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CATENARY – ITS HISTORY, SIGNIFICANCE AND APPLICATION

LANČANICA – NJENA ISTORIJA, ZNAČAJ I PRIMENA

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Ključne reči
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Abstract

The catenary as an optimal architectural form has been known since ancient times. Its mathematical definition was discovered in 17th century and since then, this planar curve has become one of the most common shapes, not only in architecture, but also in the field of engineering.

Izvod

Lančanica, kao optimalna arhitektonska forma, poznata je još od davnina. Svoju konačnu matematičku formulaciju dobila je tek u sedamnaestom veku i od tada, ova ravanska kriva, postala je jedan od najčešće korišćenih oblika, ne samo u arhitekturi, već i u polju tehničkih nauka.

INTRODUCTION

The catenary (alysoid, chainette) is a planar curve representing the form that a uniform hanging chain or cable assumes under the force of gravity (i.e. its own weight) from two supports at its ends (Fig. 1). It is superficially similar to the parabola, but unlike this algebraic curve, the catenary belongs to the group of transcendental curves. By rotating the catenary curve about an axis, one gets the catenoid, a type of minimal surface, particularly a minimal surface of revolution first described in 1744 by mathematician Leonhard Euler /1-3/.

In 1637, Rene Descartes published his *La Geometrie* where he described the ground-breaking connection between a curve's construction and its algebraic equation. Moreover, he made an algebraic classification of curves based on the degree of equations used to represent these curves. However, he omitted transcendental curves, i.e. curves with no polynomial equation, from his classification. As the fields of infinitesimal calculus and mathematical mechanics rapidly progressed, Descartes' synthesis failed and decades after, Gottfried Wilhelm von Leibniz mathematically described the catenary, /1, 4/.

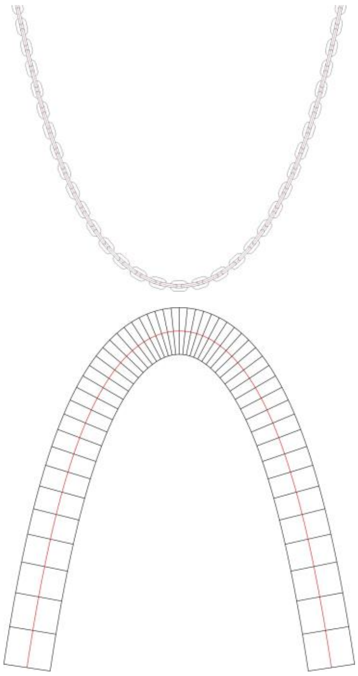


Figure 1. Chain and inverted variant.

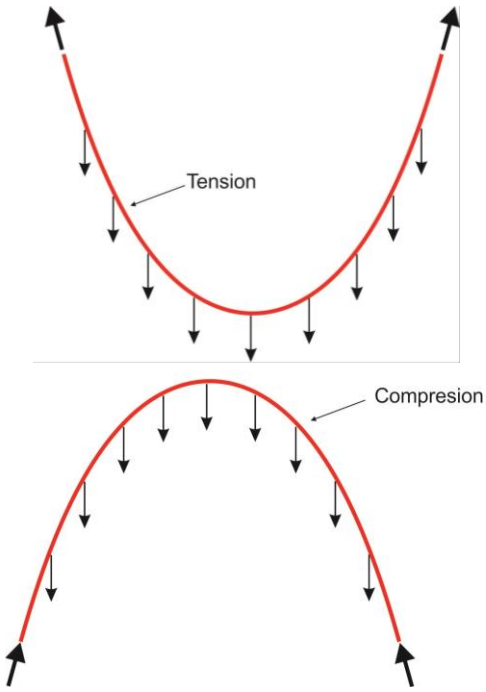


Figure 2. Hanging chain and inverted catenary under loading.

In general, the catenary is a form of a flexible, non-rigid supporting element that transfers only axial, tensile forces and when inverted, it takes form of an idealized rigid arch, which when loaded with uniform distributed load (dead load) is free from bending moments and stressed by axial compression only, Fig. 2, /2, 5/.

