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THE EXACT NATURAL FREQUENCY SOLUTION OF A FREE AXIAL-BENDING VIBRATION PROBLEM OF A NON-UNIFORM AFG CANTILEVER BEAM WITH A TIP BODY

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Abstract: In this paper we present the solution of exact equations of motion for coupled axial and bending vibrations of a non-uniform axially functionally graded (AFG) cantilever beam with a body of which mass center is eccentrically displaced in axial and transverse direction with respect to the beam's end. The Euler-Bernoulli beam theory is implemented to model behavior of the beam under axial and transverse in-plane vibrations. Based on the paper [1], that is supported by results publish in the paper [2] authors confirm the obtained results of natural frequencies of the AFG cantilever beam, when boundary conditions define the vibration coupling. The modified symbolic-numeric method of initial parameters presented in [3] is implemented in computing natural frequencies of the beam. This method is expanded to solve axial-bending vibration problems, with respect to the one presented in the literature [3] for the problem of the vibration of a cantilever beam. Some minor deviations in the obtained results may be noticed with respect to those obtained in [1], yet all within a tolerance domain due to the computational precision.

Key words: coupled vibrations, boundary conditions, natural frequency, functionally graded materials, modified symbolic-numeric method of initial parameters, cantilever beam

1. Introduction

Elastic beams, as one of the most commonly used structural elements, are omnipresent in engineering constructions. Due to increasing demands in the reduction of structural weights, while thermal or mechanical durability is not compromised, functionally graded materials have often been used, as engineering materials of choice. These materials are also more superior to steel in chemical resistance to aggressive substances and they can be highly attractive in terms of aesthetics demands. One of the main advantages of functionally graded materials to laminated composite materials is the absence of concentrated interlaminar stresses [4]. The reason of this absence is a characteristic of FGMs that changes of material characteristics occur smoothly, in a manner of gradient distribution from one end of a beam to the opposite one. Furthermore, a functionally graded material may be designed to satisfy specific constructional needs. Significant research has been implemented in producing parts of FGMs using additive manufacturing techniques [5,6,7,8]. FGMs are mainly used in those applications where combinations of two extreme properties are required in a single component for example hardness and toughness [9]. We shall not discuss in details material characteristics of functionally graded materials in this paper. Readers are encouraged to look for details in the following literature sources [10,11,12].

In order to avoid natural frequencies, so designers may avoid resonant conditions, researchers have introduced and implemented various models for the modal analysis of AFG beams. Literature offers two different approaches to solving problems of vibration of functionally graded beams. One approach is based on solving the governing partial differential equations of motion, while the second one is based on the discretization of a beam to a set of rigid segments mutually connected by elastic elements. Methods based on discretization of elastic elements shall not be discussed in this paper. For further readings in this topic we may kindly suggest a literature review and the presented approach presented by Nikolić in [1].

Several theories are presented in the literature for modeling the elastic behavior of AFG beams, namely Euler-Bernoulli Theory, Timoshenko Beam Theory and higher order shear deformation theories. The Euler-Bernoulli deformation theory will be implemented in the background of the conducted computational research, as it was implemented in [13], when shooting method was used for solving problems of vibration of a beam made of non-homogeneous material. The presented paper discusses only cases when shear deformation and rotary inertia effect may be neglected in the analysis of beam's dynamical behavior. To compare results obtained using the modified symbolic numeric method of initial parameters with those gained through the numerically efficient rigid element method as in [1]. Temperature flux influences were not included in this paper, yet it can be treated in future research activities.

In this paper we present an implementation and generalization of the modified Symbolic-Numeric method of initial parameters presented in [3] in order to make a solid foundation model for determining natural frequencies of elastic beams. The proposed method does not pose any restrictions on a type of material or a cross-sectional profile of the cantilever beam. Only the case of a cantilever beam with attached eccentric end mass is considered in this paper.

2. Formulation of the problem

The elastic cantilever beam, introduced in Figure 1 is clamped at the left end, while the motion of the right end is unconstrained. Rigid body of mass m_R is attached to the right end of cantilever beam. Its mass center is eccentrically displaced as presented in the Figure 1. We analyze the beam's motion with respect to the inertial coordinate system *Oxyz*.

