt{N}\theta\mathbb{Q}$. Therefore there are rationals p, q, r, s such that

$$\theta' = p + q\sqrt{N}\theta$$
$$\frac{\theta'\sqrt{N}}{\theta} = r + s\sqrt{N}\theta$$

and so

$$\frac{\sqrt{N}}{\theta}(p + q\sqrt{N}\theta) = r + s\sqrt{N}\theta.$$

This can be rearranged to give that

$$s(\sqrt{N}\theta)^2 + (r - qN)(\sqrt{N}\theta) - Np = 0.$$

If $s = 0$ then $\theta'\sqrt{N} = r\theta$ and so as $|\theta| = |\theta'| = 1$ we have that $\sqrt{N} = r$, a contradiction. Therefore $\sqrt{N}\theta$, the period ratio of the lattice Ω is an imaginary quadratic and so Ω has complex multiplication, a contradiction.

□

Example 4.0.4. Consider the lattice $\Omega = (\sqrt{2}(1-i)/2)\mathbb{Z} + (1+i)\mathbb{Z}$. Then we can observe that

$$i\overline{\Omega} = i((\sqrt{2}(1+i)/2)\mathbb{Z} + (1-i)\mathbb{Z}) = (\sqrt{2}(1-i)/2)\mathbb{Z} + (1+i)\mathbb{Z} = \Omega$$

and so $\Omega + i\overline{\Omega} = \Omega$ and therefore i is in the set L associated to this lattice. By part 2 of Lemma 4.0.3 the lattice Ω is isogenous to a real lattice and so Ω is an example of a non-real lattice that is isogenous to a real lattice.

From part 3 of Lemma 4.0.3 we can see that we must therefore split the proof of the theorem into two cases. Namely when the lattice Ω is isogenous to its conjugate and when it is not. In the second of these cases we may use Theorem 2.3.8 for the functions $\wp$ and $\tilde{\wp}$ as discussed. For the other case it turns out that it is possible to use Theorem 2.3.8 for one $\wp$-function. This shall be done after showing that $\tilde{\wp}(z)$ may be written as an algebraic function in $\wp(\alpha^{-1}z)$ for some $\alpha \in \mathbb{C}^\times$ witnessing the isogeny between Ω and $\overline{\Omega}$. In order to prove Theorem 4.0.1 in this case we shall also require that α is non-real. This can be shown for lattices of the form $\mathbb{Z} + \tau\mathbb{Z}$. If $\{\omega_1, \omega_2\}$ is an oriented basis for the lattice Ω we denote this by $\Omega = \langle \omega_1, \omega_2 \rangle$.

Lemma 4.0.5. *Let Ω be a complex lattice of the form $\Omega = \langle 1, \tau \rangle$ and suppose that Ω is not isogenous to a real lattice. Suppose that there is some $\alpha \in \mathbb{C}^\times$ such that $\alpha\Omega \subseteq \overline{\Omega}$. Then α is non-real.*

Proof. Assume for a contradiction that α is real. As $\alpha\Omega \subseteq \overline{\Omega}$ it is clear that $\overline{\alpha}\overline{\Omega} = \alpha\overline{\Omega} \subseteq \Omega$. Hence

$$\alpha^2\Omega \subseteq \alpha\overline{\Omega} \subseteq \Omega$$

as a sublattice. Therefore as α is real we have that $\alpha^2 \in \mathbb{Z}$ and so $\alpha = \pm\sqrt{n}$ for some $n \in \mathbb{N}$. We suppose that $\alpha = \sqrt{n}$ and write $\tau = x + iy$. Then as $\alpha\Omega \subseteq \overline{\Omega}$ we have that $\sqrt{n}\tau = a + b\bar{\tau}$ for some integers a and b. Comparing real and imaginary parts gives that

$$\begin{aligned} \sqrt{n}y + by &= 0 \\ a + bx - \sqrt{n}x &= 0. \end{aligned}$$

Consider the first of these equations,

$$y(\sqrt{n} + b) = 0.$$

As $y \neq 0$ we have that $\sqrt{n} = -b$ and so $\sqrt{n} = -b \in \mathbb{Z}$. This is a contradiction unless n is a square.

Therefore we suppose that $n = m^2$ for some integer m and so $\alpha = m$. We have that $m\Omega \subseteq \overline{\Omega}$ and $m\overline{\Omega} \subseteq \Omega$. As Ω is of the form $\mathbb{Z} + \tau\mathbb{Z}$ we have that $\{m, m\bar{\tau}\} \subseteq \mathbb{Q} + \tau\mathbb{Q}$. As m is an integer $\{1, \overline{\tau}\}$ is in $\mathbb{Q} + \tau\mathbb{Q}$. Hence $1 \in L$, by part 1 of Lemma 4.0.3 and $L \cap S^1$ is non-empty. Therefore by the second part of Lemma 4.0.3 the lattice Ω is isogenous to a real lattice, a contradiction.

□

However this shall only allow us to give the proof of Theorem 4.0.1 in the $\Omega \sim \overline{\Omega}$ case for lattices of the form $\Omega = \langle 1, \tau \rangle$. In fact the previous lemma is false for lattices not of this form. For example consider a lattice $\Omega = \omega\mathbb{Z} + \sqrt{N}\overline{\omega}\mathbb{Z}$ where N is a non-square natural number and $\overline{\omega}/\omega \in \mathbb{H}$ and Ω does not have complex multiplication. Then by part 3 of Lemma 4.0.3 the lattice Ω_τ is not isogenous to a real lattice and so Ω is not isogenous to a real lattice. However $\sqrt{N} \cdot \overline{\Omega} = \sqrt{N}\overline{\omega}\mathbb{Z} + N\omega\mathbb{Z} \subseteq \Omega$ and so Lemma 4.0.5 is false for a lattice of this form. For general lattices we overcome this obstruction in the following way.

Lemma 4.0.6. *Let Ω be a complex lattice that is not isogenous to a real lattice and suppose that there is some $\alpha \in \mathbb{R}^\times$ such that $\alpha\Omega \subseteq \overline{\Omega}$. Then Ω is isogenous to a lattice of the form $\mathbb{Z} + \sqrt{N}\theta\mathbb{Z}$ for $N \in \mathbb{N}$ and $\theta \in S^1$.*

Proof. Let ω_1 and ω_2 be an oriented basis for the lattice Ω and let $\tau = \omega_2/\omega_1$. By the proof of Lemma 4.0.5 we have that $\alpha = \sqrt{m}$ for some non-square natural number m. This part of the proof of Lemma 4.0.5 applies to general lattices. Observe that

$$\alpha\frac{\omega_1}{\overline{\omega}_1}\Omega_\tau \subseteq \Omega_{\bar{\tau}}.$$

This implies that $N = m$ and $\theta = \omega_1/\overline{\omega}_1$ and so $\sqrt{N}\theta = \alpha\omega_1/\overline{\omega}_1$. Therefore we have that,

$$\begin{aligned}
\Omega_\theta = \mathbb{Z} + \alpha\frac{\omega_1}{\overline{\omega}_1}\mathbb{Z} &\subseteq \mathbb{Z} + \alpha\frac{\omega_1}{\overline{\omega}_1}\mathbb{Z} + \alpha\frac{\omega_2}{\overline{\omega}_1}\mathbb{Z} \\
&= \mathbb{Z} + \alpha\frac{\omega_1}{\overline{\omega}_1}\left(\mathbb{Z} + \frac{\omega_2}{\omega_1}\mathbb{Z}\right) \\
&= \mathbb{Z} + \alpha\frac{\omega_1}{\overline{\omega}_1}\Omega_\tau \\
&\subseteq \mathbb{Z} + \frac{\overline{\omega}_2}{\overline{\omega}_1}\mathbb{Z} + \Omega_{\bar{\tau}} \\
&\subseteq \Omega_{\bar{\tau}} + \Omega_{\bar{\tau}} \\
&= \Omega_{\bar{\tau}}
\end{aligned}$$

and so Ω is isogenous to Ω_θ as required. □

Observe that $\overline{\Omega_\theta} = \mathbb{Z} + \sqrt{N}\theta^{-1}\mathbb{Z}$ and so

$$\sqrt{N}\theta^{-1}\Omega = \sqrt{N}\theta^{-1}\mathbb{Z} + N\mathbb{Z} \subseteq \overline{\Omega_\theta}.$$

Clearly $\sqrt{N}\theta^{-1}$ is non-real otherwise Ω_θ is not a lattice, a contradiction. Therefore for lattices Ω such that $\alpha\Omega \subseteq \overline{\Omega}$ for some $\alpha \in \mathbb{R}^\times$ it suffices to prove Theorem 4.0.1 for a lattice of the form Ω_θ. Therefore in this chapter we may assume that all lattices are of the form $\mathbb{Z} + \tau\mathbb{Z}$ in the proof of the $\Omega \sim \overline{\Omega}$ case. Now we give a well known fact on lattices with complex multiplication, deduced here from Lemma 4.0.3.

Lemma 4.0.7. *Let Ω be a complex lattice with complex multiplication. Then Ω is isogenous to a real lattice.*

Proof. We may assume that Ω is of the form $\mathbb{Z} + \tau\mathbb{Z}$. As Ω has complex multiplication we have that $|\tau|^2 = q \in \mathbb{Q}$ and we also have that

$$a\tau^2 + b\tau + c = 0$$

for integers a, b, c where $a \neq 0$. Therefore

$$a\tau|\tau|^2 + b|\tau|^2 + c\bar{\tau} = 0.$$

We may assume that $c \neq 0$ (if not then the lemma follows by a similar argument) and so

$$c\bar{\tau} = -q(a\tau + b).$$

Therefore $\{1, \bar{\tau}\} \subseteq \mathbb{Q} + \tau\mathbb{Q}$ and so by part 1 of Lemma 4.0.3 we have that $1 \in L$. By part 2 of this lemma we have that Ω is isogenous to a real lattice, as required. $\square$

Now we give the proof of Theorem 4.0.1, which completes the proof of Theorem 1.0.4. Firstly we shall prove Macintyre's observation for a lattice with complex multiplication.

4.1 Proof of Theorem 4.0.1

4.1.1 Macintyre's Lemma for all lattices with complex multiplication

Lemma 4.1.1. *Let Ω be a complex lattice with complex multiplication and let $\wp_\Omega$ be its $\wp$-function. Let I be a bounded real open interval that does not intersect Ω and whose endpoints are not in Ω. Then the restriction of $\wp$ to any complex disc which does not contain any lattice points is definable in $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$.*

Proof. As in the previous chapter this proof follows that of Macintyre for the case where $\Omega = \mathbb{Z} + i\mathbb{Z}$. Clearly the functions $\wp|_I$ and $\wp'|_I$ are definable in the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$. As the lattice Ω has complex multiplication there is a non-integer complex number α such that $\alpha\Omega \subseteq \Omega$. Let $z \in I$ and consider the function $f(z) = \wp(\alpha z)$. From the proof of Lemma 3.1.1 we know that f is an elliptic function with respect to the lattice Ω. Hence f may be written as a rational function R in $\wp$ and $\wp'$. Therefore we have that

$$f(z) = R(\wp(z), \wp'(z)).$$

Similarly the function $g(z) = \wp'(\alpha z)$ may be written as a rational function S where

$$g(z) = S(\wp(z), \wp'(z)).$$

Therefore the functions $\wp|_{\alpha I}$ and $\wp'|_{\alpha I}$ are definable in the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$. Now consider a disc $D \subseteq \{x + \alpha y : x, y \in I\}$. For any $z \in D$ we may write $z = x + \alpha y$ for $x, y \in I$ and assume that $x - \alpha y \notin \Omega$. Then by the addition formula for $\wp$,

$$\wp(z) = \wp(x + \alpha y) = \frac{1}{4}\left(\frac{\wp'(x) - S(\wp(y), \wp'(y))}{\wp(x) - R(\wp(y), \wp'(y))}\right) - \wp(x) - R(\wp(y), \wp'(y)).$$

As every function in this expression is definable in the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$ for all $z \in D$, the restriction of $\wp$ to the disc D is definable in the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$ as required. By applying the addition and duplication formulae for $\wp$ the definability of the restriction of $\wp$ to any disc that does not intersect Ω also follows. □

4.1.2 The converse of Theorem 4.0.1

Now we assume that the lattice Ω does not have complex multiplication. As described at the beginning of this chapter this proof is split into two cases, namely the case where Ω is isogenous to $\overline{\Omega}$, denoted by $\Omega \sim \overline{\Omega}$ and when it is not, denoted by $\Omega \nsim \overline{\Omega}$. Before giving the proof in each case we explain which structure the proof is completed in and why this structure may be used. The reasoning is similar to that seen in the previous chapter.

Suppose that some restriction of $\wp$ to a disc D in $\mathbb{C}$ is definable in the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I)$. By the differential equation for $\wp$ and the formula for its second derivative it is clear that for $n \geq 2$ the real and imaginary parts of the nth derivative of $\wp$ can be written as a polynomial in the real and imaginary parts of $\wp$ and $\wp'$. Therefore the set $\{\Re(\wp)|_I, \Im(\wp)|_I, \Re(\wp')|_I, \Im(\wp')|_I\}$ is closed under differentiation and so by Gabrielov's theorem, Theorem 2.4.7 the structure $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I, \Re(\wp')|_I, \Im(\wp')|_I)$ is model complete. It also has a ring of terms that is closed under differentiation. Consider the auxiliary structure $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$ where $B : \mathbb{R} \to I$ is the corresponding semialgebraic function as defined in Section 2.4.1. The structures $(\overline{\mathbb{R}}, \Re(\wp)|_I, \Im(\wp)|_I, \Re(\wp')|_I, \Im(\wp')|_I)$ and $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$ are equivalent in the sense of having the same definable sets and so the structure $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$ is model complete by a similar argument to that seen in the previous chapter and also has a ring of terms that is closed under differentiation. Now we can pass to this auxiliary structure and give the proof of Theorem 4.0.1 in each of the aforementioned cases. As we are concerned with general lattices there is no need to specify lattice shape here. As a result we do not give an explicit open interval I, we merely assume that the intersection $\overline{I} \cap \Omega$ is empty.

The $\Omega \not\sim \overline{\Omega}$ case

Assume for a contradiction that there is a disc $D \subseteq \mathbb{C}$ such that $\wp|_D$ is definable in the structure $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$. By shifting the disc D we may assume that it contains iI and so the functions $f_1, f_2 : I \to \mathbb{R}$ given by $f_1(t) = \Re(\wp(it))$ and $f_2(t) = \Im(\wp(it))$ are definable in the structure $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$. We now apply Theorem 2.4.3 to both the functions f_1 and f_2 in order to give an implicit definition. This gives non-singular systems for f_1 and f_2 in turn. As applying Theorem 2.4.3 splits I in to finitely many intervals there is some open subinterval of I on which a non-singular system for both f_1 and f_2 may be given. These are then combined into a single non-singular system. The non-singularity is clearly preserved as we can consider a 2×2 block matrix whose top left and bottom right blocks are the matrices of partial derivatives for these systems and whose remaining blocks are zero. Therefore for some integer $n \geq 1$ and some subinterval $I' \subseteq I$ there are polynomials $R_1^*, \ldots, R_n^* \in \mathbb{R}[y_1, \ldots, y_{7n+7}]$ and certain functions

$f_3, \dots, f_n : I' \to \mathbb{R}$ such that for all $t \in I'$,

$$\begin{aligned} F_1(t, f_1(t), \dots, f_n(t)) &= 0 \\ &\vdots \\ F_n(t, f_1(t), \dots, f_n(t)) &= 0 \end{aligned}$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (t, f_1(t), \dots, f_n(t)) \neq 0,$$

where for $i = 1, \dots, n$ we have that

$$\begin{aligned} F_i(t, f_1(t), \dots, f_n(t)) = R_i^*(t, f_1(t), \dots, f_n(t), \\ &\Re(\wp(B(t))), \Re(\wp(B(f_1(t)))), \dots, \Re(\wp(B(f_n(t)))), \\ &\Im(\wp(B(t))), \Im(\wp(B(f_1(t)))), \dots, \Im(\wp(B(f_n(t)))), \\ &\Re(\wp'(B(t))), \Re(\wp'(B(f_1(t)))), \dots, \Re(\wp'(B(f_n(t)))), \\ &\Im(\wp'(B(t))), \Im(\wp'(B(f_1(t)))), \dots, \Im(\wp'(B(f_n(t)))), \\ &B(t), B(f_1(t)), \dots, B(f_n(t)), \\ &B_1(t), B_1(f_1(t)), \dots, B_1(f_n(t))). \end{aligned}$$

As $\wp$ and $\wp'$ are algebraically dependent and B and B_1 are algebraic functions we have that the functions $F_1, \dots, F_n$ can be written as algebraic functions in $t, f_1(t), \dots, f_n(t), \Re(\wp(B(t))), \Re(\wp(B(f_1(t)))), \dots, \Re(\wp(B(f_n(t))))$ and $\Im(\wp(B(t))), \Im(\wp(B(f_1(t)))), \dots, \Im(\wp(B(f_n(t))))$. These algebraic functions are denoted $R_1, \dots, R_n$ and their domain is a small open set in $\mathbb{R}^{3n+3}$ which, perhaps after shrinking the interval I', contains the set

$$\{[f(t), \Re(\wp(B(f(t)))), \Im(\wp(B(f(t))))] : t \in I'\},$$

where $f(t) = (t, f_1(t), \dots, f_n(t))$ and these algebraic functions are analytic on this domain. By using the identities for the real and imaginary parts of $\wp$ these algebraic functions can be rearranged to give a system of algebraic functions in $t, f_1(t), \dots, f_n(t), \wp(B(t)), \wp(B(f_1(t))), \dots, \wp(B(f_n(t)))$ and $\tilde{\wp}(B(t)), \tilde{\wp}(B(f_1(t))), \dots, \tilde{\wp}(B(f_n(t)))$.

Therefore we have that there is some integer $n \geq 1$ and some subinterval

$I' \subseteq I$, and algebraic functions $P_1, \ldots, P_n$, certain functions $f_3, \ldots, f_n : I' \to \mathbb{R}$ such that for all $t \in I'$,

$$\begin{gathered} F_1(t, f_1(t), \ldots, f_n(t)) = 0 \\ \vdots \\ F_n(t, f_1(t), \ldots, f_n(t)) = 0 \end{gathered}$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}} (t, f_1(t), \ldots, f_n(t)) \neq 0,$$

where for $i = 1, \ldots, n$

$$\begin{aligned} F_i(t, f_1(t), \ldots, f_n(t)) = P_i[&t, f_1(t), \ldots, f_n(t), \\ &\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_n(t))), \\ &\tilde{\wp}(B(t)), \tilde{\wp}(B(f_1(t))), \ldots, \tilde{\wp}(B(f_n(t)))] = 0. \end{aligned}$$

The domain of the algebraic functions $P_1, \ldots, P_n$ is a small open subset of $\mathbb{C}^{3n+3}$ which, perhaps after shrinking the interval I', contains the set

$$\{[f(t), \wp(B(f(t))), \tilde{\wp}(B(f(t)))] : t \in I'\}$$

and these algebraic functions are analytic on this domain. Now take n to be minimal such that there is some interval I' and algebraic functions $P_1, \ldots, P_n$ in $3n+3$ variables and $F_i(x_1, \ldots, x_{n+1}) = P_i(x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \tilde{\wp}(B(x_1)), \ldots, \tilde{\wp}(B(x_{n+1})))$ and there are also functions $f_3, \ldots, f_n$ whose domains are I' such that $F_i(t, f_1(t), \ldots, f_n(t)) = 0$ and $\det(\partial F_i/\partial x_j)(t, f_1(t), \ldots, f_n(t)) \neq 0$ for all $t \in I'$ and $P_1, \ldots, P_n$ are analytic on their respective domains. Observe that the subinterval I', the functions $f_3, \ldots, f_n$ and the algebraic functions $P_1, \ldots, P_n$ may not be the same as those given above. For $i = 1, \ldots, n$ and $j = 2, \ldots, n+1$

$$\frac{\partial F_i}{\partial x_j}(\bar{x}) = \frac{\partial R_i}{\partial w_j}(\bar{w}) + B'(x_j)\Re(\wp'(B(x_j)))\frac{\partial R_i}{\partial w_{j+n+1}}(\bar{w}) \tag{4.5}$$

$$+ B'(x_j)\Im(\wp'(B(x_j)))\frac{\partial R_i}{\partial w_{j+2n+2}}(\bar{w})$$

$$= \frac{\partial P_i}{\partial y_j}(\bar{y}) + B'(x_j)\wp'(B(x_j))\frac{\partial P_i}{\partial y_{j+n+1}}(\bar{y}) + B'(x_j)\tilde{\wp}'(B(x_j))\frac{\partial P_i}{\partial y_{j+2n+2}}(\bar{y}), \tag{4.6}$$

where

$$\bar{x} = (x_1, \ldots, x_{n+1})$$

and

$$\bar{w} = [x_1, \ldots, x_{n+1}, \Re(\wp(B(x_1))), \ldots, \Re(\wp(B(x_{n+1}))), \Im(\wp(B(x_1))), \ldots, \Im(\wp(B(x_{n+1})))]$$

and

$$\bar{y} = (x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \tilde{\wp}(B(x_1)), \ldots, \tilde{\wp}(B(x_{n+1}))).$$

The functions $B \circ f_1, \ldots, B \circ f_n$ are real analytic functions and can therefore be continued to analytic functions defined on a disc D centred at $\beta \in I'$. The rest of the proof consists of finding upper and lower bounds on transcendence degree. Firstly we obtain a lower bound using Theorem 2.3.8. From the discussion at the beginning of this chapter we know that the condition on the period ratios τ and $\bar{\tau}$ is satisfied. Therefore in order to use Theorem 2.3.8 we must show that $B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly independent over $\mathbb{Q}$ where $f_0(t) = t$.

Claim 4.1.2. *We have that $B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly independent over $\mathbb{Q}$.*

Proof. It suffices to prove this claim for the restrictions of these functions to the interval I'. Suppose that $B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly dependent. Therefore there are rational $a_0, \ldots, a_n$ not all zero such that for all

$t \in I'$

$$a_0(B(t) - B(\beta)) + a_1(B(f_1) - B(f_1(\beta))) + \cdots + a_n(B(f_n(t)) - B(f_n(\beta))) = 0.$$

By the same reasoning as in the proof of the corresponding linear independence claim in the previous chapter it is clear for some $i = 2, \ldots, n$ that $a_i \neq 0$. Upon taking this to be a_n and relabelling we may write,

$$B(f_n(t)) = B(f_n(\beta)) + a_0(B(t) - B(\beta)) + \cdots + a_{n-1}(B(f_{n-1}(t)) - B(f_{n-1}(\beta)))$$

and so

$$f_n(t) = A[B(f_n(\beta)) + a_0(B(t) - B(\beta)) + \cdots + a_{n-1}(B(f_{n-1}(t)) - B(f_{n-1}(\beta)))],$$

where the rationals $a_0, \ldots, a_{n-1}$ are not all zero. Define $\tilde{A}, \tilde{B} : \mathbb{R}^n \to \mathbb{R}$ as

$$\tilde{A}(t_1, \ldots, t_n) = A[B(f_n(\beta)) + a_0(t_1 - B(\beta)) + \cdots + a_{n-1}(t_n - B(f_{n-1}(\beta)))]$$

and

$$\tilde{B}(t_1, \ldots, t_n) = B(f_n(\beta)) + a_0(t_1 - B(\beta)) + \cdots + a_{n-1}(t_n - B(f_{n-1}(\beta))).$$

Then

$$\tilde{A}(B(t), B(f_1(t)), \ldots, B(f_{n-1}(t))) = f_n(t) \tag{4.7}$$

and

$$\tilde{B}(B(t), B(f_1(t)), \ldots, B(f_{n-1}(t))) = B(f_n(t)). \tag{4.8}$$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be algebraic functions such that

$$\mathcal{V}_1(\wp(B(z_1)), \ldots, \wp(B(z_n))) = \wp(\tilde{B}(B(z_1), \ldots, B(z_n))) \tag{4.9}$$

and

$$\mathcal{V}_2(\tilde{\wp}(B(z_1)), \ldots, \tilde{\wp}(B(z_n))) = \tilde{\wp}(\tilde{B}(B(z_1), \ldots, B(z_n))). \tag{4.10}$$

As in the previous chapter the existence of these algebraic functions comes from Remark 2.1.13. Differentiating (4.9) and (4.10) with respect to z_j for $j = 2, \ldots, n$ and evaluating at $(z_1, \ldots, z_n) = (t, f_1(t), \ldots, f_{n-1}(t))$ and using the expressions (4.7) and (4.8) gives that

$$B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\frac{\partial \mathcal{V}_1}{\partial v_j}(\tilde{v}_1(t)) = a_{j-1}B'(f_{j-1}(t))\wp'(B(f_n(t))) \tag{4.11}$$

and

$$B'(f_{j-1}(t))\tilde{\wp}'(B(f_{j-1}(t)))\frac{\partial \mathcal{V}_2}{\partial v_j}(\tilde{v}_2(t)) = a_{j-1}B'(f_{j-1}(t))\tilde{\wp}'(B(f_n(t))) \tag{4.12}$$

where

$$\tilde{v}_1(t) = [\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_{n-1}(t)))]$$

and

$$\tilde{v}_2(t) = [\tilde{\wp}(B(t)), \tilde{\wp}(B(f_1(t))), \ldots, \tilde{\wp}(B(f_{n-1}(t)))].$$

Now for $i = 1, \ldots, n$ define

$$\begin{aligned} Q_i(s_1, \ldots, s_{3n}) = P_i(s_1, \ldots, s_n, \tilde{A}(B(s_1), \ldots, B(s_n)),\\ s_{n+1}, \ldots, s_{2n}, \mathcal{V}_1(s_{n+1}, \ldots, s_{2n}),\\ s_{2n+1}, \ldots, s_{3n}, \mathcal{V}_2(s_{2n+1}, \ldots, s_{3n})) \end{aligned}$$

and let

$$\begin{aligned} G_i(u_1, \ldots, u_n) = Q_i(u_1, \ldots, u_n, \wp(B(u_1)), \ldots, \wp(B(u_n)),\\ \tilde{\wp}(B(u_1)), \ldots, \tilde{\wp}(B(u_n))). \end{aligned}$$

Then for all $t \in I'$ and $i = 1, \ldots, n$,

$$G_i(t, f_1(t), \ldots, f_{n-1}(t)) = 0.$$

So we have a system of n algebraic functions $Q_1, \ldots, Q_n$ in fewer variables. The algebraic functions $Q_1, \ldots, Q_n$ have a domain that is an open subset of $\mathbb{C}^{3n}$, which contains the set

$$\{[t, f_1(t), \ldots, f_{n-1}(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_{n-1}(t))), \\ \tilde{\wp}(B(t)), \tilde{\wp}(B(f_1(t))), \ldots, \tilde{\wp}(B(f_{n-1}(t)))] : t \in I'\}.$$

If one of the $(n-1) \times (n-1)$ minors of the matrix

$$\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n}} (t, f_1(t), \ldots, f_n(t))$$

is non-zero for some $t \in I'$ then we have a contradiction to the minimality of n. Therefore assume that all such minors are zero. Now for $i = 1, \ldots, n$ and $j = 2, \ldots, n$ we have that

$$\frac{\partial G_i}{\partial u_j} = \frac{\partial Q_i}{\partial s_j} + B'(u_j)\wp'(B(u_j))\frac{\partial Q_i}{\partial s_{j+n}} + B'(u_j)\tilde{\wp}'(B(u_j))\frac{\partial Q_i}{\partial s_{j+2n}}$$

and so we have that

$$\frac{\partial Q_i}{\partial s_j} = \frac{\partial P_i}{\partial y_j} + a_{j-1}B'(u_j)A'[B(f_n(\beta)) + a_0(B(u_1) - B(\beta)) + \ldots \\ + a_{n-1}(B(u_n) - B(f_{n-1}(\beta)))]\frac{\partial P_i}{\partial y_{n+1}}$$

and

$$\frac{\partial Q_i}{\partial s_{j+n}} = \frac{\partial P_i}{\partial y_{j+n+1}} + \frac{\partial P_i}{\partial y_{2n+2}}\frac{\partial \mathcal{V}_1}{\partial v_j}$$

as well as

$$\frac{\partial Q_i}{\partial s_{j+2n}} = \frac{\partial P_i}{\partial y_{j+2n+2}} + \frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_j}.$$

Here all the partial derivatives of Q_i are evaluated at

$$(u_1,\ldots,u_n,\wp(B(u_1)),\ldots,\wp(B(u_n)),\tilde{\wp}(B(u_1)),\ldots,\tilde{\wp}(B(u_n)))$$

and the partial derivatives of P_i are evaluated at

$$\begin{aligned}&(u_1,\ldots,u_n,\tilde{A}(B(u_1),\ldots,B(u_n)),\\&\wp(B(u_1)),\ldots,\wp(B(u_n)),\mathcal{V}_1(\wp(B(u_1)),\ldots,\wp(B(u_n))),\\&\tilde{\wp}(B(u_1)),\ldots,\tilde{\wp}(B(u_n)),\mathcal{V}_2(\tilde{\wp}(B(u_1)),\ldots,\tilde{\wp}(B(u_n))).\end{aligned}$$

Therefore putting this all together and using (2.13) as well as (4.11) and (4.12) we can see that upon evaluating at $(t,f_1(t),\ldots,f_{n-1}(t))$ we have for all $i=1,\ldots,n$ and $j=2,\ldots,n$ that,

$$\begin{aligned}\frac{\partial G_i}{\partial u_j}&=\frac{\partial P_i}{\partial y_j}+a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial P_i}{\partial y_{n+1}}\\&\quad+B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\left(\frac{\partial P_i}{\partial y_{j+n+1}}+\frac{\partial P_i}{\partial y_{2n+2}}\frac{\partial \mathcal{V}_1}{\partial v_j}\right)\\&\quad+B'(f_{j-1}(t))\tilde{\wp}'(B(f_{j-1}(t)))\left(\frac{\partial P_i}{\partial y_{j+2n+2}}+\frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_j}\right)\\&=\frac{\partial F_i}{\partial x_j}+a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial P_i}{\partial y_{n+1}}+\\&a_{j-1}B'(f_{j-1}(t))\wp'(B(f_n(t)))\frac{\partial P_i}{\partial y_{2n+2}}+a_{j-1}B'(f_{j-1}(t))\tilde{\wp}'(B(f_n(t)))\frac{\partial P_i}{\partial y_{3n+3}}\\&=\frac{\partial F_i}{\partial x_j}+a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial F_i}{\partial x_{n+1}},\end{aligned}$$

where the derivatives of F_i are evaluated at $(t,f_1(t),\ldots,f_n(t))$ for $i=1,\ldots,n$. Therefore as the $(n-1)\times(n-1)$ minors for the matrix of partial derivatives

$$\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,n}}(t,f_1(t),\ldots,f_n(t))$$

are all zero for all $t\in I'$ we have that the determinant

$$\begin{vmatrix}F & \begin{matrix}\partial F_1/\partial x_{n+1}\\ \vdots\\ \partial F_n/\partial x_{n+1}\end{matrix}\end{vmatrix}(t,f_1(t),\ldots,f_n(t))=0,$$

where F is the $n \times (n-1)$ matrix

$$F = \left(\frac{\partial F_i}{\partial x_j} + a_{j-1} B'(x_j) A'(B(x_{n+1})) \frac{\partial F_i}{\partial x_n} \right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}}$$

and we can see that this is the same matrix as the one for the original system up to column operations and these column operations do not affect the determinant. In particular

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (t, f_1(t), \dots, f_n(t)) = 0$$

for all $t \in I'$, a contradiction.

□

Suppose that $i(t-\beta), B(t) - B(\beta), B \circ f_1 - B(f_1(\beta)), \dots, B \circ f_n - B(f_n(\beta))$ are linearly dependent over $\mathbb{Q}$. Then for rational $a_0, \dots, a_n$ not all zero,

$$\begin{aligned} i(t-\beta) = & a_0(B(t) - B(\beta)) + a_1(B(f_1(t)) - B(f_1(\beta))) \\ & + \dots + a_n(B(f_n(t)) - B(f_n(\beta))) \end{aligned}$$

and as the left hand side is non-real and the right hand side is real we obtain a contradiction. Applying Theorem 2.3.8 with the functions $\wp$ and $\tilde{\wp}$ to $it, B(t), B \circ f_1, \dots, B \circ f_n$ gives that,

$$\begin{aligned} \text{tr.deg}_{\mathbb{C}} \mathbb{C}[& it, B(t), B \circ f_1, \dots, B \circ f_n, \\ & \wp(it), \wp(B(t)), \wp(B(f_1)), \dots, \wp(B(f_n)), \\ & \tilde{\wp}(it), \tilde{\wp}(B(t)), \tilde{\wp}(B(f_1)), \dots, \tilde{\wp}(B(f_n))] \geq 2n + 5. \end{aligned}$$

To find an upper bound on transcendence degree we wish to follow the argument from the previous chapter and use Lemma 2.4.12. However in order to use this lemma we require a system of real valued algebraic functions in terms of functions that are also real valued. In the real lattice case in the previous chapter the invariants of $\wp$ are real numbers. So when the system of polynomials is changed to a system of algebraic functions, using in part the differential equation for $\wp$ which contains these invariants, we still have real valued functions in our system. Here as the lattice is not isogenous to a real lattice this may no longer be the case. Also as we have been working with the system of algebraic functions $P_1, \ldots, P_n$, which involves $\wp$ and $\tilde{\wp}$ we no longer have a system of algebraic functions that is evaluated at real valued functions either. Returning to the system of algebraic functions $R_1, \ldots, R_n$ involving the real and imaginary parts of $\wp$ gives a system of complex valued algebraic functions in terms of real valued functions. Now we wish to obtain a non-singular system of real valued algebraic functions. For $i = 1, \ldots, n$ we write $F_i^{\Re} = \Re(F_i)$, $F_i^{\Im} = \Im(F_i)$, $R_i^{\Re} = \Re(R_i)$ and $R_i^{\Im} = \Im(R_i)$. Therefore for all $t \in I'$ we have that,

$$\begin{gathered}
F_1^{\Re}(t, f_1(t), \ldots, f_n(t)) = 0 \\
\vdots \\
F_n^{\Re}(t, f_1(t), \ldots, f_n(t)) = 0 \\
F_1^{\Im}(t, f_1(t), \ldots, f_n(t)) = 0 \\
\vdots \\
F_n^{\Im}(t, f_1(t), \ldots, f_n(t)) = 0.
\end{gathered}$$

Consider the Jacobian matrix,

$$F = \begin{pmatrix}
\partial F_1^{\Re}/\partial x_2 & \ldots & \partial F_1^{\Re}/\partial x_{n+1} \\
\vdots & \ddots & \vdots \\
\partial F_n^{\Re}/\partial x_2 & \ldots & \partial F_n^{\Re}/\partial x_{n+1} \\
\partial F_1^{\Im}/\partial x_2 & \ldots & \partial F_1^{\Im}/\partial x_{n+1} \\
\vdots & \ddots & \vdots \\
\partial F_n^{\Im}/\partial x_2 & \ldots & \partial F_n^{\Im}/\partial x_{n+1}
\end{pmatrix}$$

and let

$$\tilde{x} = \tilde{x}(t) = (t, f_1(t), \ldots, f_n(t))$$

and

$$\begin{aligned}
\tilde{w} = \tilde{w}(t) = [&t, f_1(t), \ldots, f_n(t), \Re(\wp(B(t))), \Re(\wp(B(f_1(t)))), \ldots, \Re(\wp(B(f_n(t)))), \\
&\Im(\wp(B(t))), \Im(\wp(B(f_1(t)))), \ldots, \Im(\wp(B(f_n(t))))].
\end{aligned}$$

For $j = 1, \ldots, n$ we apply the row operation $r_j \to r_j + ir_{j+n}$ to the matrix F.