HISTORICAL BACKGROUND

The story behind the catenary is very interesting in itself. For the first time, catenary was defined by Guidobaldo Del Monte (1545-1607), one of the most prominent Italian mathematicians of the sixteenth century who was the representative of the mathematical school in Urbino /2, 6/. His studies were continued by his most notable student and friend at the same time, Galileo Galilei (1564-1642). Galileo was one of the most significant scientists in human history and in his *Discorsi* in 1638, he wrote much about strength of materials and cross-sections of beams mentioning parabola in these contexts. The trajectory of a projectile (for artillery) served as an example of this curve: ‘The other method of drawing the desired curve is the following: Drive two nails into a wall at a convenient height and at the same level. Over these two nails hang a light chain. This chain will take the form of a parabola,’ /2, 7/. Following his instructions for drawing such a curve, the resulting curve certainly looked like it could be a parabola, and for many years it was widely accepted that a hanging chain did take form of it. This assumption was disproved by Joachim Jungius (1587-1657), a German mathematician, and it was published posthumously in 1669, /2/. The name catenary was coined in 1690 by Christiaan Huygens, a Dutch physicist and mathematician, who first used that term in a letter to Gottfried Wilhelm von Leibniz. The correct equation of catenary was obtained independently by Leibniz, Huygens, and Johann Bernoulli (younger brother of Jacob Bernoulli, a mathematician who discovered the mathematical constant *e*) in 1691, Fig. 3, /2, 8/.

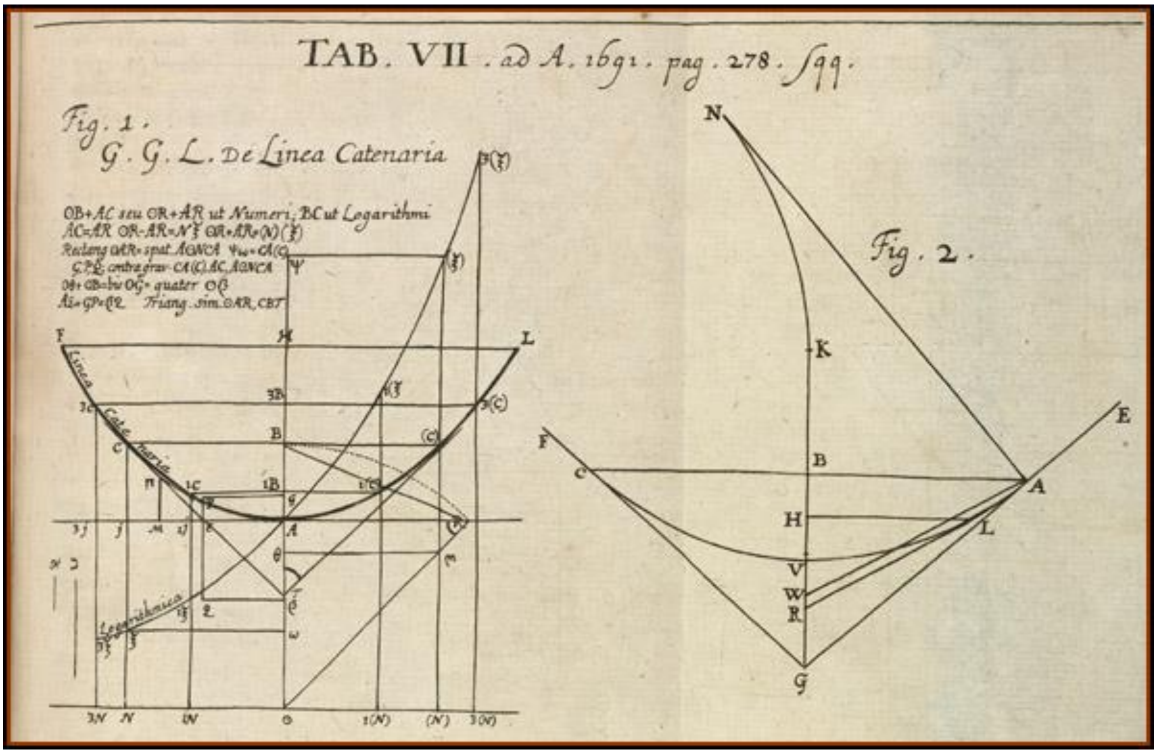


Figure 3. Leibniz’s solution on the left and Huygens’ illustration on the right (Huntington Library, San Marino, California), /9/.

EQUATION OF CATENARY

The catenary curve is mathematically described by a hyperbolic cosine curve, as follows in Cartesian coordinates:

$$y = a\cosh\left(\frac{x}{a}\right) = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \qquad (1)$$

where: *x* and *y* are Cartesian coordinates; and *a* is a scaling factor, Fig. 4.

