OPTIMAL MEAN-VARIANCE PORTFOLIO SELECTION

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This thesis studies the dynamic optimality introduced by Pedersen and Peskir [43] in the mean-variance portfolio selection problem from the dynamic programming perspective. For a self-financing portfolio, the investor aims to determine the maximal value function defined by:

$$V(t,x) = \sup_u \left[\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) \right].$$

The quadratic nonlinearity introduced by the variance term can be handled by the method of Lagrange multipliers, and the application of the HJB equation enables us to obtain the optimal solution. In this thesis, we introduce various different market settings, including (a) prohibition of short-selling, (b) margin requirements, (c) that the stock that is driven by the constant elasticity of variance model, and (d) partial information. For these problems, we investigate (i) time-inconsistent strategies (static optimality) and (ii) the time-consistent strategies (dynamic optimality), and we compare their performance. Alongside solving the original optimal problem, we also consider the other two constrained cases where we condition the size of the expectation/variance of the terminal wealth. In Chapter 2, we consider portfolio selection under a no short-selling constraint, in which a change-of variable formula from [45] is used to replace the viscosity solution to overcome the non-smoothness of the value function. Under the no short-selling constraint, both static and dynamic optimalities naturally prevent bankruptcy. However, static optimality suggests that the investor should hold all of his wealth in the riskless bond if his wealth is large enough, while dynamic optimality encourages the investor to keep holding the risky asset for a higher return. Inspired by the method applied in Chapter 2, we further consider the margin requirement for short-selling in Chapter 3. The conclusion analyses the impact of the change of margin rate on the performance of both static and dynamic optimalities and verifies that some properties of those two optimalities described in [43] are still valid in this case. In Chapter 4, we study the case where the stock price follows the constant elasticity of variance (CEV) model. The CEV model can be taken as a natural extension of geometric Brownian motion, and it has advantages such as explaining the implied volatility smile. We derive static and dynamic optimalities for both the unconstrained problem and the constrained problems. By choosing a proper value for the elasticity parameter, we can easily extend our conclusion to cases where the risky asset follows different processes such as geometric Brownian motion and the Ornstein-Uhlenbeck process. Hence, the model we set in Chapter 4 can be seen as a general solution that covers the work of [43]. Besides, the conclusion in Chapter 4 is also valid when there exist arbitrage opportunities for the stock. In Chapter 5, we consider portfolio selection under partial information, since the investor normally can only access limited information in a real financial market. The biggest difficulty is that the filtering and optimisation aspects of the problem are hard to separate, which can be handled by using the separation principle studied in [56]. Under partial information, we obtain both time-consistent and time-inconsistent solutions.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

In mathematical finance, portfolio selection has always been a core problem and has constantly driven researchers along the path of extending this problem to a broader level. In 1952, Markowitz [37] (see also [36]) introduced his pioneering mean-variance analysis for the modern portfolio theory, in which the variance of underlying assets is considered as a determinant for deciding the optimal strategy. Markowitz introduced the principle that maximises the expected return for each given and fixed value of variance which constructs the efficient frontier. Merton [38] further derives the explicit efficient frontier for different cases for the single-period case under different settings. As summarised in [18], one of the most important features of modern portfolio theory is that the investor should consider the diversification of risky assets rather than blindly choosing a risky asset by its unique features.

From the single-period case, this problem has been extended to multi-period cases (for example [40], [24], [19] and [32]). In [24], Hakansson analyses optimal consumption and investment as well as lending/borrowing strategies via utility analysis over multiple periods. Fama [19] observes that in a perfect market setting and with a strictly concave utility function, the action of the investor in any period is consistent with the conclusion in a single period. Furthermore, in [32], Li and Ng obtain the mean-variance efficient frontier under the multi-period setting. In recent years, more practical assumptions and settings have been added such as the mean-semi-variance model [59] and bankruptcy prohibition [53]. Since the multi-period case is not the target of this thesis, we will skip further details and focus on the continuous setting.

There is a series of studies that considers the mean-variance portfolio selection under a continuous formulation (for example Chapters 4, 5 and 6 of [39]). The martingale method has played an important role in solving the continuous portfolio selection problem. In [3], Bielecki, Jin, Pliska, and Zhou adopt the martingale method to study the portfolio selection under the bankruptcy prohibition. They first design the equivalent martingale measure, and under the martingale measure, the discounted wealth process is a martingale, from which the problem is split into two steps: achieving the optimal terminal wealth for the constrained problem firstly and the optimal control can be obtained by replacing the contingent claim of the corresponding optimal terminal wealth. The Ito-Clark theorem guarantees the one-to-one correspondence between the optimal control and the corresponding optimal terminal wealth. Similar methods can be seen in [42] where the bankruptcy constraint is replaced by a guaranteed value g, and the conclusion indicates that for the dynamic optimal wealth process, the terminal wealth will only take either g or the expected terminal wealth β. The martingale method is applied to handle more challenging cases. In [22], Gao, Xiong, and Li consider the mean-variance-CVaR (conditional value-at-risk) model and mean-variance-SFP (safety-first-principle) where the definition of CVaR is described by [47] in which the safety-first-principle is inspired by the work of [48] and [31] and is described by a disaster probability.

Besides the martingale approach, dynamic programming has also been wildly applied in continuous portfolio selection. For example, Zhou and Li consider the continuous mean-variance problem and handle the nonlinearity caused by variance via converting the problem into the auxiliary control problem which is solved by the stochastic linear-quadratic method (see Chapter 6 of [61] for further details about the stochastic LQ framework). Lim and Zhou [34] further extend the previous work and the LQ optimal control technique to consider the mean-variance portfolio selection with random parameters. The LQ optimal control approach has been adopted in more complicated cases. Based upon the previous work [63], Xie further introduces liability management setting into the portfolio selection problem under regime switching, in which two different Brownian motions drive the stock price process and the liability process respectively, and obtains explicit optimal solutions and the corresponding efficient frontier. In [60], Yao, Li, and Chen consider the portfolio selection when the market only consists of risky assets by applying the HJB equation in which they not only obtain the explicit efficient frontier but also verify the two fund separation principle that is first introduced by [52] and describes that the linear combination of two efficient portfolios will still be efficient. Inspired by those previous works, in this thesis, we will consider the continuous mean-variance portfolio selection problem under different conditions and assumptions from the dynamic programming perspective, in which we will apply different techniques such as the change-of-variable formula [45] and Legendre transform (cf. [27], [21] and [54]) to solve the HJB equation and investigate both time-inconsistent and time-consistent solutions.

Time-inconsistent and time-consistent solutions have been greatly considered in portfolio theory. In [50], Strotz notes the time inconsistency of the consumption plan and comes up with two concepts: the pre-commitment strategy and the 'consistent planning'. Recalling the formulation of [50], we see that the pre-commitment suggests the investor should follow the strategy he made at the beginning date while in the consistent planning, the investor will reject those strategies he will not follow and only consider the optimal plan from the strategies that he will follow. This 'consistent planning' strategy is considered by Peleg and Yaari in [44], and they point out that this 'consistent planning' only gives optimal solutions when the investor has the 'stationary preference' (i.e. the utility function depends on the consumption rate from the current time and the utility function stays the same at any point of time). Seeing the limit of the 'consistent planning', Peleg and Yaari further introduce the Nash-equilibrium control which will be able to handle the case when the investor is changing his preference and note that if there exists an optimal control under the definition of 'consistent planning', then it must be a Nash-equilibrium one. Nash-equilibrium optimal control has been considered in many studies. Björk and Murgoci [6] introduce the delicate extended HJB equation to obtain the Nash subgame perfect equilibrium controls, which is time-consistent, to overcome the time inconsistency. This extended HJB equation has been used to derive the equilibrium optimal control when the risk aversion rate is state-dependent [7] and the time consistent portfolio selection under the short-selling constraint [2]. Furthermore, in [43], Pedersen and Peskir [43] adopt the dynamic programming method to solve the nonlinear mean-variance portfolio selection problem, in which they note that optimal control depends on the initial point of the underlying wealth process. This concept is referred to as static optimality (it should be mentioned that the static optimality has been considered as pre-commitment in [50]). Furthermore, they extend the static optimality to the dynamic optimality in which the optimal control is time-consistent. Dynamic optimality claims that for each new position of the controlled wealth process, there is a new optimal control problem to solve based on overruling all the previous problems [43]. It should be mentioned that the dynamic optimality introduced by [43] is different from the time-consistent control under the Nash subgame perfect equilibrium setting. In [43], we have seen the comparison between the subgame-perfect Nash equilibrium and the dynamic optimality and [43] has shown that the dynamic optimality outperforms the subgame-perfect Nash equilibrium control. In this thesis, we will follow the definition and method of [43] to investigate the statically optimal control and dynamically optimal control under four different settings including no short-selling constraint, the existence of margin requirement, the stock price follows the constant elasticity of variance model, and portfolio selection under partial information.

In this thesis, we assume that there is a financial market that consists of two assets, a riskless bond and a risky stock, and we aim to construct a self-financing portfolio dynamically until the maturity. To optimise the performance of the portfolio, we need to decide the optimal control by evaluating the following value function:

$$V(t,x) = \sup_{u} \mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) \tag{1.1}$$

in which $c > 0$ represents the risk aversion rate. Following the idea introduced by Markowitz [37], the expected return and risk of the portfolio are represented by the expectation and variance of the value process of the portfolio respectively. Naturally, the variance term introduces the quadratic nonlinearity into this problem. The quadratic nonlinearity makes the standard optimal control theory (cf. [20]) infeasible, and instead, we follow the methodology of [43] to solve the optimal control problem by applying Lagrange multipliers and the HJB equation. In the following chapters, we will adopt different techniques to solve the HJB equation because the setting of the market is changing in each chapter. The solution of the HJB equation shows that the optimal control relates to the initial time and value of the controlled wealth process, and this time-inconsistent solution is named as static optimality in [43]. From the statically optimal control, we can derive the time-consistent control, i.e. the dynamic optimality. As [43] mentions that there is a connection between these two controls, which is, the dynamically optimal control is equivalent to the statically optimal control with the same initial status. This fact is used to derive the dynamically optimal control from the static case. Besides the unconstrained case given by (1.1), we further attempt to consider the other two constrained cases mentioned in [43] where we maximise the expectation of the terminal wealth or minimise the variance of the terminal wealth when we condition the size of the variance or expectation of the terminal wealth respectively. More details will be exhibited in the formulation section in each chapter as we may change the setting or notation.

Before we introduce the content of each chapter, we firstly introduce definitions of the static optimality and dynamic optimality, which are given in [43].

Definition 1.1 (Static optimality) [43]. For a given status $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control u_* is called a statically optimal control in the unconstrained problem $V(t,x) = \sup_u \mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)$ if there is no other control v such that:

$$\mathrm{E}_{t,x}(X_T^v) - c\,\mathrm{Var}_{t,x}(X_T^v) > \mathrm{E}_{t,x}(X_T^{u_*}) - c\,\mathrm{Var}_{t,x}(X_T^{u_*}). \tag{1.2}$$

For a given status $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control u_* is called a statically optimal control in the constrained problem $V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u) \leq \alpha} \mathrm{E}_{t,x}(X_T^u)$ if $\mathrm{Var}_{t,x}(X_T^{u_*}) \leq \alpha$ and there is no other admissible v such that $\mathrm{Var}_{t,x}(X_T^v) \leq \alpha$ and:

$$\mathrm{E}_{t,x}(X_T^v) > \mathrm{E}_{t,x}(X_T^{u_*}). \tag{1.3}$$

For a given status $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control u_* is called a statically optimal control in the constrained problem $V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u) \geq \beta} \mathrm{Var}_{t,x}(X_T^u)$ if $\mathrm{E}_{t,x}(X_T^{u_*}) \geq \beta$ and there is no other control v such that $\mathrm{E}_{t,x}(X_T^v) \geq \beta$ and:

$$\mathrm{Var}_{t,x}(X_T^v) < \mathrm{Var}_{t,x}(X_T^{u_*}). \tag{1.4}$$

Definition 1.2 (Dynamic optimality) [43]. For each given and fixed pair $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control $u_*(t,x)$ is dynamically optimal in the constrained problem $V(t,x) = \sup_u \mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)$ if, for any admissible control $v(t,x) \neq u_*(t,x)$, there exists a control w such that $w(t,x) = u_*(t,x)$ and we have:

$$\mathrm{E}_{t,x}(X_T^v) - c\,\mathrm{Var}_{t,x}(X_T^v) < \mathrm{E}_{t,x}(X_T^w) - c\,\mathrm{Var}_{t,x}(X_T^w). \tag{1.5}$$

For each given and fixed pair $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control $u_*(t,x)$ is dynamically optimal in the constrained problem $V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u) \leq \alpha} \mathrm{E}_{t,x}(X_T^u)$ if $\mathrm{Var}_{t,x}(X_T^{u_*}) \leq \alpha$ and for any admissible control $v(t,x) \neq u_*(t,x)$ such that $\mathrm{Var}_{t,x}(X_T^v) \leq \alpha$, there exists a control w such that $w(t,x) = u_*(t,x)$ and $\mathrm{Var}_{t,x}(X_T^w) \leq \alpha$ and we have:

$$\mathrm{E}_{t,x}(X_T^v) < \mathrm{E}_{t,x}(X_T^w). \tag{1.6}$$

For each given and fixed pair $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control $u_*(t,x)$ is dynamically optimal in the constrained problem $V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u) \geq \beta} \mathrm{Var}_{t,x}(X_T^u)$ if $\mathrm{E}_{t,x}(X_T^{u_*}) \geq \beta$ and for any admissible control $v(t,x) \neq u_*(t,x)$ such that $\mathrm{E}_{t,x}(X_T^v) \geq \beta$, there exists a control w such that $w(t,x) = u_*(t,x)$ and $\mathrm{E}_{t,x}(X_T^w) \geq \beta$ and we have:

$$\mathrm{Var}_{t,x}(X_T^v) > \mathrm{Var}_{t,x}(X_T^w). \tag{1.7}$$

In these two definitions, we set that u_t is the percentage of wealth invested in the stock at $t \in [t_0, T]$ and U is the set of all admissible controls. For any admissible control u, we have $u_t = u(t, X_t^u)$ where $(t,x) \mapsto u(t,x) \cdot x$ is a continuous function from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$. For completeness, following the idea of [43], we define $u(t,0) = u(t,0) \cdot 0 = \lim_{0 \neq x \mapsto 0} u(t,x)$ because the map $x \mapsto u(t,x)$ may not exist at 0.

In Chapter 2, we study nonlinear mean-variance portfolio selection under a no short-selling constraint, for which we obtain both the statically and the dynamically optimal controls. The short-selling prohibition introduces non-smoothness into the value function of the HJB equation, and we manage to use the change of variable formula with local time on the curve to replace the viscosity solution technique to achieve the optimal control, which has not been noted in other papers. The optimal controls look intuitively similar to that of [43] as they are the non-negative part of the optimal controls given by the unconstrained case in [43]. Besides the optimal control, we also derive the strong solutions of the optimal wealth process for both static and dynamic optimalities. Both strategies naturally prevent bankruptcy of the investor, however, static optimality suggests that the investor should invest all his wealth in the riskless bond if his wealth is large enough while dynamic optimality encourages the investor to keep holding the risky asset for a higher return, which is also original. Since the distribution of the stopping time of the wealth process is unknown, calculating the expectation and variance of the terminal wealth is not feasible (note there is still a possible way to calculate the expectation without knowing the distribution of the stopping time but it is out of the scope of this chapter). The conclusion of the numerical analysis indicates that the performance of those two strategies and their stopping time distributions is highly impacted by the change of risk aversion rate, and Theorem 2.3 provides the investor with different risk aversion preferences. Moreover, comparing Theorem 2.3 from Chapter 2 with Theorem 3 of the unconstrained cases in [43], we note that both static and dynamic optimalities under short-selling prohibition outperform the unconstrained case.

In Chapter 3, we introduce the margin requirement for short-selling into the mean-variance portfolio selection, which can be viewed as an extension as Chapter 2. The margin requirement also introduces the non-smoothness into the value function of the HJB equation. Hence, following the same method in Chapter 2, we apply the change-of-variable formula with local time on a curve to replace the Ito formula in the verification theorem to prove the optimality of the control. For both statically and dynamically optimal controls, the non-negative part of both controls is also consistent with the unconstrained cases in [43] while the negative part is smaller than that of [43] as the margin requirement forces the investor to short sell larger portion of the risky asset. Setting the margin rate to 0 will reduce Theorem 3.1 to the unconstrained case in [43]. Moreover, if the margin rate is large enough, the short-selling will be forbidden. In the numerical analysis, we further consider the impact of the margin rate on both of the strategies and note that increasing the margin rate will enhance the performance of both strategies. This phenomenon only exists under the perfect market assumption. Moreover, we verify the features of both optimalities observed in [43] are still valid under the margin requirement.

In the two chapters discussed above, the stock price follows a geometric Brownian motion; while in Chapter 4, we adopt a constant elasticity of variance (CEV) model to describe the stock prices in the mean-variance formulation. Due to the CEV model, the HJB equation turns into a complicated second-order nonlinear partial differential equation. In order to handle this difficulty, we introduce the Legendre transform and dual theory to transform the HJB equation to its dual function. From the solution of the dual function, we can easily derive the optimal control for the prime problem. Choosing a proper value for the constant elasticity parameter β of the CEV model can lead to various cases such as geometric Brownian motion, the Ornstein-Uhlenbeck process, etc. Hence, we can consider the model described in Chapter 4 as a general solution and we will see that the conclusion in [43] will be easily derived from Theorem 4.1 by setting $\beta = 0$. In Chapter 4, we also conduct numerical analysis to further observe the impact of changing the constant elasticity parameter to the performances of both static and dynamic optimalities. In the unconstrained case, numerical analysis indicates that under the CEV model, the static investor will have a different prediction of the future trend of the stock from the dynamic investor. In the numerical analysis section, we verify that the conclusion under the CEV model will have the same features as the previous work [43]. However, the only difference is, in the first constrained problem where the investor attempts to maximise the expectation of the terminal wealth while conditioning the size of the variance of the terminal wealth, the static optimality always outperforms the dynamic optimality if $\beta \neq 0$ as the static sample mean $\bar{\mu}^s$ of terminal wealth is larger than the dynamic case and $\beta = 0$ is the only case that the dynamic optimality outperforms the static optimality.

In Chapter 5, we consider the mean-variance portfolio selection under partial information, in which the drift rate of the geometric Brownian motion that describes the stock price is unknown and the only information available for the investor is the stock price up to the current time. The partial information setting naturally leads to a filtering problem in mean-variance portfolio selection. Adopting the Kalman-Bucy method enables us to obtain the process for the optimal estimator μ_t and its error of estimation γ_t. Moreover, in the past, the biggest challenge has been to separate the filtering and optimisation. However, [56] verifies a significant result, the separation principle, which is specifically applicable for the mean-variance portfolio selection problem, and it enables us to replace the unknown drift rate by the corresponding conditional expectation in the wealth process for X_t^u and solve the optimal control problem with respect to the wealth process as the full information case. By adopting the separation principle and solving the HJB equation by solving a Riccati-type ODE gives the closed-form solution for the statically optimal control, from which the dynamically optimal control can be easily derived. Apart from solving the unconstrained problem, we extend the conclusions to two constrained problems.

In [43], Pedersen and Peskir firstly introduce the concept of dynamic optimality for time-consistency in portfolio selection problem and manage to derive the time-consistent solution from the time-inconsistent solution. They further prove that the dynamic optimality outperforms the static case in constrained cases where they condition the size of the expectation/-variance of the terminal wealth and analyse optimal wealth process behaviour. Hence, this thesis extends the work of [43] to four different situations and compares the performance of both optimalities. In Chapter 2, portfolio selection with a no short-selling constraint, we manage to use the change of variable formula with local time on the curve to replace the viscosity solution technique to achieve the optimal control and conduct numerical analysis to compare the performance of both optimalities and verify the conclusion introduced in [43]. In Chapter 3, we follow the same method adopted in Chapter 2 to extend the work of

[43] under the existence of margin requirements and consider the impact of margin requirement to the performance of both strategies as well as verifying the features found in [43]. In Chapter 4, we generalise the work of [43] by introducing the CEV model which is solved by applying Legendre transform and from Theorem 4.1, we can easily derive the conclusion of [43]. Furthermore, by numerical analysis, we verify that most of the features of [43] are still valid except the case where the investor attempts to maximise the expectation of the terminal wealth while conditioning the size of the variance of the terminal wealth, in which the static optimality outperforms the dynamic case under the existences of the non-zero value of the constant elasticity parameter. In Chapter 5, we extend the problem to partial information case, in which the separation principle is adopted to separate the filtering and optimisation and obtain both time-inconsistent and time-consistent solutions. Note that the substance of Chapter 2 has been written up and submitted for publication. The remaining chapters have also been written so as to form the basis of a set of working papers, which we hope to submit for publication in the future.

Chapter 2

Optimal Mean-Variance Portfolio Selection with No Short-Selling Constraint

2.1 Introduction

In a real financial market, it could be difficult to conduct short-selling of the risky asset for many reasons such as borrowing a small amount of a single risky asset is not always feasible, especially for those stocks with low institutional ownership; moreover, the investor may face public moral censure, especially during a financial crisis (see [30] for further details). Seeing those restrictions inspires us to introduce a no short-selling constraint into the mean-variance portfolio selection problem and consider the performance of both the time-inconsistent and time-consistent strategies.

Various studies consider the mean-variance portfolio selection with a no short-selling constraint. For instance, in [57] and [58], Shreve and Xu introduce the duality method and utility analysis for the constrained consumption-portfolio problem under the no short-selling constraint. Specifically, [57] considers short-selling prohibition in the market with time-dependent coefficients, which can be seen as a general case, and introduces the method to construct the primal and dual functions as well as deriving the optimal strategy for the primal problem from the solution of the dual function. In [57], they rule out the short-selling of the risky asset by constraining the wealth process to be non-negative during the entire time, and it turns this problem into a path-wise constrained case, which is solved by the martingale method. Since [57] considers the linear problem, in the following part, we will see that in the non-linear problem, this path-wise constraint is not enough to rule out the short-selling case completely. Furthermore, in the second part work of Shreve and Xu [58], they further explore the duality analysis for the portfolio selection under the short-selling prohibition when the market coefficients are given constants by HJB approach. Furthermore, based upon the work of [15] which considers the log utility function and two risky stocks with uncorrelated rate of return under the short-selling constraint, [51] investigates the portfolio selection with CRRA utility function with the borrowing and short-selling constraint. It should be noted that as [33] states, the utility analysis does not reveal the relationship between the underlying return and risk explicitly, and the optimal strategy achieved by utility functions normally is not the mean-variance efficient except the case introduced in [17] where the quadratic utility function can be related to the mean-variance setting. Li, Zhou and Lim extend their work from unconstrained mean-variance portfolio selection [34] to the constrained control case [33] by using stochastic linear-quadratic control technology. In [33], the main difficulty is that the optimal control obtained by applying the Riccati equation may conflict the constraint on the admissible control, and there is no classical solution of the Hamilton-Jacobi-Bellman equation under the constraint on control. In striving for handling this issue, Li, Zhou and Lim solve the Hamilton-Jacobi-Bellman equation by constructing a viscosity solution introduced in [13] and using the verification theorem introduced in [64] to obtain the optimal control which is of time-inconsistency for each given and fixed desired terminal expected wealth, i.e. constructing the efficient frontier. Furthermore, Bensoussan, Wong and Yung [2] not only manage to extend the time-inconsistent solution of mean-variance portfolio selection under short-selling prohibition to the time consistent case but also solve the case where the risk aversion is state-dependent by applying extended Hamilton-Jacobi-Bellman equation introduced in [6]. Moreover, [2] achieves the subgame-perfect Nash optimal control for the Lagrangian problem. In Subsection 4 of Section 4 in this chapter, we further consider two constrained problems when we introduce the constraint on the size of the expectation/-variance of the terminal wealth.

We aim to construct a self-financing portfolio under the no short-selling constraint dynamically in time to achieve the highest return and the lowest risk at maturity in a financial market consisting of a riskless bond and risky stock. The variance brings the quadratic nonlinearity into this problem, and we follow the methodology of [43] to solve the optimal control problem by applying Lagrange multipliers and the Hamilton-Jacobi-Bellman equation. However, the classic method (cf. [5]) for solving the HJB equation is not feasible in the no short-selling problem because this constraint on the admissible control will introduce non-smoothness into the value function of the Hamilton-Jacobi-Bellman system that precludes the classical way to solve it. In this case, we follow the idea of [33] to construct the value function. However, instead of achieving viscosity solution and using the viscosity verification theorem to prove the optimality, we construct a change-of-variable formula with local time on the curve where the non-smoothness occurs, which can be used to replace Itô's formula. We can then apply the verification theorem described in [5] to obtain the optimal control. It should be pointed out that, based upon our best knowledge, using a change-of-variable formula instead of applying viscosity solution has not been studied in the portfolio selection problem.

The solution of the Hamilton-Jacob-Bellman equation shows that the optimal control relates to the initial time and value of the controlled wealth process, and this time-inconsistent control is named as static optimality in [43]. From the statically optimal control, we can derive the time-consistent control, i.e. the dynamic optimality, as the dynamically optimal control is equal to the statically optimal control with the same initial status. Both optimal controls have been exhibited in Theorem 2.3. Furthermore, the controlled processes corresponding to both the statically and dynamically optimal control have been achieved in Theorem 2.3, from which we note that both the statically optimal control and dynamically optimal control prevent the bankruptcy naturally. However, comparing both the statically optimal controlled wealth process and the dynamically optimal one, we can find that the statically optimal control suggests that the investor should invest all the wealth in the bond if the current wealth is large enough, which introduces the upper boundary of the controlled wealth process. However, the dynamically optimal control encourages the investor to keep holding the risky asset to achieve a higher return, which has not been observed before. Besides that, it should be pointed out that, when comparing the previous work such as [33], [57] and [58], in this chapter, the optimal control represents the optimal percentage of wealth investing in the risky asset rather than the optimal total amount of risky asset.

Besides, we investigate the optimal control problems under the constraints on the size of the expectation and variance of the terminal wealth respectively. In Corollary 2.4, we achieve the optimal control that maximises the expectation of the terminal wealth of the investor, $\mathrm{E}_{t,x}(X_T^u)$, over all admissible control u such that the variance, $\mathrm{Var}_{t,x}(X_T^u)$, is bounded above by a positive constant. Furthermore, in Corollary 2.5, we achieve the optimal control that minimises the the variance of the terminal wealth of the investor, $\mathrm{Var}_{t,x}(X_T^u)$, over all admissible control u such that the expectation, $\mathrm{E}_{t,x}(X_T^u)$, is bounded below by a positive constant.

Additionally, in Section 4, we conduct numerical analysis for the static optimality and dynamic optimality. Upon the numerical results, we note that the performance of both strategies are highly impacted by the risk aversion rate. The static optimality outperforms the dynamic optimality when the risk aversion is large enough while the dynamic optimality leads to a more favourable result when risk aversion rate is smaller, which provides the investors with different levels of risk aversion with different investment strategies and has practical meaning in the real financial market. Furthermore, changing the risk aversion rate will change the distribution of the stopping time of both the static and dynamic wealth processes. Besides, we observe that, under the no short-selling constraint, both static optimality and dynamic optimality outperform the static and dynamic strategies introduced by [43] by setting the unconstrained case as an example. It should be mentioned that in this chapter, we assume that there is no transaction cost or tax deduction.

2.2 Formulation of the problem

There is a financial market which consists of two assets, a riskless bond and a risky stock. The price of the riskless bond is indicated by B, which solves the following stochastic differential equation:

$$dB_t = rB_t dt \tag{2.1}$$

with initial value $B_{t_0} = b$, where $b > 0$ and the riskless interest rate $r \in \mathbb{R}$ are constants. The price of the risky stock, S, follows a geometric Brownian motion, which solves

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2}$$

where we have the drift rate $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$. For this risky stock, we set the initial value $S_{t_0} = s$ for a given constant $s > 0$. Furthermore, in the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, S has the same natural filtration as that of W, where W is a standard Brownian motion. It is reasonable to stipulate that the value of μ must be greater than r, which indicates a risk premium. Otherwise, if $\mu \leq r$, a wise investor will simply invest all money in the riskless bond.

Suppose an investor wants to invest in a self-financing portfolio that consists of the two assets above. If the investor invests dynamically in time until the maturity $T > 0$ with initial wealth is $x_0 > 0$, the investor's wealth can be described by the following stochastic process or controlled wealth process (see e.g. [5], Chapter 6):

$$dX_t^u = (r + (\mu - r)u_t)X_t^u dt + \sigma u_t X_t^u dW_t. \tag{2.3}$$

By self-financing, we mean there is no external fund added or withdrawn. In (2.3), u_t represents the percentage of wealth invested in the stock at a given point of time t, where $t \in [t_0, T]$ and $t_0 \in [0, T)$, and U represents the set of all admissible controls. In this chapter, we do not allow short-selling of the stock, which means for any admissible control $u \in U$, we have $u \geq 0$. However, there is no constraint for borrowing money from the financial market with the riskless interest rate r, which means $1 - u \in (-\infty, 1]$.

Furthermore, for each admissible control u in (2.3), we assume $u_t = u(t, X_t^u)$ where $(t,x) \mapsto u(t,x) \cdot x$ is a continuous function from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$. For completeness, we define $u(t,0) = u(t,0) \cdot 0 = \lim_{0 \neq x \to 0} u(t,x)$ because the map $x \mapsto u(t,x)$ may not exist at 0. In other words, we replace $u_t X_t^u$ by $u(t,0) \cdot 0$ in (2.3) when $X_t^u = 0$. Additionally, in this chapter, we only consider the situation where each admissible control leads to a unique strong solution X^u in Itô's sense. (We will omit the case when u leads to the weak solution here).

Given a probability measure $\mathbb{P}_{t,x}$, for each admissible control u, X_t^u is a strong Markov process and takes value x for a given and fixed time t where $(t,x) \in [0,T] \times \mathbb{R}$. In the first constrained problem, we will find the optimal control for the following equation:

$$V(t,x) = \sup_{u \geq 0} [\mathrm{E}_{t,x}(X_T^u) - c \, \mathrm{Var}_{t,x}(X_T^u)] \tag{2.4}$$

in which $c > 0$ is a given constant. For all $u \in U$ we have $u \geq 0$ and $\mathrm{E}_{t,x}[(X_T^u)^2] < \infty$ and a sufficient condition for this relation is given by

$$\mathrm{E}_{t,x}\left[\int_t^T (1 + u_s^2)(X_s^u)^2 ds\right] < \infty, \tag{2.5}$$

which can be proved by Jensen's inequality and the Burkholder-Davis-Gundy inequality.

Apart from solving the first constrained problem (2.4), we also solve two constrained problems, which are defined by:

$$V_1(t,x) = \sup_{u\geq 0:\mathrm{Var}_{t,x}(X_T^u)\leq\alpha} \mathrm{E}_{t,x}(X_T^u) \tag{2.6}$$

$$V_2(t,x) = \inf_{u\geq 0:\mathrm{E}_{t,x}(X_T^u)\geq\beta} \mathrm{Var}_{t,x}(X_T^u) \tag{2.7}$$

where u is the admissible control defined above, $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$ are given constants. The solution of the first constrained problem can be used in the proof of the other two constrained cases V_1 and V_2, which will be explained in Corollary 2.4 and 2.5.

In this chapter, we modify the definition of static optimality and dynamic optimality given by reference [43] to fit the no short-selling constraint.

Definition 2.1 (Static optimality) [43]. For a given status $(t,x) \in [0,T]\times\mathbb{R}$, an admissible control $u_* \geq 0$ is called a statically optimal control in the constrained problem (2.4) if there is no other control $v \geq 0$ such that:

$$\mathrm{E}_{t,x}(X_T^v) - c\,\mathrm{Var}_{t,x}(X_T^v) > \mathrm{E}_{t,x}(X_T^{u_*}) - c\,\mathrm{Var}_{t,x}(X_T^{u_*}). \tag{2.8}$$

For a given status $(t,x) \in [0,T]\times\mathbb{R}$, an admissible control $u_* \geq 0$ is called a statically optimal control in the constrained problem (2.6) if $\mathrm{Var}_{t,x}(X_T^{u_*}) \leq \alpha$ and there is no other admissible $v \geq 0$ such that $\mathrm{Var}_{t,x}(X_T^v) \leq \alpha$ and:

$$\mathrm{E}_{t,x}(X_T^v) > \mathrm{E}_{t,x}(X_T^{u_*}). \tag{2.9}$$

For a given status $(t,x) \in [0,T]\times\mathbb{R}$, an admissible control $u_* \geq 0$ is called a statically optimal control in the constrained problem (2.7) if $\mathrm{E}_{t,x}(X_T^{u_*}) \geq \beta$ and there is no other control $v \geq 0$ such that $\mathrm{E}_{t,x}(X_T^v) \geq \beta$ and:

$$\mathrm{Var}_{t,x}(X_T^v) < \mathrm{Var}_{t,x}(X_T^{u_*}). \tag{2.10}$$

Definition 2.2 (Dynamic optimality) [43]. For each given and fixed pair $(t,x) \in [0,T]\times\mathbb{R}$, an admissible control $u_*(t,x) \geq 0$ is dynamically optimal in the constrained problem (2.4) if, for any admissible control $v(t,x) \neq u_*(t,x)$, there exists a control $w \geq 0$ such that $w(t,x) = u_*(t,x)$ and we have:

$$\mathrm{E}_{t,x}(X_T^v) - c\,\mathrm{Var}_{t,x}(X_T^v) < \mathrm{E}_{t,x}(X_T^w) - c\,\mathrm{Var}_{t,x}(X_T^w). \tag{2.11}$$

For each given and fixed pair $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control $u_*(t,x) \geq 0$ is dynamically optimal in the constrained problem (2.6) if $\mathrm{Var}_{t,x}(X_T^{u_*}) \leq \alpha$ and for any admissible control $v(t,x) \neq u_*(t,x)$ such that $\mathrm{Var}_{t,x}(X_T^v) \leq \alpha$, there exists a control $w \geq 0$ such that $w(t,x) = u_*(t,x)$ and $\mathrm{Var}_{t,x}(X_T^w) \leq \alpha$ and we have:

$$\mathrm{E}_{t,x}(X_T^v) < \mathrm{E}_{t,x}(X_T^w). \tag{2.12}$$

For each given and fixed pair $(t,x) \in [0,T] \times \mathbb{R}$, an admissible control $u_*(t,x) \geq 0$ is dynamically optimal in the constrained problem (2.7) if $\mathrm{E}_{t,x}(X_T^{u_*}) \geq \beta$ and for any admissible control $v(t,x) \neq u_*(t,x)$ such that $\mathrm{E}_{t,x}(X_T^v) \geq \beta$, there exists a control $w \geq 0$ such that $w(t,x) = u_*(t,x)$ and $\mathrm{E}_{t,x}(X_T^w) \geq \beta$ and we have:

$$\mathrm{Var}_{t,x}(X_T^v) > \mathrm{Var}_{t,x}(X_T^w). \tag{2.13}$$

2.3 Solution to the constrained problems

In this chapter, we will explain the solution of the constrained problems. The main idea of the proof below follows the idea in [43].

Theorem 2.3 *Consider the control problem $V(t,x) = \sup_u [\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)]$ in which X^u is the wealth process with $X_{t_0}^u = x_0$ and solves the SDE (2.3) under the probability measure $\mathbb{P}_{t_0,x_0}$ for $(t_0,x_0) \in [0,T] \times \mathbb{R}$ given and fixed. The related risk coefficient is defined by $\delta = (\mu - r)/\sigma$ in which $\mu, r \in \mathbb{R}$, $\mu > r$ and $\sigma > 0$. We further assume that $\delta \neq 0$ and $r \neq 0$ in the following part. (The cases $\delta = 0$ or $r = 0$ follow by passage to the limit when the non-zero δ or r approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,x) = \max\left[\frac{\delta}{\sigma}\frac{1}{x}[-x + x_0 \mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} + \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}], 0\right] \tag{2.14}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$. The statically optimal controlled wealth process is given by:

If $\tau_\alpha < \tau_\beta$,

$$X_t^s = \begin{cases} x_0 \mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} + \frac{1}{2\mathrm{c}}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})}\left[\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)} - \mathrm{e}^{-\delta(\mathrm{W}_\mathrm{t}-\mathrm{W}_{\mathrm{t}_0})-\frac{\delta^2}{2}(\mathrm{t}-\mathrm{t}_0)}\right] & \text{if } t \leq \tau_\alpha \\ x_0 \mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} + \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})} & \text{if } \tau_\alpha < t \end{cases} \tag{2.15}$$

If $\tau_\beta < \tau_\alpha$,

$$X_t^s = \begin{cases} x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{(\delta^2 - \mathrm{r})(\mathrm{T-t})} \left[\mathrm{e}^{\delta^2(\mathrm{T-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t - W_{t_0}}) - \frac{\delta^2}{2}(\mathrm{t-t_0})} \right] & \text{if } t \leq \tau_\beta \\ 0 & \text{if } \tau_\beta < t \end{cases} \tag{2.16}$$

where τ_α *and* τ_β *are given by:*

$$\begin{aligned} \tau_\alpha = \inf \Big\{ t \in [t_0, T] | x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{(\delta^2 - \mathrm{r})(\mathrm{T-t})} [\mathrm{e}^{\delta^2(\mathrm{T-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t - W_{t_0}}) - \frac{\delta^2}{2}(\mathrm{t-t_0})}] \\ = x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{\delta^2(\mathrm{T-t_0}) - \mathrm{r(T-t)}} \Big\} \end{aligned} \tag{2.17}$$

$$\tau_\beta = \inf \Big\{ t \in [t_0, T] | x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{(\delta^2 - \mathrm{r})(\mathrm{T-t})} [\mathrm{e}^{\delta^2(\mathrm{T-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t - W_{t_0}}) - \frac{\delta^2}{2}(\mathrm{t-t_0})}] \leq 0 \Big\} \tag{2.18}$$

for $t \in [t_0, T]$ *respectively.*

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,x) = \max \left[\frac{\delta}{2c\sigma} \frac{1}{x} \mathrm{e}^{(\delta^2 - \mathrm{r})(\mathrm{T-t})}, 0 \right] \tag{2.19}$$

for $(t,x) \in [t_0, T] \times \mathbb{R}$. *The dynamically optimal controlled wealth process is given by:*

$$X_t^d = \begin{cases} x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{(\delta^2 - \mathrm{r})(\mathrm{T-t})} [\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - 1 + \delta \int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{e}^{\delta^2(\mathrm{t-s})} \mathrm{dW_s}] & \text{if } t \leq \tau_\gamma \\ 0 & \text{if } \tau_\gamma < t \end{cases} \tag{2.20}$$

where τ_γ *is given by:*

$$\begin{aligned} \tau_\gamma = \inf \Big\{ t \in [t_0, T] | x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{(\delta^2 - \mathrm{r})(\mathrm{T-t})} [\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - 1 \\ + \delta \int_{t_0}^{t} \mathrm{e}^{\delta^2(\mathrm{t-s})} \mathrm{dW_s}] \leq 0 \Big\} \end{aligned} \tag{2.21}$$

for $t \in [t_0, T]$.

Proof. In this proof, we claim that, for each pair of $(t_0, x_0) \in [0, T] \times \mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0, x_0}$ under which X^u is the solution of the SDE (2.3) with initial condition $X_{t_0}^u = x_0$. Furthermore, for X_t^u, $u \in U$ is any admissible control we defined in Section 2.2.

(A): The objective function $\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)$ can be re-arranged as:

$$\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) = \mathrm{E}_{t,x}(X_T^u) - c \left[\mathrm{E}_{t,x}[(X_T^u)^2] - \mathrm{E}_{t,x}(X_T^u)^2 \right] \tag{2.22}$$

in which $\mathrm{E}_{t,x}(X_T^u)^2$ brings quadratic nonlinearity into our problem. To handle this difficulty, we condition the size of $\mathrm{E}_{t,x}(X_T^u)$ by setting $\mathrm{E}_{t,x}(X_T^u) = M$, and this leads to:

$$\begin{aligned} V(t,x) &= \sup_{M\in\mathbb{R}} \sup_{u\geq 0:\mathrm{E}_{t,x}(X_T^u)=M} \big[\mathrm{E}_{t,x}(X_T^u) - c\big[\mathrm{E}_{t,x}[(X_T^u)^2] - \mathrm{E}_{t,x}(X_T^u)^2\big]\big] \\ &= \sup_{M\in\mathbb{R}} \big[M + cM^2 - \inf_{u\geq 0:\mathrm{E}_{t,x}(X_T^u)=M} \mathrm{E}_{t,x}[(X_T^u)^2]\big]. \end{aligned} \tag{2.23}$$

In equation (2.23), there is a constrained problem:

$$V_M(t,x) = \inf_{u\geq 0:\mathrm{E}_{t,x}(X_T^u)=M} \mathrm{E}_{t,x}\left[(X_T^u)^2\right] \tag{2.24}$$

where $M \in \mathbb{R}$ given and fixed and $u \geq 0$ is any admissible control. Applying Lagrange multipliers in equation (2.24), we have the following Lagrangian function:

$$L_{t,x}(u,\lambda) = \mathrm{E}_{t,x}[(X_T^u)^2] - \lambda[\mathrm{E}_{t,x}(X_T^u) - M] \tag{2.25}$$

in which $\lambda > 0$. Solving equation (2.25) gives the optimal control that minimises (2.24). To verify this, assume there exists u_*^λ that is the optimal control in (2.25) such that:

$$L_{t,x}(u_*^\lambda,\lambda) := \inf_u L_{t,x}(u,\lambda). \tag{2.26}$$

Furthermore, we assume there is a $\lambda = \lambda(M,t,x) > 0$ such that $\mathrm{E}_{t,x}(X_T^{u_*^\lambda}) = M$. Therefore,

$$V_M(t,x) = L_{t,x}(u_*^\lambda,\lambda) \leq \mathrm{E}_{t,x}[(X_T^u)^2] \tag{2.27}$$

for any admissible control $u \in U$ with $\mathrm{E}_{t,x}(X_T^u) = M$, which indicates that the optimal control u_*^λ that minimises (2.26) with $\mathrm{E}_{t,x}(X_T^{u_*^\lambda}) = M$ is optimal in (2.24).

2. To solve (2.25) and achieve the optimal control, we need to solve the following optimal control problem:

$$V^\lambda(t,x) = \inf_{u\geq 0} \mathrm{E}_{t,x}[(X_T^u)^2 - \lambda X_T^u] \tag{2.28}$$

where $u \in U$ is any admissible control. In this chapter, we use the HJB approach to achieve the optimal control.

According to (2.28) and SDE (2.3), we have the following HJB system:

$$\inf_{u\geq 0} \big[V_t^\lambda + (r + (\mu - r)u)xV_x^\lambda + \frac{1}{2}\sigma^2 u^2 x^2 V_{xx}^\lambda\big] = 0 \tag{2.29}$$

$$V^\lambda(T,x) = x^2 - \lambda x \tag{2.30}$$

on $[t_0,T]\times\mathbb{R}$. However, because we have constrained the admissible control in $[0,\infty)$, which causes non-smoothness of the value function V^{λ}, the classical method in [5] is no longer able to handle this problem. To overcome this difficult, we will follow the idea of [33] to construct a value function $V^{\lambda}(t,x)$ and prove its optimality.