This produces a matrix whose first $n \times n$ block is the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (\tilde{x})$$

and so when we consider $F(\tilde{x})$ and apply these row operations we obtain a matrix with a non-vanishing $n \times n$ minor. These row operations do not affect the determinant. Therefore the matrix $F(\tilde{x}(t))$ has maximal rank n for all $t \in I'$. We now show that the matrix

$$\left(\frac{\partial R_i^{\Re}}{\partial w_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,3n+3}} (\tilde{w}(t))$$

has maximal rank n for all $t \in I'$. For $i = 1, \dots, n$ and $j = 2, \dots, n+1$

$$\begin{aligned} \frac{\partial F_i^{\Re}}{\partial x_j}(x_1, \dots, x_{n+1}) =& \frac{\partial R_i^{\Re}}{\partial w_j}(\bar{w}) + \Re[B'(z_j)\wp'(B(z_j))]\frac{\partial R_i^{\Re}}{\partial w_{j+n+1}}(\bar{w}) \\ &+ \Im[B'(z_j)\wp'(B(z_j))]\frac{\partial R_i^{\Re}}{\partial w_{j+2n+2}}(\bar{w}), \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \bar{w} = [&z_1, \dots, z_{n+1}, \Re(\wp(B(z_1))), \dots, \Re(\wp(B(z_{n+1}))), \\ &\Im(\wp(B(z_1))), \dots, \Im(\wp(B(z_{n+1})))]. \end{aligned}$$

Therefore the matrix

$$\left(\frac{\partial F_i^{\Re}}{\partial x_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (\tilde{x}(t))$$

is given by multiplying

$$\left(\frac{\partial R_i^{\Re}}{\partial w_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,2n+2}} (\tilde{w}(t))$$

by the matrix M, where M is a $(3n+2) \times n$ matrix given by,

$$M = \begin{pmatrix} & 0 & & 0 & \\ I_n & \vdots & M_1 & \vdots & M_2 \\ & 0 & & 0 & \end{pmatrix}^T,$$

for

$$M_1 = \begin{pmatrix} \Re[B'(f_1(t))\wp'(B(f_1(t)))] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Re[B'(f_n(t))\wp'(B(f_n(t)))] \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} \Im[B'(f_1(t))\wp'(B(f_1(t)))] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Im[B'(f_n(t))\wp'(B(f_n(t)))] \end{pmatrix}.$$

As the rows of

$$\left(\frac{\partial F_i^{\Re}}{\partial x_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,n+1}} (\tilde{x}(t))$$

are linearly independent over $\mathbb{R}$ we have that the rows of

$$\left(\frac{\partial R_i^{\Re}}{\partial w_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,2n+2}} (\tilde{w}(t))$$

are also linearly independent over $\mathbb{R}$ for all $t \in I'$. Therefore the matrix

$$\left(\frac{\partial R_i^{\Re}}{\partial w_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,2n+2}} (\tilde{w}(t))$$

has maximal rank n for all $t \in I'$. So we have a system of real valued algebraic functions $R_1^{\Re}, \dots, R_n^{\Re}$ in terms of real valued functions whose Jacobian matrix has maximal rank n when evaluated at $\tilde{w}(t)$ for all $t \in I'$. Therefore, after restricting the functions $t, f_1, \dots, f_n, \Re(\wp(B(t))), \Re(\wp(B(f_1))), \dots, \Re(\wp(B(f_n))), \Im(\wp(B(t)))$, $\Im(\wp(B(f_1))), \dots, \Im(\wp(B(f_n)))$ to some subinterval $I'' \subseteq I'$ if necessary, we may apply Lemma 2.4.12 and obtain the upper bound,

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[t, f_1, \ldots, f_n, &\Re(\wp(B(t))),\\ &\Re(\wp(B(f_1))), \ldots, \Re(\wp(B(f_n))),\\ &\Im(\wp(B(t))), \Im(\wp(B(f_1))), \ldots, \Im(\wp(B(f_n)))] \leq 2n + 3.\end{aligned}$$

As the real and imaginary parts of $\wp$ may be written as polynomials in $\wp$ and $\tilde{\wp}$ by the identities given at the beginning of this chapter we have that,

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[t, f_1, \ldots, f_n, &\\ &\wp(B(t))), \wp(B(f_1)), \ldots, \wp(B(f_n)),\\ &\tilde{\wp}(B(t)), \tilde{\wp}(B(f_1)), \ldots, \tilde{\wp}(B(f_n))] \leq 2n + 3.\end{aligned}$$

As in the previous chapter this is an upper bound on the transcendence degree of a slightly different finitely generated extension of $\mathbb{C}$ than the one for which we have obtained a lower bound on transcendence degree. As it and $B(t)$ are algebraically dependent and $f_1(t) = \Re(\wp(it)), f_2(t) = \Im(\wp(it))$ and $\wp(it)$ are also algebraically dependent and B is an algebraic function we have that,

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[it, B(t), B \circ f_1, \ldots, B \circ f_n, &\\ &\wp(it), \wp(B(t)), \wp(B(f_1)), \ldots, \wp(B(f_n)),\\ &\tilde{\wp}(it), \tilde{\wp}(B(t)), \tilde{\wp}(B(f_1)), \ldots, \tilde{\wp}(B(f_n))] \leq 2n + 3.\end{aligned}$$

Therefore we have upper and lower bounds on the transcendence degree of some finitely generated extension of $\mathbb{C}$ which are contradictory as required.

The $\Omega \sim \overline{\Omega}$ case

As noted at the beginning of this chapter we assume that all the lattices in this section are of the form $\Omega_\tau = \mathbb{Z} + \tau\mathbb{Z}$. Once again we assume that $\wp|_D$ is definable in the structure $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$ for some disc $D \subseteq \mathbb{C}$. It can be assumed that the disc D contains iI and therefore the functions $f_1, f_2 : I \to \mathbb{R}$ defined by $f_1(t) = \Re(\wp(it))$ and $f_2(t) = \Im(\wp(it))$ are definable in the structure $(\overline{\mathbb{R}}, \Re(\wp \circ B), \Im(\wp \circ B), \Re(\wp' \circ B), \Im(\wp' \circ B), B, B_1)$. As in the $\Omega \not\sim \overline{\Omega}$ case we apply Theorem 2.4.3 to both of the functions f_1 and f_2 and obtain a nonsingular system of equations. This is a system of algebraic functions involving the real and imaginary parts of $\wp \circ B$, which can then be rearranged into algebraic functions involving $\wp \circ B$ and $\tilde{\wp} \circ B$ in place of these real and imaginary parts. Therefore there is some integer $n \geq 1$ and some subinterval $I' \subseteq I$, algebraic functions $P_1, \dots, P_n$, certain functions $f_3, \dots, f_n : I' \to \mathbb{R}$ such that for all $t \in I'$,

$$\begin{aligned} F_1(t, f_1(t), \dots, f_n(t)) &= 0 \\ &\vdots \\ F_n(t, f_1(t), \dots, f_n(t)) &= 0 \end{aligned}$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (t, f_1(t), \dots, f_n(t)) \neq 0,$$

where for $i = 1, \dots, n$

$$\begin{aligned} F_i(t, f_1(t), \dots, f_n(t)) = P_i[&t, f_1(t), \dots, f_n(t), \\ &\wp(B(t)), \wp(B(f_1(t))), \dots, \wp(B(f_n(t))), \\ &\tilde{\wp}(B(t)), \tilde{\wp}(B(f_1(t))), \dots, \tilde{\wp}(B(f_n(t)))] = 0. \end{aligned}$$

As in the previous case the algebraic functions $P_1, \dots, P_n$ are considered on a domain that is a small open subset of $\mathbb{C}^{3n+3}$ which, perhaps after shrinking the interval I', contains the set

$$\{[f(t), \wp(B(f(t))), \tilde{\wp}(B(f(t)))] : t \in I'\}$$

where $f(t) = (t, f_1(t), \dots, f_n(t))$ and these algebraic functions are analytic on this domain. Now we once again take n to be minimal so that there is an interval I' and algebraic functions $P_1, \dots, P_n$ in $3n+3$ variables and $F_i(x_1, \dots, x_{n+1}) = P_i(x_1, \dots, x_{n+1}, \wp(B(x_1)), \dots, \wp(B(x_{n+1})), \tilde{\wp}(B(x_1)), \dots, \tilde{\wp}(B(x_{n+1})))$ and there are also functions $f_3, \dots, f_n$ such that $F_i(t, f_1(t), \dots, f_n(t)) = 0$ and $\det(\partial F_i/\partial x_j)$ $(t, f_1(t), \dots, f_n(t)) \neq 0$ for all $t \in I'$ and $P_1, \dots, P_n$ are analytic on this domain. Once again observe that the subinterval I', the functions $f_3, \dots, f_n$ and the algebraic functions $P_1, \dots, P_n$ may not be the same as those above.

Now we show that the function $\tilde{\wp}(z)$ may be written as a rational function in terms of $\wp(\alpha^{-1}z)$ for some $\alpha \in \mathbb{C}^{\times}$ and so the system of algebraic functions involving $\wp(z)$ and $\tilde{\wp}(z)$ can be rewritten as a system of algebraic functions involving $\wp(z)$ and $\wp(\alpha^{-1}z)$. This enables us to use the theorem of Brownawell and Kubota for a single $\wp$-function in order to obtain the lower bound on transcendence degree. As $\Omega \sim \overline{\Omega}$ there is some $\alpha \in \mathbb{C}^{\times}$ such that $\alpha\Omega \subseteq \overline{\Omega}$. From the definition of the $\wp$-function, one may easily obtain the identity

$$\wp_{\Omega}(z) = \alpha^2 \wp_{\alpha\Omega}(\alpha z). \tag{4.14}$$

Now let $\omega \in \alpha\Omega$. Then,

$$\wp_{\overline{\Omega}}(z+\omega) = \wp_{\overline{\Omega}}(z)$$

as $\alpha\Omega \subseteq \overline{\Omega}$. Hence $\wp_{\overline{\Omega}}$ is an elliptic function with respect to $\alpha\Omega$ and so $\wp_{\overline{\Omega}}$ may be written as a rational function in terms of $\wp_{\alpha\Omega}$ and $\wp'_{\alpha\Omega}$. In fact as the Weierstrass $\wp$-function is an even function we have that $\wp_{\overline{\Omega}}$ is an even elliptic function with respect to $\alpha\Omega$ and so $\wp_{\overline{\Omega}}$ may be written as a rational function in terms of $\wp_{\alpha\Omega}$. This along with the identity (4.14) gives that

$$\wp_{\overline{\Omega}}(z) = \mathcal{U}(\wp_{\alpha\Omega}(z)) = \mathcal{V}(\wp_{\Omega}(\alpha^{-1}z))$$

for rational functions $\mathcal{U}$ and $\mathcal{V}$. So for $i = 1, \ldots, n$

$$\begin{aligned} F_i(x_1, \ldots, x_{n+1}) &= P_i(x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \\ &\qquad \tilde{\wp}(B(x_1)), \ldots, \tilde{\wp}(B(x_{n+1}))) \\ &= P_i[x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \\ &\qquad \mathcal{V}(\wp(\alpha^{-1}B(x_1))), \ldots, \mathcal{V}(\wp(\alpha^{-1}B(x_{n+1})))]. \end{aligned}$$

As the functions $P_1, \ldots, P_n$ are algebraic functions and $\mathcal{V}$ is a rational function this can be rearranged to give for $i = 1, \ldots, n$.

$$\begin{aligned} F_i(x_1, \ldots, x_{n+1}) &= Q_i(x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \\ &\qquad \wp(\alpha^{-1}B(x_1)), \ldots, \wp(\alpha^{-1}B(x_{n+1}))), \end{aligned}$$

where $Q_1, \ldots, Q_n$ are algebraic functions with a domain that is a small open subset of $\mathbb{C}^{3n+3}$ containing the set $\{[f(t), \wp(B(f(t))), \wp(\alpha^{-1}B(f(t)))] : t \in I'\}$ and which are analytic on this domain. Therefore for $i = 1, \ldots, n$ we have that

$$\begin{aligned} F_i(x_1, \ldots, x_{n+1}) = P_i(&x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \\ &\tilde{\wp}(B(x_1)), \ldots, \tilde{\wp}(B(x_{n+1}))) \\ = Q_i(&x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \\ &\wp(\alpha^{-1}B(x_1)), \ldots, \wp(\alpha^{-1}B(x_{n+1}))). \end{aligned}$$

Differentiating with respect to x_j for $j = 2, \ldots, n+1$ gives that,

$$\begin{aligned} \frac{\partial F_i}{\partial x_j}(x_1, \ldots, x_{n+1}) &= \frac{\partial P_i}{\partial y_j}(\bar{y}) + B'(x_j)\wp'(B(x_j))\frac{\partial P_i}{\partial y_{j+n+1}}(\bar{y}) \\ &\quad + B'(x_j)\tilde{\wp}'(B(x_j))\frac{\partial P_i}{\partial y_{j+2n+2}}(\bar{y}) \\ &= \frac{\partial Q_i}{\partial w_j}(\bar{w}) + B'(x_j)\wp'(B(x_j))\frac{\partial Q_i}{\partial w_{j+n+1}}(\bar{w}) \\ &\quad + \alpha^{-1}B'(x_j)\wp'(\alpha^{-1}B(x_j))\frac{\partial Q_i}{\partial w_{j+2n+2}}(\bar{w}) \end{aligned}$$

where

$$\bar{y} = (x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \tilde{\wp}(B(x_1)), \ldots, \tilde{\wp}(B(x_{n+1})))$$

and

$$\bar{w} = (x_1, \ldots, x_{n+1}, \wp(B(x_1)), \ldots, \wp(B(x_{n+1})), \wp(\alpha^{-1}B(x_1)), \ldots, \wp(\alpha^{-1}B(x_{n+1}))).$$

The functions $B \circ f_1, \ldots, B \circ f_n$ are real analytic and can be continued to analytic functions on a disc $D' \subseteq \mathbb{C}$ centred at $\beta \in I'$. Now we prove the corresponding linear independence claim for this case in order to apply Theorem 2.3.7. Let $f_0(t) = t$.

Claim 4.1.3. *$B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly independent over*

$\mathbb{Q}$.

Proof. Assume that $B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta))$ are linearly dependent over $\mathbb{Q}$. By the same arguments as in the corresponding claim in the previous section, we may write

$$B(f_n(t)) = B(f_n(0)) + a_0(B(t) - B(\beta)) + \cdots + a_{n-1}(B(f_{n-1}(t)) - B(\beta))$$

for all $t \in I'$ and for rationals $a_0, \ldots, a_{n-1}$ not all zero. Now we define the functions $\tilde{A} : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{B} : \mathbb{R}^n \to \mathbb{R}$ to be

$$\tilde{A}(t_1, \ldots, t_n) = A[B(f_n(0)) + a_0(t_1 - B(\beta)) + \cdots + a_{n-1}(t_n - B(\beta))]$$

and

$$\tilde{B}(t_1, \ldots, t_n) = B(f_n(0)) + a_0(t_1 - B(\beta)) + \cdots + a_{n-1}(t_n - B(\beta)),$$

so that

$$\tilde{A}(B(t), B(f_1(t)), \ldots, B(f_{n-1}(t))) = f_n(t) \tag{4.15}$$

and

$$\tilde{B}(B(t), B(f_1(t)), \ldots, B(f_{n-1}(t))) = B(f_n(t)). \tag{4.16}$$

As in the corresponding linear independence claim in the previous case we let $\mathcal{V}_1$ and $\mathcal{V}_2$ be algebraic functions in the variables $s_1, \ldots, s_n$ such that,

$$\mathcal{V}_1(\wp(B(z_1)), \ldots, \wp(B(z_n))) = \wp(\tilde{B}(B(z_1), \ldots, B(z_n))) \tag{4.17}$$

$$\mathcal{V}_2(\wp(\alpha^{-1}B(z_1)), \ldots, \wp(\alpha^{-1}B(z_n))) = \wp(\alpha^{-1}\tilde{B}(B(z_1), \ldots, B(z_n))). \tag{4.18}$$

Differentiating (4.17) and (4.18) with respect to z_j and evaluating at $(t, f_1(t), \ldots, f_{n-1}(t))$ and using the expressions (4.15) and (4.16) gives that

$$B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\frac{\partial\mathcal{V}_1}{\partial s_j}(\tilde{v}_1(t)) = a_{j-1}B'(f_{j-1}(t))\wp'(B(f_n(t))) \tag{4.19}$$

and

$$\alpha^{-1}B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\frac{\partial\mathcal{V}_2}{\partial s_j}(\tilde{v}_2(t)) = \alpha^{-1}a_{j-1}B'(f_{j-1}(t))\wp'(\alpha^{-1}B(f_n(t))). \tag{4.20}$$

where

$$\tilde{v}_1(t) = [\wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_{n-1}(t)))]$$

and

$$\tilde{v}_2(t) = [\wp(\alpha^{-1}B(t)), \wp(\alpha^{-1}B(f_1(t))), \ldots, \wp(\alpha^{-1}B(f_{n-1}(t)))].$$

Now for $i = 1, \ldots, n$ we define

$$\begin{aligned} S_i(v_1, \ldots, v_{3n}) = Q_i(&v_1, \ldots, v_n, \tilde{A}(B(v_1), \ldots, B(v_n)), \\ &v_{n+1}, \ldots, v_{2n}, \mathcal{V}_1(v_{n+1}, \ldots, v_{2n}), \\ &v_{2n+1}, \ldots, v_{3n}, \mathcal{V}_2(v_{2n+1}, \ldots, v_{3n})) \end{aligned}$$

and

$$\begin{aligned} G_i(u_1, \ldots, u_n) = S_i(&u_1, \ldots, u_n, \wp(B(u_1)), \ldots, \wp(B(u_n)), \\ &\wp(\alpha^{-1}B(u_1)), \ldots, \wp(\alpha^{-1}B(u_n))) \end{aligned}$$

where the functions $S_1, \ldots, S_n$ are algebraic. Therefore for all $t \in I'$

$$G_i(t, f_1(t), \ldots, f_{n-1}(t)) = 0.$$

The algebraic functions $S_1, \ldots, S_n$ have a domain that is an open subset of $\mathbb{C}^{3n}$ that contains the set

$$\{[t, f_1(t), \ldots, f_{n-1}(t), \wp(B(t)), \wp(B(f_1(t))), \ldots, \wp(B(f_{n-1}(t))),$$
$$\wp(\alpha^{-1}B(t)), \wp(\alpha^{-1}B(f_1(t))), \ldots, \wp(\alpha^{-1}B(f_{n-1}(t)))] : t \in I'\}.$$

This gives a system of the same number of equations in fewer variables. If one of the $(n-1) \times (n-1)$ minors of the matrix

$$\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n}} (t, f_1(t), \ldots, f_n(t))$$

is non-zero for some $t \in I'$ then we have a contradiction to the minimality of n. Therefore we assume that all these minors are zero when evaluated at such a point. For $i = 1, \ldots, n$ and $j = 2, \ldots, n+1$ we have that

$$\frac{\partial G_i}{\partial u_j} = \frac{\partial S_i}{\partial v_j} + B'(u_j)\wp'(B(u_j))\frac{\partial S_i}{\partial v_{j+n}} + \alpha^{-1}B'(u_j)\wp'(\alpha^{-1}B(u_j))\frac{\partial S_i}{\partial v_{j+2n}}$$

and so we have that

$$\frac{\partial S_i}{\partial v_j} = \frac{\partial Q_i}{\partial w_j} + a_{j-1}B'(u_j)A'[B(f_n(\beta)) + a_0(B(u_1) - B(\beta)) + \ldots$$
$$+ a_{n-1}(B(u_n) - B(f_{n-1}(\beta)))]\frac{\partial Q_i}{\partial w_{n+1}}.$$

as well as,

$$\frac{\partial S_i}{\partial v_{j+n}} = \frac{\partial Q_i}{\partial w_{j+n+1}} + \frac{\partial Q_i}{\partial w_{2n+2}}\frac{\partial \mathcal{V}_1}{\partial s_j}.$$

Furthermore differentiating S_i with respect to v_{j+2n} gives that,

$$\frac{\partial S_i}{\partial v_{j+2n}} = \frac{\partial Q_i}{\partial w_{j+2n+2}} + \frac{\partial Q_i}{\partial w_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial s_j}.$$

Here the derivatives of $S_1, \ldots, S_n$ are evaluated at

$$(u_1, \ldots, u_n, \wp(B(u_1)), \ldots, \wp(B(u_n)), \wp(\alpha^{-1}B(u_1)), \ldots, \wp(\alpha^{-1}B(u_n)))$$

and the derivatives of $Q_1, \dots, Q_n$ are evaluated at

$$
\begin{aligned}
&[u_1, \dots, u_n, \tilde{A}(B(u_1), \dots, B(u_n)),\\
&\wp(B(u_1)), \dots, \wp(B(u_n)), \mathcal{V}_1(\wp(B(u_1)), \dots, \wp(B(u_n))),\\
&\wp(\alpha^{-1}B(u_1)), \dots, \wp(\alpha^{-1}B(u_n)), \mathcal{V}_2(\wp(\alpha^{-1}B(u_1)), \dots, \wp(\alpha^{-1}B(u_n)))].
\end{aligned}
$$

Putting this all together and using (2.13) as well as (4.19) and (4.20) we can see that upon evaluating at $(t, f_1(t), \dots, f_{n-1}(t))$ we have for all $i = 1, \dots, n$ and $j = 2, \dots, n$ that,

$$
\begin{aligned}
\frac{\partial J_i}{\partial u_j} &= \frac{\partial S_i}{\partial v_j} + B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\frac{\partial S_i}{\partial v_{j+n}}\\
&\quad + \alpha^{-1}B'(f_{j-1}(t))\wp'(\alpha^{-1}B(f_{j-1}(t)))\frac{\partial S_i}{\partial v_{j+2n}}\\
&= \frac{\partial Q_i}{\partial w_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial Q_i}{\partial w_{n+1}}\\
&\quad + B'(f_{j-1}(t))\wp'(B(f_{j-1}(t)))\left(\frac{\partial Q_i}{\partial w_{j+n+1}} + \frac{\partial Q_i}{\partial w_{2n+2}}\frac{\partial \mathcal{V}_1}{\partial s_j}\right)\\
&\quad + \alpha^{-1}B'(f_{j-1}(t))\wp'(\alpha^{-1}B(f_{j-1}(t)))\left(\frac{\partial Q_i}{\partial w_{j+2n+2}} + \frac{\partial Q_i}{\partial w_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial s_j}\right)\\
&= \frac{\partial F_i}{\partial x_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial Q_i}{\partial w_{n+1}}\\
&\quad + a_{j-1}B'(f_{j-1}(t))\wp'(B(f_n(t)))\frac{\partial Q_i}{\partial w_{2n+2}}\\
&\quad + a_{j-1}B'(f_{j-1}(t))\alpha^{-1}\wp'(\alpha^{-1}B(f_n(t)))\frac{\partial Q_i}{\partial w_{3n+3}}\\
&= \frac{\partial F_i}{\partial x_j} + a_{j-1}B'(f_{j-1}(t))A'(B(f_n(t)))\frac{\partial F_i}{\partial x_{n+1}}
\end{aligned}
$$

where the partial derivatives of $F_1, \dots, F_n$ are evaluated at $(t, f_1(t), \dots, f_n(t))$. Therefore as all the minors of the matrix

$$
\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\dots,n\\ j=2,\dots,n}} (t, f_1(t), \dots, f_n(t))
$$

are zero for all $t \in I'$ we have that the determinant

$$\begin{vmatrix} F & \begin{matrix} \partial F_1/\partial x_{n+1} \\ \vdots \\ \partial F_n/\partial x_{n+1} \end{matrix} \end{vmatrix} (t, f_1(t), \ldots, f_n(t)) = 0$$

where F is the matrix

$$F = \left(\frac{\partial F_i}{\partial x_j} + a_{j-1} B'(x_j) A'(B(x_{n+1})) \frac{\partial F_i}{\partial x_{n+1}} \right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}}$$

and as in the $\Omega \nsim \overline{\Omega}$ case this is the same as the Jacobian matrix for the nonsingular system $F_1, \ldots, F_n$ evaluated at $(t, f_1(t), \ldots, f_n(t))$ up to column operations and these column operations do not affect the determinant. In particular we have that

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}} (t, f_1(t), \ldots, f_n(t)) = 0,$$

for all $t \in I'$, which is a contradiction.

□

Suppose that

$$B \circ f_0 - B(\beta), \ldots, B \circ f_n - B(f_n(\beta)),$$
$$\alpha^{-1}(B \circ f_0 - B(\beta)), \ldots, \alpha^{-1}(B \circ f_n - B(f_n(\beta)))$$

are linearly dependent over $\mathbb{Q}$. Then there are rational $a_0, \ldots, a_n, b_0, \ldots, b_n$ not all zero such that for all $t \in I'$

$$a_0(B(t) - B(\beta)) + a_1(B(f_1(t)) - B(f_1(\beta))) + \cdots + a_n(B(f_n(t)) - B(f_n(\beta))) +$$
$$\alpha^{-1} b_0(B(t) - B(\beta)) + \alpha^{-1} b_1(B(f_1(t)) - B(f_1(\beta))) + \ldots$$
$$+ \alpha^{-1} b_n(B(f_n(t)) - B(f_n(\beta))) = 0$$

and rearranging and rewriting the rationals $b_0, \ldots, b_n$ gives that,

$$
\begin{aligned}
&a_0(B(t)-B(\beta))+a_1(B(f_1(t))-B(f_1(\beta)))+\cdots+a_n(B(f_n(t))-B(f_n(\beta)))\\
&=\alpha^{-1}b_0(B(t)-B(\beta))+\alpha^{-1}b_1(B(f_1(t))-B(f_1(\beta)))+\dots\\
&+\alpha^{-1}b_n(B(f_n(t))-B(f_n(\beta))).
\end{aligned}
$$

The left hand side of this final equality is real and the right hand side is non-real as α is non-real by Lemma 4.0.5. Therefore we have a contradiction and the linear independence is preserved. As the lattice Ω does not have complex multiplication $\alpha \neq qi$ for some rational q. Therefore adding $i(B(t)-B(\beta))$ does not affect linear independence. Applying Theorem 2.3.7 to the functions $iB(t), B(t), B\circ f_1, \dots B\circ f_n, \alpha^{-1}B(t), \alpha^{-1}B\circ f_1 \dots, \alpha^{-1}B\circ f_n$ gives that

$$
\begin{aligned}
\text{tr.deg}_{\mathbb{C}}\mathbb{C}[&it, B(t), B\circ f_1, \dots, B\circ f_n,\\
&\alpha^{-1}B(t), \alpha^{-1}B\circ f_1, \dots, \alpha^{-1}B\circ f_n,\\
&\wp(it), \wp(B(t)), \wp(B(f_1)), \dots, \wp(B(f_n)),\\
&\wp(\alpha^{-1}B(t)), \wp(\alpha^{-1}B(f_1)), \dots, \wp(\alpha^{-1}B(f_n))] \geq 2n+4.
\end{aligned}
$$

Now we shall obtain an upper bound on this transcendence degree that is contradictory. As in the previous case we return to the original system of algebraic functions involving the real and imaginary parts of $\wp$ and by a repetition of the argument in this case we obtain the upper bound,

$$
\begin{aligned}
\text{tr.deg}_{\mathbb{C}}\mathbb{C}[&t, f_1, \dots, f_n,\\
&\Re(\wp(B(t))), \Re(\wp(B(f_1))), \dots, \Re(\wp(B(f_n))),\\
&\Im(\wp(B(t))), \Im(\wp(B(f_1))), \dots, \Im(\wp(B(f_n)))] \leq 2n+3.
\end{aligned}
$$

As $\wp$ and $\tilde{\wp}$ can be written as polynomials in $\Re(\wp)$ and $\Im(\wp)$ and $\tilde{\wp}(z)$ can be written as an algebraic function in $\wp(\alpha^{-1}z)$ we have the upper bound,

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[t, f_1, \ldots, f_n,\\ &\wp(B(t)), \wp(B(f_1)), \ldots, \wp(B(f_n)),\\ &\wp(\alpha^{-1}B(t)), \wp(\alpha^{-1}B(f_1)), \ldots, \wp(\alpha^{-1}B(f_n))] \leq 2n+3.\end{aligned}$$

However this an upper bound on the transcendence degree of a different extension of $\mathbb{C}$. Clearly $it, B(t), \alpha^{-1}B(t)$ are algebraically dependent over $\mathbb{C}$ as are $B(\wp(it))$ and $\wp(it)$ as well as $\wp(\alpha^{-1}B(t))$ and $\wp'(\alpha^{-1}B(t))$. For $j = 1, \ldots, n$ we have that f_j and $B(f_j)$ as well as f_j and $\alpha^{-1}B(f_j)$ are also algebraically dependent and so we have that

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[it, B(t), B\circ f_1, \ldots, B\circ f_n,\\ &\alpha^{-1}B(t), \alpha^{-1}B\circ f_1, \ldots, \alpha^{-1}B\circ f_n,\\ &\wp(it), \wp(B(t)), \wp(B(f_1)), \ldots, \wp(B(f_n)),\\ &\wp(\alpha^{-1}B(t)), \wp(\alpha^{-1}B(f_1)), \ldots, \wp(\alpha^{-1}B(f_n))] \leq 2n+3.\end{aligned}$$

This is a contradictory upper bound on this transcendence degree as required.

Chapter 5

Expanding $\overline{\mathbb{R}}$ by a restriction of $\wp$ admits no new complex functions

The results in the previous two chapters are partially motivated by an earlier non-definability result of Bianconi involving the expansion of the ordered real field $\overline{\mathbb{R}}$ by the real exponential function. In the introduction we saw that in fact in the exponential case one can go further with Theorem 1.0.2, which is due to Bianconi and is Theorem 4 in [6].

Returning to the $\wp$-function we recall that each complex lattice Ω is associated to a Weierstrass $\wp$-function denoted $\wp_\Omega$ and so varying this lattice produces a different $\wp$-function. A natural question is whether we can define restrictions of $\wp_{\Omega'}$ for some other complex lattice Ω' in the structure $(\overline{\mathbb{R}}, \wp_\Omega|_I)$ for some bounded real open interval I that does not contain any points in the lattice Ω and whose endpoints are also not in Ω. Let Ω be a real lattice which has complex multiplication and consider a complex lattice Ω' such that $\Omega \subseteq \Omega'$. Then for all $\omega \in \Omega$ it is clear that $\wp_{\Omega'}(z + \omega) = \wp_{\Omega'}(z)$ as $\omega \in \Omega'$. In particular, $\wp_{\Omega'}$ is an elliptic function with respect to the lattice Ω. Therefore $\wp_{\Omega'}(z) = R(\wp_\Omega(z), \wp'_\Omega(z))$ for a rational function R. As $\wp_\Omega|_D$ is already definable in the structure $(\overline{\mathbb{R}}, \wp_\Omega|_I)$ by Lemma 3.1.1 this proves the following lemma.

Lemma 5.0.1. *Let Ω be a real lattice with complex multiplication and let Ω' be a real lattice such that $\Omega' \supseteq \Omega$ and also let I' be a bounded real open interval that does not contain points from either lattice and whose endpoints are not in either lattice. Then there is a disc $D \subseteq \mathbb{C}$ such that $\wp_{\Omega'}|_D$ is definable in $(\overline{\mathbb{R}}, \wp_\Omega|_I)$.*

In this chapter we show that this does not hold when Ω does not have complex multiplication. This also shows that we get no new complex functions that are definable in the structure $(\overline{\mathbb{R}}, \wp_\Omega|_I)$ in this case. This theorem can be thought of as a $\wp$-function analogue of Theorem 1.0.2.

Theorem 5.0.2. *Let $D \subseteq \mathbb{R}^{2N}$ be a definable open polydisc and $u, v : D \to \mathbb{R}$ be two functions that are both definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$, where Ω is a real lattice without complex multiplication and I is some bounded real interval that does not contain a pole. Let $f(x, y) = u(x, y) + iv(x, y)$ be holomorphic in D. Then u and v are definable in $\overline{\mathbb{R}}$.*

The proof of this theorem is given in Section 5.1 and heavily adapts that of Theorem 1.0.2, which involves an argument similar to those seen in the previous two chapters. The main differences between the proof given in Section 5.1 and the proof of Bianconi involve an implicit definition which arises from a model completeness result due to Gabrielov, which is Theorem 2.4.7. The implicit definition can be seen in Proposition 2.4.10. As noted in Section 2.4.2 this is as far as I am aware the first application of this result in order to obtain an implicit definition of this kind. The other main difference comes at the end of the proof, which applies the penultimate lemma in a different way to that of Bianconi. The main reason for this difference is that it is not clear how some of Bianconi's assumptions are justified.

5.1 Proof of Theorem 5.0.2

We can assume that $N = 1$. To see this consider the $N > 1$ case. As the function $f : D \to \mathbb{C}$ is holomorphic, it is holomorphic in each variable. We fix all the variables except one and then apply the $N = 1$ case for each variable. Therefore each coordinate function of f is a semialgebraic function that is also holomorphic. Therefore the function f is an algebraic function in each variable with all other variables fixed and so by Theorem 2 in [33], the function $f : D \to \mathbb{C}$ is also algebraic and therefore definable in $\overline{\mathbb{R}}$.

By applying the addition formula for $\wp$ we may shift and shrink the interval I if necessary and assume that $I \subseteq [0, 1]$. Assume for a contradiction that v is not definable in $\overline{\mathbb{R}}$. Firstly we give a claim on the definability of u. The proof of this claim is the same as the proof of Claim 1 in the proof of Theorem 4 in [6] and so the proof given here is a simply a rewrite of the proof in [6].

Claim 5.1.1. *The function* $u(x, y)$ *is not definable in* $\overline{\mathbb{R}}$. *In fact the functions* $x, y, u(x, y), v(x, y)$ *are algebraically independent over* $\mathbb{R}$.

Proof of Claim 5.1.1. Let $z = x + iy$. From the usual identities for the real and imaginary parts of a complex function we have that

$$u(x, y) = \frac{f(x + iy) + \overline{f(x + iy)}}{2}.$$

Now let $x = z/2$ and $y = z/2i$ and so we have that

$$f(z) = 2u(z/2, z/2i) - \overline{f(0)}. \tag{5.1}$$

Hence if u is definable in $\overline{\mathbb{R}}$ and therefore semialgebraic we have that f is definable in $\overline{\mathbb{R}}$, a contradiction as v is not definable in $\overline{\mathbb{R}}$. By a similar argument we have that

$$f(z) = 2iv(z/2, z/2i) + \overline{f(0)}. \tag{5.2}$$

Therefore if x, y, u and v are algebraically dependent there is a polynomial $P \in \mathbb{R}[X_1, X_2, X_3, X_4]$ such that $P(x, y, u(x, y), v(x, y)) = 0$. Letting $x = z/2, y = z/2i$ and using the identities (5.1) and (5.2) gives that there is a complex polynomial Q such that $Q(z, f(z)) = 0$. Therefore f is algebraic and so u and v are semialgebraic and definable in $\overline{\mathbb{R}}$, a contradiction.