If assumed that a cable has the shape of a catenary, the scaling factor in Eq.(1) may be thought of as a ratio between the horizontal tension on the cable and the weight of the cable per unit length. A low scaling factor will therefore result in a deeper curve.

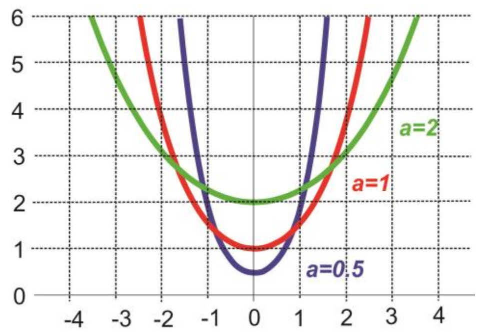


Figure 4. The shape of the catenary due to the scaling factor *a*.

NUMERICAL ANALYSIS

In the field of engineering and architecture, over hundreds of years, the imperative was to provide the covering over a large area by using less and less material. Therefore, an experiment is conducted using two software packages for 3D modelling – CATIA® (Computer Aided Three-Dimensional Interactive Application) and SolidWorks®. These programmes are leading solutions for product design and experience. The force of 100 N is applied on a thin arch of finite thickness made of steel, that has the form of catenary and parabola, respectively. It is noticed that the distribution of stress induced by the force is more uniform on the arch of the catenary form, Figs. 5-6.

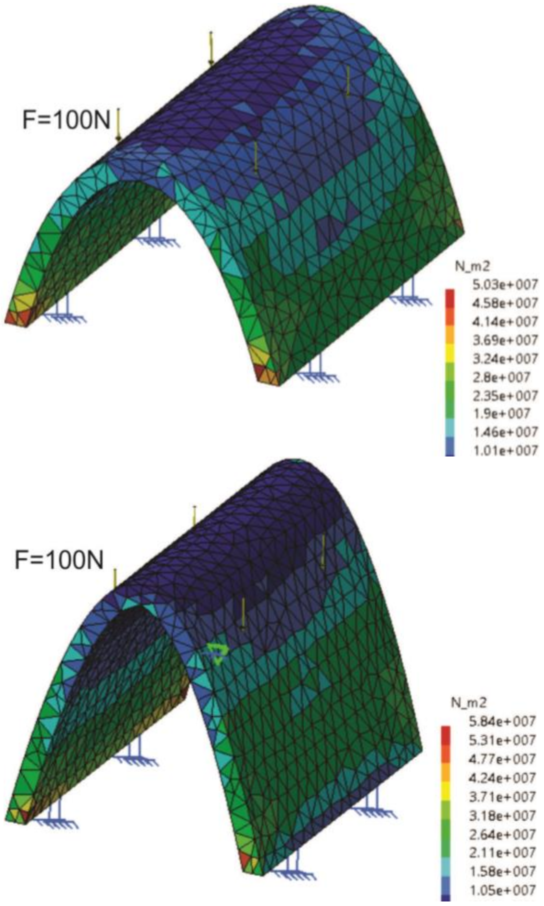
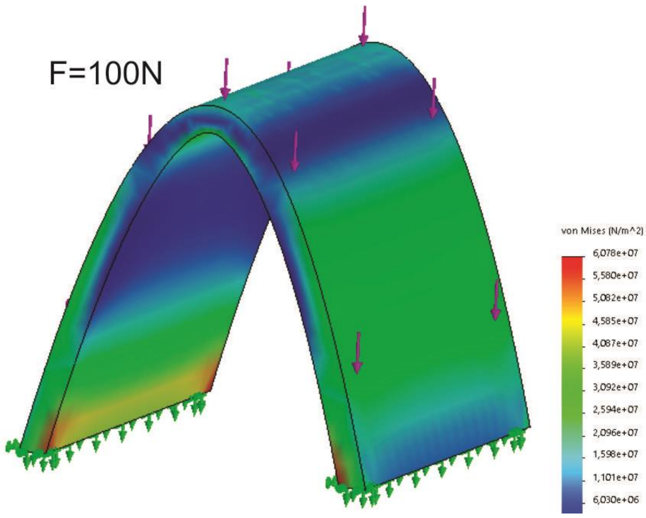


Figure 5. Stress distribution on thin arch of finite thickness in the form of catenary (top) and parabola (bottom) – CATIA®.