We define the value function $V^{\lambda}(t,x)$ as:

$$V^{\lambda}(t,x)=\begin{cases}A(t)x^2+B(t)x+C(t) & \text{if } -\frac{\delta}{\sigma}\frac{1}{x}[x-\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}]\geq 0\\ E(t)x^2+F(t)x+G(t) & \text{if } -\frac{\delta}{\sigma}\frac{1}{x}[x-\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}]<0\end{cases} \tag{2.31}$$

where

$$\begin{cases}A'(t)=(\delta^2-2r)A(t)\\ A(T)=1,\end{cases} \tag{2.32}$$

$$\begin{cases}B'(t)=(\delta^2-r)B(t)\\ B(T)=-\lambda,\end{cases} \tag{2.33}$$

$$\begin{cases}C'(t)=\frac{\delta^2}{4}\frac{B(t)^2}{A(t)}\\ C(T)=0,\end{cases} \tag{2.34}$$

and

$$\begin{cases}E'(t)=-2rE(t)\\ E(T)=1,\end{cases} \tag{2.35}$$

$$\begin{cases}F'(t)=-rF(t)\\ F(T)=-\lambda,\end{cases} \tag{2.36}$$

$$\begin{cases}G'(t)=0\\ G(T)=0.\end{cases} \tag{2.37}$$

Now, we are going to demonstrate (2.31) is the optimal value function. Assume that there is a (t,x)-plane Γ_1 such that:

$$\Gamma_1:=\left\{(t,x)\in[t_0,T]\times\mathbb{R}\mid -\frac{\delta}{\sigma}\frac{1}{x}\left[x-\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right]>0\right\}. \tag{2.38}$$

Solving (2.32)-(2.34) gives:

$$\begin{cases} A(t) = \mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t})} \\ B(t) = -\lambda \mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})} \\ C(t) = -\frac{\lambda^2}{4}[1-\mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t})}]. \end{cases} \tag{2.39}$$

Clearly, we can observe that (2.31) is sufficiently smooth in Γ_1. Hence, substituting the expressions for V_t^λ, V_x^λ and V_{xx}^λ into the HJB system (2.29) yields:

$$\begin{aligned} & V_t^\lambda + (r(1-u)+\mu u)xV_x^\lambda + \frac{1}{2}\sigma^2 u^2 x^2 V_{xx}^\lambda \qquad (2.40) \\ & = A'(t)x^2 + B'(t)x + C'(t) + rx[2A(t)x + B(t)] \\ & \quad + \inf_{u\geq 0}\left\{(\mu-r)x(2A(t)x+B(t))u + \sigma^2 x^2 A(t)u^2\right\} \\ & = (\delta^2-2r)\mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t})}x^2 - \lambda(\delta^2-r)\mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})}x + \frac{\lambda^2\delta^2}{4}\mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t})} \\ & \quad + rx[2\mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t})}\mathrm{x} - \lambda \mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})}] \\ & \quad + \inf_{u\geq 0}\left\{(\mu-r)x(2\mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t})}\mathrm{x} - \lambda \mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})})\mathrm{u} + \sigma^2\mathrm{x}^2\mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t})}\mathrm{u}^2\right\}. \end{aligned}$$

In (2.40), the last term on the right hand side is a quadratic function with respect to u. Due to the properties of quadratic functions, the unique optimal control u^* is given by:

$$u^*(t,x) = -\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right] \tag{2.41}$$

which is greater than zero in Γ_1. Substituting (2.41) back into (2.40), the right hand side equals zero, which means V^λ in (2.31) satisfies the HJB system (2.29)-(2.30) and is the optimal value function in the region Γ_1. Similarly, we define another region Γ_2 such that

$$\Gamma_2 := \left\{(t,x) \in [t_0,T]\times\mathbb{R} \mid -\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right] < 0\right\}. \tag{2.42}$$

Solving (2.35)-(2.37) gives:

$$\begin{cases} E(t) = \mathrm{e}^{2\mathrm{r}(\mathrm{T}-\mathrm{t})} \\ F(t) = -\lambda \mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t})} \\ G(t) = 0. \end{cases} \tag{2.43}$$

In region Γ_2, V^λ in (2.31) is smooth enough for the HJB system (2.29)-(2.30). Hence, substituting the equations V_t^λ, V_x^λ and V_{xx}^λ into the HJB system (2.29) yields:

$$\begin{aligned}
&V_t^\lambda + (r(1-u)+\mu u)xV_x^\lambda + \frac{1}{2}\sigma^2 u^2 x^2 V_{xx}^\lambda \\
&= E'(t)x^2 + F'(t)x + rx[2E(t)x + F(t)] \\
&\quad + \inf_{u\geq 0}\left\{(\mu - r)x(2E(t)x+F(t))u + \sigma^2 x^2 E(t)u^2\right\} \\
&= -2re^{2r(T-t)}x^2 + \lambda re^{r(T-t)}x + rx[2e^{2r(T-t)}x - \lambda e^{r(T-t)}] \\
&\quad + \inf_{u\geq 0}\left\{(\mu - r)x(2e^{2r(T-t)}x - \lambda e^{r(T-t)})u + \sigma^2 x^2 u^2 e^{2r(T-t)}\right\}.
\end{aligned} \tag{2.44}$$

The last term in (2.44) is a quadratic function of u and the global optimal control $u^*(t,x)$ is given by $u^*(t,x) = -\frac{\delta}{\sigma}\frac{1}{x}[x - \frac{\lambda}{2}e^{-r(T-t)}]$. However, in Γ_2, we have $-\frac{\delta}{\sigma}\frac{1}{x}[x - \frac{\lambda}{2}e^{-r(T-t)}] < 0$ and all admissible controls must be greater than or equal to 0. Hence, we claim the optimal control $u^*(t,x)$ is given by:

$$u^*(t,x) = 0 \tag{2.45}$$

which is the locally optimal control in Γ_2. To verify (2.45) is the solution of the HJB system, we substitute (2.45) into (2.44) which leads to zero on the right-hand side. This means (2.45) is the optimal control and V^λ in (2.31) is the optimal value function in Γ_2 for the HJB system (2.29)-(2.30). Now, we define the final region, Γ_3 such that:

$$\Gamma_3 := \left\{(t,x) \in [t_0, T] \times \mathbb{R} \mid -\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}e^{-r(T-t)}\right] = 0\right\}. \tag{2.46}$$

In this region, it can be noted that $A(t)x^2 + B(t)x + C(t) = E(t)x^2 + F(t)x + G(t) = -\lambda^2/4$, which means, for each pair $(t,x) \in \Gamma_3$, $V^\lambda(t,x)$ is continuous. Additionally, we have:

$$\begin{cases} V_t^\lambda = A'(t)x^2 + B'(t)x + C'(t) = E'(t)x^2 + F'(t)x + G'(t) \\ V_x^\lambda = 2A(2)x + B(t) = 2E(t)x + F(t) \end{cases} \tag{2.47}$$

which indicates the existence and continuity of V_t^λ and V_x^λ in Γ_3. However, V_{xx}^λ does not exist in Γ_3 because $A(t) \neq E(t)$. This indicates that V^λ given by (2.31) is not a $C^{1,2}$ function for the HJB system (2.29)-(2.30). To overcome this non-smoothness of the value function (2.31), we will show V^λ can be expressed through a change-of-variable formula including this curve, and the optimality can be achieved by verification theorem. On this plane Γ_3, where $-\frac{\delta}{\sigma}\frac{1}{x}[x - \frac{\lambda}{2}e^{-r(T-t)}] = 0$, we have:

$$x = \frac{\lambda}{2}e^{-r(T-t)} \tag{2.48}$$

where x is a function from $\mathbb{R}_+$ to $\mathbb{R}$. For any partition $0 \leq t_0 \leq t_1 \leq t_2 \leq ...... \leq t_n = T$ of $[t_0, T]$, we have:

$$\begin{aligned} BV([t_0,T]) &= \sum_{i=0}^{n-1} \left|\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t}_{\mathrm{i}+1})} - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t}_{\mathrm{i}})}\right| = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{rT}}\sum_{\mathrm{i}=0}^{\mathrm{n}-1}|\mathrm{e}^{\mathrm{rt}_{\mathrm{i}+1}} - \mathrm{e}^{\mathrm{rt}_{\mathrm{i}}}| \\ &= \frac{\lambda}{2}\mathrm{e}^{-\mathrm{rT}}[\mathrm{e}^{\mathrm{rt}_{\mathrm{n}}} - \mathrm{e}^{\mathrm{rt}_{\mathrm{n}-1}} + \mathrm{e}^{\mathrm{rt}_{\mathrm{n}-1}} - \mathrm{e}^{\mathrm{rt}_{\mathrm{n}-2}} + ... + \mathrm{e}^{\mathrm{rt}_1} - \mathrm{e}^{\mathrm{rt}_0}] \\ &= \frac{\lambda}{2}\mathrm{e}^{-\mathrm{rT}}[\mathrm{e}^{\mathrm{rT}} - \mathrm{e}^{\mathrm{rt}_0}] < \infty \end{aligned} \tag{2.49}$$

which shows that the curve $b(t) = x$ parameterising Γ_3 is of bounded variation. Additionally, V^λ_{xx} is locally bounded in both of Γ_1 and Γ_2. Hence, according to Theorem 3.1 in [45], we conclude that the change-of-variable formula can be applied to V^λ on $[t_0, T] \times \mathbb{R}$, which yields:

$$\begin{aligned} V^\lambda(t, X^u_t) =& V^\lambda(t_0, x_0) \\ &+ \int_{t_0}^{t} \left(V^\lambda_t + (r + (\mu - r)uX^u_s)V^\lambda_x + \frac{\sigma^2 u^2 X^{u2}_s}{2}V^\lambda_{xx}\right) I(X^u_s \neq b(s))ds \\ &+ \int_{t_0}^{t} (\sigma u X^u_s V^\lambda_x)dW_s \end{aligned} \tag{2.50}$$

where $b(s) = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{s})}$. Also, due to the continuity of admissible controls, we claim the optimal control is given by

$$u^*(t,x) = \max\left[-\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right], 0\right]. \tag{2.51}$$

Hence, for any admissible control $u \in U$, (2.51) holds. Furthermore, based upon (2.50), $V^\lambda(T, X^u_T)$ is given by:

$$\begin{aligned} V^\lambda(T, X^u_T) =& V^\lambda(t, x) \\ &+ \int_{t}^{T} \left(V^\lambda_t + (r + (\mu - r)uX^u_s)V^\lambda_x + \frac{\sigma^2 u^2 X^{u2}_s}{2}V^\lambda_{xx}\right) I(X^u_s \neq b(s))ds \\ &+ \int_{t}^{T} (\sigma u X^u_s V^\lambda_x)dW_s. \end{aligned} \tag{2.52}$$

As we have shown V^λ in (2.31) solves the HJB equation in Γ_1 and Γ_2, hence, we have:

$$\left(V^\lambda_t + (r + (\mu - r)uX^u_s)V^\lambda_x + \frac{\sigma^2 u^2 X^{u2}_s}{2}V^\lambda_{xx}\right) I(X^u_s \neq b(s)) \geq 0. \tag{2.53}$$

Moreover, according to the terminal condition (2.30) of V^λ at the maturity T, we can see that:

$$
\begin{aligned}
X_T^{u2} - \lambda X_T^u = & V^\lambda(t,x) \\
& + \int_t^T \left(V_t^\lambda + (r + (\mu - r) u X_s^u) V_x^\lambda + \frac{\sigma^2 u^2 X_s^{u2}}{2} V_{xx}^\lambda) I(X_s^u \neq b(s) \right) ds \\
& + \int_t^T (\sigma u X_s^u V_x^\lambda) dW_s.
\end{aligned} \tag{2.54}
$$

Equations (2.53) and (2.54) yield:

$$
V^\lambda(t,x) \leq X_T^{u2} - \lambda X_T^u - \int_t^T (\sigma u X_s^u V_x^\lambda) dW_s. \tag{2.55}
$$

Taking $\mathrm{E}_{t,x}$ on the both side of (2.55), the stochastic integral, a martingale under condition $\mathrm{E}_{t_0,x_0}[\max_{t_0 \leq t \leq T}(X_t^\nu)^2] < \infty$, disappears. Then we have the following inequality:

$$
V^\lambda(t,x) \leq \mathrm{E}_{t,x}[X_T^{u2} - \lambda X_T^u]. \tag{2.56}
$$

Equation (2.56) holds for all admissible controls, which indicates:

$$
V^\lambda(t,x) \leq \inf_u \mathrm{E}_{t,x}[X_T^{u2} - \lambda X_T^u]. \tag{2.57}
$$

For the reverse inequality, we claim the optimal control is given by (2.51) and for the optimal control u^* in Γ_1 and Γ_2, we have:

$$
V_t^\lambda + (r + (\mu - r) u^*) x V_x^\lambda + \frac{1}{2} \sigma^2 u^{*2} x^2 V_{xx}^\lambda = 0. \tag{2.58}
$$

Combining (2.58) with (2.54) yields:

$$
V^\lambda(t,x) = X_T^{u^* 2} - \lambda X_T^{u^*} - \int_t^T (\sigma u^* X_s^u V_x^\lambda) dW_s. \tag{2.59}
$$

For completeness, on the curve (i.e. $(t,x) \in \Gamma_3$), the optimal control is given by $u^* = -\frac{\delta}{\sigma} \frac{1}{x} [x - \frac{\lambda}{2} \mathrm{e}^{-\mathrm{r(T-t)}}] = 0$, and it can easily be seen that:

$$
V^\lambda(t,x) = X_T^{u^* 2} - \lambda X_T^{u^*}. \tag{2.60}
$$

Hence, for the optimal control u^* given by (2.51), we always have that:

$$
V^\lambda(t,x) = \mathrm{E}_{t,x}[X_T^{u^* 2} - \lambda X_T^{u^*}]. \tag{2.61}
$$

Therefore, there is the trivial inequality:

$$
\inf_u \mathrm{E}_{t,x}[X_T^{u2} - \lambda X_T^u] \leq \mathrm{E}_{t,x}[X_T^{u^* 2} - \lambda X_T^{u^*}]. \tag{2.62}
$$

Combining (2.57) with (2.62), we have:

$$V^{\lambda}(t,x) \leq \inf_{u} \mathrm{E}_{t,x}[X_T^{u2} - \lambda X_T^u] \leq \mathrm{E}_{t,x}[X_T^{u^*2} - \lambda X_T^{u^*}] = V^{\lambda}(t,x). \tag{2.63}$$

Therefore, according to the verification theorem (cf. [5], Theorem 19.6), we conclude that V^{λ} given by (2.31) is the optimal value function and u^* given by (2.51) is the optimal control for the HJB system (2.29)-(2.30).

3. Observing the optimal control (2.51), it can be seen that:

$$u(t,x) = -\frac{\delta}{\sigma}\frac{1}{x}[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}] > 0 \tag{2.64}$$

only if:

$$0 \leq x < \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}. \tag{2.65}$$

To describe the controlled process, we define the following two stopping times:

$$\tau_{\alpha} = \inf\left\{t \in [t_0,T] | X_t^s = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right\} \tag{2.66}$$

$$\tau_{\beta} = \inf\left\{t \in [t_0,T] | X_t^s \leq 0\right\} \tag{2.67}$$

with $\mathbb{P}_{t_0,x_0}(T < \tau_{\alpha}) > 0$ and $\mathbb{P}_{t_0,x_0}(T < \tau_{\beta}) > 0$. The stopping times τ_{α} and τ_{β} represent the first hitting time when the controlled process hits the upper curve $b(t) = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}$ and 0 respectively. In the following part, we will focus on the case when $u(t,x) = -\frac{\delta}{\sigma}\frac{1}{x}[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}] > 0$ (i.e. $t \leq (\tau_{\alpha} \wedge \tau_{\beta})$ for all $t \in [t_0,T]$) and achieving the optimal value for λ.

4. Firstly, we define a stochastic process Z_t such that

$$Z_t = K - \mathrm{e}^{-\mathrm{r(t-t_0)}}\mathrm{X_t^u} \tag{2.68}$$

where $K = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t_0)}}$ and $Z_0 = K - x_0$ under $\mathbb{P}_{t_0,x_0}$. Using Ito's formula and the SDE (2.3), we have:

$$dZ_t = -\delta^2 Z_t dt - \delta Z_t dW_t. \tag{2.69}$$

Solving SDE (2.69) and using (2.68), we have the following closed form solution:

$$X_t^u = \mathrm{e}^{\mathrm{r(t-t_0)}}\left[\mathrm{K} - (\mathrm{K} - \mathrm{x_0})\mathrm{e}^{-\delta(\mathrm{W_t} - \mathrm{W_{t_0}}) - \frac{3\delta^2}{2}(\mathrm{t-t_0})}\right] \tag{2.70}$$

which gives an unique strong solution of SDE (2.3) under $u^*(t,x)$ when $T \leq (\tau_\alpha \wedge \tau_\beta)$.

Considering the condition that $\mathrm{E}_{t,x}(X_T^u) = M$, we have:

$$\mathrm{E}_{t_0,x_0}(X_T^u) = x_0 \mathrm{e}^{-(\delta^2 - \mathrm{r})(\mathrm{T}-\mathrm{t}_0)} + \frac{\lambda}{2}\left[1 - \mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t}_0)}\right] = \mathrm{M}. \tag{2.71}$$

Rearranging (2.71) yields:

$$\lambda = 2\frac{M - x_0 \mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t}_0)}}{1 - \mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t}_0)}}. \tag{2.72}$$

Upon noting (2.70), it can be seen that:

$$\mathrm{E}_{t_0,x_0}[(X_T^u)^2] = {x_0}^2 \mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t}_0)} + \frac{\lambda^2}{4}\left[1 - \mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t}_0)}\right]. \tag{2.73}$$

Combining (2.72) with (2.73), we can find that (2.24) is given by:

$$V_M(t_0,x_0) = {x_0}^2 \mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t}_0)} + \frac{\left(\mathrm{M} - \mathrm{x}_0 \mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t}_0)}\right)^2}{1 - \mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t})}}. \tag{2.74}$$

Substituting (2.74) into (2.23) yields:

$$V(t_0,x_0) = \sup_{M\in\mathbb{R}} \left[M + cM^2 - c({x_0}^2 \mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T}-\mathrm{t}_0)} + \frac{(\mathrm{M} - \mathrm{x}_0 \mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t}_0)})^2}{1 - \mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t})}})\right] \tag{2.75}$$

which is a quadratic function with respect to M. Noting that the coefficient of the term M^2 is strictly negative, there is a unique maximum point in (2.75), which is given by:

$$M_* = x_0 \mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)} + \frac{1}{2\mathrm{c}}\left[\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)} - 1\right]. \tag{2.76}$$

Substituting (2.76) into (2.72), we can easily see that:

$$\lambda_* = 2x_0 \mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)} + \frac{1}{\mathrm{c}} \mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}. \tag{2.77}$$

Substituting (2.77) into (2.51), we achieved the statically optimal control given by (2.14). Inserting (2.77) into (2.70), we find the statically optimal controlled process for $T \leq (\tau_\alpha \wedge \tau_\beta)$. Furthermore, inserting (2.70) and (2.77) into (2.66) and (2.67) respectively, we can achieve the stopping times given by (2.17) and (2.18). In (2.17), we can see that the curve where the non-smoothness of the value function occurs is given by

$$b(t) = x_0 \mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} + \frac{1}{2\mathrm{c}} \mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}. \tag{2.78}$$

Furthermore, in the case when $\tau_\alpha < t$ for $t \in [t_0, T]$, the optimal control is $u^s_*(t,x) = 0$, and this reduces the SDE (2.3) into $dX^u_t = rX^u_t dt$ with the initial value $X^u_{\tau_\alpha}$. Solving $dX^u_t = rX^u_t dt$ with the initial value $X^u_{\tau_\alpha}$ gives:

$$X^s_t = x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}} \tag{2.79}$$

for $t \in [\tau_\alpha, T]$. Note that once X^s_t hits the upper curve $b(t)$, it will stay on the curve until the maturity and this confirms (2.15).

In another case, when $\tau_\beta < t$, X^s_t hits zero at τ_β. Once X^s_t hits zero, the wealth process will go beneath 0 with probability one almost surely, and the optimal control (3.1) gives $u^s_* = 0$. In this situation, SDE (2.3) will become $dX^u_t = rX^u_t dt$ with initial value $\lim_{h\to 0} X^u_{\tau_\beta + h} = X^u_{\tau_\beta} = 0$, which gives $X^s_t = 0$ for $t \in [\tau_\beta, T]$. Hence, we have confirmed (2.16) and completed the first part of the proof.

(B) As we claim that the dynamically optimal control is equal to the statically optimal control with the same initial state (t,x), replacing x_0 and t_0 by x and t in (2.14) gives the candidate dynamically optimal control given in (2.19). To prove the optimality of (2.19), we set that $u^d_*(t_0,x_0) = w(t_0,x_0)$, $w(t_0,x_0) = u^s_*(t_0,x_0)$, and $v(t_0,x_0)$ for any admissible such that $v(t_0,x_0) \neq u^d_*(t_0,x_0)$. For a dynamically optimal control, the following relationship must hold:

$$V_w(t_0,x_0) = \mathrm{E}_{t_0,x_0}(X^w_T) - c\,\mathrm{Var}_{t_0,x_0}(X^w_T) > \mathrm{E}_{t_0,x_0}(X^v_T) - c\,\mathrm{Var}_{t_0,x_0}(X^v_T) = V_v(t_0,x_0) \tag{2.80}$$

for any $(t_0,x_0) \in [0,T] \times \mathbb{R}$ based upon the fact that $V_w(t_0,x_0) = V(t_0,x_0)$ and $w(t_0,x_0)$ is statically optimal in (2.4).

5. To verify (2.80), we set $\mathrm{E}_{t_0,x_0}(X^w_T) = M_*$, the value of M_* is given by (2.76), and $\mathrm{E}_{t_0,x_0}(X^v_T) = M_v$. Let us consider the case when $M_* \neq M_v$ firstly. For a given pair $(t_0,x_0) \in [0,T] \times \mathbb{R}$ such that $\frac{\delta}{2c\sigma}\frac{1}{x_0}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T-t_0})} > 0$, applying (2.75)+(2.76) and (2.24)+(2.74) yields:

$$\begin{aligned} V_w(t_0,x_0) &= M_* + cM_*{}^2 - c\left(x_0{}^2 \mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T-t_0})} + \frac{\left(\mathrm{M}_* - \mathrm{x_0 e}^{-(\delta^2-\mathrm{r})(\mathrm{T-t_0})}\right)^2}{1-\mathrm{e}^{-\delta^2(\mathrm{T-t})}} \right) \\ &> M_v + cM_v{}^2 - c\left(x_0{}^2 \mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T-t_0})} + \frac{\left(\mathrm{M_v} - \mathrm{x_0 e}^{-(\delta^2-\mathrm{r})(\mathrm{T-t_0})}\right)^2}{1-\mathrm{e}^{-\delta^2(\mathrm{T-t})}} \right) \\ &= V_v(t_0,x_0) \end{aligned} \tag{2.81}$$

in which the strict inequality holds because M_* is the unique and global optimal point of (2.75). Inequality (2.81) verifies the relation we state in (2.80) when $M_* \neq M_v$.

It should be mentioned that in a special case when $x_0 = 0$, there is $w(t_0, x_0) = u_*^s(t_0, x_0) = 0$. In this case, for any other admissible controls $v(t_0, x_0) > 0$, there is $M_* = M_v$ as the expected value of $x_0 = 0$ must be 0 otherwise there is an arbitrage opportunity. To prove the optimality of $w(t_0, x_0)$ under $x_0 = 0$, we claim the following relation:

$$V_v^{\lambda_*}(t_0, x_0) = \mathrm{E}_{t_0,x_0}[(X_T^v)^2 - \lambda_* X_T^v] > V^{\lambda_*}(t_0, x_0) = \mathrm{E}_{t_0,x_0}[(X_T^w)^2 - \lambda_* X_T^w] \tag{2.82}$$

in which V^{λ_*} and λ_* were defined in (2.28) and (2.77) respectively. By using our previous proof (2.52)-(2.54), we can see that under the terminal condition of the HJB system (2.30) and the change-of-variable formula with local time on curve, there is:

$$\begin{aligned} V^{\lambda_*}(T, X_T^v) &= (X_T^v)^2 - \lambda_* X_T^v \\ &= V^{\lambda_*}(t_0, x_0) \\ &\quad + \int_{t_0}^{T} \left(V_t^{\lambda_*} + [r(1-v) + \mu v] X_s^v V_x^{\lambda_*} + \frac{\sigma^2 v^2}{2} X_s^{v2} V_{xx}^{\lambda_*} \right) I(X_s^v \neq b(s)) ds \\ &\quad + M_T \end{aligned} \tag{2.83}$$

in which $M_t = \int_{t_0}^{t} \sigma V X_s^v V_x^{\lambda_*} dW_s$, under the probability measure $\mathbb{P}_{t_0,x_0}$, is a continuous local martingale and $b(t) = \frac{\lambda_*}{2} \mathrm{e}^{-\mathrm{r(T-t)}}$ for any $t \in [t_0, T]$. By using Burkholder-Davis-Gundy Inequality and Jensen's inequality, we can see that $\mathrm{E}_{t_0,x_0}[\max_{t_0 \le t \le T} (X_t^v)^2] < \infty$ upon noting the condition $\mathrm{E}_{t_0,x_0}[\int_{t_0}^{T} (1 + v_t^2)(X_t^v)^2 dt] < \infty$. Furthermore, according to Holder's inequality, it can be verified that $\mathrm{E}_{t_0,x_0} < M, M >_T < \infty$ which is sufficient to ensure M_t is a martingale with $\mathrm{E}_{t_0,x_0}(M_T) = 0$. Hence, taking E_{t_0,x_0} on (2.83) yields:

$$\begin{aligned} \mathrm{E}_{t_0,x_0}[(X_T^v)^2 - \lambda_* X_T^v] =& V^{\lambda_*}(t_0, x_0) + \mathrm{E}_{t_0,x_0} \left[\int_{t_0}^{T} [V_t^{\lambda_*} + [r(1-v) + \mu v] X_s^v V_x^{\lambda_*} \right. \\ & \left. + \frac{\sigma^2 v^2}{2} X_s^{v2} V_{xx}^{\lambda_*}] I(X_s^v \neq b(s)) ds \right]. \end{aligned} \tag{2.84}$$

Assuming $x_0 \neq 0$ and we define a region $R_\varepsilon = [t_0, t_0 + \varepsilon] \times [x_0 - \varepsilon, x_0 + \varepsilon]$ for some $\varepsilon > 0$ small enough such that $t_0 + \varepsilon \le T$, then for all $(s, x) \in R_\varepsilon$, there is $v(s, x) \neq w(s, x)$, which can be easily seen by the continuity of v and w. Furthermore, in the previous part, we have stated that $w(t, x)$ is the unique minimum control of the value function of (2.29) for each pair of $(t, x) \in [0, T] \times \mathbb{R}$. Hence, the value of ε is sufficiently small enough to meet:

$$V_t^{\lambda_*} + [r(1-v) + \mu v] X_s^v V_x^{\lambda_*} + \frac{\sigma^2 v^2}{2} X_s^{v2} V_{xx}^{\lambda_*} I(X_s^v \neq b(s)) \ge \beta > 0 \tag{2.85}$$

where β is a positive constant given and fixed and $(s,x) \in R_\varepsilon$. Hence, setting $\tau_\varepsilon = \inf\{s \in [t_0, t_0+\varepsilon] | (s, X_s^v) \notin R_\varepsilon\}$, we can see that:

$$\mathrm{E}_{t_0,x_0}[(X_T^v)^2 - \lambda_* X_T^v] \geq V^{\lambda_*}(t_0,x_0) + \beta(\tau_\varepsilon - t_0) > V^{\lambda_*}(t_0,x_0). \tag{2.86}$$

This conclusion will hold in the case when $x_0 = 0$, in which case we identify $v(t_0,0)$ and $w(t_0,0)$ by $v(t_0,0)\cdot 0$ and $w(t_0,0)\cdot 0$. Hence, according to (2.86), we have verified (2.82).

Moreover, inequality (2.82) indicates that $\mathrm{E}_{t_0,x_0}[X_T^{w2}] - \lambda_* M_* < \mathrm{E}_{t_0,x_0}[X_T^{v2}] - \lambda_* M_v$, which yields:

$$\mathrm{E}_{t_0,x_0}[X_T^{w2}] < \mathrm{E}_{t_0,x_0}[X_T^{v2}] \tag{2.87}$$

upon recalling the assumption $M_* = M_v$ we introduced before. Therefore, according to (2.23), we have:

$$M_* + cM_*^2 - \mathrm{E}_{t_0,x_0}[X_T^{w2}] > M_v + cM_v^2 - \mathrm{E}_{t_0,x_0}[X_T^{v2}] \tag{2.88}$$

under the hypothesis $M_* = M_v$, which confirms the statement we made in (2.80). Hence, we conclude that u_*^d given by (2.19) is the dynamically optimal control.

6. Observing the dynamically optimal control given by (2.19), we can see that $u_*^d(t,x) = 0$ if and only if $x \leq 0$. Hence, to describe the optimal controlled process, we define the following stopping time:

$$\tau_\gamma = \inf\{t \in [t_0,T] | X_t^d \leq 0\} \tag{2.89}$$

with $\mathbb{P}_{t_0,x_0}(T < \tau_\gamma) > 0$, and this is the first hitting time at 0. When X_t^d hits zero (i.e. $\tau_\gamma \leq T$), the wealth process will go beneath 0 with probability one almost surely, and the optimal control (2.19) gives $u_*^d = 0$ which reduces the SDE (2.3) into $dX_t^u = rX_t^u dt$. Solving this deterministic SDE with initial wealth $X_{\tau_\gamma}^d = 0$ gives $X_t^d = 0$ for $t \in [\tau_\gamma, T]$.

In the case if $T < \tau_\gamma$, we set $Z_t = \mathrm{e}^{\mathrm{r(T-r)}}\mathrm{X_t^d}$ and apply Ito's formula so that:

$$X_t^d = x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T-t})} \big[\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - 1 + \delta \int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{e}^{\delta^2(\mathrm{t-s})} \mathrm{dW_s}\big]. \tag{2.90}$$

From (2.89) and (2.90), we have confirmed (2.20)-(2.21) and completed the proof. □

So far we have solved the first constrained problem. In [43], we have seen that the solution of (2.6) and (2.7) can be derived from the solution of (2.4) by choosing a proper

Lagrange multiplier. Similar conclusion can also be done here. In the following part, we will derive the solution for (2.6) and (2.7) respectively, and the proof is consistent with the proof of Corollary 5 and Corollary 7 of [43].

Corollary 2.4. *Consider the optimal control problem* $V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u)\leq\alpha} \mathrm{E}_{t,x}(X_T^u)$ *in which* X^u *is the wealth process with* $X_{t_0}^u = x_0$ *under the probability measure* $\mathbb{P}_{t_0,x_0}$ *for* $(t_0,x_0) \in [0,T]\times\mathbb{R}$ *given and fixed and* $\alpha \in (0,\infty)$. *The related risk coefficient is defined by* $\delta = (\mu - r)/\sigma$ *in which* $\mu, r \in \mathbb{R}$, $\mu > r$ *and* $\sigma > 0$. *We further assume that* $r \neq 0$ *in the following part. (The cases* $\delta = 0$ *or* $r = 0$ *follow by passage to the limit when the non-zero* δ *or* r *approaches to 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,x) = \max\left[\frac{\delta}{\sigma}\frac{1}{x}\left[x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - \mathrm{x} + \sqrt{\alpha}\frac{\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\right],0\right] \tag{2.91}$$

for $(t,x) \in [t_0,T]\times\mathbb{R}$. *The statically optimal controlled process is given by:*

If $\tau_\alpha^c < \tau_\beta^c$,

$$X_t^s = \begin{cases} x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \sqrt{\alpha}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T-t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\left[\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}(\mathrm{t-t_0})}\right] & \text{if } t \leq \tau_\alpha^c \\ x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{\sqrt{\alpha}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}} & \text{if } \tau_\alpha^c < t \end{cases} \tag{2.92}$$

if $\tau_\beta^c < \tau_\alpha^c$,

$$X_t^s = \begin{cases} x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \sqrt{\alpha}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T-t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\left[\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}(\mathrm{t-t_0})}\right] & \text{if } t \leq \tau_\beta^c \\ 0 & \text{if } \tau_\beta^c < t \end{cases} \tag{2.93}$$

where τ_α^c *and* τ_β^c *are given by:*

$$\tau_\alpha^c = \inf\left\{t \in [t_0,T] | x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \sqrt{\alpha}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T-t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\left[\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}(\mathrm{t-t_0})}\right]\right. \tag{2.94}$$

$$\left. = x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{\sqrt{\alpha}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}\right\}$$

$$\tau_\beta^c = \inf\left\{t \in [t_0,T] | x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \sqrt{\alpha}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T-t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T-t_0})}-1}}\left[\mathrm{e}^{\delta^2(\mathrm{t-t_0})} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}(\mathrm{t-t_0})}\right] \leq 0\right\} \tag{2.95}$$

for $t \in [t_0, T]$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,x) = \max\left[\sqrt{\alpha}\frac{\delta}{\sigma x}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t})}-1}}, 0\right] \tag{2.96}$$

for $(t,x) \in [t_0,T]\times\mathbb{R}$. *The dynamically optimal controlled process is given by:*

$$X_t^d = \begin{cases} x_0\mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} + 2\sqrt{\alpha}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}[\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1} - \sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t})}-1} + & \frac{\delta}{2}\int_{t_0}^t \frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{s})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{s})}-1}}dW_s] \\ & \text{if } t \le \tau_\gamma^c \\ 0 & \text{if } \tau_\gamma^c < t \end{cases} \tag{2.97}$$

where τ_γ^c *is given by:*

$$\tau_\gamma^c = \inf\Big\{t\in[t_0,T] | x_0\mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} + 2\sqrt{\alpha}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\big[\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1} - \sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t})}-1} \tag{2.98}$$
$$+\frac{\delta}{2}\int_{t_0}^t \frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{s})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{s})}-1}}dW_s\big] \le 0\Big\}$$

for $t \in [t_0, T]$.

Proof. In this proof, we claim that, for each pair of $(t_0, x_0) \in [0,T]\times\mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (2.3) with initial condition $X_{t_0}^u = x_0$. Furthermore, for X_t^u, $u \in U$ is any admissible control we defined in Section 2.2.

(A): Applying Lagrange multipliers in (2.6) yields:

$$L_{t,x}(u,c) = \mathrm{E}_{t,x}(X_T^u) - c[\mathrm{Var}_{t,x}(X_T^u) - \alpha] \tag{2.99}$$

for $c > 0$. Based upon Theorem 2.3, the optimal control u_*^s given by (2.14) maximises $\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)$ will be the optimal control that maximises (2.99) with $c = c(\alpha,t,x) > 0$ meeting $\mathrm{Var}_{t,x}(X_T^{u_*^s}) \le \alpha$. In this case, we state

$$L_{t,x}(u_*^c, c) = \sup_u L_{t,x}(u,c) \tag{2.100}$$

for $c > 0$. For the given pair $(t,x) \in [t_0, x_0]\times\mathbb{R}$ such that $u_*^c(t,x) > 0$, we can see that:

$$\mathrm{E}_{t,x}(X_T^{u_*^c}) = L_{t,x}(u_*^c, c) \ge \mathrm{E}_{t,x}(X_T^u) - c[\mathrm{Var}_{t,x}(X_T^u) - \alpha] \ge \mathrm{E}_{t,x}(X_T^u) \tag{2.101}$$

in which u is any admissible control that satisfies $\mathrm{Var}_{t,x}(X_T^u) \leq \alpha$. In the other case when $u_*^c(t,x) = 0$, we have $\mathrm{Var}_{t,x}(X_T^{u_*^c}) = 0 < \alpha$. This result indicates that the optimal control u_*^c given by (2.14) with $c(\alpha,t,x) > 0$ is the statically optimal control in (2.6).

According to (2.15)-(2.18), we know that the $\mathrm{Var}_{t,x}(X_T^{u_*^c})$ does not equal to zero only if $T < (\tau_\alpha \wedge \tau_\beta)$. Hence, taking E_{t_0,x_0} on the first line of (2.15) with $T < (\tau_\alpha \wedge \tau_\beta)$ gives:

$$\mathrm{Var}_{t_0,x_0}(X_T^{u_*^c}) = \frac{1}{4c^2}[\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)} - 1]. \tag{2.102}$$

Setting (2.102) equal to α, we achieve the optimal value of c, which is:

$$c = \frac{1}{2\sqrt{\alpha}}\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)} - 1}. \tag{2.103}$$

Substituting (2.103) into (2.14)-(2.18), we obtain the statically optimal control and the optimal controlled wealth process, which confirms (2.91)-(2.95) and completes the first part of the proof.

(B): Replacing t_0 and x_0 by t and x in the statically optimal control (2.91), we can obtain the control u_*^d given in (2.96). We claim this gives the dynamically optimal control for (2.5). Also, it is clear that (2.80) holds with c given by (2.103). Hence, based upon (2.80), we can see that for any pair $(t_0,x_0) \in [0,T] \times \mathbb{R}$ such that $w(t_0,x_0) = u_*^s(t_0,x_0) > 0$ and $\mathrm{Var}_{t_0,x_0}(X_T^w) \leq \alpha$, the following inequality holds:

$$\mathrm{E}_{t_0,x_0}(X_T^w) > \mathrm{E}_{t_0,x_0}(X_T^v) - c\,\mathrm{Var}_{t_0,x_0}(X_T^v) \geq \mathrm{E}_{t_0,x_0}(X_T^v) \tag{2.104}$$

in which v satisfies $\mathrm{Var}_{t_0,x_0}(X_T^v) \leq \alpha$. Hence, we can conclude that the optimal control given by (2.96) is the dynamically optimal control for (2.6).

Observing the optimal control (2.96), we can note that $u_*^d(t,x) = 0$ if and only if $x < 0$. Hence, we define the following stopping time:

$$\tau_\gamma = \inf\{t \in [t_0,T] | X_t^d \leq 0\} \tag{2.105}$$

with $\mathbb{P}_{t_0,x_0}(T < \tau_\gamma) > 0$, and this is the first hitting time of X_t^d at 0. When X_t^d hits zero (i.e. $\tau_\gamma \leq T$), the wealth process will go beneath 0 with probability one almost surely, and the optimal control (2.96) gives $u_*^d = 0$ which reduces the SDE (2.3) into $dX_t^u = rX_t^u dt$. Solving this deterministic SDE with initial wealth $X_{\tau_\gamma}^d = 0$ gives $X_t^d = 0$ for $t \in [\tau_\gamma, T]$. If $T < \tau_\gamma$, using Ito's formula to $\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t})}\mathrm{X}_\mathrm{t}^\mathrm{d}$ in which we set that $X^d = X^{u_*^d}$ and SDE (2.3), we can easily achieve the first line of (2.97). Hence, we can summarise that the dynamic optimal wealth process is given by (2.97)-(2.98) and complete the proof. □

Corollary 2.5. *Consider optimal control problem $V_2(t,x) = \sup_{u:\mathrm{E}_{t,x}(X_T^u)\geq\beta} \mathrm{Var}_{t,x}(X_T^u)$ in which X^u is the wealth process with $X_{t_0}^u = x_0$ under the probability measure $\mathbb{P}_{t_0,x_0}$ for $(t_0,x_0) \in [0,T] \times \mathbb{R}$ given and fixed and $\beta \in \mathbb{R}$. The related risk coefficient is defined by $\delta = (\mu - r)/\sigma$ in which $\mu, r \in \mathbb{R}$, $\mu > r$ and $\sigma > 0$. We further assume that $r \neq 0$ in the following part. (The cases $\delta = 0$ or $r = 0$ follow by passage to the limit when the non-zero δ or r approaches to 0.) Furthermore, we assume that the expectation of the terminal wealth, β, must satisfy $\beta > x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$. For a wise investor, if $\beta \leq x_0\mathrm{e}^{\mathrm{r(T-t_0)}}$, he can simply invest all his wealth in the riskless asset and receive zero variance at the maturity T. Hence, in the following part, we assume that $\beta > x_0\mathrm{e}^{\mathrm{r(T-t_0)}}$.*

(A) *The statically optimal control is given by:*

$$u_*^s(t,x) = \max\left[\frac{\delta}{\sigma}\frac{1}{x}\left[x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \mathrm{x} + (\beta - \mathrm{x_0e^{r(T-t_0)}})\frac{\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}} - 1}\right], 0\right] \tag{2.106}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$. The statically optimal controlled process is given by:

If $\tau_\alpha^c < \tau_\beta^c$,

$$X_t^s = \begin{cases} x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + (\beta - \mathrm{x_0e^{r(T-t_0)}})\frac{\mathrm{e}^{(\delta^2-\mathrm{r})\mathrm{(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\left[\mathrm{e}^{\delta^2\mathrm{(t-t_0)}} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}\mathrm{(t-t_0)}}\right] & \text{if } \mathrm{t} \leq \tau_\alpha^{\mathrm{c}} \\ x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{(\beta - \mathrm{x_0e^{r(T-t_0)}})}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}} & \text{if } \tau_\alpha^{\mathrm{c}} < \mathrm{t} \end{cases} \tag{2.107}$$

If $\tau_\beta^c < \tau_\alpha^c$,

$$X_t^s = \begin{cases} x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + (\beta - \mathrm{x_0e^{r(T-t_0)}})\frac{\mathrm{e}^{(\delta^2-\mathrm{r})\mathrm{(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\left[\mathrm{e}^{\delta^2\mathrm{(t-t_0)}} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}\mathrm{(t-t_0)}}\right] & \text{if } \mathrm{t} \leq \tau_\beta^{\mathrm{c}} \\ 0 & \text{if } \tau_\beta^c < t \end{cases} \tag{2.108}$$

where τ_α^c and τ_β^c are given by:

$$\begin{aligned} \tau_\alpha^c = \inf\Bigg\{ t \in [t_0,T] | x_0\mathrm{e}^{\mathrm{r(t-t_0)}} &+ (\beta - x_0\mathrm{e}^{\mathrm{r(T-t_0)}})\frac{\mathrm{e}^{(\delta^2-\mathrm{r})\mathrm{(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}} - 1}\left[\mathrm{e}^{\delta^2\mathrm{(t-t_0)}} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}\mathrm{(t-t_0)}}\right] \\ &= x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{(\beta - \mathrm{x_0e^{r(T-t_0)}})}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}} - 1}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}\Bigg\}, \end{aligned} \tag{2.109}$$

$$\begin{aligned} \tau_\beta^c = \inf\Bigg\{ t \in [t_0,T] | x_0\mathrm{e}^{\mathrm{r(t-t_0)}} &+ (\beta - \mathrm{x_0e^{r(T-t_0)}})\frac{\mathrm{e}^{(\delta^2-\mathrm{r})\mathrm{(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}} - 1} \\ &\times \left[\mathrm{e}^{\delta^2\mathrm{(t-t_0)}} - \mathrm{e}^{-\delta(\mathrm{W_t-W_{t_0}})-\frac{\delta^2}{2}\mathrm{(t-t_0)}}\right] \leq 0\Bigg\} \end{aligned} \tag{2.110}$$

for $t \in [t_0, T]$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,x) = \max\left[\frac{\delta}{\sigma}\frac{1}{x}(\beta - x e^{r(T-t)})\frac{e^{(\delta^2 - r)(T-t)}}{e^{\delta^2(T-t)} - 1}, 0\right] \tag{2.111}$$

for $(t,x) \in [t_0, T] \times \mathbb{R}$. The dynamically optimal controlled process is given by:

$$X_t^d = \begin{cases} e^{-r(T-t)}\Big[\beta - (\beta - x_0 e^{r(T-t_0)})\frac{e^{\delta^2(T-t)}-1}{e^{\delta^2(T-t_0)}-1} \\ \qquad \times \exp(-\delta\int_{t_0}^t \frac{e^{\delta^2(T-s)}}{e^{\delta^2(T-s)}-1}dW_s - \frac{\delta^2}{2}\int_{t_0}^t \frac{e^{2\delta^2(T-s)}}{(e^{\delta^2(T-s)}-1)^2}ds)\Big] & \text{if } t \le \tau_\gamma^c \\ 0 & \text{if } \tau_\gamma^c < t \end{cases} \tag{2.112}$$

and τ_γ^c is defined by:

$$\tau_\gamma^c = \inf\Big\{t \in [t_0, T) | e^{-r(T-t)}[\beta - (\beta - x_0 e^{r(T-t_0)})\frac{e^{\delta^2(T-t)} - 1}{e^{\delta^2(T-t_0)} - 1} \\ \times \exp(-\delta\int_{t_0}^t \frac{e^{\delta^2(T-s)}}{e^{\delta^2(T-s)} - 1}dW_s - \frac{\delta^2}{2}\int_{t_0}^t \frac{e^{2\delta^2(T-s)}}{(e^{\delta^2(T-s)} - 1)^2}ds)] \le 0\Big\}. \tag{2.113}$$

for all $t \in [t_0, T)$. Furthermore, if $T < \tau_\gamma^c$, then X_T^d is given by $\lim_{t \to T} X_t^d = \beta$ with $\mathbb{P}_{t_0,x_0}$-probability one.

