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Some New Observations for F-Contractions in Vector-Valued Metric Spaces of Perov's Type

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Abstract: The main purpose of this article is to improve, generalize and complement some recently established results for Perov's type F-contractions. In our approach, we use only the property (F1) of Wardowski while other authors employed all three conditions. Working only with the fact that the function F is strictly increasing on $(0,+\infty)^m$, we obtain as a consequence new families of contractive conditions in the realm of vector-valued metric spaces of Perov's type. At the end of the article, we present an example that supports obtained theoretical results and genuinely generalizes several known results in existing literature.

Keywords: fixed point; vector-valued metric; pseudometric; Perov type F-contraction

MSC: Primary 54H25; Secondary 47H10; 65F15

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Citation: Mirkov, N.; Radenović, S.; Radojević, S. Some New Observations for F-Contractions in Vector-Valued Metric Spaces of Perov's Type. *Axioms* **2021**, *10*, 127. https://doi.org/10.3390/axioms10020127

Academic Editors: Sergei Silvestrov and Wei-Shih Du

Received: 8 May 2021
Accepted: 18 June 2021
Published: 21 June 2021

1. Introduction and Preliminaries

Since Banach [1] proved his famous theorem in 1922, many hundreds of researchers have tried generalizing this result. The generalization went primarily in two main directions. One is to generalize the contractive condition and the other is to alter the axioms of metric space. Thanks to the second condition, new classes of so-called generalized metric spaces (vector-valued metric spaces) have emerged, such as, cone metric spaces ($d(x,y)$ is a vector), then b-metric spaces, partial metric spaces, metric–like spaces, and many others. Some researchers have combined these two directions. Such is the recent work of Altun and Olgun [2] in which they combine Wardowski's approach from 2012 [3] and the cone of metric space introduced by Đ. Kurepa [4] in 1933. On the other hand, in 1964, A. I. Perov [5] introduced special cone metric spaces called generalized metric space or vector-valued metric space. For more details on this subject, see ([6–26]), and regarding some recent results pertinent to F-contraction mappings, see [27–30].

The definitions of Kurepa and Perov are in fact the same, except that, in the Perov case, the vector space $\mathbb{R}^m$ is taken as the Banach space where m is a given natural number. The axioms of the cone metric and the vector-valued metric space are based on three well-known conditions, present also in the case of ordinary metric space of Frechét, but imposed on cone metric and vector-valued distance, respectively.

It should be noted, as it is useful for young researchers, that, according to many results, vector-valued metric spaces, i.e., cones of metric spaces in the sense of Perov, do not differ from ordinary metric spaces. Moreover, many results of fixed point theorems in relation to known contractive conditions are equivalent to those in ordinary metric spaces. For details, see [10,12,17].

Thus, for example, it is easy to see that, for each contractive condition of the Banach, Kannan, Chatterjee, Zamfirescu, or Hardy–Rogers type, the generalized metric space of Perov's type corresponds to the contractive conditions in ordinary metric space. Using this fact, in this section, we will show that the recent results of Altun and Olgun are equivalent to the corresponding results in ordinary metric space.

Recently, in 2020, Altun and Olgun introduced and proved the following:

Definition 1 ([2], Definition 1)**.** *Let (X,d) be a vector-valued metric space and $T : X \to X$ be a map. If there exist $F \in \mathcal{F}^m$ and $\tau = (\tau_i)_{i=1}^m \in \mathbb{R}_+^m$, such that:*

$$\tau + F(d(Tu,Tv)) \preceq F(d(u,v)), \tag{1}$$

for all $u, v \in X$ with $d(Tu,Tv) \succ \theta$, then T is called a Perov type F-contraction.

From [2], it follows that $F \in \mathcal{F}^m$ if it satisfies the next properties.