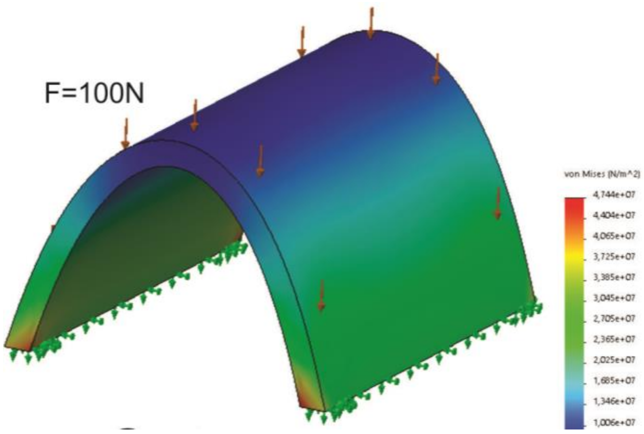


Figure 6. Stress distribution on thin arch of finite thickness in the form of catenary (top) and parabola (bottom) – SolidWorks®.

APPLICATION

The catenary, as other curves, can be found almost everywhere within our surrounding, from animals' creations to marvellous buildings and architectural wonders. One of the best known examples of this planar curve found in nature is certainly a spider web - the silk on it forms multiple elastic catenaries, Fig. 7.

Figure 7. Catenary curve on spider web, /10/.

In the offshore oil and gas industry, a steel catenary riser, a pipeline suspended between a production platform and the seabed has an approximate catenary shape. When it comes to the rail industry, the overhead wiring that transfers power to trains adopts the form of previously mentioned curve. Even the electrical wires are of this shape - due to the heat and force of gravity the electrical connection wire elongates, and due to the metal present in the wire itself it expands when exposed to sunlight, so these powerlines eventually take the form of catenary. The value of parameter a in the hyperbolic cosine equation should be greater than 2 when it comes to electrical power wires. In the field of optics and electromagnetics, the hyperbolic sine and cosine functions are fundamental solutions to Maxwell's equations (a set of coupled partial differential equations that, together with the Lorenz force law, represent the basis of classical electromagnetism and optics), /1, 2, 6/.

With regard to application of catenary curve in architecture, its inverted version represents one of architecture's perfect shapes. If the arch has no additional load or stress placed upon it, it produces no excess shear or pressure on its building materials, which means that the arch will stand on its own indefinitely with no outer support or maintenance. Moreover, the distribution of load is more uniform when the arch has the form of inverted catenary than inverted parabola, Fig. 8, /1, 2, 11/.

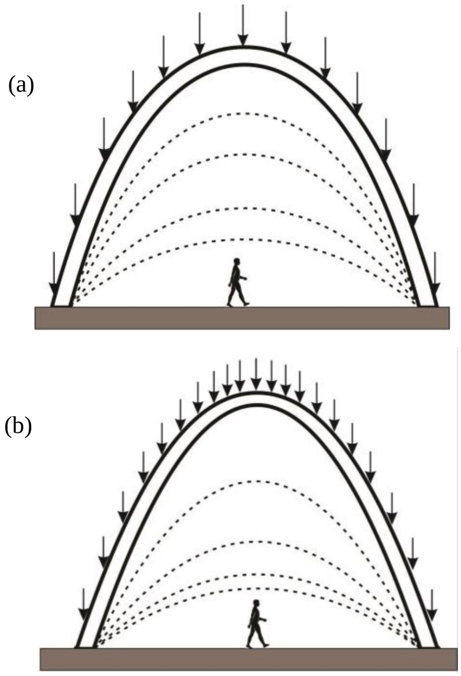


Figure 8. Load distribution on the arch of: a) inverted catenary; and b) parabola shape.

Early examples of such arches are found in Taq-i Kisra in Ctesiphon (Mesopotamia), 6th century BC, while Greek and Roman cultures opted for circular aches and semispherical vaults, where the curvature of a circle is much less statically efficient, /2/. For a long period of time, catenary remained forgotten in Europe, but endured in Islamic architecture. It is assumed that Robert Hooke, famous English mathematician, responsible for the discovery of the law of elasticity, was the one who recognized the long forgotten shape while rebuilding St Paul's Cathedral. In 1671 Hooke announced to the Royal Society that he had solved the problem of the optimal shape of an arch, and in 1675 published an encrypted solution in his *Description of Helioscopes* /2, 6/. Later on, Antonio Gaudí (1852-1926), the brilliant Catalan architect, studied thoroughly the catenary and incorporated it in his work, Fig. 9. As the representative of Art Nouveau, the style mostly inspired by natural forms, Gaudí's creations consist of various shapes expressed by the family of ruled geometrical forms (conoid as a form in tree leaves, hyperboloid as a form in femur, helicoid as a form in the trunk of the eucalyptus tree) and supported by catenary arches, where the weight is distributed evenly and is affected only by self-cancelling tangential forces. The extensive use of catenary shapes was made in the Sagrada Familia, Casa Mila and in the crypt of the Church of Colònia Güell, /1, 2, 6, 12/.