Proof. In this proof, we claim that, for each pair of $(t_0, x_0) \in [0, T] \times \mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (2.3) with initial condition $X_{t_0}^u = x_0$. Furthermore, for X_t^u, $u \in U$ is any admissible control we defined in Section 2.2.

(A): Applying Lagrangian for the constrained problem (2.7) yields:

$$L_{t,x}(u,c) = \mathrm{Var}_{t,x}(X_T^u) - c[\mathrm{E}_{t,x}(X_T^u) - \beta] \tag{2.114}$$

for $c > 0$. We can take the advantage of the previous conclusion. From (2.114), we can see that

$$\inf_{u \ge 0}(\mathrm{Var}_{t,x}(X_T^u) - c[\mathrm{E}_{t,x}(X_T^u) - \beta]) = -c\sup_{u \ge 0}[\mathrm{E}_{t,x}(X_T^u) - \frac{1}{c}\mathrm{Var}_{t,x}(X_T^u)] + c\beta \tag{2.115}$$

and we claim that the optimal control $u_*^{\frac{1}{c}}$ given by (2.14) leads to:

$$L_{t,x}(u_*^{\frac{1}{c}}, c) = \inf L_{t,x}(u,c) \tag{2.116}$$

for $c > 0$. Furthermore, if there exist $c = c(\beta, t, x) > 0$ and $(t, x) \in [t_0, T] \times \mathbb{R}$ such that $\mathrm{E}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = \beta$, it can be easily seen that:

$$\mathrm{Var}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = L_{t,x}(u_*^{\frac{1}{c}}, c) \leq \mathrm{Var}_{t,x}(X_T^u) - c[\mathrm{E}_{t,x}(X_T^u) - \beta] \leq \mathrm{Var}_{t,x}(X_T^u) \tag{2.117}$$

for any admissible u meeting $\mathrm{E}_{t,x}(X_T^u) \geq \beta$, which indicates that $u_*^{\frac{1}{c}}$ is statically optimal control for (2.6).

Using the fact that $\mathrm{E}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = \beta$ and taking E_{t_0,x_0} in the first line of (2.15) we receive:

$$\mathrm{E}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = x_0 \mathrm{e}^{\mathrm{r(T-t_0)}} + \frac{\mathrm{c}}{2}[\mathrm{e}^{\delta^2\mathrm{(T-t_0)}} - 1] = \beta \tag{2.118}$$

which gives:

$$c = \frac{2(\beta - x_0 \mathrm{e}^{\mathrm{r(T-t_0)}})}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}} - 1}. \tag{2.119}$$

Furthermore, substituting (2.119) back into (2.14)-(2.18), we receive (2.106)-(2.110) and this completes the first part of proof.

(B): Replacing t_0 and x_0 by t and x in (2.106), we can obtain the control u_*^d given in (2.111). We claim this gives the dynamically optimal control for (2.7). Also, it is clear that (2.80) holds with c given by (2.119). Since $\mathrm{E}_{t_0,x_0}(X_T^w) = \beta$, it can be easily seen that (2.80) leads to:

$$\mathrm{Var}_{t_0,x_0}(X_T^w) < \frac{1}{c}\left[\beta - \mathrm{E}_{t_0,x_0}(X_T^v) + c\,\mathrm{Var}_{t_0,x_0}(X_T^v)\right] \leq \mathrm{Var}_{t_0,x_0}(X_T^v) \tag{2.120}$$

in which $\mathrm{E}_{t_0,x_0}(X_T^v) \geq \beta$ and c is given by (2.119). This indicates that the optimal control given by (2.111) is the dynamically optimal control we are looking for.

Observing the optimal control (2.111), we can note that $u_*^d(t,x) = 0$ if and only if $x \leq 0$. Hence, we define the following stopping time:

$$\tau_\gamma^c = \inf\{t \in [t_0, T] | X_t^d \leq 0\} \tag{2.121}$$

with $\mathbb{P}_{t_0,x_0}(T < \tau_\gamma) > 0$, and this is the first hitting time at 0. When X_t^d hits zero (i.e. $\tau_\gamma \leq T$), the wealth process will go beneath 0 with probability one almost surely, and the optimal control (2.111) gives $u_*^d = 0$ which reduces the SDE (2.3) into $dX_t^u = rX_t^u dt$. Solving this deterministic SDE with initial wealth $X_{\tau_\gamma}^d = 0$ gives $X_t^d = 0$ for $t \in [\tau_\gamma, T]$.

Recalling the proof of Corollary 7 in [43], we can see that in the case if $T < \tau_\gamma^c$, applying Ito's formula to the following process Z:

$$Z_t = \beta - \mathrm{e}^{\mathrm{r(T-t)}} \mathrm{X}_\mathrm{t}^\mathrm{d} \tag{2.122}$$

in which we set that $X_t^d = X_t^{u_*^d}$ and using SDE (2.3), we can easily achieve that:

$$dZ_t = -\delta^2 \frac{Z_t}{1-\mathrm{e}^{-\delta^2(\mathrm{T-t})}} dt - \delta \frac{Z_t}{1-\mathrm{e}^{-\delta^2(\mathrm{T-t})}} dW_t \tag{2.123}$$

with initial value $Z_{t_0} = M - x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$ under probability measure $\mathbb{P}_{t_0,x_0}$. Solving (2.123) yields the following process:

$$Z_t = Z_{t_0} \exp\left(- \int_{t_0}^t \frac{\delta}{1-\mathrm{e}^{-\delta^2(\mathrm{T-s})}} dW_s - \int_{t_0}^t \left[\frac{\delta^2}{1-\mathrm{e}^{-\delta^2(\mathrm{T-s})}} + \frac{1}{2} \frac{\delta^2}{(1-\mathrm{e}^{-\delta^2(\mathrm{T-s})})^2} \right] ds \right) \tag{2.124}$$

for $t \in [t_0, T)$ under $\mathbb{P}_{t_0,x_0}$. In (2.124), the continuous martingale M:

$$M_t = -\delta \int_{t_0}^t \frac{\mathrm{e}^{\delta^2(\mathrm{T-s})}}{\mathrm{e}^{\delta^2(\mathrm{T-s})} - 1} dW_s \tag{2.125}$$

for $t \in [t_0, T)$ is a time-changed Brownian motion $\bar{W}$ in which case $M_t = \bar{W}_{<M,M>_t}$ by Dambis-Dubins-Schwarz theorem. Moreover, there is:

$$\langle M, M \rangle_t = \delta^2 \int_{t_0}^t \frac{\mathrm{e}^{2\delta^2(\mathrm{T-s})}}{(\mathrm{e}^{\delta^2(\mathrm{T-s})} - 1)^2} ds \to \infty \tag{2.126}$$

when $t \to T$. According to the sample path properties of $\bar{W}_t$, there is:

$$M_t - \frac{1}{2} \langle M, M \rangle_t = \bar{W}_{\langle M,M \rangle_t} - \langle M, M \rangle_t \to -\infty \tag{2.127}$$

as $t \to T$ with probability one. Using this fact in (2.123) we can see that $X_t^d \to \beta$ as $t \to T$ with $\mathbb{P}_{t_0,x_0}$-probability one if $T < \tau_\gamma^c$. Moreover, Z_t is always positive which guarantees that $\beta > \mathrm{e}^{\mathrm{r(T-t)}} \mathrm{X}_\mathrm{t}^\mathrm{d}$ for $t \in [t_0, T]$. Hence, we can conclude that if X_t^d does not hit 0 before the maturity, it will stay strictly below β under the maturity. □

Hence, we have obtained the solution for two constrained cases (2.6) and (2.7). In the following part, we will further analysis the performance of both static and dynamic optimalities.

2.4 Static optimality vs dynamic optimality

1. In this section, we have found both the statically and dynamically optimal controls for (2.4)-(2.7) under the no short-selling constraint. In this section, we will compare the difference between those two optimalities by setting Theorem 2.3 as an example. As we have seen in [42], the dynamically optimal control u_*^d from (2.19) rejects all past points (t_0, x_0) to consider its performance while the statically optimal control is related to the initial state, which raises the problem of making a decision between them. Clearly, the dynamically optimal control u_*^d is time consistent while the statically optimal control is time-inconsistent. In this chapter, the controlled wealth processes contain stopping times with unknown distribution, which makes the calculation of the expectation and variance of the controlled process become difficult. (There is still a way to calculate the expectation without knowing the distribution but that is beyond the scope of this chapter.) Also, another problem of the comparison between the static value $V_s(t,x)$ and the dynamic value $V_d(t,x)$ for $(t,x) \in [t_0,T] \times \mathbb{R}$ is that the optimal control process X^s and X^d may never have the same value at a given point of time t so that the comparison may not be feasible, which is consistent with the conclusion in [43]. In [43], the authors focus on the terminal value of X^s and X^d and compare the value of $\mathrm{E}_{t_0,x_0}[V_s(T,X_T^s)]$ and $\mathrm{E}_{t_0,x_0}[V_d(T,X_T^d)]$, where they find that the two expectations coincide and conclude that the optimally dynamic control u_*^d is as good as u_*^s. However, since we can not achieve the expectation of the wealth process without knowing the distribution of the wealth process, comparing the expectation of the value function at the terminal time T becomes infeasible. In this situation, we can still make an intuitive comparison. When t achieves the terminal time T, there is $V_s(t,X_T^s) = X_T^s$ and $V_d(t,X_T^d) = X_T^d$. According to (2.15)-(2.18) and (2.20)-(2.21), the terminal value of the optimally static wealth process is upper-and-lower bounded, which yields $X_T^s \in [0, x_0 \mathrm{e}^{\mathrm{r(T-t_0)}} + \frac{1}{2\mathrm{c}} \mathrm{e}^{\delta^2 \mathrm{(T-t_0)}}]$, while the optimally dynamic wealth process only has lower boundary, which gives $X_T^d \in [0,\infty)$. Intuitively, the optimally dynamic control has better potential to achieve a higher portfolio value.

Furthermore, it should be pointed out that the statically optimal control (2.14) suggests that the investor should invest all the wealth in the bond if the current wealth is large enough, which causes the upper boundary of the controlled wealth process. This fact is also observed in [33], in which they achieved time-inconsistent optimal control under the no short selling constraint. However, the dynamically optimal control (2.19) encourages the investor to keep

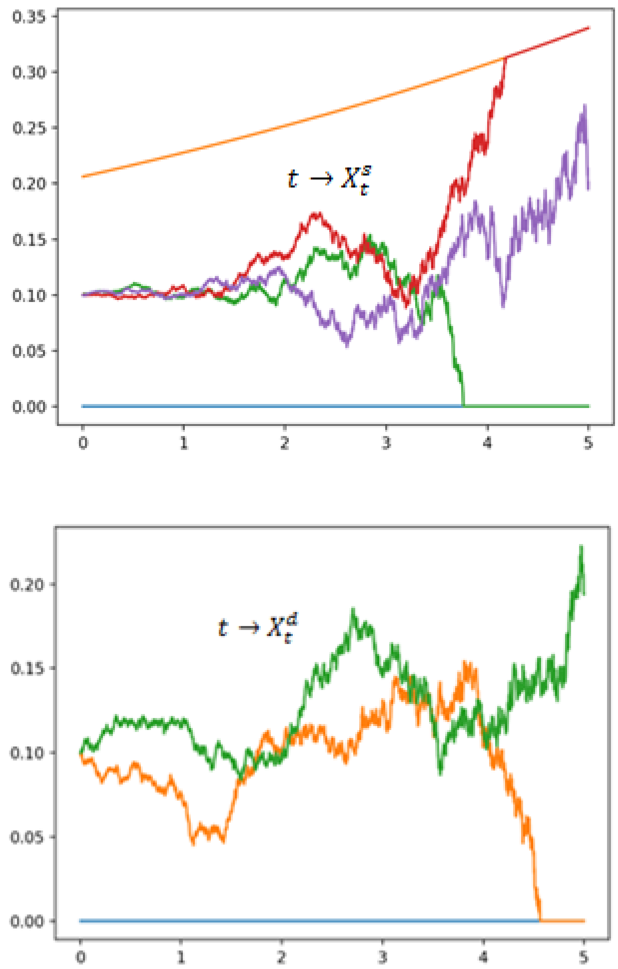

Figure 2.1: The first picture is the simulation of statically optimal controlled wealth process based upon (2.15)-(2.18), and it can be clearly seen that the wealth process is bounded by the upper boundary (orange curve) and the lower boundary (blue line). The second picture presents the simulation of dynamically optimal controlled wealth process that is only bounded below (blue line).

holding the risky asset to achieve a higher return, which has not been observed before.

Additionally, following the same idea of [43], from (2.19), it can be noted that the amount of the dynamically optimal wealth $u^d_*(t,x)\cdot x$ held in the risky asset at time t does not depend on the amount of the current wealth x. This observation is consistent with the fact that, in (2.4), the risk is measured by the variance, a quadratic function of the terminal wealth, while the expectation is a linear function of the terminal wealth which measures the return. For the stochastic movement of the large wealth, the penalisation caused by the variance will be more severe than the compensation brought by the expectation (see Remark 4 in [43] for further details). In this point, the dynamic investor is not encouraged to hold larger amounts of risky assets, and for the dynamic investor, the optimal total amount of risky asset $u_*^d(t,x)\cdot x$ is independent of x.

Under the no short-selling constraint, both of the static and dynamic optimality naturally prevent the bankruptcy of the investor. As we can see in Theorem 2.3, once the value of wealth hits 0, both of the static and dynamic processes will stay at 0 until the maturity, which is clearly indicated in Figure 2.1. Coincidentally, similar bankrupt behaviour can be also seen in [3] and [42] under pathwise constraint. In [3], Bielecki et al developed the optimal control that prevents bankruptcy by adopting the martingale method. Pedersen and Peskir [42] achieved the time consistent optimal control under a general pathwise constraint $X_s^u \geq \mathrm{e}^{-\mathrm{r(T-s)}}\mathrm{g}$ for any $s \in [t,T]$ and $g \in \mathbb{R}$.

2. The observation of the wealth process behaviour leads us to analyse those two strategies in a numerical way. For the numerical analysis, the key idea is to focus on analysing the terminal wealth. Since $V(T,X_T^u) = X_T^u$, comparing the $\mathrm{E}_{t_0,x_0}(X_T^s)$ and $\mathrm{E}_{t_0,x_0}(X_T^d)$ will be the target in the following part. Hence, we firstly simulate both wealth processes and collect the value of X_T^s and X_T^d respectively and form the sample base, from which we can receive the sample mean $\bar{\mu}$, sample variance $\bar{m}$, and the distribution of the stopping time. In this section, we will focus on comparing the value of sample means to decide which strategy will lead to a better performance. Note that in the case if X_T is not well-defined, we take the value of X at the second last point of time to the maturity T. In the following part, we will have a more intuitive understanding about the performance of those strategies.

In Tables 2.1 and 2.2, we simulate the static optimality and dynamic optimality under the no short-selling constraint with respect to different values of risk aversion rate c. Surprisingly, Table 2.1 and Table 2.2 show the opposite results; the static optimality outperforms the dynamic optimality when c gets larger while the dynamic optimality has a better performance when c is smaller. For example, when risk aversion rate $c = 0.1$, we can see that the sample mean of the static optimality is $\bar{\mu}^s \approx 1.12$, which is greatly smaller than the dynamic optimality $\bar{\mu}^d \approx 2.98$. When c increases to 1.6, we can see there is $\bar{\mu}^s \approx 1.76 > 1.23 \approx \bar{\mu}^d$. Moreover, the distribution of the stopping time for both static and dynamic wealth process also changes with respect to the value of c. In Tables 2.1 and 2.2, we can see that the chance of the statically optimal wealth process hits the lower boundary 0 or hitting the upper curve increases alongside with the increase of c while in the dynamic case, the possibility of hitting 0 decreases as c increases. In Figure 2.2, we apply the kernel density estimation (KDE) to illustrate this change in a more intuitive way. In the static case in Figure 2.2, we see that the peaks on both sides increase when c moves from 1 to 2, which indicates that the increase of the probability of hitting the upper/lower boundary. In the dynamic case, we see that the probability of hitting 0 disappears when c moves to 2. This observation is consistent with Tables 2.1 and 2.2. When comparing Table 2.1 and Table 2.2, we can tell that Theorem 2.3 in this chapter gives different strategies to investors with different risk aversion levels, and this has practical meaning in the real financial market.

Furthermore, we have seen that the optimal wealth processes are bounded, which naturally leads to another problem: will the investor receive a better result under the short-selling constraint? To compare with the unconstrained case, we simulate Theorem 3 introduced in [43] in Table 2.3. Overall, the static optimality under the short-selling constraint outperforms the static optimality in the unconstrained case, and the difference between $\bar{\mu}^s$ and $\bar{\mu}^{us}$ increases as c increases. The reasons are: when c is small such as $c = 0.1$, the chance of the wealth process hits upper/lower boundary is relatively small so that the effect of bankruptcy prevention or the limit of the growth of wealth is very small, which leads to the fact that $\bar{\mu}^s \approx \bar{\mu}^{us}$; when c gets larger, the probability of the statically optimal wealth process hits the upper/lower boundary increases and the chance of hitting 0 is much larger than that of hitting the upper boundary, which means that the effect of bankruptcy prevention is more obvious than the limit of the growth of wealth and leads to $\bar{\mu}^s > \bar{\mu}^{us}$. It should be pointed out that for the unconstrained case in [43], the wealth process is unbounded so that it still can reach very high terminal wealth in the extreme case, which is not common during the simulation. Hence, from this observation, we can conclude that, for a static investor, choosing the static optimality under no short-selling constraint will be a safer choice and it may overall outperform the unconstrained strategy.

For the dynamic optimality, we have seen in Figure 2.1 that the optimal wealth process is only bounded by 0, which makes the dynamic optimality in Theorem 2.3 more attractive. The numerical analysis confirms this guess. Comparing Tables 2.2 and 2.3, we can see that, under the no short-selling constraint, the dynamic optimality in Theorem 2.3 outperforms the unconstrained case in [43]. The reasons are: when c is small, the wealth process has a higher chance to hit 0 and the effect of bankruptcy is more obvious; in the constrained case, the wealth process has the same potential to reach the high value at the maturity as the unconstrained case, which makes the dynamic optimality under no short-selling constraint more favourable than the unconstrained case. The only special case is that, when c gets large enough such as $c = 1.6$ and $c = 2.0$, and in this situation, the dynamic optimality wealth process will only have a very low chance to hit the lower boundary 0, which makes $\bar{\mu}^d \approx \bar{\mu}^{ud}$ and indicates there is no difference between the constrained and unconstrained cases. Hence, for a dynamic investor, it will be optimal for him to choose the strategy under no short-selling constraint no matter the level of his risk aversion.

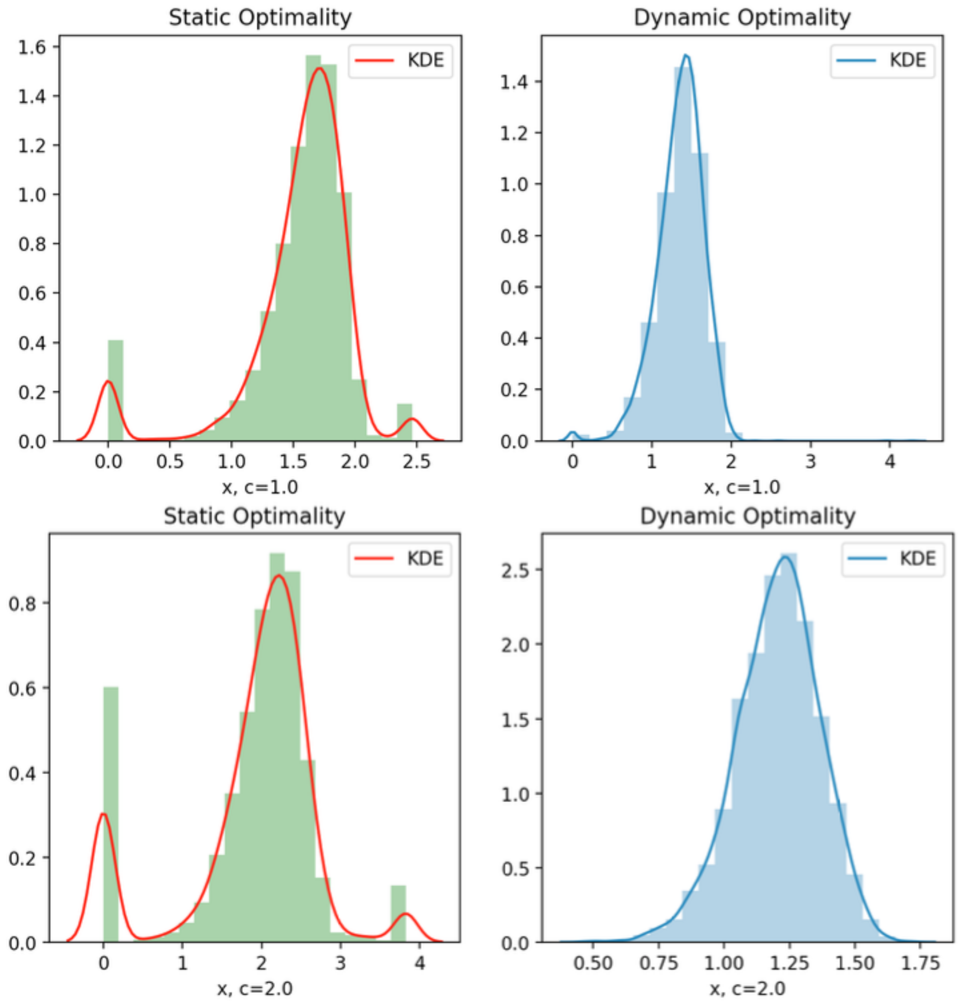

Figure 2.2: The kernel density estimation (KDE) for the terminal wealth of static optimality and dynamic optimality with different values of risk aversion rate c.

3. Naturally, we consider if the findings for V_1 and V_2 in [43] (cf. Remarks 6 and 8 in [43]) will still be valid for (2.91)-(2.96). For V_1, in Table 2.5, we compare the sample mean and sample variance, from which we can see that the dynamic optimality always leads to a larger sample mean. This indicates that dynamic optimality outperforms the static optimality no matter what value α takes. A wise investor should always follow the dynamic optimality in this situation. Furthermore, similar to the first constrained problem we discussed above,

Risk aversion	Sample mean $\bar{\mu}^s$	No. of hitting curve	No. of hitting 0
0.1	1.11688875	1	0
0.4	1.26921424	9	18
0.8	1.45521204	17	66
1.2	1.61320726	37	139
1.6	1.75307588	41	191
2.0	1.94263783	66	203

Table 2.1: Simulation for the static optimality under no short selling constraint with respect to different values of c.

Risk aversion	Sample mean $\bar{\mu}^d$	No. of hitting 0
0.1	2.98313262	516
0.4	1.74596633	144
0.8	1.43388909	27
1.2	1.30790685	5
1.6	1.24797581	2
2.0	1.20147576	0

Table 2.2: Simulation for the dynamic optimality under no short selling constraint with respect to different values of c.

Risk aversion	Sample mean $\bar{\mu}^{us}$	Sample mean $\bar{\mu}^{ud}$
0.1	1.11838796	2.70643823
0.4	1.27795929	1.65555340
0.8	1.38743068	1.37788403
1.2	1.45383901	1.30356596
1.6	1.67188472	1.24672175
2.0	1.80920598	1.20803812

Table 2.3: Simulation for static optimality and dynamic optimality for the unconstrained problem of [43] with respect to different values of c.

(Note the related parameters are given by $\mu = 0.4$, $\sigma = 0.3$, $\delta = 1$, $x_0 = 1$, $r = 0.1$, $T = 1$, $t_0 = 0$, and the sample size is 2000.)

α	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$	No. of hitting curve	No. of hitting 0
0.4	1.52587802	0.16473329	25	76
0.8	1.67132281	0.35295135	40	153
1.2	1.78574262	0.44878562	32	172
1.6	1.89849498	0.60161303	44	207
2.0	1.96355465	0.76313842	64	232

α	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$	No. of hitting 0
0.4	1.83241904	2.02759367	182
0.8	2.17311476	14.61036095	250
1.2	2.29215003	11.43652527	306
1.6	2.48901354	11.12743796	322
2.0	2.88505624	78.01683357	355

Table 2.5: Simulation for static and dynamic optimalities under the controls given by (2.91) and (2.96) respectively with respect to different values of α.

(Note the related parameters are given by $\mu = 0.4$, $\sigma = 0.3$, $\delta = 1$, $x_0 = 1$, $r = 0.1$, $T = 1$, and $t_0 = 0$, and the sample size is 2,000.)

we can see that the chance of the statically optimal wealth process hits upper boundary or lower boundary increases as α increases, which indicates that both the effect of bankruptcy and limit of the growth of the wealth of the static optimality have been enhanced as α increases. And similar observation can be noted for the dynamic optimality.

Besides, different from [43], our simulation for the dynamic optimality does not show infinitely large variance. The possible reason is (2.97) is not well-defined at T and our simulation can not achieve the actual limit of X_t when $t \uparrow T$. Alternatively, we the value of X at the second last point of time to the maturity T. However, the sample variance of the dynamic case is still significantly larger than the static case.

In Remark 8 in [43], we see that the dynamic wealth process attains to β as $t \uparrow T$ with probability one. Similar results can be also noted in this chapter. Simulating the wealth process with respect to (2.106) and (2.111) respectively, we obtain Figure 2.3. Furthermore, for the numerical analysis in Table 2.7, we can see that the static optimality leads to a much smaller variance, which indicates that the static optimality outperforms the dynamic optimality before reaching the maturity. However, at the maturity, if the dynamically optimal wealth process does not hit 0, it will converge to β at T with probability one, which makes the variance equal to 0 at the maturity and makes the dynamic optimality outperform the static case. This fact is also consistent with [43].

β	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$	No. of hitting curve	No. of hitting 0
1.5	1.29947227	0.04236852	13	16
2.0	1.55416621	0.21447791	37	103
2.5	1.77437248	0.45841363	52	172
3.0	1.99439075	0.72966848	46	226
3.5	2.18880178	1.01684948	51	239

β	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$	No. of hitting 0
1.5	6.14219774	25424.941	223
2.0	2.39631266	51.461699	332
2.5	4.37193665	2187.0591	420
3.0	4.20123011	632.97101	509
3.5	5.17520506	4058.9352	574

Table 2.7: Simulation for static and dynamic optimalities under the controls given by (2.106) and (2.111) respectively with respect to different values of β.

(Note the related parameters are given by $\mu = 0.4$, $\sigma = 0.3$, $\delta = 1$, $x_0 = 1$, $r = 0.1$, $T = 1$, and $t_0 = 0$, and the sample size is 2,000.)

4. In [43], we have seen the comparison between the subgame-perfect Nash equilibrium and the dynamic optimality and [43] has shown that the dynamic optimality outperforms the subgame-perfect Nash equilibrium control. Furthermore, [2] has achieved the subgame perfect Nash optimal control under the short-selling constraint, and this naturally leads to the question: will the dynamic optimality still outperform the subgame-perfect Nash control? However, there is no simple way to compare these two kinds of controls under no short-selling constraint. The reasons are: (I) [2] also considers the existence of wealth-dependent risk aversion rate, which cannot be ignored in the numerical comparison; (II) [2] not only constrains short-selling of the stock but also constrains borrowing from the market. The admissible control of [2] is constrained by upper and lower boundaries p_tX and q_tX, where p_t and q_t are finite and the short-selling prohibition is achieved by setting $p_t = 0$ and $q_t = 1$. In our chapter, there is no upper boundary for the admissible control. Moreover, setting a borrowing constraint in our chapter is not possible as it will lead to the discontinuity of the value function for the HJB equation, which will violate our conclusion. Moreover, it should be pointed out that in Corollary 2.4 and 2.5 in this chapter, we have obtained the dynamically optimal control under no short-selling constraint for two constrained case (2.6) and (2.7) while the subgame-perfect Nash optimal control for those constrained cases has not been presented so far.

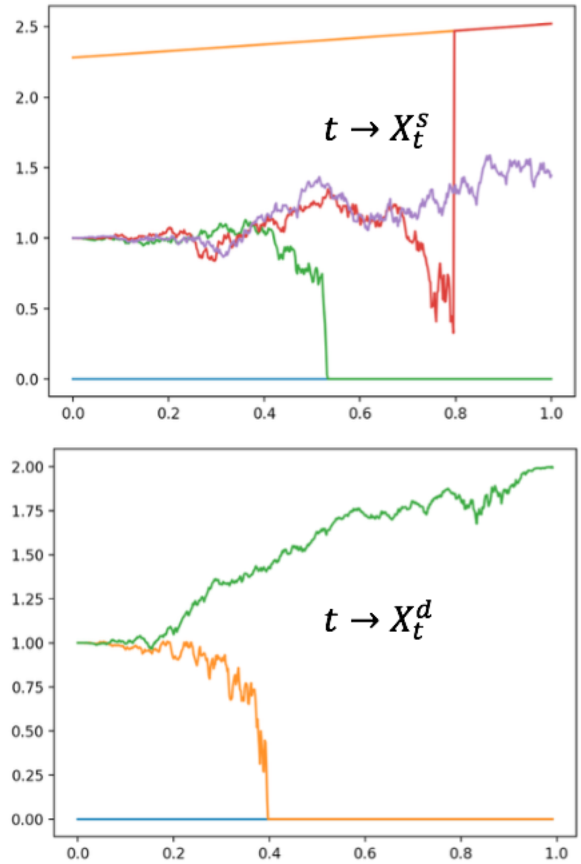

Figure 2.3: The first picture is the simulation of statically optimal controlled wealth process based upon (2.107), and the wealth process is bounded by the upper boundary (orange curve) and the lower boundary (blue line). The second picture presents the simulation of dynamically optimal controlled wealth process upon (2.112) that is only bounded below (blue line).

(Note the related parameters are given by $\mu = 0.4$, $\sigma = 0.3$, $\delta = 1$, $x_0 = 1$, $r = 0.1$, $T = 1$, and $t_0 = 0$, and $\beta = 2$.)

Chapter 3

Dynamic Mean-Variance Portfolio Selection under a Margin Requirement

3.1 Introduction

In the previous chapter, we have seen that a change-of-variable with local time on curves, following the method of [45], can replace the viscosity solution to overcome the non-smoothness of the value function under the short-selling constraint. This inspires us to apply this technique in solving more complicated cases. In real financial markets, the broker-dealer agent normally requires a maintenance margin for short-selling (on account of credit risk), and the consideration of margin requirements in portfolio selection has a practical meaning. Hence, in this chapter, we shall consider portfolio selection under margin requirements and investigate both static optimality and dynamic optimality.

The margin requirement has been considered a series of financial studies. In [14], Cuoco and Liu introduce margin requirements in the problem of optimal consumption and investment for portfolio optimisation as well as considering the minimal cost of hedging European contingent claims. By adopting duality techniques, utility analysis, and a martingale approach, they obtain an explicit solution for optimal investment/consumption for both logarithmic utility and CRRA utility. In [25], they consider the impact of margin requirement on the valuation of call options and prove that the margin requirement will rule out the arbitrage opportunity as well as verifying that the Black-Scholes model for call options is still valid. Most of the studies involving the margin requirements are in the asset pricing area, and the margin requirement has not been wildly considered in portfolio selection from the dynamic programming perspective. Zhou and Wu [65] consider this problem by applying the HJB equation to obtain the explicit solution and construct the efficient frontier. The existence of the margin requirement leads to an issue that the positive and negative controls have different coefficients, which is the main difficulty in [65]. To handle this difficult, they firstly decompose the original problem into two sub-problems and apply the HJB equation to achieve the optimal solution for each of them, from which the viscosity solution technique is applied to obtain the optimal control. Since margin requirements have practical implications, in this chapter, we shall follow the work of [65] to consider the portfolio selection problem and achieve both time-inconsistent solutions and time-consistent solutions.

For the investor who aims to invest dynamically in time in a self-financing portfolio and uses variance as the risk measure. In this problem, the variance introduces the quadratic nonlinearity into this problem, which can be handled by applying Lagrange multipliers and turns into a set of linear problems. The linear problem can be solved by HJB equation. However, the existence of the margin requirement introduces non-smoothness into this problem. Following the idea of [65], we decompose the problem into two sub-problems for three independent regions and achieve the optimal control for each of them respectively. Different from [65], we adopt the change-of-variable formula with local time on curve [45] to overcome the non-smoothness on the value function like the last chapter instead of applying viscosity solution. The optimality of the solution of the HJB equation can be verified by the verification theorem described in [5]. The solution of the HJB equation relies on the initial status (t_0, x_0), this fact is consistent with the last chapter and this kind of optimal control is named as the statically optimal control. From the statically optimal control, we derive the dynamically optimal control which purely depends on the current status (t, x) i.e. time-consistent. The definition and property of these two optimal control have been introduced in the introduction and the last chapter. Hence, we will omit the related details here. To the best of our knowledge, the time-consistent solution of optimal portfolio selection under a margin requirement has not been considered before.

Like the last chapter, we also consider the other two constrained cases where we constrain the size of the expectation/variance of the terminal wealth. The solution of these two problems can be easily derived from the first case. In Section 4, we further analyse the path of the optimal wealth process by numerical analysis. The numerical result indicates that increasing the margin rate will enhance the performance of both strategies under the perfect market assumption. Moreover, under the margin requirement, the following Theorem 3.1, Corollary 3.3 and Corollary 3.4 will have the same properties as the work of [43] .

3.2 Formulation of the problem

Consider an investor who aims to construct a self-financing portfolio consisting of the following riskless bond and risky asset:

$$dB_t = rB_t dt \tag{3.1}$$

with initial value $B_{t_0} = b$, where $b > 0$ and the riskless interest rate $r \in \mathbb{R}$ are constants; the price of the risky stock, S solves:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.2}$$

where we have the drift rate $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$. For the risky stock, we set the initial value $S_{t_0} = s_0$ as a constant $s_0 > 0$, and we condition that the drift rate μ must be strictly higher than the riskfree rate r, which guarantees the investor will receive risk premium for bearing risk.

In this chapter, short-selling of the risky asset is allowed; however, to avoid default, the investor is requested to deposit a part of his wealth with a given and fixed rate. In the following part, we will follow the setting of [65] to construct the model and denote this margin rate as θ and $\theta \in [0, +\infty]$. Recalling [5] and (3.1) and (3.2), we can derive the wealth process for the self-financing portfolio:

$$dX_t = (r + (\mu - r)u_t - r\theta u_t^-)X_t^u dt + X_t^u u_t \sigma dW_t \tag{3.3}$$

where $u^- = \max\{0, -u\}$ and $X_{t_0}^u = x_0$. In (3.3), the admissible control $u(t,x) = u(t, X_t^u)$ represents the percentage of wealth investing in the risky asset and U represents the set of all admissible controls. Moreover, following the setting of [43], for the admissible control u_t, the mapping $(t,x) \mapsto u(t,x) \cdot x$ is a continuous function from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$, and there is $u(t,0) = 0 \cdot u(t,0)$ for completeness as $u(t,x)$ may not well-defined at 0.

Under the probability measure $\mathbb{P}_{t,x}$, X_t^u is a strong Markov process for each admissible control and gives value x at t where $(t,x) \in [0,T] \times \mathbb{R}$. In this chapter, we will consider the following problem:

$$V(t,x) = \sup_u \left[\mathrm{E}_{t,x}(X_T^u) - c \, \mathrm{Var}_{t,x}(X_T^u) \right] \tag{3.4}$$

in which risk aversion rate $c > 0$ is a given constant. For the admissibility condition of u, we set that $\mathrm{E}_{t,x}\left[\int_t^T (1+u_s^2)(X_s^u)^2 ds\right] < \infty$.

Similar to the previous chapter, we will attempt to extend our conclusion to the other constrained cases:

$$V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u)\leq\alpha} \mathrm{E}_{t,x}(X_T^u) \tag{3.5}$$

$$V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u)\geq\beta} \mathrm{Var}_{t,x}(X_T^u) \tag{3.6}$$

where u is the admissible control, and $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$ are given constants.

The definition for the static and dynamic optimalities are introduced in the Introduction chapter and we will omit this part here.

3.3 Solution to the problems

In the following part, we will follow the idea and method of [43] to obtain the solution.

Theorem 3.1. *Consider the optimal problem $V(t,x) = \sup_u[\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)]$ in which X^u represents the wealth process and is the solution of the SDE (3.3) with $X_{t_0}^u = x_0$ under the probability measure $\mathbb{P}_{t_0,x_0}$ for $(t_0,x_0) \in [0,T]\times\mathbb{R}$ given and fixed. The related risk coefficient is defined by $\delta = (\mu - r)/\sigma$ in which $\mu, r \in \mathbb{R}$, and the under the margin requirement, there is $\rho = (\mu - r + r\theta)/\sigma$. Note that we assume that $\delta \neq 0$ and $r \neq 0$. (The cases $\delta = 0$, $\rho = 0$ or $r = 0$ follow by passage to the limit when the non-zero δ, ρ or r approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,x) = \begin{cases} \frac{\delta}{\sigma}\frac{1}{x}[-x + x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}] & \text{if (t,x)} \in \Gamma_1\cup\Gamma_3 \\ \frac{\rho}{\sigma}\frac{1}{x}[-x + x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}] & \text{if (t,x)} \in \Gamma_2 \end{cases} \tag{3.7}$$

where Γ_1, Γ_2, and Γ_3 are given by:

$$\begin{cases} \Gamma_1 := \left\{(t,x)\in[t_0,T]\times\mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}\right] < 0\right\} \\ \Gamma_2 := \left\{(t,x)\in[t_0,T]\times\mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}\right] > 0\right\} \\ \Gamma_3 := \left\{(t,x)\in[t_0,T]\times\mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}\right] = 0\right\} \end{cases} \tag{3.8}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,x) = \begin{cases} \frac{\delta}{2c\sigma}\frac{1}{x}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})} & \text{if } x \geq 0 \\ \frac{\rho}{2c\sigma}\frac{1}{x}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})} & \text{if } x < 0 \end{cases} \tag{3.9}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$.

Proof. In this chapter, we claim that, for each pair of $(t_0,x_0) \in [0,T] \times \mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (4.3) with initial condition $X^u_{t_0} = x_0$. Furthermore, for X^u_t, $u \in U$ is any admissible control we defined in Section 3.2.

(A): Similar to what we have done in the previous chapter, we firstly convert the non-linear problem into a set of a linear problems. Re-arranging the value function gives:

$$\mathrm{E}_{t,x}(X^u_T) - c\,\mathrm{Var}_{t,x}(X^u_T) = \mathrm{E}_{t,x}(X^u_T) - c\big[\mathrm{E}_{t,x}[(X^u_T)^2] - \mathrm{E}_{t,x}(X^u_T)^2\big]. \tag{3.10}$$

By setting $\mathrm{E}_{t,x}(X^u_T) = M$ and $M \geq x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}$, equation (3.10) leads to:

$$\begin{aligned} V(t,x) &= \sup_{M\in\mathbb{R}} \sup_{u:\mathrm{E}_{t,x}(X^u_T)=M} \Big[\mathrm{E}_{t,x}(X^u_T) - c\big[\mathrm{E}_{t,x}[(X^u_T)^2] - \mathrm{E}_{t,x}(X^u_T)^2\big]\Big] \\ &= \sup_{M\in\mathbb{R}} \Big[M + cM^2 - c \inf_{u:\mathrm{E}_{t,x}(X^u_T)=M} \mathrm{E}_{t,x}[(X^u_T)^2]\Big] \end{aligned} \tag{3.11}$$

in which there exists a linear and constrained problem:

$$V_M(t,x) = \inf_{u:\mathrm{E}_{t,x}(X^u_T)=M} \mathrm{E}_{t,x}[(X^u_T)^2]. \tag{3.12}$$

1. As what we have done in the previous chapter, we firstly introduce Lagrange multipliers in (3.12) and receive:

$$L_{t,x}(u,\lambda) = \mathrm{E}_{t,x}[(X^u_T)^2] - \lambda[\mathrm{E}_{t,x}(X^u_T) - M] \tag{3.13}$$

where $\lambda > 0$. Furthermore, we have seen that there exists $\lambda = \lambda(t,x,M)$ such that $\mathrm{E}_{t,x}(X^{u^\lambda_*}_T) = M$ and

$$V_M(t,x) = L_{t,x}(u^\lambda_*,\lambda) \leq E_{t,x}[(X^u_T)^2] \tag{3.14}$$

where $u \in U$ is any admissible control with $\mathrm{E}_{t,x}(X^u_T) = M$. Hence, solving (3.13) will give the optimal control that minimises (3.12).

2. To handle (3.13), let us adopt the HJB equation in the following optimal control problem:

$$V^{\lambda}(t,x) = \inf_{u} \mathrm{E}_{t,x}(X_T^{u2}) - \lambda \mathrm{E}_{t,x}(X_T^u). \tag{3.15}$$

Recalling the SDE (3.3), the corresponding HJB equation is given by:

$$\inf_{u}\{V_t^{\lambda} + (r + (\mu - r)u - r\theta u^-)xV_x^{\lambda} + \frac{1}{2}x^2\sigma^2u^2V_{xx}^{\lambda}\} = 0 \tag{3.16}$$

with terminal condition:

$$V^{\lambda}(T,x) = x^2 - \lambda x. \tag{3.17}$$

3. In (3.16), we see that the HJB equation contains term $u^- = \max\{0, -u\}$, which indicates that if $u < 0$, then there is an extra charge for the short-selling action. In [65], Zhou and Wu handles this case with general market coefficient by decomposing (3.16) into a few of sub-problems and achieve the optimal solution the HJB equation in each disjoint region. Hence, we will follow the idea of [65] but conduct a simplified method. Hence, in this case, we need to consider the following two cases:

$$\begin{cases} \inf_u\{V_t^{\lambda} + (r + (\mu - r)u)xV_x^{\lambda} + \frac{1}{2}x^2\sigma^2u^2V_{xx}^{\lambda}\} = 0 & \text{if } u(t,x) \geq 0 \\ \inf_u\{V_t^{\lambda} + (r + (\mu - r)u + r\theta u)xV_x^{\lambda} + \frac{1}{2}x^2\sigma^2u^2V_{xx}^{\lambda}\} = 0 & \text{if } u(t,x) < 0 \end{cases} \tag{3.18}$$

with the same terminal condition as (3.17). This piece-wise condition in (3.18) naturally leads to non-smoothness in $V^{\lambda}(t,x)$. These two HJB equations can be seen as the function of u, hence, according to the quadratic function property, we further assume that $V_{xx}^{\lambda} > 0$. We firstly design the value function of $V^{\lambda}(t,x)$ by:

$$V^{\lambda}(t,x) = \begin{cases} a(t)x^2 + b(t)x + c(t) & \text{if } u(t,x) \geq 0 \\ A(t)x^2 + B(t)x + C(t) & \text{if } u(t,x) < 0 \end{cases} \tag{3.19}$$

with

$$\begin{cases} a'(t) = (\delta^2 - 2r)a(t) \\ a(T) = 1, \end{cases} \tag{3.20}$$

$$\begin{cases} b'(t) = (\delta^2 - r)b(t) \\ b(T) = -\lambda, \end{cases} \tag{3.21}$$

$$\begin{cases} c'(t) = \frac{\delta^2}{4}\frac{b(t)^2}{a(t)} \\ c(T) = 0 \end{cases} \tag{3.22}$$

where $\delta = (\mu - r)/\sigma$, and

$$\begin{cases} A'(t) = (\rho^2 - 2r)A(t) \\ A(T) = 1, \end{cases} \tag{3.23}$$

$$\begin{cases} B'(t) = (\rho^2 - r)B(t) \\ B(T) = -\lambda, \end{cases} \tag{3.24}$$

$$\begin{cases} C'(t) = \frac{\rho^2}{4}\frac{B(t)^2}{A(t)} \\ c(T) = 0, \end{cases} \tag{3.25}$$

where $\rho = (\mu - r + r\theta)/\sigma$. Equations (3.20)-(3.25) can be achieved by substituting (3.19) into (3.18) and comparing the coefficients for each term. Solving (3.20)-(3.25), we receive:

$$\begin{cases} a(t) = \mathrm{e}^{-(\delta^2 - 2\mathrm{r})(\mathrm{T}-\mathrm{t})} \\ b(t) = -\lambda \mathrm{e}^{-(\delta^2 - \mathrm{r})(\mathrm{T}-\mathrm{t})} \\ c(t) = -\frac{\lambda^2}{4}[1 - \mathrm{e}^{-\delta^2(\mathrm{T}-\mathrm{t})}], \end{cases} \tag{3.26}$$

and

$$\begin{cases} A(t) = \mathrm{e}^{-(\rho^2 - 2\mathrm{r})(\mathrm{T}-\mathrm{t})} \\ B(t) = -\lambda \mathrm{e}^{-(\rho^2 - \mathrm{r})(\mathrm{T}-\mathrm{t})} \\ C(t) = -\frac{\lambda^2}{4}[1 - \mathrm{e}^{-\rho^2(\mathrm{T}-\mathrm{t})}]. \end{cases} \tag{3.27}$$

To further consider the case when $u(t,x) \geq 0$, we define the following region:

$$\Gamma_1 := \left\{ (t,x) \in [t_0, T] \times \mathbb{R} \mid \frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right] < 0 \right\} \tag{3.28}$$

in which we have $V^\lambda(t,x) = a(t)x^2 + b(t)x + c(t)$ and V^λ is smooth enough in Γ_1. Substituting V_t^λ, V_x^λ, and V_{xx}^λ back into the first HJB equation of (3.19), we see that:

$$\begin{aligned}
&V_t^\lambda + (r + (\mu - r)u)xV_x^\lambda + \frac{1}{2}\sigma^2 u^2 x^2 V_{xx}^\lambda \qquad (3.29)\\
&= a'(t)x^2 + b'(t)x + c'(t) + rx[2a(t)x + b(t)]\\
&\quad + \inf_u \{(\mu - r)x(2a(t)x + b(t))u + \sigma^2 x^2 a(t)u^2\}\\
&= (\delta^2 - 2r)e^{-(\delta^2-2r)(T-t)}x^2 - \lambda(\delta^2 - r)e^{-(\delta^2-r)(T-t)}x + \frac{\lambda^2\delta^2}{4}e^{-\delta^2(T-t)}\\
&\quad + rx[2e^{-(\delta^2-2r)(T-t)}x - \lambda e^{-(\delta^2-r)(T-t)}]\\
&\quad + \inf_u \{(\mu - r)x(2e^{-(\delta^2-2r)(T-t)}x - \lambda e^{-(\delta^2-r)(T-t)})u + \sigma^2 x^2 e^{-(\delta^2-2r)(T-t)}u^2\}.
\end{aligned}$$

From the last term of (3.28), we can see that:

$$u^*(t,x) = -\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}e^{-r(T-t)}\right] \qquad (3.30)$$

which is positive in Γ_1. Substituting (3.30) back into (3.29) we can see that (3.29) equals to 0, which verifies that the control given by (3.30) and the first value function of (3.19) is optimal for the first HJB equation in (3.18) in Γ_1.