(F1) F is strictly increasing in each variable, i.e., for all $(-\infty,+\infty)^m = \mathbb{R}^m \ni u = (u_i)_{i=1}^m, (-\infty,+\infty)^m = \mathbb{R}^m \ni v = (v_i)_{i=1}^m$, such that $u \prec v$, and, then, $F(u) \prec F(v)$,

(F2) For each sequence, $\{u_n\} = \left\{\left(u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(m)}\right)\right\}$ of $\mathbb{R}_+^m = (0,+\infty)^m$

$$\lim_{n\to+\infty} u_n^{(i)} = 0 \text{ if and only if } \lim_{n\to+\infty} v_n^{(i)} = -\infty \tag{2}$$

for each $i \in \{1,2,\ldots,m\}$, where:

$$F\left(\left(u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(m)}\right)\right) = \left(v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}\right). \tag{3}$$

(F3) There exists $k \in (0,1)$, such that $\lim_{u_i\to0^+} u_i^k v_i = 0$ for each $i \in \{1,2,\ldots,m\}$, where:

$$F((u_1,u_2,\ldots,u_m)) = (v_1,v_2,\ldots,v_m). \tag{4}$$

Authors in [2] denote by $\mathcal{F}^m$ the set of all functions F satisfying **(F1)–(F3)**.

Theorem 1 ([2], Theorem 3)**.** *Let (X,d) be a complete vector-valued metric space and $T : X \to X$ be a Perov type F-contraction. Then, T has a unique fixed point.*

Remark 1. *It is not difficult to notice that, in the case $m = 1$, the vector valued metric space (X,d) is actually an ordinary metric space, while the properties **(F1)–(F3)** are equal to those introduced by Wardowski in his work of 2012 [3]. Note also that it is sufficient to consider only the case $m = 2$, i.e., Banach space $\left(\mathbb{R}^2, \|.\|_e\right)$ where the norm $\|.\|_e$ is Euclidean, and it is ordered by the cone $P = \{(a,b) : a \geq 0 \text{ and } b \geq 0\}$. Cases $m = 3,4,\ldots$ add only the burden of writing but do not change the essence of the results. Thus, in all announced articles for vector-valued metric spaces of Perov's type, $d(u,v)$ can be written as an ordered pair $d(u,v) = (d_1(u,v), d_2(u,v))$ of non-negative real numbers $d_1(u,v)$ and $d_2(u,v)$. The functions $d_i : X^2 \to [0,+\infty)$ are in fact pseudometrics defined on the non-empty set X (for more details, see [16], Proposition 2.1).*

Hence, if $X \neq \emptyset$ and $d : X^2 \to \mathbb{R}^m$ is a generalized metric i.e., vector–valued metric, then $d(u,v) = (d_i(u,v))_{i=1}^m$, where all $d_i : X^2 \to [0,+\infty), i \in \{1,2,\ldots,m\}$ are pseudometrics. In addition, it is worth knowing that at least one d_i is an ordinary metric (see [16], Proposition 2.1). Furthermore, if $F \in \mathcal{F}^m$, then $F = (F_i)_{i=1}^m$, where $F_i : [0,+\infty) \to (-\infty,+\infty)$ for all $i \in \{1,2,\ldots,m\}$.

It is well known that vector space $\mathbb{R}^m$ is ordered by: $u = (u_i)_{i=1}^m, v = (v_i)_{i=1}^m \in \mathbb{R}^m, u \preceq v$ if and only if $u_i \leq v_i$ for all $i \in \{1,2,\ldots,m\}$. In addition, in $\mathbb{R}^m$, the following three standard norms are given with $u = (u_i)_{i=1}^m$:

$$\|u\|_1 := \sum_{i=1}^m |u_i|, \|u\|_2 := \left(\sum_{i=1}^m u_i^2\right)^{\frac{1}{2}}, \text{ and } \|u\|_\infty := \max_{1\leq i\leq m} |u_i|. \tag{5}$$

It is easily seen that each of these norms is monotone with respect to the partial ordering defined by $u \preceq v$ if and only if $u_i \leq v_i$ for all $i \in \{1, 2, \ldots, m\}$ if we restrict to vectors in $\mathbb{R}^m$ with non-negative coordinates, i.e.,

$$\text{if } u, v \in \mathbb{R}^m \text{ and } \theta \preceq u \preceq v \text{ then } \|u\|_k \leq \|v\|_k \text{ for } k \in \{1, 2, +\infty\}. \tag{6}$$

Hence, we may infer that, if (X, d) is a vector-valued metric spaces, and, for $u, v \in X$ and $k \in \{1, 2, +\infty\}$,

$$\rho^{(k)}(u, v) = \|d(u, v)\|_k, \tag{7}$$

then each $\rho^{(k)}$ is a metric in X.