□

Consider the graph $X = \Gamma(u, v) \subseteq \mathbb{R}^4$. Clearly $\dim X = 2$. We now wish to apply Proposition 2.4.10 to the function (u, v) in order to obtain a non-singular system of equations.

Firstly we may translate the disc D and replace D with a smaller disc in order to assume that $D \subseteq I^2 \subseteq [0, 1]^2$. If f is algebraic on this smaller disc then it will be algebraic on the original disc and therefore it suffices to prove the theorem on this smaller disc. The images of u and v restricted to this disc will be bounded and therefore by translating and scaling if necessary we may suppose that these images are also contained in I. Furthermore we may assume that the interval I is an open interval whose endpoints are not in Ω. As discussed in earlier chapters we have that the set $\{\wp|_I, \wp'|_I\}$ is closed under differentiation and so by Theorem 2.4.7 the structure $(\overline{\mathbb{R}}, \wp|_I, \wp'|_I)$ is model complete and we may apply Proposition 2.4.10.

Let $f_2(x,y) = u(x,y)$ and $f_3(x,y) = v(x,y)$. By Proposition 2.4.10, for some integer $n \geq 1$ and an open box $B \subseteq D$ there are polynomials $P_2, \ldots, P_n : \mathbb{R}^{3n+3} \to \mathbb{R}$ in $\mathbb{C}[y_0, \ldots, y_{3n+2}]$ and non-zero rationals $a_0, \ldots, a_n$, certain functions $f_4, \ldots, f_n : I' \to I$ such that for all $(x,y) \in B$,

$$F_2(x, y, f_2(x,y), \ldots, f_n(x,y)) = 0$$
$$\vdots$$
$$F_n(x, y, f_2(x,y), \ldots, f_n(x,y)) = 0$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} (x, y, f_2(x,y), \ldots, f_n(x,y)) \neq 0,$$

where for $i = 2, \ldots, n$ we have that

$$F_i(x_0, \ldots, x_n) = P_i(x_0, \ldots, x_n, \wp(a_0x_0), \ldots, \wp(a_nx_n), \wp'(a_0x_0), \ldots, \wp'(a_nx_n)).$$

Therefore for all $i = 2, \ldots, n$ and $j = 0, \ldots, n$

$$\frac{\partial F_i}{\partial x_j}(x_0, \ldots, x_n) = \frac{\partial P_i}{\partial y_j}(\bar{y}) + a_j\wp'(a_jx_j)\frac{\partial P_i}{\partial y_{j+n+1}}(\bar{y}) + a_j\wp''(a_jx_j)\frac{\partial P_i}{\partial y_{j+2n+2}}(\bar{y}). \tag{5.3}$$

where

$$\bar{y} = (x_0, \ldots, x_n, \wp(a_0x_0), \ldots, \wp(a_nx_n), \wp'(a_0x_0), \ldots, \wp'(a_nx_n)).$$

Now we take n to be minimal such that there exists an open box B, some non-zero rationals $a_0, \ldots, a_n$ and polynomials $P_2, \ldots, P_n$ in $3n+3$ variables and $F_i(x_0, \ldots, x_n) = P_i(x_0, \ldots, x_n, \wp(a_0x_0), \ldots, \wp(a_nx_n), \wp'(a_0x_0), \ldots, \wp'(a_nx_n))$ and there are also some functions $f_4, \ldots f_n$ whose domain is I' such that $F_i(f_0(x,y), \ldots, f_n(x,y)) = 0$ and $\det(\partial F_i/\partial x_j)(f_0(x,y), \ldots, f_n(x,y)) \neq 0$ for all $x, y \in I'$. Note once again that the rationals $a_0, \ldots, a_n$, the functions $f_4, \ldots, f_n$ and the open box B may not be those given here.

The functions $f_0, \ldots, f_n$ are real analytic on a disc $D' \subseteq B$ centred at some $\alpha \in B$. The rest of the proof involves finding contradictory upper and lower bounds on transcendence degree. Before applying Ax's theorem to obtain a lower bound we prove the corresponding linear independence claim. For notational convenience we write $f_i(\alpha) = f_i(\alpha, 0)$ for all $i = 0, \ldots, n$.

Claim 5.1.2. *Over $\mathbb{Q}$ we have that $f_0 - f_0(\alpha), \ldots, f_n - f_n(\alpha)$ are linearly independent.*

Proof. Suppose that $f_0 - f_0(\alpha), \ldots, f_n - f_n(\alpha)$ are linearly dependent over $\mathbb{Q}$. Then we have that for some integers $b_0, \ldots, b_n$ not all zero and for all $(x, y) \in B$

$$b_0(f_0(x, y) - f_0(\alpha)) + \cdots + b_n(f_n(x, y) - f_n(\alpha)) = 0. \tag{5.4}$$

Now suppose that for some integers b_0, b_1, b_2, b_3 that are not all zero we have

$$b_0(f_0(x, y) - f_0(\alpha)) + \cdots + b_3(f_3(x, y) - f_3(\alpha)) = 0.$$

Then we have an algebraic relation between f_0, f_1, f_2 and f_3, contradicting Claim 5.1.1. Therefore for some $i = 4, \ldots, n$ the integer b_i is non-zero. We take this to be b_n. As $b_i = b_i a_i / a_i$ for all $i = 0, \ldots, n$ we can multiply both sides of (5.4) by a common denominator for the rationals $a_0, \ldots, a_n$, which are all non-zero and change the b_i to the product of the original b_i with this common denominator and get that

$$a_0 b_0(f_0(x, y) - f_0(\alpha)) + \cdots + a_n b_n(f_n(x, y) - f_n(\alpha)) = 0$$

for integers $b_0, \ldots, b_n$ where b_n remains non-zero. This can be rearranged to give that

$$a_n f_n(x, y) = a_n f_n(\alpha) + \frac{a_0 b_0}{b_n}(f_0(x, y) - f_0(\alpha)) + \cdots + \frac{a_{n-1} b_{n-1}}{b_n}(f_{n-1}(x, y) - f_{n-1}(\alpha)). \tag{5.5}$$

Observe that there exist rational functions $\mathcal{U}_1$ and $\mathcal{U}_2$ (depending on b_n) such that

$$\mathcal{U}_1(\wp(a_i z / b_n), \wp'(a_i z / b_n)) = \wp(a_i z)$$

and

$$\mathcal{U}_2(\wp(a_i z/b_n), \wp'(a_i z/b_n)) = \wp'(a_i z)$$

for all $i = 0, \ldots, n-1$. So we have that for all $i = 0, \ldots, n-1$

$$\wp(a_i f_i(x, y)) = \mathcal{U}_1(\wp(a_i f_i(x, y)/b_n), \wp'(a_i f_i(x, y)/b_n))$$

and

$$\wp'(a_i f_i(x, y)) = \mathcal{U}_2(\wp(a_i f_i(x, y)/b_n), \wp'(a_i f_i(x, y)/b_n)).$$

Define rational functions $Q_2, \ldots, Q_n$ by

$$\begin{aligned} Q_i(w_0, \ldots, w_{3n+2}) = P_i(w_0, \ldots, w_n, &\mathcal{U}_1(w_{n+1}, w_{2n+2}), \ldots, \mathcal{U}_1(w_{2n}, w_{3n+1}), w_{2n+1}, \\ &\mathcal{U}_2(w_{n+1}, w_{2n+2}), \ldots, \mathcal{U}_2(w_{2n}, w_{3n+1}), w_{3n+2}). \end{aligned}$$

Therefore,

$$\begin{aligned} F_i(x_0, \ldots, x_n) &= P_i(x_0, \ldots, x_n, \wp(a_0 x_0), \ldots, \wp(a_n x_n), \wp'(a_0 x_0), \ldots, \wp'(a_n x_n)) \\ &= Q_i(x_0, \ldots, x_n, \wp(a_0 x_0/b_n), \ldots, \wp(a_{n-1} x_{n-1}/b_n), \wp(a_n x_n), \\ &\qquad \wp'(a_0 x_0/b_n), \ldots, \wp'(a_{n-1} x_{n-1}/b_n), \wp'(a_n x_n)). \end{aligned}$$

The rational functions Q_i may be written as $Q_i = R_i/S_i$ for polynomials R_i, S_i. We show that the system of polynomials R_i has a corresponding non-singularity condition. Differentiating gives that

$$\frac{\partial Q_i}{\partial w_j} = \frac{1}{S_i}\frac{\partial R_i}{\partial w_j} + \frac{R_i}{S_i^2}\frac{\partial S_i}{\partial w_j}.$$

Upon evaluating at the points

$$\begin{aligned}\tilde{w} = \tilde{w}(x,y) = \Big(& f_0(x,y), \ldots, f_{n-1}(x,y), f_n(x,y), \\ & \wp(a_0 f_0(x,y)/b_n), \ldots, \wp(a_{n-1} f_{n-1}(x,y)/b_n), \wp(a_n f_n(x,y)), \\ & \wp'(a_0 f_0(x,y)/b_n), \ldots, \wp'(a_{n-1} f_{n-1}(x,y)/b_n), \wp'(a_n f_n(x,y))\Big)\end{aligned}$$

for $(x,y) \in B$ the second term vanishes. Then at the points $\tilde{w}(x,y)$ we have that,

$$\frac{\partial R_i}{\partial w_j}(\tilde{w}) = S_i(\tilde{w}) \frac{\partial Q_i}{\partial w_j}(\tilde{w})$$

where $S_i(\tilde{w}(x,y))$ is non-zero for all $i = 2, \ldots, n$ and for all $(x,y) \in B$. For $i = 2, \ldots, n$ let

$$\begin{aligned}H_i(z_0, \ldots, z_n) = R_i(& z_0, \ldots, z_n, \wp(a_0 z_0/b_n), \ldots, \wp(a_{n-1} z_{n-1}/b_n), \wp(a_n z_n), \\ & \wp'(a_0 z_0/b_n), \ldots, \wp'(a_{n-1} z_{n-1}/b_n), \wp'(a_n z_n))\end{aligned}$$

where $R_2, \ldots, R_n$ are the polynomials above. Therefore for any fixed $(x,y) \in B$ the matrix

$$\left(\frac{\partial H_i}{\partial z_j}\right)_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} (f_0(x,y), \ldots, f_n(x,y))$$

is the same as the matrix

$$\left(\frac{\partial F_i}{\partial y_j}\right)_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} (f_0(x,y), \ldots, f_n(x,y))$$

under the row operations which multiply the kth row by the non-zero constant $S_{k+1}(\tilde{w}(x,y))$ for all $k = 1, \ldots, n-1$. Therefore if the determinant of

$$\left(\frac{\partial H_i}{\partial z_j}\right)_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} (f_0(x,y), \ldots, f_n(x,y))$$

is zero then the determinant of

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=2,\ldots,n \\ j=2,\ldots,n}} (f_0(x,y), \ldots, f_n(x,y))$$

is also zero, a contradiction. By the minimality condition, n is minimal such that there is an open box $B \subseteq I^2$, polynomials $R_2, \ldots, R_n$ as above and functions $f_4, \ldots, f_n$ and rationals $a_0/b_n, \ldots, a_{n-1}/b_n, a_n$ where $H_i(f_0(x,y), \ldots, f_n(x,y)) = 0$ and $\det(\partial H_i/\partial z_j)(f_0(x,y), \ldots, f_n(x,y)) \neq 0$ for $i = 2, \ldots, n$ and $j = 2, \ldots, n$. Now using this new system $H_2, \ldots, H_n$ we complete this linear independence claim using a similar method to that used for the corresponding claims in earlier chapters. The above discussion means that we may write $a_n f_n(x,y) - a_n f_n(\alpha)$ as a linear combination in $a_i(f_i(x,y) - f_i(\alpha))/b_n$ for $i = 0, \ldots, n-1$ with integer coefficients $b_0, \ldots, b_{n-1}$. We shall see that we may therefore write $\wp(a_n f_n(x,y))$ and $\wp'(a_n f_n(x,y))$ as rational functions in $\wp(a_i f_i(x,y)/b_n), \wp'(a_i f_i(x,y)/b_n)$ for all $i = 0, \ldots, n-1$, which enables us to obtain systems of polynomials in fewer variables as opposed to the systems of algebraic functions that were obtained in the proofs of earlier linear independence results. Here we give the details.

Now we define the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(s_1, \ldots, s_n) = a_n f_n(\alpha) + \frac{a_0 b_0}{b_n}(s_1 - f_0(\alpha)) + \cdots + \frac{a_{n-1} b_{n-1}}{b_n}(s_n - f_{n-1}(\alpha))$$

and so by (5.5) we have that $\tilde{f}(f_0(x,y), \ldots, f_{n-1}(x,y)) = a_n f_n(x,y)$. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be rational functions in the variables $v_1, \ldots, v_{2n}$ such that,

$$\begin{aligned} &\mathcal{V}_1(\wp(a_0 z_1/b_n), \ldots, \wp(a_{n-1} z_n/b_n), \wp'(a_0 z_1/b_n), \ldots, \wp'(a_{n-1} z_n/b_n)) \\ &\quad = \wp(\tilde{f}(z_1, \ldots, z_n)) \qquad (5.6) \\ &\mathcal{V}_2(\wp(a_0 z_1/b_n), \ldots, \wp(a_{n-1} z_n/b_n), \wp'(a_0 z_1/b_n), \ldots, \wp'(a_{n-1} z_n/b_n)) \\ &\quad = \wp'(\tilde{f}(z_1, \ldots, z_n)). \qquad (5.7) \end{aligned}$$

In particular,

$$\mathcal{V}_1(\tilde{v}) = \wp(a_n f_n(x,y))$$

and

$$\mathcal{V}_2(\tilde{v}) = \wp'(a_n f_n(x,y))$$

where

$$\begin{aligned}\tilde{v} = (&\wp(a_0 f_0(x,y)/b_n), \ldots, \wp(a_{n-1} f_{n-1}(x,y)/b_n),\\ &\wp'(a_0 f_0(x,y)/b_n), \ldots, \wp'(a_{n-1} f_{n-1}(x,y)/b_n)).\end{aligned}$$

Differentiating (5.6) and (5.7) with respect to z_j for $j = 1, \ldots, n$ and evaluating at $(z_1, \ldots, z_n) = (x, y, f_2(x,y), \ldots, f_{n-1}(x,y))$ gives that,

$$\begin{aligned}&\frac{a_{j-1}\wp'(a_{j-1} f_{j-1}(x,y))}{b_n} \frac{\partial \mathcal{V}_1}{\partial v_j}(\tilde{v}) + \frac{a_{j-1}\wp''(a_{j-1} f_{j-1}(x,y))}{b_n} \frac{\partial \mathcal{V}_1}{\partial v_{j+n}}(\tilde{v})\\ &= \frac{a_{j-1} b_{j-1}}{b_n} \wp'(a_n f_n(x,y))\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}&\frac{a_{j-1}\wp'(a_{j-1} f_{j-1}(x,y))}{b_n} \frac{\partial \mathcal{V}_2}{\partial v_j}(\tilde{v}) + \frac{a_{j-1}\wp''(a_{j-1} f_{j-1}(x,y))}{b_n} \frac{\partial \mathcal{V}_2}{\partial v_{j+n}}(\tilde{v})\\ &= \frac{a_{j-1} b_{j-1}}{b_n} \wp''(a_n f_n(x,y)).\end{aligned} \tag{5.9}$$

Now for $i = 2, \ldots, n$ define,

$$\begin{aligned}Q_i^*(t_1, \ldots, t_{3n}) = R_i(&t_1, \ldots, t_n, \tilde{f}(t_1, \ldots, t_n)/a_n,\\ &t_{n+1}, \ldots, t_{2n}, \mathcal{V}_1(t_{n+1}, \ldots, t_{3n}),\\ &t_{2n+1}, \ldots, t_{3n}, \mathcal{V}_2(t_{n+1}, \ldots, t_{3n}))\end{aligned}$$

and also define

$$\begin{aligned}G_i^*(u_1, \ldots, u_n) = Q_i^*(&u_1, \ldots, u_n, \wp(a_0 u_1/b_n), \ldots, \wp(a_{n-1} u_n/b_n),\\ &\wp'(a_0 u_1/b_n), \ldots, \wp'(a_{n-1} u_n/b_n)).\end{aligned}$$

Therefore for all $i = 2, \ldots, n$ and $(x, y) \in B$ we have that

$$G_i^*(x, y, f_2(x,y), \ldots, f_{n-1}(x,y)) = 0.$$

Hence we have a system of rational functions $Q_2^*, \ldots, Q_n^*$ in fewer variables than the system of polynomials $R_2, \ldots, R_n$. In other words for $i = 2, \ldots, n$

$$Q_i^* = \frac{R_i^*}{S_i^*} \tag{5.10}$$

for polynomials R_i^* and S_i^*. Let,

$$\begin{aligned} H_i^*(u_1, \ldots, u_n) = R_i^*(u_1, \ldots, u_n, &\wp(a_0 u_1/b_n), \ldots, \wp(a_{n-1} u_n/b_n), \\ &\wp'(a_0 u_1/b_n), \ldots, \wp'(a_{n-1} u_n/b_n)). \end{aligned}$$

Then for all $(x, y) \in B$

$$H_i^*(x, y, f_2(x, y), \ldots, f_{n-1}(x, y)) = 0.$$

The polynomials $R_2^*, \ldots, R_n^*$ are a system of polynomials in fewer variables. Therefore if an $(n-2) \times (n-2)$ minor of the matrix

$$\left(\frac{\partial H_i^*}{\partial u_j}\right)_{\substack{i=2,\ldots,n \\ j=3,\ldots,n}} (x, y, f_2(x, y), \ldots, f_{n-1}(x, y))$$

is non-zero for some $(x, y) \in B$ we have a contradiction to the minimality of n. Therefore we assume that all such minors are zero. Differentiating (5.10) with respect to t_j for $j = 1, \ldots, n$ gives that

$$\frac{\partial Q_i^*}{\partial t_j} = \frac{1}{S_i^*}\frac{\partial R_i^*}{\partial t_j} + \frac{R_i^*}{(S_i^*)^2}\frac{\partial S_i^*}{\partial t_j}.$$

Upon evaluating at

$$\begin{aligned} \tilde{t} = \tilde{t}(x, y) = (&f_0(x, y), \ldots, f_{n-1}(x, y), \\ &\wp(a_0 f_0(x, y)/b_n), \ldots, \wp(a_{n-1} f_{n-1}(x, y)/b_n), \\ &\wp'(a_0 f_0(x, y)/b_n), \ldots, \wp'(a_{n-1} f_{n-1}(x, y)/b_n)) \end{aligned}$$

the second term on the right hand side vanishes and the polynomial S_i^* does not. In particular when evaluated at $\tilde{t}$,

$$\frac{\partial R_i^*}{\partial t_j}(\tilde{t}) = S_i^*(\tilde{t})\frac{\partial Q_i^*}{\partial t_j}(\tilde{t}).$$

For $i = 2, \ldots, n$ and $j = 3, \ldots, n$ we have that

$$\frac{\partial G_i^*}{\partial u_j} = \frac{\partial Q_i^*}{\partial t_j} + \frac{a_{j-1}\wp'(a_{j-1}u_j/b_n)}{b_n}\frac{\partial Q_i^*}{\partial t_{j+n}} + \frac{a_{j-1}\wp''(a_{j-1}u_j/b_n)}{b_n}\frac{\partial Q_i^*}{\partial t_{j+2n}}.$$

Now differentiating Q_i^* with respect to t_j, t_{j+n}, t_{j+2n}, for $j = 3, \ldots, n$, in turn gives that

$$\frac{\partial Q_i^*}{\partial t_j} = \frac{\partial R_i}{\partial w_{j-1}} + \frac{a_{j-1}b_{j-1}}{a_n b_n}\frac{\partial R_i}{\partial w_n},$$

$$\frac{\partial Q_i^*}{\partial t_{j+n}} = \frac{\partial R_i}{\partial w_{j+n}} + \frac{\partial R_i}{\partial w_{2n+1}}\frac{\partial \mathcal{V}_1}{\partial v_j} + \frac{\partial R_i}{\partial w_{3n+2}}\frac{\partial \mathcal{V}_2}{\partial v_j}$$

and

$$\frac{\partial Q_i^*}{\partial t_{j+2n}} = \frac{\partial R_i}{\partial w_{j+2n+1}} + \frac{\partial R_i}{\partial w_{2n+1}}\frac{\partial \mathcal{V}_1}{\partial v_{j+n}} + \frac{\partial R_i}{\partial w_{3n+2}}\frac{\partial \mathcal{V}_2}{\partial v_{j+n}}.$$

Here all the partial derivatives of Q_i^* are evaluated at

$$\begin{aligned}&(f_0(x,y), \ldots, f_n(x,y), \wp(a_0 f_0(x,y)/b_n), \ldots, \wp(a_{n-1}f_{n-1}(x,y)/b_n),\\ &\wp'(a_0 f_0(x,y)/b_n), \ldots, \wp'(a_{n-1}f_{n-1}(x,y)/b_n))\end{aligned}$$

and the derivatives of R_i are evaluated at

$$\begin{aligned}&(f_0(x,y), \ldots, f_n(x,y), \tilde{f}(f_0(x,y), \ldots, f_n(x,y))/a_n,\\ &\wp(a_0 f_0(x,y)/b_n), \ldots, \wp(a_{n-1}f_{n-1}(x,y)/b_n),\\ &\mathcal{V}_1(\wp(a_0 f_0(x,y)/b_n), \ldots, \wp(a_{n-1}f_{n-1}(x,y)/b_n)),\\ &\wp'(a_0 f_0(x,y)/b_n), \ldots, \wp'(a_{n-1}f_{n-1}(x,y)/b_n),\\ &\mathcal{V}_2(\wp'(a_0 f_0(x,y)/b_n), \ldots, \wp'(a_{n-1}f_{n-1}(x,y)/b_n))).\end{aligned}$$

Using the expressions (5.8) and (5.9) and evaluating at $(x, y, f_2(x,y), \ldots, f_{n-1}(x,y))$ gives that for all $i = 2, \ldots, n$ and $j = 3, \ldots, n$,

$$
\begin{aligned}
\frac{\partial G_i^*}{\partial u_j} &= \frac{\partial R_i}{\partial w_{j-1}} + \frac{a_{j-1}b_{j-1}}{a_n b_n}\frac{\partial R_i}{\partial w_{n+1}} + \frac{a_{j-1}\wp'(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\Big(\frac{\partial R_i}{\partial w_{j+n}} \\
&\quad + \frac{\partial R_i}{\partial w_{2n+1}}\frac{\partial \mathcal{V}_1}{\partial v_j} + \frac{\partial R_i}{\partial w_{3n+2}}\frac{\partial \mathcal{V}_2}{\partial v_j}\Big) + \frac{a_{j-1}\wp''(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\Big(\frac{\partial R_i}{\partial w_{j+2n+1}} \\
&\quad + \frac{\partial R_i}{\partial w_{2n+1}}\frac{\partial \mathcal{V}_1}{\partial v_{j+n}} + \frac{\partial R_i}{\partial w_{3n+2}}\frac{\partial \mathcal{V}_2}{\partial v_{j+n}}\Big) \\
&= \frac{\partial R_i}{\partial w_{j-1}} + \frac{a_{j-1}\wp'(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\frac{\partial R_i}{\partial w_{j+n}} \\
&\quad + \frac{a_{j-1}\wp''(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\frac{\partial R_i}{\partial w_{j+2n+1}} \\
&\quad + \frac{a_{j-1}b_{j-1}}{a_n b_n}\frac{\partial R_i}{\partial w_n} + \Big(\frac{a_{j-1}\wp'(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\frac{\partial \mathcal{V}_1}{\partial v_j} \\
&\quad + \frac{a_{j-1}\wp''(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\frac{\partial \mathcal{V}_1}{\partial v_{j+n}}\Big)\frac{\partial R_i}{\partial w_{2n+1}} \\
&\quad + \frac{b_{j-1}}{b_n}\frac{\partial R_i}{\partial w_n} + \Big(\frac{a_{j-1}\wp'(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\frac{\partial \mathcal{V}_2}{\partial v_j} \\
&\quad + \frac{a_{j-1}\wp''(a_{j-1}f_{j-1}(x,y)/b_n)}{b_n}\frac{\partial \mathcal{V}_2}{\partial v_{j+n}}\Big)\frac{\partial R_i}{\partial w_{3n+2}} \\
&= \frac{\partial H_i}{\partial z_{j-1}} + \frac{a_{j-1}b_{j-1}}{a_n b_n}\frac{\partial R_i}{\partial w_n} + \frac{a_{j-1}b_{j-1}\wp'(a_n f_n(x,y))}{b_n}\frac{\partial R_i}{\partial w_{2n+1}} \\
&\quad + \frac{a_{j-1}b_{j-1}\wp''(a_n f_n(x,y))}{b_n}\frac{\partial R_i}{\partial w_{3n+2}} \\
&= \frac{\partial H_i}{\partial z_{j-1}} + \frac{a_{j-1}b_{j-1}}{a_n b_n}\frac{\partial H_i}{\partial z_n}
\end{aligned}
$$

where the derivatives of H_i are evaluated at $\tilde{x} = \tilde{x}(x,y) = (x, y, f_2(x,y), \ldots, f_n(x,y))$. Therefore for all $(x,y) \in B$ we have that

$$
\frac{\partial H_i^*}{\partial u_j}(x, y, f_2(x,y), \ldots, f_{n-1}(x,y)) = S_i^*(\tilde{t}(x,y))\left(\frac{\partial H_i}{\partial z_{j-1}}(\tilde{x}) + \frac{a_{j-1}b_{j-1}}{a_n b_n}\frac{\partial H_i}{\partial z_n}(\tilde{x})\right).
$$

As all the $(n-2)\times(n-2)$ minors of the matrix

$$
\left(\frac{\partial H_i^*}{\partial u_j}\right)_{\substack{i=2,\ldots,n\\ j=3,\ldots,n}} (x, y, f_2(x,y), \ldots, f_{n-1}(x,y))
$$

are zero for all $(x,y) \in B$ we have that the determinant

$$\begin{vmatrix} S_2^*(\tilde{t})\left(\frac{\partial H_2}{\partial z_2}+\frac{a_2b_2}{a_nb_n}\frac{\partial H_2}{\partial z_n}\right) & \dots & S_2^*(\tilde{t})\left(\frac{\partial H_2}{\partial z_{n-1}}+\frac{a_{n-1}b_{n-1}}{a_nb_n}\frac{\partial H_2}{\partial z_n}\right) & \frac{\partial H_2}{\partial z_n} \\ \vdots & \ddots & \vdots & \\ S_n^*(\tilde{t})\left(\frac{\partial H_n}{\partial z_2}+\frac{a_2b_2}{a_nb_n}\frac{\partial H_n}{\partial z_n}\right) & \dots & S_n^*(\tilde{t})\left(\frac{\partial H_n}{\partial z_{n-1}}+\frac{a_{n-1}b_{n-1}}{a_nb_n}\frac{\partial H_n}{\partial z_n}\right) & \frac{\partial H_n}{\partial z_n} \end{vmatrix}(\tilde{x}(x,y))=0.$$

This determinant is the same as the determinant of the Jacobian matrix of the non-singular system $H_2,\dots,H_n$ up to a non-zero constant. In particular we have that

$$\begin{vmatrix} \partial H_2/\partial z_2 & \dots & \partial H_2/\partial z_n \\ \vdots & \ddots & \vdots \\ \partial H_n/\partial z_2 & \dots & \partial H_n/\partial z_n \end{vmatrix}(x,y,f_2(x,y),\dots,f_{n-1}(x,y))=0$$

for all $(x,y)\in B$, a contradiction.

□

Therefore by applying Theorem 2.3.7 to the functions $f_0,\dots,f_n$ we have that,

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[f_0,\dots,f_n,\wp(f_0),\dots,\wp(f_n)]\geq n+2.$$

The rest of the proof of Theorem 5.0.2 consists of finding a contradictory upper bound on this transcendence degree. Let

$$\tilde{x}=\tilde{x}(x,y)=(x,y,f_2(x,y),\dots,f_n(x,y))$$

and

$$\begin{aligned}\tilde{y}=\tilde{y}(x,y)=(&x,y,f_2(x,y),\dots,f_n(x,y),\\ &\wp(a_0x),\wp(a_1y),\wp(a_2f_2(x,y)),\dots,\wp(a_nf_n(x,y)),\\ &\wp'(a_0x),\wp'(a_1y),\wp'(a_2f_2(x,y)),\dots,\wp'(a_nf_n(x,y)))\end{aligned}$$

for all $(x,y)\in B$.

Firstly we show that the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=2,\dots,n\\ j=2,\dots,3n+2}}(\tilde{y}(x,y))$$

has the maximal rank of $n-1$ for all $(x,y)\in B$. From (5.3) it is clear that

$$\begin{pmatrix} \partial F_2/\partial x_2 & \dots & \partial F_2/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial F_n/\partial x_2 & \dots & \partial F_n/\partial x_n \end{pmatrix}(\tilde{x}) = \begin{pmatrix} \partial P_2/\partial y_2 & \dots & \partial P_2/\partial y_{3n+2} \\ \vdots & \ddots & \vdots \\ \partial P_n/\partial y_2 & \dots & \partial P_n/\partial y_{3n+2} \end{pmatrix}(\tilde{y}) \cdot M,$$

where M is the $(3n+1)\times(n-1)$ matrix

$$M = \begin{pmatrix} & 0 & 0 & & 0 & 0 & \\ I_{n-1} & \vdots & \vdots & M_1 & \vdots & \vdots & M_2 \\ & 0 & 0 & & 0 & 0 & \end{pmatrix}^T$$

where

$$M_1 = \begin{pmatrix} a_2\wp'(a_2 f_2(x,y)) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n\wp'(a_n f_n(x,y)) \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} a_2\wp''(a_2 f_2(x,y)) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n\wp''(a_n f_n(x,y)) \end{pmatrix}.$$

The rows of

$$\begin{pmatrix} \partial F_2/\partial x_2 & \dots & \partial F_2/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial F_n/\partial x_2 & \dots & \partial F_n/\partial x_n \end{pmatrix}(\tilde{x})$$

are linearly independent over $\mathbb{R}$ and so the rows of

$$\begin{pmatrix} \partial P_2/\partial y_2 & \dots & \partial P_2/\partial y_{3n+2} \\ \vdots & \ddots & \vdots \\ \partial P_n/\partial y_2 & \dots & \partial P_n/\partial y_{3n+2} \end{pmatrix}(\tilde{y})$$

are also linearly independent over $\mathbb{R}$. Therefore the matrix

$$\begin{pmatrix} \partial P_2/\partial y_2 & \dots & \partial P_2/\partial y_{3n+2} \\ \vdots & \ddots & \vdots \\ \partial P_n/\partial y_2 & \dots & \partial P_n/\partial y_{3n+2} \end{pmatrix} (\tilde{y})$$

has maximal rank $n-1$. As in earlier chapters this upper bound is the number of variables minus the number of equations. However this produces an upper bound on the transcendence degree of some finitely generated extension of $\mathbb{C}$ of $2n+4$, which is not contradictory. In order to lower this upper bound we add $n+3$ new equations to the non-singular system. Two of these equations arise from the Cauchy-Riemann equations for the functions u and v and the final $n+1$ equations correspond to the differential equation for $\wp$ in each of our $n+1$ variables. Namely for each $i = 0, \dots, n$ we define

$$P_{i+n+1}(y_{i+n+1}, y_{i+2n+2}) = y_{i+2n+2}^2 - 4y_{i+n+1}^3 + g_2 y_{i+n+1} + g_3,$$

where g_2 are g_3 are real numbers depending on the lattice Ω known as the invariants of $\wp$ as seen in the background chapter.

Clearly for all $(x, y) \in B$ and $i = 0, \dots, n$,

$$P_{i+n+1}(\wp(a_i f_i(x, y)), \wp'(a_i f_i(x, y))) = 0.$$

Lemma 5.1.3. *For all $i = 0, \dots, n$ the expression*

$$\frac{\partial P_{i+n+1}}{\partial y_i}(y_{i+n+1}, y_{i+2n+2}) + a_i \wp'(a_i f_i(x, y)) \frac{\partial P_{i+n+1}}{\partial y_{i+n+1}}(y_{i+n+1}, y_{i+2n+2}) + a_i \wp''(a_i f_i(x, y)) \frac{\partial P_{i+n+1}}{\partial y_{i+2n+2}}(y_{i+n+1}, y_{i+2n+2}) \quad (5.11)$$

equals zero when evaluated at $(y_{i+n+1}, y_{i+2n+2}) = (\wp(a_i f_i(x, y)), \wp'(a_i f_i(x, y)))$ for all $(x, y) \in B$.

Proof. For all $i = 0, \dots, n+1$ we have that,

$$\frac{\partial P_{i+n+1}}{\partial y_{i+n+1}}(y_{i+n+1}, y_{i+2n+2}) = -12 y_{i+n+1}^2 + g_2$$

and

$$\frac{\partial P_{i+n+1}}{\partial y_{i+2n+2}}(y_{i+n+1}, y_{i+2n+2}) = 2 y_{i+2n+2}.$$

Substituting this into (5.11) and evaluating at

$$(y_{i+n+1}, y_{i+2n+2}) = (\wp(a_i f_i(x,y)), \wp'(a_i f_i(x,y))),$$

which we denote $\hat{y}$, gives that

$$\begin{aligned}
&a_i\wp'(a_i f_i(x,y))\frac{\partial P_{i+n+1}}{\partial y_{i+n+1}}(\hat{y}) + a_i\wp''(a_i f_i(x,y))\frac{\partial P_{i+n+1}}{\partial y_{i+2n+2}}(\hat{y})\\
&= a_i\wp'(a_i f_i(x,y))(-12\wp^2(a_i f_i(x,y)) + g_2)\\
&\qquad + a_i\wp''(a_i f_i(x,y))2\wp'(a_i f_i(x,y))\\
&= a_i\wp'(a_i f_i(x,y))(-12\wp^2(a_i f_i(x,y)) + g_2 + 2\wp''(a_i f_i(x,y)))\\
&= 0,
\end{aligned}$$

by the formula for $\wp''$.

□

By the definition of P_{i+n+1} for $i = 0, \dots, n$ it is clear that all the other derivatives of these functions are zero. Now we show that the $(2n) \times (3n+1)$ matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=2,\dots,2n+1\\ j=2,\dots,3n+2}} (\tilde{y}(x,y))$$

has maximal rank $2n$ for all $(x,y) \in B$.

Firstly note that the lower $(n+1) \times (3n+1)$ block of this matrix is simply the matrix

$$P' = \begin{pmatrix} & \frac{\partial P_{n+1}}{\partial y_{n+1}} & \dots & 0 & \frac{\partial P_{n+1}}{\partial y_{2n+3}} & \dots & 0\\ 0_{n-1} & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ & 0 & \dots & \frac{\partial P_{2n+1}}{\partial y_{2n+2}} & 0 & \dots & \frac{\partial P_{2n+1}}{\partial y_{3n+3}} \end{pmatrix}$$

the rows of which are clearly linearly independent over the reals. In particular $P'(\tilde{y}(x,y))$ has maximal rank $n+1$ for all $(x,y) \in B$. The upper $(n-1) \times (3n+1)$ block is the matrix

$$P = \left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=2,\dots,n\\ j=2,\dots,3n+2}}$$

which we have already shown has maximal rank $n-1$ when evaluated at the points $\tilde{y}(x,y)$ for all $(x,y)\in B$.

