8th International Congress of Serbian Society of Mechanics
Kragujevac, Serbia, June 28-30, 2021

MASS MINIMIZATION OF AN AFG TIMOSHENKO CANTILEVER BEAM WITH A LARGE BODY PLACED ECCENTRICALLY AT THE BEAM END

Aleksandar M. Obradović[1], Zoran S. Mitrović[1], Nemanja D. Zorić[1]

[1] Faculty of Mechanical Engineering,
The University of Belgrade, Kraljice Marije 16, 11 120 Belgrade 35, Serbia
e-mail: aobradovic@mas.bg.ac.rs, zmitrovic@mas.bg.ac.rs, nzoric@mas.bg.ac.rs

Abstract:

Shape optimization of an AFG Timoshenko cantilever beam of a variable cross-sectional area, with a specified fundamental frequency, is considered. The cantilever beam has a finite-dimensional body placed eccentrically at the right end. Optimization is performed in terms of beam mass minimization. Considerations involve the case of coupled axial and bending oscillations, where contour conditions are the cause of coupling, which exist at the place of junction between the cantilever beam and the body. The problem is solved applying Pontryagin's maximum principle, with the beam cross-sectional area being taken for control. The two-point boundary value problem is obtained, and the shooting method is applied to solving it. The property of self-adjoint systems is employed, where all costate variables are expressed by state variables, which facilitates solving the appropriate differential equations. Also, the percent saving of the beam mass is determined, achieved by using the cantilever beam of an optimum variable square cross-section compared to the cantilever beam of a constant cross-section at specified value of the fundamental frequency.

Key words: coupled vibrations, axially functionally graded beams, Timoshenko beam, Pontryagin's maximum principle, mass minimization, axial vibration, bending vibration, cantilever beam

1. Introduction

Mass minimization of the oscillating bodies is one of the significant optimization requirements. It is needed to define the body optimum shape so as to preserve some of the oscillating processes characteristics, the fundamental frequency value being the most important. The subject of the present paper are Timoshenko beams [1] in which for the case of a variable cross-section it is impossible to obtain even the frequency equations in the analytical form. Considerations involve the case of complex boundary conditions that lead to the coupling between axial and bending oscillations, although the differential equations themselves are not mutually coupled. Besides, the problem is additionally complex here for the case of applying axially functionally graded materials, where the material characteristics such as density, Young's modulus of elasticity and the shear modulus change along the beam axis. Pontryagin's maximum principle is used in this paper [2], which is reduced to the two-point boundary value problem of the system of ordinary differential equations. Numerical solving of this problem is significantly facilitated by the fact that these are self-adjoint systems, where the variables are coupled to a constant proportional to the magnitudes of the state.

The Timoshenko beam model must be used instead of a simplified Euler-Bernoulli model in the case of relatively thicker beams. The present paper is a bidirectional generalization of paper [3]. Here, instead of a homogeneous material we consider the case of AFG material and instead of Euler-Bernoulli beams we have a more complex case of Timoshenko beams.

In order to show the percent saving of the beam mass compared to the beam of a constant cross-section, the numerical results from paper [4] are used for comparison. It defines a constant cross-sectional area that ensures the same value of the fundamental frequency. Mentioned paper is based on the results reported in [5,6], where numerical solving is grounded on boundary conditions transfer and reduction to the Cauchy problem.

Chapter 2 presents formulation of the optimal control problem that includes derivation of the differential equations of state, contour conditions and the functional to be minimized. Chapter 3 describes the procedure of solving the optimal control problem applying Pontryagin's principle, while Chapter 4 gives an appropriate numerical solution and defined percent saving of the beam mass.

2. Problem formulation

The Timoshenko cantilever beam [1] (Fig.1) of length *L*, variable cross-sectional area $A(z)$ and axial moment of inertia $I_x(z) = sA(z)^2$, here coefficient *s* depends on the cross-sectional shape (for the square cross-section $s = 1/12$, while in the case of circular cross-section $s = 1/4\pi$), is considered. In the case of AFG material, the density $\rho(z)$, Young's modulus of elasticity $E(z)$ and the shear modulus $G(z)$, are variable along the beam axis. At the right end a body of mass *M* and moment of inertia J_{Cx} is fixed eccentrically to the central axis, where the position of the center of mass is defined by quantities *e* and *h*.