2. Main Results

Now, according to the previous results, the contractive condition (1.1) from [2] became:

$$\tau_i + F_i(d_i(Tu, Tv)) \leq F_i(d_i(u, v)), \tag{8}$$

where $\tau_i > 0, d_i(T(u), T(v)) > 0, F_i : (0, +\infty) \to (-\infty, +\infty)$ for all $i = 1, 2, \ldots, m$. It is clear that F_i satisfies **(F1)–(F3)** from [3] for all $i = 1, 2, \ldots, m$.

Since at least one of d_i, $i \in \{1, 2, \ldots, m\}$ is an ordinary metric on X, the proof of Theorem 3 from [2] follows.

However, Theorem 3 from [2] is true if $F \in \mathcal{F}^m$ satisfies only the property **(F1)**. Indeed, this follows from ([18], Corollary 2) or from ([22], Theorem 2.1).

In the sequel of this section, we will try to generalize, complement, unify, enrich, and extend all results recently established in [2]. First, we introduce and prove the following:

Definition 2. *Let (X, d) be a generalized metric space. A mapping $T : X \to X$ is called an F-contraction of the Hardy–Rogers type if there exists $\tau \succ \theta$ and strictly increasing mapping $F : (0, +\infty)^m \to (-\infty, +\infty)^m$ such that*

$$\tau + F(d(T(u), T(v))) \preceq F(A((u), (v))) \tag{9}$$

holds for any $u, v \in X$ with $d(T(u), T(v)) \succ \theta$, where $A(u, v) = a \cdot d(u, v) + b \cdot d(u, T(u)) + c \cdot d(v, T(v)) + \delta \cdot d(u, T(v)) + e \cdot d(v, T(u)), a, b, c, \delta, e$ are non-negative numbers, $\delta < \frac{1}{2}, c < 1$, $a + b + c + 2\delta = 1$ and $0 < a + \delta + e \leq 1$.

Theorem 2. *Let (X, d) be a complete vector-valued metric space. Then, each F-contraction of Hardy–Rogers type defined in it has a unique fixed point $u^* \in X$ and, for every $u \in X$, the sequence $\{T^n(u)\}_{n \in \mathbb{N}}$ converges to u^*.*

Proof. Since $\tau = (\tau_1, \cdots, \tau_m), F = (F_1, \cdots, F_m), A = (A_1, \cdots, A_m)$ and $d = (d_1, \cdots, d_m)$, then the inequality (9) became

$$\tau_i + F_i(d_i(T(u), T(v))) \leq F_i(A_i(u, v)), \tag{10}$$

where $A_i(u, v) = a \cdot d_i(u, v) + b \cdot d_i(u, T(u)) + c \cdot d_i(v, T(v)) + \delta \cdot d_i(u, T(v)) + e \cdot d_i(v, T(u))$, $i \in \{1, 2, \cdots, m\}$. According to Proposition 2.1 from [16], it follows that there exists $i_0 \in \{1, 2, \cdots, m\}$ such that the mapping $d_{i_0} : X^2 \to [0, +\infty)$ is the ordinary metric. Hence, now we have obtained

$$\tau_{i_0} + F_{i_0}\big(d_{i_0}(T(u), T(v))\big) \leq F_{i_0}\big(A_{i_0}(u, v)\big), \tag{11}$$

where (X, d_{i_0}) is a complete metric space, $A_{i_0}(u, v) = a \cdot d_{i_0}(u, v) + b \cdot d_{i_0}(u, T(u)) + c \cdot d_{i_0}(v, T(v)) + \delta \cdot d_{i_0}(u, T(v)) + e \cdot d_{i_0}(v, T(u)), a, b, c, \delta, e$ are non-negative numbers, $\delta < \frac{1}{2}, c < 1, a + b + c + 2\delta = 1$ and $0 < a + \delta + e \leq 1$. Furthermore, by Theorem 5 from [18], the condition (11) yields that T has a unique fixed point in X. The theorem is proved. $\square$

