5th Serbian-Greek Symposium on Advanced Mechanics
Kragujevac, Serbia, June 28-30, 2021

FREE VIBRATION OF AXIALLY FUNCTIONALLY GRADED TIMOSHENKO CANTILEVER BEAM WITH A LARGE RIGID BODY ATTACHED AT ITS FREE END

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Abstract:

The problem of natural frequency computing of an axially functionally graded (AFG) Timoshenko cantilever beam of a constant cross-sectional area is considered. The longitudinal and transverse vibrations of the cantilever beam are coupled because of the boundary conditions at its right end, where the eccentrically displaced rigid body is attached. Once the variables are separated, the procedure of solving the system of partial differential equations is reduced to solving the two-point boundary value problem of ordinary linear differential equations with nonlinear coefficients and linear boundary conditions. In this case, it is possible to transfer boundary conditions and to reduce the problem to the Cauchy problem of initial conditions. Also, it is possible to analyze the influence of various parameters on the values of natural frequencies. The impact of the cross-sectional area on natural frequency is analyzed and illustrated by a numerical example.

Key words: free vibrations, longitudinal and transverse vibrations, Timoshenko beam, axially functionally graded cantelever beam

1. Introduction

Numerous engineering structures may be modeled as cantilever beams of a constant cross-sectional area with a large eccentrically displaced rigid body with its free end. In the case of large aspect ratios, the Euler Bernoulli beam model does not provide satisfying results. Thus, it is necessary to apply the Timoshenko beam theory [1], which considers both the rotation and shear effects in the cross-section. The coupling between longitudinal and transverse vibrations is caused by the boundary conditions, while the differential equations remain uncoupled. In the analysis of free vibration of the elastic structures, the natural frequencies are of uttermost interest. That is why it is significant to define the computation procedure. The problem is more complex if beams were designed of axially functionally graded materials. It is the type of material where the mechanical characteristics change along the longitudinal axes of an object. The possibility of finding the analytical solutions of Euler Bernoulli beams of a constant cross-sectional area enabled the authors of this paper to extend the procedure to the cases of inserted rigid bodies among elastic bodies and at the end of a structure in their references [2,3]. Also, the procedure of finding analytical solutions of the governing equations of vibrations of all structural elements was introduced. This procedure is preceded by the derivation of proper orthogonality conditions of mode shapes. There is an absence of a closed-form solution of the system of differential equations of vibrations when elasticity modulus, shear modulus, and mass density of the material vary along the axial axes of a cantilever beam. The equations have to be solved numerically, as it is presented in this paper, or by discretization that requires a model with a sufficiently large number of degrees of freedom. This problem is always present in the case of nonhomogeneous, axially functionally graded materials for both Euler Bernoulli and Timoshenko beams. This paper is the sequel of the research published in papers [4,5,6] and dissertation [7], where the vibrations of coupled Euler Bernoulli beams and frames as well as the transverse vibrations of Timoshenko beams of functionally graded materials were considered. The structural elements have complex boundary conditions due to the attached rigid bodies or elastic elements. Also, their cross-sectional area characteristics are variable. The structural elements have complex boundary conditions due to the attached rigid bodies or elastic supports. In paper [4], the longitudinal and transverse vibrations were uncoupled, thus they were solved separately. In papers [5,6] and dissertation [7] the coupling is analyzed and the closed-form solution is obtained. The given procedure will be extended to the Timoshenko cantilever beam with coupled longitudinal and transverse vibrations.

The system of coupled partial differential equations and boundary conditions are introduced in Chapter 2 of this paper. Also, the procedure of separation of variables and reduction to the system of ordinary differential equations is implemented.

Chapter 3 covers the modal analysis. It can show how the system parameters influence the natural frequencies and mode shapes.

A numerical example in Chapter 4 illustrates the procedure given for various boundary conditions.