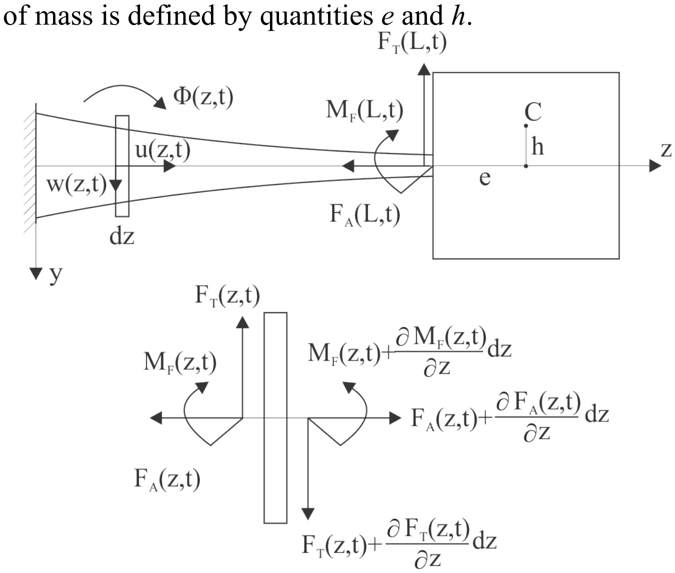

Fig.1. AFG Timoshenko cantilever beam of variable cross-section with a body eccentrically located at the beam end

Differential equations of Timoshenko beams, oscillating in the axial and bending direction, in the case of the linear theory, can be derived based on dynamic equations of the elementary particle of the beam of mass $dm = \rho(z)A(z)dz$ and a corresponding moment of inertia of masses $dJ_x = \rho(z)I_x(z)dz$:

$$
\begin{aligned}
&(\rho(z)A(z)dz)\frac{\partial^2 u(z,t)}{\partial t^2} = \frac{\partial}{\partial z}\left[F_A(z,t)\right]dz,\\
&(\rho(z)A(z)dz)\frac{\partial^2 w(z,t)}{\partial t^2} = \frac{\partial}{\partial z}\left[F_T(z,t)\right]dz,\\
&(\rho(z)I_x(z)dz)\frac{\partial^2 \phi(z,t)}{\partial t^2} = \left[F_T(z,t)\right]dz - \frac{\partial}{\partial z}\left[M_F(z,t)\right]dz,
\end{aligned}
\tag{1}
$$

where $u(z,t)$ and $w(z,t)$ are axial and transverse displacements, $\phi(z,t)$ is the cross-sectional angle of rotation, $F_A(z,t)$ represents the axial force:

$$
F_A(z,t) = E(z)A(z)\frac{\partial u(z,t)}{\partial z},
\tag{2}
$$

the bending moment is given by the expression:

$$
M_F(z,t) = -E(z)I_x(z)\frac{\partial \phi(z,t)}{\partial z},
\tag{3}
$$

where for Timoshenko beams [1] the slope angle of the elastic line is:

$$
\frac{\partial w(z,t)}{\partial z} = \phi(z,t) + \frac{F_T(z,t)}{kA(z)G(z)},
\tag{4}
$$

where $F_T(z,t)$ is the transverse force and *k* represents the Timoshenko coefficient.

Differential equations (1) can be found in the literature reference [1] derived based on Hamilton's principle.

In our paper [4], in the expressions (1-4), instead of the variable values of cross-section $A(z)$ and axial moment of inertia $I_x(z)$ there are constant quantities A and I_x. Like in the cited paper, the system of linear differential equations (1-4) is solved by the method of separation of variables [1]:

$$
\begin{aligned}
&w(z,t) = W(z)T(t),\\
&\Phi(z,t) = \varphi(z)T(t),\, u(z,t) = U(z)T(t),\, F_A(z,t) = F_a(z)T(t),\\
&F_T(z,t) = F_t(z)T(t),\, M_F(z,t) = M_f(z)T(t),
\end{aligned}
\tag{5}
$$

where $\frac{\partial^2 T(t)}{\partial t^2} = -\omega^2 T(t)$ and ω represents a circular frequency. If we want all functions (5) to retain their physical dimensions and units, the function $T(t)$ will be considered to be dimensionless.

