Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 187–198.
DOI: 10.3934/math.2022011
Received: 25 June 2021
Accepted: 26 September 2021
Published: 09 October 2021

Research article

Some significant remarks on multivalued Perov type contractions on cone metric spaces with a directed graph

Ana Savić[1], Nicola Fabiano[2], Nikola Mirkov[2,*], Aleksandra Sretenović[3] and Stojan Radenović[3]

[1] School of Electrical and Computer Engineering, Academy of Technical and Art Applied Studies, Belgrade, Serbia

[2] University of Belgrade, 'Vinča' Institute of Nuclear Sciences-National Institute of the Republic of Serbia, Mike Petrovića Alasa 12–14, 11351 Belgrade, Serbia

[3] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade 35, Serbia

* **Correspondence:** Email: nmirkov@vin.bg.ac.rs; Tel: +381655381771.

Abstract: Using the approach of so-called c-sequences introduced by the fifth author in his recent work, we give much simpler and shorter proofs of multivalued Perov's type results with respect to the ones presented in the recently published paper by M. Abbas et al. Our proofs improve, complement, unify and enrich the ones from the recent papers. Further, in the last section of this paper, we correct and generalize the well-known Perov's fixed point result. We show that this result is in fact equivalent to Banach's contraction principle.

Keywords: common fixed point; Perov's type results; multivalued mapping; directed graph; graphic contraction; cone metric space; c-sequence
Mathematics Subject Classification: 47H10, 54H25, 54E50

1. Introduction and preliminaries

In 2018, M. Abbas et al., ([4], Remark 16.) formulated and proved the following:

Remark 1.1. *Let $\mathcal{P} \subseteq \mathcal{E}$ be a solid cone in Banach space $\mathcal{E}$ and $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ a linear operator with $\|\mathcal{A}\| < 1$ and $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. If*

(**a**) for any u in $\mathcal{P}$, we have

$$u \preceq \mathcal{A}(u), \tag{1}$$

then $u = \theta$.

(**b**) for any u, v in $\mathcal{P}$, we have

$$u \preceq \mathcal{A}\left(\frac{u+v}{2}\right) = \frac{1}{2}\mathcal{A}(u) + \frac{1}{2}\mathcal{A}(v), \tag{2}$$

then $u \preceq \mathcal{A}(v)$.

To prove (**b**), the authors assume on the contrary that $u \succ \mathcal{A}(v)$. However, this is wrong. It is possible only in a totally ordered vector space. The only such possibility is $(\mathbb{R}, \leq)$. In that case, the cone metric space (X, d) in [4] becomes an ordinary metric space. Since $\mathcal{E} \neq \mathbb{R}$ then the incomparable elements exist. For example, if $\mathcal{E} = \mathbb{R}^2$ with the cone $\mathcal{P} = \{(u, v) : u, v \geq 0\}$, then the pairs $(1, 2)$ and $(2, 1)$ are such elements. Therefore, the claim that (2) yields $u \preceq A(v)$ is in doubt. It is clear that the condition (2) implies the next relation:

$$u \preceq \left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)^{-1} \frac{1}{2}\mathcal{A}(v). \tag{3}$$

This is true because $\left\|\frac{1}{2}\mathcal{A}\right\| = \frac{1}{2}\|\mathcal{A}\| < \frac{1}{2} \cdot 1 = \frac{1}{2} < 1$, i.e., $\left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)^{-1}$ exists.

Also, it is worth mentioning that the negation of $u \preceq \mathcal{A}(v)$ is not $u \succ \mathcal{A}(v)$, in general. That is, the negation of $u \preceq \mathcal{A}(v)$ implies that either $u \succ \mathcal{A}(v)$ or that u and $\mathcal{A}(v)$ are incomparable.

Other than that, the paper [4] has several weaknesses which are listed in the following:

- The authors do not faithfully convey the formulation of the famous Perov's theorem [21]. They even add and make up some points of the theorem not present in the original text. It is not clear why they did such a thing.
- Many proofs either look dubious or are long, while it is well known there are shorter, simpler and more elegant proofs (pages 13 and 14, [4]). Those proofs stem from the so-called method of c–sequences introduced by the fifth author (S. Radenović) in the last years. Briefly, the sequence x_n in the cone $\mathcal{P}$ of ordered Banach space $\mathcal{E}$ is called a c–sequence if for any internal point $c \in int\mathcal{P}$ there exists a natural number k such that $x_n \ll c$ when $n > k$. For further details see [1, 5, 6, 11–17, 20, 22, 23, 26, 27].
- The authors used the following statement: Let the linear map $\mathcal{A}$ be positive and the cone $\mathcal{P}$ be solid. According to [6] it follows that the map $\mathcal{A}$ is automatically continuous (bounded) and has a defined spectral radius. The authors, beyond the hypothesis of a solid cone, assume that $r(\mathcal{A}) < 1$.
- In their Theorem 2.3. authors make an assumption that $\mathcal{A}_4(v) \preceq \mathcal{A}_5(v)$ for any $v \in \mathcal{P}$. In our approach this assumption is not necessary. We must also add that the Theorem 2.3 is actually the famous Hardy–Rogers contraction [10] in the context of cone metric spaces.
- The authors have used the symbol $\leq$ in place of $\preceq$ in a few instances. This practice cannot be considered a trivial typo.