So the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=2,\ldots,n+1\\ j=2,\ldots,3n+2}}(\tilde{y})$$

consists of two blocks which are $(n-1)\times(3n+1)$ and $(n+1)\times(3n+1)$ matrices. Both of these matrices have maximal rank. Therefore to show the desired maximality of the rank of the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=2,\ldots,n+1\\ j=2,\ldots,3n+2}}(\tilde{y})$$

it suffices to show that the rows of the matrix $P(\tilde{y})$ can not be written in terms of the rows of $P'(\tilde{y})$. It suffices to check this for a single row of the matrix $P'(\tilde{y})$, which we take to be the first. Assume for a contradiction that this row and the rows of the matrix $P(\tilde{y})$ are linearly dependent over $\mathbb{R}$. Therefore there are reals $b_2,\ldots,b_n$ not all zero such that,

$$\begin{pmatrix}\partial P_{n+1}/\partial y_2\\ \vdots\\ \partial P_{n+1}/\partial y_{3n+2}\end{pmatrix}^T(\tilde{y}) = b_2\begin{pmatrix}\partial P_2/\partial y_2\\ \vdots\\ \partial P_2/\partial y_{3n+2}\end{pmatrix}^T(\tilde{y}) + \cdots + b_n\begin{pmatrix}\partial P_n/\partial y_2\\ \vdots\\ \partial P_n/\partial y_{3n+2}\end{pmatrix}^T(\tilde{y}).$$

Hence for $j=2,\ldots,n$

$$\begin{aligned} 0 &= b_2\frac{\partial P_2}{\partial y_j}(\tilde{y}) + \cdots + b_n\frac{\partial P_n}{\partial y_j}(\tilde{y})\\ \frac{\partial P_{n+1}}{\partial y_{j+n+1}}(\tilde{y}) &= b_2\frac{\partial P_2}{\partial y_{j+n+1}}(\tilde{y}) + \cdots + b_n\frac{\partial P_n}{\partial y_{j+n+1}}(\tilde{y})\\ \frac{\partial P_{n+1}}{\partial y_{j+2n+2}}(\tilde{y}) &= b_2\frac{\partial P_2}{\partial y_{j+2n+2}}(\tilde{y}) + \cdots + b_n\frac{\partial P_n}{\partial y_{j+2n+2}}(\tilde{y}). \end{aligned}$$

By multiplying these equations by the appropriate factors and using Lemma 5.1.3 we have that

$$0_{1\times n-1} = b_2 \begin{pmatrix} \partial F_2/\partial x_2 \\ \vdots \\ \partial F_2/\partial x_n \end{pmatrix}^T (\tilde{x}) + \cdots + b_n \begin{pmatrix} \partial F_n/\partial x_2 \\ \vdots \\ \partial F_n/\partial x_n \end{pmatrix}^T (\tilde{x}).$$

As the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{i=2,\dots,n} (x, y, f_2(x,y), \dots, f_n(x,y))$$

has maximal rank n then its rows are linearly independent over $\mathbb{R}$ and in particular $b_2 = \cdots = b_n = 0$, a contradiction. Therefore we have the desired linear independence. Hence we have that for all $(x,y) \in B$ the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=2,\dots,2n+1 \\ j=2,\dots,3n+2}} (\tilde{y}(x,y))$$

has maximal rank $2n$. So by Lemma 2.4.12 we have that,

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[f_0, \dots, f_n, \wp(a_0 f_0), \dots, \wp(a_n f_n), \wp'(a_0 f_0), \dots, \wp'(a_n f_n)] \leq n+3.$$

Now we define two further equations using the Cauchy-Riemann equations for u and v. These are then added to the system and we may then obtain a contradictory upper bound.

By the implicit function theorem the derivatives of $f_i(x_0, x_1)$ for $i = 2, \dots, n$ are given by

$$\begin{pmatrix} \frac{\partial f_2}{\partial x_k} \\ \vdots \\ \frac{\partial f_n}{\partial x_k} \end{pmatrix} = -\Delta^{-1} \begin{pmatrix} \frac{\partial F_2}{\partial x_k} \\ \vdots \\ \frac{\partial F_n}{\partial x_k} \end{pmatrix},$$

where $k = 0, 1$ and $\Delta = (\partial F_i/\partial x_j)$ and the right hand side is evaluated at $(x_0, \dots, x_n) = (f_0, \dots, f_n)$. Multiplying both sides by the determinant of Δ and using the Cauchy-Riemann equations for f_2 and f_3 gives two new equations, F_0 and F_1. These are of the form,

$$\begin{aligned}F_0 = [&\text{First line of } -\det\Delta\cdot\left(\Delta^{-1}(\partial F_i/\partial x_0)\right)\\ &\text{minus the second line of } -\det\Delta\cdot\left(\Delta^{-1}(\partial F_i/\partial x_1)\right)]\end{aligned}$$

and

$$\begin{aligned}F_1 =[&\text{First line of } -\det\Delta\cdot\left(\Delta^{-1}(\partial F_i/\partial x_1)\right)\\ &\text{plus the second line of } -\det\Delta\cdot\left(\Delta^{-1}(\partial F_i/\partial x_0)\right)],\end{aligned}$$

which each have corresponding polynomials P_0 and P_1. Once these equations are added to the system $P_2, \dots, P_{2n+1}$ the upper bound on transcendence degree can be further reduced by two giving a contradictory upper bound of $n+1$ as we now show, which thus completes the proof of Theorem 5.0.2. In order to lower this upper bound we first require a lemma.

Lemma 5.1.4. *For each $k = 0, 1$ there is a point $z \in \mathbb{C}^{3n+3}$ such that $P_k(z) \neq 0$ and $P_{1-k}(z) = 0$ and $P_i(z) = 0$ for all $i = 2, \dots, 2n+1$.*

Proof. This adapts the proofs of Claims 5 and 6 in the proof of Theorem 4 in [6]. Let V be the subset of $\mathbb{R}^{3n+3}$ defined by

$$V = \{(x, y, z) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : y = \wp(ax), z = \wp'(ax)\}$$

where $\wp(ax) = (\wp(a_0x_0), \dots, \wp(a_nx_n))$ and $\wp'(ax) = (\wp'(a_0x_0), \dots, \wp'(a_nx_n))$. Also let W be the subset of $\mathbb{R}^{3n+3}$ defined by

$$\begin{aligned}W = \{z \in \mathbb{R}^{3n+3} : P_2(z) = 0, \dots, P_{2n+1}(z) = 0 \text{ and } (\partial P_i/\partial y_j)(z) \neq 0\\ \text{for } i = 2, \dots, 2n+1, j = 2, \dots, 3n+2 \text{ has maximal rank }\}.\end{aligned}$$

Let X be the subset of $\mathbb{R}^{3n+3}$ defined by $\{\tilde{y}(x, y) | (x, y) \in B\}$. Then it is clear that $X \subseteq V \cap W$.

The subset V may also be written as

$$V = \{(x, y, z) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \hat{F}_0(x, y, z) = \cdots = \hat{F}_{2n+1}(x, y, z) = 0\},$$

where for $i = 0, \ldots, n$

$$\begin{aligned}\hat{F}_i(x,y,z) &= y_i - \wp(a_i x_i)\\ \hat{F}_{i+n+1}(x,y,z) &= z_i - \wp'(a_i x_i).\end{aligned}$$

We denote the Jacobian matrix for this system by Φ and this is a $(2n+2) \times (3n+3)$ matrix given by

$$\Phi = \begin{pmatrix} -a_0\wp'(a_0x_0) & \ldots & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & -a_n\wp'(a_nx_n) & 0 & \ldots & 1 & 0 & \ldots & 0 \\ -a_0\wp''(a_0x_0) & \ldots & 0 & 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & -a_n\wp''(a_nx_n) & 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

The normal space to V at a point is generated by the rows of Φ evaluated at this point. Recall the matrix M,

$$M = \begin{pmatrix} & 0 & 0 & & 0 & 0 & \\ I_{n-1} & \vdots & \vdots & M_1 & \vdots & \vdots & M_2 \\ & 0 & 0 & & 0 & 0 & \end{pmatrix}^T$$

where

$$M_1 = \begin{pmatrix} a_2\wp'(a_2f_2(x,y)) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a_n\wp'(a_nf_n(x,y)) \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} a_2\wp''(a_2f_2(x,y)) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a_n\wp''(a_nf_n(x,y)) \end{pmatrix}.$$

Let M' be the matrix

$$M' = \begin{pmatrix} 0 & 0 & \\ \vdots & \vdots & M^T \\ 0 & 0 & \end{pmatrix}.$$

Then the matrix product $M' \cdot (\Phi(\tilde{y}))^T$ gives the $(n-1) \times (2n+2)$ zero matrix. Therefore the kernel of the linear transformation from $\mathbb{R}^{3n+3}$ to $\mathbb{R}^{2n+2}$ given by the matrix M' is generated by the rows of the matrix $\Phi(\tilde{y})$. Let P be the matrix

$$P = \begin{pmatrix} \partial P_2/\partial y_0 & \dots & \partial P_n/\partial y_0 \\ \vdots & \ddots & \vdots \\ \partial P_2/\partial y_{3n+2} & \dots & \partial P_n/\partial y_{3n+2} \end{pmatrix} (\tilde{y}).$$

Then we have that

$$M' \cdot P = \begin{pmatrix} \partial F_2/\partial x_2 & \dots & \partial F_n/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial F_2/\partial x_n & \dots & \partial F_n/\partial x_n \end{pmatrix} (\tilde{x}).$$

The columns of the matrix on the right hand side of this equation are linearly independent over $\mathbb{R}$. Therefore the subspace of $\mathbb{R}^{3n+3}$ generated by the columns of P has trivial intersection with the kernel of the linear transformation given by M'. As the normal space to W at a point is generated by the columns of P evaluated at this point we have that in particular the normal spaces to V and W at each point in X have trivial intersection and so the intersection of V and W is transversal.

Therefore if the subspace V is shifted locally then the intersection of V and W is still transversal. We shall now give such a shift explicitly. For real numbers η and ξ we let $V_{\eta,\xi}$ be the subset given by applying the following operations to V. In other words $V_{\eta,\xi} = \Psi(V)$ for $\Psi : \mathbb{R}^{3n+3} \to \mathbb{R}^{3n+3}$ where Ψ does the following, for $(y_0, \dots, y_{3n+2}) \in \mathbb{R}^{3n+3}$

$$y_2 \mapsto y_2 + \eta y_0 + \xi y_1$$

$$y_{2+n+1} \mapsto \frac{1}{4} \left(\frac{y_{2+2n+2} - \wp'(a_2(\eta y_0 + \xi y_1))}{y_{2+n+1} - \wp(a_2(\eta y_0 + \xi y_1))} \right)^2 - y_{2+n+1} - \wp(a_2(\eta y_0 + \xi y_1))$$

and

$$y_{2+2n+2} \mapsto \frac{\begin{array}{c}\Big(\wp(a_2(\eta y_0 + \xi y_1))y_{2+2n+2} - \wp'(a_2(\eta y_0 + \xi y_1))y_{2+n+1} \\ -\wp(a_2(y_2 + \eta y_0 + \xi y_1))(y_{2+2n+2} - \wp'(a_2(\eta y_0 + \xi y_1))\Big)\end{array}}{y_{2+n+1} - \wp(a_2(\eta y_0 + \xi y_1))}$$

and the rest of the variables are fixed. The projection of W onto the variables y_0, y_1, y_2, y_3 contains the set

$$\{(f_0, f_1, f_2(f_0, f_1), f_3(f_0, f_1)) | f_0, f_1 \in B\}$$

in its interior. If it did not then as $\dim \pi W = 4$ we have $\dim \partial W \leq 3$ and so there is an algebraic relation between f_0, f_1, f_2 and f_3 contradicting Claim 5.1.1. So for each real η and ξ there is a positive real number δ such that for all real f_0 and f_1 with $f_0^2 + f_1^2 < \delta^2$ the intersection of X with $V_{\eta,\xi}$ is non-empty. The effect of Ψ on the subset X is the following.

$$\begin{aligned} &f_2 \to f_2 + \eta f_0 + \xi f_1 \\ &\wp(a_2 f_2) \to \wp(a_2(f_2 + \eta f_0 + \xi f_1)) \\ &\wp'(a_2 f_2) \to \wp'(a_2(f_2 + \eta f_0 + \xi f_1)). \end{aligned}$$

The real numbers η and ξ may be chosen so that at least one of the Cauchy-Riemann equations for u and v are not satisfied. Therefore there is a point $z \in \mathbb{R}^{3n+3}$ such that $P_k(z) \neq 0$ for some $k = 0, 1$ and $P_{1-k}(z) = P_j(z) = 0$ for $j = 2, \ldots, 2n+1$ and so the lemma is proved. □

Now we may lower the upper bound on transcendence degree and therefore obtain a contradiction.

Lemma 5.1.5.

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}[f_0, \ldots, f_n, \wp(a_0 f_0), \ldots, \wp(a_n f_n), \wp'(a_0 f_0), \ldots, \wp'(a_n f_n)] \leq n + 1.$$

Proof. Recall that for all $(x, y) \in B$,

$$F_2(x, y, f_2(x, y), \dots, f_n(x, y)) = 0$$
$$\vdots$$
$$F_{2n+1}(x, y, f_2(x, y), \dots, f_n(x, y)) = 0.$$

Also for all $i = 2, \dots, 2n+1$,

$$\begin{aligned} F_i(f_0(x, y), \dots, f_n(x, y)) = P_i(f_0(x, y), \dots, f_n(x, y), \\ \wp(a_0 f_0(x, y)), \dots, \wp(a_n f_n(x, y)), \\ \wp'(a_0 f_0(x, y)), \dots, \wp'(a_n f_n(x, y))), \end{aligned}$$

where $P_2, \dots, P_{2n+1}$ are polynomials and $f_0(x, y) = x$ and $f_1(x, y) = y$. By shrinking and shifting the disc D if necessary we may assume that all the points

$$\begin{aligned} \tilde{y}(x, y) = (x, y, f_2(x, y), \dots, f_n(x, y), \\ \wp(a_0 x), \wp(a_1 y), \wp(a_2 f_2(x, y)), \dots, \wp(a_n f_n(x, y)), \\ \wp'(a_0 x), \wp'(a_1 y), \wp'(a_2 f_2(x, y)), \dots, \wp'(a_n f_n(x, y))) \end{aligned}$$

such that the above system is satisfied are contained in a single irreducible component of the variety $\mathcal{V}(\langle P_2, \dots, P_{2n+1} \rangle)$ denoted $\mathcal{W}$. Now we shall add each of the polynomials P_0 and P_1 to the system $P_2, \dots, P_{2n+1}$ and consider the variety corresponding to the ideal generated by each of these new systems in turn. We shall then show that the dimension of each of these varieties decreases. This lowers the transcendence degree, proving the lemma.

Suppose that $\dim(\mathcal{W} \cap \mathcal{V}(\langle P_0 \rangle)) = \dim \mathcal{W}$. Then $\mathcal{W} \cap \mathcal{V}(\langle P_0 \rangle) = \mathcal{W}$ as $\mathcal{W}$ is irreducible. By Lemma 5.1.4 we may find a point $z \in \mathbb{C}^{3n+3}$ such that $P_2(z) = \cdots = P_{2n+1}(z) = 0$ and $P_0(z) \neq 0$, a contradiction. Now by once again shifting and shrinking the disc D we may suppose that all of the points $\tilde{y}(x, y)$ satisfying the above system are contained in an irreducible component of the variety $\mathcal{V}(\langle P_0, P_2, \dots, P_{2n+1} \rangle)$, denoted $\mathcal{W}'$.

Suppose that $\dim(\mathcal{W}' \cap \mathcal{V}(\langle P_1 \rangle)) = \dim \mathcal{W}'$, then again as $\mathcal{W}'$ is irreducible we have that $\mathcal{W}' \cap \mathcal{V}(\langle P_1 \rangle) = \mathcal{W}'$. By Lemma 5.1.4 we may also find a point $z \in \mathbb{C}^{3n+3}$ such that only one of $P_0(z)$ and $P_1(z)$ equals zero and $P_2(z) = \cdots = P_{2n+1}(z) = 0$. Therefore there is a point $z \in \mathbb{C}^{3n+3}$ such that $z \in \mathcal{W}'$ and $z \notin \mathcal{V}(\langle P_1 \rangle)$, a contradiction as required.

□

Therefore we have a lower bound

$$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}[f_0, \ldots, f_n, \wp(f_0), \ldots, \wp(f_n)] \geq n + 2$$

and an upper bound

$$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}[f_0, \ldots, f_n, \wp(a_0 f_0), \ldots, \wp(a_n f_n), \wp'(a_0 f_0), \ldots, \wp(a_n f_n)] \leq n + 1$$

on the transcendence degree of different finitely generated extensions of $\mathbb{C}$. However as $a_0, \ldots, a_n$ are rational we may write $\wp(a_i z)$ and $\wp'(a_i z)$ as algebraic functions in $\wp(z)$ for all $i = 0, \ldots, n$ after perhaps shrinking to smaller interval as discussed in Remark 2.1.13. Hence we have the upper bound,

$$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}[f_0, \ldots, f_n, \wp(f_0), \ldots, \wp(f_n)] \leq n + 1.$$

In particular we have upper and lower bounds on the transcendence degree of a finitely generated extension of $\mathbb{C}$ which are contradictory, as required.

Chapter 6

Nondefinability for the modular j-function

In this chapter we prove the following theorem, which is essentially a version of Theorem 1.0.1 of Bianconi for the modular j-function. Recall that the restriction of j to the intersection of the upper half plane and the imaginary axis is a real valued function.

Theorem 6.0.1. *Let $I \subseteq \mathbb{R}^{>0}$ be an open interval that is bounded away from zero and let $D \subseteq \mathbb{H}$ be an non-empty disc. Then the restriction of j to the disc D is not definable in the structure $(\overline{\mathbb{R}}, j|_{iI})$.*

6.1 Proof of Theorem 6.0.1

Assume for a contradiction that there is a disc $D \subseteq \mathbb{H}$ such that the restriction $j|_D$ is definable in the structure $(\overline{\mathbb{R}}, j|_{iI})$. For notational convenience we can suppose that the disc D contains the horizontal line segment $i + I$ and so the real and imaginary parts of the function $j|_{i+I}$ are definable in the structure $(\overline{\mathbb{R}}, j|_{iI})$. Rearranging the differential equation satisfied by j given in (2.8) gives that

$$ij'''(it) = \frac{-3}{2}\frac{(j''(it))^2}{ij'(it)} + \left(\frac{j^2(it) - 1968j(it) + 2654208}{2j^2(it)(j(it) - 1728)^2}\right)(ij'(it))^3 \qquad (6.1)$$

and so $ij'''(it)$ may be written as a polynomial in $j(it), ij'(it), j''(it), (ij'(it))^{-1}$ and $(2j^2(it)(j(it) - 1728)^2)^{-1}$. By shrinking the interval I if necessary we may assume that the denominators do not vanish for any $t \in I$. Therefore by differentiating this equation with respect to t we can see that all the higher derivatives of $j(it)$ may also be given as polynomials in these functions. Consider the auxiliary structure given by expanding $\overline{\mathbb{R}}$ by the functions $j_B(t) = j(iB(t)), j'_B(t) = ij'(iB(t)), j''_B(t) = j''(iB(t)), j_1(t) = (ij'(B(t)))^{-1}$ and $j_2(t) = (2j(iB(t))^2(j(iB(t))-1728)^2)^{-1}$ as well as B and B_1 where $B : \mathbb{R} \to I$ is the semi-algebraic function as described in Chapters 3 and 4. These structures are equivalent in the sense of having the same definable sets. They also have the same universally and existentially definable sets. Therefore the real and imaginary parts of the function $j|_{i+I}$ are definable in the structure $(\overline{\mathbb{R}}, j_B, j'_B, j''_B, j_1, j_2, B, B_1)$. As in previous chapters it suffices to prove Theorem 6.0.1 in this auxiliary structure. It is clear from construction that the set $\{j_B, j'_B, j''_B, j_1, j_2, B, B_1\}$ is closed under differentiation. By the theorem of Gabrielov, Theorem 2.4.7, and the same argument as in Chapter 3 we have that the auxiliary structure $(\overline{\mathbb{R}}, j_B, j'_B, j''_B, j_1, j_2, B, B_1)$ is model complete. Again by a similar argument to that seen in Chapter 3 we have that the ring of terms of this auxiliary structure is closed under differentiation.

Let $f_1, f_2 : I \to \mathbb{R}$ be defined by $f_1(t) = \Re(j(i+t))$ and $f_2(t) = \Im(j(i+t))$. By applying Theorem 2.4.3 to both f_1 and f_2, we have that for some integer $n \geq 1$ and a subinterval $I' \subseteq I$ there are polynomials $P_1^*, \ldots, P_n^* : \mathbb{R}^{8n+8} \to \mathbb{R}$ in $\mathbb{R}[y_1, \ldots, y_{8n+8}]$, certain functions $f_3, \ldots, f_n : I' \to \mathbb{R}$ such that for all $t \in I'$,

$$\begin{gathered} F_1(t, f_1(t), \ldots, f_n(t)) = 0 \\ \vdots \\ F_n(t, f_1(t), \ldots, f_n(t)) = 0 \end{gathered}$$

and

$$\det \left(\frac{\partial F_i}{\partial x_j} \right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}} (t, f_1(t), \ldots, f_n(t)) \neq 0,$$

where for $i = 1, \ldots, n$ we have that

$$\begin{aligned}F_i(t, f_1(t), \ldots, f_n(t)) = P_i^*(t, f_1(t), \ldots, f_n(t), & \\ & j(iB(t)), j(iB(f_1(t))), \ldots, j(iB(f_n(t))), \\ & ij'(iB(t)), ij'(iB(f_1(t))), \ldots, ij'(iB(f_n(t))), \\ & j''(iB(t)), j''(iB(f_1(t))), \ldots, j''(iB(f_n(t))), \\ & j_1(t), j_1(f_1(t)), \ldots, j_1(f_n(t)), \\ & j_2(t), j_2(f_1(t)), \ldots, j_2(f_n(t)) \\ & B(t), B(f_1(t)), \ldots, B(f_n(t)), \\ & B_1(t), B_1(f_1(t)), \ldots, B_1(f_n(t))).\end{aligned}$$

By the definition of the functions j_1 and j_2 as well as B and B_1 we may write $F_1, \ldots, F_n$ as algebraic functions in $t, f_1(t), \ldots, f_n(t), j(iB(t)), j(iB(f_1(t))), \ldots,$ $j(iB(f_n(t)))$ and $ij'(iB(t)), ij'(iB(f_1(t))), \ldots, ij'(iB(f_n(t)))$ as well as $j''(iB(t))$, $j''(iB(f_1(t))), \ldots, j''(iB(f_n(t)))$. In defining these algebraic functions square roots are introduced from the definition of B, which may affect the analyticity of these algebraic functions. The domain of these algebraic functions is a small open subset of $\mathbb{R}^{4n+4}$ containing the set

$$\Gamma_j = \{[f(t), j(iB(f(t))), ij'(iB(f(t))), j''(iB(f(t)))] : t \in I'\}$$

where $f(t) = (t, f_1(t), \ldots, f_n(t))$ and the algebraic functions are taken to be analytic on this domain. Hence for $i = 1, \ldots, n$ we have that

$$\begin{aligned}F_i(x_1, \ldots, x_{n+1}) = P_i(x_1, \ldots, x_{n+1}, j(iB(x_1)), \ldots, j(iB(x_{n+1})), & \\ ij'(iB(x_1)), \ldots, ij'(iB(x_{n+1})), j''(iB(x_1)), \ldots, j''(iB(x_{n+1}))) &\end{aligned}$$

for algebraic functions $P_1, \ldots, P_n$ and in particular for all $t \in I'$,

$$\begin{aligned}F_i(t, f_1(t), \ldots, f_n(t)) = P_i[&t, f_1(t), \ldots, f_n(t),\\ &j(iB(t)), j(iB(f_1(t))), \ldots, j(iB(f_n(t))),\\ &ij'(iB(t)), ij'(iB(f_1(t))), \ldots, ij'(iB(f_n(t))),\\ &j''(iB(t)), j''(iB(f_1(t))), \ldots, j''(iB(f_n(t)))] = 0.\end{aligned}$$

Now we take n to be minimal such that there is some interval I' and algebraic functions $P_1, \ldots, P_n$ in $4n+4$ variables (possibly taking complex values) and for all $i = 1, \ldots, n$ we have that $F_i(x_1, \ldots, x_{n+1}) = P_i(x_1, \ldots, x_{n+1}, j(iB(x_1)), \ldots, j(iB(x_{n+1})), ij'(iB(x_1)), \ldots, ij'(iB(x_{n+1})), j''(iB(x_1)), \ldots, j''(iB(x_{n+1})))$ and there are also functions $f_3, \ldots, f_n$ whose domain is I' such that $F_i(t, f_1(t), \ldots, f_n(t)) = 0$ and $\det(\partial F_i/\partial x_j)(t, f_1(t), \ldots, f_n(t)) \neq 0$ for all $t \in I'$ and $P_1, \ldots, P_n$ are analytic on their respective domains. Observe that the subinterval I', the functions $f_3, \ldots, f_n$ and the algebraic functions $P_1, \ldots, P_n$ may be different to those given above.

For $i = 1, \ldots, n$ and $j = 2, \ldots, n+1$ differentiating F_i with respect to x_j gives that,

$$\begin{aligned}\frac{\partial F_i}{\partial x_j}(\bar{x}) = \frac{\partial P_i}{\partial y_j}(\bar{y}) &+ iB'(x_j)j'(iB(x_j))\frac{\partial P_i}{\partial y_{j+n+1}}(\bar{y})\\ &- B'(x_j)j''(iB(x_j))\frac{\partial P_i}{\partial x_{j+2n+2}}(\bar{y}) + iB'(x_j)j'''(iB(x_j))\frac{\partial P_i}{\partial y_{j+3n+3}}(\bar{y}),\end{aligned} \tag{6.2}$$

where,

$$\bar{x} = (x_1, \ldots, x_{n+1})$$

and

$$\begin{aligned}\bar{y} = (&x_1, \ldots, x_{n+1}, j(iB(x_1)), \ldots, j(iB(x_{n+1})),\\ &ij'(iB(x_1)), \ldots, ij'(iB(x_{n+1})), j''(iB(x_1)), \ldots, j''(iB(x_{n+1}))).\end{aligned}$$

For the lower bound on transcendence degree we shall use Theorem 2.3.9. In order to apply this theorem we show that there is no positive integer M and no $k, l = 0, \ldots, n$ such that $\Phi_M(j(iB(f_k)), j(iB(f_l))) = 0$, where $k \neq l$ and $f_0(t) = t$.

Lemma 6.1.1. *There is no integer M and no k and l for $k, l = 0, \ldots, n$ with $k \neq l$ such that*

$$\Phi_M(j(iB(f_k)), j(iB(f_l))) = 0.$$

Proof. Suppose for a contradiction that there are such a M, k and l. For convenience we suppose that k and l are $n-1$ and n. Therefore we have that,

$$\Phi_M[j(iB(f_{n-1}(t))), j(iB(f_n(t)))] = 0 \tag{6.3}$$

for all $t \in I'$ and that there is some $\gamma \in GL_2^+(\mathbb{Q})$ such that $\gamma(iB(f_{n-1}(t))) = iB(f_n(t))$ where $M(\gamma)\gamma \in GL_2(\mathbb{Z})$ and $\det(M(\gamma)\gamma) = M$. Therefore $j(iB(f_n(t)))$ $= j(\gamma(iB(f_{n-1}(t))))$. Let $\mathcal{V}_0$ be a rational function such that $\mathcal{V}_0(z) = \gamma z$.

From (6.3) we have upon shrinking the interval I' if necessary that there is an algebraic analytic function $\mathcal{V}_1$ such that

$$\mathcal{V}_1(j(iB(f_{n-1}(t)))) = j(\mathcal{V}_0(iB(f_{n-1}(t)))). \tag{6.4}$$

Differentiating both sides of (6.4) with respect to t and cancelling the $B'(f_{n-1}(t))f'_{n-1}(t)$ factor that appears on both sides gives that

$$ij'(iB(f_{n-1}(t)))\mathcal{V}_1'(j(iB(f_{n-1}(t)))) = \mathcal{V}_0'(iB((f_{n-1}(t))))[ij'(\mathcal{V}_0(iB(f_{n-1}(t))))], \tag{6.5}$$

which may be rearranged to give that

$$\mathcal{V}_2(f_{n-1}(t), j(iB(f_{n-1}(t))), ij'(iB(f_{n-1}(t)))) = ij'(\mathcal{V}_0(iB(f_{n-1}(t))))$$

for an algebraic analytic function $\mathcal{V}_2$ in variables v_1, v_2, v_3. Differentiating both sides with respect to t and cancelling the $f'_{n-1}(t)$ factor that appears on both sides gives that,

$$\begin{aligned}
&\frac{\partial \mathcal{V}_2}{\partial v_1}(\tilde{v}_2(t)) + iB'(f_{n-1}(t))j'(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_2}{\partial v_2}(\tilde{v}_2(t)) \\
&\quad - B'(f_{n-1}(t))j''(iB(f_n(t)))\frac{\partial \mathcal{V}_2}{\partial v_3}(\tilde{v}_2(t)) \\
&= -B'(f_{n-1}(t))\mathcal{V}_0'(iB((f_{n-1}(t))))j''(\mathcal{V}_0(iB(f_{n-1}(t)))) \qquad (6.6)
\end{aligned}$$

where

$$\tilde{v}_2(t) = (f_{n-1}(t), j(iB(f_{n-1}(t))), ij'(iB(f_{n-1}(t)))).$$

We can rearrange (6.6) to give an algebraic analytic function $\mathcal{V}_3$ such that

$$\mathcal{V}_3(\tilde{v}_3(t)) = j''(\mathcal{V}_0(iB(f_{n-1}(t))))$$

where

$$\tilde{v}_3(t) = (f_{n-1}(t), j(iB(f_{n-1}(t))), ij'(iB(f_{n-1}(t))), j''(iB(f_{n-1}(t)))).$$

Finally differentiating both sides of this expression with respect to t and cancelling the $f'_{n-1}(t)$ factor that appears on both sides gives that,

$$\begin{aligned}
&\frac{\partial \mathcal{V}_3}{\partial v_1}(\tilde{v}_3(t)) + iB'(f_{n-1}(t))j'(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_3}{\partial v_2}(\tilde{v}_3(t)) \\
&- B'(f_{n-1}(t))j''(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_3}{\partial v_3}(\tilde{v}_3(t)) + iB'(f_{n-1}(t))j'''(iB(f_n(t)))\frac{\partial \mathcal{V}_3}{\partial v_4}(\tilde{v}_3(t)) \\
&= iB'(f_{n-1}(t))\mathcal{V}_0'(iB(f_{n-1}(t)))j'''(\mathcal{V}_0(iB(f_{n-1}(t)))). \qquad (6.7)
\end{aligned}$$

Now we rewrite the nonsingular system $F_1, \ldots, F_n$ as a new system of equations in fewer variables. Observe that

$$A(-i\mathcal{V}_0(iB(f_{n-1}(t)))) = A(-i(iB(f_n(t)))) = A(B(f_n(t))) = f_n(t)$$

as A is the compositional inverse of B as defined in Section 2.4.1. For $i = 1, \ldots, n$ define

$$\begin{aligned} Q_i(w_1,\ldots,w_{4n}) = P_i(&w_1,\ldots,w_n, A(-i\mathcal{V}_0(iB(w_n))),\\ &w_{n+1},\ldots,w_{2n},\mathcal{V}_1(w_{2n}),\\ &w_{2n+1},\ldots,w_{3n},\mathcal{V}_2(w_n,w_{2n},w_{3n}),\\ &w_{3n+1},\ldots,w_{4n},\mathcal{V}_3(w_n,w_{2n},w_{3n},w_{4n})) \end{aligned}$$

and then define,

$$\begin{aligned} G_i(u_1,\ldots,u_n) = Q_i(&u_1,\ldots,u_n, j(iB(u_1)),\ldots,j(iB(u_n)),\\ &ij'(iB(u_1)),\ldots,ij'(iB(u_n)), j''(iB(u_1)),\ldots,j''(iB(u_n))). \end{aligned}$$

Hence for all $t \in I'$,

$$G_i(t, f_1(t),\ldots,f_{n-1}(t)) = 0$$

for $i = 1,\ldots,n$. Hence we have a system of algebraic functions $Q_1,\ldots,Q_n$ in fewer variables. These algebraic functions have a domain which is an open set in $\mathbb{R}^{4n}$ that contains the set

$$\{[\tilde{f}(t), j(iB(\tilde{f}(t))), ij'(iB(\tilde{f}(t))), j''(iB(\tilde{f}(t)))] : t \in I'\}$$

where $\tilde{f}(t) = (t, f_1(t),\ldots,f_{n-1}(t))$ and these functions are analytic on this domain. If one of the $(n-1)\times(n-1)$ minors of the matrix of partial derivatives for this new system of equations is non-zero when evaluated at $(t, f_1(t),\ldots,f_{n-1}(t))$ for some $t \in I'$ then we have a contradiction to the minimality of n. Therefore we may assume that all such minors are zero. For $i = 1,\ldots,n$ we have that

$$\begin{aligned} \frac{\partial G_i}{\partial u_n} =& \frac{\partial Q_i}{\partial w_n} + iB'(u_n)j'(iB(u_n))\frac{\partial Q_i}{\partial w_{2n}} - B'(u_n)j''(iB(u_n))\frac{\partial Q_i}{\partial w_{3n}}\\ &+ iB'(u_n)j'''(iB(u_n))\frac{\partial Q_i}{\partial w_{4n}}. \end{aligned}$$

Also we have that

$$\begin{aligned}
\frac{\partial Q_i}{\partial w_n} &= \frac{\partial P_i}{\partial y_n} + B'(iu_n)\mathcal{V}_0'(iB(u_n))A'(-i\mathcal{V}_0(iB(u_n)))))\frac{\partial P_i}{\partial y_{n+1}} \\
&\quad + \frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_1} + \frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_1} \\
\frac{\partial Q_i}{\partial w_{2n}} &= \frac{\partial P_i}{\partial y_{2n+1}} + \frac{\partial P_i}{\partial y_{2n+2}}\frac{d\mathcal{V}_1}{dv} + \frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_2} + \frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_2} \\
\frac{\partial Q_i}{\partial w_{3n}} &= \frac{\partial P_i}{\partial y_{3n+2}} + \frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_3} + \frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_3} \\
\frac{\partial Q_i}{\partial w_{4n}} &= \frac{\partial P_i}{\partial y_{4n+4}} + \frac{\partial P_i}{\partial y_{4n+3}}\frac{\partial \mathcal{V}_3}{\partial v_4}.
\end{aligned}$$