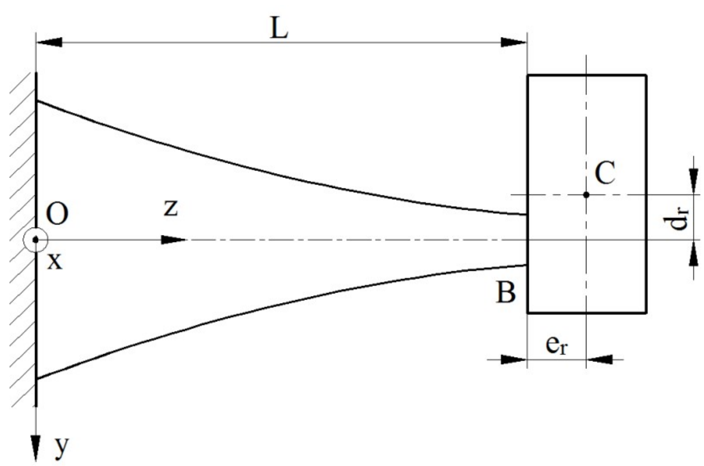

Fig. 1. An undeformed elastic cantilever beam with a tip mass presented schematically as in [1], except the orientation of coordinate system

Governing equations of motion in axial and transverse direction of presented cantilever beam under the assuption of validity of Euler-Bernouli beam theory, read as

$$\frac{\partial}{\partial z}\left[F_t(z,t)\right]-\rho(z)A(z)\frac{\partial^2 w(z,t)}{\partial t^2}=0 \tag{1}$$

$$\frac{\partial}{\partial z}\left[F_a(z,t)\right]-\rho(z)A(z)\frac{\partial^2 u(z,t)}{\partial t^2}=0 \tag{2}$$

After implementing the method of separation of variables on displacements, as presented in [14] , one reads:

$$w(z,t)=W(z)T(t) \tag{3}$$

$$u(z,t)=U(z)T(t) \tag{4}$$

The axial, transverse force and the bending moment [14], using the method of separation of variables are respectively presented analytically by equations

$$F_a(z,t)=E(z)A(z)\frac{dU(z)}{dz}T(t)=F_a(z)T(t) \tag{5}$$

$$F_t(z,t)=\frac{d}{dz}\left(-E(z)I_x(z)\frac{d^2w(z)}{dz^2}\right)T(t)=F_t(z)T(t) \tag{6}$$

$$M_f(z,t)=-E(z)I_x(z)\frac{d^2W(z)}{dz^2}T(t)=M_f(z)T(t) \tag{7}$$

Let us introduce a vector ***X(z)*** and a vector of derivatives of arguments of ***X(z)*** and a matrix ***T(z)*** that connects them in the similarly to the one presented in [3]

$$\boldsymbol{X}(\mathrm{z})=\begin{bmatrix}U(z) & W(z) & W'(z) & F_a(z) & F_t(z) & M_f(z)\end{bmatrix}^T \tag{8}$$

$$\frac{d}{dz}\boldsymbol{X}(z)=\begin{bmatrix}\frac{dU(z)}{dz} & \frac{dW(z)}{dz} & \frac{dW'(z)}{dz} & \frac{dF_a(z)}{dz} & \frac{dF_t(z)}{dz} & \frac{dM_f(z)}{dz}\end{bmatrix}^T \tag{9}$$

$$\boldsymbol{T}(z)=\begin{bmatrix}0 & 0 & 0 & \frac{1}{E(z)A(z)} & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{E(z)I_x(z)}\\ -\omega^2\rho(z)A(z) & 0 & 0 & 0 & 0 & 0\\ 0 & -\omega^2\rho(z)A(z) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \tag{10}$$

Now, we may write a system of linear differential equations in matrix form as

$$\frac{d}{dz}\boldsymbol{X}(z)=\boldsymbol{T}(z)\times\boldsymbol{X}(z) \tag{11}$$