Remark 1.2. *According to the observations above, it follows that several results in [4] (Theorem 2.1., Theorem 2.3.) are in fact incorrect. Indeed, the authors use the incorrect implication: (2) yields* $u \preceq \mathcal{A}(v)$ *in all proofs, which is evidently wrong. In this paper we will use the implication: (2) yields (3) to obtain correct results.*

As it is mentioned already, the correct parts of the proofs in [4] can be made simpler and shorter. It is useful to point readers to the recent survey [5], where the authors describe in detail all known papers on cone metric spaces with a new approach: by means of the Minkowski functional the problems can usually be reduced to the case of a solid normal cone with normality constant $K = 1$. This is the most recent result in this field, and allows the relaxation of most of the results and considerations given in the last years.

Similarly as in [2, 3, 19, 25] for ordinary metric spaces, authors in [4] (Definition 1.17. (I), (II)) introduced the so-called cone graphic $\mathcal{P}_1-$contraction pairs and cone graphic $\mathcal{P}_2-$contraction pair in the setting of cone metric spaces as follows:

Definition 1.1. *Let $T_1, T_2 : X \to \mathcal{P}_{cl}(X)$ be two multivalued mappings. Suppose that for every vertex x in G and for every $u_x \in T_i(x), i \in \{1, 2\}$ we have $(x, u_x) \in E(G)$. A pair (T_1, T_2) is said to form:*

(**I**) a cone graphic $\mathcal{P}_1-$contraction pair if there exists a linear bounded operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ with $\|\mathcal{A}\| < 1$ and $\mathcal{A}(\mathcal{P}) \subseteq \mathcal{P}$ such that for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_i(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $\left(u_x, u_y\right) \in E(G)$ and

$$d\left(u_x, u_y\right) \preceq \mathcal{A}\left(M_1\left(x, y; u_x, u_y\right)\right), \tag{4}$$

hold, where

$$M_1\left(x, y; u_x, u_y\right) \in \left\{d(x, y), d(x, u_x), d\left(y, u_y\right), \frac{d(x, u_x) + d\left(y, u_y\right)}{2}, \frac{d\left(x, u_y\right) + d(y, u_x)}{2}\right\}.$$

(**II**) a cone graphic $\mathcal{P}_2-$contraction pair if there exist linear bounded operators $\mathcal{A}_k : \mathcal{E} \to \mathcal{E}$ for $k = 1, 2, 3, 4, 5$ with $\sum_{k=1}^{5} \|\mathcal{A}_k\| < 1$, $\mathcal{A}_k(\mathcal{P}) \subseteq \mathcal{P}$ for $k = 1, 2, 3, 4, 5$ and $\mathcal{A}_4(v) \preceq \mathcal{A}_5(v)$ for all $v \in \mathcal{P}$ such that for any $x, y \in X$ with $(x, y) \in E(G)$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that $\left(u_x, u_y\right) \in E(G)$ and

$$d\left(u_x, u_y\right) \preceq M_2\left(x, y; u_x, u_y\right) \tag{5}$$

hold, where

$$\begin{aligned} M_2\left(x, y; u_x, u_y\right) = {} & \mathcal{A}_1(d(x, y)) + \mathcal{A}_2(d(x, u_x)) + \mathcal{A}_3\left(d\left(y, u_y\right)\right) \\ & + \mathcal{A}_4\left(d\left(x, u_y\right)\right) + \mathcal{A}_5(d(y, u_x)). \end{aligned}$$

Further, authors in [4] proved the following two results:

Theorem 1.1. *([4], Theorem 2.1.) Let (X, d) be a complete cone metric space endowed with a directed graph G such that $V(G) = X$ and $E(G) \supseteq \triangle$. If mappings $T_1, T_2 : X \to \mathcal{P}_{cl}(X)$ form a cone graphical $\mathcal{P}_1-$contraction pair, then following statements hold:*

(i) $Fix(T_1) \neq \emptyset$ or $Fix(T_2) \neq \emptyset$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.

(ii) $X_{T_1,T_2} \neq \emptyset$ provided that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.

(iii) If $X_{T_1,T_2} \neq \emptyset$ and G is a weakly connected, then $Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that graph G has property (P).

(iv) $Fix(T_1) \cap Fix(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix(T_1) \cap Fix(T_2)$ is a singleton.

Theorem 1.2. *([4], Theorem 2.3.) Let (X,d) be a complete cone metric space endowed with a directed graph G such that $V(G)=X$ and $E(G)\supseteq\triangle$. If mappings $T_1,T_2: X\to\mathcal{P}_{cl}(X)$ form a cone graphical $\mathcal{P}_2-$contraction pair, then following statements hold:*

(i) $Fix(T_1)\neq\emptyset$ or $Fix(T_2)\neq\emptyset$ if and only if $Fix(T_1)=Fix(T_2)\neq\emptyset$.

(ii) $X_{T_1,T_2}\neq\emptyset$ provided that $Fix(T_1)\cap Fix(T_2)\neq\emptyset$.

(iii) If $X_{T_1,T_2}\neq\emptyset$ and G is a weakly connected, then $Fix(T_1)=Fix(T_2)\neq\emptyset$ provided that graph G has property (P).

(iv) $Fix(T_1)\cap Fix(T_2)$ is a clique of $\widetilde{G}$ if and only if $Fix(T_1)\cap Fix(T_2)$ is a singleton.

It is worth noticing that the authors suppose that (X,d) is a cone metric space with solid cone $\mathcal{P}$ throughout the paper [4]. Also, the following remark is significant.