2. Problem formulation

The Timoshenko cantilever beam of length *L*, constant cross-section *A* and axial moment of inertia I_x for the case of axially functionally graded material, Fig. 1, is considered. In such materials the density $\rho(z)$, Young's modulus of elasticity $E(z)$ and the shear modulus $G(z)$ are variable along the beam axis. At the right beam end a body of mass *M* and moment of inertia J_{Cx} is eccentrically displaced to the central axis, where the position of the center of mass is defined by the quantities *e* and *h*.

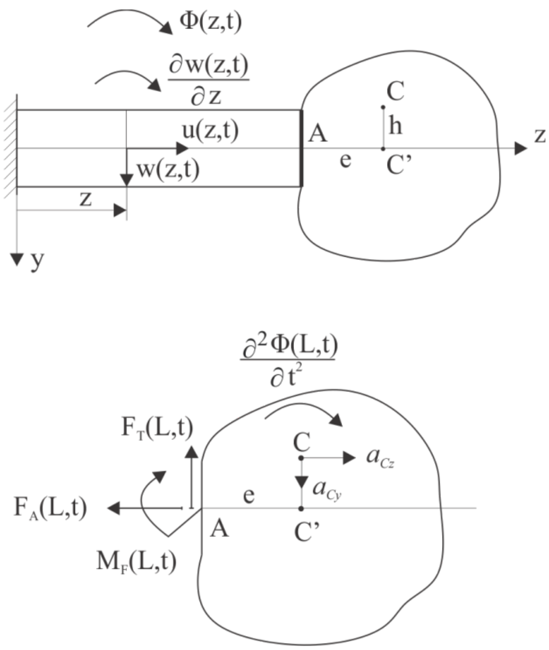

Fig.1. AFG Timoshenko cantilever beam with eccentrically displaced body at the beam end

Differential equations of the Timoshenko beams, oscillating in the longitudinal and transverse direction, have the following form [1,7]:

$$\frac{\partial}{\partial z}\left[F_T(z,t)\right]-\rho(z)A\frac{\partial^2 w(z,t)}{\partial t^2}=0,\ \frac{\partial}{\partial z}\left[M_F(z,t)\right]-\left[F_T(z,t)\right]+\rho(z)I_x\frac{\partial^2\phi(z,t)}{\partial t^2}=0,$$
$$\frac{\partial}{\partial z}\left[F_A(z,t)\right]-\rho(z)A\frac{\partial^2 u(z,t)}{\partial t^2}=0, \tag{1}$$

where $u(z,t)$ and $w(z,t)$ are axial and transverse displacements, $\phi(z,t)$ is the cross-sectional angle of rotation, $F_A(z,t)$ represents the axial force:

$$F_A(z,t)=E(z)A\frac{\partial u(z,t)}{\partial z}, \tag{2}$$

the bending moment is given by the expression:

$$M_F(z,t)=-E(z)I_x\frac{\partial\phi(z,t)}{\partial z}, \tag{3}$$

where the slope angle of the elastic line for Timoshenko beams is $\frac{\partial w(z,t)}{\partial z}$, due to shear impact, different from the cross-sectional angle of rotation $\phi(z,t)$:

$$\frac{\partial w(z,t)}{\partial z}=\phi(z,t)+\frac{F_T(z,t)}{kAG(z)} \tag{4}$$

where $F_T(z,t)$ is the transverse force and k represents the Timoshenko coefficient. The Euler-Bernoulli case, which can be applied with satisfactory accuracy to comparatively thinner beams, is obtained when instead of (4) the slope angle of the elastic line $\frac{\partial w(z,t)}{\partial z}$ is considered to be equal to the cross-sectional angle of rotation $\phi(z,t)$ and when in (1) inertia is neglected at the cross-sectional rotation ($I_x = 0$).