Further procedure yields the following differential equations:

$$\frac{\partial U(z)}{\partial z}=\frac{F_a(z)}{E(z)A(z)},\frac{\partial W(z)}{\partial z}=\varphi(z)+\frac{F_t(z)}{kA(z)G(z)},\frac{\partial \varphi(z)}{\partial z}=-\frac{M_f(z)}{E(z)sA(z)^2},$$
$$\frac{\partial F_a(z)}{\partial z}=-\omega^2\rho(z)A(z)U(z),\frac{\partial F_t(z)}{\partial z}=-\omega^2\rho(z)A(z)W(z),\frac{\partial M_f(z)}{\partial z}=F_t(z)+\omega^2\rho(z)sA(z)^2\varphi(z). \tag{6}$$

Optimization problem, to be considered in this paper, includes defining the function of change of the cross-sectional area $A(z)$ that will lead to the Timoshenko cantilever beam mass minimization, where the fundamental frequency of oscillation $\omega_1=\omega^*$ is specified. In that regard, the functional that is minimized has the form:

$$J=\int_0^L \rho(z)A(z)dz \tag{7}$$

differential equations (6) represent the equations of state, with the contour conditions [4] at the left end:

$$U(0)=0, W(0)=0, \varphi(0)=0, \tag{8}$$

whereas at the right end:

$$M\omega^2(U(L)+h\varphi(L))-F_a(L)=0,$$
$$M\omega^2(W(L)+e\varphi(L))-F_t(L)=0, \tag{9}$$
$$J_{Cx}\omega^2\varphi(L)+M_f(L)+eF_t(L)+hF_a(L)=0.$$

3. Shape optimization of a cantilever beam by applying Pontryagin's maximum principle

The optimal control problem (6-9) will be solved by applying Pontryagin's maximum principle [2]. To this end, let us write Pontryagin's function:

$$H=\lambda_0\rho(z)A(z)+\lambda_U(z)\frac{F_a(z)}{E(z)A(z)}+\lambda_W(z)(\varphi(z)+\frac{F_t(z)}{k\,A(z)G(z)})+\lambda_\varphi(z)\frac{-M_f(z)}{E(z)sA(z)^2}$$
$$-\lambda_{F_a}(z)\omega^2\rho(z)A(z)U(z)-\lambda_{F_t}(z)\omega^2\rho(z)A(z)W(z)+\lambda_{M_f}(z)(F_t(z)+\omega^2\rho(z)sA(z)^2) \tag{10}$$

where $\lambda_0,\lambda_U(z),\lambda_W(z),\lambda_\varphi(z),\lambda_{F_a}(z),\lambda_{F_t}(z),\lambda_{M_f}(z)$ are costate variables, which satisfy the coupled system of equations:

$$\frac{\partial \lambda_U(z)}{\partial z}=\lambda_{F_a}(z)\omega^2\rho(z)A(z),\frac{\partial \lambda_W(z)}{\partial z}=\lambda_{F_t}(z)\omega^2\rho(z)A(z),\frac{\partial \lambda_\varphi(z)}{\partial z}=-\lambda_W(z),$$
$$\frac{\partial \lambda_{F_a}(z)}{\partial z}=\frac{-\lambda_U(z)}{E(z)A(z)},\frac{\partial \lambda_{F_t}(z)}{\partial z}=\frac{-\lambda_W(z)}{kA(z)G(z)}-\lambda_{M_f}(z),\frac{\partial \lambda_{M_f}(z)}{\partial z}=\frac{\lambda_\varphi(z)}{E(z)sA(z)^2}, \tag{11}$$

where λ_0 is an arbitrary non-positive constant. In problems of this type it is commonly taken that $\lambda_0=-1$.