Remark 1.3. *Let us notice that an operator $\mathcal{A}$ in Definition 1.1 (I), (II) is automatically continuous, and its spectral radius $r(\mathcal{A})$ is well-defined. Indeed, by ([8], Proposition 19.1) every solid cone in $\mathcal{E}$ is generating, i.e., $\mathcal{P}-\mathcal{P}=\mathcal{E}$, so by ([6], Theorem 2.32), every linear positive operator $\mathcal{A}$ from $\mathcal{E}$ to $\mathcal{E}$ is continuous. This further means that the assumption $\|\mathcal{A}\|<1$ in whole paper [4] is superfluous.*

2. The improved results

In this section we discuss, complement and improve some results given in [4]. By using the method of c-sequences we get much simpler and shorter proofs than the ones presented in the paper of Abbas et al. in [4]. The main motivation behind the present effort is setting the proper level of mathematical rigour, not attained in the mentioned paper, as seen in several previous remarks. We will, therefore, try to give new and correct proofs of both theorems from [4] stated above.

First of all we will give the proof of **(iii)** of ([4], Theorem 2.1.).

Let x_0, as in [4], be any point in X. If $x_0\in T_1(x_0)$ or $x_0\in T_2(x_0)$, then by (i) the result follows. Therefore, assume that $x_0\notin T_i(x_0)$ for both $i=1$ and $i=2$. Further, for $i,j\in\{1,2\}$ with $i\neq j$, if $x_1\in T_i(x_0)$, then there exists $x_2\in T_j(x_1)$ with $(x_1,x_2)\in E(G)$ such that

$$d(x_1,x_2)\preceq\mathcal{A}(M_1(x_0,x_1;x_1,x_2)),\tag{6}$$

where

$$\begin{aligned}M_1(x_0,x_1;x_1,x_2)&\in\Big\{d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),\\&\quad\frac{d(x_0,x_1)+d(x_1,x_2)}{2},\frac{d(x_0,x_2)+d(x_1,x_1)}{2}\Big\}\\&=\left\{d(x_0,x_1),d(x_1,x_2),\frac{d(x_0,x_1)+d(x_1,x_2)}{2},\frac{d(x_0,x_2)}{2}\right\}.\end{aligned}$$

Now, we have the following four possibilities:

1. $M_1(x_0,x_1;x_1,x_2)=d(x_0,x_1)$;

2. $M_1(x_0,x_1;x_1,x_2)=d(x_1,x_2)$;

3. $M_1(x_0,x_1;x_1,x_2)=\frac{d(x_0,x_1)+d(x_1,x_2)}{2}$;

4. $M_1(x_0,x_1;x_1,x_2)=\frac{d(x_0,x_2)}{2}$.

In the case 1, we get $d(x_1, x_2) \preceq \mathcal{A}(d(x_0, x_1))$. The second case obviously yields that $x_1 = x_2$, that is, by (i) the proof is finished. If $M_1(x_0, x_1; x_1, x_2) = \frac{d(x_0,x_1)+d(x_1,x_2)}{2}$ we get

$$d(x_1, x_2) \preceq \frac{d(x_0, x_1) + d(x_1, x_2)}{2}, \tag{7}$$

or $\left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)(d(x_1, x_2)) \preceq \frac{1}{2}\mathcal{A}(d(x_0, x_1))$. That is, $d(x_1, x_2) \preceq \mathcal{B}(d(x_0, x_1))$, where $\mathcal{B} = \left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)^{-1} \frac{1}{2}\mathcal{A}$. It is evident that $\|\mathcal{B}\| < 1$. In case 4 we get the same as in case 3 because $d(x_0, x_2) \preceq d(x_0, x_1) + d(x_1, x_2)$. Hence, according to the above we can prove that for each $n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \preceq \mathcal{C}(d(x_{n-1}, x_n)) \preceq \mathcal{C}^n(d(x_0, x_1)), \tag{8}$$

where $\mathcal{C} = \mathcal{A}$ or $\mathcal{C} = \mathcal{B}$.

The condition (8) yields by routine method that the sequence $\{x_n\}$ is a Cauchy sequence. Indeed, if $n < m$ we have

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq \left(\mathcal{C}^n + \mathcal{C}^{n+1} + \cdots + \mathcal{C}^{m-1}\right)(d(x_0, x_1)) \\ &\prec \mathcal{C}^n(\mathcal{I} - \mathcal{C})^{-1}(d(x_0, x_1)). \end{aligned} \tag{9}$$

Since, $\mathcal{C}^n(\mathcal{I} - \mathcal{C})^{-1}(d(x_0, x_1)) \to \theta$ as $n \to +\infty$, we conclude that $d(x_n, x_m)$ is a c-sequence (for more details on application of c-sequences see [5, 9, 18, 23]). This means that the sequence $\{x_n\}$ is a Cauchy sequence in complete cone metric space (X, d). By completeness of X, there exists an element $x^* \in X$ such that $d(x_n, x^*)$ is a c-sequence in an ordered Banach space $\mathcal{E}$. Since $d(x_{2n}, x^*)$ is also a c-sequence in $\mathcal{E}$ and $(x_{2n}, x_{2n+1}) \in E(G)$, we have that $(x_{2n}, x^*) \in E(G)$. For $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that $(x_{2n}, u_n) \in E(G)$. Because (T_1, T_2) form a graphic $\mathcal{P}_1-$contraction, then

$$d(x_{2n}, u_n) \preceq \mathcal{A}(M_1(x_{2n-1}, x^*; x_{2n}, u_n)), \tag{10}$$

where

$$\begin{aligned} M_1(x_{2n-1}, x^*; x_{2n}, u_n) \in \{&d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n}), d(x^*, u_n), \\ &\left.\frac{d(x_{2n-1}, x_{2n}) + d(x^*, u_n)}{2}, \frac{d(x_{2n-1}, u_n) + d(x^*, x_{2n})}{2}\right\}. \end{aligned}$$