Here the derivatives of the functions $Q_1, \dots, Q_n$ and $P_1, \dots, P_n$ are evaluated at

$$\begin{aligned}
&(u_1, \dots, u_n, j(iB(u_1)), \dots, j(iB(u_n)), j'(iB(u_1)), \dots, j'(iB(u_n)), \\
&j''(iB(u_1)), \dots, j''(iB(u_n)))
\end{aligned}$$

and

$$\begin{aligned}
&(u_1, \dots, u_n, A(-i\mathcal{V}_0(iB(u_n))), j(iB(u_1)), \dots, j(iB(u_n)), \mathcal{V}_1(j(iB(u_n))), \\
&j'(iB(u_1)), \dots, j'(iB(u_n)), \mathcal{V}_2(u_n, j(iB(u_n)), j'(iB(u_n))), \\
&j''(iB(u_1)), \dots, j''(iB(u_n)), \mathcal{V}_3(u_n, j(iB(u_n)), j'(iB(u_n)), j''(iB(u_n))))
\end{aligned}$$

respectively. Putting this all together and using (2.13), evaluating at $(t, f_1(t), \dots, f_{n-1}(t))$ and using the equations (6.5), (6.6) and (6.7) gives for all $i = 1, \dots, n$ that

$$\begin{aligned}
\frac{\partial G_i}{\partial u_n} &= \frac{\partial P_i}{\partial y_n} + B'(f_{n-1}(t))\mathcal{V}_0'(iB(f_{n-1}(t)))A'(B(f_n(t)))\frac{\partial P_i}{\partial y_{n+1}} + \frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_1} \\
&+ \frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_1} + iB'(f_{n-1}(t))j'(iB(f_{n-1}(t)))\Big(\frac{\partial P_i}{\partial y_{2n+1}} + \frac{\partial P_i}{\partial y_{2n+2}}\frac{d\mathcal{V}_1}{dv} + \\
&\frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_2} + \frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_2}\Big) - B'(f_{n-1}(t))j''(iB(f_{n-1}(t)))\Big(\frac{\partial P_i}{\partial y_{3n+2}} + \\
&\frac{\partial P_i}{\partial y_{3n+3}}\frac{\partial \mathcal{V}_2}{\partial v_3} + \frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_3}\Big) + iB'(f_{n-1}(t))j'''(iB(f_{n-1}(t)))\Big(\frac{\partial P_i}{\partial y_{4n+3}} + \\
&\frac{\partial P_i}{\partial y_{4n+4}}\frac{\partial \mathcal{V}_3}{\partial v_4}\Big)
\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial G_i}{\partial u_n} &= \frac{\partial F_i}{\partial x_n} + B'(f_{n-1}(t))\mathcal{V}_0'(iB(f_{n-1}(t)))A'(B(f_n(t)))\frac{\partial P_i}{\partial y_{n+1}} + \\ &iB'(f_{n-1}(t))j'(iB(f_{n-1}(t)))\frac{d\mathcal{V}_1}{dv}\frac{\partial P_i}{\partial y_{2n+2}} \\ &+ \Big(\frac{\partial \mathcal{V}_2}{\partial v_1} + iB'(f_{n-1}(t))j'(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_2}{\partial v_2} \\ &- B'(f_{n-1}(t))j''(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_2}{\partial v_3}\Big)\frac{\partial P_i}{\partial y_{3n+3}} \\ &+ \Big(\frac{\partial \mathcal{V}_3}{\partial v_1} + iB'(f_{n-1}(t))j'(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_3}{\partial v_2} - B'(f_{n-1}(t))j''(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_3}{\partial v_3} \\ &+ iB'(f_{n-1}(t))j'''(iB(f_{n-1}(t)))\frac{\partial \mathcal{V}_3}{\partial v_4}\Big)\frac{\partial P_i}{\partial y_{4n+4}} \\ &= \frac{\partial F_i}{\partial x_n} + B'(f_{n-1}(t))\mathcal{V}_0'(iB(f_{n-1}(t)))A'(B(f_n(t)))\frac{\partial F_i}{\partial x_{n+1}},\end{aligned}$$

where the partial derivatives of F_i are evaluated at $(t, f_1(t), \ldots, f_n(t))$. Therefore as all the $(n-1) \times (n-1)$ minors of the matrix

$$\left(\frac{\partial G_i}{\partial u_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n}}(t, f_1(t), \ldots, f_{n-1}(t))$$

are zero and $\partial G_i/\partial u_j = \partial F_i/\partial x_j$ for all $i = 1, \ldots, n$ and $j = 2, \ldots, n-1$ we have that

$$\begin{vmatrix} \frac{\partial F_1}{\partial x_2} & \ldots & \frac{\partial F_1}{\partial x_n} + \mathcal{V}_0'(iB(f_{n-1}(t)))A'(B(f_n(t)))\frac{\partial F_1}{\partial x_{n+1}} & \frac{\partial F_1}{\partial x_{n+1}} \\ \vdots & \ddots & \vdots & \\ \frac{\partial F_n}{\partial x_2} & \ldots & \frac{\partial F_n}{\partial x_n} + \mathcal{V}_0'(iB(f_{n-1}(t)))A'(B(f_n(t)))\frac{\partial F_n}{\partial x_{+1}} & \frac{\partial F_n}{\partial x_{n+1}} \end{vmatrix}(t, f_1(t), \ldots, f_n(t))$$

equals zero. This is the same as the matrix of the partial derivatives for the system $F_1, \ldots, F_n$, up to column operations, which do not affect the determinant. In particular

$$\det\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,n \\ j=2,\ldots,n+1}}(t, f_1(t), \ldots, f_n(t)) = 0$$

for some $t \in I'$, a contradiction and so the lemma is proved. □

In order to obtain a large enough lower bound we wish to apply Theorem 2.3.10 to $i+t, iB(t), iB(f_1), \ldots, iB(f_n)$. Suppose that there is some non-zero positive integer M such that $\Phi_M(j(i+t), j(iB(f_k))) = 0$ for some $k = 0, \ldots, n$. We may assume that $k = 1$ and so there is some $\gamma \in GL^+(\mathbb{Q})$ such that $i+t = \gamma(iB(f_1(t)))$ for all $t \in I'$. Therefore

$$i+t = \frac{aiB(f_1(t)) + b}{ciB(f_1(t)) + d}$$

for $a, b, c, d \in \mathbb{Q}$ and $ad - bc > 0$. Rearranging and comparing real and imaginary parts gives that

$$\begin{aligned} dt - cB(f_1(t)) &= b \\ d + cB(f_1(t))t &= aB(f_1(t)) \end{aligned}$$

and so

$$cdt^2 - (ad+bc)t + ba + cd = 0.$$

Therefore there are at most two values of $t \in I'$ for which we can not apply Theorem 2.3.10. We therefore shrink the interval I' to avoid these values. For all $t \in I'$ we have that $i+t, iB(t), iB(f_1(t)), \ldots, iB(f_n(t)) \in \mathbb{H}$ and we may assume that $j(i+t), j(iB(t)), j(iB(f_1)), \ldots, j(iB(f_n))$ are non-constant. Applying Theorem 2.3.10 to $i+t, iB(t), iB(f_1), \ldots, iB(f_n)$ gives that,

$$\begin{aligned} \text{tr.deg}_{\mathbb{C}}\mathbb{C}[&i+t, iB(t), iB(f_1), \ldots, iB(f_n), \\ &j(i+t), j(iB(t)), j(iB(f_1)), \ldots, j(iB(f_n)), \\ &j'(i+t), j'(iB(t)), j'(iB(f_1)), \ldots, j'(iB(f_n)), \\ &j''(i+t), j''(iB(t)), j''(iB(f_1)), \ldots, j''(iB(f_n))] \geq 3n+7. \end{aligned}$$

The rest of the proof consists of obtaining a contradictory upper bound. For all $t \in I'$ define,

$$\tilde{x} = \tilde{x}(t) = (t, f_1(t), \ldots, f_n(t))$$

and

$$\begin{aligned}\tilde{y} = \tilde{y}(t) = [t, f_1(t), \ldots, f_n(t), j(iB(t)), j(iB(f_1(t))), \ldots, j(iB(f_n(t))),\\ ij'(iB(t)), ij'(iB(f_1(t))), \ldots, ij'(iB(f_n(t))),\\ j''(iB(t)), j''(iB(f_1(t))), \ldots, j''(iB(f_n(t)))].\end{aligned}$$

Firstly it must be shown that the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,4n+4}} (\tilde{y}(t))$$

has maximal rank n for all $t \in I'$. By (6.2) we can see that the matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,n+1}} (\tilde{x}(t))$$

is given by multiplying the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\ldots,n\\ j=2,\ldots,4n+4}} (\tilde{y}(t))$$

by a $(4n+3) \times n$ matrix M, where M is the matrix

$$M = \begin{pmatrix} & 0 & & 0 & & 0 & \\ I_n & \vdots & M_1 & \vdots & M_2 & \vdots & M_3 \\ & 0 & & 0 & & 0 & \end{pmatrix}^T$$

for

$$M_1 = \begin{pmatrix} iB'(f_1(t))j'(iB(f_1(t))) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & iB'(f_n(t))j'(iB(f_n(t))) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -B'(f_1(t))j''(iB(f_1(t))) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & -B'(f_n(t))j''(iB(f_n(t))) \end{pmatrix}$$

and

$$M_3 = \begin{pmatrix} iB'(f_1(t))j'''(iB(f_1(t))) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & iB'(f_n(t))j'''(iB(f_n(t))) \end{pmatrix}.$$

As the rows of

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (\tilde{x}(t))$$

are linearly independent over $\mathbb{R}$ the rows of

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (\tilde{y}(t))$$

are also linearly independent over $\mathbb{R}$. Therefore the matrix

$$\left(\frac{\partial P_i}{\partial y_j}\right)_{\substack{i=1,\dots,n \\ j=2,\dots,n+1}} (\tilde{y}(t))$$

has maximal rank n as required. Now after restricting to a subinterval $I'' \subseteq I'$ if necessary we apply Lemma 2.4.12 as in Chapter 2 to obtain a contradictory upper bound on transcendence degree. Namely we have that,

$$\begin{aligned} \text{tr.deg}_{\mathbb{C}}\mathbb{C}[t, f_1, \dots, f_n,& \\ & j(iB(t)), j(iB(f_1)), \dots, j(iB(f_n)), \\ & ij'(iB(t)), ij'(iB(f_1)), \dots, ij'(iB(f_n)), \\ & j''(iB(t)), j''(iB(f_1)), \dots, j''(iB(f_n))] \leq 4n + 4 - n = 3n + 4. \end{aligned}$$

Recall the lower bound,

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[&i+t, iB(t), iB(f_1), \ldots, iB(f_n),\\ &j(i+t), j(iB(t)), j(iB(f_1)), \ldots, j(iB(f_n)),\\ &j'(i+t), j'(iB(t)), j'(iB(f_1)), \ldots, j'(iB(f_n)),\\ &j''(i+t), j''(iB(t)), j''(iB(f_1)), \ldots, j''(iB(f_n))] \geq 3n+7.\end{aligned}$$

Clearly these are upper and lower bounds on different finitely generated extensions of $\mathbb{C}$ and so to complete the proof we must obtain these contradictory upper and lower bounds on the transcendence degree of the same finitely generated extension. Firstly we can observe that removing $j'(i+t)$ and $j''(i+t)$ from the lower bound expression gives that

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[&i+t, iB(t), iB(f_1), \ldots, iB(f_n),\\ &j(i+t), j(iB(t)), j(iB(f_1)), \ldots, j(iB(f_n)),\\ &j'(iB(t)), j'(iB(f_1)), \ldots, j'(iB(f_n)),\\ &j''(iB(t)), j''(iB(f_1)), \ldots, j''(iB(f_n))] \geq 3n+5.\end{aligned}$$

As B is an algebraic function we have that $t, B(t), B_1(t), iB(t), i+t$ are algebraically dependent and that f_k and $iB(f_k)$ are algebraically dependent for $k = 1, \ldots, n$. From the construction of our non-singular system we know that the real and imaginary parts of the function $j(i+t)$ are among the functions $f_1, \ldots, f_n$. Therefore adding $j(i+t)$ does not affect the transcendence degree. Hence we have that,

$$\begin{aligned}\text{tr.deg}_{\mathbb{C}}\mathbb{C}[&i+t, iB(t), iB(f_1), \ldots, iB(f_n),\\ &j(i+t), j(iB(t)), j(iB(f_1)), \ldots, j(iB(f_n)),\\ &j'(iB(t)), j'(iB(f_1)), \ldots, j'(iB(f_n)),\\ &j''(iB(t)), j''(iB(f_1)), \ldots, j''(iB(f_n))] \leq 3n+4.\end{aligned}$$

Therefore we have obtained contradictory upper and lower bounds on the transcendence degree of some finitely generated extension of $\mathbb{C}$ and so we have a contradiction as desired.

Chapter 7

Interdefinability of restrictions of the exponential maps of abelian varieties

Throughout this thesis we have considered definability questions on certain restrictions of the Weierstrass $\wp$-function in structures given by expanding the ordered real field, $\overline{\mathbb{R}}$, by restrictions of the same $\wp$-function to an interval. In Chapter 5 we also considered the definability of different $\wp$-functions in this structure before proving a theorem showing that there are no new complex functions definable in these structures. Now we turn to a different question.

Consider the $N+1$ complex lattices $\Omega_1, \ldots, \Omega_{N+1}$ and let $\wp_1, \ldots, \wp_{N+1}$ be the associated Weierstrass $\wp$-functions for some integer $N \geq 1$. The question we consider is the following. When is some restriction of the real and imaginary parts of $\wp_{N+1}$ definable in the structure given by expanding $\overline{\mathbb{R}}$ by the real and imaginary parts of some restriction of $\wp_1, \ldots, \wp_N$. More precisely the definable restrictions considered here are the following. Let $\mathcal{F}$ be a set of real valued maps each defined on an open subset of $\mathbb{R}^n$ for some n. Recall that a function $f \in \mathcal{F}$ is said to be *locally definable* in some expansion of $\overline{\mathbb{R}}$ if the restriction of f to some neighbourhood of each point in its domain is definable. Then the smallest expansion of $\overline{\mathbb{R}}$ in which all of these maps are locally definable is denoted $\mathbb{R}_{PR(\mathcal{F})}$.

Recall from the introduction to this thesis the following theorem of Jones, Kirby and Servi which answers this question and is Theorem 1.2 in [21].

Theorem 7.0.1. *Let $\mathcal{F}_1$ consist of complex exponentiation and some Weierstrass $\wp$-functions and let $\mathcal{F}_2$ consist of Weierstrass $\wp$-functions. Suppose that none of* *the functions in* $\mathcal{F}_2$ *is isogenous to any* $\wp$*-function from* $\mathcal{F}_1$ *or isogenous to the Schwarz reflection of a* $\wp$*-function in* $\mathcal{F}_1$*. Then any set which is definable in both* $\mathbb{R}_{PR(\mathcal{F}_1)}$ *and in* $\mathbb{R}_{PR(\mathcal{F}_2)}$ *is definable in* $\overline{\mathbb{R}}$*.*

In the introduction we observed that the Weierstrass $\wp$-function arises in the exponential map of elliptic curves over $\mathbb{C}$. In fact elliptic curves are the abelian varieties of dimension 1. An abelian variety is a complete algebraic group. It is therefore natural to ask whether the previous theorem can be extended to the exponential maps of general abelian varieties. Recall from the introduction that in order to formulate this question more precisely we must choose the set $\mathcal{F}$ more carefully. Here all abelian varieties are defined over $\mathbb{C}$.

Let G be an abelian variety over $\mathbb{C}$ and let $\mathcal{F}_G$ be the set consisting of: the exponential map of G, the exponential maps of all abelian subvarieties of G, the exponential maps of all abelian varieties isogenous to an abelian subvariety of G, the exponential maps of the products of these abelian varieties and the exponential maps of all abelian varieties isogenous to an abelian subvariety of these products as well as the Schwarz reflections of all these exponential maps. Then we have the following theorem, which is joint work with Gareth Jones and Jonathan Kirby.

Theorem 7.0.2. *Let* G *and* H *be abelian varieties and let* $\mathcal{F}_G$ *and* $\mathcal{F}_H$ *be their associated sets of exponential maps. Suppose that* $\mathcal{F}_G \cap \mathcal{F}_H = \emptyset$*.*

Then any set definable in both $\mathbb{R}_{PR(\mathcal{F}_G)}$ *and* $\mathbb{R}_{PR(\mathcal{F}_H)}$ *is semialgebraic.*

The format of this chapter is the following. The general strategy in the proof of Theorem 7.0.2 is to follow that used by Jones, Kirby and Servi to prove Theorem 7.0.1 and use the method of predimensions due to Hrushovski in [19]. Here the predimension is different to that used in [21]. Instead we introduce the notion of $G^{\max}$, an abelian variety that is defined up to isogeny and determines which points on the graphs of the exponential maps in $\mathcal{F}$ are in B and not in C. Here $C \subseteq B$ are subfields of $\mathbb{C}$ and C is closed under a pregeometry that we shall define. This $G^{\max}$ is defined more precisely in Lemma 7.3.2 and it is here that we use an Ax type result, which follows from a result due to Kirby in [22].

In the next section we give some background material on local definability and its connections to linear relations between differential forms. This involves recalling a pregeometry introduced by Wilkie in [40] and giving Wilkie's characterisation of this pregeometry in terms of differential forms. In Section 7.2 we define the set of exponential maps $\mathcal{F}$ explicitly and explain how we may define certain differential forms. These are vector spaces of differential forms spanned by the forms associated to the exponential map of an abelian variety and some point on its graph. In Section 7.3 we define and explain the notions of $G^{\max}$, predimension, self-sufficiency and hull and prove results on the interactions between these notions. These results can then be used to prove Theorem 7.4.1, which is the main technical theorem needed in the proof of Theorem 7.0.2. The rest of Section 7.4 consists of the proof of Theorem 7.0.2, which follows that of Theorem 7.0.1, where Theorem 7.4.1 takes the place of Theorem 7.1 in [21].

The material in this chapter is in collaboration with my supervisor Gareth Jones and Jonathan Kirby. It also uses some ideas from earlier unfinished work in the case of Weierstrass elliptic functions that come from the exponential maps of groups of Jones, Kirby and Schmidt.

7.1 Background on local definability and differential forms

In this section we give some background on local definability as well as holomorphic closure and differential forms. This follows the presentation in Section 2 of [21]. Firstly we discuss local definability, where definability is now taken to mean 0-definable. Recall that $\overline{\mathbb{R}}$ denotes the first order structure $\overline{\mathbb{R}} := (\mathbb{R}, +, \times, 0, 1, <)$.

Definition 7.1.1. Let $U \subseteq \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}$ a function. The function f is said to be *locally definable* with respect to an expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$ if for every $a \in U$ there is a neighbourhood U_a of a such that the function $f|_{U_a}$ is definable in $\mathcal{R}$.

Now we turn to a more general setting, namely that of definable manifolds. The following definition can be seen in [13], however the version given here is from Section 2 of [15].

Definition 7.1.2. An (abstract) definable manifold, of dimension n, is a triple $(M, M_i, \phi_i)_{i\in I}$ where $\{M_i : i \in I\}$ is a finite cover of the set X and for each $i \in I$:

1. we have injective maps $\phi_i : M_i \to \mathbb{R}^n$ such that $\phi_i(M_i)$ is an open definably connected definable set.

2. each $\phi_i(M_i \cap M_j)$ is an open definable subset of $\phi_i(M_i)$

3. the map $\phi_{ij} : \phi_i(M_i \cap M_j) \to \phi_j(M_i \cap M_j)$ given by $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is a definable homeomorphism for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$.

If we have a definable manifold M, then a map $f : U \to M$ is *locally definable* if for every $a \in U$ there is a neighbourhood U_a of a, an open set $W \subseteq M$ containing $f(U_a)$ and a definable chart $\phi : W \to \mathbb{R}^m$ such that each of the components of the restriction $(\phi \circ f)|_{U_a}$ is definable.

Upon making the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$, we can see that a complex function is locally definable if and only if its real and imaginary parts are locally definable.

Definition 7.1.3. Let $U \subseteq \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}^m$ a map. Also let $\Delta = (r_1, s_1) \times \cdots \times (r_n, s_n)$ be an open rectangular box whose corners are rational and where the closure $\overline{\Delta} \subseteq U$. A *proper restriction* of f is a restriction of the form $f|_\Delta$ and if $a \in \Delta$ we say that $f|_\Delta$ is a *proper restriction of* f *around* a.

Let $\mathcal{F}$ be a set of maps each of which is defined on an open subset of $\mathbb{R}^n$ and takes values in $\mathbb{R}^m$ for some positive integers n and m. We write $PR(\mathcal{F})$ for the set of all proper restrictions of maps in $\mathcal{F}$ and $\mathbb{R}_{PR(\mathcal{F})}$ for the expansion of the real field $\overline{\mathbb{R}}$ by the graphs of all the component functions of the maps in $PR(\mathcal{F})$.

Using this terminology we can consider $\mathbb{R}_{an}$ to be the expansion of $\overline{\mathbb{R}}$ by all proper restrictions of all real-analytic functions. (This is equivalent to the usual definition of $\mathbb{R}_{an}$, which was given in the introduction, in the sense of having the same definable sets.)

We finish this discussion on local definability by stating a lemma from [21].

Lemma 7.1.4 (Lemma 2.3 in [21])**.** *A function $f : U \to \mathbb{R}$ is locally definable in an expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$ if and only if all of its proper restrictions are definable in $\mathcal{R}$.*

This shows that $\mathbb{R}_{PR(\mathcal{F})}$ is the smallest expansion of $\overline{\mathbb{R}}$ in which all functions from $\mathcal{F}$ are locally definable and therefore this definition agrees with the one given at the beginning of this chapter, in the discussion before Theorem 7.0.2. Now we discuss holomorphic closure.

Definition 7.1.5. Let $A \subseteq \mathbb{C}$ and $b \in \mathbb{C}$. Suppose that there is an $n \in \mathbb{N}$, an open set $U \subseteq \mathbb{C}^n$, a holomorphic function $f : U \to \mathbb{C}$ which is definable in $\mathbb{R}_{PR(\mathcal{F})}$ and a point $a \in U \cap A^n$ such that $f(a) = b$. Then we say that b is in the *holomorphic closure* of A with respect to $\mathcal{F}$. This is denoted by $b \in \mathrm{hcl}_{\mathcal{F}}(A)$.

The following is an improved version of Proposition 3.3 in [21].

Proposition 7.1.6. *Let $\mathcal{F}$ be a countable set of holomorphic functions, which is closed under partial differentiation and under Schwarz reflection. Let $C \subseteq \mathbb{C}$ be a countable subfield. Let $U \subseteq \mathbb{C}^n$ be open and $f : U \to \mathbb{C}$ be a holomorphic function. Assume that we have $f(a) \in hcl_{\mathcal{F}}(C(a))$ for all $a \in U$ except in some subset $X \subseteq U$ of measure 0. Then the subset $U' := \{a \in U | f$ is locally definable at $a\}$ is open in U and $U \setminus U'$ has measure 0. Furthermore, if $n = 1$ and X is countable then $U \setminus U'$ is countable.*

Proof. From the definition of local definability it is clear that U' is open in U. Now we show that $U \setminus U'$ has measure zero.

Since C is countable we can enumerate the pairs $(U_i, g_i)_{i \in \mathbb{N}}$ where U_i is a C-definable connected open subset of U and $g_i : U_i \to \mathbb{C}$ is a C-definable holomorphic function. Now define $J := \{i \in \mathbb{N} : g_i \neq f|_{U_i}\}$ and for each $i \in J$ let

$$V_i := \{a \in U_i | g_i(a) = f(a)\}.$$

Then V_i is locally the zero set of the holomorphic function $g_i - f$ and is therefore an analytic set with a well defined $\mathbb{C}$-dimension. Furthermore, since $f \neq g_i$, the set V_i is a proper closed subset of U_i and so this dimension is less than n and V_i has measure 0. If $n = 1$ then each V_i is countable.

Now suppose that $a \in U \setminus U'$. Then either $a \in X$ or $f(a) \in \mathrm{hcl}_{\mathcal{F}}(C(a))$ and in the latter case the point $a \in V_i$ for some $i \in J$. So $U \setminus U' \subseteq X \cup \bigcup_{i \in J} V_i$, a set measure 0. If $n = 1$ then this set is countable as X is countable and V_i is countable for all $i \in J$. □

We next consider differential forms and the connection between local definability and differential forms following Wilkie. Let $C \subseteq B \subseteq \mathbb{C}$ be fields. Associated to the field extension B/C in $\mathbb{C}$ we have the B-vector space $\Omega(B/C)$ of Kähler differential forms together with the universal derivation $d : B \to \Omega(B/C)$. We define $\Omega(B) := \Omega(B/\mathbb{Q})$. For $B \subseteq \mathbb{C}$ we have inclusion maps $\Omega(B) \subseteq \Omega(B) \otimes \mathbb{C} \subseteq \Omega(\mathbb{C})$. Let $\mathcal{F}$ be a collection of holomorphic maps.

Definition 7.1.7. Let $U \subseteq \mathbb{C}^n$ for some integer $n \geq 1$ and let $a \in U$. Let $f : U \to \mathbb{C}^m$ be in $\mathcal{F}$, with component functions $f_1, \ldots, f_m$. Let $C \subseteq \mathbb{C}$. For $j = 1, \ldots, m$ we define $\omega_{f_j,a} \in \Omega(\mathbb{C}/C)$ by

$$\omega_{f_j,a} = d(f_j(a)) - \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(a) da_i.$$

Definition 7.1.8. With $\mathcal{F}, B$ and C as before, we define $W_{\mathcal{F}}(B/C)$ to be the subspace of $\Omega(B/C)$ spanned by those $\omega_{f_j,a}$ such that there is $U \subseteq \mathbb{C}^n$ in $\mathcal{F}$, and $a \in U \cap B^n$ with $f(a) \in B^m$ and all partial derivatives $(\partial f_j/\partial x_i)(a) \in B$. Then we define $\Omega_{\mathcal{F}}(B/C)$ to be the quotient of $\Omega(B/C)$ by $W_{\mathcal{F}}(B/C)$.

Now we define a closure operator related to this space of differential forms.

Definition 7.1.9. Let $A \subseteq \mathbb{C}$ and $b \in \mathbb{C}$. Define $b \in \mathrm{Dcl}_{\mathcal{F}}(A)$ if db is in the span of $\{da | a \in A\}$ in $\Omega_{\mathcal{F}}(\mathbb{C})$. We say that $C \subseteq \mathbb{C}$ is $\mathrm{Dcl}_{\mathcal{F}}$-closed in $\mathbb{C}$ if $C = \mathrm{Dcl}_{\mathcal{F}}(C)$.

The critical fact that connects local definability with these differential forms is due to Wilkie. This fact can be seen in [40] and is the following.

Fact 7.1.10. Suppose that $\mathcal{F}$ is closed under partial differentiation and Schwarz reflection. Then $\mathrm{Dcl}_{\mathcal{F}}$ is a pregeometry on $\mathbb{C}$ and this pregeometry coincides with $\mathrm{hcl}_{\mathcal{F}}$.

Throughout the rest of this chapter the sets $\mathcal{F}$ are closed under partial differentiation.

Definition 7.1.11. We write $\dim_{\mathcal{F}}$ for the dimension function associated with the pregeometry $\mathrm{Dcl}_{\mathcal{F}}$.

The following lemma gives some useful consequences.

Lemma 7.1.12. *1. For $C \subseteq A \subseteq \mathbb{C}$, $\dim_{\mathcal{F}}(A/C)$ is equal to the $\mathbb{C}$-linear dimension of the image of $\Omega_{\mathcal{F}}(A/C)$ in $\Omega_{\mathcal{F}}(\mathbb{C}/C)$.*

2. $\dim_{\mathcal{F}}(A/C) \leq \dim \Omega_{\mathcal{F}}(A/C)$.

Proof. The second part of this lemma follows from the first as the function from $\Omega_{\mathcal{F}}(A/C)$ to the image of the natural map between $\Omega_{\mathcal{F}}(A/C)$ and $\Omega_{\mathcal{F}}(\mathbb{C}/C)$ is surjective. For the first part, let $\dim_{\mathcal{F}}(A/C) = n$. Then there is a basis $A_0 \subseteq A$ with cardinality n such that for all $a \in A$ we have that $a \in \mathrm{Dcl}_{\mathcal{F}}(A_0 \cup C)$ and A_0

is independent over C. By definition we have for all $a \in A$ that $da \in \text{span}\{da' : a' \in A_0 \cup C\}$ in $\Omega_{\mathcal{F}}(\mathbb{C})$. Therefore, $da \in \text{span}\{da' : a' \in A_0\}$ in $\Omega_{\mathcal{F}}(\mathbb{C}/C)$. So the dimension of the image of $\Omega_{\mathcal{F}}(A/C)$ in $\Omega_{\mathcal{F}}(\mathbb{C}/C)$ is equal to $\dim_{\mathcal{F}}(A/C)$ as required.

□

Definition 7.1.13. We define $\text{Der}_{\mathcal{F}}(\mathbb{C}/C)$ to be the space of $\mathcal{F}$-derivations on $\mathbb{C}$ which vanish on C. These are the maps $\partial : \mathbb{C} \to \mathbb{C}$ which are additive, vanish on C, satisfy the Leibniz rule, and for each component f_j of each $f : U \to \mathbb{C}^m$ in $\mathcal{F}$ and each $a \in U$ satisfy

$$\partial(f_j(a)) - \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(a)\partial a_i = 0.$$

Note that $\text{Der}_{\mathcal{F}}(\mathbb{C}/C)$ is canonically isomorphic to the dual space of $\Omega_{\mathcal{F}}(\mathbb{C}/C)$. Also if $C \subseteq A \subseteq \mathbb{C}$ then we may may define $\text{Der}_{\mathcal{F}}(A/C)$ by simply replacing $\mathbb{C}$ with A and considering all points $a \in U \cap A$ in the above definition.

7.2 Abelian varieties and differential forms

In this section we explain precisely how the set of exponential maps $\mathcal{F}$ seen in the statement of Theorem 7.0.2 are defined. Firstly we give some background material on abelian varieties that we shall need. The following definition can be seen Definition 3.7 in [31]. Everything here is defined over an algebraically closed subfield $C \subseteq \mathbb{C}$.

Definition 7.2.1. An *(abstract) variety* is a set G with a covering $V_1, \ldots, V_m$ and an atlas of charts $\phi_i : V_i \to U_i$ where for each $i = 1, \ldots, m$ we have that $U_i \subseteq \mathbb{C}^{m_i}$ is an affine variety (and therefore definable in $\overline{\mathbb{R}}$) and ϕ_i is a bijection and we require that for all $1 \leq i, j \leq m$,

1. $U_{ij} = \phi_i(V_i \cap V_j) \subseteq U_i$ is open.
2. The map $\phi_{i,j} = \phi_j \circ \phi_i^{-1} : U_{ij} \to U_{ji}$ is an isomorphism of quasi-affine varieties (and is definable in $\overline{\mathbb{R}}$).

Remark 7.2.2. On $V_i \cap V_j$ we have that $\phi_i \circ \phi_j^{-1} \circ \phi_j = \phi_i$.

The following definitions can be seen in 3.15, 4.1 and 4.6 of [31] respectively.

Definition 7.2.3. A variety X is said to be *complete* if for any variety Y the projection map $\pi : X \times Y \to Y$ is a closed map in the Zariski topology.

Definition 7.2.4. An *algebraic group* G is by definition a variety V together with a pair of morphisms $\mu : V \times V \to V$ and $\rho : V \to V$, such that μ yields a group operation on V and ρ is the map $x \to x^{-1}$. If V, μ, ρ are defined over C then we say that G is defined over C. Notationally we identify G with V.

Definition 7.2.5. An *abelian variety* is a connected algebraic group whose underlying variety is complete. An abelian variety is *simple* if it has no proper abelian subvarieties that are not the zero variety.

An abelian variety is an example of a Lie group. In this chapter we take the exponential map of an abelian variety G to be the exponential map of G as a Lie group. The following definition can be seen in Chapter 8 of [8].

Definition 7.2.6. Let G be a Lie group and let LG be its Lie algebra. For $X \in LG$ let $p : \mathbb{R} \to G$ be an integral curve for the left-invariant vector field X with $p(0) = 0$. Then we define $\exp_G(X) = p(1)$.

The following is Proposition 8.2 in [8].

Proposition 7.2.7. *Let G, H be Lie groups and let LG, LH be their respective Lie algebras. Let $f : G \to H$ be a homomorphism and $df : LG \to LH$ be the induced map on the Lie algebras. Then* $\exp_H \circ df = f \circ \exp_G$.

Remark 7.2.8. Over the complex numbers abelian varieties can be defined using an alternative approach. One considers them as complex tori, which can be embedded into projective space. This gives rise to a map $\exp : \mathbb{C}^n \to \mathbb{C}^m$, which arises from the composition of the abstract exponential map for G with this projective embedding. This allows us to consider the definability of the exponential map in this setup. Using Definition 7.1.7 we may then define forms associated to exp at a point $a \in \mathbb{C}^n$. However there are many possible choices for this projective embedding. The advantage of the abstract variety definition is that we may talk about the unique exponential map for the abelian variety G denoted $\exp_G$ and differential forms associated to $\exp_G$ at a point. This uniqueness allows us to talk about definability in this setup. These differential forms shall be described in more detail later in this section.

Throughout the rest of this chapter we take an abelian variety G to be an abelian variety with a fixed choice of atlas, unless specified otherwise. Recall that G is defined over the algebraically closed subfield $C \subseteq \mathbb{C}$. The following fact can be seen in Example 4.6 in [31].

Fact 7.2.9. If f is a morphism from G to H where G is an abelian variety and H is an algebraic group then f is the translate of a homomorphism.

This next definition can be seen in page 81 of [36].

Definition 7.2.10. Let G_1 and G_2 be abelian varieties. A homomorphism $f : G_1 \to G_2$ is an *isogeny* if it is surjective, has finite kernel and $\dim G_1 = \dim G_2$.

The following theorem is due to Poincaré and can be seen in Theorem 1 of Section 19 in [29].

Theorem 7.2.11 (Poincaré's Reducibility Theorem)**.** *If G is an abelian variety and H is an abelian subvariety of G there is an abelian subvariety H' such that $H \cap H'$ is finite and $H + H' = G$. In other words, G is isogenous to $H \times H'$.*

The next theorem can be seen in Corollary 1 in Section 19 of [29].

Theorem 7.2.12 (Poincaré's Complete Reducibility Theorem)**.** *Given an abelian variety G there is an isogeny defined over C*

$$G \to G_1^{n_1} \times \cdots \times G_r^{n_r},$$

where the abelian varieties $G_1, \ldots, G_r$ are simple and the integers $n_1, \ldots, n_r$ are uniquely determined up to isogenies defined over C and permutations.

Definition 7.2.13. Let $G_1, \ldots, G_r$ be non-isogenous simple abelian varieties and let $\mathcal{F}$ be the set of exponential maps for $G_1, \ldots, G_r$ composed with each of the charts in their atlas. The set of complex numbers definable without parameters in $\mathbb{R}_{PR(\mathcal{F})}$ is denoted $C_{\mathcal{B}}$. Then the set of basic abelian varieties $\mathcal{B} = \mathcal{B}(G_1, \ldots, G_r)$ is the set containing all the simple abelian varieties G such that G is defined over $C_{\mathcal{B}}$ and G is isogenous to either G_i or the Schwarz reflection of G_i for some $i = 1, \ldots, r$.