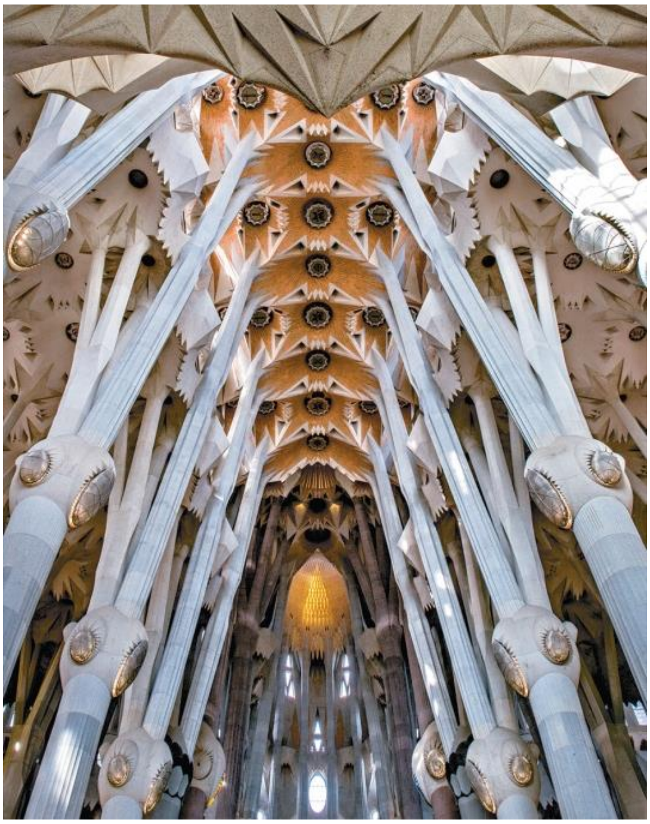


Figure 9. Catenary arches in Sagrada Familia, /13/.

A catenoid, a type of surface that arises by rotating a catenary curve about an axis, was discovered in the 18th century. Catenoids are the only type of minimal surfaces that are also surfaces of revolution, and the soap film that is stretched between two circular rings is a typical example of them. Soap films naturally tend to minimise their mean curvature. When two rings move apart, the radius of the neck of the soap film will decrease until it reaches zero and the soap film is split, Fig. 10, /3/.

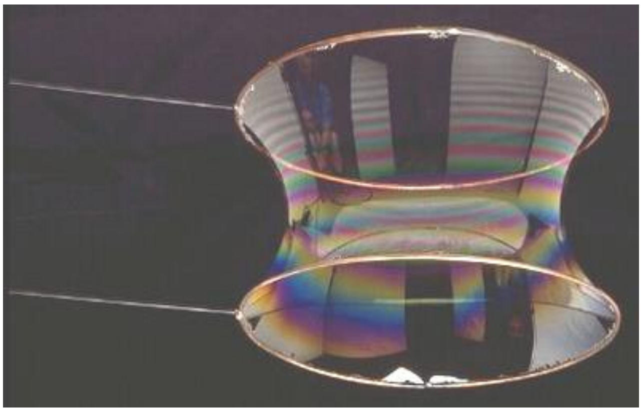


Figure 10. Soap bubble with wire boundary as the example of a catenoid, /14/.

CONCLUSION

Importance of geometry on structural integrity is emphasized in this paper, as it was done before, /15, 16/. It is shown that the catenary and its variants represent very important curves in architecture and engineering. Starting from the 17th century, when the catenary was introduced mathematically for the first time, it has been applied in numerous architectural works. However, many ancient structures, such as Brunelleschi's dome, contain this curve, long before its formal discovery, thus approving that the builders knew this shape was the best way to bear weight of buildings. Long standing structures certainly prove great contribution of the catenary to their integrity, not to mention the beauty.

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