In the next part we will prove that $d(u_n, x^*)$ is a c-sequence in $\mathcal{E}$. To prove this we use the relation:

$$\begin{aligned} d(u_n, x^*) &\preceq d(u_n, x_{2n}) + d(x_{2n}, x^*) \\ &\preceq \mathcal{A}(M_1(x_{2n}, x^*; x_{2n+1}, u_n)) + d(x_{2n}, x^*). \end{aligned} \tag{11}$$

Now for $M_1(x_{2n}, x^*; x_{2n+1}, u_n)$ there are following five possibilities:

1. $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = d(x_{2n-1}, x^*)$.

2. $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = d(x_{2n-1}, x_{2n})$.

3. $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = d(x^*, u_n)$.

4. $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = \frac{d(x_{2n-1},x_{2n})+d(x^*,u_n)}{2}$.

5. $M_1(x_{2n}, x^*; x_{2n+1}, u_n) = \frac{d(x_{2n-1},u_n)+d(x^*,x_{2n})}{2}$.

For the first two cases (11) becomes

$$d(u_n, x^*) \preceq \mathcal{A}(d(x_{2n-1}, x^*)) + d(x_{2n}, x^*) \tag{12}$$

$$\text{i.e., } d(u_n, x^*) \preceq \mathcal{A}(d(x_{2n-1}, x_{2n})) + d(x_{2n}, x^*). \tag{13}$$

Since the continuous image of a c-sequence is a c-sequence ($\mathcal{A}$ is a continuous linear operator) we get that in both cases $d(u_n, x^*)$ is also a c-sequence. For the case 3 we get

$$(\mathcal{I} - \mathcal{A})(d(u_n, x^*)) \preceq d(x_{2n}, x^*) \text{ or } d(u_n, x^*) \preceq (\mathcal{I} - \mathcal{A})^{-1}(d(x_{2n}, x^*)). \tag{14}$$

Since $(\mathcal{I} - \mathcal{A})^{-1}$ is a positive linear operator on $\mathcal{E}$, it is continuous (by Remark 1.3). Hence, $(\mathcal{I} - \mathcal{A})^{-1}(d(x_{2n}, x^*))$ is a c-sequence, i.e., $d(u_n, x^*)$ is a c-sequence, i.e., $u_n \to x^*$ as $n \to +\infty$. For the case 4 we get,

$$d(u_n, x^*) \preceq \mathcal{A}\left(\frac{d(x_{2n-1}, x_{2n}) + d(x^*, u_n)}{2}\right) + d(x_{2n}, x^*)$$

$$= \frac{1}{2}\mathcal{A}(d(x_{2n-1}, x_{2n})) + \frac{1}{2}\mathcal{A}(d(x^*, u_n)) + d(x_{2n}, x^*),$$

or further,

$$\left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)(d(u_n, x^*)) \preceq v_n, \tag{15}$$

where $v_n = \frac{1}{2}\mathcal{A}(d(x_{2n-1}, x_{2n})) + d(x_{2n}, x^*)$ is obviously a c-sequence. Since, $\left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)^{-1}$ exists and it is a continuous linear operator we get $d(u_n, x^*) \preceq \left(\mathcal{I} - \frac{1}{2}\mathcal{A}\right)^{-1}(v_n)$, that is, $d(u_n, x^*)$ is a c-sequence. Finally, for the last case, case 5, (11) becomes:

$$d(u_n, x^*) \preceq \mathcal{A}\left(\frac{d(x_{2n-1}, u_n) + d(x^*, x_{2n})}{2}\right) + d(x_{2n}, x^*)$$

$$\preceq \frac{1}{2}\mathcal{A}(d(x_{2n-1}, x^*)) + \frac{1}{2}A(x^*, u_n) + \frac{1}{2}\mathcal{A}(x^*, x_{2n}) + d(x_{2n}, x^*),$$

that is,

$$\left(\mathcal{I} - \frac{1}{2}A\right)(d(u_n, x^*)) \preceq w_n, \tag{16}$$

where $w_n = \frac{1}{2}\mathcal{A}(d(x_{2n-1}, x^*)) + \frac{1}{2}\mathcal{A}(x^*, x_{2n}) + d(x_{2n}, x^*)$ is an obvious c-sequence. In a similar way as in the case 4, we get that $d(u_n, x^*)$ is a c-sequence. The rest of the proof is as in [4].□

Next follows the complete and correct proof for (iii) of ([4], Theorem 2.3.). Our approach goes without the assumption $A_4(v) \preceq A_5(v)$ for all $v \in \mathcal{P}$. For the case of usual metric spaces see [10], [24].