At the left end the boundary conditions have the form:

$$u(0,t)=0,\ldots w(0,t)=0,\ldots \phi(0,t)=0. \tag{5}$$

Boundary conditions at the right end of the structure, based on dynamic equations of the planar motion of the rigid body passing through the equilibrium position (Fig. 1), are of the form:

$$\begin{aligned}
&Ma_{Cz}=M(\frac{\partial^2 u(L,t)}{\partial t^2}+h\frac{\partial^2 \phi(L,t)}{\partial t^2})=-F_A(L,t)\\
&Ma_{Cy}=M(\frac{\partial^2 w(L,t)}{\partial t^2}+e\frac{\partial^2 \phi(L,t)}{\partial t^2})=-F_T(L,t)\\
&J_x\frac{\partial^2 \phi(L,t)}{\partial t^2}=M_F(L,t)+eF_T(L,t)+hF_A(L,t)
\end{aligned} \tag{6}$$

where a_{Cy} and a_{Cz} are a corresponding projection of the acceleration of point *C*.

The system of linear differential equations (1-4) is solved by the method of separation of variables[1,8]:

$$\begin{aligned}
&w(z,t)=W(z)T(t),\\
&\Phi(z,t)=\varphi(z)T(t), u(z,t)=U(z)T(t), F_A(z,t)=F_a(z)T(t),\\
&F_T(z,t)=F_t(z)T(t), M_F(z,t)=M_f(z)T(t),
\end{aligned} \tag{7}$$

where the coupling of longitudinal and transverse vibrations in boundary conditions (6) leads to the same function of time $T(t)$ for longitudinal and transverse oscillations, where [1,8]:

$$\frac{\partial^2 T(t)}{\partial t^2}=-\omega^2 T(t), \tag{8}$$

where ω is the natural frequency.

Further procedure yields a system of six linear ordinary differential equations with variable coefficients

$$\begin{aligned}
&\frac{\partial U(z)}{\partial z}=\frac{F_a(z)}{E(z)A}, \frac{\partial W(z)}{\partial z}=\varphi(z)+\frac{F_t(z)}{kAG(z)}, \frac{\partial \varphi(z)}{\partial z}=-\frac{M_f(z)}{E(z)I_x},\\
&\frac{\partial F_a(z)}{\partial z}=-\omega^2\rho(z)AU(z), \frac{\partial F_t(z)}{\partial z}=-\omega^2\rho(z)AW(z), \frac{\partial M_f(z)}{\partial z}=F_t(z)+\omega^2\rho(z)I_x\varphi(z).
\end{aligned} \tag{9}$$

with boundary conditions at the left end (5) becoming

$$U(0)=0, W(0)=0, \varphi(0)=0\,, \tag{10}$$

while at the right end:

$$\begin{aligned} &M\omega^2(U(L)+h\varphi(L))-F_A(L)=0\\ &M\omega^2(W(L)+e\varphi(L))-F_T(L)=0\\ &J_x\omega^2\varphi(L)+M_f(L)+eF_T(L)+hF_A(L)=0 \end{aligned} \tag{11}$$

they are also linear. Therefore, it is possible to reduce the problem to numerical solving of the Cauchy problem [8] that is applied instead of numerical solving of the system of 6 differential equations (9) with equal number of boundary conditions. Thereafter, in each concrete case a frequency equation would be simply developed and solved and natural frequencies and mode shapes would be determined. That procedure is described in Chapter 3 of the paper.

Note that, in the general case, numerical solving of this problem is necessary, however solving the Cauchy problem, frequency equations and computing the vibration mode shapes are a part of standard numerical procedures that can be found within various program packages. In this paper, we opted for using the MATHEMATICA package [9].

3. Modal analysis of coupled longitudinal and transverse vibrations

Considering the linearity (9-11), the solution can be sought, depending on the natural frequency value ω , as a linear combination of three independent solutions of the system (9,10):

$$\boldsymbol{X}(z)=C_1\boldsymbol{X}^1(z)+C_2\boldsymbol{X}^2(z)+C_3\boldsymbol{X}^3(z),\,, \tag{12}$$

where:

$$\begin{aligned} &\boldsymbol{X}(z)=\left[U(z),W(z),\varphi(z),F_a(z),F_t(z),M_f(z)\right]^T,\\ &\boldsymbol{X}^k(z)=\left[U^k(z),W^k(z),\varphi^k(z),F^k_a(z),F^k_t(z),M^k_f(z)\right]^T,\quad k=1,2,3. \end{aligned} \tag{13}$$