Since the systems (6) and (11) are non-autonomous, after formally introducing an additional state quantity Z such that:

$$\frac{\partial Z}{\partial z}=1, \quad Z(0)=0, \tag{12}$$

the transversality conditions [2] can be represented in the form as follows:

$$\begin{aligned}&(\lambda_U(z)\Delta U(z)+\lambda_W(z)\Delta W(z)+\lambda_\varphi(z)\Delta\varphi(z)+\lambda_{F_a}(z)\Delta F_a(z)+\\&\lambda_{F_t}(z)\Delta F_t(z)+\lambda_{M_f}(z)\Delta M_f(z)+\lambda_Z(z)\Delta Z(z))\ \Big|_0^L=0,\end{aligned} \tag{13}$$

where $\Delta(\cdot)$ is an asynchronous variation, λ_Z is a state variable of the additional state quantity Z, with initial and final conditions (8,9) and (12) leading to the following variation dependencies at the left:

$$\Delta U(0)=0, \Delta W(0)=0, \Delta\varphi(0)=0, \Delta Z(0)=0, \tag{14}$$

and at the right end:

$$\begin{aligned}&M\omega^2(\Delta U(L)+h\Delta\varphi(L))-\Delta F_a(L)=0\\&M\omega^2(\Delta W(L)+e\Delta\varphi(L))-\Delta F_t(L)=0\\&J_{Cx}\omega^2\Delta\varphi(L)+\Delta M_f(L)+e\Delta F_t(L)+h\Delta F_a(L)=0\end{aligned} \tag{15}$$

Substituting (14) and (15) in (13) the transversality conditions are obtained:

$$\begin{aligned}&\lambda_{F_a}(0)=0, \quad \lambda_{F_t}(0)=0, \quad \lambda_{M_f}(0)=0,\\&M\omega^2(\lambda_{F_a}(L)-h\lambda_{M_f}(L))+\lambda_U(L)=0,\\&M\omega^2(\lambda_{F_t}(L)-e\lambda_{M_f}(L))+\lambda_W(L)=0,\\&-J_{Cx}\omega^2\lambda_{M_f}(L)+\lambda_\varphi(L)-e\lambda_W(L)-h\lambda_U(L)=0,\end{aligned} \tag{16}$$

as well as that $\lambda_Z(L)=0$. This coordinate of the conjugate vector is no longer needed to be analyzed for solving such formulated optimal control problem.

If the costate vector coordinates are expressed via state quantities using the scalar parameter p:

$$\lambda_U=pF_a,\ \lambda_W=pF_t,\lambda_\varphi=-pM_f,\ \lambda_{M_f}=p\varphi,\ \lambda_{F_t}=-pW,\ \lambda_{F_a}=-pU, \tag{17}$$

it can be noted that differential equations of the coupled system (11) are reduced to the governing system (6), and that the transversality conditions (16) are satisfied in the case when the conditions at the left end (8) and at the right end (9) are satisfied. This has been noted in the shape optimization problems reported by Atanacković et al. [7,8,9] and has facilitated the application of Pontryagin's maximum principle. It is very well known that numerical difficulties related to computations of the costate variables are those that are limiting the application of maximum principle.

Optimal controls $A(z)$ are defined from the maximum condition of Pontryagin's function (10):

$$\frac{\partial H}{\partial A(z)}=0,\ \frac{\partial^2 H}{\partial A(z)^2}<0, \tag{18}$$

which, considering (17), is reduced to the conditions:

$$\begin{aligned}&-\frac{pF_a(z)^2}{E(z)A(z)^2}-\frac{pF_t(z)^2}{kG(z)A(z)^2}-\frac{2pM_f(z)^2}{sA(z)^3E(z)}+\\&+2ps\omega^2A(z)\varphi(z)^2\rho(z)+(-1+p\omega^2U(z)^2+p\omega^2W(z)^2)\rho(z)=0,\end{aligned} \tag{19}$$

$$2p\left(\frac{F_a(z)^2}{E(z)A(z)^3}+\frac{F_t(z)^2}{kG(z)A(z)^3}+\frac{3M_f(z)^2}{sA(z)^4E(z)}+s\omega^2A(z)\varphi(z)^2\rho(z)\right)<0. \tag{20}$$