Remark 2. *It is obvious that Theorem 2 and Theorem 5 from [18] are equivalent. In addition, our Theorem 2 is a proper generalization of main results (Theorem 3) from [2]. Indeed, putting in (9) $a = 1, b = c = \delta = e = 0$, Theorem 3 follows from [2]. Furthermore, since we use only the property* **(F1)** *in Theorem 2, we get a second direction of generalization of main results from [2]. In our approach, the method of the system of inequalities (11) shows that many (and maybe all) well-known results in the setting of ordinary metric spaces with known contractive conditions (see [31–33]) are equivalent to the corresponding ones in generalized metric spaces of Perov's type.*

In the sequel, we shall consider the main result of A. I. Perov ([5], Theorem 3). In 1964, A. I. Perov proved the following result:

Theorem 3. *Let (X, d) be a complete generalized metric space, $f : X \to X$ and $A \in M_{m\times m}(\mathbb{R}^+)$ a matrix convergent to zero. If, for any $u, v \in X$, we have*

$$d(f(u), f(v)) \preceq A(d(u, v)). \tag{12}$$

Then, the following statements hold:

1. f has a unique fixed point $u^* \in X$;
2. The Picard iterative sequence $u_n = f^n(u), n \in \mathbb{N}$ converges to u^* for all $u \in X$;
3. $d(u_n, u^*) \preceq A^n(I_m - A)^{-1}(d(u_0, u_1)), n \in \mathbb{N}$;
4. if $g : X \to X$ satisfies the condition $d(f(u), g(u)) \preceq c$ for all $u \in X$ and some $c \in \mathbb{R}^m$, then, for the sequence $v_n = g^n(u_0), n \in \mathbb{N}$, the following inequality

$$d(v_n, u^*) \preceq (I_m - A)^{-1}(c) + A^n(I_m - A)^{-1}(d(u_0, u_1)) \tag{13}$$

is valid for all $n \in \mathbb{N}$.

It is worth mentioning that the statements **3.** and **4.** are not given in [5]. For details, see [7].

Now, we give our proof of Perov's famous theorem. We will use the following:

Proof. Each matrix $A \in M_{m\times m}(\mathbb{R}^+)$ is in fact a bounded linear operator on the Banach space $(\mathbb{R}^m, ||.||)$, where $||.||$ is one of all equivalent norms on a finite-dimensional vector space $\mathbb{R}^m$. Otherwise, on $\mathbb{R}^m$, all norms are equivalent. Thus, we take the matrix A as a bounded linear operator in space $(\mathbb{R}^m, ||.||)$. Using its coordinate notation having the form $A = (A_1, \ldots, A_m)$, we have that

$$A((u_1, \ldots, u_m)) = (A_1(u_1), \ldots, A_m(u_m)), \tag{14}$$

and because $\|A\| < 1, A_i(u_i) = \lambda_i u_i, i = 1, 2, \ldots, m$ with $\lambda_i \in [0, 1)$. Therefore, the condition (12) reduces to a system of m inequalities of the form

$$d_i(T(u), T(v)) \leq A_i(d_i(u, v)) = \lambda_i \cdot d_i(u, v), i = 1, 2, \ldots, m. \tag{15}$$

Since, there exists $i_0 \in \{1, 2, \ldots, m\}$ such that the mapping $d_{i_0} : X^2 \to [0, +\infty)$ is the ordinary metric, then $\left(X, d_{i_0}\right)$ is a complete metric space. The results are further yielded by the Banach contraction principle for the mapping $T : X \to X$, i.e., there is a unique fixed point $u^* \in X$ for the mapping T. The proof of Perov's theorem is finished. □

The proofs of points 2, 3, and 4 are immediate consequences of the proof for 1.

As our first significant applications of Theorem 1 and our new Theorems 2 and 3 are new contractive conditions in the setting of generalized metric spaces in Perov's sense, they complement some very well known contractive conditions in the setting of ordinary metric spaces as well as cone metric spaces ([6–26,31–33]).