Let $x_0 \in X$ be an arbitrary element. For $i, j \in \{1, 2\}$, with $i \neq j$, take $x_1 \in T_i(x_0)$, there exists $x_2 \in T_j(x_1)$ with $(x_1, x_2) \in E(G)$ such that

$$d(x_1, x_2) \preceq M_2(x_0, x_1; x_1, x_2), \tag{17}$$

where

$$M_2(x_0, x_1; x_1, x_2) = \{\mathcal{A}_1(d(x_0, x_1)) + \mathcal{A}_2(d(x_0, x_1)) + \mathcal{A}_3(d(x_1, x_2))$$

$$+ \mathcal{A}_4 (d (x_0, x_2)) + \mathcal{A}_5 (d (x_1, x_1))\} .$$

Then (17) becomes

$$(\mathcal{I} - (A_3 + \mathcal{A}_4)) (d (x_1, x_2)) \preceq (\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_4) (d (x_0, x_1)) , \tag{18}$$

because

$$d (x_2, x_1) \preceq M_2 (x_1, x_0; x_2, x_1) , \tag{19}$$

where

$$M_2 (x_1, x_0; x_2, x_1) = \{\mathcal{A}_1 (d (x_1, x_0)) + \mathcal{A}_2 (d (x_1, x_2)) + \mathcal{A}_3 (d (x_0, x_1)) \\ + \mathcal{A}_4 (d (x_0, x_0)) + \mathcal{A}_5 (d (x_0, x_1))\} .$$

Hence, (18) becomes

$$(\mathcal{I} - (\mathcal{A}_2 + \mathcal{A}_5)) (d (x_1, x_2)) \preceq (\mathcal{A}_1 + \mathcal{A}_3 + \mathcal{A}_5) (d (x_0, x_1)) . \tag{20}$$

Adding (18) and (20) we get

$$(2\mathcal{I} - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5)) (d (x_1, x_2)) \preceq (2\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5) (d (x_0, x_1)) ,$$

i.e.,

$$d (x_1, x_2) \preceq C (d (x_0, x_1)) , \tag{21}$$

where $C = (2\mathcal{I} - (\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5))^{-1} (2\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5)$. For the norm $\|C\|$ we have:

$$\begin{aligned} \|C\| &\leq \frac{1}{2} \cdot \frac{1}{1 - \left\| \frac{\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{A}_5}{2} \right\|} \cdot (2 \|\mathcal{A}_1\| + \|\mathcal{A}_2\| + \|\mathcal{A}_3\| + \|\mathcal{A}_4\| + \|\mathcal{A}_5\|) \\ &= \frac{1}{2 - \|\mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5\|} \cdot (2 \|\mathcal{A}_1\| + \|\mathcal{A}_2\| + \|\mathcal{A}_3\| + \|\mathcal{A}_4\| + \|\mathcal{A}_5\|) \\ &= \frac{2 \|\mathcal{A}_1\| + \|\mathcal{A}_2\| + \|\mathcal{A}_3\| + \|\mathcal{A}_4\| + \|\mathcal{A}_5\|}{2 - (\|\mathcal{A}_2\| + \|\mathcal{A}_3\| + \|\mathcal{A}_4\| + \|\mathcal{A}_5\|)} < 1, \cdot \end{aligned} \tag{22}$$

because $\|\mathcal{A}_1\| + \|\mathcal{A}_2\| + \|\mathcal{A}_3\| + \|\mathcal{A}_4\| + \|\mathcal{A}_5\| < 1$. In the same way one can obtain the next relation:

$$d (x_n, x_{n+1}) \preceq C (d (x_{n-1}, x_n)) , \tag{23}$$

for all $n \in \mathbb{N}$. By routine (and well-known) method (23) implies that the sequence $\{x_n\}$ is a Cauchy sequence in a cone complete metric space (X, d) . Since (X, d) is cone complete, there exists a point $x^* \in X$ in it such that the sequence $d (x_n, x^*)$ is a c-sequence, i.e., $x_n \to x^*$ as $n \to +\infty$. In the sequel we will prove also that $u_n \to x^*$ as $n \to +\infty$, that is the sequence $d (x^*, u_n)$ is a c-sequence. For this we have

$$\begin{aligned} d (x^*, u_n) &\preceq d (x^*, x_{2n}) + d (x_{2n}, u_n) \\ &\preceq d (x^*, x_{2n}) + M_2 (x_{2n-1}, x^*; x_{2n}, u_n) \\ &= \{d (x^*, x_{2n}) + \mathcal{A}_1 (d (x_{2n-1}, x^*)) + \mathcal{A}_2 (d (x_{2n-1}, x_{2n})) \\ &\quad + \mathcal{A}_3 (d (x^*, u_n)) + \mathcal{A}_4 (d (x_{2n-1}, u_n)) + \mathcal{A}_5 (d (x^*, x_{2n}))\} . \end{aligned} \tag{24}$$

Further, it yields that

$$(\mathcal{I}-\mathcal{A}_3)\left(d\left(x^*,u_n\right)\right) \preceq \{d\left(x^*,x_{2n}\right)++\mathcal{A}_1\left(d\left(x_{2n-1},x^*\right)\right)+\mathcal{A}_2\left(d\left(x_{2n-1},x_{2n}\right)\right)$$
$$+\mathcal{A}_4\left(d\left(x_{2n-1},u_n\right)\right)+\mathcal{A}_5\left(d\left(x^*,x_{2n}\right)\right)\}=z_n, \tag{25}$$

where z_n is, obviously, a c-sequence. Since, also obviously, $(\mathcal{I}-\mathcal{A}_3)^{-1}$ exists then (25) becomes

$$d\left(x^*,u_n\right) \preceq (\mathcal{I}-\mathcal{A}_3)^{-1}\left(z_n\right), \tag{26}$$