Based on (20), without loss of generality, it can be taken that $p=-1\frac{s^2}{m^2}$, so that the expression for defining optimal control is reduced to the 4th degree polynomial with respect to *A(z)*:

$$\begin{aligned}&\left(\frac{F_a(z)^2}{E(z)}+\frac{F_t(z)^2}{kG(z)}\right)A(z)+\frac{2M_f(z)^2}{sE(z)}-\\&-2s\omega^2A(z)^4\varphi(z)^2\rho(z)-(1+\omega^2U(z)^2+\omega^2W(z)^2)\rho(z)A(z)^3=0.\end{aligned} \tag{21}$$

The procedure of numerical solving of the two-point boundary value problem consists of the three-parameter shooting that involves selecting $F_a(0), F_t(0), M_f(0)$ and with initial conditions (8) the system of differential equations is solved, with the condition (21), so that three final conditions (9) are satisfied. If numerical solving is performed in the program package *WolframMathematica* [10] using function NDSolve[…], it is not necessary to express *A(z)* from (21) in analytical form via state quantity, because this function contains in itself the procedure for numerical solving of the system of differential and ordinary equations.

4. Numerical example

The shape optimization procedure will be presented using the example of a cantilever beam of a square cross-section, length L=1m, with a rigid body placed eccentrically at the free end, as shown in Fig. 1. The rigid body has mass *M=10kg* and moment of inertia $J_{Cx}=2.5kgm^2$. Axial and transverse eccentricities of the rigid body amount to $e=h=0.5m$. In AFG material considered herein the laws of change in density and modulus of elasticity are taken as in [4]:

$$\begin{aligned}&\rho(z)=\rho_0(1-0.8\cos(\pi z)),\ \rho_0=7850\frac{kg}{m^3},\\&E(z)=E_0(1-0.2\cos(\pi z)),\ E_0=2.068\cdot10^{11}\frac{N}{m^2}.\end{aligned} \tag{22}$$

For Timoshenko beams of a square cross-section $s=1/12$. The Timoshenko coefficient, in this case, amounts approximately to $k=\frac{5}{6}$. The shear modulus is defined using the Poisson coefficient ν from the expression $G(z)=\dfrac{E(z)}{2(1+\nu)}$, where for its value it is taken here that $\nu = 0.3$.

Also, let the required value of the fundamental frequency be $f = 10Hz$ which leads to the fundamental circular frequency $\omega^* = 20\pi\ Hz$.

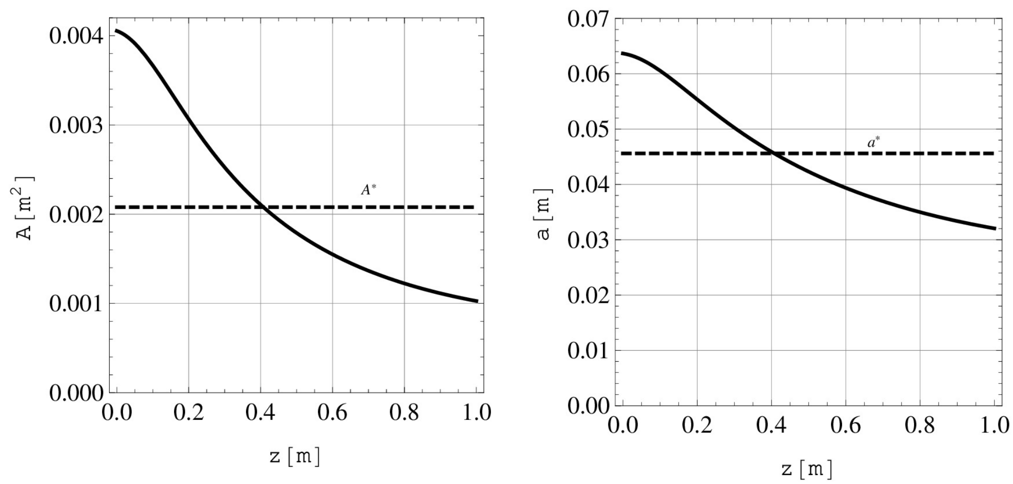

Fig. 2. Optimum cross-sectional area *A(z)* and the side of a square *a(z)*

When performing a three-parameter shooting in the program package *WolframMathematica* [10], three missing values of the three parameters at the left end are obtained:

$$F_a(0) = -219.631N,\quad F_t(0) = -451.863N,\quad M_f(0) = 848.826Nm, \tag{23}$$