The set $C_{\mathcal{B}}$ is an algebraically closed field that is the algebraic closure of the set $\{x + iy : x, y \in \mathbb{R}$ such that x, y are 0-definable in $\mathbb{R}_{PR(\mathcal{F})}.\}$.

Definition 7.2.14. Given abelian varieties $G_1, \dots, G_r$ as in Definition 7.2.13, we define $\mathcal{S} = \mathcal{S}(\mathcal{B})$ to be the smallest set of abelian varieties containing $\mathcal{B}$ that satisfies the following conditions,

- If G and H are in $\mathcal{S}$ then the product $G \times H \in \mathcal{S}$.
- If $G \in \mathcal{S}$ and H is an abelian subvariety of G defined over $C_{\mathcal{B}}$, then $H \in \mathcal{S}$.
- If $G \in \mathcal{S}$ and H is an abelian variety defined over $C_{\mathcal{B}}$ which is isogenous to G, then $H \in \mathcal{S}$.

Remark 7.2.15. Suppose that G and H are abelian varieties where $G \in \mathcal{S}$ and H is defined over $C_{\mathcal{B}}$. Let $q : G \to H$ be a surjective algebraic homomorphism defined over $C_{\mathcal{B}}$. Then there is an abelian subvariety G' of G such that $G/G' \simeq H$. Then by Theorem 7.2.11 there is a complimentary abelian subvariety H' of G defined over $C_{\mathcal{B}}$ such that $G' \times H' \sim G$ and the induced map $f : H' \to H$ is an isogeny and so $H' \sim H$. Therefore by the above definition we have that $H \in \mathcal{S}$. Also it follows from the reducibility theorem that quotients of abelian varieties are isogenous to abelian subvarieties.

Now consider two abelian varieties G and H with corresponding exponential maps $\exp_G$ and $\exp_H$. The exponential map of the product $G \times H$ can be defined as $\exp_{G\times H}(z, w) = (\exp_G(z), \exp_H(w))$. Similarly if G and H are isogenous then one may define the exponential map $\exp_H$ by using $\exp_G$ and this isogeny. The same may be done for the Schwarz reflection of G and abelian subvarieties of G. Therefore we have the following definition.

Definition 7.2.16. Given non-isogenous simple abelian varieties $G_1, \dots, G_r$ we define $\mathcal{F} = \mathcal{F}(\mathcal{B})$ to be the set of maps given by composing the exponential maps built from the exponential maps $\exp_{G_1}, \dots, \exp_{G_r}$ with each chart in the atlas of charts associated to the corresponding abelian variety. We shall refer to this as the set of exponential maps for the abelian varieties in $\mathcal{S}(\mathcal{B})$.

Now consider an abelian variety G and the set of maps $\mathcal{F}_G$ given by composing $\exp_G$ with the charts in the corresponding atlas for G and consider the set of complex numbers definable without parameters in the structure $\mathbb{R}_{PR(\mathcal{F}_G)}$, denoted C_G. By Theorem 7.2.12 applied over the algebraically closed field C_G, we have that G is isogenous to a product of simple abelian varieties $G_1^{n_1} \times \cdots \times G_r^{n_r}$. Here each of the simple abelian varieties $G_1, \ldots, G_r$ is defined over C_G. Using the above definitions we may define the sets of abelian varieties $\mathcal{B}(G_1, \ldots, G_r), \mathcal{S}(G_1, \ldots, G_r)$ and exponential maps $\mathcal{F}(G_1, \ldots, G_r)$ associated to these simple abelian varieties. Then the corresponding sets of abelian varieties and exponential maps associated to G are then defined to be $\mathcal{B}(G) = \mathcal{B}(G_1, \ldots, G_r), \mathcal{S}(G) = \mathcal{S}(G_1, \ldots, G_r)$ and $\mathcal{F}(G) = \mathcal{F}(G_1, \ldots, G_r)$ respectively.

Now we discuss the differential forms that we associate to the exponential map of an abelian variety evaluated at a certain point. Firstly we give a useful background lemma.

Lemma 7.2.17. *If $\theta : U \to V$ for open subsets U and V of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, is a regular map (algebraic morphism) defined over C and $\theta(a) = b$ then*

$$db_j = \sum_{i=1}^{n} \frac{\partial \theta_j}{\partial x_i}(a) da_i$$

and setting $J_\theta = \left(\frac{\partial \theta_j}{\partial x_i}\right)_{j,i}$, the Jacobian matrix we have the vector and matrix equation

$$db = J_\theta(a) da.$$

Furthermore if f and g are any analytic (or continuously differentiable) maps then

$$J_{g \circ f}(a) = J_g(f(a)) J_f(a).$$

Proof. As θ is regular, its coordinate functions are rational functions and so the above expression follows by the Leibniz rule for d. The second part follows from the usual chain rule. □

Let G be an abelian variety of dimension n and denote its cover by $V_1, \ldots, V_r$ and collection of charts by $\phi_i : V_i \to U_i \subseteq \mathbb{C}^{m_i}$ for some integer $m_i \geq 1$ for all $i = 1, \ldots, r$. Let $\alpha = (a, \exp_G(a))$ be a point in the graph $\mathcal{G}_G$ of $\exp_G$. Consider the chart ϕ_i such that $\exp_G(a) \in V_i$. Then we can define the Kähler differentials for $e_i = \phi_i \circ \exp_G$ at a using Definition 7.1.7. The space generated by these differential forms over C is denoted $\mathcal{V}_{e_i,\alpha}$ and is a subspace of $\Omega(\mathbb{C}/C)$. However it is likely that there is some $j \neq i$ such that $\exp_G(a) \in V_j$. Therefore for some neighbourhood of $\exp_G(a)$ in V_j we may define the differentials for $e_j = \phi_j \circ \exp_G$

at a and therefore define the vector space $\mathcal{V}_{e_j,\alpha}$. In order for this vector space to be well defined we must show that it does not depend on this choice of chart. In other words that $\mathcal{V}_{e_i,\alpha}$ and $\mathcal{V}_{e_j,\alpha}$ are isomorphic. It turns out that we can go further.

Claim 7.2.18. *For $1 \leq i, j \leq m$ with $i \neq j$ and $\alpha = (a, \exp_G(a)) \in \mathcal{G}_G$,*

$$\mathcal{V}_{e_i,\alpha} = \mathcal{V}_{e_j,\alpha}.$$

Proof. We may assume that $i = 1$ and $j = 2$. If α is a C-point then both $\mathcal{V}_{e_1,\alpha}$ and $\mathcal{V}_{e_s,\alpha}$ are $\{0\}$ and we are done and so we may assume that α is not a C-point. By definition and the above remark we have that $\phi_2 = \phi_{1,2} \circ \phi_1$ on $V_1 \cap V_2$ and that $\phi_{1,2}$ is an isomorphism. Also $\phi_{1,2}$ and e_1 are continuously differentiable and so we may apply Lemma 7.2.17. For $i = 1, 2$ we denote the tuple of the differential forms associated to e_i at a by $\omega_{e_i,a}$.

$$\begin{aligned}
\omega_{e_2,a} &= d(e_2(a)) - J_{e_2}(a)da \\
&= d(\phi_{1,2} \circ e_1(a)) - J_{\phi_{1,2} \circ e_1}(a)da \\
&= J_{\phi_{1,2}}(e_1(a))d(e_1(a)) - J_{\phi_{1,2}}(e_1(a))J_{e_1}(a)da \\
&= J_{\phi_{1,2}}(e_1(a))\left(d(e_1(a)) - J_{e_1}(a)da\right) \\
&= J_{\phi_{1,2}}(e_1(a))\omega_{e_1,a}
\end{aligned}$$

and so $\mathcal{V}_{e_2,\alpha} \subseteq \mathcal{V}_{e_1,\alpha}$. Similarly we get that $\mathcal{V}_{e_1,\alpha} \subseteq \mathcal{V}_{e_2,\alpha}$.

□

Before developing the theory of these spaces of differential forms further we prove a lemma, relating the spaces of differential forms and isomorphisms of abelian varieties. Here everything is defined over an algebraically closed subfield $C \subseteq \mathbb{C}$.

Lemma 7.2.19. *Let G, H be abelian varieties and let $f : G \to H$ be an isomorphism such that $f(\exp_G(a)) = \exp_H(b)$. Let $\tilde{f} : LG \to LH$ be the induced map and assume that $\tilde{f}(a) = b$. Also let $\alpha = (a, \exp_G(a))$ and $\beta = (b, \exp_H(b))$ be the corresponding points on the graphs of $\exp_G$ and $\exp_H$ respectively and assume that neither of these points are C-points. Then, $\mathcal{V}_{e_G,\alpha} = \mathcal{V}_{e_H,\beta}$.*

Proof. Let ϕ_G, ϕ_H be charts such that the functions e_G, e_H are defined at a, b respectively. Therefore,

$$\begin{aligned}
\omega_{e_H,b} &= d(e_H(b)) - J_{e_H}(b)db \\
&= d(\phi_H(\exp_H(b))) - J_{e_H}(b)db \\
&= d(\phi_H(f(\exp_G(a)))) - J_{e_H}(\tilde{f}(a))d\tilde{f}(a) \\
&= d(\phi_H \circ f \circ \phi_G^{-1}(e_G(a))) - J_{e_H}(\tilde{f}(a))J_{\tilde{f}}(a)da \\
&= J_{\phi_H \circ f \circ \phi_G^{-1}}(e_G(a))d(\exp_G(a)) - J_{e_H \circ \tilde{f}}(a)da,
\end{aligned}$$

where the last two lines follow from Lemma 7.2.17 applied to e_H, $\tilde{f}$ and $\phi_H \circ f \circ \phi_G^{-1}$. By Proposition 7.2.7, we have that $\exp_H \circ \tilde{f} = f \circ \exp_G$ and so

$$e_H \circ \tilde{f} = \phi_H \circ \exp_H \circ \tilde{f} = \phi_H \circ f \circ \exp_G = \phi_H \circ f \circ \phi_G^{-1} \circ e_G.$$

Therefore,

$$\begin{aligned}
\omega_{e_H,b} &= de_H(b) - J_{e_H}(b)db \\
&= J_{\phi_H \circ f \circ \phi_G^{-1}}(e_G(a))d(e_G(a)) - J_{\phi_H \circ f \circ \phi_G^{-1} \circ e_G}(a)da \\
&= J_{\phi_H \circ f \circ \phi_G^{-1}}(e_G(a))d(e_G(a)) - J_{\phi_H \circ f \circ \phi_G^{-1}}(e_G(a))J_{e_G}(a)da \\
&= J_{\phi_H \circ f \circ \phi_G^{-1}}(e_G(a))\omega_{e_G,a}
\end{aligned}$$

where the penultimate equality follows from applying Lemma 7.2.17 to $\phi_H \circ f \circ \phi_G^{-1} \circ e_G$. Therefore $\mathcal{V}_{e_H,\beta} \subseteq \mathcal{V}_{e_G,\alpha}$. Similarly we get the reverse containment and so $\mathcal{V}_{e_G,\alpha} = \mathcal{V}_{e_H,\beta}$ as required. □

Corollary 7.2.20. *Let* $(G; V_1, \ldots, V_m, \phi_1, \ldots, \phi_m)$ *and* $(G; V_1^*, \ldots, V_l^*, \phi_1^*, \ldots, \phi_r^*)$ *be abelian varieties of dimension* n *and assume that* id_G *is an isomorphism between these abelian varieties. Let* $\exp_G(a) \in V_i \cap V_j^*$ *for some* $i = 1, \ldots, m$ *and* $j = 1, \ldots, r$ *and let* $e_G = \phi_i \circ \exp_G$ *and* $e_G^* = \phi_j^* \circ \exp_G$ *and* $\alpha = (a, \exp_G(a)) \in \mathcal{G}_G$*. Then* $\mathcal{V}_{e_G,\alpha} = \mathcal{V}_{e_G^*,\alpha}$*.*

Proof. This follows immediately from the previous lemma, using the isomorphism id_G. □

Therefore the space of differential forms associated to the exponential map at a particular point does not depend on the choice of atlas.

Lemma 7.2.21. *Let G be an abelian variety in $\mathcal{S} = \mathcal{S}(\mathcal{B})$ of dimension n and let $\alpha = (a, \exp_G(a))$. We fix a chart ϕ such that the function $e_G = \phi \circ \exp_G$ is defined at a as discussed above. Let C be a subfield of the complex numbers containing $C_{\mathcal{B}}$. Then $\mathcal{V}_{e_G,\alpha} \subseteq W_{\mathcal{F}}(C(\alpha)/C)$.*

Proof. Let $e_G = \phi \circ \exp_G : \mathbb{C}^n \to \mathbb{C}^m$ for some $m \geq 1$ where ϕ is a chart such that e_G is defined at a. Throughout the rest of this proof we write $e = e_G$. By the definition of $W_{\mathcal{F}}(C(\alpha)/C)$ and the differential forms $\omega_{e_1,a}, \dots, \omega_{e_m,a}$ it suffices to show that $\omega_{e_i,a}$ are in $\Omega(C(\alpha)/C)$. Therefore we must show that $(\partial e_j/\partial x_i)(a)$ are in the field $C(\alpha)$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. From the end of Chapter 2 of [36] we have that G is isomorphic to a complex torus $T = \mathbb{C}^n/\Omega$ where Ω is a complex lattice in $\mathbb{C}^n$. The derivatives $\partial e_j/\partial x_i$ are meromorphic functions and are in the field of meromorphic functions on T, denoted F. By Theorem 28 in [36] the field F is finitely generated with transcendence degree at most n. Let $F = \mathbb{C}(f_1, \dots, f_r)$ for some integer $r \geq 1$ and so for each $j = 1, \dots, m$ and $i = 1, \dots, n$ there exist polynomials $P, Q \in \mathbb{C}[X_1, \dots, X_r]$ such that,

$$\frac{\partial e_j}{\partial x_i} = \frac{P(f_1, \dots, f_r)}{Q(f_1, \dots, f_r)}.$$

One can quantify over polynomials which have bounded degree as P and Q do. Therefore the structure $\mathbb{R}_{PR(\mathcal{F})}$ models the sentence

"There exist polynomials $P, Q \in \mathbb{C}[X_1, \dots, X_r]$ of degree at most s such that for all $\bar{x} = (x_1, \dots, x_n)$ in some fixed fundamental domain

$$\frac{\partial e_j}{\partial x_i}(\bar{x}) = \frac{P(f_1(\bar{x}), \dots, f_r(\bar{x}))}{Q(f_1(\bar{x}), \dots, f_r(\bar{x}))}.\text{"} \tag{7.1}$$

Let $\mathcal{M}$ be the definable closure of the empty set. Then $\mathcal{M}$ is an elementary substructure of $\mathbb{R}_{PR(\mathcal{F})}$ and also models the sentence (7.1). So as $C_{\mathcal{B}}$ is the algebraic closure of M we may identify $C_{\mathcal{B}}$ with M^2. Therefore we have that for all $j = 1, \dots, m$ and $i = 1, \dots, n$ there exist polynomials P and Q in $C_{\mathcal{B}}[X_1, \dots, X_r]$ such that $\partial e_j/\partial x_i = (P/Q)(f_1, \dots, f_r)$. Therefore as C contains $C_{\mathcal{B}}$ the expressions $(\partial e_j/\partial x_i)(a)$ are all contained in the field $C(\alpha)$ as required.

□

7.3 Applications of Ax-Schanuel for differential forms and abelian varieties

In this section we fix a set of abelian varieties $\mathcal{B}$ and its associated set of exponential maps as described in the previous section in Definitions 7.2.16 and 7.2.14 and also fix a $\mathrm{Dcl}_{\mathcal{F}}$ closed subfield $C \subseteq \mathbb{C}$. Observe that this subfield C contains the algebraically closed field $C_{\mathcal{B}}$. For an abelian variety G of dimension n we write LG for its Lie algebra as in the previous section. Finally, we write $TG = LG \times G$ and denote the graph of $\exp_G$ by $\mathcal{G}_G = \{(z, \exp_G(z)) : z \in LG\} \subseteq TG$.

Now we state an Ax-type result that shall be used to prove the first lemma in this section, Lemma 7.3.2. This lemma introduces the notion of $G^{\max}$. For any fields $B \supseteq C$ with $\mathrm{tr.deg}_C B$ finite, the abelian variety $G^{\max}_{\mathcal{S},B/C}$ is defined up to isogeny and encodes all the points on the graph of the exponential maps of all the abelian varieties in $\mathcal{S}$ that appear in B but not C.

Theorem 7.3.1. *Let G be an abelian variety of dimension n. Suppose that $\alpha \in \mathcal{G}_G$ is such that $\dim \mathcal{V}_{e_G,\alpha} < n$. Then there is a proper abelian subvariety H of G such that $\alpha \in \mathcal{G}_H + \mathcal{G}_G(C)$.*

Proof. This theorem follows from Proposition 3.7 in [22]. For $j = 1, \dots, m$ let

$$\omega_j(\alpha) = d((\exp_G)_j(a)) - \sum_{i=1}^{n} \frac{\partial(\exp_G)_j}{\partial x_i}(a)da_i$$

be the forms associated to $\exp_G$ at a. These are the differential forms on the graph $\mathcal{G}_G$ evaluated at the point a and are defined using the map $\exp_G : \mathbb{C}^n \to \mathbb{C}^m$ for some integer $m \geq 1$ as described in Remark 7.2.8. Now let $\Delta = \mathrm{Der}_{\mathcal{F}}(C(\alpha)/C)$ the set of all $\mathcal{F}$-derivations. As C is $\mathrm{Dcl}_{\mathcal{F}}$ closed in $\mathbb{C}$ it is also $\mathrm{Dcl}_{\mathcal{F}}$ closed in $C(\alpha)$. Therefore C is the intersection of all constant fields of the derivations in Δ. Hence Γ_G is the solution set to the differential equations $\bigcap_{D\in\Delta} D^*\omega(x) = 0$. Therefore we are in the setting of Proposition 3.7 in [22]. Finally we show that $\omega_1(\alpha), \dots, \omega_n(\alpha)$ are in $\mathcal{V}_{e_G,\alpha}$ and so are linearly dependent and we are done. Let ϕ be a chart such that $e_G = \phi \circ \exp_G$ is defined. We have that

$$\begin{aligned}\omega(\alpha) &= d(\exp_G(a)) - J_{\exp_G}(a)da \\ &= d(\phi^{-1} \circ e_G(a)) - J_{\phi^{-1}\circ e_G}(a)da \\ &= J_{\phi^{-1}}(a)\left(d(e_G(a)) - J_{e_G}(a)da\right) \\ &= J_{\phi^{-1}}(a)\omega_{e_G,a}\end{aligned}$$

by Lemma 7.2.17 as ϕ^{-1} is regular and so $\omega_1(\alpha), \ldots, \omega_n(\alpha)$ are in $\mathcal{V}_{e_G,\alpha}$ as required. □

Lemma 7.3.2. *Let $C \subseteq B$ be subfields of $\mathbb{C}$, where $\text{tr.deg}_C B$ is finite and C is $Dcl_{\mathcal{F}}$ closed. Then there is an abelian variety G in $\mathcal{S}$ of maximal dimension such that there is a point $\beta \in \mathcal{G}_G(B)$ not in a C-coset of $\mathcal{G}_H$ for any proper abelian subvariety H of G.*

Moreover for any $S \in \mathcal{S}$ and $\alpha \in \mathcal{G}_S(B)$ there is an algebraic group homomorphism $q : G \to S$ and non-zero integer N such that $N\alpha - Tq(\beta) \in \mathcal{G}_S(C)$.

Furthermore such a G is unique up to isogeny.

Proof. Suppose $G \in \mathcal{S}$ has dimension n say and is such that there is a point β in $\mathcal{G}_G(B)$ that is not in a C-coset of TH for any proper abelian subvariety H of G.

Then by Ax's theorem, Theorem 7.3.1, the space of forms $\mathcal{V}_{e_G,\beta}$ has dimension at least n in $\Omega(C(\beta)/C)$. By Theorem 16.14 in [16] we have that $\dim(\Omega(C(\beta/C))) = \text{tr.deg}_C C(\beta)$. Putting this all together gives that

$$n \leq \dim \mathcal{V}_{e_G,\beta} \leq \text{tr.deg}_C C(\beta) \leq \text{tr.deg}_C B.$$

Since the trivial group is in $\mathcal{S}$ and has such a point, the set of all such G is non-empty. In particular there is such an abelian variety G of maximal dimension as required, of dimension n say. Now assume that $\dim G = n$ is maximal.

Let $S \in \mathcal{S}$ be an abelian variety such that $\dim S \geq 1$ and let $\alpha \in \mathcal{G}_S(B)$. Then by the definition of $\mathcal{S}$ the abelian variety $G \times S$ is in $\mathcal{S}$ and we also have that $(\beta, \alpha) \in \mathcal{G}_{G\times S}(B)$. By the maximality of $\dim G$, there is a proper abelian subvariety J of $G \times S$, and $(\gamma_1, \gamma_2) \in \mathcal{G}_{G\times S}(C)$ such that $(\beta, \alpha) - (\gamma_1, \gamma_2) \in \mathcal{G}_J \subseteq \mathcal{G}_G \times \mathcal{G}_S$. Moreover, $\dim J \leq n$ again by maximality. We write $\beta' = \beta - \gamma_1$ and $\alpha' = \alpha - \gamma_2$. Then β' does not lie in a C-coset of $\mathcal{G}_H$ for some proper abelian subvariety H of G as if it did then so would β.

Let $p : J \to G$ be the projection map. Then $p(J) = G$ as β' is not in a C-coset of $\mathcal{G}_H$ for any proper abelian subvariety H of G. Therefore $\dim J = \dim G = n$ and so p is an isogeny. Hence there is another isogeny $p' : G \to J$ such that the composition $p' \circ p : J \to J$ is the multiplication by N map on J, denoted $[N]$, for some non-zero integer $N \geq 1$. The graphs $\mathcal{G}_G$ and $\mathcal{G}_J$ are divisible and torsion free and so the restrictions of Tp and Tp' respectively to these graphs are mutually inverse modulo the integer N.

Let $q' : J \to S$ be the projection and let $q = q' \circ p'$. Then $Tq(\beta') = N\alpha'$ and so $N\alpha - Tq(\beta) = N\gamma_2 - Tq(\gamma_1)$ and therefore $N\alpha - Tq(\beta) \in G_S(C)$ as required.

For the uniqueness of G suppose $\dim S = \dim G = n$ and that α is not in a C-coset of $\mathcal{G}_K$ for any abelian subvariety K of S. By a repetition of the above argument we have the functions p' and q' as above. Then α' does not lie in a C-coset of $\mathcal{G}_{K'}$ for any abelian subvariety K' of S and so q' is surjective and as p' is surjective we have that the function $q = q' \circ p'$ is surjective. Then the map $q : G \to S$ is in fact an isogeny and G is unique up to isogeny as required. □

Definition 7.3.3. If G and β are as in Lemma 7.3.2 then β is said to be an $\mathcal{S}$-*basis* for B/C and we write G as $G^{\max}_{\mathcal{S},B/C}$. If the set of abelian varieties $\mathcal{S}$ is clear, it is dropped from the notation. We also note that if G has dimension n then $n \leq \dim \mathcal{V}_{e_G,\beta}$.

Lemma 7.3.4. *Let $G = G^{\max}_{B/C}$ and let β be an $\mathcal{S}$-basis. Let H be an abelian variety in $\mathcal{S}$ and let $\alpha \in \mathcal{G}_H(B)$. Then $\mathcal{V}_{e_H,\alpha} \subseteq \mathcal{V}_{e_G,\beta}$.*

Proof. By Lemma 7.3.2 there is an algebraic homomorphism $q : G \to H$ such that $N\alpha - Tq(\beta) \in \mathcal{G}_H(C)$ for some integer $N \geq 1$. In other words there is some $\gamma = (c, \exp_H(c)) \in \mathcal{G}_H(C)$ such that $N\alpha = Tq(\beta) + \gamma$ and

$$\begin{aligned}(Na, \exp_H(Na)) &= \Big(\tilde{q}(b), q(\exp_G(b))\Big) + \Big(c, \exp_H(c)\Big) \\ &= \Big(\tilde{q}(b) + c, q(\exp_G(b)) \oplus \exp_H(c)\Big).\end{aligned}$$

Here $\oplus$ denotes the group law on the abelian variety H. Also $\tilde{q}$ denotes the induced map on LG and the addition for points on the graph is the group law on the tangent bundle. Define $f : H \to H$ by $f(z) = z \oplus \exp_H(c)$ and $\tilde{f} : LH \to LH$ by $\tilde{f}(z) = z + c$. Also define $g : H \to H$ by $g(z) = z \oplus \exp_H(-(N-1)a)$ and $\tilde{g} : LH \to LH$ by $\tilde{g}(z) = z - (N-1)a$. The spaces of differential forms for the functions e_G, e_H at b, a respectively can be defined in the usual way. Let ϕ_G, ϕ_H

be charts so that $e_G = \phi_G \circ \exp_G$ and $e_H = \phi_H \circ \exp_H$ are defined at b and a respectively. We wish to show that $\omega_{(e_H)_k,a} \in \mathcal{V}_{e_G,\beta}$ for all $k = 1, \dots, m_H$. This is done by considering the vector given by the forms $\omega_{(e_H)_1,a}, \dots, \omega_{(e_H)_{m_H},a}$ as in Claim 7.2.18, we denote this vector by $\omega_{e_H,a}$. We have,

$$\begin{aligned}
\omega_{e_H,a} &= d(e_H(a)) - J_{e_H}(a)da \\
&= d(\phi_H(\exp_H(a))) - J_{e_H}(a)da \\
&= d(\phi_H(g(\exp_H(Na)))) - J_{e_H}(\tilde{g}(Na))d(\tilde{g}(Na)) \\
&= d(\phi_H(g(q(\exp_G(b)) \oplus \exp_H(c)))) - J_{e_H}(\tilde{g}(\tilde{q}(b)+c))d(\tilde{g}(\tilde{q}(b)+c)), \quad (7.2)
\end{aligned}$$

where the last two lines of this expression follow from substitution. Considering the left hand term in (7.2) gives that

$$\begin{aligned}
d(\phi_H(g(q(\exp_G(b))) \oplus \exp_H(c))) &= d(\phi_H \circ g \circ f \circ q(\exp_G(b))) \\
&= d(\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}(e_G(b))) \\
&= J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}}(e_G(b))d(e_G(b)),
\end{aligned}$$

by Lemma 7.2.17 as the map $\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}$ is regular. Considering the right hand term in (7.2) gives that

$$\begin{aligned}
J_{e_H}(\tilde{g}(\tilde{q}(b)+c))d(\tilde{g}(\tilde{q}(b)+c)) &= J_{e_H}\left((\tilde{g}(\tilde{f}(\tilde{q}(b)))\right) d\left(\tilde{g}(\tilde{f}(\tilde{q}(b)))\right) \\
&= J_{e_H}\left(\tilde{g} \circ \tilde{f} \circ \tilde{q}(b)\right) J_{\tilde{g}\circ\tilde{f}\circ\tilde{q}}(b)db \\
&= J_{e_H \circ \tilde{g}\circ\tilde{f}\circ\tilde{q}}(b)db.
\end{aligned}$$

From Proposition 7.2.7 we have that $\exp_H \circ \tilde{q} = q \circ \exp_G$. Let $z \in \mathbb{C}^r$. Then,

$$\begin{aligned}\exp_H \circ \tilde{g} \circ \tilde{f} \circ \tilde{q}(z) &= \exp_H(\tilde{g}(\tilde{q}(z) + c)) \\ &= \exp_H(\tilde{q}(z) + c - (N-1)a) \\ &= \exp_H(\tilde{q}(z)) \oplus \exp_H(c) \oplus \exp_H(-(N-1)a) \\ &= q(\exp_G(z)) \oplus \exp_H(c) \oplus \exp_H(-(N-1)a) \\ &= f(q(\exp_G(z))) \oplus \exp_H(-(N-1)a) \\ &= g \circ f \circ q \circ \exp_G(z)\end{aligned}$$

and so $\exp_H \circ \tilde{g} \circ \tilde{f} \circ \tilde{q} = g \circ f \circ q \circ \exp_G$. Therefore we have that,

$$\begin{aligned}e_H \circ \tilde{g} \circ \tilde{f} \circ \tilde{q} = \phi_H \circ \exp_H \circ \tilde{g} \circ \tilde{f} \circ \tilde{q} &= \phi_H \circ g \circ f \circ q \circ \exp_G \\ &= \phi_H \circ g \circ f \circ q \circ \phi_G^{-1} \circ \phi_G \circ \exp_G \\ &= \phi_H \circ g \circ f \circ q \circ \phi_G^{-1} \circ e_G.\end{aligned}$$

Therefore,

$$\begin{aligned}J_{e_H \circ \tilde{g} \circ \tilde{f} \circ \tilde{q}}(b) &= J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1} \circ e_G}(b) \\ &= J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}}(e_G(b)) J_{e_G}(b)\end{aligned}$$

by Lemma 7.2.17. Putting this all together gives that,

$$\begin{aligned}\omega_{e_H,a} &= J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}}(e_G(b)) d(e_G(b)) - J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}}(e_G(b)) J_{e_G}(b) db \\ &= J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}}(e_G(b)) \Big(d(e_G(b)) - J_{e_G}(b) db \Big) \\ &= J_{\phi_H \circ g \circ f \circ q \circ \phi_G^{-1}}(e_G(b)) \omega_{e_G,b}.\end{aligned}$$

Hence $\mathcal{V}_{e_H,\alpha} \subseteq \mathcal{V}_{e_G,\beta}$, as required. □

Corollary 7.3.5. *Let $G = G^{\max}(B/C)$ be of dimension n and let $\beta \in \mathcal{G}_G$ be an $\mathcal{S}$-basis. Then $\mathcal{V}_{e_G,\beta} = W_{\mathcal{F}}(B/C)$.*

Proof. By Lemma 7.2.21 we have that $\mathcal{V}_{e_G,\beta} \subseteq W_{\mathcal{F}}(C(\beta)/C) \subseteq W_{\mathcal{F}}(B/C)$. Let

$\eta \in W_{\mathcal{F}}(B/C)$. By definition $\eta \in \mathcal{V}_{e_H,\alpha}$ for some abelian variety $H \in \mathcal{S}$ of dimension r and $\alpha \in \mathcal{G}_H(B)$. By Lemma 7.3.2 there is an algebraic homomorphism $q : G \to H$ such that $N\alpha - Tq(\beta) \in \mathcal{G}_H(C)$ for some integer $N \geq 1$. By Lemma 7.3.4 we have that $\mathcal{V}_{e_H,\alpha} \subseteq \mathcal{V}_{e_G,\beta}$ and so $\eta \in \mathcal{V}_{e_G,\beta}$ and $W_{\mathcal{F}}(B/C) \subseteq \mathcal{V}_{e_G,\beta}$ as required. □

In order to define this $G^{\max}$ the fact that our base field C is $\mathrm{Dcl}_{\mathcal{F}}$ closed is crucial as this allows us to use the Ax statement, Theorem 7.3.1. However this reliance on C being $\mathrm{Dcl}_{\mathcal{F}}$ closed does cause a problem. Consider for example the tower of fields $C \subseteq A \subseteq B \subseteq \mathbb{C}$ and suppose that C is $\mathrm{Dcl}_{\mathcal{F}}$ closed. Then we may define the abelian varieties $G = G^{\max}_{B/C}$ and $S = G^{\max}_{A/C}$ and define and compare their group rank. Using this construction we cannot define $G^{\max}_{B/A}$, as A may not be $\mathrm{Dcl}_{\mathcal{F}}$ closed. However by using Poincaré's reducibility theorem we can in fact define $G^{\max}_{B/A}$ in this case.

Lemma 7.3.6. *Let $C \subseteq A \subseteq B \subseteq \mathbb{C}$ where C is $\mathrm{Dcl}_{\mathcal{F}}$ closed and suppose that $\mathrm{tr.deg}_C B$ is finite. Let $G = G^{\max}_{B/C}$ and $S = G^{\max}_{A/C}$. Then there is an abelian variety $S' \in \mathcal{S}$ such that there is no abelian variety H in $\mathcal{S}$ with dimension greater than $\dim S'$ such that there is a point $\gamma \in \mathcal{G}_H(B)$ not in an A-coset of $\mathcal{G}_K$ for any proper subvariety K of H. Let $\beta \in \mathcal{G}_G(B)$ and $\alpha \in \mathcal{G}_S(A)$ be $\mathcal{S}$-bases for B/C and A/C respectively. Then in fact $G = S \times S'$ and $\beta = (\alpha, \alpha')$ where $\alpha' \in \mathcal{G}_{S'}(B)$ is not in an A-coset of $\mathcal{G}_{H'}$ for any proper subvariety H' of S'.*

Also for any $S'' \in \mathcal{S}$ and $\epsilon \in G_{S''}(B)$ there is an algebraic homomorphism $q : S' \to S''$ and non-zero integer N such that $N\epsilon - Tq(\alpha') \in G_{S''}(A)$. In particular the abelian variety S' is unique up to isogeny.

The abelian variety S' therefore satisfies all the properties of $G^{\max}$ given in Lemma 7.3.2 and so S' is defined to be $G^{\max}_{B/A}$. Recall that as the abelian varieties G, S and S' are all in the set $\mathcal{S}$ they are therefore all defined over C.

Proof. Write $n = \dim G, m = \dim S$ and $r = n - m$. As $\alpha \in \mathcal{G}_S(B)$ we have by Lemma 7.3.4 that $\mathcal{V}_{e_S,\alpha} \subseteq \mathcal{V}_{e_G,\beta}$ and so $\mathrm{grk}_{\mathcal{F}}(A/C) \leq \mathrm{grk}_{\mathcal{F}}(B/C)$. As α is an $\mathcal{S}$-basis it is not in a C-coset of H for any proper subvariety H of S. Therefore by a repetition of the argument in the proof of Lemma 7.3.2 the algebraic homomorphism $q : G \to S$ is surjective and so the abelian variety S is a quotient of G. As S is defined up to isogeny and as up to isogeny quotients of abelian varieties are abelian subvarieties we can assume that S is an abelian subvariety of G. If

$G = S$ we are done and so we may assume that S is a proper abelian subvariety of G. By the Poincaré Reducibility Theorem, Theorem 7.2.11, applied over the subfield C, there is a complementary abelian subvariety S' with $\dim S' = r$ such that G is isogenous to $S \times S'$. By the definition of the set $\mathcal{S}$, this S' is also in $\mathcal{S}$. We may write $\beta = (\alpha, \alpha')$ for some $\alpha' \in \mathcal{G}_{S'}$. Now we show that S' is $G^{\max}_{B/A}$, in the sense of the definition of $G^{\max}$ given in the statement. This requires two steps, the first of which is to show that there is no subvariety H of G with $\dim H > r$ such that there is a $\gamma \in \mathcal{G}_H(B)$ not in an A-coset of $\mathcal{G}_K$ for any subvariety K of H. Then it is shown that the point α' is not in an A-coset of $\mathcal{G}_{H'}$ for any proper subvariety H' of S'.