and $d\left(x^*,u_n\right)$ is a c-sequence. The proof of (iii) is complete. □

3. Some remarks on Perov's type results

Throughout this section we denote by $\mathcal{M}_{m,m}$ the set of all $m\times m$ matrices, and by $\mathcal{M}_{m,m}(\mathbb{R}^+)$ the set of all $m\times m$ matrices with non-negative elements. It is well known that if $M\in\mathcal{M}_{m,m}$ then $M(\mathcal{P})\subseteq\mathcal{P}$ if and only if $M\in\mathcal{M}_{m,m}(\mathbb{R}^+)$. We write Θ for the zero $m\times m$ matrix and $I_{m,m}$ for the identity $m\times m$ matrix. For the sake of simplicity we will identify row and column vector in $\mathbb{R}^m$. A matrix $M\in\mathcal{M}_{m,m}(\mathbb{R}^+)$ is said to be convergent to Θ if $M^n\to\Theta$ as $n\to+\infty$.

In 1964 A.I. Perov formulated his main result as follows:

Theorem 3.1. *([21], Theorem 3). Let the operator $U:F\to F$ where F is a closed subset of a generalized metric space (R,p) with $p\,(Ux,Uy)<+\infty$, for $x,y\in F$. Suppose further that the following contractive condition holds:*

$$p\,(Ux,Uy)\leq S\,p\,(x,y)\,,\ \ (x,y\in UF)\,, \tag{27}$$

where matrix $S\in\mathcal{M}_{m,m}(\mathbb{R}^+)$ is an $a-$matrix. Then the operator U has a unique fixed point $x^*\in F$ that can be obtained by method of successive approximation given with the sequence $x^{(m+1)}=Ux^{(m)}$, $(m=0,1,\dots)$ where the beginning point $x^{(0)}\in F$. In this case the next estimate holds:

$$p\left(x^{(m+1)},x^*\right)\leq(I-S)^{-1}\,S^m p\left(x^{(1)},x^{(2)}\right),(m=0,1,\dots)\,. \tag{28}$$

In this section we will prove Perov's type theorem in the framework of complete cone metric spaces over solid cone. Our approach is much simpler and shorter than ones in recently announced papers. We use only the property that the linear operator $\mathcal{A}$ is continuous with the spectral radius $r(\mathcal{A})<1$. Because of this it follows that mapping $T:X\to X$ is continuous. Firstly, we give the formulation of A.I. Perov's result in a new way:

Theorem 3.2. *Let (X,d) be a complete cone metric space over solid cone $\mathcal{P}$ and let T be a self-mapping on X. Suppose that there exists a linear continuous operator $\mathcal{A}$ on Banach space $\mathcal{E}$ such that $r(\mathcal{A})<1$ and for all $x,y\in X$, $d(T\,(x)\,,T\,(y))\preceq\mathcal{A}(d(x,y))$ holds true. Then T has a unique fixed point in X say $x^*$and for each x from X the corresponding Picard sequence $T^n(x)$ tends to point x^*. Further, $d(x_n,x^*)\preceq\mathcal{A}^n(I-\mathcal{A})^{-1}(d(x_0,x_1))$.*

Proof. If $\mathcal{A}$ is a continuous linear operator then it also implies that T is continuous mapping from X to X. Indeed, if x_n belongs to X and x_n tends to x in X, then from the contractive condition it follows that

$d(T(x_n), T(x)) \preceq \mathcal{A}(d(x_n, x)$. From this, we get that $d(T(x_n), T(x))$ is a c-sequence, i.e., $T(x_n)$ tends to $T(x)$ or T is continuous. We used the fact that a continuous image of a c-sequence is a c-sequence. Also, if x, y are two different fixed points of T we have: $d(x, y) = d(T(x), T(y)) \preceq \mathcal{A}(d(x, y))$ or $(I - \mathcal{A})(d(x, y)) \preceq \theta$. Since $r(\mathcal{A}) < 1$ then there exists $(I - \mathcal{A})^{-1}$ and then we get $d(x, y) \preceq (I - \mathcal{A})^{-1}(\theta) = \theta$, that is $x = y$.

Now, we prove the existence of a fixed point of T.

Let x_0 be an arbitrary point in X. Consider a Picard sequence: $x_n = T(x_{n-1}), n = 1, 2, \ldots$. If $x_k = x_{k-1}$ for some k from $\mathbb{N}$ then x_{k-1} is a unique fixed point of T. Suppose that x_n is different from x_{n-1} for each n from $\mathbb{N}$. For $d(x_n, x_{n+1})$ we have:

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \preceq \mathcal{A}(d(x_{n-1}, x_n)) \preceq \mathcal{A}^n(d(x_0, x_1)).$$

Now for $n < m$ we get

$$d(x_n, x_m) \preceq (\mathcal{A}^n + \mathcal{A}^{n+1} + \cdots + \mathcal{A}^{m-1} + \mathcal{A}^m)(d(x_0, x_1)) \preceq (\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1})(d(x_0, x_1)).$$

Because $r(A) < 1$ we obtain that $(\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1})(d(x_0, x_1)) \to \theta$ as $n \to +\infty$. This further means that $d(x_n, x_m)$ is a c-sequence, i.e., $\{x_n\}$ is a Cauchy sequence in complete cone metric space (X, d). Therefore, x_n tends to some point in X say x^*. Since T is a continuous self-mapping we get that $T(x_n)$ tends to $T(x^*)$. Since a sequence in cone metric space has a unique limit, we get that $T(x^*) = x^*$, that is x^* is a unique fixed point of T.