Furthermore, we define the second region:

$$\Gamma_2 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - \frac{\lambda}{2}e^{-r(T-t)}\right] > 0\right\} \qquad (3.31)$$

with $V^\lambda(t,x) = A(t)x^2 + B(t)x + C(t)$. Similarly, substituting V_t^λ, V_x^λ and V_{xx}^λ back into (3.18), we have:

$$\begin{aligned}
&V_t^\lambda + (r + (\mu - r)u + r\theta u)xV_x^\lambda + \frac{1}{2}\sigma^2 u^2 x^2 V_{xx}^\lambda \qquad (3.32)\\
&= A'(t)x^2 + B'(t)x + C'(t) + rx[2A(t)x + B(t)]\\
&\quad + \inf_u \{(\mu - r + r\theta)x(2A(t)x + B(t))u + \sigma^2 x^2 A(t)u^2\}\\
&= (\rho^2 - 2r)e^{-(\rho^2-2r)(T-t)}x^2 - \lambda(\rho^2 - r)e^{-(\rho^2-r)(T-t)}x + \frac{\lambda^2\rho^2}{4}e^{-\rho^2(T-t)}\\
&\quad + rx[2e^{-(\rho^2-2r)(T-t)}x - \lambda e^{-(\rho^2-r)(T-t)}]\\
&\quad + \inf_u \{(\mu - r + r\theta)x(2e^{-(\rho^2-2r)(T-t)}x - \lambda e^{-(\rho^2-r)(T-t)})u + \sigma^2 x^2 e^{-(\rho^2-2r)(T-t)}u^2\}.
\end{aligned}$$

In this case, we can see that the optimal control u is given by:

$$u^*(t,x) = -\frac{\rho}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}e^{-r(T-t)}\right] \qquad (3.33)$$

Moreover, since $\rho := (\mu - r + r\theta)/\sigma > 0$, we can see that:

$$-\frac{\rho}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right] < 0. \tag{3.34}$$

The first inequality in (3.34) holds as in Γ_2 there is $\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right] > 0$. Furthermore, substituting (3.33) back into (3.32), we can see that (3.32) equals 0, which verifies the control given in (3.33) and the value function are the optimal in Γ_2.

For the final region where the non-smoothness occurs:

$$\Gamma_3 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right] = 0\right\} \tag{3.35}$$

Hence, under $x = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}$, we can easily verify that:

$$a(t)x^2 + b(t)x + c(t) = A(t)x^2 + B(t)x + C(t) = -\frac{\lambda^2}{4} \tag{3.36}$$

which indicates that $V^{\lambda}(t,x)$ is continuous on Γ_3. Moreover, there are:

$$\begin{cases} V_t^{\lambda} = A'(t)x^2 + B'(t)x + C'(t) = a'(t)x^2 + b'(t)x + c'(t) = 0 \\ V_x^{\lambda} = 2A(t)x + B(t) = 2a(t)x + b(t) = 0 \end{cases} \tag{3.37}$$

which indicates the existence and continuity of V_t^{λ} and V_x^{λ} in Γ_3. However, in Γ_3, we can see that V_{xx}^{λ} does not exist on this curve, i.e. Γ_3, as $A(t) \neq a(t)$, which indicates that $V^{\lambda}(t,x)$ is not a $C^{1,2}$ function for the HJB equation (3.18). In the previous chapter, we have seen that the change-of-variable formula [45] enables us to express the value function V^{λ} through the curve, and the optimality of the candidate control can be achieved by verification theorem [5]. In Chapter 2, we have shown that the curve $b(t) = x = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}$ parameterising Γ_3 is of bounded variation, and V_{xx}^{λ} is locally bounded in both Γ_1 and Γ_2. Furthermore, we can summarise the candidate control by:

$$u^*(t,x) = \begin{cases} -\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right] & \text{if (t,x)} \in \Gamma_1 \cup \Gamma_3 \\ -\frac{\rho}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right] & \text{if (t,x)} \in \Gamma_2. \end{cases} \tag{3.38}$$

Again, recalling Theorem 3.1 in [45] and SDE (3.3), we can express the value function $V^{\lambda}(T, X_T^u)$ on $[0,T] \times \mathbb{R}$, which gives:

$$\begin{aligned} V^{\lambda}(T,X_T^u) =& V^{\lambda}(t,x) \\ &+ \int_t^T \left(V_t^{\lambda} + (rX_s^u + (\mu - r)u - r\theta u^-)V_x^{\lambda} + \frac{\sigma^2 u^2 X_s^{u2}}{2} V_{xx}^{\lambda}\right) I(X_s^u \neq b(s))ds \\ &+ \int_t^T \sigma u X_s^u V_x^{\lambda} dW_s \end{aligned} \tag{3.39}$$

with $b(t) = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}$. We have seen that in Γ_1 and Γ_2, there is:

$$\big(V_t^\lambda + (rX_s^u + (\mu - r)u - r\theta u^-)V_x^\lambda + \frac{\sigma^2 u^2 X_s^{u2}}{2} V_{xx}^\lambda\big) I(X_s^u \neq b(s)) \geq 0. \tag{3.40}$$

Since $V^\lambda(T, X_T^u) = X_T^{u2} - \lambda X_T^u$, equation (3.39) leads to:

$$V^\lambda(t,x) \geq V^\lambda(T, X_T^u) = X_T^{u2} - \lambda X_T^u - \int_t^T \sigma u X_s^u V_x^\lambda dW_s. \tag{3.41}$$

Taking $\mathrm{E}_{t,x}$ on both sides of (3.41), we can see that the stochastic integral term is actually a martingale under the admissibility condition $\mathrm{E}_{t_0,x_0}[\max_{t_0 \leq t \leq T}(X_T^u)^2] < \infty$ and the term $\mathrm{E}_{t,x}\left[\int_t^T \sigma u X_s^u V_x^\lambda dW_s\right]$ vanishes. Therefore, there is:

$$V^\lambda(t,x) \geq \mathrm{E}_{t,x}\left[X_T^{u2} - \lambda X_T^u\right] \tag{3.42}$$

which holds for all admissible controls. Hence, we can see that:

$$V^\lambda(t,x) \geq \inf_u \mathrm{E}_{t,x}\left[X_T^{u2} - \lambda X_T^u\right]. \tag{3.43}$$

Furthermore, we further claim that the control given by (3.38) is optimal, hence, for (t,x) in Γ_1 and Γ_2, we have:

$$V_t^\lambda + (rX_t^{u^*} + (\mu - r)u^* - r\theta u^{*-})V_x^\lambda + \frac{\sigma^2 u^{*2} X_t^{u^*2}}{2} V_{xx}^\lambda = 0 \tag{3.44}$$

and in Γ_3, the integrand term in (3.39) vanishes. Hence, for the optimal control u^*, there is:

$$V^\lambda(t,x) = X_T^{u^*2} - \lambda X_T^{u^*} - \int_t^T (\sigma u^* X_s^{u^*} V_x^\lambda) dW_s. \tag{3.45}$$

Taking expectation $\mathrm{E}_{t,x}$ on the both side of (3.45) yields:

$$V^\lambda(t,x) = \mathrm{E}_{t,x}\left[X_T^{u^*2} - \lambda X_T^{u^*}\right]. \tag{3.46}$$

Hence, we have the following trivial inequality:

$$\inf_u \mathrm{E}_{t,x}\left[X_T^{u2} - \lambda X_T^u\right] \geq \mathrm{E}_{t,x}\left[X_T^{u^*2} - \lambda X_T^{u^*}\right]. \tag{3.47}$$

Upon (3.43) and (3.47) we have:

$$V^\lambda(t,x) \leq \mathrm{E}_{t,x}\left[X_T^{u2} - \lambda X_T^u\right] \leq \mathrm{E}_{t,x}\left[X_T^{u^*2} - \lambda X_T^{u^*}\right] \leq V^\lambda(t,x) \tag{3.48}$$

Hence, we have verified that the control given by (3.38) is the optimal control for V^λ.

4. Now, the target is to achieve the optimal λ for (3.38). For $(t,x) \in \Gamma_1 \cup \Gamma_3$, we have $u^*(t,x) = -\frac{\delta}{\sigma}\frac{1}{x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\right]$. This case is consistent with the conclusion in previous chapter (2.68)-(2.77), hence, we will omit the details and take the advantage from the last chapter. For $(t,x) \in \Gamma_1 \cup \Gamma_3$, the corresponding wealth process is given by:

$$X_t^u = \mathrm{e}^{\mathrm{r(t-t_0)}}\left[\mathrm{K} - (\mathrm{K} - \mathrm{x_0})\mathrm{e}^{-\delta(\mathrm{W_t} - \mathrm{W_{t_0}}) - \frac{3\delta^2}{2}(\mathrm{t-t_0})}\right] \tag{3.49}$$

where $K = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t_0)}}$. Recalling $\mathrm{E}_{t,x}(X_T^u) = M$, there is:

$$\mathrm{E}_{t_0,x_0}(X_T^u) = x_0\mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T-t_0})} + \frac{\lambda}{2}\left[1 - \mathrm{e}^{-\delta^2(\mathrm{T-t_0})}\right] = \mathrm{M}. \tag{3.50}$$

Rearranging (3.50) yields:

$$\lambda = 2\frac{M - x_0\mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T-t_0})}}{1 - \mathrm{e}^{-\delta^2(\mathrm{T-t_0})}}. \tag{3.51}$$

Hence, we can easily calculate $\mathrm{E}_{t_0,x_0}(X_T^{u2})$ and $V_M(t_0,x_0)$, which gives the the value function $V(t_0,x_0)$ as a quadratic function of M:

$$V(t_0,x_0) = \sup_{M\in\mathbb{R}}\left[M + cM^2 - c(x_0{}^2\mathrm{e}^{-(\delta^2-2\mathrm{r})(\mathrm{T-t_0})} + \frac{(\mathrm{M} - \mathrm{x_0}\mathrm{e}^{-(\delta^2-\mathrm{r})(\mathrm{T-t_0})})^2}{1-\mathrm{e}^{-\delta^2(\mathrm{T-t})}})\right]. \tag{3.52}$$

Calculating the M_* directly gives the optimal value of λ_*, which is

$$\lambda_* = 2x_0\mathrm{e}^{\mathrm{r(T-t_0)}} + \frac{1}{\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})} \tag{3.53}$$

which is independent of θ. Substituting (3.53) back into (3.38) gives the optimal control:

$$u_*^s(t,x) = \begin{cases} \frac{\delta}{\sigma}\frac{1}{x}[-x + x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}] & \text{if (t,x)} \in \Gamma_1 \cup \Gamma_3 \\ \frac{\rho}{\sigma}\frac{1}{x}[-x + x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}] & \text{if (t,x)} \in \Gamma_2 \end{cases} \tag{3.54}$$

where Γ_1, Γ_2, and Γ_3 are given by:

$$\begin{cases} \Gamma_1 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}\right] < 0\right\} \\ \Gamma_2 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}\right] > 0\right\} \\ \Gamma_3 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} - \frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2(\mathrm{T-t_0})-\mathrm{r(T-t)}}\right] = 0\right\} \end{cases} \tag{3.55}$$

which confirms (3.7). Hence, we have obtained the statically optimal control for (3.4)

(B): In striving for the dynamically optimal control, we firstly replace the t_0 and x_0 in (3.54) as we have claimed that the dynamically optimal control is equivalent to the statically optimal control with the same initial status, which gives:

$$u_*^d(t,x) = \begin{cases} \frac{\delta}{2c\sigma}\frac{1}{x}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})} & \text{if (t,x)} \in \Gamma_1 \cup \Gamma_3 \\ \frac{\rho}{2c\sigma}\frac{1}{x}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})} & \text{if (t,x)} \in \Gamma_2. \end{cases} \tag{3.56}$$

Furthermore, we replace t_0 and x_0 in three regions Γ_1, Γ_2, and Γ_3, we noted that conducting short-selling or not purely depends on the current wealth. Hence, we can further simplify $\Gamma_1 \cup \Gamma_3$ to $\{(t,x) \in [t_0,T] \times \mathbb{R} \mid x \geq 0\}$ and $\Gamma_2 := \{(t,x) \in [t_0,T] \times \mathbb{R} \mid x < 0\}$. To verify the optimality of (3.56), we follow the previous chapter and set $u_*^d(t_0,x_0) = w(t_0,x_0)$, $w(t_0,x_0) = u_*^s(t_0,x_0)$, and $v(t_0,x_0)$ for any admissible such that $v(t_0,x_0) \neq u_*^d(t_0,x_0)$. For a dynamically optimal control, the following relationship must hold:

$$V_w(t_0,x_0) = \mathrm{E}_{t_0,x_0}(X_T^w) - c\,\mathrm{Var}_{t_0,x_0}(X_T^w) > \mathrm{E}_{t_0,x_0}(X_T^v) - c\,\mathrm{Var}_{t_0,x_0}(X_T^v) = V_v(t_0,x_0) \tag{3.57}$$

for any $(t_0,x_0) \in [0,T] \times \mathbb{R}$ based upon the fact that $V_w(t_0,x_0) = V(t_0,x_0)$ and $w(t_0,x_0)$ is statically optimal.

5. To verify the optimality, we need to consider the following cases when $x_0 > 0$, $x_0 = 0$, and $x_0 < 0$. If $x_0 > 0$, there is $u_*^d(t_0,x_0) = \frac{\delta}{2c\sigma}\frac{1}{x_0}\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t_0})} > 0$, and if $x_0 = 0$, we have $u_*^d(t_0,x_0) = x_0 \cdot u_*^d(t_0,x_0) = 0$. Those two cases are consistent with the conclusion in the proof of Theorem 2.3 in Chapter 2. The proof of exhibited in (2.81)-(2.88) which proves (3.56) is the dynamically optimal control when $(t_0,x_0) \in \Gamma_1 \cup \Gamma_3$ still holds in this case. Hence, we will omit this part and focus on proving that (3.56) is dynamically optimal when $x_0 < 0$. For any other admissible control $v(t_0,x_0) \neq w(t_0,x_0)$, we set $\mathrm{E}_{t_0,x_0}(X_T^v) := M_v \neq M_w =: \mathrm{E}_{t_0,x_0}(X_T^w)$. Recalling (3.15) and (3.27), we can see that for $w(t_0,x_0)$ there is:

$$\begin{aligned} V_*^\lambda(t_0,x_0) &= A(t_0)x_0^2 + B(t_0)x_0 + C(t_0) \\ &= \mathrm{e}^{-(\rho^2-2\mathrm{r})(\mathrm{T}-\mathrm{t_0})}\mathrm{x_0^2} - \lambda_*\mathrm{e}^{-(\rho^2-\mathrm{r})(\mathrm{T}-\mathrm{t_0})}\mathrm{x_0} - \frac{\lambda_*^2}{4}[1-\mathrm{e}^{-\rho^2(\mathrm{T}-\mathrm{t_0})}] \end{aligned} \tag{3.58}$$

where λ_* is given by (3.53). Since $V^{\lambda_*}(t_0,x_0) = \mathrm{E}_{t_0,x_0}(X_T^{w2}) - \lambda_*\mathrm{E}_{t_0,x_0}(X_T^w)$ and $\mathrm{E}_{t_0,x_0}(X_T^w) = M_w$, there is:

$$\begin{aligned} \mathrm{E}_{t_0,x_0}(X_T^{w2}) &= V^{\lambda_*}(t_0,x_0) + \lambda_* M_w \\ &= \mathrm{e}^{-(\rho^2-2\mathrm{r})(\mathrm{T}-\mathrm{t_0})}\mathrm{x_0^2} - \lambda_*\mathrm{e}^{-(\rho^2-\mathrm{r})(\mathrm{T}-\mathrm{t_0})}\mathrm{x_0} - \frac{\lambda_*^2}{4}[1-\mathrm{e}^{-\rho^2(\mathrm{T}-\mathrm{t_0})}] + \lambda_*\mathrm{M_w}. \end{aligned} \tag{3.59}$$

Substituting (3.59) back into (3.11) yields:

$$
\begin{aligned}
&V_w(t_0,x_0) \qquad (3.60)\\
&= M_w + cM_w^2 - c\left(\mathrm{e}^{-(\rho^2-2r)(T-t_0)}x_0^2 - \lambda_*\mathrm{e}^{-(\rho^2-r)(T-t_0)}x_0 - \frac{\lambda_*^2}{4}[1-\mathrm{e}^{-\rho^2(T-t_0)}] + \lambda_* M_w\right)\\
&> M_v + cM_v^2 - c\left(\mathrm{e}^{-(\rho^2-2r)(T-t_0)}x_0^2 - \lambda_*\mathrm{e}^{-(\rho^2-r)(T-t_0)}x_0 - \frac{\lambda_*^2}{4}[1-\mathrm{e}^{-\rho^2(T-t_0)}] + \lambda_* M_v\right)\\
&= V_v(t_0,x_0)
\end{aligned}
$$

in which the inequality holds as $w(t_0,x_0)$ is also statically optimal when $x_0 < 0$, i.e. in Γ_2. Hence, we have verified the optimal control given by (3.56) is dynamically optimal. □

Remark 3.2. In the proof above, the optimal value of λ_* is given by (3.53), which is independent of θ and we will explore the reason behind it. Recalling (3.15), (3.26) and (3.27), there exists:

$$
\begin{aligned}
V^\lambda(t_0,x_0) &= a(t_0)x_0^2 + b(t_0)x_0 + c(t_0) \qquad (3.61)\\
&= \mathrm{e}^{-(\delta^2-2r)(T-t_0)}x_0^2 - \lambda\mathrm{e}^{-(\delta^2-r)(T-t_0)}x_0 - \frac{\lambda^2}{4}[1-\mathrm{e}^{-\delta^2(T-t_0)}]
\end{aligned}
$$

when $\frac{1}{x_0}\left[x_0 - \frac{\lambda}{2}\mathrm{e}^{-r(T-t_0)}\right] \le 0$. Recalling $L_{t_0,x_0}(u,\lambda) = \mathrm{E}_{t_0,x_0}({X_T^u}^2) - \lambda\mathrm{E}_{t_0,x_0}(X_T^u) + \lambda M$, we can see that:

$$
L_{t_0,x_0}(u,\lambda) = \mathrm{e}^{-(\delta^2-2r)(T-t_0)}x_0^2 + \lambda\left(M - \mathrm{e}^{-(\delta^2-r)(T-t_0)}x_0\right) - \frac{\lambda^2}{4}\left[1-\mathrm{e}^{-\delta^2(T-t_0)}\right] \qquad (3.62)
$$

which gives an quadratic function of λ. Moreover the condition $\frac{1}{x_0}\left[x_0 - \frac{\lambda}{2}\mathrm{e}^{-r(T-t_0)}\right] \le 0$ implies that $\lambda \ge 2x_0\mathrm{e}^{r(T-t_0)}$. According to the quadratic function property, we have:

$$
\lambda_1^* = 2\frac{M - x_0\mathrm{e}^{-(\delta^2-r)(T-t_0)}}{1-\mathrm{e}^{-\delta^2(T-t_0)}} \qquad (3.63)
$$

and we can easily verify that:

$$
\lambda_1^* - 2x_0\mathrm{e}^{r(T-t_0)} = \frac{2\left[M - x_0\mathrm{e}^{r(T-t_0)}\right]}{1-\mathrm{e}^{-\delta^2(T-t_0)}} \ge 0 \qquad (3.64)
$$

as $M \ge x_0\mathrm{e}^{r(T-t_0)}$, which indicates that λ_* given in (3.63) is optimal point of (3.62) under $\lambda \ge 2x_0\mathrm{e}^{r(T-t_0)}$, and $L_{t_0,x_0}(u,\lambda)$ achieves optimal value:

$$
L_{t_0,x_0}(u,\lambda_1^*) = \mathrm{e}^{-(\delta^2-2r)(T-t_0)}x_0^2 + \frac{(M - x_0\mathrm{e}^{-(\delta^2-r)(T-t_0)})^2}{1-\mathrm{e}^{-\delta^2(T-t_0)}} > 0. \qquad (3.65)
$$

On the other hand, when $\frac{1}{x_0}\left[x_0 - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}\right] \geq 0$, there is $\lambda \leq 2x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}$, and $L_{t_0,x_0}(u,\lambda)$ is given by:

$$\begin{aligned} L_{t_0,x_0}(u,\lambda) &= A(t_0)x_0^2 + B(t_0)x_0 + C(t_0) + \lambda M \\ &= \mathrm{e}^{-(\rho^2-2\mathrm{r})(\mathrm{T}-\mathrm{t}_0)}\mathrm{x}_0^2 + \lambda(\mathrm{M} - \mathrm{e}^{-(\rho^2-\mathrm{r})(\mathrm{T}-\mathrm{t}_0)}\mathrm{x}_0) - \frac{\lambda^2}{4}[1 - \mathrm{e}^{-\rho^2(\mathrm{T}-\mathrm{t}_0)}] \end{aligned} \tag{3.66}$$

with optimal λ_2^*:

$$\lambda_2^* = 2\frac{M - x_0\mathrm{e}^{-(\rho^2-\mathrm{r})(\mathrm{T}-\mathrm{t}_0)}}{1 - \mathrm{e}^{-\rho^2(\mathrm{T}-\mathrm{t}_0)}} \tag{3.67}$$

However, we can see that:

$$\lambda_2^* - 2x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)} = \frac{2\left[\mathrm{M} - \mathrm{x}_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}\right]}{1 - \mathrm{e}^{-\rho^2(\mathrm{T}-\mathrm{t}_0)}} \geq 0 \tag{3.68}$$

which violates the condition $0 < \lambda \leq 2x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}$ except $\lambda = 2x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}$. Hence, for (3.66), λ can only reach its local optimal point at $2x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}$. It is easily seen that:

$$L_{t_0,x_0}(u,\lambda_2^*) = 2x_0\mathrm{e}^{\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}\mathrm{M} - \mathrm{x}_0^2\mathrm{e}^{2\mathrm{r}(\mathrm{T}-\mathrm{t}_0)}. \tag{3.69}$$

Since the target is to achieve the optimal value of $L_{t_0,x_0}(u,\lambda)$, then we can see that:

$$L_{t_0,x_0}(u,\lambda_1^*) \geq L_{t_0,x_0}(u,\lambda_2^*). \tag{3.70}$$

Hence, λ_1^* is the optimal solution we are looking for, and this is consistent with (3.51). □

Similar to the two corollaries in Chapter 2, we can still extend our conclusion from the (3.4) to (3.5) and (3.6). As [43] states, V_1 and V_2 can be derived from the (3.4) by choosing a proper value of Lagrange multiplier. Hence, we will following the work of proofs of Corollary 5 and Corollary 7 of [43] to derive the solution for (3.5) and (3.6). Since the work of this part is very similar to the work of [43], we will only briefly introduce the idea and omit most of the detailed proof.

Corollary 3.3. *Consider the optimal problem* $V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u)\leq\alpha}[\mathrm{E}_{t,x}(X_T^u)]$ *in which* X^u *represents the wealth process and is the solution of the SDE* (3.3) *with* $X_{t_0}^u = x_0$ *under the probability measure* $\mathbb{P}_{t_0,x_0}$ *for* $(t_0,x_0) \in [0,T] \times \mathbb{R}$ *given and fixed. The related risk coefficient is defined by* $\delta = (\mu - r)/\sigma$ *in which* $\mu, r \in \mathbb{R}$*, and the under the margin requirement* $\rho = (\mu - r + r\theta)/\sigma$*. Note that we assume that* $\delta \neq 0$ *and* $r \neq 0$*. (The cases* $\delta = 0$*,* $\rho = 0$ *or* $r = 0$ *follow by passage to the limit when the non-zero* δ*,* ρ *or* r *approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,x) = \begin{cases} \frac{\delta}{\sigma}\frac{1}{x}\left[x_0 \mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} - \mathrm{x} + \sqrt{\alpha}\frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1}}\right] & \text{if } (\mathrm{t},\mathrm{x}) \in \Gamma_1^\alpha \cup \Gamma_3^\alpha \\ \frac{\rho}{\sigma}\frac{1}{x}\left[x_0 \mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} - \mathrm{x} + \sqrt{\alpha}\frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1}}\right] & \text{if } (\mathrm{t},\mathrm{x}) \in \Gamma_2^\alpha \end{cases} \tag{3.71}$$

where Γ_1, Γ_2, and Γ_3 are given by:

$$\begin{cases} \Gamma_1^\alpha := \left\{(t,x) \in [t_0,T]\times\mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} - \sqrt{\alpha}\frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1}}\right] < 0\right\} \\ \Gamma_2^\alpha := \left\{(t,x) \in [t_0,T]\times\mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} - \sqrt{\alpha}\frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1}}\right] > 0\right\} \\ \Gamma_3^\alpha := \left\{(t,x) \in [t_0,T]\times\mathbb{R} \mid \frac{1}{x}\left[x - x_0\mathrm{e}^{\mathrm{r}(\mathrm{t}-\mathrm{t}_0)} - \sqrt{\alpha}\frac{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)-\mathrm{r}(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)}-1}}\right] = 0\right\} \end{cases} \tag{3.72}$$

for $(t,x) \in [t_0,T]\times\mathbb{R}$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,x) = \begin{cases} \frac{\delta}{\sigma}\frac{1}{x}\sqrt{\alpha}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t})}-1}} & \text{if } x \geq 0 \\ \frac{\rho}{\sigma}\frac{1}{x}\sqrt{\alpha}\frac{\mathrm{e}^{(\delta^2-\mathrm{r})(\mathrm{T}-\mathrm{t})}}{\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t})}-1}} & \text{if } x < 0 \end{cases} \tag{3.73}$$

for $(t,x) \in [t_0,T]\times\mathbb{R}$.

Proof. In this chapter, we claim that, for each pair of $(t_0,x_0) \in [0,T]\times\mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (3.3) with initial condition $X_{t_0}^u = x_0$ and $u \in U$ is any admissible control we defined in Section 3.2.

(A): Applying Lagrange multipliers in $V_1(t,x)$, there is:

$$\begin{aligned} L_{t,x}^1(u,c) &= \mathrm{E}_{t,x}(X_T^u) - c[\mathrm{Var}_{t,x}(X_T^u) - \alpha] \\ &= \mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) + c\alpha. \end{aligned} \tag{3.74}$$

We can see that the statically optimal control given by (3.7) meeting $\mathrm{Var}_{t,x}(X_T^u) = \alpha$ will maximise (3.74) and is the statically optimal control for (3.5). To determine the optimal Lagrange multiplier $c(\alpha,t,x) > 0$, we further assume that for a given pair of $(t,x) \in [t_0,x_0]\times\mathbb{R}$ such that $u_*^c(t,x) > 0$. Recalling (3.49) and (3.53), we can calculate that:

$$\mathrm{Var}_{t_0,x_0}(X_T^{u_*^c}) = \frac{1}{4c^2}[\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)} - 1]. \tag{3.75}$$

Setting (3.75) equal to α yields:

$$c = \frac{1}{2\sqrt{\alpha}}\sqrt{\mathrm{e}^{\delta^2(\mathrm{T}-\mathrm{t}_0)} - 1}. \tag{3.76}$$

Substituting (3.76) back into (3.7), we receive the statically optimal control given by (3.71).

(B) In striving for the dynamically optimal control, we first replace t_0 and x_0 by t and x in (3.71), which gives the candidate dynamically optimal control given by (3.72). To prove its optimality, we set $w(t_0,x_0) = u^d_*(t_0,x_0)$ and $w = u^s_*(t_0,x_0)$, and $v(t_0,x_0) \neq u^d_*(t_0,x_0)$ is any other admissible control in U. We can see that (3.57) is still valid with c given by (3.76), and under the constraint that $\mathrm{Var}_{t_0,x_0}(X^w_T) \leq \alpha$ and $\mathrm{Var}_{t_0,x_0}(X^v_T) \leq \alpha$, we can easily see that:

$$\mathrm{E}_{t_0,x_0}(X^w_T) - c\,\mathrm{Var}_{t_0,x_0}(X^w_T) \geq \mathrm{E}_{t_0,x_0}(X^w_T) - c\alpha > \mathrm{E}_{t_0,x_0}(X^v_T) - c\,\mathrm{Var}_{t_0,x_0}(X^v_T) \tag{3.77}$$

which leads to:

$$\mathrm{E}_{t_0,x_0}(X^w_T) > \mathrm{E}_{t_0,x_0}(X^v_T) - c(\mathrm{Var}_{t_0,x_0}(X^v_T) - \alpha) > \mathrm{E}_{t_0,x_0}(X^u_T) \tag{3.78}$$

and this confirms that (3.79) and finishes the proof. □

Corollary 3.4. *Consider the optimal problem* $V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X^u_T)\geq\beta}[\mathrm{Var}_{t,x}(X^u_T)]$ *in which* X^u *represents the wealth process and is the solution of the SDE* (3.3) *with* $X^u_{t_0} = x_0$ *under the probability measure* $\mathbb{P}_{t_0,x_0}$ *for* $(t_0,x_0) \in [0,T] \times \mathbb{R}$ *given and fixed. The related risk coefficient is defined by* $\delta = (\mu - r)/\sigma$ *in which* $\mu, r \in \mathbb{R}$*, and the under the margin requirement* $\rho = (\mu - r + r\theta)/\sigma$*. Note that we assume that* $\delta \neq 0$ *and* $r \neq 0$*. (The cases* $\delta = 0$*,* $\rho = 0$ *or* $r = 0$ *follow by passage to the limit when the non-zero* δ*,* ρ *or* r *approaches 0.)* Furthermore, we assume that the expectation of the terminal wealth, β, must satisfy $\beta > x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$. For a wise investor, if $\beta \leq x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$, he can simply invest all his wealth in the riskless asset and receive zero variance at the maturity T. Hence, in the following part, we assume that $\beta > x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$.

(A) *The statically optimal control is given by:*

$$u^s_*(t,x) = \begin{cases} \frac{\delta}{\sigma}\frac{1}{x}\left[x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - \mathrm{x} + (\beta - \mathrm{x_0 e^{r(T-t_0)}})\frac{\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\right] & \text{if (t,x)} \in \Gamma_1 \cup \Gamma_3 \\ \frac{\rho}{\sigma}\frac{1}{x}\left[x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - \mathrm{x} + (\beta - \mathrm{x_0 e^{r(T-t_0)}})\frac{\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\right] & \text{if (t,x)} \in \Gamma_2 \end{cases} \tag{3.79}$$

where Γ_1*,* Γ_2*, and* Γ_3 *are given by:*

$$\begin{cases} \Gamma^\beta_1 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - (\beta - \mathrm{x_0 e^{r(T-t_0)}})\frac{\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\right] < 0\right\} \\ \Gamma^\beta_2 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - (\beta - \mathrm{x_0 e^{r(T-t_0)}})\frac{\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\right] > 0\right\} \\ \Gamma^\beta_3 := \left\{(t,x) \in [t_0,T] \times \mathbb{R} \mid \frac{1}{x}\left[x - x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - (\beta - \mathrm{x_0 e^{r(T-t_0)}})\frac{\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}}{\mathrm{e}^{\delta^2\mathrm{(T-t_0)}}-1}\right] = 0\right\} \end{cases} \tag{3.80}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,x) = \begin{cases} \frac{\delta}{\sigma}\frac{1}{x}(\beta - x e^{r(T-t)})\frac{e^{(\delta^2-r)(T-t)}}{e^{\delta^2(T-t)}-1} & \text{if } x \geq 0 \\ \frac{\rho}{\sigma}\frac{1}{x}(\beta - x e^{r(T-t)})\frac{e^{(\delta^2-r)(T-t)}}{e^{\delta^2(T-t)}-1} & \text{if } x < 0 \end{cases} \tag{3.81}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$.

Proof. In this chapter, we claim that, for each pair of $(t_0,x_0) \in [0,T] \times \mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (3.3) with initial condition $X^u_{t_0} = x_0$ and $u \in U$ is any admissible control we defined in Section 3.2.

(A): As we have seen that applying Lagrange multipliers in (3.6) gives:

$$\inf_u \left[\mathrm{Var}_{t,x}(X_T^u) - c(\mathrm{E}_{t,x}(X_T^u) - \beta)\right] = -c \sup_u \left[\mathrm{E}_{t,x}(X_T^u) - \frac{1}{c}\mathrm{Var}_{t,x}(X_T^u)\right] + c\beta \tag{3.82}$$

from which we can see that the statically optimal control given by (3.7) with $\mathrm{E}_{t,x}(X_T^{u^*}) = \beta$ and risk aversion rate $1/c$ is the statically optimal control for (3.6). To obtain the optimal value of $1/c$, we further assume that for each $(t,x) \in [t_0,T] \times \mathbb{R}$, $u_*^s(t,x)$ given by (3.7) is greater than 0. Hence, recalling (3.49) and (3.53), we can calculate that:

$$\beta = x_0 e^{r(T-t_0)} + \frac{c}{2}[e^{\delta^2(T-t_0)} - 1] \tag{3.83}$$

which gives:

$$c = \frac{2(\beta - x_0 e^{r(T-t_0)})}{e^{\delta^2(T-t_0)} - 1}. \tag{3.84}$$

Substituting (3.48) back into (3.7) gives the statically optimal control given by (3.79).

(B) Replacing t_0 and x_0 by t and x in (3.79) gives the dynamically optimal control. Similar to the proof in the last corollary, we see that (3.57) is still valid with c given by (3.84). To prove the optimality of (3.81), we set $w(t_0,x_0) = u_*^d(t_0,x_0)$ and $w = u_*^s(t_0,x_0)$, and $v(t_0,x_0) \neq u_*^d(t_0,x_0)$ is any other admissible control in U. For w and v, there are $\mathrm{E}_{t,x}(X_T^w) \geq \beta$ and $\mathrm{E}_{t,x}(X_T^v) \geq \beta$. We can easily derive from (3.57) that:

$$\mathrm{E}_{t_0,x_0}(X_T^w) - c\,\mathrm{Var}_{t_0,x_0}(X_T^w) \geq \beta - c\,\mathrm{Var}_{t_0,x_0}(X_T^w) > \mathrm{E}_{t_0,x_0}(X_T^v) - c\,\mathrm{Var}_{t_0,x_0}(X_T^v) \tag{3.85}$$

from which we can derive that:

$$\begin{aligned} &-c\,\mathrm{Var}_{t_0,x_0}(X_T^w) > \mathrm{E}_{t_0,x_0}(X_T^v) - c\,\mathrm{Var}_{t_0,x_0}(X_T^v) - \beta \Rightarrow \\ &\mathrm{Var}_{t_0,x_0}(X_T^w) < \mathrm{Var}_{t_0,x_0}(X_T^v) - \frac{1}{c}\left[\mathrm{E}_{t_0,x_0}(X_T^v) - \beta\right] < \mathrm{Var}_{t_0,x_0}(X_T^v). \end{aligned} \tag{3.86}$$

Hence, we have verified that (3.81) is dynamically optimal and completed the proof. □

3.4 Analysis of the optimal wealth processes

1. In Section 3.3, we have achieved both statically and dynamically optimal controls for (3.4), (3.5) and (3.6) respectively, in which the margin requirement introduces the curve for the value function as well as both optimal strategies. The existence of the curve inspires us to further consider the optimal wealth processes for those two strategies.

In Theorem 3.1, the statically optimal control (3.7) changes when (t,x) moves from $\Gamma_1 \cup \Gamma_3$ to Γ_2. Hence, for each $(t,x) \in \Gamma_1 \cup \Gamma_3$, the wealth process is consistent with the cases when $t \leq \tau_\alpha$ given by (2.15) in Chapter 2, and we will take the advantage of the previous conclusion and omit details. Therefore, for $(t,x) \in \Gamma_1 \cup \Gamma_3$, the statically optimal wealth is

$$X_t^{s1} = x_0 e^{r(t-t_0)} + \frac{1}{2c} e^{(\delta^2-r)(T-t)} \left[e^{\delta^2(T-t_0)} - e^{-\delta(W_t - W_{t_0}) - \frac{\delta^2}{2}(t-t_0)} \right]. \tag{3.87}$$

Note that the steps of achieving (3.87) can be seen between (2.68)-(2.77) (cf. [43], page 9-10). Since $x_0 > 0$, we can see that the wealth process starts from $\Gamma_1 \cup \Gamma_3$ until the wealth process hits the upper curve or 0. Once it crosses the upper or lower boundary, the statically optimal control gives negative value, i.e. conducting short-selling, and the new wealth process will be driven by the optimal control in the second line of (3.7), which gives a new process starts from this point. To further illustrate this, we assume that at $t' \in (t_0, T)$, X_t^{s1} hits the upper curve or 0 with value $X_{t'}^{s1} = x'$, and the wealth process will across the curve with probability 1 almost surely. Hence, the static optimality will conduct short-selling and the new corresponding wealth process starts from x'. The corresponding new optimal wealth process is given by:

$$X_t^{s2} = x' e^{r(t-t')} + \frac{1}{2c} e^{\delta^2(T-t')-r(T-t)} \left[1 - e^{-\rho(W_t - W_{t'}) - \frac{3\rho^2}{2}(t-t')} \right] \tag{3.88}$$

where $t \in [t', T]$ (note that the step of calculating (3.88) is the the same as the step of obtaining (3.87). Similar thing will be observed when X_t^{s2} hits the boundary again, and the optimal control will gives the process (3.87) but with new initial value. This behaviour will keep happening until either X_t^{s1} or X_t^{s2} hits the maturity time T.

To illustrate this in a more intuitive way, we simulate this process and plot it in Figure 3.1. And in Figure 3.1, we exhibit the case when X_t^{s1} across the lower boundary 0. The behaviour is the same when it hits the upper boundary and we will omit this plot.

For the dynamically optimal control given by (3.9), there is only a lower boundary at 0. In Chapter 2, we have obtained the explicit solution for the dynamically optimal wealth

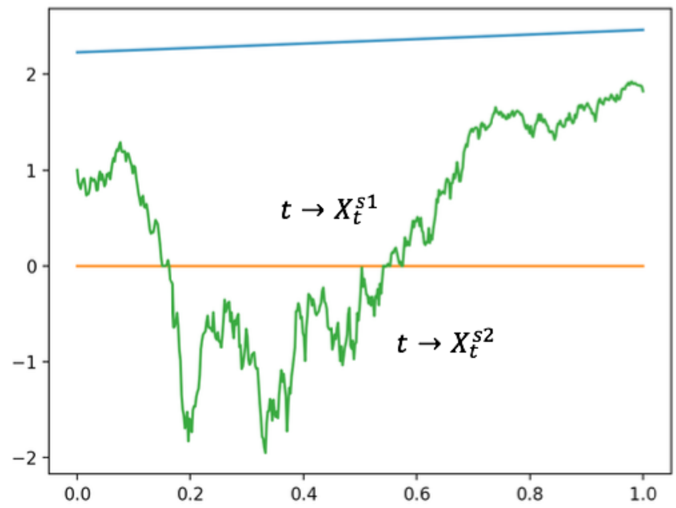

Figure 3.1: Simulation of statically optimal controlled wealth process for (3.7) based on the analysis of (3.87)-(3.88). It can be clearly seen that there exist an upper boundary (blue curve) and a lower boundary (orange line).

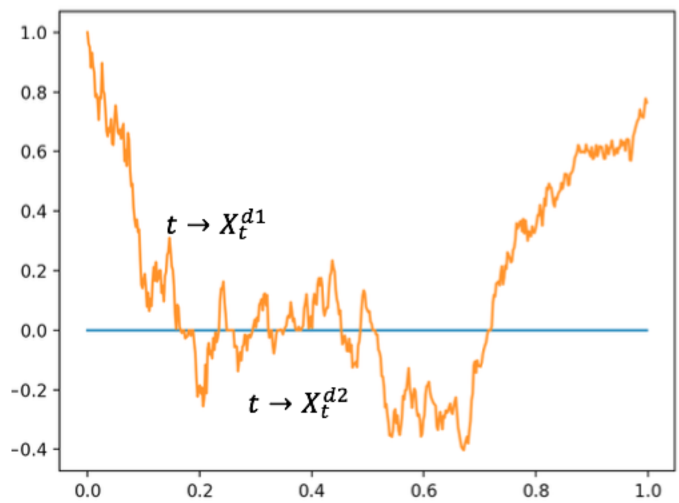

Figure 3.2: Simulation of dynamically optimal controlled wealth process for (3.9). It can be clearly seen that there exists a lower boundary (blue line).

process (2.20). Similar conclusions can be made in the dynamic case, and we plot the dynamic wealth process in Figure 3.2, in which X_t^{d1} is the process for $u^d \geq 0$ and X_t^{d2} is the process for $u^d < 0$.

Besides the unconstrained case, in [43] we have seen that for the constrained case where we constrain the size of the expectation of the terminal wealth, the dynamically optimal wealth process will converge to the targeted expected terminal wealth with probability one (see Corollary 7 of [43]). This phenomenon also exists in the previous chapter in Corollary 2.5 in Chapter 2 if the dynamically optimal wealth process does not hit 0 before it approaches to the maturity time. Hence, it will be interesting to consider if this phenomenon still exists under the margin requirement or not. In Figure 3.3, we simulate the dynamically optimal wealth process for the constrained problem (3.6). In the first figure of Figure 3.3, the wealth process does not hit 0 during the entire time, and this case is the same as Figure 1 in [43]. Furthermore, in the second figure of Figure 3.3, the wealth process passes through 0 and the part of process below 0 is driven by the control (3.81) when $u_*^d < 0$. However, we can see that even if in this chapter the wealth process is driven by two different processes, the dynamically optimal wealth process will still converge to the expected terminal wealth at the maturity T. This fact can also be proved by investigating the wealth process, and this part of proof will be consistent with the proof in (2.122)-(2.127). Hence, we will omit further details here.