Boundary conditions at the left end of the cantilever beam (*z=0*) shown in Figure 1 read

$$U(0)=0,\ W(0)=0,\ W'(0)=0 \tag{12}$$

Boundary conditions at the right end of the cantilever beam (*z=L*) shown in Figure 1 read

$$-\omega^2 J_{C_r} W'(L)=M_f(L)+F_a(L)d_r+F_t(L)e_r \tag{13}$$

$$m_r\omega^2\left[U(L)+W'(L)d_r\right]=F_a(L) \tag{14}$$

$$m_r\omega^2\left[W(L)+W'(L)e_r\right]=F_t(L) \tag{15}$$

3. Numerical procedure

Similarly to the model presented in [3], based on the linearity of the presented system and the implementation of the method of initial parameters in differential form [15] one may write a solution as a sum of particular solutions as

$$\boldsymbol{X}(z,\omega)=C_1\boldsymbol{X_1}(z,\omega)+C_2\boldsymbol{X_2}(z,\omega)+C_3\boldsymbol{X_3}(z,\omega) \tag{16}$$

Where, C_1, C_2 and C_3 are integration constants, while vectors of particular solutions with respect to unknown natural frequency, as a symbolic variable, are defined as

$$\boldsymbol{X_i}(z,\omega)=\begin{bmatrix}U_i(z,\omega) & W_i(z,\omega) & W_i'(z,\omega) & F_{ai}(z,\omega) & F_{ti}(z,\omega) & M_{fi}(z,\omega)\end{bmatrix}^T \quad i=1,2,3 \tag{17}$$

Presented particular solutions shall satisfy the folowing boundary conditions

$$\boldsymbol{X}_1(0,\omega)=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^T \tag{18}$$

$$\boldsymbol{X}_2(0,\omega)=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}^T \tag{19}$$

$$\boldsymbol{X}_3(0,\omega)=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}^T \tag{20}$$

Constants C_1, C_2 and C_3 are determined form the condition that solution (16) satisfies boundary conditions (13-15). In that manner, let us define expressions $a_{ji}(\omega)$, i,j=1,2,3 based on the boundary conditions (13-15)

$$a_{1i}(\omega)=-F_{ai}(L,\omega)+m_r\omega^2\left(U_i(L,\omega)+W_i'(L,\omega)d_r\right),\ i=1,2,3 \tag{21}$$

$$a_{2i}(\omega)=-F_{ti}(L,\omega)+m_r\omega^2\left(W_i(L,\omega)+W_i'(L,\omega)e_r\right),\ i=1,2,3 \tag{22}$$

$$a_{3i}(\omega)=M_{fi}(L,\omega)+\omega^2 J_{Cr}W_i'(\mathrm{L},\omega)+F_{ai}(\mathrm{L},\omega)d_r+F_{ti}(\mathrm{L},\omega)e_r,\ i=1,2,3 \tag{23}$$

Thus, we obtain a system of linear equations as one may read

$$\begin{bmatrix} a_{11}(\omega) & a_{12}(\omega) & a_{13}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) & a_{23}(\omega) \\ a_{31}(\omega) & a_{32}(\omega) & a_{33}(\omega) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \boldsymbol{0} \tag{24}$$

The latter system of equations will have nontrivial solutions if

$$h(\omega) \equiv \begin{vmatrix} a_{11}(\omega) & a_{12}(\omega) & a_{13}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) & a_{23}(\omega) \\ a_{31}(\omega) & a_{32}(\omega) & a_{33}(\omega) \end{vmatrix} = 0 \tag{25}$$

Using numerical procedures natural frequencies may be obtained by solving the equation (25), as it is explained in [3]. Constants C_1, C_2 and C_3 are not uniquely determined and it is not in the scope of this paper.

4. Numerical Example

The example for numerical calculations is based on the AFG cantilever beam with an eccentrically displaced mass as in Figure 1. The beam's characteristics are taken as in a paper published by Nikolic in [1].

The results presented in Table 1 correspond to those from Table 5 in [1].