Firstly suppose that there is an abelian variety $H \in \mathcal{S}$, with $\dim H > r$ such that there is a point $\gamma \in \mathcal{G}_H(B)$ not in an A-coset of $\mathcal{G}_K$ for any proper abelian subvariety K of H. Consider $(\alpha, \gamma) \in \mathcal{G}_{S\times H}(B)$. By the definition of α and γ there is no proper abelian subvariety H' of $S \times H$ such that $(\alpha + c, \gamma + c') \in \mathcal{G}_{H'}(B)$, where $(c, c') \in \mathcal{G}_{S\times H}(C)$. Hence by Lemma 7.3.2 there is a map $q : G \to S \times H$ such that $N(\alpha, \gamma) = Tq(\beta)$ for some non-zero integer N. As γ is not in an A-coset we have that $p_H(S \times H) = H$ and as α is not in a C-coset we have that $p_S(S \times H) = S$. Therefore $q_S = p_S \circ q$ and $q_H = p_H \circ q$ are surjective and so $q = (q_S, q_H)$ is surjective. So

$$\dim(S \times H) \leq \dim G = m + r < \dim S + \dim H = \dim(S \times H),$$

a contradiction.

Now suppose that α' is in an A-coset of $\mathcal{G}_{H'}$ for some proper subvariety H' of S'. Then there is $a' \in \mathcal{G}_{S'}(A)$ such that $\alpha' + a' \in \mathcal{G}_{H'}(B)$. As H' is a proper subvariety of S', by the Poincaré Reducibility Theorem there is a complementary abelian subvariety $H'' \in \mathcal{S}$ such that S' is isogenous to $H' \times H''$. So we may assume that $S' = H' \times H''$. Let $a' = (\gamma', \gamma'')$ and $\alpha' = (\delta', \delta'')$. As $\alpha' + a' = (\delta' + \gamma', \delta'' + \gamma'') \in \mathcal{G}_{H'}(B)$ we get that $\delta'' + \gamma'' = 0$. Therefore as γ'' is an A point so is δ''. By the maximality of S the point $(\alpha, \delta'') \in G_{S\times H''}(A)$ is in a C-coset. Therefore $\beta = (\alpha, \delta', \delta'')$ is in a C-coset, a contradiction.

Using the maximality of the abelian variety S' a simple reproduction of the argument at the end of the proof of Lemma 7.3.2 gives the rest of the statement.

□

Now we consider $W_{\mathcal{F}}(B/A)$ and give a version of Corollary 7.3.5 for $W_{\mathcal{F}}(B/A)$.

Remark 7.3.7. Let $C \subseteq A \subseteq B \subseteq \mathbb{C}$ be a tower of subfields where C is $\mathrm{Dcl}_{\mathcal{F}}$ closed. Define $W_{\mathcal{F}}(B/A)$ in the usual way as in Definition 7.1.8. Then we have that $W_{\mathcal{F}}(B/A) \simeq W_{\mathcal{F}}(B/C)/W_{\mathcal{F}}(A/C)$.

Lemma 7.3.8. *Let G and H be abelian varieties with points $\alpha = (a, \exp_G(a)) \in \mathcal{G}_G(B)$ and $\beta = (b, \exp_H(b)) \in \mathcal{G}_H(B)$. Then $\mathcal{V}_{e_{G\times H},(\alpha,\beta)} = \mathcal{V}_{e_G,\alpha} \oplus \mathcal{V}_{e_H,\beta}$.*

Proof. Consider the varieties G and H and let $\alpha = (a, \exp_G(a))$ and $\beta = (b, \exp_H(b))$. By definition these are equipped with covers $V_1, \ldots, V_{m_G}$ and $V_1^*, \ldots, V_{m_H}^*$ and atlases of charts $f_1, \ldots, f_{m_G}$ and $f_1^* \ldots, f_{m_H}^*$ respectively. Then $G \times H$ has a cover $V_i \times V_j^*$ and atlas of charts (f_i, f_j^*) for $i = 1, \ldots, m_G$ and $j = 1, \ldots, m_H$ respectively. In particular $e_{G\times H} = (e_G, e_H)$.

Therefore differentiating $(e_{G\times H})_l$ with respect to x_i for $l = 1, \ldots, m_G$ and $i = m_G + 1, \ldots, m_G + m_H$ gives zero as does differentiating $(e_{G\times H})_l$ with respect to x_i for $l = m_G + 1, \ldots, m_G + m_H$ and $i = 1, \ldots, m_G$. Hence for $l = 1, \ldots, m_G$,

$$\omega_{(e_{G\times H})_l,(a,b)} = d((e_G)_l(a)) - \sum_{i=1}^{m_G} \frac{\partial (e_G)_l}{\partial x_i}(a) da_i = \omega_{(e_G)_l,a}$$

and for $l = m_G + 1, \ldots, m_G + m_H$

$$\omega_{(e_{G\times H})_l,(a,b)} = d((e_H)_l(b)) - \sum_{i=1}^{m_H} \frac{\partial (e_H)_l}{\partial x_i}(b) db_i = \omega_{(e_H)_l,b}$$

where $k + m_G = l$ for $k = 1, \ldots, m_H$. Therefore the differential forms for $e_{G\times H}$ at (a, b) are simply the ones for e_G at a and together with the ones for e_H at b respectively. Hence $\mathcal{V}_{e_{G\times H},(\alpha,\beta)} = \mathcal{V}_{e_G,\alpha} \oplus \mathcal{V}_{e_H,\beta}$, as required. □

Lemma 7.3.9. *Let $C \subseteq A \subseteq B \subseteq \mathbb{C}$ be a tower of subfields where C is $\mathrm{Dcl}_{\mathcal{F}}$ closed and $\mathrm{tr.deg}_C B$ is finite. Let $G = G^{\max}_{B/C}, S = G^{\max}_{A/C}$ and $S' = G^{\max}_{B/A}$ and let β, α and α' be $\mathcal{S}$-bases for $B/C, A/C$ and B/A respectively. Then $W_{\mathcal{F}}(B/A)$ is isomorphic to $\mathcal{V}_{e_{S'},\alpha'}$.*

Proof. By Corollary 7.3.5, $W_{\mathcal{F}}(B/C) = \mathcal{V}_{e_G,\beta}$ and $W_{\mathcal{F}}(A/C) = \mathcal{V}_{e_S,\alpha}$. We may take G to be $S \times S'$ and β to be (α, α'). By Lemma 7.3.8, we have that $\mathcal{V}_{e_{S\times S'},(\alpha,\alpha')} = \mathcal{V}_{e_S,\alpha} \oplus \mathcal{V}_{e_{S'},\alpha'}$ and so

$$W_{\mathcal{F}}(B/A) \simeq W_{\mathcal{F}}(B/C)/W_{\mathcal{F}}(A/C) = \mathcal{V}_{e_G,\beta}/\mathcal{V}_{e_S,\alpha} \simeq \mathcal{V}_{e_{S'},\alpha'}$$

as required. □

Definition 7.3.10. Let $C \subseteq B \subseteq \mathbb{C}$ be subfields and C be $\mathrm{Dcl}_{\mathcal{F}}$ closed. Then the *group rank* $\mathrm{grk}_{\mathcal{F}}(B/C)$ is defined to be the dimension of $\mathcal{V}_{e_G,\beta}$. The *predimension* of B over C is defined to be

$$\delta_{\mathcal{F}}(B/C) := \mathrm{tr.deg}_C B - \mathrm{grk}_{\mathcal{F}}(B/C).$$

Remark 7.3.11. Therefore in particular if $G = G^{\max}(B/C)$ and $\beta \in \mathcal{G}_G$ is an $\mathcal{S}$-basis we have that $\dim W_{\mathcal{F}}(B/C) = \mathrm{grk}_{\mathcal{F}}(B/C)$. Consider a tower of subfields $C \subseteq A \subseteq B \subseteq \mathbb{C}$ where C is $\mathrm{Dcl}_{\mathcal{F}}$-closed and $\mathrm{tr.deg}_C B$ is finite. Then we may define $G = G^{\max}_{B/C}$ and $S = G^{\max}_{A/C}$ and $\mathcal{S}$-bases $\beta \in \mathcal{G}_G(B)$ and $\alpha \in \mathcal{G}_S(A)$ for B/C and A/C respectively. As $\alpha \in \mathcal{G}_S(B)$ Lemma 7.3.4 gives that $\mathcal{V}_{e_S,\alpha} \subseteq \mathcal{V}_{e_G,\beta}$ and by the definition of the group rank we have that $\mathrm{grk}_{\mathcal{F}}(A/C) \leq \mathrm{grk}_{\mathcal{F}}(B/C)$.

Now we show that the group rank is upper semi-modular and the predimension is submodular on the lattice of algebraically closed fields.

Lemma 7.3.12. *Let $B_1, B_2 \subseteq \mathbb{C}$ be algebraically closed field extensions of C with finite transcendence degree over C. Let $B_0 = B_1 \cap B_2$ and $B_3 = \overline{(B_1 \cup B_2)}^{alg}$. Then,*

$$grk_{\mathcal{F}}(B_0/C) + grk_{\mathcal{F}}(B_3/C) \geq grk_{\mathcal{F}}(B_1/C) + grk_{\mathcal{F}}(B_2/C)$$

and

$$\delta_{\mathcal{F}}(B_0/C) + \delta_{\mathcal{F}}(B_3/C) \leq \delta_{\mathcal{F}}(B_1/C) + \delta_{\mathcal{F}}(B_2/C)$$

Proof. As transcendence degree is submodular we need only show that the group rank is upper semi-modular and then apply the definition of the predimension to complete the proof. For $i = 1, 2, 3, 4$ let $G_i = G^{\max}_{B_i/C}$ and let β_i be an $\mathcal{S}$-basis for B_i/C for all $i = 1, 2, 3, 4$. Consider $(\beta_1, \beta_2) \in \mathcal{G}_{G_1 \times G_2}$. Clearly $(\beta_1, \beta_2) \in \mathcal{G}_{G_1 \times G_2}(B_3)$. By Lemma 7.3.4,

$$\dim \mathcal{V}_{e_{G_1 \times G_2},(\beta_1,\beta_2)} \leq \dim \mathcal{V}_{e_{G_3},\beta_3}.$$

By Lemma 7.3.8 we have that

$$\dim \mathcal{V}_{e_{G_1},\beta_1} + \dim \mathcal{V}_{e_{G_2},\beta_2} = \dim \mathcal{V}_{e_{G_1 \times G_2},(\beta_1,\beta_2)}.$$

Therefore,

$$\begin{aligned}\dim \mathcal{V}_{e_{G_1},\beta_1} + \dim \mathcal{V}_{e_{G_2},\beta_2} &\leq \dim \mathcal{V}_{e_{G_3},\beta_3} \\ &\leq \dim \mathcal{V}_{e_{G_3},\beta_3} + \dim \mathcal{V}_{e_{G_0},\beta_0}\end{aligned}$$

as required. □

Remark 7.3.13. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be finite sets of pairwise non-isogenous simple abelian varieties and suppose that $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Then as in Definition 7.2.13 we can construct sets of basic abelian varieties $\mathcal{B}_1$ and $\mathcal{B}_2$ and similarly we may use Definition 7.2.14 to construct sets of abelian varieties $\mathcal{S}_1$ and $\mathcal{S}_2$. Each of these sets has an associated set of exponential maps $\mathcal{F}_1$ and $\mathcal{F}_2$ as in Definition 7.2.16. It is clear from this construction that $\mathcal{B}_1 \subseteq \mathcal{B}_2, \mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Let $G_1 = G^{\max}_{\mathcal{S}_1,B/C}$ and $G_2 = G^{\max}_{\mathcal{S}_2,B/C}$ and also let β_1 be an $\mathcal{S}_1$-basis for B/C and β_2 be an $\mathcal{S}_2$ basis for B/C respectively. By Lemma 7.3.2 there is an algebraic homomorphism $q : G_2 \to G_1$ such that $N\beta_1 - Tq(\beta_2) \in \mathcal{G}_{G_1}(C)$ for some integer $N \geq 1$ and as β_1 does not lie in a C-coset of $\mathcal{G}_H$ for any abelian subvariety of G_1 we have that q is surjective and therefore G_1 is a quotient of G_2. This quotient is isogenous to an abelian subvariety and as $G^{\max}$ is defined up to isogeny we may take G_1 to be an abelian subvariety of G_2. Also as $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and β_1 is a B-point then by Lemma 7.3.4 we have that $\mathcal{V}_{e_{G_1},\beta_1} \subseteq \mathcal{V}_{e_{G_2},\beta_2}$ and in particular $\mathrm{grk}_{\mathcal{F}_1}(B/C) \leq \mathrm{grk}_{\mathcal{F}_2}(B/C)$. However it is not possible in general to compare the predimensions $\delta_{\mathcal{F}_1}(B/C)$ and $\delta_{\mathcal{F}_2}(B/C)$. Similarly for a chain of subfields $C \subseteq A \subseteq B$ we know that $\mathrm{grk}_{\mathcal{F}}(A/C) \leq \mathrm{grk}_{\mathcal{F}}(B/C)$ and $\mathrm{tr.deg}_C A \leq \mathrm{tr.deg}_C B$. However it is also not possible to compare the predimensions $\delta_{\mathcal{F}}(A/C)$ and $\delta_{\mathcal{F}}(B/C)$ in general.

We now consider extensions $A \supseteq C$ where these predimensions can be compared, for any extension B of the subfield A. Such extensions A are called *self-sufficient*.

Definition 7.3.14. Let A be as above. Then A is said to be *self-sufficient* if for every $B \supseteq A$ where $B \subseteq \mathbb{C}$ and $\mathrm{tr.deg}_C B$ is finite we have that $\delta_{\mathcal{F}}(A/C) \leq \delta_{\mathcal{F}}(B/C)$.

Now we give an alternative condition for self-sufficiency for algebraically closed A. This uses the submodularity of the predimension.

Lemma 7.3.15. *Let $C \subseteq A \subseteq \mathbb{C}$, where C is $Dcl_{\mathcal{F}}$ closed and $\text{tr.deg}_C A$ is finite. Also assume that A is algebraically closed. Then A is self-sufficient if and only if for all $C \subseteq X \subseteq \mathbb{C}$, with X algebraically closed and $\text{tr.deg}_C X$ is finite, we have that $\delta_{\mathcal{F}}(X \cap A) \leq \delta_{\mathcal{F}}(X)$.*

Proof. Firstly we suppose that A is self-sufficient and let X be as given. By the self-sufficiency of A we have that $\delta_{\mathcal{F}}(A) \leq \delta_{\mathcal{F}}(\overline{(X \cup A)}^{alg})$. Also by the submodularity of the predimension, (Lemma 7.3.12)

$$\delta_{\mathcal{F}}(X \cap A) + \delta_{\mathcal{F}}(\overline{(X \cup A)}^{alg}) \leq \delta_{\mathcal{F}}(A) + \delta_{\mathcal{F}}(X).$$

Putting these together gives that

$$\delta_{\mathcal{F}}(X) - \delta_{\mathcal{F}}(X \cap A) \geq \delta_{\mathcal{F}}(\overline{(X \cup A)}^{alg}) - \delta_{\mathcal{F}}(A) \geq 0$$

and so $\delta_{\mathcal{F}}(X \cap A) \leq \delta_{\mathcal{F}}(X)$ as required.

For the converse suppose that $\delta_{\mathcal{F}}(X \cap A) \leq \delta_{\mathcal{F}}(X)$ for all algebraically closed X and let $A \subseteq Y$ for some $Y \subseteq \mathbb{C}$. Let $X = \text{acl}(Y)$. Then $X \cap A = A$ and so $\delta_{\mathcal{F}}(A) \leq \delta_{\mathcal{F}}(X)$ by our assumption. Also by definition we have that $\text{tr.deg}_C Y = \text{tr.deg}_C X$ and $\text{grk}_{\mathcal{F}}(X/C) \geq \text{grk}_{\mathcal{F}}(Y/C)$ and therefore $\delta_{\mathcal{F}}(X) \leq \delta_{\mathcal{F}}(Y)$ as required. □

Lemma 7.3.16. *Let $C \subseteq \mathbb{C}$ be $Dcl_{\mathcal{F}}$-closed and I be a non-empty index set. For all i in I suppose that B_i are subfields of $\mathbb{C}$ that contain C and have finite transcendence degree over C and are also algebraically closed and self sufficient. Then $\bigcap_{i \in I} B_i$ is self-sufficient.*

Proof. This intersection is also algebraically closed and so we use the previous lemma. Let $X \subseteq \mathbb{C}$ be algebraically closed and suppose that $\text{tr.deg}_C X$ is finite. Every time a B_i is added to the intersection, either the transcendence degree goes down or the intersection does not change. Therefore we have a finite subset $I_0 \subseteq I$ such that $X \cap \bigcap_{i \in I} B_i = X \cap \bigcap_{i \in I_0} B_i$. Hence we can consider finite intersections and so binary intersections $B_1 \cap B_2$ say. By the self-sufficiency of B_1 and B_2 and Lemma 7.3.15 we get that

$$\delta_{\mathcal{F}}(X \cap (B_1 \cap B_2)) = \delta_{\mathcal{F}}((X \cap B_1) \cap B_2) \leq \delta_{\mathcal{F}}(X \cap B_1) \leq \delta_{\mathcal{F}}(X).$$

□

Definition 7.3.17. Let $C \subseteq A \subseteq \mathbb{C}$ with $\mathrm{tr.deg}_C A$ is finite. The *hull* of A, denoted $\lceil A \rceil_{\mathcal{F}}$, is the intersection of all algebraically closed and self-sufficient $B \subseteq \mathbb{C}$ such that $A \subseteq B$ and $\mathrm{tr.deg}_C B$ is finite.

Lemma 7.3.18. *Let $C \subseteq A \subseteq \mathbb{C}$ with $\mathrm{tr.deg}_C A$ is finite. Then $\lceil A \rceil_{\mathcal{F}}$ has finite transcendence degree over C.*

Proof. Let B vary over all algebraically closed extensions of A that are of finite transcendence degree. Then the predimension $\delta_{\mathcal{F}}(B/C)$ takes values in $\mathbb{N}$ and therefore there is a least such value for this predimension. In particular the extension B where $\delta_{\mathcal{F}}(B/C)$ takes this smallest value is self-sufficient by definition. Therefore self-sufficient extensions exist and we may take the intersection of all the algebraically closed self-sufficient extensions which has finite transcendence degree as required. $\square$

Lemma 7.3.19. *Let $C \subseteq B \subseteq \mathbb{C}$ be fields where C is $Dcl_{\mathcal{F}}$ closed and $\mathrm{tr.deg}_C B$ is finite. Then*

$$\delta_{\mathcal{F}}(B/C) \geq \dim_{\mathcal{F}}(B/C).$$

Proof. From Corollary 7.3.5 we know that $\mathrm{grk}_{\mathcal{F}}(B/C) = \dim W_{\mathcal{F}}(B/C)$. So we have that,

$$\begin{aligned} \delta_{\mathcal{F}}(B/C) &= \mathrm{tr.deg}_C B - \mathrm{grk}_{\mathcal{F}}(B/C) \\ &= \dim \Omega(B/C) - \dim W_{\mathcal{F}}(B/C) \\ &= \dim \Omega_{\mathcal{F}}(B/C) \end{aligned}$$

by the definition of $\Omega_{\mathcal{F}}(B/C)$, and Lemma 7.1.12 we have that

$$\dim \Omega_{\mathcal{F}}(B/C) \geq \dim_{\mathcal{F}}(B/C).$$

$\square$

Now consider the fields $C \subseteq A \subseteq \mathbb{C}$ where C is $\mathrm{Dcl}_{\mathcal{F}}$ closed and $\mathrm{tr.deg}_C A$ is finite. Let $B = \lceil A \rceil_{\mathcal{F}}$. Then as $\dim_{\mathcal{F}}(B/C) \geq \dim_{\mathcal{F}}(A/C)$ we can apply the previous lemma to get that $\delta_{\mathcal{F}}(B/C) \geq \dim_{\mathcal{F}}(A/C)$. Now we would like to show the reverse inequality. This is done by showing that every derivation in $\mathrm{Der}_{\mathcal{F}}(B/C)$ may be extended to one in $\mathrm{Der}_{\mathcal{F}}(\mathbb{C}/C)$.

Consider two extensions $A \subseteq B$ of some $\mathrm{Dcl}_{\mathcal{F}}$ closed subfield C of $\mathbb{C}$. Also suppose that $\mathrm{tr.deg}_C B$ is finite and A is self-sufficient over C. The difference $\mathrm{tr.deg}_C B - \mathrm{tr.deg}_C A$ is also finite. As A is self-sufficient the difference $\delta_{\mathcal{F}}(B/C) - \delta_{\mathcal{F}}(A/C)$ is at least zero. We shall now show that it is possible to reduce to the case where this difference is zero or one providing that B is also self sufficient over C and is algebraically closed. We also show that any $\mathcal{F}$-derivation on A can be extended to one on B in each of these cases. This involves an analogue of Lemma 6.4 and Propositions 6.5 and 6.6 in [21] and the argument is similar. Firstly we recall some details on $G^{\max}$ and the associated space of differential forms.

Remark 7.3.20. Let $C \subseteq \mathbb{C}$ be $\mathrm{Dcl}_{\mathcal{F}}$ closed and suppose that $C \subseteq A \subseteq B \subseteq \mathbb{C}$. Then there are abelian varieties in $\mathcal{S}$ associated to the field extensions $B/C, A/C$ and B/A. These are $G^{\max}_{B/C}, G^{\max}_{A/C}$ and $G^{\max}_{B/A}$ as defined in Lemmas 7.3.2 and 7.3.6 and throughout the rest of this section these are denoted G, S and S' respectively. By Lemmas 7.3.2 and 7.3.6 each of these abelian varieties is equipped with a certain point on the graph of its exponential map known as an $\mathcal{S}$-basis. Let β, α and α' be $\mathcal{S}$-bases of $B/C, A/C$ and B/A respectively. These $\mathcal{S}$-bases are points in $\mathcal{G}_G(B), \mathcal{G}_S(A)$ and $\mathcal{G}_{S'}(B)$ respectively.

Using the work in Section 7.2 one may define the space of differential forms associated to each of these $\mathcal{S}$-bases. The spaces of differential forms associated to the exponential maps of G, S and S' evaluated at β, α and α' respectively are denoted by $\mathcal{V}_{e_G,\beta}, \mathcal{V}_{e_S,\alpha}$ and $\mathcal{V}_{e_{S'},\alpha'}$ respectively. Finally the abelian variety G is isogenous to $S \times S'$ via an isogeny whose induced map from $\mathcal{G}_G$ to $\mathcal{G}_{S\times S'}$ sends β to (α, α'). As G is only defined up to isogeny we may assume that G is in fact $S \times S'$ and $\beta = (\alpha, \alpha')$.

Lemma 7.3.21. *Let $C \subseteq A \subseteq B \subseteq \mathbb{C}$ where C is $\mathrm{Dcl}_{\mathcal{F}}$ closed and both A and B are self-sufficient over C. Also suppose that B is algebraically closed with finite transcendence degree over C. Then there is an ordinal λ and a chain of subfields $(A_\alpha)_{\alpha\leq\lambda}$ such that $A_0 = A$ and $A_\lambda = B$ and*

1. *for each $\alpha \leq \lambda$ either $\delta_{\mathcal{F}}(A_{\alpha+1}/C) - \delta_{\mathcal{F}}(A_\alpha/C) = 1$ and $\mathrm{tr.deg}_{A_\alpha} A_{\alpha+1} = 1$ or $\delta_{\mathcal{F}}(A_{\alpha+1}/C) - \delta_{\mathcal{F}}(A_\alpha/C) = 0$.*
2. *If $0 \leq \alpha \leq \lambda$ then A_α is self-sufficient over C.*

Proof. Enumerate B as $(b_\alpha)_{\alpha<\lambda}$ for some limit ordinal λ and assume inductively that we have A_β for $\beta < \alpha$ satisfying both of the conditions in the statement and such that $b_\gamma \in A_\beta$ whenever $\gamma < \beta$. If α is a limit we take $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

Now we suppose that α is a successor say $\alpha = \gamma + 1$. If there is some field extension $F \subseteq B$ of A_γ that contains b_γ such that $\delta_{\mathcal{F}}(F/C) - \delta_{\mathcal{F}}(A_\gamma/C) = 0$ then we take A_α to be F. Otherwise take A_α to be the algebraic closure of $A_\gamma \cup \{b_\gamma\}$. As A_γ is self sufficient over C we have that $\delta_{\mathcal{F}}(A_\alpha/C) - \delta_{\mathcal{F}}(A_\gamma/C) \geq 0$ and as this difference is not zero we have that

$$\text{tr.deg}_{A_\gamma} A_\alpha - (\text{grk}_{\mathcal{F}}(A_\alpha/C) - \text{grk}_{\mathcal{F}}(A_\gamma/C)) \geq 1.$$

By Remark 7.3.11 we have that $\text{grk}_{\mathcal{F}}(A_\gamma/C) \leq \text{grk}_{\mathcal{F}}(A_\alpha/C)$ and so

$$0 \leq \text{grk}_{\mathcal{F}}(A_\alpha/C) - \text{grk}_{\mathcal{F}}(A_\gamma/C) + 1 \leq \text{tr.deg}_{A_\gamma} A_\alpha = 1.$$

Therefore $\text{grk}_{\mathcal{F}}(A_\alpha/C) - \text{grk}_{\mathcal{F}}(A_\gamma/C) = 0$ and

$$\delta_{\mathcal{F}}(A_\alpha/C) - \delta_{\mathcal{F}}(A_\gamma/C) = \text{tr.deg}_{A_\gamma} A_\alpha = 1$$

as required. It remains to be shown that A_α is self-sufficient over C. Suppose for a contradiction that there is some field extension K of A_α such that $\text{tr.deg}_C K$ is finite and $\delta_{\mathcal{F}}(K/C) < \delta_{\mathcal{F}}(A_\alpha/C)$. Suppose that $K \subseteq B$. As A_γ is self-sufficient over C we have that

$$\delta_{\mathcal{F}}(A_\gamma/C) \leq \delta_{\mathcal{F}}(K/C) < \delta_{\mathcal{F}}(A_\alpha/C)$$

and so

$$0 \leq \delta_{\mathcal{F}}(K/C) - \delta_{\mathcal{F}}(A_\gamma/C) < \delta_{\mathcal{F}}(A_\alpha/C) - \delta_{\mathcal{F}}(A_\gamma/C).$$

By construction we have that $\delta_{\mathcal{F}}(A_\alpha/C) - \delta_{\mathcal{F}}(A_\gamma/C)$ is 0 or 1. If $\delta_{\mathcal{F}}(A_\alpha/C) - \delta_{\mathcal{F}}(A_\gamma/C) = 0$ then we have that $0 < 0$, a contradiction. If $\delta_{\mathcal{F}}(A_\alpha/C) - \delta_{\mathcal{F}}(A_\gamma/C) = 1$ then $\delta_{\mathcal{F}}(K/C) - \delta_{\mathcal{F}}(A_\gamma/C) = 0$ and K is a field extension $F \subseteq B$ containing b_γ such that $\text{tr.deg}_{A_\gamma} F$ is finite and $\delta_{\mathcal{F}}(F/C) - \delta_{\mathcal{F}}(A_\gamma/C) = 0$. But by construction there is no such F and so we again have a contradiction and A_α is self-sufficient over C as required. Now suppose that K is not contained in B. We show that we may reduce to the above case. By Remark 7.3.11 we have that $\text{grk}_{\mathcal{F}}(K/C) \leq \text{grk}_{\mathcal{F}}(\overline{K}^{\text{alg}}/C)$. As $\text{tr.deg}_C K = \text{tr.deg}_C \overline{K}^{\text{alg}}$ we have by the definition of the predimension that $\delta_{\mathcal{F}}(\overline{K}^{\text{alg}}/C) \leq \delta_{\mathcal{F}}(K/C)$ and so

$$\delta_{\mathcal{F}}(A_\gamma/C) \leq \delta_{\mathcal{F}}(\overline{K}^{\text{alg}}/C) < \delta_{\mathcal{F}}(A_\alpha/C).$$

As B is self-sufficient and algebraically closed we have by Lemma 7.3.15 that $\delta_{\mathcal{F}}(\overline{K}^{\text{alg}} \cap B/C) \leq \delta_{\mathcal{F}}(\overline{K}^{\text{alg}}/C)$ and so

$$\delta_{\mathcal{F}}(A_\gamma/C) \leq \delta_{\mathcal{F}}(\overline{K}^{\text{alg}} \cap B/C) < \delta_{\mathcal{F}}(A_\alpha/C),$$

where the inequality on the left continues to hold by the self-sufficiency of A_γ. Therefore by a repetition of the argument in the previous case we have a contradiction as required.

□

Proposition 7.3.22. *Let $C \subseteq A \subseteq B$ be subfields of $\mathbb{C}$ and assume that* $\mathrm{tr.deg}_C B$ *is finite and that both A and B are self-sufficient over C. Also suppose that $\delta_{\mathcal{F}}(B/C) = \delta_{\mathcal{F}}(A/C)$ and let $\partial \in \mathit{Der}_{\mathcal{F}}(A/C)$. Then ∂ extends uniquely to $\partial' \in \mathit{Der}_{\mathcal{F}}(B/C)$.*

Proof. Let $\partial \in \mathrm{Der}_{\mathcal{F}}(A/C)$. Let $\Omega(B/\partial)$ be the quotient of $\Omega(B/C)$ by relations $\sum a_i db_i = 0$ for $a_i, b_i \in A$ such that $\sum a_i \partial b_i = 0$. Note that this includes relations of the form $\eta = 0$ for all $\eta \in W_{\mathcal{F}}(A/C)$. We have the chain of quotient maps

$$\Omega(B/C) \twoheadrightarrow \Omega(B/\partial) \twoheadrightarrow \Omega(B/A),$$

which induces the chain

$$W_{\mathcal{F}}(B/C) \twoheadrightarrow W_{\mathcal{F}}(B/\partial) \twoheadrightarrow W_{\mathcal{F}}(B/A)$$

where $W_{\mathcal{F}}(B/\partial)$ is defined to be the image of $W_{\mathcal{F}}(B/C)$ under this quotient map. Now define $\mathrm{Der}(B/\partial) = \{\eta \in \mathrm{Der}(B/C) : (\exists \lambda \in B)[\eta|_A = \lambda\partial]\}$, a B-vector subspace of $\mathrm{Der}(B/C)$ that is the dual of $\Omega(B/\partial)$. Hence we have a sequence dual to the one above, given by inclusion maps. Namely,

$$\mathrm{Der}(B/A) \hookrightarrow \mathrm{Der}(B/\partial) \hookrightarrow \mathrm{Der}(B/C).$$

Now we define $\mathrm{Der}_{\mathcal{F}}(B/\partial) = \mathrm{Der}(B/\partial) \cap \mathrm{Der}_{\mathcal{F}}(B/C)$. As noted in Section 7.1 the set of $\mathcal{F}$-derivations $\mathrm{Der}_{\mathcal{F}}(B/A)$ is isomorphic to the dual of $\Omega_{\mathcal{F}}(B/A)$ and so $\mathrm{Der}_{\mathcal{F}}(B/A)$ is isomorphic to $\mathrm{Ann}(W_{\mathcal{F}}(B/A))$. Similarly we have that $\mathrm{Der}_{\mathcal{F}}(B/C) \simeq \mathrm{Ann}(W_{\mathcal{F}}(B/C))$. Therefore $\mathrm{Der}_{\mathcal{F}}(B/\partial)$ is isomorphic to the intersection of $\mathrm{Ann}(W_{\mathcal{F}}(B/C))$ and the dual of $\Omega(B/\partial)$, which are subspaces of $\mathrm{Der}(B/C)$. A derivation in this intersection is a linear functional on $\Omega(B/\partial)$ that vanishes on $W_{\mathcal{F}}(B/C)$. More precisely this is a linear functional on $\Omega(B/\partial)$ that vanishes on the image of $W_{\mathcal{F}}(B/C)$ in $\Omega(B/\partial)$. By definition this is an element of $\mathrm{Ann}(W_{\mathcal{F}}(B/\partial))$ and so this intersection is contained in $\mathrm{Ann}(W_{\mathcal{F}}(B/\partial))$. Now let $\delta \in \mathrm{Ann}(W_{\mathcal{F}}(B/\partial))$. By definition δ is a linear functional on $\Omega(B/\partial)$, an element of $\mathrm{Der}(B/\partial)$, that vanishes on the image of $W_{\mathcal{F}}(B/C)$ in $\Omega(B/\partial)$ and so by definition $\delta \in \mathrm{Der}_{\mathcal{F}}(B/C)$. Therefore $\delta \in \mathrm{Der}_{\mathcal{F}}(B/\partial)$ and so $\mathrm{Der}_{\mathcal{F}}(B/\partial) \simeq \mathrm{Ann}(W_{\mathcal{F}}(B/\partial))$.

As noted in Remark 7.3.20 we let $G = G^{\max}_{B/C}, S = G^{\max}_{A/C}$ and $S' = G^{\max}_{B/A}$ and let β, α and α' be $\mathcal{S}$-bases for $B/C, A/C$ and B/A respectively. Also let $\dim \mathcal{V}_{e_G,\beta} = n, \dim \mathcal{V}_{e_S,\alpha} = m$ and $\dim \mathcal{V}_{e_{S'},\alpha'} = r$. We can take $G = S \times S'$ and $\beta = (\alpha, \alpha')$. By Corollary 7.3.5 and Lemma 7.3.8 we can take $W_{\mathcal{F}}(B/C) = \mathcal{V}_{e_{S\times S'},(\alpha,\alpha')} = \mathcal{V}_{e_S,\alpha} \oplus \mathcal{V}_{e_{S'},\alpha'}$ and $W_{\mathcal{F}}(A/C) = \mathcal{V}_{e_S,\alpha}$. Recall from Remark 7.3.7 that $W_{\mathcal{F}}(B/A) \simeq W_{\mathcal{F}}(B/C)/W_{\mathcal{F}}(A/C)$ and so $W_{\mathcal{F}}(B/A) \simeq \mathcal{V}_{e_{S'},\alpha'}$. By a repetition of the argument in the proof of Lemma 7.3.8 we can see that if we consider a basis for $\mathcal{V}_{e_S,\alpha}$ and a basis for $\mathcal{V}_{e_{S'},\alpha'}$ coming from the differential forms associated to S and S' and the points α and α' respectively then the union of these bases forms a basis for $W_{\mathcal{F}}(B/C)$. We denote these bases by $\eta_1(\alpha), \dots, \eta_m(\alpha)$ and $\xi_1(\alpha'), \dots, \xi_r(\alpha')$ respectively. We denote the images of $\xi_k(\alpha')$ in $\Omega(B/\partial)$ and $\Omega(B/A)$ by $\tilde{\xi}_k(\alpha')$ and $\hat{\xi}_k(\alpha')$ respectively for all $k = 1, \dots, r$. The corresponding images of $\eta_1(\alpha), \dots, \eta_m(\alpha)$ vanish. As $\mathcal{V}_{e_{S'},\alpha'} \simeq W_{\mathcal{F}}(B/A)$ the images $\hat{\xi}_k(\alpha')$ form a basis for the quotient $W_{\mathcal{F}}(B/A)$ and so are linearly independent. Therefore their preimages $\tilde{\xi}_k(\alpha')$ are linearly independent in $\Omega(B/\partial)$ and we denote their span by $W \subseteq W_{\mathcal{F}}(B/\partial)$, where $\dim W = \dim W_{\mathcal{F}}(B/A)$. As the images of $\eta_1(\alpha), \dots, \eta_m(\alpha)$ in $\Omega(B/\partial)$ vanish it is clear that $W = W_{\mathcal{F}}(B/\partial)$.