2. To further investigate the impact of the margin requirement on the performance of both strategies, we conduct the sample path analysis by simulating the wealth processes for each optimal control and collecting the terminal wealth X_T^s and X_T^d to form the sample base, from which we can obtain the sample mean $\bar{\mu}^s$ and $\bar{\mu}^d$ as well as the corresponding sample variance $\bar{m}^s$ and $\bar{m}^d$. Note that in the case if X_T is not well-defined, we take the value of X at the second last point of time to the maturity T. In the following part, we firstly consider the unconstrained case (3.4) as an example.

In Table 3.1, we can see that the sample mean $\bar{\mu}^s$ is highly impacted by the change of risk aversion rate, which has been observed in the previous chapter. However, comparing the sample mean $\bar{\mu}^s$ under the fixed risk aversion rate with different values of margin rate, it is surprising to note that the impact of the margin rate is not obvious. Changing the margin rate slightly increases the sample mean. Since in Table 3.1 the margin rate is relatively small, we further increase the value of θ to consider some extreme cases. For instance, we consider the case when the risk aversion $c = 2.0$ is given and fixed and the margin rate θ increases to some large values. In Table 3.3, we can see that when θ increases from 1.0 to 15.0, the sample mean increases. However, if we further increase the value of θ we note that the sample mean $\bar{\mu}^s$ takes a negative value, which indicates the investor may suffer an infinite loss. Hence, we may conclude that short-selling should not be conducted when θ is large enough as it will lead to bankruptcy almost surely. This conclusion is consistent with [65] in which they claim that if the margin rate is infinitely large, the short-selling will be forbidden. A similar conclusion can be observed in the dynamic case. In Table 3.5, the

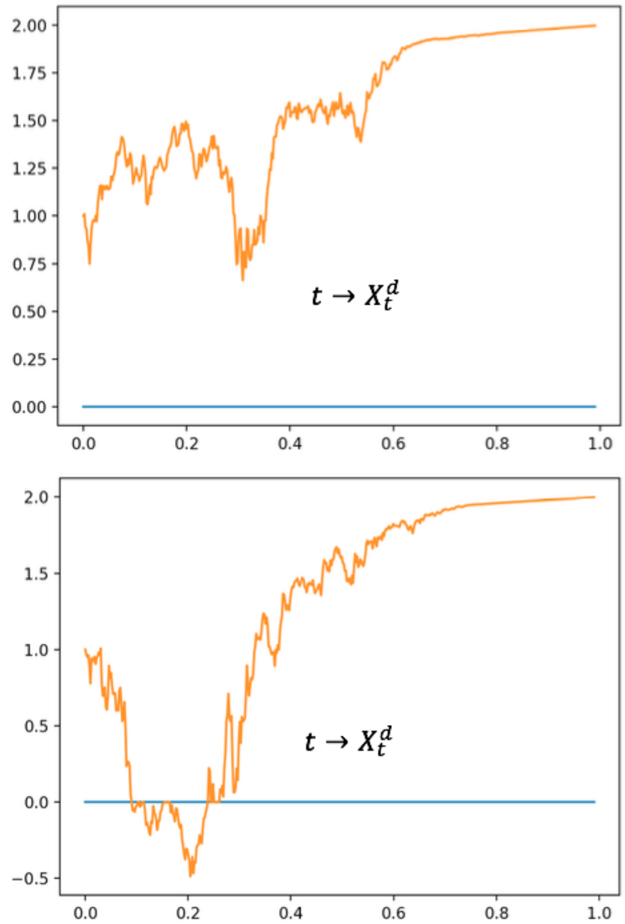

Figure 3.3: Simulation of the dynamically optimal controlled wealth process for (3.81), in which the blue line represents the boundary for short-selling.

only intuitive pattern is the sample mean $\bar{\mu}^d$ reduces as the risk aversion rate increases. The impact of the margin rate θ is not obvious as there exists the error of simulation. We further consider the extreme cases when we increase θ to large values in Table 3.7, which exhibits a more obvious pattern. From Table 3.7, we can see that increasing the value of margin rate will increase the performance of the dynamic optimality. However, it should be pointed out that the extreme value of θ is not possible in the real financial industry. Hence, we can conclude that changing the margin rate will not bring obvious impact on the performance of both static and dynamic optimalities when θ is small.

3. In Remark 6 of [43], we have seen the dynamic optimality always outperforms the static optimality, and this conclusion is valid in the previous chapter under the short-selling constraint. Naturally, it will be interesting to verify this conclusion under the margin requirement and observe if margin rate will affect this conclusion.

As we stated, calculating the analytical expectation can be difficult in this chapter.

Margin rate θ	Risk aversion rate	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
0	0.5	2.02300794	0.20356879
	1.0	2.79877920	0.70378819
	1.5	3.51006684	3.19572603
	2.0	4.38865317	4.29249344
0.1	0.5	2.00628372	0.48156230
	1.0	2.85193313	0.43348756
	1.5	3.66542053	1.19945378
	2.0	4.38802536	3.04788508
0.2	0.5	2.01203512	0.14163037
	1.0	2.82807101	0.57508790
	1.5	3.57955776	2.18991246
	2.0	4.40385168	3.87789854
0.4	0.5	2.00621573	0.25380631
	1.0	2.84881132	0.41474374
	1.5	3.65336612	1.16306604
	2.0	4.46565957	1.68974319

Table 3.1: Simulation of static optimality for the unconstrained problem (3.4) with respect to different values of θ and c.

(Note the related parameters are given by $\mu = 0.5$, $\sigma = 0.25$, $\delta = 1.2$, $x_0 = 1$, $r = 0.2$, $T = 1$, and $t_0 = 0$, and the sample size is 1,000.)

Margin rate θ	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
1.0	4.82014336	1.52749700
3.0	4.88100671	0.54567696
5.0	5.01756950	0.36420870
7.0	5.04423417	0.33173649
10.0	5.05005605	0.34274342
15.0	5.05668864	0.32547489
20.0	$-4.02\times e^{27}$	$1.61 \times e^{58}$

Table 3.3: Simulation for static optimality for the unconstrained problem (3.4) with respect to large values of θ.

(Note the related parameters are given by $c = 2$, $\mu = 0.5$, $\sigma = 0.25$, $\delta = 1.2$, $x_0 = 1$, $r = 0.2$, $T = 1$, and $t_0 = 0$, and the sample size is 1,000.)

Margin rate θ	Risk aversion rate	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
0	0.5	4.35473294	8.60838696
	1.0	2.79877920	2.05356981
	1.5	2.25720724	0.99281305
	2.0	2.03061625	0.52015353
0.1	0.5	4.40609172	8.24263509
	1.0	2.84947736	2.04956431
	1.5	2.29791417	1.00831042
	2.0	2.04731600	0.52691648
0.2	0.5	4.38611524	8.20322457
	1.0	2.84035364	2.09775305
	1.5	2.27582767	0.94816332
	2.0	2.04153863	0.50540597
0.4	0.5	4.36924374	7.75298679
	1.0	2.84921284	1.99734659
	1.5	2.29810646	0.90868216
	2.0	2.01822466	0.53486249

Table 3.5: Simulation for dynamic optimality for the unconstrained problem (3.4) with respect to different values of θ and c.

(Note the related parameters are given by $\mu = 0.5$, $\sigma = 0.25$, $\delta = 1.2$, $x_0 = 1$, $r = 0.2$, $T = 1$, and $t_0 = 0$, and the sample size is 1,000.)

Margin rate θ	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
1.0	4.49562744	7.49773070
3.0	4.64751518	6.60952526
10.0	4.74286547	6.38234598
30.0	4.78002478	6.30199539
50.0	4.82368995	6.87078233
80.0	4.90232811	12.8843884
150.0	5.24329565	76.4745865

Table 3.7: Simulation for dynamic optimality for the unconstrained problem (3.4) with respect to large values of θ.

(Note the related parameters are given by $c = 0.5$, $\mu = 0.5$, $\sigma = 0.25$, $\delta = 1.2$, $x_0 = 1$, $r = 0.2$, $T = 1$, and $t_0 = 0$, and the sample size is 1,000.)

Hence, we conduct numerical analysis to receive more insights. From Table 3.9 and Table 3.10, we can see that increasing the value of β will strengthen the performance of both strategies, the sample mean $\bar{\mu}$, if we fix the value of θ and increasing the value of θ will also improve the sample mean $\bar{\mu}$ of both static and dynamic optimalities for the given and fixed value of β. Moreover, comparing Table 3.9 and Table 3.10, it can be easily seen that the dynamic optimality outperforms the static optimality all the time, and we can conclude that under the existence of margin requirement, the conclusion made in Remark 6 [43] is still valid.

Besides, we further need to consider the Remark 8 of [43] where they prove that the dynamically optimal control of the constrained case outperforms the statically optimal control. Similar results can be also observed under the margin requirement. In the first part of this section, we have seen that the dynamically optimal wealth process of (3.6) will converge to β when t approaches to T. However, since the dynamically optimal control for constrained problem (3.6) is not well-defined at T, Python cannot simulate the terminal value. Upon the observation of Figure 3.3, we may conclude that $X_T^d = \beta$ at the maturity. This fact means when $t = T$, the variance of the terminal wealth of the dynamic wealth process will equal to 0. In Table 3.12 and Table 3.13, we see that the variance of the static strategy is significantly less than that of the dynamic strategy. From this perspective, we can summarise that the static strategy outperforms the dynamic strategy before the maturity time T. And at the maturity T, dynamic optimality gives the variance as 0, which makes it outperforms the static optimality.

Also, in Remark 10 of [43], Pedersen and Peskir prove that $\text{Var}_{t_0,x_0}(X_t^d) \to \infty$ when $t \to T$. We can also observe this fact under the existence of the margin requirement. The variance in Table 3.13, we can see that the variance is very large but not infinite. The reason is when we simulate the wealth process, we can only divide the time interval into a limited number of steps. If we can further divide the time interval into more steps, we will receive a larger value of variance in the dynamic case. Since Table 3.13 is enough to verify the conclusion in Remark 10, we will skip this step.

4. In the analysis above, we have seen that increasing the margin rate will strengthen the performance of both static and dynamic optimalities. This fact violates common sense as the margin requirement should be considered as an extra cost and should limit the performance of the portfolio. However, Theorem 3.1 is set under the perfect market assumption and

there is no constraint for the investor to borrow money from the market by the risk-free rate. Setting static optimality as an example. If the broker-dealer requests the margin requirement for short-selling, Theorem 3.1 gives

$$\frac{\rho}{\sigma}\frac{1}{x}[-x+x_0\mathrm{e}^{\mathrm{r(t-t_0)}}+\frac{1}{2\mathrm{c}}\mathrm{e}^{\delta^2\mathrm{(T-t_0)-r(T-t)}}]<0 \tag{3.89}$$

for short-selling. Comparing with the control given by the unconstrained case in Theorem 3 in [43], we note that (3.7) will suggest the investor to short sell larger amounts of risk asset. In this situation, the investor can borrow money from the market with the risk-free rate to deposit in the margin account and short sell a larger amount of risky asset to invest in the bond. The time value of money of the margin account will be hedged by the interest from investing the bond. Moreover, the larger short position will lead to better performance when the stock price goes down, which yields better performance. However, it should be kept in mind that this conclusion only exists in the perfect market. In the real financial market, there exist many constraints for conducting short-selling such as the investor can not borrow unlimited money from the bank, the short position is constrained, etc. Any of those constraints will violate Theorem 3.1 and considering other constraints under the margin requirement will be left for future study.

Margin rate θ	Value of α	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
0.1	0.5	2.46484527	0.58768332
	1.0	2.99972554	0.68905681
	1.5	3.39374216	1.34149399
	2.0	3.63191419	3.85247734
1.0	0.5	2.50308404	0.35920460
	1.0	3.03992670	0.63366782
	1.5	3.52830478	0.83906928
	2.0	3.81881015	1.17325717
5.0	0.5	2.62946088	0.09607987
	1.0	3.26394946	0.15422624
	1.5	3.72830443	0.26067631
	2.0	4.16795097	0.31048006

Table 3.9: Simulation for the static optimality for the constrained problem (3.5) with respect to different values of θ and α.

Margin rate θ	Value of α	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
0.1	0.5	3.58925300	4.08310046
	1.0	4.43799281	8.21504228
	1.5	5.34829753	11.9444946
	2.0	5.91638425	15.5421517
1.0	0.5	3.58995719	4.04360358
	1.0	4.54296061	7.82952493
	1.5	5.33156890	12.1885934
	2.0	6.14613367	15.5421517
5.0	0.5	3.60192126	3.79639666
	1.0	4.64491418	6.89650475
	1.5	5.50823356	10.2143093
	2.0	6.37682644	13.8118228

Table 3.10: Simulation for the dynamic optimality for the constrained problem (3.5) with respect to different values of θ and α.

(Note the related parameters are given by $\mu = 0.5$, $\sigma = 0.25$, $\delta = 1.2$, $x_0 = 1$, $r = 0.2$, $T = 1$, and $t_0 = 0$, and the sample size is 1,000.)

Margin rate θ	Value of β	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
0.1	0.5	0.86225163	0.61685684
	1.0	1.34047954	1.14101472
	1.5	1.83410985	2.06447222
	2.0	2.22716223	3.26000806
1.0	0.5	0.89015532	1.28193829
	1.0	1.40715629	0.73702737
	1.5	1.90851713	1.37623514
	2.0	2.42685763	1.87788770
5.0	0.5	1.07577665	0.14240407
	1.0	1.68344999	0.35783756
	1.5	2.24464729	0.35783756
	2.0	2.79822920	1.21824660

Table 3.12: Simulation for the static optimality for the constrained problem (3.6) with respect to different values of θ and β.

Margin rate θ	Value of β	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
0.1	0.5	-3.8670093	7177.2189276
	1.0	-4.9800826	33449.688192
	1.5	1.98351747	7333.4211276
	2.0	3.83660568	9758.2559762
1.0	0.5	0.37111224	72.864568828
	1.0	6.58788445	17751.168875
	1.5	5.07432139	13771.648638
	2.0	1.62512695	11931.423438
5.0	0.5	5.74371476	21148.534799
	1.0	44.7618918	1397820.5968
	1.5	5.14232131	96075.979379
	2.0	957.191529	906579607.43

Table 3.13: Simulation for the dynamic optimality for the constrained problem (3.6) with respect to different values of θ and β.

(Note the related parameters are given by $\mu = 0.5$, $\sigma = 0.25$, $\delta = 1.2$, $x_0 = 1$, $r = 0.2$, $T = 1$, and $t_0 = 0$, and the sample size is 1,000.)

Chapter 4

Dynamic Mean-Variance Portfolio Selection under the Constant Elasticity of Variance Model

4.1 Introduction

It can be seen that in many works of portfolio selection, the stock price follows geometric Brownian motion. In this chapter, we will attempt to extend the previous work in [43] to the constant elasticity of variance model from the dynamic programming perspective and investigate both the time-inconsistent and time-consistent solutions respectively.

The constant elasticity of variance model was first introduced by Cox in 1975 in a short note about deriving an option pricing model ([11] and [12]). This model has been widely studied in the option pricing area ([11], [12], and [16]). One of the advantages of this model is that it can explain the implied volatility skew and volatility smile of option pricing. However, comparing the rich studies in the option pricing field, this model has not attracted great attention in the portfolio area. Shen, Zhang, and Siu [49] attempt to achieve the explicit solution of the portfolio with a single risky asset under the constant elasticity of variance model by applying the stochastic linear-quadratic control approach applied in [34] and backward stochastic Riccati equation technique. Zhao and Rong [62] consider the case when there are multiple risky assets in the portfolio, in which they maximise the constant absolute risk aversion utility function by applying the Hamilton-Jacobi-Bellman technique. Furthermore, Chang and Rong [8] introduce a constraint on borrowing rate to mean-variance portfolio selection, in which borrowing money from the market is penalised by a higher rate and the risky asset follows a constant elasticity variance process. Within the process of achieving the optimal control, they note the Hamilton-Jacobi-Bellman equation for the value function is a non-linear second-order partial differential equation, and they adopt Legendre transform and dual theory to obtain the closed-form solution for each given and fixed expected terminal wealth, and this forms the efficient frontier. In this chapter, we introduce the constant elasticity of variance model into the mean-variance portfolio selection problem and follow the methodology in [8] to handle the difficulty caused by Hamilton-Jacobi-Bellman equation and achieve both of the time-inconsistent and time-consistent solutions.

In this chapter, we assume there is a financial market with a riskless bond and a risky stock that following the constant elasticity of variance model. We aim to construct a self-financing portfolio dynamically in time to achieve the highest return and the lowest risk at maturity. Following the idea introduced by Markowitz [37], the expected return and risk of the portfolio will be represented by the expectation and variance of the value process of the portfolio respectively, in which the variance brings the quadratic nonlinearity into this problem. The quadratic nonlinearity makes the standard optimal control theory (cf. [20]) infeasible, and instead we follow the methodology of [43] to solve the optimal control problem by applying Lagrange multipliers and the Hamilton-Jacobi-Bellman equation. However, under the constant elasticity of variance model, the Hamilton-Jacobi-Bellman equation turns out to be a nonlinear second-order partial differential equation. In striving to handle this difficulty, we follow the idea of [8] (Also cf. [9]) and apply Legendre transform and dual theory to transform the HJB equation to its dual function. Since the HJB system only leads to a candidate solution, we can then apply the verification theorem described in [5] to prove the optimality of the solution.

The solution of the Hamilton-Jacob-Bellman equation shows that the optimal control relates to the initial time and value of the controlled wealth process. Following the same idea as in [43], we refer to this optimality as the static optimality to distinguish it from the dynamic optimality in which the optimal control only depends on the current position of the controlled process. It should be pointed out that, based upon our best knowledge, the time-consistent optimality has not been studied in the portfolio selection problem under the constant elasticity of variance model. Moreover, since the constant elasticity of variance model can be taken as a natural extension or general model of geometric Brownian motion, in this chapter, Theorem 4.1 can be seen as a general theorem that covers the conclusion in [43]. For instance, setting $\beta = 0$, the elasticity parameter, we can easily derive the conclusion of Theorem 3 in [43]. Moreover, by choosing a proper elasticity parameter, Theorem 4.1 gives the optimal solutions for different cases such as portfolio selection where the stock price follows Ornstein-Uhlenbeck process, and investigating this general model is the main motivation of this chapter.

Additionally, we investigate the optimal control problems under the constraints on the size of the expectation and variance of the terminal wealth respectively. In Corollary 4.2, we achieve the optimal control that maximises the expectation of the terminal wealth of the investor, $\mathrm{E}_{t,x}(X_T^u)$, over all admissible control u such that the variance, $\mathrm{Var}_{t,x}(X_T^u)$, is bounded above by a positive constant. Furthermore, in Corollary 4.3, we achieve the optimal control that minimises the variance of the terminal wealth, $\mathrm{Var}_{t,x}(X_T^u)$, over all admissible control u such that the expectation, $\mathrm{E}_{t,x}(X_T^u)$, is bounded below by a positive constant. In this chapter, we assume that there is no transaction cost or tax deduction.

In Section 4, we conduct numerical analysis and verify that, under the CEV model, the features of Theorem 4.1 and Corollary 4.3 will be consistent with the previous work [43]. However, in Corollary 4.2, we note that the statically optimal control outperforms the dynamically optimal control for $\beta \neq 0$, and the only case the dynamically optimal control dominates the statically optimal control is $\beta = 0$.

4.2 Formulation of the problem

Consider an investor who aims to construct a self-financing portfolio consisting of two assets, a riskless bond and a risky stock. In the financial market, the price of the riskless bond, B, is described by:

$$dB_t = rB_t dt \tag{4.1}$$

with initial value $B_{t_0} = b$, where $b > 0$ and the riskless interest rate $r \in \mathbb{R}$ are constants. The price of the risky stock, S, follows the constant elasticity of variance model, which solves:

$$\frac{dS_t}{S_t} = \mu dt + \sigma S_t^{\beta} dW_t \tag{4.2}$$

where we have the drift rate $\mu \in \mathbb{R}$ and the volatility $\sigma > 0$. In (4.2), β is named as the elasticity parameter of the risky stock and $S_t^{\beta}\sigma$ represents the instantaneous volatility of the risky stock for a given point of time t. It should be pointed out that if $\beta = 0$, process (4.2) reduces to a geometric Brownian motion; if $\beta < 0$, the decreasing stock price leads to the increasing value of instantaneous volatility $S_t^{\beta}\sigma$, which generates a contribution with a fatter left tail; if $\beta > 0$, we will see reversed and unrealistic situation [21]. In fact, if $\beta > 0$, the instantaneous volatility of the stock price increases when the stock price increases, and this phenomenon is named as inverse leverage effect in [23]. Furthermore, under the risk-neutral measure, if $\beta > 0$, S_t is a strictly local martingale, which may lead to arbitrage opportunity [23] (also cf. [49]). There are studies that consider the value of elasticity parameter such as [26] in which they analyse the existence of arbitrage opportunities and price bubbles in the case when β is positive. However, the value of β is not the main purpose of this chapter, and in the following part, we will see that the model in this chapter will still provide the investor with the optimal strategies regardless of the existence of the arbitrage opportunity or not.

As we know, the constant elasticity of variance model can be converted into different models. Except for geometric Brownian motion, there are several examples that can be derived from (4.2) by choosing a proper value of β. For instance, if we set $\beta = -1$, (4.2) leads to the Ornstein-Uhlenbeck process and if we set $\beta = -1/2$, SDE (4.2) leads to a diffusion process introduced in [12]. Hence, the study of this chapter will help provide us with a better understanding of mean-variance portfolio selection under different models.

For the risky stock, we set the initial value $S_{t_0} = s_0$ as a constant $s_0 > 0$. Furthermore, in the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, S has the same natural filtration as that of W, where W is a standard Brownian motion. Besides, it is reasonable to stipulate that the value of μ must be greater than r, which indicates the risk premium. Otherwise, a wise investor will simply invest all money in the riskless bond and receive a riskless return.

Under (4.1) and (4.2), we can derive the SDE for the wealth process of the self-financing portfolio. Following ([[5], Chapter 6], we can see that the investor's wealth follows:

$$dX_t^u = (r + (\mu - r)u_t)X_t^u dt + \sigma u_t X_t^u S^{\beta} dW_t \tag{4.3}$$

with initial value $x_0 > 0$. In (4.3), we set that u_t is the percentage of wealth invested in the stock at $t \in [t_0, T]$ and U is the set of all admissible controls. For any admissible control u in (4.3), we have $u_t = u(t, X_t^u)$ where $(t,x) \mapsto u(t,x) \cdot x$ is a continuous function from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$. For completeness, following the idea of [43], we define $u(t,0) = u(t,0) \cdot 0 = \lim_{0 \neq x \mapsto 0} u(t,x)$ because the map $x \mapsto u(t,x)$ may not exist at 0.

For probability measure $\mathbb{P}_{t,x}$, X_t^u is a strong Markov process for each admissible control

u and takes value x for a given and fixed time t where $(t,x) \in [0,T] \times \mathbb{R}$. This chapter will focus on the following unconstrained problem:

$$V(t,x) = \sup_u \left[\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) \right] \tag{4.4}$$

in which $c > 0$ is a given constant. For all $u \in U$, the admissibility condition is given by:

$$0 < \mathrm{E}_{t,x} \left[\max_{t_0 \leq t \leq T} (X_t^{u2} - \lambda X_t^u) \right] < \infty \tag{4.5}$$

in which λ any positive constant.

Besides, in this chapter, we attempt to consider the other two constrained cases, which are given by:

$$V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u) \leq \alpha} \mathrm{E}_{t,x}(X_T^u) \tag{4.6}$$

$$V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u) \geq \gamma} \mathrm{Var}_{t,x}(X_T^u) \tag{4.7}$$

where u is the admissible control, and $\alpha \in (0,\infty)$ and $\gamma \in \mathbb{R}$ are given constants. In the following part, we will see that solving (4.4) will naturally leads to the solution of those two constrained problems, and we will see this in the following part.

In this chapter, definitions of static optimality and dynamic optimality are consistent with those in [43] and we have exhibited them in the Introduction.

4.3 Solution to the optimal control problem

In this chapter, we will explain the solution of the constrained problems. The main idea of the proof below follows the idea in [43].

Theorem 4.1. *Consider the optimal problem* $V(t,x) = \sup_u [\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)]$ *in which* X^u *represents the wealth process and is the solution of the SDE* (4.3) *with* $X_{t_0}^u = x_0$ *under the probability measure* $\mathbb{P}_{t_0,x_0}$ *for* $(t_0,x_0) \in [0,T] \times \mathbb{R}$ *given and fixed. The related risk coefficient is defined by* $\delta = (\mu - r)/\sigma$ *in which* $\mu, r \in \mathbb{R}$, $\sqrt{2}r > \mu > r$ *and* $\sigma > 0$. *Note that we assume that* $\delta \neq 0$ *and* $r \neq 0$. *(The cases* $\delta = 0$ *or* $r = 0$ *follow by passage to the limit when the non-zero* δ *or* r *approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\left[x - x_0 e^{r(t-t_0)} + \frac{1}{2c}\frac{(1-e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})e^{-r(T-t)}}{e^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}(1-e^{\int_{t_0}^{T}\theta^2(s)K^2(s)ds})}\right]K(t) \quad (4.8)$$

for $(t,x) \in [t_0,T]\times\mathbb{R}$. *In* (4.8), *related parameters are given by:*

$$\theta(t) = \frac{\delta}{S_t^{\beta}}, \quad (4.9)$$

$$K(t) = [1 + \frac{2\beta\sigma B(t)}{\delta}], \quad (4.10)$$

$$B(t) = \frac{z_1 z_2(1-e^{-2\beta^2\sigma^2(z_1-z_2)(T-t)})}{z_1 - z_2 e^{-2\beta^2\sigma^2(z_1-z_2)(T-t)}}, \quad (4.11)$$

$$z_1 = \frac{-(\mu-2r)+\sqrt{2r^2-\mu^2}}{2\beta\sigma^2}, \quad (4.12)$$

and

$$z_2 = \frac{-(\mu-2r)-\sqrt{2r^2-\mu^2}}{2\beta\sigma^2} \quad (4.13)$$

for $t \in [t_0,T]$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}}\frac{1}{x}\frac{1}{2c}\frac{(1-e^{\int_{t}^{T}-\theta^2(s)K(s)ds})e^{-r(T-t)}}{e^{\int_{t}^{T}-2\theta^2(s)K(s)ds}(1-e^{\int_{t}^{T}\theta^2(s)K^2(s)ds})}K(t) \quad (4.14)$$

for $(t,x) \in [t_0,T]\times\mathbb{R}$ *and the related parameters are defined above.*

Proof. In this proof, we claim that, for each pair of $(t_0,x_0) \in [0,T]\times\mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (4.3) with initial condition $X_{t_0}^u = x_0$. Furthermore, for X_t^u, $u \in U$ is any admissible control we defined in Section 4.2.

(A): Recalling the objective function, $\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)$, we note that it can be rearranged as:

$$\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) = \mathrm{E}_{t,x}(X_T^u) - c\big[\mathrm{E}_{t,x}[(X_T^u)^2] - \mathrm{E}_{t,x}(X_T^u)^2\big] \quad (4.15)$$

in which term $\mathrm{E}_{t,x}(X_T^u)^2$ introduces quadratic non-linearity into this problem. Hence, to overcome this difficulty, we condition the size of $\mathrm{E}_{t,x}(X_T^u)$ by assuming $\mathrm{E}_{t,x}(X_T^u) = M$ where $M \in \mathbb{R}$. This naturally leads to:

$$\begin{aligned} V(t,x) &= \sup_{M\in\mathbb{R}} \sup_{u:\mathrm{E}_{t,x}(X_T^u)=M} \left[\mathrm{E}_{t,x}(X_T^u) - c\big[\mathrm{E}_{t,x}[(X_T^u)^2] - \mathrm{E}_{t,x}(X_T^u)^2\big] \right] \\ &= \sup_{M\in\mathbb{R}} \left[M + cM^2 - c \inf_{u:\mathrm{E}_{t,x}(X_T^u)=M} \mathrm{E}_{t,x}[(X_T^u)^2] \right]. \end{aligned} \tag{4.16}$$

Equation (4.16) leads to a constrained problem:

$$V_M(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u)=M} \mathrm{E}_{t,x}\left[(X_T^u)^2\right] \tag{4.17}$$

where $M \in \mathbb{R}$ given and fixed and u is any admissible control.

1. Applying Lagrange multipliers in equation (4.17), we have the following Lagrangian function:

$$L_{t,x}(u,\lambda) = \mathrm{E}_{t,x}\left[(X_T^u)^2\right] - \lambda\big[\mathrm{E}_{t,x}(X_T^u) - M\big] \tag{4.18}$$

in which $\lambda > 0$. Solving equation (4.18) gives the optimal solution (4.17). To verify this, we assume there exists u_*^λ that minimises (4.18), which leads to:

$$L_{t,x}(u_*^\lambda,\lambda) := \inf_u L_{t,x}(u,\lambda). \tag{4.19}$$

Furthermore, we assume there exists a $\lambda = \lambda(M,t,x) > 0$ such that $\mathrm{E}_{t,x}(X_T^{u_*^\lambda}) = M$. In this situation, we can see that:

$$V_M(t,x) = L_{t,x}(u_*^\lambda,\lambda) \le \mathrm{E}_{t,x}\left[(X_T^u)^2\right] \tag{4.20}$$

for any admissible control $u \in U$ with $\mathrm{E}_{t,x}(X_T^u) = M$, which indicates that the optimal control u_*^λ that minimises (4.18) with $\mathrm{E}_{t,x}(X_T^{u_*^\lambda}) = M$ is optimal in (4.17).

2. To solve (4.18) and achieve the optimal control, we need to consider the following optimal control problem:

$$V^\lambda(t,s,x) = \inf_u \mathrm{E}_{t,x}\left[(X_T^u)^2 - \lambda X_T^u | S_t = s, X_t = x\right] \tag{4.21}$$

where $u \in U$ is any admissible control we defined in the previous section. In this chapter, we apply the HJB approach to achieve the candidate optimal control and then prove the optimality of the candidate solution by using verification theorem described in [5]. According to dynamic programming principle, $V^{\lambda}(t,s,x)$ can be taken as a smooth enough solution of the following HJB system (in the following part we will see the explicit form of V^{λ} exists).

Upon the SDE (4.3), the HJB system is given by:

$$\inf_u \left[V_t^{\lambda} + \mu s V_s^{\lambda} + \frac{1}{2}\sigma^2 s^{2\beta+2} V_{ss}^{\lambda} + (r + (\mu - r)u_t) x V_x^{\lambda} + \sigma^2 s^{2\beta+1} x u V_{xs}^{\lambda} + \frac{1}{2}\sigma^2 s^{2\beta} u^2 x^2 V_{xx}^{\lambda}\right] = 0 \tag{4.22}$$

with terminal condition:

$$V^{\lambda}(T,s,x) = x^2 - \lambda x. \tag{4.23}$$

In striving for the solution, we will follow the idea of [8] to introduce the technique of Legendre transform and dual theory to convert the HJB equation into its dual equation, from which we will be able to derive the candidate solution.

3. According to the property of quadratic equation, we obtain:

$$u = -\frac{(\mu - r)V_x^{\lambda} + \sigma^2 s^{2\beta+1} V_{xs}^{\lambda}}{\sigma^2 s^{2\beta} x V_{xx}^{\lambda}} \tag{4.24}$$

Substituting (4.24) back into HJB equation (4.22) we can see that:

$$V_t^{\lambda} + \mu s V_s^{\lambda} + \frac{1}{2}\sigma^2 s^{2\beta+2} V_{ss}^{\lambda} + r x V_x^{\lambda} - \frac{1}{2\sigma^2 s^{2\beta} V_{xx}^{\lambda}}\left[(\mu - r)V_x^{\lambda} + \sigma^2 s^{2\beta+1} V_{xs}^{\lambda}\right]^2 = 0 \tag{4.25}$$

4. In (4.25), we see a nonlinear second order partial differential equation. Hence, upon the strictly convexity of the value function V^{λ}, we follow the work described in [27] (also cf. [46]) to define the following Legendre transform:

$$G(t,s,z) = \sup_x \left[V^{\lambda}(t,s,x) - zx\right] \tag{4.26}$$

in which z is the dual variable to x. Furthermore, as [54] mentioned that, for strictly convex function V^{λ}, the maximum point of equation (4.26) will only be attained at the unique solution of $V_x^{\lambda}(t,s,x) = z$. Furthermore, the optimal value of x that maximises G in (4.26) is denoted by $g(t,s,z)$ such that:

$$g(t,s,z) = \inf\{x | V^{\lambda}(t,s,x) \geq zx + G(t,s,z)\} \tag{4.27}$$

Functions g and G are related and either of them can be seen as the dual function of V^{λ}. It should be pointed out that $g(t,s,z) = -G_z(t,s,z)$, and this relationship will help us to identify the explicit form of V^{λ}. In the following part, we will focus on analysing g to obtain the solution. According to terminal condition at maturity, $V^{\lambda}(T,s,x) = x^2 - \lambda x$, there are:

$$G(T,s,z) = \sup_x \left[x^2 - \lambda x - zx\right] \tag{4.28}$$

and

$$g(T,s,z) = \inf\{x | x^2 - \lambda x \geq zx + G(T,s,z)\}. \tag{4.29}$$

From (4.28) and (4.29), we can see that the optimal value x_* that maximises (4.28) is attained at $V_x^{\lambda}(T,s,x) = z$ which leads to:

$$g(T,s,z) = \frac{1}{2}z + \frac{1}{2}\lambda. \tag{4.30}$$

We have stated that the optimal value x_* in (4.26) is denoted by

$$g(t,s,z) = x_* \tag{4.31}$$

and this leads to:

$$G(t,s,z) = V^{\lambda}(t,s,g) - zg. \tag{4.32}$$

According to the transformation rules described in [27], [21] and [54], there is:

$$\begin{aligned} &V_t^{\lambda} = G_t, \quad V_x^{\lambda} = z, \quad V_{xx}^{\lambda} = -\frac{1}{G_{zz}} \\ &V_s^{\lambda} = G_s, \quad V_{ss}^{\lambda} = G_{ss} - \frac{G_{sz}^2}{G_{zz}}, \quad V_{xs}^{\lambda} = -\frac{G_{sz}}{G_{zz}} \end{aligned} \tag{4.33}$$

Substituting (4.33) back into (4.25), there is:

$$G_t + \mu s G_s + \frac{1}{2}\sigma^2 s^{2\beta+2} G_{ss} + rgz + \frac{(\mu - r)^2 z^2 G_{zz}}{2\sigma^2 s^{2\beta}} - (\mu - r) sz G_{sz} = 0. \tag{4.34}$$

Using the relation that $g(t,s,z) = -G_z(t,s,z)$ and differentiating $G(t,s,z)$ with respect to z, there is:

$$g_t + rsg_s + \frac{1}{2}\sigma^2 s^{2\beta+2} g_{ss} + \left[\frac{(\mu - r)^2}{\sigma^2 s^{2\beta}} - r\right] z g_z + \frac{(\mu - r)^2}{2\sigma^2 s^{2\beta}} z^2 g_{zz} - (\mu - r) sz g_{sz} - rg = 0 \tag{4.35}$$

which gives a second-order linear partial differential equation.

5. To handle (4.35), we assume the solution $g(t,s,z)$ takes the following form:

$$g(t,s,z) = f(t,y)z + h(t) \quad \text{and} \quad y = s^{-2\beta} \tag{4.36}$$

for $t \in [t_0, T]$ with terminal condition:

$$f(T,y) = \frac{1}{2}, \qquad h(T) = \frac{1}{2}\lambda. \tag{4.37}$$

Hence, differentiating (4.36) with respect to t, s, and z respectively, we can easily receive:

$$\begin{aligned} & g_t = f_t z + h_t, \quad g_s = f_y(-2\beta)s^{-2\beta-1}z, \quad g_{sz} = f_y(-2\beta)s^{-2\beta-1}, \\ & g_{ss} = f_{yy}((-2\beta)s^{-2\beta-1})^2 z + f_y(-2\beta)(-2\beta-1)s^{-2\beta-2}z, \\ & g_z = f, \quad g_{zz} = 0. \end{aligned} \tag{4.38}$$

Substituting (4.38) back into (4.35), there is

$$\begin{aligned} & \big[f_t + \big(2(\mu-2r)\beta y + \beta(2\beta+1)\sigma^2\big) f_y + 2\beta^2\sigma^2 y f_{yy} \\ & \quad + \big(\frac{(\mu-r)^2}{\sigma^2} y - 2r\big) f\big] z + h_t - rh = 0 \end{aligned} \tag{4.39}$$

from which we can easily see that:

$$f_t + \left[2(\mu-2r)\beta y + \beta(2\beta+1)\sigma^2\right] f_y + 2\beta^2\sigma^2 y f_{yy} + \left[\frac{(\mu-r)^2}{\sigma^2} y - 2r\right] f = 0 \tag{4.40}$$

and

$$h_t - rh = 0. \tag{4.41}$$

Solving (4.41) with the terminal condition $h(T) = \lambda/2$ gives:

$$h(t) = \frac{\lambda}{2} e^{-r(T-t)} \tag{4.42}$$

for $t \in [t_0, T]$.

To achieve the explicit solution for $f(t,y)$, we follow the idea of [8] and assume that:

$$f(t,y) = A(t) e^{B(t)y} \tag{4.43}$$

for $t \in [t_0, T]$ with $A(T) = 1/2$ and $B(T) = 0$. Differentiating (4.43) and connecting it with (4.40) gives:

$$\begin{aligned} & \left[A(t)\frac{dB(t)}{dt} + 2\beta(\mu-2r)A(t)B(t) + 2\beta^2\sigma^2 A(t)B^2(t) + \frac{(\mu-r)^2}{\sigma^2} A(t)\right] y \\ & \quad + \frac{dA(t)}{dt} + \beta(2\beta+1)\sigma^2 A(t)B(t) - 2rA(t) = 0. \end{aligned} \tag{4.44}$$

The coefficient term of y and the constant term must be equal to 0. Hence, comparing and re-arranging the coefficients of (4.44), we can find that:

$$\frac{dA(t)}{dt} + \beta(2\beta+1)\sigma^2 A(t)B(t) - 2rA(t) = 0 \tag{4.45}$$

with $A(T) = 1/2$. For the coefficient term of y, we can further move $A(t)$ out of the square brackets, which gives:

$$\frac{dB(t)}{dt} + 2\beta(\mu-2r)B(t) + 2\beta^2\sigma^2 B^2(t) + \frac{(\mu-r)^2}{\sigma^2} = 0 \tag{4.46}$$

with $B(T) = 0$. We first consider (4.46) and it leads to:

$$\frac{dB(t)}{dt} = -2\beta(\mu-2r)B(t) - 2\beta^2\sigma^2 B^2(t) - \frac{(\mu-r)^2}{\sigma^2} \tag{4.47}$$

with $B(T) = 0$. The right-hand side can be seen as a quadratic function with respect to $B(t)$ and hence the discriminant of this quadratic equation is given by:

$$\Delta = 4\beta^2(2r^2-\mu^2). \tag{4.48}$$

In order to achieve the solution of $B(t)$, in this chapter, we assume that $\beta \neq 0$ and $\Delta > 0$ and this leads to $r < \mu < \sqrt{2}r$. The two distinct real roots of the quadratic equation are given by:

$$z_1 = \frac{-(\mu-2r)+\sqrt{2r^2-\mu^2}}{2\beta\sigma^2}, \tag{4.49}$$

and

$$z_2 = \frac{-(\mu-2r)-\sqrt{2r^2-\mu^2}}{2\beta\sigma^2}. \tag{4.50}$$

Hence, (4.47) leads to:

$$\frac{dB(t)}{dt} = -2\beta^2\sigma^2\big(B(t)-z_1\big)\big(B(t)-z_2\big) \tag{4.51}$$

and there is:

$$\frac{1}{z_1-z_2}\int_t^T \left[\frac{1}{B(s)-z_1} - \frac{1}{B(s)-z_2}\right] dB(s) = -2\beta^2\sigma^2(T-t) \tag{4.52}$$

which is a simple ODE and can be easily solved. Hence, we have:

$$B(t) = \frac{z_1 z_2\left(1-\mathrm{e}^{-2\beta^2\sigma^2(z_1-z_2)(T-t)}\right)}{z_1 - z_2\mathrm{e}^{-2\beta^2\sigma^2(z_1-z_2)(T-t)}} \tag{4.53}$$

for $t \in [t_0, T]$. In this case, substituting (4.53) back into (4.45), we can solve (4.45) and obtain:

$$A(t) = \frac{1}{2}\mathrm{e}^{-\int_t^T [2r-\beta(2\beta+1)\sigma^2 B(\tau)]d\tau} \tag{4.54}$$

for $t \in [t_0, T]$. Combining (4.43) and (4.36) gives the explicit solution for $g(t,s,z)$. Moreover, using the relationship $G(t,s,z) = V^\lambda(t,s,g) - zg$, and $g(t,s,z) = x = -G_z(t,s,z)$, we can see the existence of V^λ (the solution of the HJB system exists only if the value function V^λ exists).

The last step is to find the optimal control for the HJB system. Under the transformation rules given by (4.33) above, the control given by (4.24) leads to:

$$\begin{aligned} u &= -\frac{(\mu-r)V_x^\lambda + \sigma^2 s^{2\beta+1} V_{xs}^\lambda}{\sigma^2 s^{2\beta} x V_{xx}^\lambda} \\ &= -\frac{(\mu-r)\frac{V_x^\lambda}{V_{xx}^\lambda} + \sigma^2 s^{2\beta+1}\frac{V_{xs}^\lambda}{V_{xx}^\lambda}}{\sigma^2 s^{2\beta} x} \\ &= \frac{(\mu-r)zG_{zz} - \sigma^2 s^{2\beta+1} G_{sz}}{\sigma^2 s^{2\beta} x} \\ &= \frac{-(\mu-r)zg_z + \sigma^2 s^{2\beta+1} g_s}{\sigma^2 s^{2\beta} x}. \end{aligned} \tag{4.55}$$

Since $g(t,s,z) = f(t,y)z + h(t)$ with $y = s^{-2\beta}$, there are:

$$g_z = f, \quad g_s = zf_y(-2\beta)s^{-2\beta-1}. \tag{4.56}$$

Hence, substituting (4.56) into (4.55) gives:

$$\begin{aligned} u(t,s,x) &= \frac{-(\mu-r)zg_z + \sigma^2 s^{2\beta+1} g_s}{\sigma^2 s^{2\beta} x} \\ &= \frac{-(\mu-r)zg_z + \sigma^2 s^{2\beta+1}(-2\beta)zs^{-2\beta-1} f_y}{\sigma^2 s^{2\beta} x} \\ &= \frac{-(\mu-r)[g(t,s,z)-h(t)] - 2\beta\sigma^2 zA(t)B(t)\mathrm{e}^{B(t)y}}{\sigma^2 s^{2\beta} x} \\ &= \frac{-(\mu-r)[g(t,s,z)-h(t)] - 2\beta\sigma^2 B(t)[g(t,s,z)-h(t)]}{\sigma^2 s^{2\beta} x}. \end{aligned} \tag{4.57}$$

As we know that $h(t) = \frac{\lambda}{2}\mathrm{e}^{-r(T-t)}$ and $g(t,s,z) = x$ hence:

$$\begin{aligned} u(t,s,x) &= -\frac{(\mu-r)(x-\frac{\lambda}{2}\mathrm{e}^{-r(T-t)}) + \sigma^2 \mathrm{B(t)} 2\beta(\mathrm{x}-\frac{\lambda}{2}\mathrm{e}^{-r(T-t)})}{\sigma^2 s^{2\beta} x} \\ &= -\frac{(\mu-r)+2\beta\sigma^2 B(t)}{\sigma^2 s^{2\beta} x}\left[x - \frac{\lambda}{2}\mathrm{e}^{-r(T-t)}\right] \end{aligned} \tag{4.58}$$

for $(t,s,x) \in [t_0,T] \times \mathbb{R}_+ \times \mathbb{R}$.