Natural frequencies are presented in non-dimensional form computed by the expression

$$\overline{\omega}_i = \omega_i L^2 \sqrt{\frac{\rho_0 A_0}{E_0 I_{x0}}} \tag{26}$$

Natural frequencies of a cantilever beam with a tip mass are determined for material, cross-sectional characteristics, length of a beam and various values of a tip mass as defined in the Case 3 of [1]

$$\begin{gathered} A(z) = A_0(1-0.5z)\,,\; A_0 = 0.0025m^2 \\ I_x(z) = I_0(1-0.5z)\,,\; I_{x0} = 5.20833\times10^{-7}m^4 \\ \rho(z) = \rho_0(1-0.8\cos(\pi z))\,,\; \rho_0 = 7850\frac{kg}{m^3} \\ E(z) = E_0(1-0.2\cos(\pi z))\,,\; E_0 = 2{,}068\cdot10^{11}\frac{N}{m^2} \end{gathered} \tag{27}$$

L=1m.

Results of exact solutions of non-dimensional natural frequencies obtained by the presented method are presented in Table 1. These results correspond to approximate solutions obtained by the discretization of the beam in [1].

m_0	e_x/L	e_y/L	$\overline{\omega_1}$	$\overline{\omega_2}$	$\overline{\omega_3}$
0.5	0	0	1.77924	16.31864	53.67654
		0.05	1.77588	15.63776	44.02430
		0.1	1.76583	13.80094	33.71129
		0.5	1.48856	5.06105	25.07258
	0.05	0	1.68764	14.17096	43.90148
		0.05	1.68463	13.69108	39.59564
		0.1	1.67565	12.43438	33.55737
		0.5	1.42882	5.18285	25.24057
	0.1	0	1.59647	11.93163	36.13141
		0.05	1.59381	11.64142	34.73042
		0.1	1.58586	10.87365	32.04515
		0.5	1.36782	5.25911	25.41614
	0.5	0	1.01764	5.84198	27.58303
		0.05	1.01680	5.82958	27.55965
		0.1	1.01430	5.79308	27.49181
		0.5	0.94204	5.00015	26.28668
1	0	0	1.36739	15.67383	51.89784
		0.05	1.36427	14.51755	36.39552
		0.1	1.35495	11.80404	28.46110
		0.5	1.10859	3.83966	23.80774
	0.05	0	1.28435	13.23746	40.48897
		0.05	1.28163	12.48766	34.28830
		0.1	1.27352	10.73735	28.96219
		0.5	1.05915	3.96814	23.90551
	0.1	0	1.20390	10.69414	32.42530
		0.05	1.20156	10.29798	30.83229
		0.1	1.19458	9.32437	28.37917
		0.5	1.00941	4.06588	24.01268
	0.5	0	0.73539	4.92549	25.73294
		0.05	0.73475	4.91250	25.71144
		0.1	0.73284	4.87441	25.64955
		0.5	0.67813	4.09147	24.66745

Table 1. Calculated non-dimensional natural frequencies for Case 3 of [1]

4.1 The influence of r_0 and m_0 to natural frequencies

When solving system of ODEs (11), we introduce symbolic variables r_0 and m_0 and modify boundary conditions to modify the system (21-23) with the following system of equations

$$a_{1i}\left(\omega,\rho_0\right) = -F_{ai}\left(L,\omega,\rho_0\right) + m_r\omega^2\left(U_i\left(L,\omega,\rho_0\right) + W_i^{'}\left(L,\omega,\rho_0\right)d_r\right) \text{ , } i=1,2,3 \tag{28}$$

$$a_{2i}\left(\omega,\rho_0\right) = -F_{ti}\left(L,\omega,\rho_0\right) + m_r\omega^2\left(W_i\left(L,\omega,\rho_0\right) + W_i^{'}\left(L,\omega,\rho_0\right)e_r\right) \text{ , } i=1,2,3 \tag{29}$$

$$a_{3i}\left(\omega,\rho_0\right)=M_{fi}\left(L,\omega,\rho_0\right)+\omega^2 J_{Cr}W_i^{'}\left(\mathrm{L},\omega,\rho_0\right)+F_{ai}\left(\mathrm{L},\omega,\rho_0\right)d_r+F_{ti}\left(\mathrm{L},\omega,\rho_0\right)e_r \quad i=1,2,3 \tag{30}$$