Fig. 2 shows values of the optimum cross-sectional area shape and its corresponding sides of a square. The dashed line denotes values corresponding to a constant cross-sectional area A^* and the side of a square a^*, respectively:

$$A^* = 0.00207910m^2,\quad a^* = 0.0455971m \tag{24}$$

for which the fundamental circular frequency is $\omega^* = 20\pi\ Hz$, and which were obtained in paper [4].

Relative material saving compared to the cantilever beam of a constant cross-section corresponding to the same circular frequency amounts to:

$$\Delta = \frac{\left(\int_0^L \rho(z)\,A_1 \mathrm{d}z - \int_0^L \rho(z)A(z)\mathrm{d}z\right)}{\int_0^L \rho(z)\,A_1 \mathrm{d}z} \mathrm{x}100\% = 23.38\%, \tag{25}$$

where numerical integration was done in (25).

5. Conclusions

This paper demonstrates the performance of shape optimization of AFG Timoshenko cantilever beam of a square cross-section with coupled axial and bending vibrations, where the cantilever beam mass minimization is done at specified fundamental frequency. In solving this optimization problem Pontryagin's maximum principle is applied. So far, Pontryagin's maximum principle has been practically used for solving optimization problems in buckling so that in this paper its application is extended to optimization problems in oscillating body. Mass minimization of the cantilever beam is accomplished by taking the cross-sectional area for the control quantity. The procedure described can be used with slight modifications when the cross-sectional area is limited. The lower limit may be defined based on beam strength, whereas the upper limit may correspond to validity limits of the Timoshenko beam theory. The above procedure can be also applied to another case of contour conditions at the beam ends, including bodies eccentrically positioned at both ends, different types of supports at beam ends, as well as clamping of the bodies with different springs.

Acknowledgment

Support for this research was provided by the Ministry of Education, Science and Technological Development of the Republic of Serbia under Grants No. 451-03-9/2021-14/200105. This support is gratefully acknowledged.

References

[1] Rao S., *Vibration of continuous systems,* John Wiley & Sons, New Jersey, 2007.

[2] Pontryagin L., Boltyanskii V., Gamkrelidze R., Mishchenko E., *The mathematical theory of optimal processes,* New York: Interscience, 1962.

[3] Obradović A, Šalinić S, Grbović A., *Mass minimization of an Euler-Bernoulli beam with coupled bending and axial vibrations at prescribed fundamental frequency*, Engineering Structures, Vol. 228, 111538, 2021.

[4] Obradović A., Šalinić S., Tomović A., *Free vibration of axially functionally graded timoshenko cantilever beam with a large rigid body attached at its free end*, 8th International Serbian Congress on Theoretical and Applied Mechanics, Serbaian Society of Mechanics, Kragujevac, Serbia, 28.-30. Jun, 2021.

[5] Šalinić S., Obradović A., Tomović A., *Free vibration analysis of axially functionally graded tapered, stepped, and continuously segmented rods and beams*, Compos. B. Eng., Vol. 150, 135-143, 2018.

[6] Tomović A., *Coupled transverse and longitudinal vibrations of Euler-Bernoulli and Timoshenko beams of functionally graded materials (in Serbian)*. Doctoral Dissertation, University of Belgrade, Faculty of Mechanical Engineering, Belgrade, 2019.

[7] Atanackovic T., Glavardanov V., *Optimal shape of a heavy compressed column*, Structural and Multidisciplinary Optimization, Vol. 28, 388-396, 2004.

[8] Atanackovic T., *Optimal shape of column with own weight: bi and single modal optimization*, Meccanica, Vol. 41, 173–196, 2006.

[9] Atanackovic T., Seyranian A., *Application of Pontryagin's principle to bimodal optimization problems*, Structural and Multidisciplinary Optimization, Vol. 37, 1-12, 2008.

[10] Wolfram S., *The Mathematica Book*, 5th ed. Champaign, IL: Wolfram Media, 2003.