If x is some given point in X then by previous method we obtain that the corresponding Picard sequence $T^n(x)$ converges to the already obtained fixed point x^* (because of uniqueness of fixed point).

Now, we will estimate $d(x_n, x^*)$. Firstly we have:

$$d(x_n, x^*) \preceq d(x_n, x_m) + d(x_m, x^*) \preceq (\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}))(d(x_0, x_1)) + d(x_m, x^*).$$

Since, $d(x_m, x^*)$ is a c-sequence the result follows, i.e., $d(x_n, x^*) \preceq (\mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1})(d(x_0, x_1))$.

The proof of theorem is complete. □

Remark 3.1. *If $a \preceq b + x_n$ then $a \preceq b$ whenever x_n is a c-sequence. Indeed, by the definition of c-sequence we get that for any c from int$\mathcal{P}$ there is n_0 in $\mathbb{N}$ such that $x_n \preceq c$ for $n > n_0$. Hence, for $n > n_0$ we have that $a \preceq b + c$. Putting further $\frac{1}{n} \cdot c$ instead of c and taking the limit as $n \to +\infty$ we obtain that $a \preceq b$. Indeed, since $\theta \preceq b - a + \frac{1}{n} \cdot c$ we have that* $\lim_{n \to +\infty} \left(b - a + \frac{1}{n} \cdot c\right) = b - a + 0 \cdot c = b - a \in \mathcal{P}$ *because $\mathcal{P}$ is closed.*

Hence (X, d) is a complete generalized metric space over normal (and clearly) solid cone $\mathcal{P} = \{(u_1, \ldots, u_m) : u_i \geq 0 \text{ for } i = 1, 2, \ldots, m\}$**,** with the coefficient of normality $K = 1$. This further means that there is an ordinary metric D on X such that (X, D) is complete metric space where $D(x, y) = \|d(x, y)\|$ (for more details see for example [5], [14]). Since M^n tends to Θ as $n \to +\infty$ then there exists n_0 in $\mathbb{N}$ such that $\|M^{n_0}\| < 1$. Putting $\|M^{n_0}\| = k$ we get that the given contractive condition $d(T(x), T(y)) \preceq M \cdot d(x, y)$ becomes $D(T^{n_0}(x), T^{n_0}(y)) \leq kD(x, y)$. The last contractive condition yields that T^{n_0} has a unique fixed point x^* in X (by Banach contraction principle [7]), i.e., T has a unique fixed point x^*. The Perov's theorem is proved.

Corollary 3.1. *Perov's mapping T has a property (P), that is, T and any of its iterations T^n have the same set of fixed points $(Fix(T) = Fix(T^n))$. For more details see [15, 16].*

Proof. Let x^* be a fixed point of T. Then it is a fixed point for T^{n_0} where n_0 is a given natural number. The given contractive condition $d(T(x), T(y)) \preceq Md(x, y)$ implies $d(T^{n_0}(x), T^{n_0}(y)) \preceq M^{n_0} \cdot d(x, y)$. Putting $T^{n_0} = S$, $M^{n_0} = B$ we get Perov's type contractive condition in the form: $d(S(x), S(y)) \preceq B \cdot d(x, y)$. Estimate $\|B^n\| : \|(M^{n_0})^n\| = \|(M^n)^{n_0}\| \leq |\|M^n\|^{n_0} < 1^{n_0} = 1$. This means that according to the previous Theorem $S = T^{n_0}$ has a unique fixed point as T, i.e., $Fix(T) = Fix(T^{n_0}) = \{x^*\}$. □

Remark 3.2. *Our proof is completely different from the one in A.I. Perov's paper from 1964. We did not use the fact that the cone $\mathcal{P}$ is solid. The same method can be used if (X, d) is any complete cone metric space over normal non-solid cone $\mathcal{P}$.*

The following result generalizes A.I. Perov's theorem:

Theorem 3.3. *Let (X, d) be a complete cone metric space over solid cone $\mathcal{P}$ in Banach space $\mathcal{E}$. Further, let T be a self-mapping on X such that there exists a continuous linear operator $\mathcal{A}$ on $\mathcal{E}$ with $r(\mathcal{A}) < 1$. Then T has a unique fixed point (say x^*) in X and for each x from X the corresponding Picard sequence $T^n(x)$ converges to x^*. Further, for $n < m$ we get $d(x_n, x^*) \preceq \mathcal{A}^n(\mathcal{I} - \mathcal{A})^{-1}(d(x_0, x_1))$.*

Proof. First of all, the given contractive condition yields that T is a continuous mapping. Indeed, if x_n tends to x from X under the cone metric d, we get: $d(T(x_n), T(x)) \preceq \mathcal{A}(d(x_n, x))$. Since $\mathcal{A}$ is continuous then $\mathcal{A}(d(x_n, x))$ is a c-sequence and therefore $d(T(x_n), T(x))$ is also a c-sequence, which means that $T(x_n)$ tends to $T(x)$ under the cone metric d. Hence, T is a continuous self-mapping. The rest it similar to previous proofs. □

4. Conclusions

In this paper we gave much simpler and shorter proofs of multivalued Perov's type results with respect to the ones presented in the recently published paper by M. Abbas et al. [4], by using the approach of so-called c-sequences. Further, in the last section of this paper, we corrected and generalized the well-known Perov's fixed point result. We showed that this result is in fact equivalent to Banach's contraction principle.