The frequency equation is then a determinant as a function of two symbolic variables

$$h\left(\omega,\rho_0\right)\equiv\begin{vmatrix} a_{11}\left(\omega,\rho_0\right) & a_{12}\left(\omega,\rho_0\right) & a_{13}\left(\omega,\rho_0\right) \\ a_{21}\left(\omega,\rho_0\right) & a_{22}\left(\omega,\rho_0\right) & a_{23}\left(\omega,\rho_0\right) \\ a_{31}\left(\omega,\rho_0\right) & a_{32}\left(\omega,\rho_0\right) & a_{33}\left(\omega,\rho_0\right) \end{vmatrix}=0 \tag{31}$$

Thus, we obtain a function of two symbolic variables, namely r_0 and ω and using CountourPlot[] command a relation of natural frequency change to the mass density of the beginning constituent material may be observed, as presented in Figure 2.

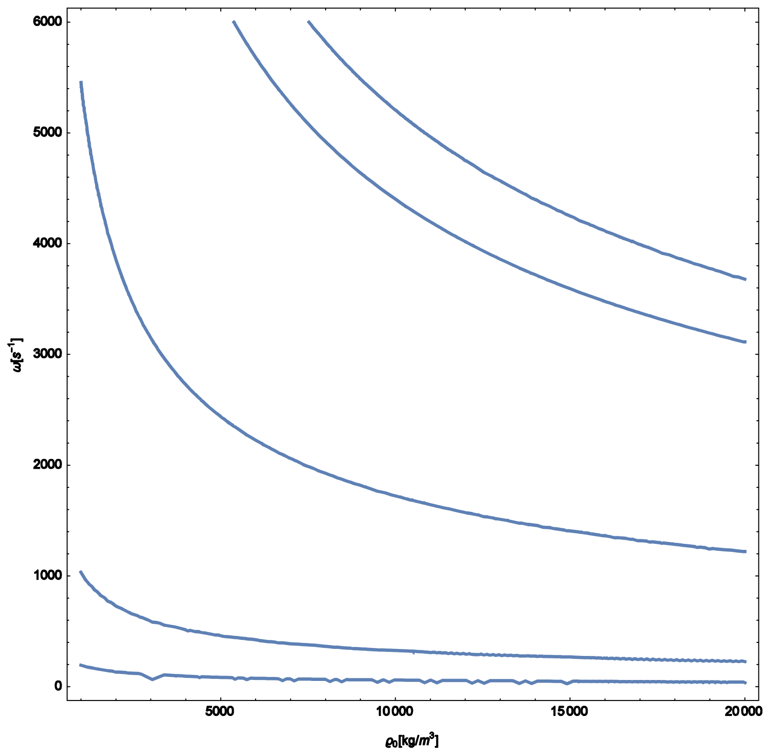

Fig. 2. The graphical correlation ω-r_0

Similarly to the previous case, we introduce parameters $a_{ij}(\omega,m_0)$ as

$$a_{1i}\left(\omega,m_0\right)=-F_{ai}\left(L,\omega,m_0\right)+m_r\omega^2\left(U_i\left(L,\omega,m_0\right)+W_i^{'}\left(L,\omega,m_0\right)d_r\right)\ ,\ i=1,2,3 \tag{32}$$

$$a_{2i}\left(\omega,m_0\right)=-F_{ti}\left(L,\omega,m_0\right)+m_r\omega^2\left(W_i\left(L,\omega,m_0\right)+W_i^{'}\left(L,\omega,m_0\right)e_r\right)\ ,\ i=1,2,3 \tag{33}$$

$$a_{3i}\left(\omega,m_0\right)=M_{fi}\left(L,\omega,m_0\right)+\omega^2 J_{Cr}W_i^{'}\left(\mathrm{L},\omega,m_0\right)+F_{ai}\left(\mathrm{L},\omega,m_0\right)d_r+F_{ti}\left(\mathrm{L},\omega,m_0\right)e_r, \quad i=1,2,3 \tag{34}$$