It should be noted that, in the previous part, we assume that $\beta \neq 0$. As we have stated that the constantly elasticity of variance model can be seen as a general extension of geometric Brownian motion, we are going to show that our solution for the HJB equation holds when $\beta = 0$. If $\beta = 0$, then $y = 1$, under which equation (4.39) reduces to:

$$\big[f_t + (\frac{(\mu-r)^2}{\sigma^2} - 2r)f\big]z + h_t - rh = 0 \tag{4.59}$$

and it can be easily seen that:

$$f(t) = \frac{1}{2}\mathrm{e}^{-\left(2\mathrm{r} - \frac{(\mu-\mathrm{r})^2}{\sigma^2}\right)(\mathrm{T}-\mathrm{t})} \quad \text{and} \quad \mathrm{h(t)} = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}. \tag{4.60}$$

for $t \in [t_0,T]$. Hence, when $\beta = 0$, $g = fz + h$ does not contain s anymore, which leads to $g_s = 0$ in (4.55). Hence, equation (4.55) can be simplified to:

$$u = -\frac{(\mu-r)zg_z}{\sigma^2 x} = -\frac{(\mu-r)}{\sigma^2 x}\big[x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}\big] \tag{4.61}$$

which is the candidate optimal solution for the HJB system (4.22)-(4.23) with $\beta = 0$ (when $\beta = 0$, V_s^λ, V_{ss}^λ and V_{xs}^λ vanish in (4.22)). Moreover, the optimal control given by (4.61) is consistent with the conclusion in [43] in which the stock price follows geometric Brownian motion. Hence, we can conclude that (4.58) holds for all values of $\beta \in \mathbb{R}$.

6. So far, we have achieved the candidate solution for the HJB system (4.22)-(4.23). Since (4.58) only gives the candidate solution, we are going to prove it is the optimal control to (4.22)-(4.23) so as to (4.21). To prove the optimality, using ito formula to $V^\lambda(t,s,x)$, which leads to:

$$\begin{aligned} V^\lambda(T,S_T,X_T^u) = V^\lambda(t,s,x) + \int_t^T &\big[V_t^\lambda(t+p,S_p,X_p^u) + \mu S_p V_s^\lambda(t+p,S_p,X_p^u) \\ &+ \frac{1}{2}\sigma^2 {S_p}^{2\beta+2} V_{ss}^\lambda(t+p,S_p,X_p^u) + (r+(\mu-r)u_p)X_p^u V_x^\lambda(t+p,S_p,X_p^u) \\ &+ \sigma^2 {S_p}^{2\beta+1} X_p^u u_p V_{xs}^\lambda(t+p,S_p,X_p^u) + \frac{1}{2}\sigma^2 {S_p}^{2\beta} u_p^2 X_p^{u2} V_{xx}^\lambda(t+p,S_p,X_p^u)\big]dp \\ &+ \int_t^T (\sigma {S_p}^{\beta+1} V_s^\lambda(t+p,S_p,X_p^u + \sigma u_p X_p^u {S_p}^\beta V_x^\lambda(t+p,S_p,X_p^u)dW_p \end{aligned} \tag{4.62}$$

in which $u \in U$ is any admissible control. As we have shown that V^λ solves the HJB equation (4.22)-(4.23), there is:

$$\begin{aligned} \big[V_t^\lambda + \mu S_t V_s^\lambda + \frac{1}{2}\sigma^2 {S_t}^{2\beta+2} V_{ss}^\lambda + (r+(\mu-r)u_t)X_t^u V_x^\lambda \\ + \sigma^2 {S_t}^{2\beta+1} X_t^u u_t V_{xs}^\lambda + \frac{1}{2}\sigma^2 {S_t}^{2\beta} u_t^2 X_t^{u2} V_{xx}^\lambda\big] \geq 0 \end{aligned} \tag{4.63}$$

for $t \in [t_0, T]$. According to the terminal condition (4.23) of V^λ at the maturity T, we have:

$$\begin{aligned} X_T^{u2} - \lambda X_T^u = V^\lambda(t,s,x) + \int_t^T \big[& V_t^\lambda(t+p, S_p, X_p^u) + \mu S_p V_s^\lambda(t+p, S_p, X_p^u) \\ & + \frac{1}{2}\sigma^2 {S_p}^{2\beta+2} V_{ss}^\lambda(t+p, S_p, X_p^u) + (r + (\mu - r)u_p) X_p^u V_x^\lambda(t+p, S_p, X_p^u) \\ & + \sigma^2 {S_p}^{2\beta+1} X_p^u u_p V_{xs}^\lambda(t+p, S_p, X_p^u) + \frac{1}{2}\sigma^2 {S_p}^{2\beta} u_p^2 X_p^{u2} V_{xx}^\lambda(t+p, S_p, X_p^u) \big] dp \\ & + \int_t^T (\sigma {S_p}^{\beta+1} V_s^\lambda(t+p, S_p, X_p^u) + \sigma u_p X_p^u {S_p}^\beta V_x^\lambda(t+p, S_p, X_p^u)) dW_p. \end{aligned} \tag{4.64}$$

Hence, there exists:

$$\begin{aligned} V^\lambda(t,s,x) \leq & X_T^{u2} - \lambda X_T^u \\ & - \int_t^T (\sigma {S_p}^{\beta+1} V_s^\lambda(t+p, S_p, X_p^u) + \sigma u_p X_p^u {S_p}^\beta V_x^\lambda)(t+p, S_p, X_p^u) dW_p. \end{aligned} \tag{4.65}$$

Noting that the continuous local martingale $M_t = \int_t^T (\sigma {S_p}^{\beta+1} V_s^\lambda + \sigma u_p X_p^u {S_p}^\beta V_x^\lambda) dW_p$ is a strictly local martingale when $\beta > 0$. Hence, in this case, there exists a sequence of stopping time τ_n such that $\tau_n \uparrow T$ as $n \uparrow \infty$. Then, for each $t' \in [t, T]$, the stopped process:

$$M_{t' \wedge \tau_n} = \int_t^{t' \wedge \tau_n} (\sigma {S_p}^{\beta+1} V_s^\lambda(t+p, S_p, X_p^u) + \sigma u_p X_p^u {S_p}^\beta V_x^\lambda(t+p, S_p, X_p^u)) dW_p \tag{4.66}$$

is a martingale. From (4.65), we can see there is:

$$\begin{aligned} V^\lambda(t,s,x) \leq & {X_{t' \wedge \tau_n}^u}^2 - \lambda X_{t' \wedge \tau_n}^u \\ & - \int_t^{t' \wedge \tau_n} (\sigma {S_p}^{\beta+1} V_s^\lambda(t+p, S_p, X_p^u) + \sigma u_p X_p^u {S_p}^\beta V_x^\lambda(t+p, S_p, X_p^u)) dW_z. \end{aligned} \tag{4.67}$$

Taking expectation on the both sides of (4.67), the martingale term vanishes, which leads to:

$$V^\lambda(t,s,x) \leq \mathrm{E}_{t,x}\left[{X_{t' \wedge \tau_n}^u}^2 - \lambda X_{t' \wedge \tau_n}^u \right]. \tag{4.68}$$

Taking $\lim_{n \uparrow \infty}$ in the right-hand side of (4.68), there is:

$$\mathrm{E}_{t,x}\left[X_{t'}^{u2} - \lambda X_{t'}^u \right] = \mathrm{E}_{t,x}\left[\lim_{n \uparrow \infty} ({X_{t' \wedge \tau_n}^u}^2 - \lambda X_{t' \wedge \tau_n}^u) \right]. \tag{4.69}$$

Recalling the admissibility condition:

$$0 < \mathrm{E}_{t,x}[\max_{t_0 \leq t \leq T} (X_t^{u2} - \lambda X_t^u)] < \infty \tag{4.70}$$

for $\lambda > 0$, we can apply Fatou's lemma, which states if there exists $\mathrm{E}(Z) < \infty$ and for all $n \geq 1$ there is $X_n \leq Z$, we have:

$$\mathrm{E}(\limsup_{n \to \infty} X_n) \geq \limsup_{n \to \infty} \mathrm{E}(X_n) \tag{4.71}$$

and this leads to:

$$\mathrm{E}_{t,x}\left[\lim_{n\uparrow\infty}({X^u_{t'\wedge\tau_n}}^2-\lambda X^u_{t'\wedge\tau_n})\right]\geq\lim_{n\uparrow\infty}\mathrm{E}_{t,x}\left[({X^u_{t'\wedge\tau_n}}^2-\lambda X^u_{t'\wedge\tau_n})\right]. \tag{4.72}$$

Upon (4.68), inequality (4.72) leads to:

$$\lim_{n\uparrow\infty}\mathrm{E}_{t,x}\left[({X^u_{t'\wedge\tau_n}}^2-\lambda X^u_{t'\wedge\tau_n})\right]\geq V^\lambda(t,s,x). \tag{4.73}$$

Hence, we can conclude that:

$$V^\lambda(t,s,x)\leq\mathrm{E}_{t,x}\left[X^{u2}_{t'}-\lambda X^u_{t'}\right] \tag{4.74}$$

which holds for any $t'\in[t,T]$. Hence, we can conclude that:

$$V^\lambda(t,s,x)\leq\mathrm{E}_{t,x}\left[X^{u2}_T-\lambda X^u_T|S_t=s,X^u_t=x\right]. \tag{4.75}$$

Equation (4.75) holds for all admissible controls $u\in U$, which means:

$$V^\lambda(t,s,x)\leq\inf_u\mathrm{E}_{t,x}[(X^u_T)^2-\lambda X^u_T|S_t=s,X_t=x]. \tag{4.76}$$

For the reverse inequality, we claim the optimal control is given by (4.58), and for the optimal control there is:

$$\begin{aligned}\big[V^\lambda_t+\mu S_tV^\lambda_s+\frac{1}{2}\sigma^2{S_t}^{2\beta+2}V^\lambda_{ss}+(r+(\mu-r){u^*}_t)X^{u^*}_tV^\lambda_x\\+\sigma^2{S_t}^{2\beta+1}X^{u^*}_t{u^*}_tV^\lambda_{xs}+\frac{1}{2}\sigma^2{S_t}^{2\beta}{u^*}^2_tX^{u2}_tV^\lambda_{xx}\big]=0\end{aligned} \tag{4.77}$$

so that:

$$\begin{aligned}V^\lambda(t,s,x)=&{X^{u^*}_T}^2-\lambda X^{u^*}_T\\&-\int_t^T(\sigma{S_p}^{\beta+1}V^\lambda_s(t+p,S_p,X^u_p)+\sigma{u^*}_pX^{u^*}_pV^\lambda_x(t+p,S_p,X^u_p))dW_p.\end{aligned} \tag{4.78}$$

Recalling (4.66)-(4.74) and taking expectation on the both sides of (4.78), we can see that for the optimal control u^*, we always have that:

$$V^\lambda(t,s,x)=\mathrm{E}_{t,x}[{X^{u^*}_T}^2-\lambda X^{u^*}_T|S_t=s,X^{u^*}_t=x]. \tag{4.79}$$

Therefore, we have the following trivial inequality:

$$\inf_u\mathrm{E}_{t,x}[X^{u2}_T-\lambda X^u_T|S_t=s,X_t=x]\leq\mathrm{E}_{t,x}[{X^{u^*}_T}^2-\lambda X^{u^*}_T|S_t=s,X^{u^*}_t=x] \tag{4.80}$$

which leads to:

$$\begin{aligned} V^{\lambda}(t,s,x) &\leq \inf_u \mathrm{E}_{t,x}[X_T^{u2} - \lambda X_T^u | S_t = s, X_t = x] \\ &\leq \mathrm{E}_{t,x}[X_T^{u^*2} - \lambda X_T^{u^*} | S_t = s, X_t^{u^*} = x] = V^{\lambda}(t,s,x). \end{aligned} \tag{4.81}$$

Therefore, upon on the verification theorem described in [5], we can conclude that the control given by (4.58) is optimal to the HJB system (4.22)-(4.23).

7. Before we attempt to achieve the optimal value of M, we need to further simplify the optimal control:

$$\begin{aligned} u(t,s,x) &= -\frac{(\mu - r) + 2\beta\sigma^2 B(t)}{\sigma^2 s^{2\beta} x}(x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}) \\ &= -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}(x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})})\mathrm{K}(\mathrm{t}) \end{aligned} \tag{4.82}$$

in which:

$$K(t) = [1 + \frac{2\beta\sigma B(t)}{\delta}] \tag{4.83}$$

for $t \in [t_0, T]$. Substituting (4.82) into SDE (4.3), there is:

$$dX_t^u = (rX_t^u - \frac{\delta^2}{s^{2\beta}}K(t)[X_t^u - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}])\mathrm{dt} - \frac{\delta}{\mathrm{s}^{\beta}}\mathrm{K}(\mathrm{t})[\mathrm{X}_\mathrm{t}^\mathrm{u} - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}]\mathrm{dW}_\mathrm{t} \tag{4.84}$$

with $X_{t_0}^u = x_0 > 0$. We further assume that $\theta = \delta/s^{\beta}$, which leads to:

$$dX_t^u = \left[rX_t^u - \theta^2(t)K(t)X_t^u + \theta^2(t)K(t)\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right]\mathrm{dt} - \theta(\mathrm{t})\mathrm{K}(\mathrm{t})\left[\mathrm{X}_\mathrm{t}^\mathrm{u} - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right]\mathrm{dW}_\mathrm{t}. \tag{4.85}$$

Taking expectation E_{t_0,x_0} on both side of (4.85) gives:

$$d\,\mathrm{E}_{t_0,x_0}(X_t^u) = \left[\left(r - \theta^2(t)K(t)\right)\mathrm{E}_{t_0,x_0}(X_t^u) + \theta^2(t)K(t)\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right]\mathrm{dt} \tag{4.86}$$

which can be taken as an ODE with respect to $\mathrm{E}_{t_0,x_0}(X_t^u)$. Solving this ODE with the initial condition $\mathrm{E}_{t_0,x_0}(X_{t_0}^u) = x_0 > 0$ leads to:

$$\mathrm{E}_{t_0,x_0}(X_t^u) = x_0\mathrm{e}^{\int_{t_0}^{\mathrm{t}}(\mathrm{r}-\theta^2(\mathrm{s})\mathrm{K}(\mathrm{s}))\mathrm{ds}} + \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\left[1 - \mathrm{e}^{-\int_{t_0}^{\mathrm{t}}\theta^2(\mathrm{s})\mathrm{K}(\mathrm{s})\mathrm{ds}}\right] \tag{4.87}$$

for $t \in [t_0, T]$ and it is easy to verify that:

$$\mathrm{E}_{t_0,x_0}(X_T^u) = x_0\mathrm{e}^{\int_{t_0}^{\mathrm{T}}(\mathrm{r}-\theta^2(\mathrm{s})\mathrm{K}(\mathrm{s}))\mathrm{ds}} + \frac{\lambda}{2}\left[1 - \mathrm{e}^{-\int_{t_0}^{\mathrm{T}}\theta^2(\mathrm{s})\mathrm{K}(\mathrm{s})\mathrm{ds}}\right]. \tag{4.88}$$

According to $E_{t,x}(X_T^u) = M$, there is:

$$M = x_0 e^{\int_{t_0}^{T}(r-\theta^2(s)K(s))ds} + \frac{\lambda}{2}\left[1 - e^{-\int_{t_0}^{T}\theta^2(s)K(s)ds}\right] \tag{4.89}$$

which gives:

$$\lambda = \frac{2[M - x_0 e^{\int_{t_0}^{T}(r-\theta^2(s)K(s))ds}]}{1 - e^{-\int_{t_0}^{T}\theta^2(s)K(s)ds}}. \tag{4.90}$$

To determine the optimal value of λ, we need to achieve $E_{t_0,x_0}(X_T^{u2})$. Hence, applying Ito's formula to (4.85) gives:

$$\begin{aligned} dX_t^{u2} = &\Big[[2r + \theta^2(t)(K^2(t) - 2K(t))]X_t^{u2} \\ &- \theta^2(t)\lambda e^{-r(T-t)}(K^2(t) - K(t))X_t^u + \theta^2(t)(\frac{\lambda}{2}e^{-r(T-t)}K(t))^2\Big]dt \\ &- 2\theta(t)(X_t^u - \frac{\lambda}{2}e^{-r(T-t)})K(t)X_t^u dW_t \end{aligned} \tag{4.91}$$

where $X_{t_0}^{u2} = x_0^2$. In the chapter, we only need to consider the case the time is at the maturity T, hence, setting $t = T$ and taking expectation on the both side of (4.91), and solving the corresponding ODE gives:

$$\begin{aligned} E_{t_0,x_0}(X_T^{u2}) = &e^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}(x_0 e^{r(T-t_0)} - \frac{\lambda}{2})^2 \\ &+ \lambda e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds}(x_0 e^{r(T-t_0)} - \frac{\lambda}{2}) + \frac{\lambda^2}{4}. \end{aligned} \tag{4.92}$$

Inserting (4.90) into (4.92), we can easily see that (4.17) is given by:

$$\begin{aligned} V_M(t_0,x_0) = &\frac{1}{(1 - e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\Big[\big(e^{\int_{t_0}^{T}\theta^2(s)(K(s)^2-2K(s))ds} - 2e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds} + 1\big)M^2 \\ &+ \big(2x_0 e^{r(T-t_0)}e^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds} - 2x_0 e^{r(T-t_0)}e^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}\big)M \\ &+ x_0^2 e^{2r(T-t_0)}\big(e^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds} - e^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}\big)\Big]. \end{aligned} \tag{4.93}$$

Substituting (4.93) into (4.16), there is:

$$\begin{aligned} V(t_0,x_0) = &c\left[1 - \frac{1}{(1 - e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\big(e^{\int_{t_0}^{T}\theta^2(s)(K(s)^2-2K(s))ds} - 2e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds} + 1\big)\right]M^2 \\ &+ \left[1 - \frac{c2x_0 e^{r(T-t_0)}}{(1 - e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\big(e^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds} - e^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}\big)\right]M \\ &- \frac{cx_0^2 e^{2r(T-t_0)}}{(1 - e^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\big(e^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds} - e^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}\big). \end{aligned} \tag{4.94}$$

The value function given by (4.94) is a quadratic function with respect to M. Recalling the property of quadratic function, we can see that the optimal value of $V(t_0,x_0)$ is attainable only if

$$c\big[1-\frac{1}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\big(\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K(s)^2-2K(s))ds}-2\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds}+1\big)\big]<0 \tag{4.95}$$

otherwise the optimal value of $V(t_0,x_0)$ is not attainable. Inequality (4.95) implies that:

$$\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)K^2(s)ds}>1 \tag{4.96}$$

which always holds and (4.94) has a strictly negative coefficient for the quadratic term M^2. By using the property of quadratic function, we see that the optimal value of M is achieved at:

$$M_*=x_0\mathrm{e}^{r(T-t_0)}-\frac{1}{2c}\frac{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}{\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}[1-\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)K^2(s)ds}]}. \tag{4.97}$$

Inserting (4.97) into (4.90), we obtain that:

$$\lambda_*=2x_0\mathrm{e}^{r(T-t_0)}-\frac{1}{c}\frac{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})}{\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}(1-\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)K^2(s)ds})}. \tag{4.98}$$

Recalling (4.98) and (4.82), we receive the optimal control given by:

$$u_*^s(t,s,x)=-\frac{\delta}{\sigma}\frac{1}{s^{2\beta}}\frac{1}{x}\left[x-x_0\mathrm{e}^{r(t-t_0)}+\frac{1}{2c}\frac{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})\mathrm{e}^{-r(T-t)}}{\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}(1-\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)K^2(s)ds})}\right]K(t) \tag{4.99}$$

which confirms (4.8). In (4.99), it is clear that the optimal control is related to the initial status (t_0,x_0), and following the previous definition, we name this optimal control as the statically optimal control.

(B) In the following part, we are going to consider the dynamically optimal control. As we claim that the dynamically optimal control is equal to the statically optimal control with the same initial state (t,x), replacing x_0 and t_0 by x and t in (4.99) gives the candidate dynamically optimal control:

$$u^d(t,s,x)=-\frac{\delta}{\sigma}\frac{1}{s^{2\beta}}\frac{1}{x}\frac{1}{2c}\frac{(1-\mathrm{e}^{\int_{t}^{T}-\theta^2(s)K(s)ds})\mathrm{e}^{-r(T-t)}}{\mathrm{e}^{\int_{t}^{T}-2\theta^2(s)K(s)ds}(1-\mathrm{e}^{\int_{t}^{T}\theta^2(s)K^2(s)ds})}K(t). \tag{4.100}$$

and $u^d(T,s,x):=\lim_{t\mapsto T}u^d(t,s,x)$. To prove the optimality of (4.100), we set that $u_*^d(t_0,x_0)=w(t_0,x_0)$, $w(t_0,x_0)=u_*^s(t_0,x_0)$, and $v(t_0,x_0)$ for any admissible such that $v(t_0,x_0)\neq u_*^d(t_0,x_0)$. For a dynamically optimal control, the following relationship must hold:

$$V_w(t_0,x_0):=\mathrm{E}_{t_0,x_0}(X_T^w)-c\,\mathrm{Var}_{t_0,x_0}(X_T^w)>\mathrm{E}_{t_0,x_0}(X_T^v)-c\,\mathrm{Var}_{t_0,x_0}(X_T^v)=:V_v(t_0,x_0) \tag{4.101}$$

for any $(t_0,x_0) \in [0,T]\times\mathbb{R}$ based upon the fact that $V_w(t_0,x_0) = V(t_0,x_0)$ and $w(t_0,x_0)$ is statically optimal in (4.8).

8. To verify (4.101), we set $\mathrm{E}_{t_0,x_0}(X_T^v) = M_v$. For w, there is $\mathrm{E}_{t_0,x_0}(X_T^w) = M_*$ and the value of M_* is given by (4.97). Let us consider the case when $M_* \neq M_v$ firstly. Equation (4.94) is a quadratic function with respect to M, which indicates that the optimal value of M_* is uniquely determined. Hence, there is:

$$
\begin{aligned}
&V_w(t_0,x_0) \qquad (4.102)\\
&= c\left[1-\frac{1}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\left(\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K(s)^2-2K(s))ds}-2\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds}+1\right)\right]M_*^2\\
&+\left[1-\frac{c2x_0\mathrm{e}^{r(T-t_0)}}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\left(\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}-\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}\right)\right]M_*\\
&-\frac{cx_0^2\mathrm{e}^{2r(T-t_0)}}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\left(\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}-\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}\right)\\
&> c\left[1-\frac{1}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\left(\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K(s)^2-2K(s))ds}-2\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds}+1\right)\right]M_v^2\\
&+\left[1-\frac{c2x_0\mathrm{e}^{r(T-t_0)}}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\left(\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}-\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}\right)\right]M_v\\
&-\frac{cx_0^2\mathrm{e}^{2r(T-t_0)}}{(1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(s)K(s)ds})^2}\left(\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds}-\mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}\right)\\
&= V_v(t_0,x_0).
\end{aligned}
$$

In (4.102), the strictly inequality always holds since M_* is the unique maximum point of the quadratic function of M. Hence, we can conclude that (4.101) exists when $M_* \neq M_v$. Next, we need to consider optimality of w when $M_v = M_*$. Recalling (4.21) and (4.98), we first claim the following relation:

$$
\begin{aligned}
V_v^{\lambda_*}(t_0,s_0,x_0) &:= E_{t_0,x_0}\left[(X_T^v)^2-\lambda_*X_T^v|S_{t_0}=s_0,X_{t_0}^v=x_0\right] \qquad (4.103)\\
&> E_{t_0,x_0}\left[(X_T^w)^2-\lambda_*X_T^w|S_{t_0}=s_0,X_{t_0}^w=x_0\right] =: V^{\lambda_*}(t_0,s_0,x_0).
\end{aligned}
$$

Recalling the terminal condition (4.23) of the HJB equation and applying Ito formula, we receive that:

$$
\begin{aligned}
X_T^{v\,2}-\lambda_* X_T^v &= V^{\lambda_*}(T,S_T,X_T^v) \qquad (4.104)\\
&= V^{\lambda_*}(t_0,s_0,x_0)+\int_{t_0}^{T}\big[V_t^{\lambda}(p,S_p,X_p^v)+\mu S_p V_s^{\lambda}(p,S_p,X_p^v)\\
&+\frac{1}{2}\sigma^2 {S_p}^{2\beta+2}V_{ss}^{\lambda}(p,S_p,X_p^v)+(r+(\mu-r)v_p)X_p^v V_x^{\lambda}(p,S_p,X_p^v)\\
&+\sigma^2 {S_p}^{2\beta+1}X_p^v v_p V_{xs}^{\lambda}(p,S_p,X_p^v)+\frac{1}{2}\sigma^2 {S_p}^{2\beta}v_p^2 X_p^{v2}V_{xx}^{\lambda}(p,S_p,X_p^v)\big]dp\\
&+\int_{t_0}^{T}(\sigma {S_p}^{\beta+1}V_s^{\lambda}(p,S_p,X_p^v)+\sigma v_p X_p^v {S_p}^{\beta}V_x^{\lambda}(p,S_p,X_p^v))dW_p
\end{aligned}
$$

in which λ_* is given by (4.98). In (4.104), the integrand term:

$$
\begin{aligned}
A_T = \int_{t_0}^{T} & \big[V_t^{\lambda}(p,S_p,X_p^v)+\mu S_p V_s^{\lambda}(p,S_p,X_p^v) \qquad (4.105)\\
&+\frac{1}{2}\sigma^2 {S_p}^{2\beta+2}V_{ss}^{\lambda}(p,S_p,X_p^v)+(r+(\mu-r)v_p)X_p^v V_x^{\lambda}(p,S_p,X_p^v)\\
&+\sigma^2 {S_p}^{2\beta+1}X_p^v v_p V_{xs}^{\lambda}(p,S_p,X_p^v)+\frac{1}{2}\sigma^2 {S_p}^{2\beta}v_p^2 X_p^{v2}V_{xx}^{\lambda}(p,S_p,X_p^v)\big]dp
\end{aligned}
$$

is non-negative because of (4.22) with $\lambda=\lambda_*$. Taking E_{t_0,x_0} on the both side of (4.104), there is:

$$
\begin{aligned}
\mathrm{E}_{t_0,x_0}\big[X_T^{v\,2}-\lambda_* X_T^v\big] &= V_v^{\lambda_*}(t_0,s_0,x_0) \qquad (4.106)\\
&= V^{\lambda_*}(t_0,s_0,x_0)+\mathrm{E}_{t_0,x_0}\int_{t_0}^{T}\big[V_t^{\lambda}(p,S_p,X_p^v)+\mu S_p V_s^{\lambda}(p,S_p,X_p^v)\\
&+\frac{1}{2}\sigma^2 {S_p}^{2\beta+2}V_{ss}^{\lambda}(p,S_p,X_p^v)+(r+(\mu-r)v_p)X_p^v V_x^{\lambda}(p,S_p,X_p^v)\\
&+\sigma^2 {S_p}^{2\beta+1}X_p^v v_p V_{xs}^{\lambda}(p,S_p,X_p^v)+\frac{1}{2}\sigma^2 {S_p}^{2\beta}v_p^2 X_p^{v2}V_{xx}^{\lambda}(p,S_p,X_p^v)\big]dp\\
&+\mathrm{E}_{t_0,x_0}\int_{t_0}^{T}(\sigma {S_p}^{\beta+1}V_s^{\lambda}(p,S_p,X_p^v)+\sigma v_p X_p^v {S_p}^{\beta}V_x^{\lambda}(p,S_p,X_p^v))dW_p.
\end{aligned}
$$

Since we have known that $v(t_0,s_0,x_0)\neq w(t_0,s_0,x_0)$, we can further define a region $R_\varepsilon := [t_0,t_0+\varepsilon]\times[s_0-\varepsilon,s_0+\varepsilon]\times[x_0-\varepsilon,x_0+\varepsilon]$ for some $\varepsilon>0$ small enough such that $t_0+\varepsilon\le T$ and $s_0-\varepsilon\ge 0$. Upon the continuity of v and w, there is $v(z,s,x)\neq w(z,s,x)$ for any choice of $(z,s,x)\in R_\varepsilon$. Moreover, from (4.22), a quadratic function of u, we can see that $w(t,s,x)$ is the unique minimum point with $\lambda=\lambda_*$ evaluated at each set of $(t,s,x)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}$. Hence, we can see that the value of ε can be chosen small enough to meet:

$$
\begin{aligned}
&\big[V_t^{\lambda_*}+\mu s V_s^{\lambda_*}+\frac{1}{2}\sigma^2 s^{2\beta+2}V_{ss}^{\lambda_*}+(r+(\mu-r)v)xV_x^{\lambda_*} \qquad (4.107)\\
&+\sigma^2 s^{2\beta+1}xvV_{xs}^{\lambda_*}+\frac{1}{2}\sigma^2 s^{2\beta}v^2x^2V_{xx}^{\lambda_*}\big]\ge\xi>0
\end{aligned}
$$

where ξ is a constant given and fixed and $(t,s,x) \in R_\varepsilon$. Hence, setting $\tau_\varepsilon = \inf\{z \in [t_0, t_0 + \varepsilon] | (z, S_z, X_z^v) \notin R_\varepsilon\}$, we can see that:

$$V_v^{\lambda_*}(t_0,s_0,x_0) \geq V^{\lambda_*}(t_0,s_0,x_0) + \xi(\tau_\varepsilon - t_0) \qquad (4.108)$$
$$+ \mathrm{E}_{t_0,x_0} \int_{t_0}^{\tau_\varepsilon} (\sigma {S_p}^{\beta+1} V_s^\lambda(p,S_p,X_p^v) + \sigma v_p X_p^v {S_p}^\beta V_x^\lambda(p,S_p,X_p^v)) dW_p.$$

As we have seen in the previous part that $M_t = \int_{t_0}^t (\sigma {S_p}^{\beta+1} V_s^\lambda + \sigma v_p X_p^v {S_p}^\beta V_x^\lambda) dW_p$ is a local martingale for $t \in [t_0, T]$. Hence, there exists a sequence of stopping time τ_n such that $\tau_n \uparrow T$ as $n \uparrow \infty$. Then, in this case the stopped process $M_{\tau_\varepsilon \wedge \tau_n}$ is a martingale. Hence, inequality (4.108) leads to:

$$V_v^{\lambda_*}(t_0,s_0,x_0) \geq V^{\lambda_*}(t_0,s_0,x_0) + \xi(\tau_\varepsilon \wedge \tau_n - t_0). \qquad (4.109)$$

Taking $\lim_{n\uparrow\infty}$ in (4.109) and recalling the dominated convergence theorem, we can see that:

$$V_v^{\lambda_*}(t_0,s_0,x_0) \geq V^{\lambda_*}(t_0,s_0,x_0) + \xi(\tau_\varepsilon - t_0) > V^{\lambda_*}(t_0,s_0,x_0) \qquad (4.110)$$

in which the strict inequality exists as $\tau_\varepsilon > t_0$ with $\mathbb{P}_{t_0,x_0}$−probability one because of the continuity of X^v. It should be pointed that in the case when $x_0 = 0$, we will take $v(t_0, 0)$ and $w(t_0,x_0)$ as $v(t_0,0) \cdot 0$ and $w(t_0,x_0) \cdot 0$, and (4.110) still holds in this case. Therefore, we have verified (4.103).

9. Recalling (4.18)-(4.20) and the assumption that $M_v = M_*$, there exists:

$$V^{\lambda_*}(t_0,x_0) = \mathrm{E}_{t_0,x_0}[X_T^{w2}] - \lambda_* M_* < \mathrm{E}_{t_0,x_0}[X_T^{v2}] - \lambda_* M_v. \qquad (4.111)$$

Hence, according to equation (4.16), there exists:

$$M_* + cM_*^2 - \mathrm{E}_{t_0,x_0}[X_T^{w2}] > M_v + cM_v^2 - \mathrm{E}_{t_0,x_0}[X_T^{v2}] \qquad (4.112)$$

as $M_* = M_v$. Inequality (4.112) confirms the statement that $V_w(t_0,x_0) > V_v(t_0,x_0)$. Hence, we conclude that $u_*^d = w$ is the dynamically optimal control as claimed. □

In the following part, we are going to follow the idea of [43] to solve the two constrained problems given by (4.6) and (4.7) respectively. We will see that Theorem 43.1 will play an important part in the following proof. The main part of the following proof is consistent with Corollary 5 and Corollary 7 in [43].

Corollary 4.2. *Consider the constrained problem* $V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u)\leq\alpha} \mathrm{E}_{t,x}(X_T^u)$ *in which* X^u *represents the wealth process and is the solution of the SDE* (4.3) *with* $X_{t_0}^u = x_0$

under the probability measure $\mathbb{P}_{t_0,x_0}$ *for* $(t_0,x_0) \in [0,T] \times \mathbb{R}$ *given and fixed and* $\alpha \in (0,\infty)$. *The related risk coefficient is defined by* $\delta = (\mu - r)/\sigma$ *in which* $\mu, r \in \mathbb{R}$, $\sqrt{2}r > \mu > r$ *and* $\sigma > 0$. *We further assume that* $\delta \neq 0$ *and* $r \neq 0$ *in the following part. (The cases* $\delta = 0$ *or* $r = 0$ *follow by passage to the limit when the non-zero* δ *or* r *approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\Bigg[x - x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} - \sqrt{\alpha}\frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{\sqrt{\left(\mathrm{e}^{\int_{t_0}^{T}\theta^2(s)(K^2(s)-2K(s))ds} - \mathrm{e}^{\int_{t_0}^{T}-2\theta^2(s)K(s)ds}\right)}}\Bigg]K(t) \tag{4.113}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$. *In* (4.113), *the related parameters are given by:*

$$\theta(t) = \frac{\delta}{S_t^{\beta}}, \tag{4.114}$$

$$K(t) = [1 + \frac{2\beta\sigma B(t)}{\delta}], \tag{4.115}$$

$$B(t) = \frac{z_1 z_2 (1 - \mathrm{e}^{-2\beta^2\sigma^2(z_1 - z_2)(T-t)})}{z_1 - z_2 \mathrm{e}^{-2\beta^2\sigma^2(z_1-z_2)(T-t)}}, \tag{4.116}$$

$$z_1 = \frac{-(\mu - 2r) + \sqrt{2r^2 - \mu^2}}{2\beta\sigma^2}, \tag{4.117}$$

and

$$z_2 = \frac{-(\mu - 2r) - \sqrt{2r^2 - \mu^2}}{2\beta\sigma^2} \tag{4.118}$$

for $t \in [t_0, T]$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,s,x) = \frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\Bigg[\sqrt{\alpha}\frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{\sqrt{\left(\mathrm{e}^{\int_{t}^{T}\theta^2(s)(K^2(s)-2K(s))ds} - \mathrm{e}^{\int_{t}^{T}-2\theta^2(s)K(s)ds}\right)}}\Bigg]K(t) \tag{4.119}$$

for $(t,x) \in [t_0,T] \times \mathbb{R}$.

Proof. In this proof, we claim that, for each pair of $(t_0,x_0) \in [0,T] \times \mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (4.3) with initial condition $X_{t_0}^u = x_0$. Furthermore, for X_t^u, $u \in U$ is any admissible control we defined in Section 4.2.

(A): By using Lagrange multipliers in (4.6), there exists:

$$L_{t,x}(u,c) = \mathrm{E}_{t,x}(X_T^u) - c\left[\mathrm{Var}_{t,x}(X_T^u) - \alpha\right] \tag{4.120}$$

for $c > 0$. Based upon Theorem 4.1, we have known that the optimal control u_*^s given by (4.8) maximises $\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)$ under the constant elasticity of variance model. Hence, we claim (4.8) with $c > 0$ maximises (4.120). In this case, we state:

$$L_{t,x}(u_*^c, c) = \sup_u L_{t,x}(u,c) \tag{4.121}$$

for $c > 0$. Furthermore, there exists $c = c(\alpha, t, s, x) > 0$ such that:

$$\mathrm{Var}_{t,x}(X_T^{u_*^s}) = \alpha. \tag{4.122}$$

Then, for any admissible control u that satisfies $\mathrm{Var}_{t,x}(X_T^u) \leq \alpha$, the following relation holds:

$$\mathrm{E}_{t,x}(X_T^{u_*^c}) = L_{t,x}(u_*^c, c) \geq \mathrm{E}_{t,x}(X_T^u) - c[\mathrm{Var}_{t,x}(X_T^u) - \alpha] \geq \mathrm{E}_{t,x}(X_T^u). \tag{4.123}$$

This result indicates that the optimal control u_*^c given by (4.8) with $c(\alpha, t, x) > 0$ is the statically optimal control in (4.6).

According to (4.92)-(4.98), we can calculate that the variance $\mathrm{Var}_{t_0,x_0}(X_T^{u_*^c})$ is given by:

$$\mathrm{Var}_{t_0,x_0}(X_T^{u_*^c}) = \frac{1}{4c^2} \frac{\left(1 - \mathrm{e}^{\int_{t_0}^T -\theta^2(s)K(s)ds}\right)^2}{\left(\mathrm{e}^{\int_{t_0}^T \theta^2(s)(K^2(s)-2K(s))ds} - \mathrm{e}^{\int_{t_0}^T -2\theta^2(s)K(s)ds}\right)}. \tag{4.124}$$

Setting (4.124) equal to α, we achieve the value of c, which is:

$$c = \frac{1}{2\sqrt{\alpha}} \frac{\left(1 - \mathrm{e}^{\int_{t_0}^T -\theta^2(s)K(s)ds}\right)}{\sqrt{\left(\mathrm{e}^{\int_{t_0}^T \theta^2(s)(K^2(s)-2K(s))ds} - \mathrm{e}^{\int_{t_0}^T -2\theta^2(s)K(s)ds}\right)}}. \tag{4.125}$$

Substituting (4.125) into (4.8), we obtain the statically optimal control:

$$u_*^s(t,s,x) = -\frac{\delta}{\sigma} \frac{1}{s^{2\beta} x} \left[x - x_0 \mathrm{e}^{r(t-t_0)} - \sqrt{\alpha} \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{\left(\mathrm{e}^{\int_{t_0}^T \theta^2(s)(K^2(s)-2K(s))ds} - \mathrm{e}^{\int_{t_0}^T -2\theta^2(s)K(s)ds}\right)}} \right] K(t) \tag{4.126}$$

which confirms (4.113) and completes the first part of the proof.

(B) Replacing t_0 and x_0 by t and x in the statically optimal control (4.126), we can obtain the candidate control u_*^d given in (4.119). We claim this gives the dynamically optimal control for (4.6). To prove its optimality, we take any other admissible control v such that $v(t_0,x_0) \neq u_*^d(t_0,x_0)$ and further assume that there exist $w(t_0,x_0) = u_*^s(t_0,x_0)$ and $w(t_0,x_0) = u_*^d(t_0,x_0)$. With c given by (4.125), it is clear that (4.101) holds, from which we can see that, for $\mathrm{Var}_{t_0,x_0}(X_T^w) = \alpha$, the following inequality holds:

$$\mathrm{E}_{t_0,x_0}(X_T^w) > \mathrm{E}_{t_0,x_0}(X_T^v) - c\,\mathrm{Var}_{t_0,x_0}(X_T^v) \geq \mathrm{E}_{t_0,x_0}(X_T^v) \tag{4.127}$$

in which v satisfies $\mathrm{Var}_{t_0,x_0}(X_T^v) \leq \alpha$. Hence, we can conclude that the optimal control given by (4.119) is the dynamically optimal control for (4.6). □

Corollary 4.3. *Consider optimal control problem* $V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u)\geq\gamma} \mathrm{Var}_{t,x}(X_T^u)$ *in which* X^u *represents the wealth process and is the solution of the SDE* (4.3) *with* $X_{t_0}^u = x_0$ *under the probability measure* $\mathbb{P}_{t_0,x_0}$ *for* $(t_0,x_0) \in [0,T] \times \mathbb{R}$ *given and fixed and* $\gamma \in \mathbb{R}$*. The related risk coefficient is defined by* $\delta = (\mu - r)/\sigma$ *in which* $\mu, r \in \mathbb{R}$*,* $\sqrt{2}r > \mu > r$ *and* $\sigma > 0$*. We further assume that* $r \neq 0$ *in the following part. (The cases* $\delta = 0$ *or* $r = 0$ *follow by passage to the limit when the non-zero* δ *or* r *approaches to 0.) Furthermore, we assume that the expectation of the terminal wealth,* γ*, must satisfy* $\gamma > x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$*. For a wise investor, if* $\gamma \leq x_0\mathrm{e}^{\mathrm{r(T-t_0)}}$*, he can simply invest all his wealth in the riskless asset and receive zero variance at the maturity* T*. Hence, in the following part, we assume that* $\gamma > x_0\mathrm{e}^{\mathrm{r(T-t_0)}}$*.*

(A) *The statically optimal control is given by:*

$$u_*^s(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\left[x - x_0\mathrm{e}^{\mathrm{r(t-t_0)}} + (\mathrm{x_0}\mathrm{e}^{\mathrm{r(T-t_0)}} - \gamma)\frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{1-\mathrm{e}^{\int_{t_0}^{T}-\theta^2(\mathrm{s})\mathrm{K(s)ds}}}\right]\mathrm{K(t)} \tag{4.128}$$

for $(t,s,x) \in [t_0,T] \times \mathbb{R}_+ \times \mathbb{R}$*. In* (4.8)*, the related parameters are given by:*

$$\theta(t) = \frac{\delta}{S_t^{\beta}}, \tag{4.129}$$

$$K(t) = [1 + \frac{2\beta\sigma B(t)}{\delta}], \tag{4.130}$$

$$B(t) = \frac{z_1 z_2(1-\mathrm{e}^{-2\beta^2\sigma^2(\mathrm{z_1-z_2})(\mathrm{T-t})})}{z_1 - z_2\mathrm{e}^{-2\beta^2\sigma^2(\mathrm{z_1-z_2})(\mathrm{T-t})}}, \tag{4.131}$$

$$z_1 = \frac{-(\mu-2r)+\sqrt{2r^2-\mu^2}}{2\beta\sigma^2}, \tag{4.132}$$

and

$$z_2 = \frac{-(\mu - 2r) - \sqrt{2r^2 - \mu^2}}{2\beta\sigma^2} \tag{4.133}$$

for $t \in [t_0, T]$.

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\left[(x\mathrm{e}^{\mathrm{r(T-t)}} - \gamma)\frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{1 - \mathrm{e}^{\int_\mathrm{t}^\mathrm{T} -\theta^2(\mathrm{s})\mathrm{K(s)ds}}}\right]\mathrm{K(t)} \tag{4.134}$$

for $(t,s,x) \in [t_0, T] \times \mathbb{R}_+ \times \mathbb{R}$.

Proof. In this chapter, we claim that, for each pair of $(t_0, x_0) \in [0, T] \times \mathbb{R}$ given and fixed, there exists a probability measure $\mathbb{P}_{t_0,x_0}$ under which X^u is the solution of the SDE (4.3) with initial condition $X_{t_0}^u = x_0$. Furthermore, for X_t^u, $u \in U$ is any admissible control we defined in Section 4.2.