These solutions $\boldsymbol{X}^k(z)$ are obtained by numerical solving of the system (9), where the initial conditions $\boldsymbol{X}^k(0)$ must be chosen to satisfy conditions (10). For the case considered herein, this can be done in the following way:

$$\boldsymbol{X}^1(0)=\left[0,0,0,1,0,0\right]^T,\ \boldsymbol{X}^2(0)=\left[0,0,0,0,1,0\right]^T,\ \boldsymbol{X}^3(0)=\left[0,0,0,0,0,1\right]^T. \tag{14}$$

The set of Cauchy problems, considering that the form of differential equations (9) for specified ω has a unique solution, can use the command NDSolve[…] in the MATHEMATICA package [9]. If we seek these solutions for different values of ω, the command ParametricNDSolve[...,{ ω }] is available. However, we are only interested in those solutions that satisfy boundary conditions at the right end of the structure (13), which leads to the homogeneous system of linear equations:

$$\boldsymbol{D}(\omega)\boldsymbol{C} = \mathbf{0}, \quad \boldsymbol{D}(\omega) = \begin{bmatrix} d_{11}(\omega) & d_{12}(\omega) & d_{13}(\omega) \\ d_{21}(\omega) & d_{22}(\omega) & d_{23}(\omega) \\ d_{31}(\omega) & d_{32}(\omega) & d_{33}(\omega) \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \tag{15}$$

where the matrix elements $\boldsymbol{D}(\omega)$ have the form:

$$\begin{aligned} d_{1k}(\omega) &= M\omega^2(U^k(L) + h\varphi^k(L)) - F_a^k(L), \\ d_{2k}(\omega) &= M\omega^2(W^k(L) + e\varphi^k(L)) - F_t^k(L), \cdots\cdots\cdots\cdots k = 1,2,3 \\ d_{3k}(\omega) &= J_x\omega^2\varphi^k(L) + M_f^k(zL) + eF_t^k(L) + hF_a^k(L). \end{aligned} \tag{16}$$

Now, the frequency equation reads:

$$F(\omega) = Det(\boldsymbol{D}(\omega)) = 0\,. \tag{17}$$

This function $F(\omega)$ can be also simply graphically represented and the finite number of its zeros $\omega_\alpha, \alpha = 1,\ldots,\infty$ can be solved numerically too.

If it is of interest to analyze the impact of some parameter p on the dynamical behavior of this structure, primarily on the values of its frequencies, this procedure can be slightly modified and dependency of the frequences of that parameter can be obtained. In that case, all numerical solutions would be sought in the function of two parameters, ω and p (ParametricNDSolve[...,{ ω, p }]), and the sought dependency would follow from the frequency equation (17) that now has the form:

$$F(\omega, p) = Det(\boldsymbol{D}(\omega, p)) = 0\,. \tag{18}$$

4. Numerical example

To illustrate the above procedure, we will consider a cantilever beam of a constant square cross-sectional area *A*. Let us define the dependency of circular frequencies of the cantilever beam on the cross-sectional area ($p = A$) if the material density and Young's modulus of elasticity change according to the laws:

$$E(z) = E_0\left(1 - 0.2\cos(\pi z)\right)\rho(z) = \rho_0\left(1 - 0.8\cos(\pi z)\right). \tag{19}$$

For the Timoshenko beams of a square cross-section, the axial moment of inertia amounts to $I_x = A^2/12$. The Timoshenko coefficient, in this case, is approximately $k = \frac{5}{6}$, although some other values can be found, which give more accurate results, as reported in [10]. The shear modulus is defined based on the Poisson coefficient ν from the expression $G(z) = \frac{E(z)}{2(1+\nu)}$.