By Remark 7.3.7 it is clear that

$$\dim W_{\mathcal{F}}(B/A) = \dim W_{\mathcal{F}}(B/C) - \dim W_{\mathcal{F}}(A/C).$$

As $\mathrm{Der}_{\mathcal{F}}(B/A)$ is isomorphic to $\mathrm{Ann}(W_{\mathcal{F}}(B/A))$ we have that

$$\dim \mathrm{Der}_{\mathcal{F}}(B/A) = \dim \Omega(B/A) - \dim W_{\mathcal{F}}(B/A) = \dim \mathrm{Der}(B/A) - \dim W_{\mathcal{F}}(B/A)$$

and this together with the fact that $\dim \mathrm{Der}(B/A) = \mathrm{tr.deg}_A B$ gives that,

$$\begin{aligned}
\dim \mathrm{Der}_{\mathcal{F}}(B/A) &= \dim \mathrm{Der}(B/A) - \dim W_{\mathcal{F}}(B/A) \\
&= \mathrm{tr.deg}_A B - (\dim W_{\mathcal{F}}(B/C) - \dim W_{\mathcal{F}}(A/C)) \\
&= \mathrm{tr.deg}_C B - \mathrm{tr.deg}_C A - (\mathrm{grk}_{\mathcal{F}}(B/C) - \mathrm{grk}_{\mathcal{F}}(A/C)) \\
&= \mathrm{tr.deg}_C B - \mathrm{grk}_{\mathcal{F}}(B/C) - (\mathrm{tr.deg}_C A - \mathrm{grk}_{\mathcal{F}}(A/C)) \\
&= \delta_{\mathcal{F}}(B/C) - \delta_{\mathcal{F}}(A/C) \\
&= 0.
\end{aligned}$$

Therefore $\mathrm{Der}_{\mathcal{F}}(B/A) = \{0\}$. If $\partial = 0$ then $\mathrm{Der}_{\mathcal{F}}(B/\partial) = \mathrm{Der}_{\mathcal{F}}(B/A)$ and the derivation ∂ will only extend to the zero derivative on B.

If $\partial \neq 0$ we may choose some $a \in A \setminus C$ such that $\partial a = 1$. By the above chain of quotient maps we have that $\Omega(B/A)$ is the image of $\Omega(B/\partial)$ under some quotient map. We now show exactly which relations we must quotient $\Omega(B/\partial)$ by in order to obtain $\Omega(B/A)$. It is clear that $\Omega(B/A)$ is obtained by quotienting out $\Omega(A/C)$ from $\Omega(B/C)$. Here we wish to show what else must be quotiented from $\Omega(B/\partial)$ in order to obtain $\Omega(B/A)$. Note that for any $\omega \in \Omega(A/C)$ we may write $\omega = \sum b_i da_i$ where $a_i, b_i \in A$. Consider $\sum b_i \partial a_i$ which by definition equals k for some $k \in A$. From the construction of $\Omega(B/\partial)$ we may assume that k is non-zero. Then we have that

$$\sum \frac{b_i}{k} \partial a_i = 1 = \partial a.$$

Therefore as ω was arbitrary we can see that quotienting $\Omega(B/\partial)$ by the relation $da = 0$ gives $\Omega(B/A)$ and so $\dim \Omega(B/\partial) \leq \Omega(B/A) + 1$ and therefore $\dim \mathrm{Der}(B/\partial) \leq \dim \mathrm{Der}(B/A) + 1$. However ∂ does extend to B as a field derivation by Theorem 5.1 in Chapter 8 of [24] and therefore $\dim \mathrm{Der}(B/\partial) \geq \dim \mathrm{Der}(B/A) + 1$ and so

$$\dim \mathrm{Der}(B/\partial) = \dim \mathrm{Der}(B/A) + 1.$$

As $\dim(W_{\mathcal{F}}(B/\partial)) = \dim W = \dim(W_{\mathcal{F}}(B/A))$ and $\dim \mathrm{Der}(B/A) = \mathrm{tr.deg}_A B$ we have that,

$$\begin{aligned}\dim \mathrm{Der}_{\mathcal{F}}(B/\partial) &= \dim \mathrm{Der}(B/\partial) - \dim W_{\mathcal{F}}(B/\partial)\\ &= \mathrm{tr.deg}_A B + 1 - \dim W_{\mathcal{F}}(B/A)\\ &= \delta_{\mathcal{F}}(B/C) - \delta_{\mathcal{F}}(A/C) + 1\\ &= 1.\end{aligned}$$

Therefore there is some η in $\mathrm{Der}_{\mathcal{F}}(B/\partial)$ not in $\mathrm{Der}_{\mathcal{F}}(B/A)$. This η is unique up to scalar multiplication. So $\eta|_A = \lambda\partial$ for some non-zero λ and therefore $\lambda^{-1}\eta$ is the unique element of $\mathrm{Der}_{\mathcal{F}}(B/C)$ that extends ∂ as required.

□

Proposition 7.3.23. *Suppose that $C \subseteq A \subseteq B \subseteq \mathbb{C}$ are subfields of $\mathbb{C}$ where B is algebraically closed and* $\mathrm{tr.deg}_C B$ *is finite and both A and B are self-sufficient over C and* $\mathrm{tr.deg}(B/A) = \delta_{\mathcal{F}}(B/C) - \delta_{\mathcal{F}}(A/C) = 1$. *Let $b \in B$ be transcendental over A and let $\partial \in Der_{\mathcal{F}}(A/C)$. Then for each $c \in B$, the derivation ∂ extends uniquely to $\partial' \in Der_{\mathcal{F}}(B/C)$ such that $\partial' b = c$. In particular there is $\partial' \in Der_{\mathcal{F}}(B/A)$ such that $\partial' b = 1$.*

Proof. Again by Theorem 5.1 in Chapter 8 of [24] we have that the derivation $\partial \in \mathrm{Der}_{\mathcal{F}}(A/C)$ extends uniquely to $\partial' \in \mathrm{Der}(B/C)$ such that $\partial' b = c$. As $\mathrm{tr.deg}(B/A) = \delta_{\mathcal{F}}(B/C) - \delta_{\mathcal{F}}(A/C) = 1$ we have that $\mathrm{grk}_{\mathcal{F}}(B/C) = \mathrm{grk}_{\mathcal{F}}(A/C)$. By Corollary 7.3.5 $\dim W_{\mathcal{F}}(B/C) = \dim W_{\mathcal{F}}(A/C)$ and so $\dim W_{\mathcal{F}}(B/A) = 0$. Therefore $\mathrm{Der}(B/A) = \mathrm{Der}_{\mathcal{F}}(B/A)$ and so $\partial' \in \mathrm{Der}_{\mathcal{F}}(B/C)$ as required. □

Corollary 7.3.24. *Let $C \subseteq B \subseteq \mathbb{C}$ and ∂ be as before and let B' be a field extension of B such that* $\mathrm{tr.deg}_C B'$ *is finite and B' is algebraically closed and is self-sufficient over C. Then ∂ can be extended uniquely to a derivation in $Der_{\mathcal{F}}(B'/C)$.*

Proof. This follows from combining the previous two propositions with Lemma 7.3.21.

□

Lemma 7.3.25. *Let $C \subseteq A \subseteq \mathbb{C}$ be subfields where C is $Dcl_{\mathcal{F}}$ closed and* $\mathrm{tr.deg}_C A$ *is finite. Then,*

$$\delta_{\mathcal{F}}(\lceil A \rceil / C) = \dim_{\mathcal{F}}(\lceil A \rceil / C).$$

Proof. Let $B = \lceil A \rceil$. This is equivalent to showing that the natural map from $\Omega_{\mathcal{F}}(B/C)$ to $\Omega_{\mathcal{F}}(\mathbb{C}/C)$ is injective. For if this map is injective then the function defined from $\Omega_{\mathcal{F}}(B/C)$ to the image of this natural map in $\Omega_{\mathcal{F}}(\mathbb{C}/C)$ is a bijection. By part 1 of Lemma 7.1.12 we have that the dimension of this image is equal to $\dim_{\mathcal{F}}(B/C)$ and so we have that

$$\delta_{\mathcal{F}}(B/C) = \dim \Omega_{\mathcal{F}}(B/C) = \dim_{\mathcal{F}}(B/C).$$

That the natural map from $\Omega_{\mathcal{F}}(B/C)$ to $\Omega_{\mathcal{F}}(\mathbb{C}/C)$ is injective is in turn equivalent to showing that any derivation in $\mathrm{Der}_{\mathcal{F}}(B/C)$ may be extended uniquely to one in $\mathrm{Der}_{\mathcal{F}}(\mathbb{C}/C)$, which we now show. Let $\partial \in \mathrm{Der}_{\mathcal{F}}(B/C)$ and let $b \in \mathbb{C}$ and consider $B' = \lceil B(b) \rceil_{\mathcal{F}}$, which is algebraically closed and self-sufficient over C by definition. Then by the previous corollary ∂ may be extended uniquely to an $\mathcal{F}$-derivation on B'. Now suppose that B'' is some other algebraically closed extension of B containing b that is self-sufficient and has finite transcendence degree over C. Then ∂ may be extended uniquely to an $\mathcal{F}$-derivation on B''. By uniqueness these derivations agree at b and so ∂ may be extended uniquely to a derivation in $\mathrm{Der}_{\mathcal{F}}(\mathbb{C}/C)$ as required. □

Before stating the penultimate proposition in this section we give some notation. For $i = 1, 2, 3$ we write Dcl_i for $\mathrm{Dcl}_{\mathcal{F}_i}$, δ_i for $\delta_{\mathcal{F}_i}$ and so on.

Proposition 7.3.26. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets of abelian varieties in the sense of Definition 7.2.14 such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Let $C \subseteq \mathbb{C}$ be a Dcl_2 closed subfield and let $a \in \mathbb{C}^d$ be a tuple that is Dcl_2-independent over C. Suppose $b \in Dcl_1(Ca)$ and let $B = \lceil Cab \rceil_1$. Then we have that $B = \lceil Cab \rceil_2$ and $G^{max}_{1,B/C} = G^{max}_{2,B/C}$.*

Proof. Let $A = C(a)^{\mathrm{alg}}$. As a is Dcl_2-independent it is immediate that $\dim_2(A/C) = d$ and so $\dim_1(A/C) = d$. As b is in the Dcl_1 closure of Ca it is clear that $\dim_1(B/C) = d$. Finally as the quantity $\dim_{\mathcal{F}}(X/C)$ is monotone increasing in X and monotone decreasing in $\mathcal{F}$ then we have that $\dim_2(A/C) \leq \dim_2(B/C) \leq \dim_1(B/C)$ and so

$$\dim_2(B/C) = d. \tag{7.3}$$

By Lemma 7.3.25,

$$\dim_1(B/C) = \delta_1(B/C) = \mathrm{tr.deg}_C B - \mathrm{grk}_1(B/C)$$

and so $\text{grk}_1(B/C) = \text{tr.deg}_C B - d$. As a is Dcl_2-independent we have that $\text{tr.deg}_C A = d$. Hence by Lemma 7.3.19

$$\delta_2(A/C) = \text{tr.deg}_C A - \text{grk}_2(A/C) \geq \dim_2(A/C).$$

Therefore $d - \text{grk}_2(A/C) \geq d$ and so $\text{grk}_2(A/C) \leq 0$. In particular, $\text{grk}_2(A/C) = 0$ and therefore $\delta_2(A/C) = d$. Now take any extension A' of A. By definition $\dim_2(A'/C) \geq \dim_2(A/C) = d$ and again by Lemma 7.3.19

$$\delta_2(A'/C) \geq \dim_2(A'/C) \geq d = \delta_2(A/C)$$

and therefore A is $\mathcal{F}_2$ self-sufficient. In particular $\delta_2(B/C) \geq \delta_2(A/C) = d$. In other words,

$$\text{tr.deg}_C B - \text{grk}_2(B/C) \geq d = \text{tr.deg}_C B - \text{grk}_1(B/C)$$

and so $\text{grk}_2(B/C) \leq \text{grk}_1(B/C)$. As $\mathcal{S}_1 \subseteq \mathcal{S}_2$ we know from Remark 7.3.13 that $\text{grk}_1(B/C) \leq \text{grk}_2(B/C)$ and so $\text{grk}_1(B/C) = \text{grk}_2(B/C)$. Therefore by definition $\dim \mathcal{V}_{e_{G_1,\beta_1}} = \dim \mathcal{V}_{e_{G_2,\beta_2}}$ and $\delta_1(B/C) = \delta_2(B/C)$.

By Remark 7.3.13, $G_1 = G^{\max}_{1,B/C}$ is an abelian subvariety of $G_2 = G^{\max}_{2,B/C}$. Suppose that this is a proper abelian subvariety. Then there is an abelian subvariety G_1' such that $G_1 \times G_1' \sim G_2$ and as G_2 is defined up to isogeny we may take G_2 to be $G_1 \times G_1'$ and β_2 to be (β_1, β_1') for some $\beta_1' \in \mathcal{G}_{G_1'}(B)$, which is not a C-point. By Lemma 7.3.8 we have that

$$\dim \mathcal{V}_{e_{G_1,\beta_1}} = \dim \mathcal{V}_{e_{G_2,\beta_2}} = \dim \mathcal{V}_{e_{G_1,\beta_1}} + \dim \mathcal{V}_{e_{G_1',\beta_1'}}$$

and so $\dim \mathcal{V}_{e_{G_1',\beta_1'}} = 0$ and β_1' is a C-point, a contradiction. Therefore $G^{\max}_{1,B/C}$ and $G^{\max}_{2,B/C}$ are equal.

It remains to be shown that $B = \lceil Cab \rceil_2$. Let $B' = \lceil Cab \rceil_2$, firstly we show that $B' \subseteq B$, by showing that B is $\mathcal{F}_2$ self-sufficient. Suppose for a contradiction that B is not $\mathcal{F}_2$ self-sufficient. Then there is some $L \supseteq B$ such that $\delta_2(L/C) < \delta_2(B/C) = \delta_1(B/C)$. By Lemma 7.3.19 we have that $\dim_2(L/C) \leq \delta_2(L/C)$ and by Lemma 7.3.25 we have that $\delta_1(B/C) = \dim_1(B/C) = d$. This together with (7.3) gives that

$$\dim_2(L/C) < \dim_2(B/C).$$

We know that $\dim_i X$ is increasing in X, in particular

$$\dim_2(B/C) \leq \dim_2(L/C)$$

and so $\dim_2(L/C) < \dim_2(L/C)$, a contradiction.

Now we show that $G^{\max}_{1,B'/C} = G^{\max}_{2,B'/C}$. Let $G = G_2 = G^{\max}_{2,B/C}$ and $S = G^{\max}_{2,B'/C}$ and let β and α be $\mathcal{S}$-bases for B/C and B'/C respectively. Then $\beta \in \mathcal{G}_G(B)$ and $\alpha \in \mathcal{G}_S(B') \subseteq \mathcal{G}_S(B)$. Let $\dim G = n$. Then by the maximality of G there is a proper abelian subvariety $J \subseteq G \times S$ such that (β, α) is in a C-coset of $\mathcal{G}_J$. Furthermore J has dimension at most n. Let $(\gamma_1, \gamma_2) \in \mathcal{G}_{G\times S}(C)$ be such that $(\beta, \alpha) - (\gamma_1, \gamma_2) \in \mathcal{G}_{G\times S}$ and write $\tilde{\beta} = \beta - \gamma_1$ and $\tilde{\alpha} = \alpha - \gamma_2$. Then $\tilde{\beta}$ does not lie in a C-coset of $\mathcal{G}_H$ for any $H \subseteq G$.

Consider the projection $p : J \to G$, then $p(J) = G$ and so $\dim(J) = \dim G = n$. As p is an isogeny there is an isogeny $p' : G \to J$ such that the composition $p' \circ p : J \to J = [m]$ the multiplication by m map on J, where m is some integer. Let $q' : J \to S$ be the projection. Then there is an $m \in \mathbb{Z}$ such that $q = q' \circ p' : G \to S$ and $Tq(\tilde{\beta}) = m\tilde{\alpha}$.

We claim that q is surjective. Let $S_q = \mathrm{Im}(q) \subseteq S$. As $m\tilde{\alpha} \in \mathcal{G}_{S_q}$ and $\mathcal{G}_{S_q}$ is divisible we have that $\tilde{\alpha} \in \mathcal{G}_{S_q}$. Hence $\alpha \in \mathcal{G}_{S_q} + \gamma_2$ and so α is in a C-coset of the subgroup $S_q \subseteq S$. Hence by the definition of α we have that $S_q = S$.

As $G_1 = G_2$ we have that $G \in \mathcal{S}_1$ and so by Remark 7.2.15 we have that $S \in \mathcal{S}_1$ and therefore the pair (α, S) is among those considered in the construction of $G^{\max}_{1,B'/C}$. As S has maximal dimension since $S = G^{\max}_{2,B'/C}$ this maximum is S. Therefore $S = G^{\max}_{1,B'/C}$ (up to isogeny).

Finally we show that $B \subseteq B'$. It suffices to show that B' is $\mathcal{F}_1$ self-sufficient. Suppose for a contradiction that B' is not $\mathcal{F}_1$ self-sufficient, then there is some $L \supseteq B'$ such that $\delta_1(L/C) < \delta_1(B'/C)$. Let β'_1 be an $\mathcal{S}_1$ basis for B'/C and let β'_2 be an $\mathcal{S}_2$-basis for B'/C. Then as we have shown that $G^{\max}_{1,B'/C} = G^{\max}_{2,B'/C}$ we have $\beta'_1 = \beta'_2$ and so $\mathcal{V}_{e_{G'_1}, \beta'_1} = \mathcal{V}_{e_{G'_2}, \beta'_2}$. Therefore, $\mathrm{grk}_1(B'/C) = \mathrm{grk}_2(B'/C)$ and $\delta_1(B'/C) = \delta_2(B'/C)$. Hence,

$$\delta_2(L/C) \leq \delta_1(L/C) < \delta_1(B'/C) = \delta_2(B'/C),$$

which contradicts B' being $\mathcal{F}_2$ self-sufficient. Therefore $B = B'$ as required.

□

Consider two sets of non-isogenous simple abelian varieties $\{G_1, \ldots, G_r\}$ and $\{H_1, \ldots, H_s\}$. In Section 7.2 we saw that we can build basic sets of abelian varieties denoted $\mathcal{B}_1$ and $\mathcal{B}_2$ say. Using Definition 7.2.14 one can construct sets of abelian varieties $\mathcal{S}_1$ and $\mathcal{S}_2$ say. Here we do not assume that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. In Section 7.2 the corresponding sets of functions $\mathcal{F}_1$ and $\mathcal{F}_2$ were also defined. Let $\mathcal{F}$ be the set of functions given by composing the exponential maps for $G_1, \ldots, G_r, H_1, \ldots, H_s$ with their corresponding charts. Then $C_{\mathcal{B}_3}$ denotes the set of complex numbers definable without parameters in $\mathbb{R}_{PR(\mathcal{F})}$. Let $\mathcal{B}_3$ be the set of basic abelian varieties arising from the union of $\{G_1, \ldots, G_r\}$ and $\{H_1, \ldots, H_s\}$ as in Definition 7.2.13 and let $\mathcal{S}_3 = \mathcal{S}(\mathcal{B}_3)$ be the corresponding set of abelian varieties as in Definition 7.2.14 and also let $\mathcal{F}_3 = \mathcal{F}(\mathcal{B}_3)$ be the corresponding set of exponential maps.

Proposition 7.3.27. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets of abelian varieties as described above and let $\mathcal{S}_0 = \mathcal{S}_1 \cap \mathcal{S}_2$. Let $\mathcal{S}_3$ denote the set of abelian varieties as is also described above.*

Let $C \subseteq \mathbb{C}$ be a Dcl_3-closed subfield and $a \in \mathbb{C}^d$ be Dcl_3-independent over C. Then $Dcl_1(C, a) \cap Dcl_2(C, a) = Dcl_0(C, a)$.

Proof. It is clear that $\mathrm{Dcl}_0(C, a) \subseteq \mathrm{Dcl}_1(C, a) \cap \mathrm{Dcl}_2(C, a)$. Let b be a point in $\mathrm{Dcl}_1(C, a) \cap \mathrm{Dcl}_2(C, a)$. We must show that $b \in \mathrm{Dcl}_0(C, a)$.

Let $B = \lceil Cab \rceil_1$. As $\mathcal{S}_1$ and $\mathcal{S}_2$ are both contained in $\mathcal{S}_3$ we have by the previous proposition that $B = \lceil Cab \rceil_3 = \lceil Cab \rceil_2$. In particular $G^{\max}_{1,B/C} = G^{\max}_{3,B/C} = G^{\max}_{2,B/C}$ and so $G^{\max}_{3,B/C} \in \mathcal{S}_0$. Let β_3 be an $\mathcal{S}_3$-basis for B/C. Therefore the pair $(G^{\max}_{3,B/C}, \beta_3)$ is among those considered in the construction of $G^{\max}_{0,B/C}$ and as $G^{\max}_{3,B/C}$ has maximal dimension $G^{\max}_{0,B/C} = G^{\max}_{3,B/C}$. Therefore $\mathcal{V}_{e_{G_3}, \beta_3} = \mathcal{V}_{e_{G_0}, \beta_0}$ and so $\delta_3(B/C) = \delta_0(B/C) = d$ and therefore $\dim_0(b/C, a) = 0$. Hence $b \in \mathrm{Dcl}_0(C, a)$ as required. □

7.4 Proof of Theorem 7.0.2

Theorem 7.4.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets of abelian varieties as in Definition 7.2.14 with corresponding sets of exponential maps $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $\mathcal{F}_0 = \mathcal{F}_1 \cap \mathcal{F}_2$ and let $U \subseteq \mathbb{C}^n$ be an open subset and also let $f : U \to \mathbb{C}$ be a holomorphic function.*

Suppose that f is locally definable with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$. Then f is locally definable almost everywhere in U with respect to $\mathcal{F}_0$.

Proof. We denote $\mathcal{F}_1 \cup \mathcal{F}_2$ by $\mathcal{F}_3$. Let C be a countable subfield of $\mathbb{C}$ that contains $C_{\mathcal{B}_3}$, where $C_{\mathcal{B}_3}$ is as described above Proposition 7.3.27. Then C contains all the parameters needed to define f with respect to both $\mathcal{F}_1$ and $\mathcal{F}_2$. Assume that C is also hcl_3 closed.

Let $a \in U$ be an hcl_3 generic point over the subfield C and denote $f(a)$ by b. By Wilkie's fact, namely Fact 7.1.10 we have that for $i = 1, 2, 3, 4$ the pregeometries hcl_i and Dcl_i coincide. As f can be locally defined with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$ we have that $b \in \mathrm{hcl}_1(C(a)) \cap \mathrm{hcl}_2(C(a))$. Therefore by Proposition 7.3.27 the point b is in $\mathrm{hcl}_0(C(a))$.

The set of all points $X \subseteq U$ not hcl_3 generic over C has measure 0. Therefore the set of all points $X \subseteq U$ not hcl_0 generic over C also has measure 0. Applying Proposition 7.1.6 with the set of functions $\mathcal{F}_0$ gives us that the set $U' = \{a \in U : f \text{ is locally definable at } a\}$ is open in U and $U \setminus U'$ has measure 0. Therefore f is locally definable almost everywhere with respect to $\mathcal{F}_0$ as required. □

Now we are in a position to prove Theorem 7.0.2. The proof of this theorem follows that of Theorem 7.0.1, which is Theorem 1.2 in [21]. Here Theorem 7.4.1 takes the place of Theorem 7.2 in that paper. Before giving the proof, we state some facts from o-minimality that are needed. These are Proposition 8.1 and Lemmas 8.2 and 8.3 in [21].

Proposition 7.4.2. *Suppose that $\mathcal{F}$ is a collection of holomorphic functions. If $f : U \to \mathbb{R}$ is an analytic function definable in the structure $\mathbb{R}_{PR(\mathcal{F})}$, on U, an open subset of $\mathbb{R}^n$, then there exists an $\mathbb{R}_{PR(\mathcal{F})}$-definable subset X of U of dimension at most $n-1$, an open subset W of $\mathbb{C}^n$ with $W \cap \mathbb{R}^n = U \setminus X$ and an $\mathbb{R}_{PR(\mathcal{F})}$-definable holomorphic $F : W \to \mathbb{C}$ such that $F|_{U\setminus X} = f$.*

Lemma 7.4.3. *Suppose that $\mathcal{R}$ is an expansion of $\overline{\mathbb{R}}$ in which every definable set is also definable in $\mathbb{R}_{an}$. Then $\mathcal{R}$ has analytic cell decomposition.*

Lemma 7.4.4. *Suppose that $C \subseteq \mathbb{R}^n$ is an analytic cell definable in an o-minimal expansion of $\overline{\mathbb{R}}$. Then there an open analytic cell $U \subseteq \mathbb{R}^n$ such that $C \subseteq U$ and an analytic retraction $\theta : U \to C$ which are definable in the structure $(\overline{\mathbb{R}}, C)$.*

Proof of Theorem 7.0.2. Recall that G and H are abelian varieties with associated sets of exponential maps $\mathcal{F}_G$ and $\mathcal{F}_H$ and that $\mathcal{F}_G \cap \mathcal{F}_H = \emptyset$. Suppose that

$X \subseteq \mathbb{R}^n$ is definable in both $\mathbb{R}_{PR(\mathcal{F}_G)}$ and $\mathbb{R}_{PR(\mathcal{F}_H)}$. In the structure $(\overline{\mathbb{R}}, X)$ there is an analytic cell decomposition of X by Lemma 7.4.3. This is also an analytic cell decomposition in both $\mathbb{R}_{PR(\mathcal{F}_G)}$ and $\mathbb{R}_{PR(\mathcal{F}_H)}$ and it is therefore sufficient to prove the theorem in the case where the set X is an analytic cell. This is done by induction on n and the $n = 1$ case is trivial.

Now consider the case where $X = \mathrm{gr}(f)$ where $f : X' \to \mathbb{R}$ for an analytic cell $X' \subseteq \mathbb{R}^{n-1}$. By induction X' is a semialgebraic cell. By Lemma 7.4.4 there is a subset $U' \subseteq X'$ and an analytic retraction $\theta : U' \to X'$, which is definable in the structures $\mathbb{R}_{PR(\mathcal{F}_G)}$ and $\mathbb{R}_{PR(\mathcal{F}_H)}$. Therefore the function $f \circ \theta : U' \to \mathbb{R}$ is both analytic and definable in the structures $\mathbb{R}_{PR(\mathcal{F}_G)}$ and $\mathbb{R}_{PR(\mathcal{F}_H)}$. Hence the holomorphic extension of $f \circ \theta$ to some open $W \subseteq \mathbb{C}$, denoted g, is definable in the structures $\mathbb{R}_{PR(\mathcal{F}_G)}$ and $\mathbb{R}_{PR(\mathcal{F}_H)}$ by Proposition 7.4.2. As $\mathcal{F}_0 = \mathcal{F}_G \cap \mathcal{F}_H = \emptyset$ we have $\mathbb{R}_{PR(\mathcal{F}_0)} = \overline{\mathbb{R}}$ and so by Theorem 7.4.1 some restriction of g to an open set is definable in $\overline{\mathbb{R}}$ and is therefore a semialgebraic function. However g is holomorphic and so it is in fact an algebraic function as are $f \circ \theta$ and f. Hence X is a semialgebraic cell as required.

Now consider the case where $X = (f, g)_{X'}$ for some analytic cell $X' \subseteq \mathbb{R}^{n-1}$. By the previous case the functions f and g are semialgebraic on X' and therefore X is also semialgebraic as required.

□

Chapter 8

Conclusion

Throughout this thesis we have seen various definability results for restrictions of some transcendental functions and the exponential maps of abelian varieties. The results in the first few chapters adapt essentially the same method of Bianconi used in [5] and [6]. To finish this thesis we give a discussion on the ways that some of these results could have been proved differently and discuss potential further results that I will consider in future work.

Firstly we consider the definability results discussed in Chapters 3,4 and 6 and observe that these results could all have been proved in a different way, although this does not significantly affect the structure of the proofs. More precisely, instead of using the implicit definition due to Wilkie and Jones, Theorem 2.4.3, in Chapters 3,4 and 6 we could have used the implicit definition due to Gabrielov, Proposition 2.4.10, that was used in Chapter 5. There is no obstruction to using this Gabrielov definition in these chapters however it does not appear that the proof of the theorem in Chapter 5 can be made to work using the method which involves the initial implicit definition as done in Chapter 3 say. Suppose that we used this implicit definition in the proof of Theorem 3.0.1 for the auxiliary structure $(\overline{\mathbb{R}}, \wp \circ B, \wp' \circ B, B, B_1)$. Then we would obtain a system of $n-1$ polynomials in the variables $x_0, \ldots, x_n$ and $\wp(B(x_0)), \ldots, \wp(B(x_n)), \wp'(B(x_0)), \ldots, \wp'(B(x_n))$ as well as $B(x_0), \ldots, B(x_n), B_1(x_0), \ldots, B_1(x_n)$ giving an initial upper bound of $4n + 5$. In Chapter 5 the corresponding upper bound was initially lowered by adding an extra $n + 1$ equations to the system, which arose from the differential equation for $\wp$ in each variable. One can add the same $n + 1$ equations here. A further $2(n+1)$ equations can be added to this system which are defined by using the definition of B_1 and the fact that A is the compositional inverse of B for each variable. By a repetition of the argument in Chapter 5 one can prove that the corresponding matrix of partial derivatives for this new system has maximal rank at the appropriate points. The issue with this approach arises when we try to reprove a corresponding version of Lemma 5.1.4 for this setup. As there is no addition formula for the algebraic function, we need to shift $B(y_2)$ instead of y_2 in this lemma. However in order to obtain a point $z \in \mathbb{C}^{5n+5}$ that satisfies the original system and does not satisfy the corresponding Cauchy-Riemann equations as in the proof of the Lemma 5.1.4, we require Cauchy-Riemann equations for $B(u)$ and $B(v)$. However this is where the problem arises. If these Cauchy-Riemann equations hold then one can show that $B'(u) = B'(v)$. Therefore as B is an algebraic function we have that u and v are algebraically dependent, a contradiction to Claim 5.1.1.

Another approach one could take in Chapters 3 and 4 would be to consider systems of polynomials rather than algebraic functions. The main alteration to the proof would be to the linear independence claim as explained in Chapter 5, which allows us to give this claim for a system of polynomials. This approach does not depend on the choice of implicit definition. Therefore in Chapter 3 for example the choice to consider a system of algebraic functions is essentially a convenience. Otherwise, we would be working with a system of polynomials in $6n + 6$ variables.

However when we work with the modular j-function it appears to be no longer possible to work with a system of polynomials. In fact from the corresponding independence claim we can see that algebraic functions are necessary, as the expression for $j(iB(f_n(t)))$ in terms of $j(iB(f_{n-1}(t)))$ arising from the modular polynomial is certainly an algebraic function and may not be a polynomial. Therefore, it is not apparent how we may reproduce a proof of the theorem in Chapter 6 using a system of polynomials. This is the main obstruction to an immediate extension of some of the work in this thesis, namely a version of the theorem in Chapter 5 for the modular j-function. In order to reproduce the proof in this chapter for the modular j-function we must construct a version of the argument at the end of this proof for a system of algebraic functions instead of a system of polynomials.

Another possible future direction for the work in this thesis would be to extend the work in Chapter 7 on the exponential maps of abelian varieties to the exponential maps of semiabelian varieties and even general commutative algebraic groups. In order to achieve this one would need to remove the dependence on Poincaré's reducibility theorem from the work in Chapter 7. The main source of this dependence arises when we consider the tower of subfields $C \subseteq A \subseteq B \subseteq \mathbb{C}$ and wish to define the abelian variety $G^{\max}_{B/A}$ for example. The motivation for considering such a tower of subfields is in Lemma 7.3.25 where we wish to uniquely extend an $\mathcal{F}$-derivation on $B = \lceil A \rceil$ to an $\mathcal{F}$-derivation on $\mathbb{C}$. Upon removing the need for this extending derivations result or providing an alternative proof of it we should be able to extend Theorem 7.0.2 to the semiabelian case. As there does not appear to be any other such restriction we should be able to extend this further to the case of general commutative algebraic groups after first replacing the version of Ax's Theorem seen in Theorem 7.3.1 by one for general commutative algebraic groups which can be seen in [3].

Finally, another direction that the work in this thesis could take in the future would be to expand some of these nondefinability results for the Weierstrass $\wp$-function to other transcendental functions that are related to $\wp$. An example of such a function is the Weierstrass ζ-function. For a complex lattice Ω the Weierstrass ζ-function is defined as

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

a meromorphic function with simple poles at precisely the points in Ω. Observe that $\zeta'(z) = \wp(z)$ and so the restriction of ζ' to a bounded real interval I that does not intersect Ω is definable in the structure $(\overline{\mathbb{R}}, \wp|_I)$. However it seems unlikely that such a restriction of ζ is definable in $(\overline{\mathbb{R}}, \wp|_I)$ and this should be possible to prove using similar methods to those seen in Chapter 3 for example. In order to use these methods we would require an Ax statement which features the ζ function as well as the $\wp$-function. This can also be seen in [7]. If we let $D \subseteq \mathbb{C}$ be a disc such that $D \cap \Omega = \emptyset$, then $\zeta|_D$ is not definable in $(\overline{\mathbb{R}}, \wp|_I)$ by Theorem 5.0.2 provided that the lattice Ω does not have complex multiplication. However if we add the restriction of ζ to the interval I to the structure $(\overline{\mathbb{R}}, \wp|_I)$ then we can answer some of these definability questions positively. Consider as we did at the start of this thesis the lattice $\Omega = \mathbb{Z} + i\mathbb{Z}$. As Ω is real then the restriction of ζ to the interval I is also real valued for the same reason as $\wp$ as described in Section 18 of [38]. By using the definition of the ζ function one can show that

$\zeta(iz) = -i\zeta(z)$. From the corollary in Section 3 of Chapter 5 in [9] we have the formula

$$\zeta(u+v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} + \zeta(u) + \zeta(v).$$

Using these formulae and a repetition of the argument in the proof of Lemma 3.1.1 we can show that the restriction of ζ to a disc D in $\mathbb{C}$ is definable in $(\overline{\mathbb{R}}, \wp|_I, \zeta|_I)$ when the associated lattice is $\mathbb{Z} + i\mathbb{Z}$. A natural next step would be to extend this to all complex lattices which have complex multiplication. A converse to this result should follow from similar methods to those seen in Chapter 3 3. Extending this converse from the real lattice to the general lattice case should follow by adapting methods from Chapter 4. Future research on nondefinability results for these functions could also consist of obtaining analogues of Theorem 7.0.2 in Chapter 7 for the structure $(\overline{\mathbb{R}}, \wp|_I, \zeta|_I)$.

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