(A): By using Lagrange multipliers in the constrained problem (4.7), there exists:

$$L_{t,x}(u,c) = \mathrm{Var}_{t,x}(X_T^u) - c\big[\mathrm{E}_{t,x}(X_T^u) - \gamma\big] \tag{4.135}$$

for $c > 0$. Re-arranging equation (4.135), the following relation holds:

$$\inf_u \big[\mathrm{Var}_{t,x}(X_T^u) - c\big[\mathrm{E}_{t,x}(X_T^u) - \gamma\big]\big) = -c\sup_u \big[\mathrm{E}_{t,x}(X_T^u) - \frac{1}{c}\mathrm{Var}_{t,x}(X_T^u)\big] + c\gamma. \tag{4.136}$$

Recalling Theorem 4.1, we can see that the optimal control $u_*^{\frac{1}{c}}$ given by (4.8) maximises the right-hand side of (4.136), which leads to:

$$L_{t,x}(u_*^{\frac{1}{c}}, c) = \inf L_{t,x}(u,c) \tag{4.137}$$

for $c > 0$. Furthermore, we assume exist $c = c(\gamma, t, s, x) > 0$ such that:

$$\mathrm{E}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = \gamma. \tag{4.138}$$

Upon (4.138), we can easily see that the following relation holds:

$$\mathrm{Var}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = L_{t,x}(u_*^{\frac{1}{c}}, c) \le \mathrm{Var}_{t,x}(X_T^u) - c[\mathrm{E}_{t,x}(X_T^u) - \gamma] \le \mathrm{Var}_{t,x}(X_T^u) \tag{4.139}$$

for any admissible u such that $\mathrm{E}_{t,x}(X_T^u) \ge \gamma$, which indicates that $u_*^{\frac{1}{c}}$ is statically optimal control for (4.7). Recalling the condition that $\mathrm{E}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = \gamma$, there is:

$$\mathrm{E}_{t,x}(X_T^{u_*^{\frac{1}{c}}}) = x_0\mathrm{e}^{\mathrm{r(T-t_0)}} - \frac{\mathrm{c}}{2}\frac{\big(1 - \mathrm{e}^{\int_{\mathrm{t_0}}^\mathrm{T} -\theta^2(\mathrm{s})\mathrm{K(s)ds}}\big)^2}{\mathrm{e}^{\int_{\mathrm{t_0}}^\mathrm{T} -2\theta^2(\mathrm{s})\mathrm{K(s)ds}}\big[1 - \mathrm{e}^{\int_{\mathrm{t_0}}^\mathrm{T} \theta^2(\mathrm{s})\mathrm{K}^2(\mathrm{s})\mathrm{ds}}\big]} = \gamma \tag{4.140}$$

which leads to:

$$c = 2(x_0 e^{r(T-t_0)} - \gamma)\frac{e^{\int_{t_0}^{T} -2\theta^2(s)K(s)ds}\left[1 - e^{\int_{t_0}^{T}\theta^2(s)K^2(s)ds}\right]}{(1 - e^{\int_{t_0}^{T} -\theta^2(s)K(s)ds})^2}. \tag{4.141}$$

Furthermore, substituting (4.141) back into (4.8), we receive:

$$u_*^s(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\left[x - x_0 e^{r(t-t_0)} + (x_0 e^{r(T-t_0)} - \gamma)\frac{e^{-r(T-t)}}{1 - e^{\int_{t_0}^{T} -\theta^2(s)K(s)ds}}\right]K(t) \tag{4.142}$$

which confirms (4.128) and completes the first part of proof.

(B): Replacing t_0 and x_0 by t and x in the statically optimal control (4.142), we can obtain the control u_*^d given by:

$$u_*^d(t,s,x) = -\frac{\delta}{\sigma}\frac{1}{s^{2\beta}x}\left[(x e^{r(T-t)} - \gamma)\frac{e^{-r(T-t)}}{1 - e^{\int_{t}^{T} -\theta^2(s)K(s)ds}}\right]K(t). \tag{4.143}$$

We claim this gives the dynamically optimal control for (4.7). To prove its optimality, we take any other admissible control v such that $v(t_0,x_0) \neq u_*^d(t_0,x_0)$ and $E_{t_0,x_0}(X_T^v) \geq \gamma$ and assume $w(t_0,x_0) = u_*^s(t_0,x_0)$ and $w(t_0,x_0) = u_*^d(t_0,x_0)$ such that $E_{t_0,x_0}(X_T^w) = \gamma$. Hence, we can see that (4.101) holds with c given by (4.141). Since $E_{t_0,x_0}(X_T^w) = \gamma$, it can be easily seen that (4.101) leads to:

$$\mathrm{Var}_{t_0,x_0}(X_T^w) < \frac{1}{c}\left[\gamma - E_{t_0,x_0}(X_T^v) + c\,\mathrm{Var}_{t_0,x_0}(X_T^v)\right] \leq \mathrm{Var}_{t_0,x_0}(X_T^v) \tag{4.144}$$

in which $E_{t_0,x_0}(X_T^v) \geq \gamma$ and c is given by (4.141). This indicates that the optimal control given by (4.143) is the dynamically optimal control we are looking for. □

Remark 4.4. As we have mentioned that when $\beta > 0$, the solution of (4.2) is a strict local martingale. A similar conclusion can be also observed in the solution of (4.3), X_T^u, which indicates that the self-financing portfolio may admit arbitrage opportunity. However, Theorem 4.1 will give the statically optimal control and the dynamically optimal control regardless of the existence or not of arbitrage opportunity which, to the best of our knowledge, has not been discussed previously in the time-consistent control area.

Remark 4.5. In Theorem 4.1, we obtain both the statically and dynamically optimal controls, which are time-inconsistent and time-consistent respectively. Moreover, Theorem 4.1 gives a general solution for different values of the elasticity parameter of the CEV model in the mean-variance portfolio selection problem. As we stated above, in the case when $\beta = 0$, the CEV model (4.2) reduces to geometric Brownian motion, under which Theorem

4.1 will give the same solution as [43]. However, comparing with the conclusion in [43], we can note that Theorem 4.1 in this chapter constrains the value of μ, i.e. $0 < \mu < \sqrt{2}r$, which is not observed in [43].

Moreover, it should be noted that in the case $\beta \neq 0$, both the static and dynamic optimalities depend on the current value of the risky asset as there exists a term $1/S^{2\beta}$ in (4.8) and (4.14). If $\beta < 0$, the investor will be suggested that keep increasing the weight of the risky asset if the price of the risky asset goes up. Contrarily, the investor will reduce the weight of risky asset alongside with the decrease of the risky asset price. Moreover, if $\beta > 0$, the optimal strategies in Theorem 4.1 would suggest that the investor reduces the weight on the risky asset when the stock price increases and increase the weight as the stock price decreases. This phenomenon is not observed in the previous study [43] under geometric Brownian motion.

The reason behind this phenomenon is that in the constant elasticity of variance model (4.2), σS_t^{β} represents the instantaneous volatility of the stock, and the change of the instantaneous volatility will impact the investor behaviour. By setting the case where $\beta < 0$ as an example, we can see that the instantaneous volatility reduces alongside the increase of the stock price, which leads to the decrease of risk of investing in this risky asset and makes the investor more willing to invest [10]. Contrarily, the decrease of the stock price will lead to the higher instantaneous volatility, which makes the risky asset less favourable to the investor.

4.4 Numerical analysis

In this section, we are providing numerical results for both the static optimality and dynamic optimality under the constant elasticity of variance model.

1. Firstly, we simulate both the statically optimal control and dynamically optimal control from Theorem 4.1 with respect to different values of β and compare the difference when we change the values of the current wealth of the portfolio and the current stock price. In Figure 4.1, we set $X_t^u = 1.8$ and $S_t = 1.2$ and plot both controls, (4.8) and (4.14), with $\beta \in [-3, 7]$. Both strategies exhibit monotonicity and approach to 0 when β gets large enough. It is evident that both strategies suggest the investor avoid investing in the risky asset when β gets larger. This fact is consistent with the analysis we made in Remark 4.5

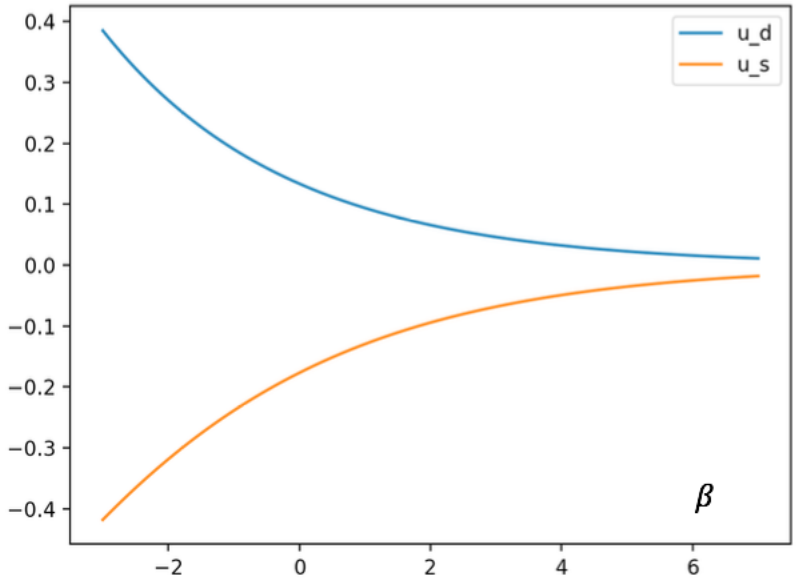

Figure 4.1: This is the simulation of statically optimal control (4.8) and dynamically optimal control (4.14) with respect to different values of $\beta \in [-3,7]$ when $\delta = 0.15$, $t_0 = 0$, $T = 1$, $t = 0.5$, $x_0 = 1$, $c = 1.5$, $s = 1.2$, $\mu = 0.13$, $r = 0.1$, $\sigma = 0.2$, $x = 1.8$ and $s_0 = 0.8$.

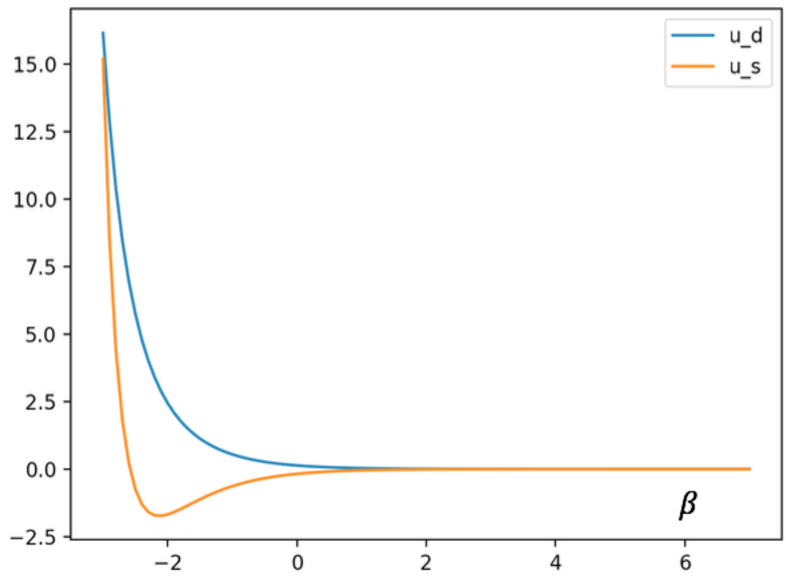

Figure 4.2: This is the simulation of statically optimal control (4.8) and dynamically optimal control (4.14) with respect to different values of $\beta \in [-3,7]$ when $\delta = 0.15$, $t_0 = 0$, $T = 1$, $t = 0.5$, $x_0 = 1$, $c = 1.5$, $s = 2$, $\mu = 0.13$, $r = 0.1$, $\sigma = 0.2$, $x = 1.8$ and $s_0 = 0.8$.

as larger positive values of β will lead to larger instantaneous volatility of the risky assets, which makes the risky asset less favourable to the investor. However, the dynamic optimality encourages the investor to hold large a amount of risky asset when β is small while the static optimality suggests that the investor should short-sell the risky asset. This indicates that the static investor and dynamic investor have opposite views of the future trend of the risky asset when the current wealth of the portfolio is sufficiently larger than the current stock price.

However, the static optimality changes its speculation on the decline of the risky stock when the current price of the risky asset exceeds the current value of the wealth process. This is reflected in Figure 4.2, in which we still set $X_t^u = 1.8$ but increase S_t to 2.0. We can see that the static optimality changes its short-selling strategy and greatly encourages the investor to hold a long position in the risky asset when β is small enough. Furthermore, in Figure 4.2 and we can see that the plots of statically optimal control with respect to different values of β lose the monotonicity, and the plot will never become monotonic again when we keep increasing the value of S_t. Figure 4.2 also indicates that the static optimality will suggest the investor take a long position only if the value of β is small enough when the current stock price is higher than the current portfolio wealth.

Besides, it should be pointed out that the dynamic optimality holds the monotonicity no matter which value we choose for X_t^u and S_t. Moreover, the dynamically optimal control will always give positive value as long as the current wealth process stays positive, which can also be seen in the dynamic optimality of [Theorem 3, [43]].

Note that $0 < S_t < 1$ will lead to a reversed conclusion and we will observe the positive value of β will increase the value of both controls. This phenomenon is exhibited in Figure 4.3 below and we will omit further discussion here.

2. Theorem 4.1 naturally comes up with a question about which strategy will lead to better performance. In [43], they mentioned that comparing $\mathrm{E}_{t_0,x_0}(X_T^s)$ and $\mathrm{E}_{t_0,x_0}(X_T^d)$ gives the insight into the performance of those two strategies. Since we do not achieve the explicit solution for the wealth process in this problem, a theoretical comparison will not be feasible. By using Python, we can still have an intuitive comparison by simulating the wealth process and taking the terminal value X_T (in the case if X_T is not well-defined, we take the value of X_t at the second to the last point of time). In Table 4.1 and Table 4.2, we display the sample $\bar{\mu}$ and sample variance $\bar{m}$ for both strategies respectively when we fix the value of risk aversion

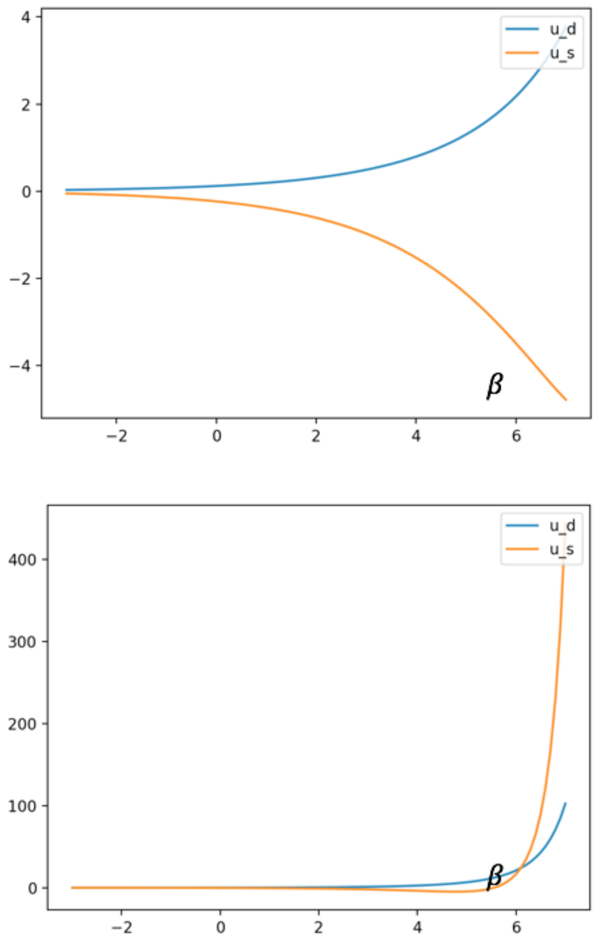

Figure 4.3: The first figure the simulation of statically optimal control (4.8) and dynamically optimal control (4.14) with respect to different values of $\beta \in [-3,7]$ when $S_t = 0.8$. And the second figure is the simulation of both strategies when $S_t = 0.7$. The rest parameters are given by $\delta = 0.15$, $t_0 = 0$, $T = 1$, $t = 0.5$, $x_0 = 1$, $c = 1.5$, $s = 1.2$, $\mu = 0.13$, $r = 0.1$, $\sigma = 0.2$, $x = 1.8$ and $s_0 = 0.8$.

rate and only consider the impact of changing the value of β. When $\beta < 0$, we can see that the static sample mean $\bar{\mu}^s$ and the dynamic sample mean $\bar{\mu}^d$ increase as β decreases. This indicates that reducing the value of β will enhance the performance of both strategies. Also, for both optimalities, we see that the sample variance $\bar{m}^s$ and $\bar{m}^d$ increases as β decreases, which indicates that the portfolio has larger fluctuation when β is small. However, when β is positive, we do not observe the obvious pattern. In the data we exhibited in Table 4.1, the performance of static optimality seems to increase as β increases, however, during our numerical analysis, we also observed the sample mean $\bar{\mu}^s$ is always around 2.62 no matter the change of the value of β. For the dynamic optimality, we can observe a similar conclusion that $\bar{\mu}^d$ fluctuates around 2.62-2.63, in which the small fluctuation of the sample mean can be the estimation error and does not show the clear trend in Table 4.2. A possible reason is that, in Figure 4.1, we have seen that when we set other parameters fixed, increasing the value of β will leads to less portion of wealth invested in the risky asset. From Figure 4.1, we can see that the difference between each value of controls with respect to different values of positive values of β is getting smaller and smaller as β increases. This action naturally leads to the fact that the impact of β becomes less important as β gets positively large.

Besides analysing the impact of β, we need to consider the impact of the risk aversion rate. In the previous two chapters, we have seen that the statically optimal control gives better performance when risk aversion rate is large while the dynamically optimal control performs better when risk aversion rate is small. Hence, we can verify if this is still valid under the existence of β. In Table 4.4 and Table 4.5, it is clear this pattern is still valid under the CEV model.

3. As we have stated the conclusion of this chapter is connected to the previous work in [43]. Hence, it would be interesting to further consider whether the remaining features will be consistent or not. The Remark 6 of [43] states that the dynamically optimal control outperforms the statically optimal control no matter the value of α. Hence, we simulate the static and dynamic optimalities for $\beta = 0.5$ and $\beta = -0.5$ respectively as the result may be different. Overall, in Table 4.7 and 4.8, we can see that for both cases where $\beta > 0$ and $\beta < 0$, the sample mean $\bar{\mu}^s$ increases as α increases while in Table 4.9 and Table 4.10, the sample mean $\bar{\mu}^d$ decreases as α increases. It should be pointed out that during the simulation we note some outliers because the variance gets larger when α increases, especially when $\alpha > 5$. Overall, from the repeated simulation, we can say that the pattern we summarised

Constant elasticity parameter β	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
0.5	2.61062483	0.01483325
1.0	2.61446869	0.00993458
1.5	2.62925411	0.01138159
2.0	2.61053020	0.01377599
-0.5	2.68152529	0.01943475
-1.0	2.69867464	0.03021396
-1.5	2.70934189	0.03925163
-2.0	2.78258761	0.06728610

Table 4.1: Simulation for the static optimality for the unconstrained problem (4.4) with respect to different values of β.

Constant elasticity parameter β	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
0.5	2.62470524	0.01356741
1.0	2.62734534	0.01381014
1.5	2.63923725	0.01402407
2.0	2.63204450	0.01775903
-0.5	2.66738856	0.01943141
-1.0	2.67906846	0.02403912
-1.5	2.73516401	0.03520119
-2.0	2.74193316	0.05436461

Table 4.2: Simulation for the dynamic optimality for the unconstrained problem (4.4) with respect to different values of β.

(Note the related parameters are given by $c = 1$, $\mu = 0.42$, $\sigma = 0.5$, $\delta = 0.24$, $x_0 = 2$, $r = 0.3$, $T = 1$, and $t_0 = 0$, and the sample size is 500.)

Risk aversion rate c	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
1.0	2.64599609	0.01957113
2.0	2.69329501	0.08572060
3.0	2.72763762	0.17225738
4.0	2.74301822	0.33851057
5.0	2.81776247	0.54094099

Table 4.4: Simulation for the static optimality for the unconstrained problem (4.4) with respect to different values of risk aversion rate c.

Risk aversion rate c	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
1.0	2.62814150	0.02193587
2.0	2.61783110	0.00428679
3.0	2.61741930	0.00213929
4.0	2.61255116	0.00158992
5.0	2.61116038	0.00090385

Table 4.5: Simulation for the dynamic optimality for the unconstrained problem (4.4) with respect to different values of risk aversion rate c.

(Note the related parameters are given by $\beta = -0.5$, $\mu = 0.42$, $\sigma = 0.5$, $\delta = 0.24$, $x_0 = 2$, $r = 0.3$, $T = 1$, and $t_0 = 0$, and the sample size is 500.)

above exists. Furthermore, different from the Corollary 5 and Remark 6 of [43], Table 4.7-Table 4.10 indicate that the statically optimal control leads to a larger sample mean than the dynamically optimal control for each α given and fixed under the existence of β (no matter $\beta > 0$ or $\beta < 0$). If $\beta = 0$, i.e. we have the same conclusion as Remark 6 in [43], the performance of dynamically optimal control will be enhanced as the value of α increases and the dynamic optimality will always outperform the static optimality. To illustrate this, we can see that $\bar{\mu}^s = 2.96160378 < 3.09161328 = \bar{\mu}^d$ where $\alpha = 1.0$ and the rest parameters are the same as Table 4.7 (further details can be seen in Remark 6 of [43]).

In Table 4.12 and Table 4.13, we compare the performance of both strategies when $\alpha = 1.5$ given and fixed. We can see that the sample mean of static optimality $\bar{\mu}^s$ increases as β decreases when $\beta < 0$ while under the same condition, the dynamic sample mean $\bar{\mu}^d$ decreases as β decreases. This observation is different from the unconstrained cases we discussed above as in the unconstrained case, the dynamic sample mean will increase as β decreases (See Table 4.2). However, similar to the unconstrained case, the impact of changing the value of β is not clear when $\beta > 0$. Both static and dynamic optimalities do not exhibit a clear trend when β changes.

4. One of the signature features in [43] is they found that in the second constrained

Value of α	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
1.0	2.86602056	1.18481662
2.0	3.16744231	4.41832150
3.0	3.70824406	9.36560057
4.0	4.00497455	15.2273346
5.0	4.42380723	25.2066747

Table 4.7: Simulation for the static optimality for the constrained problem (4.6) with respect to different values of α when $\beta = -0.5$.

Value of α	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
1.0	2.75094761	0.90511299
2.0	3.18443270	3.55295247
3.0	3.17571402	8.55327903
4.0	3.59177485	13.1009796
5.0	3.80025886	21.4758423

Table 4.8: Simulation for the static optimality for the constrained problem (4.6) with respect to different values of α when $\beta = 0.5$.

α	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
1.0	2.23340726	2.97032191
2.0	1.98558919	5.07249573
3.0	1.87565230	8.05671545
4.0	1.41277075	14.8796356
5.0	1.41752098	13.3131525

Table 4.9: Simulation for the dynamic optimality for the constrained problem (4.6) with respect to different values of α when $\beta = -0.5$.

α	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
1.0	2.41409850	3.42276108
2.0	2.19476116	7.22315804
3.0	2.16837756	11.5996044
4.0	1.64161181	12.7549176
5.0	1.29378455	15.4335927

Table 4.10: Simulation for the dynamic optimality for the constrained problem (4.6) with respect to different values of α when $\beta = 0.5$.

(Note the related parameters are given by $\mu = 0.42$, $\sigma = 0.5$, $\delta = 0.24$, $x_0 = 2$, $r = 0.3$, $T = 1$, and $t_0 = 0$, and the sample size is 500.)

Constant elasticity parameter β	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
0.5	2.87239412	2.10452625
1.0	2.86719573	2.15100396
1.5	2.95696529	2.15138499
2.0	2.97882478	2.09266052
2.5	2.89095134	1.93238464
-0.5	3.07163719	2.14170722
-1.0	3.09705684	2.10295568
-1.5	3.29234736	1.93498371
-2.0	3.46335088	1.75473028
-2.5	3.58664467	1.92066183

Table 4.12: Simulation for the static optimality for the constrained problem (4.6) with respect to different values of β.

Constant elasticity parameter β	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
0.5	2.27878837	4.68378032
1.0	2.27102544	6.02839513
1.5	2.25863679	6.62607662
2.0	2.03398082	7.57049577
2.5	2.44191787	9.35754574
-0.5	2.03462623	4.20363618
-1.0	1.95745851	4.18232359
-1.5	1.62370682	3.27018855
-2.0	1.61124399	3.94882739
-2.5	1.61752095	3.09954539

Table 4.13: Simulation for the dynamic optimality for the constrained problem (4.6) with respect to different values of β.

(Note the related parameters are given by $\alpha = 1.5$, $\mu = 0.42$, $\sigma = 0.5$, $\delta = 0.24$, $x_0 = 2$, $r = 0.3$, $T = 1$, and $t_0 = 0$, and the sample size is 500.)

problem, the dynamically optimal control drives the wealth strictly below γ for $t \in [t_0, T]$ and achieves $X_T^d = \gamma$ at maturity (cf. Corollary 7 of [43]). Actually, in Corollary 4.5, the dynamically optimal control will also exhibit the similar behaviour under the CEV model. In Figure 4.4, we simulate both optimal wealth processes. We can clearly see that the dynamically wealth process will stay below γ strictly and converges to γ only when t approaches to the maturity time T. For the static case, X_t^s can move with no restriction, and this observation is consistent with Remark 8 in [43]. Upon Figure 4.4, we may summarise that the dynamically optimal wealth process will approach β at maturity, and this fact will be important to evaluate the performance of both strategies in the following part.

In [43], they prove that the dynamically optimal control will outperforms the statically optimal control for any $\gamma > x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}$. In striving for verifying this feature, we simulate both wealth processes and collect the value of X_T^u at the point of time that is close to the maturity (note we can not obtain the exact value of X_T^d at the maturity as Python can not handle the calculation that includes $1/0$. hence, we can only take the X^u that is close to the maturity T). Observing Table 4.15 and Table 4.16, we can note that the dynamically optimal wealth process has significantly large sample variance $\bar{m}^d$, and this confirms the Remark 10 of [43] in which they prove that the variance of the dynamic wealth process fails to converge and will approach to infinity when t approaches to T. Furthermore, from those tables, the sample variance of the static strategy is much smaller than the dynamic case, which indicates that the static strategy outperforms the dynamic strategy before the maturity time T. However, as we discussed above, the dynamic wealth process will be γ at the maturity almost surely, which makes the variance of the terminal wealth equal to 0 in the dynamic cases. Hence, this leads to the conclusion that the static strategy will always outperform the dynamic strategy no matter the value of γ before the maturity while at the maturity the dynamic case will outperform the static case. It should be mentioned that the conclusions we made above are also valid when $\beta > 0$, and we will omit the related details here.

Value of γ	Sample mean $\bar{\mu}^s$	Sample variance $\bar{m}^s$
1.5	1.49551189	0.20610046
2.0	1.96235656	4.58959539
2.5	2.36699180	15.5058445
3.0	2.43396702	58.2153450
3.5	3.92593882	103.569144

Table 4.15: Simulation for the static optimality for the constrained problem (4.7) with respect to different values of γ when $\beta = -0.5$.

Value of γ	Sample mean $\bar{\mu}^d$	Sample variance $\bar{m}^d$
1.5	-112.509967	$3.4138\times e^6$
2.0	1812.66101	$2.4539\times e^8$
2.5	-267103.793	$1.2698\times e^{13}$
3.0	-532276.116	$5.0491\times e^{13}$
3.5	860653.124	$6.2851\times e^{13}$

Table 4.16: Simulation for the dynamic optimality for the constrained problem (4.7) with respect to different values of γ when $\beta = -0.5$.

(Note the related parameters are given by $\mu = 0.42$, $\sigma = 0.5$, $\delta = 0.24$, $x_0 = 2$, $r = 0.3$, $T = 1$, and $t_0 = 0$, and the sample size is 500.)

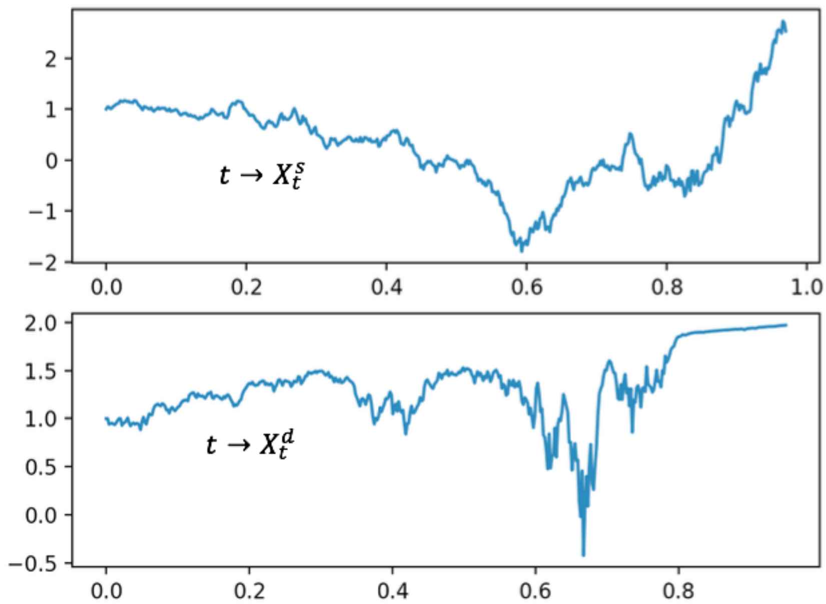

Figure 4.4: This is the simulation of the statically optimal wealth process X_t^s and dynamically optimal wealth process X_t^d when $\beta = -1$, $\delta = 0.24$, $t_0 = 0$, $T = 1$, $x_0 = 2$, $c = 1.01$, $\mu = 0.42$, $r = 0.3$, $\sigma = 0.5$, $\gamma = 2$, and $s_0 = 0.8$.

Chapter 5

Optimal Mean-Variance Portfolio Selection under Partial Information

5.1 Introduction

In financial markets, the investor, especially the individual investor, can only access limited information to make decisions. For the individual investor, it may not be possible for them to fully analyse and access financial statements and other such materials. Most of the time, they can only observe the stock price up to the current time. Observing this fact motivates us to further explore the application portfolio theory in such a situation. Hence, in this chapter, we are going to consider another scenario where the only information available for the investor is the stock price. This condition leads to the portfolio selection under partial information and it has financial meaning. Like the previous chapters, the target of this chapter is to solve the nonlinear mean-variance portfolio selection problem and obtain both time inconsistent and time-consistent solutions.

There are some previous studies that consider the portfolio selection under partial information. Studies [28] and [29] consider the optimal consumption and investment problem under the partial information by focusing on the utility function. In [29], Lakner finds the optimal strategies for the logarithmic utility function and the power utility function by martingale method and notes that the optimal strategy of the logarithmic utility function can be derived from the corresponding strategy under the full information by replacing the drift rate by the conditional expectation of the drift under partial information, while the optimal strategy of the power utility function cannot be derived by this way (cf. p86-88 of [29] for further details). Different from adopting martingale method, the work of [1] applies the dynamic programming approach, i.e. the Hamilton-Jacobi-Bellman equation, to obtain the optimal solution of the corresponding utility function under the constraint of short-selling and borrowing prohibition. It should be noted that the utility analysis does not give mean-variance efficient strategies. For partial information in the mean-variance portfolio selection, one of the key difficulties is that the filtering and optimisation are hard to separate, however, a significant achievement is studied in [56], in which they come up with the separation principle that is specifically applicable for the mean-variance portfolio problem. As stated in [56], the separation principle enables us to replace the unknown drift rate with its optimal conditional expectation in the wealth process and then solve the optimal problem with respect to the wealth process as the full information case. The separation principle has been widely applied in other works such as [55] and [41]. The work of [41] considers the case when the market consists of multiple assets under partial information and handles the optimal control from the HJB system of a linear problem by solving the corresponding Riccati equation as well as building efficient frontier. In this chapter we will follow the idea of [41] to solve the HJB equation and then derive both time-inconsistent and time-consistent solution for the nonlinear problem.

Consider a financial market that offers a riskless bond and a risky stock, for which the drift rate of the risky asset is unknown. For the investor, the only available information is the stock price up to the current time, which forms the partial information problem. By using the separation principle, we derive the innovation process which enables us to obtain the wealth process that is adapted to the same filtration as the stock price process. Furthermore, the investor aims to construct a self-financing portfolio and uses the variance as the risk measure and variance naturally bring the quadratic nonlinearity into this problem. To handle this issue, we firstly employ Lagrange multiplier to reduce the nonlinear problem into a set of linear problems, which can be solved by HJB equation. Under the partial information condition, the HJB equation leads to a complicated second-order partial differential equation. Different from applying the Legendre transform in the previous work, the work of [41] describes a method which involves solving a Riccati-type ODE gives a simpler solution, and we will follow this idea. Moreover, as we have done in the previous chapter, the solution of the HJB equation only gives the candidate solution and its optimality will be proved by the verification theorem described in [5]. By solving the HJB equation, we receive the optimal control that depends on the initial status (t_0, x_0), and this kind of optimal control is named as static optimality in [43]. Furthermore, from the static optimality, we derive the time-consistent control that only depends on the current status (t, x), which is firstly introduced in [43].

Similarly, in this chapter, we also consider the other two constrained cases where we constrain the size of the expectation/variance of the wealth process respectively.

5.2 Formulation of the problem

Consider a scene where the the only available information the investor can access is the stock price up to the current time. This naturally leads to the partial information problem. In this case, the stock price follows:

$$dS_t = \mu_t S_t dt + \sigma S_t dW_t \tag{5.1}$$

with $\sigma > 0$ and μ_t represents the appreciation rate process for $t \in [t_0, T]$. For this risky asset, we set $S_{t_0} = s_0$ for some constants $s_0 > 0$, and the standard Brownian motion W is completely defined on $(\Omega, \mathbb{F}, \mathbb{P})$. For the appreciation rate process μ_t in (5.1), we follow the setting in [4] (also cf. [41]) and consider that the appreciation rate process μ_t is a Gaussian process that follows:

$$d\mu_t = \theta \mu_t dt + \xi d\tilde{W}_t \tag{5.2}$$

in which θ and ξ are given constants and $\tilde{W}_t$ is a standard Brownian motion defined on $(\Omega, \mathbb{F}, \mathbb{P})$. Under partial information constraint, the investor can only observe the stock price, which indicates that the only information for the investor is the stock price up to t. Hence, we define:

$$\mathbb{G}_t = \sigma(S(\tau); \tau \leq t). \tag{5.3}$$

It should be pointed out that μ_t in (5.2) can be a general process that does not need to be $\mathbb{G}_t-$adapted or even $\mathbb{F}_t-$adapted and hence, for the investor, he can not observe the value of μ_t [56]. Hence, we further assume that $\tilde{W}_t$ is independent of W_t. And in this chapter, we will only consider the case when μ_t is unknown and assume σ is given and fixed.

Besides the risky stock, there is a riskless bond available for the investor, which is described by:

$$dB_t = rB_t dt \tag{5.4}$$

with initial value $B_{t_0} = b_0$, where $b_0 > 0$ and the riskless interest rate $r \in \mathbb{R}$ are constants.

This investor with initial wealth $x_0 > 0$ aims to construct a self-financing portfolio by investing these two assets and expects to maximise his wealth up to the maturity time $T > 0$. In [5], they describe that the wealth of this self-financing portfolio will follow:

$$dX_t^u = (r + (\mu_t - r)u_t)X_t^u dt + \sigma u_t X_t^u dW_t \tag{5.5}$$

in which u_t represents the percentage of wealth invested in the risky asset. To obtain the optimal control which is also $\mathbb{G}_t-$measurable, we apply the separation principle studied in [56] to convert (5.5) into:

$$dX_t^u = (r + (\hat{\mu}_t - r)u_t)X_t^u dt + \sigma u_t X_t^u dZ_t. \tag{5.6}$$

in which $\hat{\mu}_t := \mathrm{E}(\mu_t|\mathbb{G}_t)$ is the optimal estimator for μ_t and the process Z_t is called the innovation process and it is defined by:

$$dZ_t = \frac{1}{\sigma} d\log S_t - \frac{1}{\sigma}(\hat{\mu}_t - \frac{1}{2}\sigma^2)dt \tag{5.7}$$

where:

$$d\log S_t = (\mu_t - \frac{1}{2}\sigma^2)dt + \sigma dW_t. \tag{5.8}$$

Upon (5.2) and (5.8), we can apply Kalman-Bucy menthod described in Theorem 3 in [35], which enables us to obtain the process for the optimal estimator μ_t. Hence, there is:

$$\begin{aligned} d\hat{\mu}_t &= \theta\hat{\mu}_t dt + \frac{1}{\sigma^2}\big(\sigma\xi + \gamma_t\big)\big[d\log S_t - (\hat{\mu}_t - \frac{1}{2}\sigma^2)dt\big] \\ &= \theta\hat{\mu}_t dt + \big(\xi + \frac{\gamma_t}{\sigma}\big)dZ_t \end{aligned} \tag{5.9}$$

for $t \in [t_0, T]$ and in (5.9), term γ_t represents the error of estimation and it is described by:

$$d\gamma_t = 2\theta\gamma_t + \xi^2 - \frac{\big(\sigma\xi + \gamma_t\big)^2}{\sigma^2} \tag{5.10}$$

for $t \in [t_0, T]$. Under those settings, we are facing the optimal filtering problem in the portfolio selection.

For the investor, the main problem is to consider:

$$V(t,x) := \sup_u \mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) \tag{5.11}$$

For the admissible control $u \in U$, it must meet the admissibility condition

$$0 < \max_{t\le\tau\le T} \mathrm{E}_{t,x}\left[X_\tau^u - \frac{\lambda}{2}\right]^2 < \infty \tag{5.12}$$

where $\lambda > 0$ is any given constant. Furthermore, for completeness, we define that when $X_t = 0$, $u(t,0) := u(t,0)\cdot 0$ as $u(t,x)$ may not well defined on $X_t = 0$. Besides the unconstrained problem (5.11), we also attempt to consider the following two constrained cases:

$$V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u)\le\alpha} \mathrm{E}_{t,x}(X_T^u) \tag{5.13}$$

$$V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u)\ge\beta} \mathrm{Var}_{t,x}(X_T^u) \tag{5.14}$$

in which $\alpha \in (0,+)$ and $\beta \in (xe^{\mathrm{r(T-t)}},+\infty)$. In [43], it was proved that the solution of (5.11) leads to the solution for both (5.13) and (5.14), which can be verified by applying the method of Lagrange multipliers. In this case, choosing proper values of c in (5.11) will lead to the solution of (5.13) and (5.14). In this chapter, definitions of static optimality and dynamic optimality are consistent with those in [43].

5.3 Solution to the unconstrained problem

In this chapter, we will explain the solution of the constrained problems. The main idea of the proof below follows the idea in [43].

Theorem 5.1. *Consider the optimal problem $V(t,x) = \sup_u[\mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u)]$ in which X^u represents the wealth process and is the solution of the SDE (5.6) with $X_{t_0}^u = x_0$ under the probability measure $\mathbb{P}_{t_0,x_0}$ for $(t_0,x_0)\in[0,T]\times\mathbb{R}$ given and fixed. The related risk coefficient is defined by $\delta = (\hat{\mu}-r)/\sigma$ in which $r\in\mathbb{R}$, $\sigma > 0$, and $\hat{\mu}$ is the optimal estimator defined in (5.9) and $\gamma(t)$ is the error of estimation given by (5.10). Note that we assume that $\delta \neq 0$ and $r \neq 0$. (The cases $\delta = 0$ or $r = 0$ follow by passage to the limit when the non-zero δ or r approaches 0.)*

(A) *The statically optimal control is given by:*

$$u^s_*(t,\hat{\mu}_t,x) = -\frac{1}{x\sigma^2}(x - x_0 e^{r(t-t_0)} + \frac{1}{2c}\frac{(1-e^{\int_{t_0}^T -H(s)ds})}{e^{\int_{t_0}^T -2H(s)ds}(1-e^{\int_{t_0}^T \frac{h(s)^2}{\sigma^2}ds})}e^{-r(T-t)}) \tag{5.15}$$
$$\times \left[(\hat{\mu}_t - r) + \sigma K(t)(E(t) + F(t)\hat{\mu}_t)\right]$$

for $(t,\hat{\mu}_t,x) \in [t_0,T]\times\mathbb{R}\times\mathbb{R}$. *The related parameters are given by:*

$$H(t) = \frac{(\hat{\mu}_t - r)h(t)}{\sigma^2}, \tag{5.16}$$

$$h(t) = [(\hat{\mu}_t - r) + \sigma K(t)(E(t) + F(t)\hat{\mu}_t)] \tag{5.17}$$

in which $F(t)$ *and* $E(t)$ *are continuous solutions of:*

$$\frac{1}{2}F_t + \theta F(t) - \frac{1}{2}K^2(t)F^2(t) - \frac{1}{\sigma^2} - \frac{2K(t)}{\sigma}F(t) = 0 \tag{5.18}$$

and

$$E_t + \theta E(t) - K^2(t)E(t)F(t) + \frac{2r}{\sigma^2} - \frac{2K(t)}{\sigma}E(t) + \frac{2rK(t)}{\sigma}F(t) = 0 \tag{5.19}$$

with

$$K(t) = \xi + \frac{\gamma_t}{\sigma} \tag{5.20}$$

(B) *The dynamically optimal control is given by:*

$$u^d_*(t,\hat{\mu}_t,x) = -\frac{1}{x\sigma^2}\frac{1}{2c}\frac{(1-e^{\int_t^T -H(s)ds})}{e^{\int_t^T -2H(s)ds}(1-e^{\int_t^T \frac{h(s)^2}{\sigma^2}ds})}e^{-r(T-t)} \tag{5.21}$$
$$\times \left[(\hat{\mu}_t - r) + \sigma K(t)(E(t) + F(t)\hat{\mu}_t)\right]$$

for $(t,\hat{\mu}_t,x) \in [t_0,T]\times\mathbb{R}\times\mathbb{R}$. *The rest parameters are given above.*

Proof. In the first part of proof, we will use t and x to replace t_0 and x_0 to simplify the proof, and any admissible control $u \in U$ follows the definition above.

(A): Firstly, we need to handle the quadratic nonlinearity introduced by the variance term. Hence, we assume the condition $\mathrm{E}_{t,x}(X^u_T) = M$, and we can see that:

$$\begin{aligned} V(t,x) &= \sup_u \left\{ \mathrm{E}_{t,x}(X^u_T) - c\,\mathrm{Var}_{t,x}(X^u_T) \right\} \\ &= \sup_{u:\mathrm{E}_{t,x}(X^u_T)=M} \left\{ M - c\,\mathrm{E}_{t,x}[(X^u_T)^2] + cM^2 \right\} \\ &= M + cM^2 - c \inf_{u:\mathrm{E}_{t,x}(X^u_T)=M} \left\{ \mathrm{E}_{t,x}[(X^u_T)^2] \right\} \end{aligned} \tag{5.22}$$

where $M \in \mathbb{R}$. For an optimal strategy, u^*, the corresponding values of M_* and $\mathrm{Var}_{t,x}(X_T^{u^*)})$ form the efficient frontier when $M_* \in [xe^{\mathrm{r(T-t)}}, \infty)$. From (5.22), we can see that solving the following constrained problem will naturally leads to the optimal solution for the unconstrained problem, which is:

$$V_M(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u)=M} \left\{ \mathrm{E}_{t,x}\left[(X_T^u)^2\right]\right\}. \tag{5.23}$$

Applying Lagrange multiplier in (5.23) leads to:

$$L_{t,x}(u,\lambda) = \mathrm{E}_{t,x}[(X_T^u)^2] - \lambda[\mathrm{E}_{t,x}(X_T^u) - M] \tag{5.24}$$

in which $\lambda > 0$. It can be easily seen that the optimal control $u^*(t,x)$ with $\lambda(M,t,x) > 0$ such that $\mathrm{E}_{t,x}(X_T^u) = M$ will lead to:

$$L_{t,x}(u^*,\lambda) = \inf_u L_{t,x}(u,\lambda). \tag{5.25}$$

Hence, achieving the control that minimises (5.24) will gives the desired optimal control for (5.22).

1. From (5.24), we can see that the key step is to handle the following problem:

$$V^\lambda(t,\hat{\mu},x) = \inf_u \mathrm{E}_{t,x}\left[(X_T^u)^2 - \lambda X_T^u \middle| \mu_t = \hat{\mu}, X_t = x\right] \tag{5.26}$$

where $u \in U$ is any admissible control we defined above. To further simplify the following calculation, we see that equation (5.26) is equivalent to:

$$V^\lambda(t,\hat{\mu},x) = \inf_u \mathrm{E}_{t,x}\left[(X_T^u - \frac{\lambda}{2})^2 \middle| \mu_t = \hat{\mu}, X_t = x\right]. \tag{5.27}$$

In the following part, we mainly focus on two things: (I) solving (5.27) by using Hamilton-Jacobi-Bellman equation; (II) proving the optimality of the candidate solution we achieve from the HJB equation by using verification theorem described in [5]. We firstly assume there exists $V^\lambda(t,\hat{\mu},x)$, a smooth enough solution of the following HJB system (in the following part we will see the explicit form of V^λ exists).