Numerical parameters of this problem are:

$$L = 1m,\, E_0 = 2.068 * 10^{11}\, N/m^2,\, \rho_0 = 7800\, kg/m^3,$$
$$\nu = 0.3,\, M = 10kg,\, J_{Cx} = 2.5kgm^2,\, e = h = 0.5m. \tag{20}$$

The application of the procedure described enables numerical computation and graphic representation of dependency (18) of the circular frequencies on a chosen parameter $p = A$. Fig. 2 shows dependency of the first two circular frequencies on the cross-sectional area.

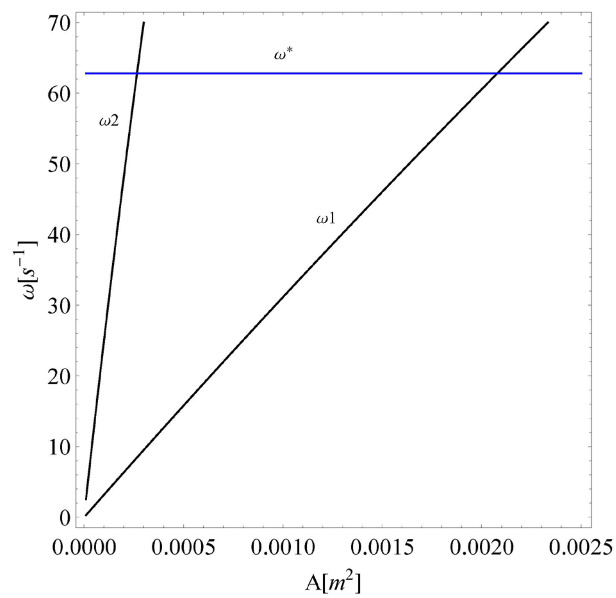

Fig. 2. Dependency of the first two circular frequencies on the cross-sectional area

Besides an undeniable fact that frequencies increase as the cross-sectional area increases, it can be noted that this dependency too is approximately a linear function.

Additionally, from above dependencies in the numerical form the values of the cross-sectional area can be chosen to provide the desired frequency value. Let us take that the desired frequency value is $f^* = 10Hz\, (\omega^* = 2\pi f^*)$. The required values of the cross-sectional area are obtained numerically by solving the equation (18), so that:

$$F(\omega^*, A^*) = 0.. \tag{21}$$

If we want the stated desired value to refer to the first frequency, the solution $A^* = 0.0020791m^2$ is obtained, whereas for the case of the second frequency $A^* = 0.000266558m^2$ is obtained. These solutions are matched by the sides of a square $a^* = 0.0455971m$ and $a^* = 0.0163266m$.

5. Conclusions

The paper presents a modified symbolic-numeric method of initial parameters for the AFG Timoshenko cantilever beam that has been applied in our earlier papers [5-7] to solving the problem of coupled longitudinal and transverse vibrations of the AFG Euler-Bernoulli beams and frames. When slightly modified, the method can be used for other cases of the boundary conditions of Timoshenko beams and frames, including different types of springs at the beam ends and embedded rigid bodies between elastic beams. Also, for the case of a variable cross-section. Based on obtained frequency dependency on some parameter, it is possible to choose the parameter to obtain the desired frequency (first or some of the higher frequencies). This method can also indicate the impact of more than one parameter on the frequency value.

The proposed method allows to avoid iterative procedures in computing eigen-frequencies, and based on obtained diagrams, that impact can be qualitatively analyzed.

The results obtained in this paper by numerical solving of correct differential equations of the Timoshenko beams can be used as references for different discretization procedures, or for the FEM analyses-based procedures.

The research continuation implies deriving the orthogonality conditions and, based on them, obtaining solutions for the closed-form final equations of oscillation in the Timoshenko beams. This has been done in our earlier papers for Euler-Bernoulli beams for the case of a homogeneous complex system of elastic and rigid bodies of a constant cross-section [3] and for the case of planar nonhomogeneous variable cross-section frames [5].

Acknowledgment

Support for this research was provided by the Ministry of Education, Science and Technological Development of the Republic of Serbia under Grants Nos. 451-03-9/2021-14/200105 and 451-03-9/2021-14/200108. This support is gratefully acknowledged.

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