The frequency equation is then a determinant of two symbolic variables

$$h(\omega,m_0) \equiv \begin{vmatrix} a_{11}(\omega,m_0) & a_{12}(\omega,m_0) & a_{13}(\omega,m_0) \\ a_{21}(\omega,m_0) & a_{22}(\omega,m_0) & a_{23}(\omega,m_0) \\ a_{31}(\omega,m_0) & a_{32}(\omega,m_0) & a_{33}(\omega,m_0) \end{vmatrix} = 0 \tag{35}$$

Once again, we obtain a function of two symbolic variables, but now m_0 and ω. By implementing the CountourPlot[] command a relation of natural frequency change to the mass of a tip body ($m = m_0\rho_0 A_0 L$) may be observed, as presented in Figure 3.

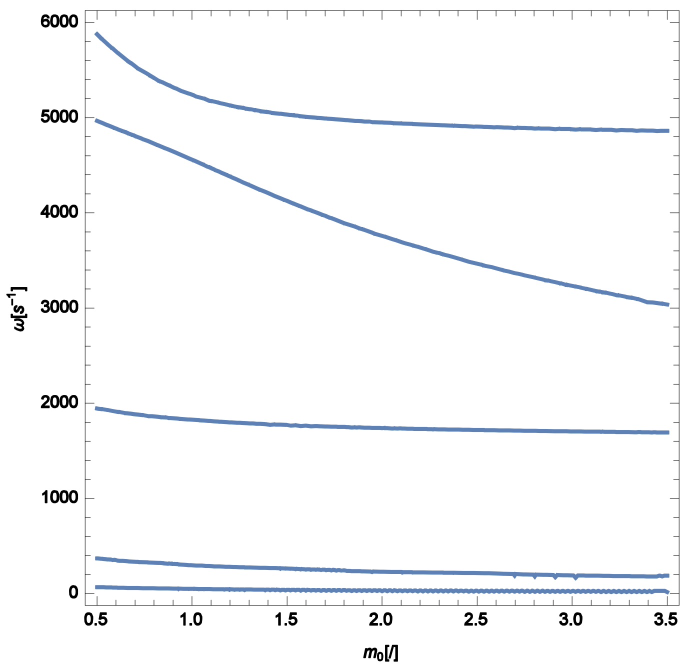

Fig. 3. The graphical correlation $\omega - m_0$

Modern software tools enable us to easily discuss complex analytical problems. Symbolic computations allow us to discuss complex problems, as the case of vibration of AFG cantilever beams with the tip rigid body. Furthermore, parametric plot of a function of two variables give as a chance to discuss the relation between natural frequency of the cantilever beam and its material characteristics. As the results Figures 2 and 3 are obtained. Based on the graphs shown in presented figures, we observe that natural frequencies decrease as parameters x_0 and m_0 increases. This is what we could expect due to the nature of the discussed problem. It is notably to emphasize that the fourth natural frequency of the cantilever beam presented in the discussed problem decreases with the higher slope with the increase in m_0 that may be observed from Figure 3. This however is not so easy to depict from the data shown in Table 1.

5. Conclusions

In this paper we have presented some possibilities of the modified symbolic-numeric method of initial parameters presented in [3] and used the method to solve problems of coupled axial and bending vibration of the AFG cantilever beam. The presented problem was discussed in [1] for the first time and we have confirmed results presented in [1] for the Case 3 of material and geometrical characteristic of the cantilever beam. The presented problem of vibration of axially functionally graded beams is described by the system of differential equation with variable parameters that depend on geometrical characteristics of the cantilever beam and its material properties that may vary along the longitudinal axes. The method, presented in [3] is expanded to solve the problem of coupled vibration of AFG cantilever beam. First three natural frequencies of a cantilever beam with a tip mass eccentrically displaced are calculated. The influence of the tip mass to natural frequencies is discussed, as well as the influence of mass density of one of the constituent materials. By using the proposed method iterative procedures in computing of natural frequencies for the presented problem are avoided so the computations are significantly faster, also no discretization technique was used. This is enabled by the possibility to use modern computer tools, where variables are treated in the symbolic manner.

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