2. Recalling the SDE (5.6) and (5.9), we can derive the following HJB equation:

$$\begin{aligned}\inf_u \big[V_t^\lambda + \hat{\mu}\theta V_{\hat{\mu}}^\lambda + \frac{1}{2}(\xi + \frac{\gamma}{\sigma})^2 V_{\hat{\mu}\hat{\mu}}^\lambda + (r + (\hat{\mu} - r)u_t)xV_x^\lambda + \sigma ux(\xi + \frac{\gamma}{\sigma})V_{x\hat{\mu}}^\lambda \\ + \frac{1}{2}\sigma^2 u^2 x^2 V_{xx}^\lambda\big] = 0\end{aligned} \tag{5.28}$$

with the terminal condition:

$$V^{\lambda}(T,\hat{\mu},x)=(x-\frac{\lambda}{2})^2 \tag{5.29}$$

in which $\gamma_t=\gamma$ is the error of estimation of $\hat{\mu}_t$ and we will omit the subscript t in this part.

3. Seeing (5.28) is a quadratic function of u, we can easily see that the optimal control is given by:

$$u=-\frac{(\hat{\mu}-r)V_x^{\lambda}+\sigma(\xi+\frac{\gamma}{\sigma})V_{x\hat{\mu}}^{\lambda}}{x\sigma^2V_{xx}^{\lambda}}. \tag{5.30}$$

To further simplify the calculation, we set that $K=\xi+\gamma/\sigma$. Substituting (5.30) back into (5.28), we can see that:

$$V_t^{\lambda}+\hat{\mu}\theta V_{\hat{\mu}}^{\lambda}+\frac{1}{2}K^2V_{\hat{\mu}\hat{\mu}}^{\lambda}+rxV_x^{\lambda}-\frac{1}{2\sigma^2V_{xx}^{\lambda}}\left[(\hat{\mu}-r)V_x^{\lambda}+\sigma KV_{x\hat{\mu}}^{\lambda}\right]^2=0. \tag{5.31}$$

Equation (5.31) is a nonlinear second-order partial differential equation. Hence, to handle this case, we further assume that:

$$z=x-\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}} \tag{5.32}$$

under which there exists:

$$H(t,\hat{\mu},z)=V^{\lambda}(t,\hat{\mu},z+\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}). \tag{5.33}$$

Differentiating $H(t,\hat{\mu},z)$, we can see that:

$$H_t=V_t^{\lambda}+\frac{\lambda}{2}r\mathrm{e}^{-\mathrm{r(T-t)}}\mathrm{V}_{\mathrm{x}}^{\lambda}, \tag{5.34}$$

and

$$H_z=V_x^{\lambda},\quad H_{zz}=V_{xx}^{\lambda},\quad H_{\hat{\mu}}=V_{\hat{\mu}}^{\lambda},\quad H_{\hat{\mu}\hat{\mu}}=V_{\hat{\mu}\hat{\mu}}^{\lambda},\quad H_{\hat{\mu}z}=V_{\hat{\mu}x}^{\lambda} \tag{5.35}$$

Hence, substituting (5.34) and (5.35) back into (5.31):

$$H_t+\hat{\mu}\theta H_{\hat{\mu}}+\frac{1}{2}K^2H_{\hat{\mu}\hat{\mu}}+rzH_z-\frac{1}{2\sigma^2}H_{zz}\left[(\hat{\mu}-r)\frac{H_z}{H_{zz}}+\sigma K\frac{H_{z\hat{\mu}}}{H_{zz}}\right]^2=0. \tag{5.36}$$

We further set that:

$$H(t,\hat{\mu},x)=f(t,\hat{\mu})z^2 \tag{5.37}$$

with $f(T) = 1$ for all values of $\hat{\mu}$. Differentiating (5.37) with respect to t, $\hat{\mu}$, and z respectively, there is:

$$H_t = f_t z^2,\ H_{\hat{\mu}} = f_{\hat{\mu}} z^2,\ H_{\hat{\mu}\hat{\mu}} = f_{\hat{\mu}\hat{\mu}} z^2,\ H_z = 2zf,\ H_{zz} = 2f,\ H_{\hat{\mu}z} = 2zf_{\hat{\mu}} \tag{5.38}$$

Substituting (5.38) into (5.36), there is:

$$f_t + \hat{\mu}\theta f_{\hat{\mu}} + \frac{1}{2}K^2 f_{\hat{\mu}\hat{\mu}} + 2rf - \frac{f}{\sigma^2}\left[(\hat{\mu} - r) + \sigma K \frac{\partial lnf}{\partial \hat{\mu}}\right]^2 = 0. \tag{5.39}$$

We set that $f(t, \hat{\mu})$ is of the following form:

$$f(t, \hat{\mu}) = \mathrm{e}^{\mathrm{D(t)+E(t)}\hat{\mu}+\frac{1}{2}\mathrm{F(t)}\hat{\mu}^2} \tag{5.40}$$

with $D(T) = 0$, $E(T) = 0$, and $F(T) = 0$. Hence, differentiating (5.40) with respect to t and $\hat{\mu}$ respectively gives:

$$f_t = (D_t + E_t\hat{\mu} + \frac{1}{2}F_t\hat{\mu}^2)f, \tag{5.41}$$

$$f_{\hat{\mu}} = (E(t) + F(t)\hat{\mu})f, \tag{5.42}$$

$$f_{\hat{\mu}\mu} = [F(t) + (E(t) + F(t)\hat{\mu})^2]f, \tag{5.43}$$

and

$$\frac{\partial lnf}{\partial \hat{\mu}} = E(t) + F(t)\hat{\mu}. \tag{5.44}$$

Inserting (5.41)-(5.44) into (5.39) leads to:

$$\begin{aligned} &D_t + E_t\hat{\mu} + \frac{1}{2}F_t\hat{\mu}^2 + \theta E(t)\hat{\mu} + \theta F(t)\hat{\mu}^2 + \frac{1}{2}K^2F(t) - \frac{1}{2}K^2(E(t) + F(t)\hat{\mu})^2 \\ &\quad + 2r - \frac{1}{\sigma^2}(\hat{\mu} - r)^2 - (\frac{2K}{\sigma})(\hat{\mu} - r)(E(t) + F(t)\hat{\mu}) = 0. \end{aligned} \tag{5.45}$$

Re-arranging (5.45) we can see there is a quadratic function of $\hat{\mu}$. Hence, in (5.45), the coefficient of term $\hat{\mu}^2$ is given by:

$$\frac{1}{2}F_t + \theta F(t) - \frac{1}{2}K^2F^2(t) - \frac{1}{\sigma^2} - \frac{2K}{\sigma}F(t) = 0 \tag{5.46}$$

which is an ODE with terminal condition $F(T) = 0$. Also, the coefficient term of $\hat{\mu}$ is given by:

$$E_t + \theta E(t) - K^2E(t)F(t) + \frac{2r}{\sigma^2} - \frac{2K}{\sigma}E(t) + \frac{2rK}{\sigma}F(t) = 0 \tag{5.47}$$

which is an ODE with terminal condition $E(T)=0$. The coefficient term of (5.45) is given by:

$$D_t+\frac{1}{2}K^2F(t)+2r+\frac{1}{2}K^2\mathrm{e}^2(\mathrm{t})-\frac{1}{\sigma^2}\left(\mathrm{r}-\sigma\mathrm{KE(t)}\right)^2=0 \tag{5.48}$$

with $D(T)=0$. The continuous solution of (5.46), (5.47) and (5.48) forms $f(t,\hat{\mu})$. It should be pointed out that (5.40) leads to $V^{\lambda}_{xx}>0$, which confirms our assumption above. Furthermore, from (5.40), (5.37) and (5.33), we can easily see the existence of V^{λ}, which confirms our assumption for the HJB system (5.28) and (5.29).

4. Recalling (5.35) and (5.40) For the candidate optimal control (5.30), there is:

$$\begin{aligned} u(t,\hat{\mu},x) &= -\frac{(\hat{\mu}-r)V^{\lambda}_x+\sigma(\xi+\frac{\gamma}{\sigma})V^{\lambda}_{x\hat{\mu}}}{x\sigma^2V^{\lambda}_{xx}} \\ &= -\frac{(\hat{\mu}-r)H_z+\sigma(\xi+\frac{\gamma}{\sigma})H_{z\hat{\mu}}}{x\sigma^2H_{zz}} \\ &= -\frac{1}{x\sigma^2}(x-\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}})\left[(\hat{\mu}-\mathrm{r})+\sigma\mathrm{K}(\mathrm{E(t)}+\mathrm{F(t)}\hat{\mu})\right]. \end{aligned} \tag{5.49}$$

5. The next step is to prove the optimality of (5.49). Firstly, applying Ito formula to the value function V^{λ}

$$\begin{aligned} V^{\lambda}(T,\hat{\mu}_T,X^u_T) =& V^{\lambda}(t,\hat{\mu},x)+\int_t^T\left[V^{\lambda}_t(t+s,\hat{\mu}_s,X^u_s)+\theta\hat{\mu}_sV^{\lambda}_{\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s)\right. \\ &+(r+(\hat{\mu}_s-r)u_s)X^u_sV^{\lambda}_x(t+s,\hat{\mu}_s,X^u_s)+\frac{1}{2}K(s)^2V^{\lambda}_{\hat{\mu}\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s) \\ &\left.+\sigma K(s)u_sX^u_sV^{\lambda}_{x\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s)+\frac{1}{2}\sigma^2u_s^2X^{u2}_sV^{\lambda}_{xx}(t+s,\hat{\mu}_s,X^u_s)\right]ds \\ &+\int_t^T[K(s)V^{\lambda}_{\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s)+\sigma u_sX^u_sV^{\lambda}_x(t+s,\hat{\mu}_s,X^u_s)]dZ_s \end{aligned} \tag{5.50}$$

in which u is any admissible control we defined above. Since V^{λ} solves the HJB equation (5.28) and (5.29), we can see there is:

$$V^{\lambda}_t+\theta\hat{\mu}_sV^{\lambda}_{\hat{\mu}}+(r+(\hat{\mu}_s-r)u_s)X^u_sV^{\lambda}_x+\frac{1}{2}K(s)^2V^{\lambda}_{\hat{\mu}\hat{\mu}}+\sigma K(s)u_sX^u_sV^{\lambda}_{x\hat{\mu}}+\frac{1}{2}\sigma^2u_s^2X^{u2}_sV^{\lambda}_{xx}\geq 0. \tag{5.51}$$

Re-calling the terminal condition (5.29), there is:

$$(X^u_T-\frac{\lambda}{2})^2\geq V^{\lambda}(t,\hat{\mu},x)+\int_t^T[K(s)V^{\lambda}_{\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s)+\sigma u_sX^u_sV^{\lambda}_x(t+s,\hat{\mu}_s,X^u_s)]dZ_s \tag{5.52}$$

in which there exists a local martingale term $M_t=\int_t^T[K(s)V^{\lambda}_{\hat{\mu}}+\sigma u_sX^u_sV^{\lambda}_x]dZ_s$. For this local martingale, there exists a sequence of stopping time τ_n such that $\tau_n\uparrow T$ as $n\uparrow\infty$, under which for each $t' \in [t,T]$ the stopped process

$$M_{t'\wedge\tau_n} = \int_t^{t'\wedge\tau_n} [K(s)V^{\lambda}_{\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s) + \sigma u_s X^u_s V^{\lambda}_x(t+s,\hat{\mu}_s,X^u_s)]dZ_s \tag{5.53}$$

is a martingale. Hence, re-calling (5.52), we can see that:

$$\begin{aligned} V^{\lambda}(t,\hat{\mu},x) \leq & (X^u_{t'\wedge\tau_n} - \frac{\lambda}{2})^2 \\ & - \int_t^{t'\wedge\tau_n} [K(s)V^{\lambda}_{\hat{\mu}}(t+s,\hat{\mu}_s,X^u_s) + \sigma u_s X^u_s V^{\lambda}_x(t+s,\hat{\mu}_s,X^u_s)]dZ_s. \end{aligned} \tag{5.54}$$

Taking expectation on the both side of (5.54), the martingale term vanishes, which leads to:

$$V^{\lambda}(t,\hat{\mu},x) \leq \mathrm{E}_{t,x}\left[(X^u_{t'\wedge\tau_n} - \frac{\lambda}{2})^2\right]. \tag{5.55}$$

Taking $\lim_{n\uparrow\infty}$ in the right-hand side we could easily see that:

$$\mathrm{E}_{t,x}\left[(X^u_{t'} - \frac{\lambda}{2})^2\right] = \mathrm{E}_{t,x}\left[\lim_{n\uparrow\infty}(X^u_{t'\wedge\tau_n} - \frac{\lambda}{2})^2\right]. \tag{5.56}$$

Recalling the admissibility condition (5.12), we can apply the Fatou lemma and receive:

$$\mathrm{E}_{t,x}\left[\lim_{n\uparrow\infty}(X^u_{t'\wedge\tau_n} - \frac{\lambda}{2})^2\right] \geq \lim_{n\uparrow\infty}\mathrm{E}_{t,x}\left[(X^u_{t'\wedge\tau_n} - \frac{\lambda}{2})^2\right]. \tag{5.57}$$

Hence, upon (5.57), we can conclude that:

$$V^{\lambda}(t,\hat{\mu},x) \leq \mathrm{E}_{t,x}\left[(X^u_{t'} - \frac{\lambda}{2})^2\right] \tag{5.58}$$

holds for all $t' \in [t,T]$. Hence, at the maturity, there exists:

$$V^{\lambda}(t,\hat{\mu},x) \leq \mathrm{E}_{t,x}[(X^u_T - \frac{\lambda}{2})^2 | \hat{\mu}_t = \hat{\mu}, X^u_t = x] \tag{5.59}$$

which holds for any admissible control $u \in U$. Hence, we can conclude that:

$$V^{\lambda}(t,\hat{\mu},x) \leq \inf_u \mathrm{E}_{t,x}[(X^u_T - \frac{\lambda}{2})^2 | \hat{\mu}_t = \hat{\mu}, X^u_t = x]. \tag{5.60}$$

For the reverse inequality, we first claim that the control u^* given by (5.49) is the optimal control for the HJB sytem (5.28)-(5.29). Hence, we have

$$\begin{aligned} & V^{\lambda}_t + \hat{\mu}\theta V^{\lambda}_{\hat{\mu}} + \frac{1}{2}K^2 V^{\lambda}_{\hat{\mu}\hat{\mu}} + (r + (\hat{\mu} - r)u^*_t)X^{u^*}_t V^{\lambda}_x + \sigma u^* X^{u^*}_t K V^{\lambda}_{x\hat{\mu}} \\ & + \frac{1}{2}\sigma^2 u^2 (X^{u^*}_t)^2 V^{\lambda}_{xx} = 0 \end{aligned} \tag{5.61}$$

in which $K = \xi + \gamma/\sigma$. Recalling (5.50), we can easily find that:

$$V^{\lambda}(t,\hat{\mu},x) = (X_T^{u^*} - \frac{\lambda}{2})^2 - \int_t^T [K(s)V_{\hat{\mu}}^{\lambda}(t+s,\hat{\mu}_s,X_s^{u^*}) + \sigma u_s^* X_s^{u^*} V_x^{\lambda}(t+s,\hat{\mu}_s,X_s^{u^*})]dZ_s. \tag{5.62}$$

Recalling (5.52)-(5.60) and taking expectation on both side of (5.62), there is:

$$V^{\lambda}(t,\hat{\mu},x) = \mathrm{E}_{t,x}[(X_T^{u^*} - \frac{\lambda}{2})^2 | \hat{\mu}_t = \hat{\mu}, X_t^{u^*} = x] \tag{5.63}$$

and this naturally leads to the trivial inequality:

$$\inf_u \mathrm{E}_{t,x}[(X_T^u - \frac{\lambda}{2})^2 | \hat{\mu}_t, X_t^u = x] = \hat{\mu}, X_t^u = x] \leq \mathrm{E}_{t,x}[(X_T^{u^*} - \frac{\lambda}{2})^2 | \hat{\mu}_t = \hat{\mu}, X_t^u = x] \tag{5.64}$$

which leads to:

$$\begin{aligned} V^{\lambda}(t,\hat{\mu},x) =& \inf_u \mathrm{E}_{t,x}[(X_T^u - \frac{\lambda}{2})^2 | \hat{\mu}_t = \hat{\mu}, X_t^u = x] \\ & \leq \mathrm{E}_{t,x}[(X_T^{u^*} - \frac{\lambda}{2})^2 | \hat{\mu}_t = \hat{\mu}, X_t^u = x] = V^{\lambda}(t,\hat{\mu},x). \end{aligned} \tag{5.65}$$

Therefore, according to the verification described in [5], we can conclude that the control given by (5.49) is optimal for the HJB system (5.28) and (5.29).

6. For (5.49), we still need to further determine the optimal value of λ^*. To achieve that, substituting (5.49) back into (5.6) gives:

$$\begin{aligned} dX_t^u =& (rX_t^u + (\hat{\mu}_t - r)(-\frac{1}{\sigma^2}(X_t^u - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}})[(\hat{\mu} - \mathrm{r}) + \sigma \mathrm{K(t)}(\mathrm{E(t)} + \mathrm{F(t)}\hat{\mu}))\mathrm{dt} \\ &+ \sigma(-\frac{1}{\sigma^2}(x - \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}})[(\hat{\mu} - \mathrm{r}) + \sigma \mathrm{K(t)}(\mathrm{E(t)} + \mathrm{F(t)}\hat{\mu})])\mathrm{dZ_t}. \end{aligned} \tag{5.66}$$

Taking expectation E_{t_0,x_0} on both sides of (5.66) and re-arranging both sides gives:

$$\begin{aligned} \frac{d\,\mathrm{E}_{t_0,x_0}(X_T^u)}{dt} =& (r - \frac{[(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu})]}{\sigma^2})\mathrm{E}_{t_0,x_0}(X_t^u) \\ &+ \frac{[(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu})]}{\sigma^2}\frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}} \end{aligned} \tag{5.67}$$

which is an ODE with initial value $\mathrm{E}_{t_0,x_0}(X_{t_0}^u) = x_0$. Solving this ODE gives:

$$\mathrm{E}_{t_0,x_0}(X_t^u) = \frac{\lambda}{2}\mathrm{e}^{-\mathrm{r(T-t)}}(1 - \mathrm{e}^{\int_{\mathrm{t_0}}^{\mathrm{t}} -\mathrm{H(s)ds}}) + \mathrm{x_0}\mathrm{e}^{\mathrm{r(t-t_0)}}\mathrm{e}^{\int_{\mathrm{t_0}}^{\mathrm{t}} -\mathrm{H(s)ds}} \tag{5.68}$$

where $H(t) := \frac{(\hat{\mu}-r)h(t)}{\sigma^2}$ and $h(t) = [(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu})]$for $t \in [t_0, T]$. At the maturity time T, there is:

$$\mathrm{E}_{t_0,x_0}(X_T^u) = \frac{\lambda}{2}(1 - \mathrm{e}^{\int_{\mathrm{t_0}}^{\mathrm{T}} -\mathrm{H(s)ds}}) + \mathrm{x_0}\mathrm{e}^{\mathrm{r(T-t_0)}}\mathrm{e}^{\int_{\mathrm{t_0}}^{\mathrm{T}} -\mathrm{H(s)ds}}. \tag{5.69}$$

Under the assumption we made above, $\mathrm{E}_{t,x}(X_T^u) = M$, there is:

$$M = \frac{\lambda}{2}(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds}) + x_0 \mathrm{e}^{r(T-t_0)} \mathrm{e}^{\int_{t_0}^{T} -H(s)ds} \tag{5.70}$$

so that:

$$\lambda = \frac{2\left[M - x_0 \mathrm{e}^{r(T-t_0)} \mathrm{e}^{\int_{t_0}^{T} -H(s)ds}\right]}{1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds}}. \tag{5.71}$$

To further find the optimal value of λ, we need to consider $\mathrm{E}_{t_0,x_0}(X_T^{u2})$. Applying Ito formula to X^2 and taking expectation, we can calculate that:

$$\mathrm{E}_{t_0,x_0}(X_T^{u2}) = (x_0 \mathrm{e}^{r(T-t_0)} - \frac{\lambda}{2})^2 \mathrm{e}^{\int_{t_0}^{T} -2H(s) + \frac{h^2(s)}{\sigma^2} ds} + \frac{\lambda^2}{4} + \lambda(x_0 \mathrm{e}^{r(T-t_0)} - \frac{\lambda}{2}) \mathrm{e}^{\int_{t_0}^{T} -H(s)ds}. \tag{5.72}$$

Hence, substituting (5.71) into (5.72) gives:

$$\begin{aligned} V_M(t_0,x_0) = & \frac{1}{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})^2} \left[\mathrm{e}^{\int_{t_0}^{T} -2H(s) + \frac{h(s)^2}{\sigma^2}} + 1 - 2\mathrm{e}^{\int_{t_0}^{T} -H(s)ds} \right] M^2 \\ & + \frac{1}{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})^2} \left[2x_0 \mathrm{e}^{r(T-t_0)} \mathrm{e}^{\int_{t_0}^{T} -2H(s)ds} (1 - \mathrm{e}^{\int_{t_0}^{T} \frac{h(s)^2}{\sigma^2} ds}) \right] M \\ & + \frac{1}{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})^2} x_0^2 \mathrm{e}^{2r(T-t_0)} \left[\mathrm{e}^{\int_{t_0}^{T} -2H(s) + \frac{h(s)^2}{\sigma^2}} - 1 \right]. \end{aligned} \tag{5.73}$$

Hence, substituting (5.73) back into (5.22), we receive:

$$\begin{aligned} V(t_0,x_0) = & \left[c - \frac{c}{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})^2} \left[\mathrm{e}^{\int_{t_0}^{T} -2H(s) + \frac{h(s)^2}{\sigma^2}} + 1 - 2\mathrm{e}^{\int_{t_0}^{T} -H(s)ds} \right] \right] M^2 \\ & \left[1 - \frac{c}{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})^2} \left[2x_0 \mathrm{e}^{r(T-t_0)} \mathrm{e}^{\int_{t_0}^{T} -2H(s)ds} (1 - \mathrm{e}^{\int_{t_0}^{T} \frac{h(s)^2}{\sigma^2} ds}) \right] \right] M \\ & - \frac{c}{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})^2} x_0^2 \mathrm{e}^{2r(T-t_0)} \left[\mathrm{e}^{\int_{t_0}^{T} -2H(s) + \frac{h(s)^2}{\sigma^2}} - 1 \right]. \end{aligned} \tag{5.74}$$

Upon the quadratic function property, we can see that the optimal value of M_* is achieved at:

$$M_* = x_0 \mathrm{e}^{r(T-t_0)} - \frac{1}{2c} \frac{\left(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds}\right)^2}{\mathrm{e}^{\int_{t_0}^{T} -2H(s)ds} \left(1 - \mathrm{e}^{\int_{t_0}^{T} \frac{h(s)^2}{\sigma^2} ds}\right)} \tag{5.75}$$

which leads to the optimal value of λ_*:

$$\lambda_* = 2x_0 \mathrm{e}^{r(T-t_0)} - \frac{1}{c} \frac{(1 - \mathrm{e}^{\int_{t_0}^{T} -H(s)ds})}{\mathrm{e}^{\int_{t_0}^{T} -2H(s)ds} (1 - \mathrm{e}^{\int_{t_0}^{T} \frac{h(s)^2}{\sigma^2} ds})}. \tag{5.76}$$

Hence, this naturally lead to the optimal control, which is given by:

$$u_*^s(t,\hat{\mu},x) = -\frac{1}{x\sigma^2}(x - x_0 e^{r(t-t_0)} + \frac{1}{2c}\frac{(1-e^{\int_{t_0}^T -H(s)ds})}{e^{\int_{t_0}^T -2H(s)ds}(1-e^{\int_{t_0}^T \frac{h(s)^2}{\sigma^2}ds})}e^{-r(T-t)}) \\ \times \left[(\hat{\mu}-r) + \sigma K(t)(E(t)+F(t)\hat{\mu})\right] \tag{5.77}$$

and this confirms (5.15).

(B): In the following part, we are going to consider the dynamically optimal control. As we claim that the dynamically optimal control is equivalent to the statically optimal control with the same initial state (t,x), replacing x_0 and t_0 by x and t in (5.77) gives the candidate dynamically optimal control:

$$u_*^d(t,\hat{\mu},x) = -\frac{1}{x\sigma^2}\frac{1}{2c}\frac{(1-e^{\int_t^T -H(s)ds})}{e^{\int_t^T -2H(s)ds}(1-e^{\int_t^T \frac{h(s)^2}{\sigma^2}ds})}e^{-r(T-t)} \\ \times \left[(\hat{\mu}-r) + \sigma K(t)(E(t)+F(t)\hat{\mu})\right] \tag{5.78}$$

and $u^d(T,\hat{\mu},x) = \lim_{t\uparrow T} u^d(t,\hat{\mu},x)$. Following the idea of [43] and recalling the definition of dynamic optimality, we set there is a control w such that $u_*^d(t_0,x_0) = w(t_0,x_0)$ and $w(t_0,x_0) = u_*^s(t_0,x_0)$. For any admissible control such that $v(t_0,x_0) \neq u_*^d(t_0,x_0)$, we claim the following relationship must hold:

$$V_w(t_0,x_0) = \mathrm{E}_{t_0,x_0}(X_T^w) - c\,\mathrm{Var}_{t_0,x_0}(X_T^w) > \mathrm{E}_{t_0,x_0}(X_T^v) - c\,\mathrm{Var}_{t_0,x_0}(X_T^v) = V_v(t_0,x_0) \tag{5.79}$$

for any choice of $(t_0,x_0) \in [0,T]\times\mathbb{R}$.

7. To verify (5.79), we firstly consider the case when $M_* \neq M_v$. Since (5.74) is a

quadratic function and the value of M_* is uniquely determined, we can see that:

$$\begin{aligned} V_w(t_0,x_0) =& \Big[c - \frac{c}{(1-e^{\int_{t_0}^T -H(s)ds})^2}\big[e^{\int_{t_0}^T -2H(s)+\frac{h(s)^2}{\sigma^2}} + 1 - 2e^{\int_{t_0}^T -H(s)ds}\big]\Big] M_*^2 \\ & \Big[1 - \frac{c}{(1-e^{\int_{t_0}^T -H(s)ds})^2}\big[2x_0 e^{r(T-t_0)} e^{\int_{t_0}^T -2H(s)ds}(1-e^{\int_{t_0}^T \frac{h(s)^2}{\sigma^2}ds})\big]\Big] M_* \\ & - \frac{c}{(1-e^{\int_{t_0}^T -H(s)ds})^2} x_0^2 e^{2r(T-t_0)}\Big[e^{\int_{t_0}^T -2H(s)+\frac{h(s)^2}{\sigma^2}} - 1\Big] \\ & > \Big[c - \frac{c}{(1-e^{\int_{t_0}^T -H(s)ds})^2}\big[e^{\int_{t_0}^T -2H(s)+\frac{h(s)^2}{\sigma^2}} + 1 - 2e^{\int_{t_0}^T -H(s)ds}\big]\Big] M_v^2 \\ & \Big[1 - \frac{c}{(1-e^{\int_{t_0}^T -H(s)ds})^2}\big[2x_0 e^{r(T-t_0)} e^{\int_{t_0}^T -2H(s)ds}(1-e^{\int_{t_0}^T \frac{h(s)^2}{\sigma^2}ds})\big]\Big] M_v \\ & - \frac{c}{(1-e^{\int_{t_0}^T -H(s)ds})^2} x_0^2 e^{2r(T-t_0)}\Big[e^{\int_{t_0}^T -2H(s)+\frac{h(s)^2}{\sigma^2}} - 1\Big] = V_v(t_0,x_0) \end{aligned} \tag{5.80}$$

which confirms (5.79) when $M_* \neq M_v$.

We further consider the case when $M_* = M_v$, we claim that:

$$\begin{aligned} V_v^{\lambda_*}(t_0,\hat{\mu}_0,x_0) :=& E_{t_0,x_0}\big[(X_T^v - \frac{\lambda_*}{2})^2 | \hat{\mu}_{t_0} = \hat{\mu}_0, X_{t_0}^v = x_0\big] \\ & > E_{t_0,x_0}\big[(X_T^w - \frac{\lambda_*}{2})^2 | \hat{\mu}_{t_0} = \hat{\mu}_0, X_{t_0}^w = x_0\big] =: V^{\lambda_*}(t_0,\hat{\mu}_0,x_0) \end{aligned} \tag{5.81}$$

in which λ_* is given by (5.76). Recalling the terminal condition of HJB equation (5.29) and applying Ito formula, there is:

$$\begin{aligned} (X_T^v - \frac{\lambda_*}{2}X_T^v)^2 =& V^\lambda(t_0,\hat{\mu}_0,x_0) + \int_{t_0}^T \big[V_t^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \theta\hat{\mu}_s V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) \\ & + (r + (\hat{\mu}_s - r)v_s)X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \frac{1}{2}K(s)^2 V_{\hat{\mu}\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) \\ & + \sigma K(s) v_s X_s^v V_{x\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \frac{1}{2}\sigma^2 v_s^2 X_s^{v2} V_{xx}^{\lambda_*}(s,\hat{\mu}_s,X_s^v)\big]ds \\ & + \int_{t_0}^T [K(s)V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \sigma u_s X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v)]dZ_s \end{aligned} \tag{5.82}$$

in which the ingredient term:

$$\begin{aligned} A_T =& \int_{t_0}^T \big[V_t^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \theta\hat{\mu}_s V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) \\ & + (r + (\hat{\mu}_s - r)v_s)X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \frac{1}{2}K(s)^2 V_{\hat{\mu}\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) \\ & + \sigma K(s) v_s X_s^v V_{x\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \frac{1}{2}\sigma^2 v_s^2 X_s^{v2} V_{xx}^{\lambda_*}(s,\hat{\mu}_s,X_s^v)\big]ds \end{aligned} \tag{5.83}$$

with $\lambda = \lambda_*$ is non-negative because of the HJB equation (5.28). Taking expectation on the both sides of (5.82) leads to:

$$\begin{aligned}
\mathrm{E}_{t_0,x_0}[(X_T^v - \frac{\lambda_*}{2}X_T^v)^2] =& V^\lambda(t_0,\hat{\mu}_0,x_0) + \mathrm{E}_{t_0,x_0}\int_{t_0}^{T} \big[V_t^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \theta\hat{\mu}_s V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) \\
&+ (r+(\hat{\mu}_s - r)v_s)X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \frac{1}{2}K(s)^2 V_{\hat{\mu}\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) \\
&+ \sigma K(s) v_s X_s^v V_{x\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \frac{1}{2}\sigma^2 v_s^2 X_s^{v2} V_{xx}^{\lambda_*}(s,\hat{\mu}_s,X_s^v)\big]\,ds \\
&+ \mathrm{E}_{t_0,x_0}\int_{t_0}^{T}[K(s)V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \sigma v_s X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v)]dZ_s.
\end{aligned} \tag{5.84}$$

Since we have known that $v(t_0,\hat{\mu}_0,x_0) \neq w(t_0,\hat{\mu}_0,x_0)$, we can further define a region $R_\varepsilon := [t_0,t_0+\varepsilon]\times[\hat{\mu}_0-\varepsilon,\hat{\mu}_0+\varepsilon]\times[x_0-\varepsilon,x_0+\varepsilon]$ for some $\varepsilon > 0$ small enough such that $t_0+\varepsilon \leq T$. Upon the continuity of v and w, there is $v(z,\hat{\mu},x) \neq w(z,\hat{\mu},x)$ for any choice of $(z,\hat{\mu},x) \in R_\varepsilon$. Moreover, from (5.28), a quadratic function of u, we can see that $w(t,\hat{\mu},x)$ is the unique minimum point with $\lambda = \lambda_*$ evaluated at each set of $(t,s,x) \in [0,T]\times\mathbb{R}\times\mathbb{R}$. Hence, we can see that the value of ε can be chosen small enough to meet:

$$\begin{aligned}
&V_t^{\lambda_*}(s,\hat{\mu}_s,X_s^u) + \theta\hat{\mu}_s V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^u) + (r+(\hat{\mu}_s-r)u_s)X_s^u V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^u) \\
&+ \frac{1}{2}K(s)^2 V_{\hat{\mu}\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^u) + \sigma K(s)u_s X_s^u V_{x\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^u) + \frac{1}{2}\sigma^2 u_s^2 X_s^{u2} V_{xx}^{\lambda_*}(s,\hat{\mu}_s,X_s^u) \geq \beta > 0
\end{aligned} \tag{5.85}$$

where β is a constant given and fixed and $(t,\hat{\mu},x) \in R_\varepsilon$. Hence, setting $\tau_\varepsilon = \inf\{z \in [t_0,t_0+\varepsilon] | (z,\hat{\mu}_z,X_z^v) \notin R_\varepsilon\}$, we can see that:

$$\begin{aligned}
V_v^{\lambda_*}(t_0,\hat{\mu}_0,x_0) \geq& V^\lambda(t_0,\hat{\mu}_0,x_0) + \beta(\tau_\varepsilon - t_0) \\
&+ \mathrm{E}_{t_0,x_0}\int_{t_0}^{\tau_\varepsilon}[K(s)V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \sigma v_s X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v)]dZ_s.
\end{aligned} \tag{5.86}$$

In the previous part that $M_t = \int_{t_0}^{\tau_\varepsilon}[K(s)V_{\hat{\mu}}^{\lambda_*}(s,\hat{\mu}_s,X_s^v) + \sigma v_s X_s^v V_x^{\lambda_*}(s,\hat{\mu}_s,X_s^v)]dZ_s$ is a local martingale for $t \in [t_0,T]$. Hence, there exists a sequence of stopping time τ_n such that $\tau_n \uparrow T$ as $n \uparrow \infty$. Then, in this case the stopped process $M_{\tau_\varepsilon\wedge\tau_n}$ is a martingale. Hence, inequality (5.86) leads to:

$$V_v^{\lambda_*}(t_0,\hat{\mu}_0,x_0) \geq V^\lambda(t_0,\hat{\mu}_0,x_0) + \beta(\tau_\varepsilon - t_0) > V^\lambda(t_0,\hat{\mu}_0,x_0) \tag{5.87}$$

which verifies (5.81). Recalling (5.24), we have:

$$V^{\lambda_*}(t_0,x_0) = \mathrm{E}_{t_0,x_0}[X_T^{w2}] - \lambda_* M_* < \mathrm{E}_{t_0,x_0}[X_T^{v2}] - \lambda_* M_v. \tag{5.88}$$

Hence, according to equation (5.22), there exists:

$$M_* + cM_*^2 - \mathrm{E}_{t_0,x_0}[X_T^{w2}] > M_v + cM_v^2 - \mathrm{E}_{t_0,x_0}[X_T^{v2}] \tag{5.89}$$

as $M_* = M_v$. Inequality (5.89) confirms the statement that $V_w(t_0,x_0) > V_v(t_0,x_0)$. Hence, we conclude that $u_*^d = w$ is the dynamically optimal control as claimed. □

5.4 Solution to the constrained problems

In the previous chapter, we have extended the conclusion from the unconstrained case to the constrained case by choosing the proper value of the Lagrange multiplier. Similar work can be done under in the partial information. Since the following proof will be consistent with the work exhibited in [43] (see the proof of Corollary 5 and Corollary 7 for further details), we will skip the details of the proof and focus on the core part.

Rearranging the constrained problem (5.13), we can see that:

$$\begin{aligned} L_{t,x}^1(u,c) &= \mathrm{E}_{t,x}(X_T^u) - c[\mathrm{Var}_{t,x}(X_T^u) - \alpha] \\ &= \mathrm{E}_{t,x}(X_T^u) - c\,\mathrm{Var}_{t,x}(X_T^u) + c\alpha. \end{aligned} \tag{5.90}$$

In (5.90), we can see that the optimal control given by Theorem 5.1 meeting $\mathrm{Var}_{t_0,x_0}(X_T^u) = \alpha$ is the optimal control that maximises (5.13). Hence, recalling (5.69), (5.72), and (5.76), we see that the variance is given by:

$$\mathrm{Var}_{t_0,x_0}(X_T^u) = \frac{1}{4c^2}\frac{\left(1 - \mathrm{e}^{\int_{t_0}^{T} -\mathrm{H(s)ds}}\right)^2}{\left(\mathrm{e}^{\int_{t_0}^{T} -2\mathrm{H(s)ds}} - \mathrm{e}^{\int_{t_0}^{T} -2\mathrm{H(s)} + \frac{\mathrm{h}^2(\mathrm{s})}{\sigma^2}\mathrm{ds}}\right)}. \tag{5.91}$$

Setting (5.91) equal to α, there is:

$$c = \frac{1}{2\sqrt{\alpha}}\frac{1 - \mathrm{e}^{\int_{t_0}^{T} -\mathrm{H(s)ds}}}{\sqrt{\mathrm{e}^{\int_{t_0}^{T} -2\mathrm{H(s)ds}} - \mathrm{e}^{\int_{t_0}^{T} -2\mathrm{H(s)} + \frac{\mathrm{h}^2(\mathrm{s})}{\sigma^2}\mathrm{ds}}}}. \tag{5.92}$$

Substituting (5.92) into Theorem 5.1 we receive the following Corollary.

Corollary 5.2. *Consider the optimal problem $V_1(t,x) = \sup_{u:\mathrm{Var}_{t,x}(X_T^u)\leq\alpha} \mathrm{E}_{t,x}(X_T^u)$ in which X^u represents the wealth process and is the solution of the SDE (5.6) with $X_{t_0}^u = x_0$ under the probability measure $\mathbb{P}_{t_0,x_0}$ for $(t_0,x_0) \in [0,T] \times \mathbb{R}$ given and fixed. The related risk coefficient is defined by $\delta = (\hat{\mu} - r)/\sigma$ in which $r \in \mathbb{R}$, $\sigma > 0$, and $\hat{\mu}$ is the optimal estimator* *defined in (5.9) and $\gamma(t)$ is the error of estimation given by (5.10). Note that we assume that $\delta \neq 0$ and $r \neq 0$. (The cases $\delta = 0$ or $r = 0$ follow by passage to the limit when the non-zero δ or r approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,\hat{\mu},x) = -\frac{1}{x\sigma^2}\left(x - x_0 \mathrm{e}^{\mathrm{r(t-t_0)}} + \sqrt{\alpha}\frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{\sqrt{\mathrm{e}^{\int_{t_0}^{T} -2H(s)ds} - \mathrm{e}^{\int_{t_0}^{T} -2H(s)+\frac{h^2(s)}{\sigma^2}ds}}}\right) \times \left[(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu})\right] \tag{5.93}$$

for $(t,\hat{\mu}_t,x) \in [t_0,T] \times \mathbb{R} \times \mathbb{R}$. The related parameters are given by:

$$H(t) = \frac{(\hat{\mu} - r)h(t)}{\sigma^2}, \tag{5.94}$$

and

$$h(t) = [(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu})] \tag{5.95}$$

in which $F(t)$ and $E(t)$ are continuous solutions of:

$$\frac{1}{2}F_t + \theta F(t) - \frac{1}{2}K(t)^2 F^2(t) - \frac{1}{\sigma^2} - \frac{2K(t)}{\sigma}F(t) = 0 \tag{5.96}$$

and

$$E_t + \theta E(t) - K(t)^2 E(t)F(t) + \frac{2r}{\sigma^2} - \frac{2K(t)}{\sigma}E(t) + \frac{2rK(t)}{\sigma}F(t) = 0 \tag{5.97}$$

with

$$K(t) = \xi + \frac{\gamma_t}{\sigma}. \tag{5.98}$$

(B) *The dynamically optimal control is given by:*

$$u_*^d(t,\hat{\mu},x) = -\frac{1}{x\sigma^2}\left(\sqrt{\alpha}\frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{\sqrt{\mathrm{e}^{\int_{t}^{T} -2H(s)ds} - \mathrm{e}^{\int_{t}^{T} -2H(s)+\frac{h^2(s)}{\sigma^2}ds}}}\right) \times \left[(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu})\right] \tag{5.99}$$

for $(t,\hat{\mu}_t,x) \in [t_0,T] \times \mathbb{R} \times \mathbb{R}$. The rest parameters are given above.

For the other constrained problem (5.14), applying Lagrangian multiplier in (5.14) yields:

$$L_{t,x}(u,c) = \mathrm{Var}_{t,x}(X_T^u) - c\left[\mathrm{E}_{t,x}(X_T^u) - \beta\right] \tag{5.100}$$

for $c > 0$. Re-arranging equation (5.100) yields:

$$\inf_u \left[\mathrm{Var}_{t,x}(X_T^u) - c\left[\mathrm{E}_{t,x}(X_T^u) - \beta\right]\right) = -c\sup_u \left[\mathrm{E}_{t,x}(X_T^u) - \frac{1}{c}\mathrm{Var}_{t,x}(X_T^u)\right] + c\beta. \qquad (5.101)$$

In (5.101), it can be seen that the optimal control Theorem 5.1 meeting $\mathrm{E}_{t_0,x_0}(X_T^u) = \beta$ will be the optimal control for (5.14). Recalling (5.75), we receive that:

$$\frac{1}{c} = -2(\beta - x_0 e^{r(T-t_0)}) \frac{\left(e^{\int_{t_0}^T -2H(s)ds} - e^{\int_{t_0}^T -2H(s)+\frac{h^2(s)}{\sigma^2}ds}\right)}{(1-e^{\int_{t_0}^T -H(s)ds})^2}. \qquad (5.102)$$

Substituting (5.102) back into (5.15)-(5.21) leads to the following corollary.

Corollary 5.3. *Consider the optimal problem $V_2(t,x) = \inf_{u:\mathrm{E}_{t,x}(X_T^u)\geq\beta} \mathrm{Var}_{t,x}(X_T^u)$ in which X^u represents the wealth process and is the solution of the SDE (5.6) with $X_{t_0}^u = x_0$ under the probability measure $\mathbb{P}_{t_0,x_0}$ for $(t_0,x_0) \in [0,T]\times\mathbb{R}$ given and fixed. The related risk coefficient is defined by $\delta = (\hat{\mu}-r)/\sigma$ in which $r \in \mathbb{R}$, $\sigma > 0$, and $\hat{\mu}$ is the optimal estimator defined in (5.9) and $\gamma(t)$ is the error of estimation given by (5.10). Note that we assume that $\delta \neq 0$ and $r \neq 0$. (The cases $\delta = 0$ or $r = 0$ follow by passage to the limit when the non-zero δ or r approaches 0.)*

(A) *The statically optimal control is given by:*

$$u_*^s(t,\hat{\mu},x) = -\frac{1}{x\sigma^2}(x - x_0 e^{r(t-t_0)} - (\beta - x_0 e^{r(T-t_0)})\frac{e^{-r(T-t)}}{(1-e^{\int_{t_0}^T -H(s)ds})}) \qquad (5.103)$$
$$\times\left[(\hat{\mu}-r) + \sigma K(t)(E(t)+F(t)\hat{\mu})\right]$$

for $(t,\hat{\mu}_t,x) \in [t_0,T]\times\mathbb{R}\times\mathbb{R}$. The related parameters are given by:

$$H(t) = \frac{(\hat{\mu}-r)h(t)}{\sigma^2}, \qquad (5.104)$$

and

$$h(t) = [(\hat{\mu}-r) + \sigma K(t)(E(t)+F(t)\hat{\mu})] \qquad (5.105)$$

in which $F(t)$ and $E(t)$ are continuous solutions of:

$$\frac{1}{2}F_t + \theta F(t) - \frac{1}{2}K(t)^2F^2(t) - \frac{1}{\sigma^2} - \frac{2K(t)}{\sigma}F(t) = 0 \qquad (5.106)$$

and

$$E_t + \theta E(t) - K(t)^2E(t)F(t) + \frac{2r}{\sigma^2} - \frac{2K(t)}{\sigma}E(t) + \frac{2rK(t)}{\sigma}F(t) = 0 \qquad (5.107)$$

with

$$K(t) = \xi + \frac{\gamma_t}{\sigma}. \tag{5.108}$$

(B) *The dynamically optimal control is given by:*

$$\begin{aligned} u_*^d(t,\hat{\mu},x) = & \frac{1}{x\sigma^2}(\beta - x_0 \mathrm{e}^{\mathrm{r(T-t_0)}}) \frac{\mathrm{e}^{-\mathrm{r(T-t)}}}{(1-\mathrm{e}^{\int_{\mathrm{t_0}}^{\mathrm{T}} -\mathrm{H(s)ds}})} \\ & \times \left[(\hat{\mu} - r) + \sigma K(t)(E(t) + F(t)\hat{\mu}) \right] \end{aligned} \tag{5.109}$$

for $(t,\hat{\mu}_t,x) \in [t_0,T] \times \mathbb{R} \times \mathbb{R}$. *The rest parameters are given above.*

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