Acknowledgments

The research of author N.M. was funded by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

Conflict of interest

Authors state no conflict of interest.

References

1. M. Abbas, G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric space, *J. Math. Anal. Appl.*, **341** (2008), 416–420. doi: 10.1016/j.jmaa.2007.09.070.

2. M. Abbas, B. E. Rhoades, T. Nazir, Common fixed points of generalized contractive multivalued mappings in cone metric spaces, *Math. Commun.*, **14** (2009), 365–378. Available from: `https://hrcak.srce.hr/44029`.
3. M. Abbas, B. E. Rhoades, Fixed point theorems for two new classes of multivalued mappings, *Appl. Math. Lett.*, **22** (2009), 1364–1368. doi: 10.1016/j.aml.2009.04.002.
4. M. Abbas, T. Nazir, V. Rakočević, Common fixed points results of multivalued Perov type contractions on cone metric spaces with a directed graph, *Bull. Belg. Math. Soc. Simon Stevin*, **25** (2018), 331–354. doi: 10.36045/bbms/1536631231.
5. S. Aleksić, Z. Kadelburg, Z. D. Mitrović, S. Radenović, A new survey: Cone metric spaces, *J. Int. Math. Virtual Inst.*, **9** (2019), 93–121. doi: 10.7251/JIMVI1901093A.
6. C. D. Aliprantis, R. Tourky, *Cones and Duality,* Graduate Studies in Mathematics 84, American Mathematical Society, Providence, RI, (2007).
7. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.*, **3** (1922), 133–181. doi: 10.4064/fm-3-1-133-181.
8. K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, 1985.
9. F. Vetro, S. Radenović, Some results of Perov type on rectangular cone metric spaces, *J. Fixed Point Theory Appl.*, **20**, Article number: 41 (2018). doi: 10.1007/s11784-018-0520-y.
10. G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, *Canad. Math. Bull. Vol.*, **16** (1973), 201–206. doi: 10.4153/CMB-1973-036-0.
11. L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. doi: 10.1016/j.jmaa.2005.03.087.
12. J. Jachymski, J. Klima, Cantor's intersection theorem for K-metric spaces with a solid cone and a contraction principle, *J. Fixed Point Theory Appl.*, **18** (2016), 445–463. doi: 10.1007/s11784-016-0312-1.
13. J. Jachymski, J. Klima, Around Perov's fixed point theorem for mappings on generalized metric spaces, *Fixed Point Theory*, **17** (2016), 367–380.
14. S. Janković, Z. Kadelburg, S. Radenović, On cone metric spaces: A survey, *Nonlinear Anal.*, **74** (2011), 2591–2601. doi: 10.1016/j.na.2010.12.014.
15. G. S. Jeong, B. E. Rhoades, Maps for which $F(T) = F(T^n)$, In: *Fixed Point Theory and Applications, Volume 6* (Eds. Yeol Je Cho et al.), Nova Science Publishers, Inc., (2007), 71–105. ISBN 1-59454-873-0.
16. G. S. Jeong, B. E. Rhoades, More maps for which $F(T) = F(T^n)$, *Demonstr. Math.*, **40** (2006), 671–680. doi: 10.1515/dema-2007-0317.
17. G. Jungck, S. Radenović, S. Radojević, V. Rakočević, Common fixed point theorems for weakly compatible pairs on cone metric spaces, *Fixed Point Theory Appl.*, **2009**, Article ID 643840, 1–13. doi: 10.1155/2009/643840.
18. Z. Kadelburg, S. Radenović, A note on various types of cones and fixed point results in cone metric spaces, *Asian J. Math. Appl.*, **2013**, Article ID ama0104.

19. A. Latif, I. Beg, Geometric fixed points for single and multivalued mappings, *Demonstr. Math.*, **30** (1997), 791–800. doi: 10.1515_dema-1997-0410.

20. N. Mirkov, S. Radenović, S. Radojević, Some new observations for F-contractions in vector-valued metric spaces of Perov's type, *Axioms*, **10** (2021), 127. doi: 10.3390/axioms10020127.

21. A. I. Perov, On Cauchy problem for a system of ordinary differential equations, *Pviblizhen. Met. Reshen. Differ. Uravn.*, **2** (1964), 115–134.

22. S. Radenović, F. Vetro, S. Xu, Some results of Perov type mappings, *J. Adv. Math. Stud.*, **10** (2017), 396–409.

23. S. Radenović, F. Vetro, Some remarks on Perov type mappings in cone metric spaces, *Mediterr. J. Math.*, **14** (2017), Article number: 240 (2017). doi: 10.1007/s00009-017-1039-y.

24. B. E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, **226** (1997), 257–290. doi: 10.1090/S0002-9947-1977-0433430-4.

25. I. A. Rus, A. Petrusel, A. Sintamarian, Data dependence of fixed point set of some multivalued weakly Picard operators, *Nonlinear Anal.*, **52** (2003), 1944–1959. doi: 10.1016/S0362-546X(02)00288-2.

26. S. Xu, Ć. Doličanin, S. Radenović, *Some remarks on results of Perov type, J. Adv. Math. Stud.*, **9** (2016), 361–369.

27. P. P. Zabrejko, K-metric and K-normed linear spaces; survey, *Collect. Math.,* **48** (1997), 825–859. Available from: `http://eudml.org/doc/41051`.

AIMS Press