Corollary 1. *Let (X,d) be a complete generalized metric space in the sense of Perov and $T : X \to X$ be a self mapping. Suppose that there exists $C_k \succ \theta, k = \overline{1,6}$ such that, for all $x, y \in X$, the following inequalities hold true:*

$$C_1 + d(T(u), T(v)) \preceq d(u,v) \tag{16}$$

$$C_2 + \exp(d(T(u), T(v))) \preceq \exp(d(u,v)) \tag{17}$$

$$C_3 - \frac{1}{d(T(u), T(v))} \preceq -\frac{1}{d(u,v)} \tag{18}$$

$$C_4 - \frac{1}{d(T(u), T(v))} + d(T(u), T(v)) \preceq -\frac{1}{d(u,v)} + d(u,v) \tag{19}$$

$$C_5 + \frac{1}{1 - \exp(d(T(u), T(v)))} \preceq \frac{1}{1 - \exp(d(u,v))} \tag{20}$$

$$C_6 + \frac{1}{\exp(-d(T(u), T(v))) - \exp(d(T(u), T(v)))} \preceq \frac{1}{\exp(-d(u,v)) - \exp(d(u,v))}. \tag{21}$$

Then, in each of these cases, there exists $\overline{u} \in X$ such that $T\overline{u} = \overline{u}$ and, for every $u \in X$, the sequence $\{T^n(u)\}_{n\in\mathbb{N}}$ converges to $\overline{u}$.

Proof. Since all of the functions $r \mapsto r$, $r \mapsto \exp r$, $r \mapsto -\frac{1}{r}$, $r \mapsto -\frac{1}{r} + r$, $r \mapsto \frac{1}{1-\exp r}$, $r \mapsto \frac{1}{\exp(-r)-\exp r}$ are strictly increasing on $(0, +\infty)$, the proof immediately follows by Theorem 1. In addition, it yields that the proofs for (18), (19), (20) and (21) are corollaries of Theorem 1. In the end, it is worth to noticing that, in all cases (16)–(21), we have:

$C_k = \left(C_1^{(k)}, \ldots, C_m^{(k)}\right)$, where $C_i^{(k)} > 0$ for $k = \overline{1,6}, i = \overline{1,m}$

$d = (d_1, \ldots, d_m)$, where d_i are pseudometrics for $i = \overline{1,m}$

$F = (F_1, \ldots, F_m)$, where F_i are strictly increasing on $(0, +\infty)$ for $i = \overline{1,m}$

For example, for the left-hand side of (16), we have:

$$\begin{aligned}
C_1 + F(d(T(u), T(v))) &= \left(C_1^{(1)}, \ldots, C_m^{(1)}\right) + F(d_1(T(u), T(v)), \ldots, d_m(T(u), T(v))) \\
&= \left(C_1^{(1)}, \ldots, C_m^{(1)}\right) + (F_1(d_1(T(u), T(v))), \ldots, F_m(d_m(T(u), T(v)))) \\
&= \left(C_1^{(1)}, \ldots, C_m^{(1)}\right) + (d_1(T(u), T(v)), \ldots, d_m(T(u), T(v))) \\
&= \left(C_1^{(1)} + d_1(T(u), T(v)), \ldots, C_m^{(1)} + d_m(T(u), T(v))\right) \\
&= C_1 + d(T(u), T(v)),
\end{aligned}$$

while, for its right-hand side, it yields

$$\begin{aligned}
d(u,v) &= (d_1(u,v), \ldots, d_m(u,v)) \\
&= (F_1(d_1(u,v)), \ldots, F_m(d_m(u,v))) \\
&= F(d_1(u,v), \ldots, d_m(u,v)) \\
&= d(u,v),
\end{aligned}$$

because $F = (F_1, \ldots, F_m)$ where $F_i(r) = r$ for $i = \overline{1,m}$. Therefore, $F(d(u,v)) = d(u,v)$. □

Our second new corollary of Theorems 2 and 3 are the following contractive conditions in the setting of generalized metric spaces of the Perov type:

Corollary 2. *Let (X,d) be a complete generalized metric space in the sense of Perov and $T : X \to X$ be a self mapping. Suppose that there exists $D_k \succ \theta, k = \overline{1,6}$ such that, for all $u,v \in X$, the following inequalities hold true:*

$$D_1 + d(T(u),T(v)) \preceq A(u,v), \tag{22}$$

$$D_2 + \exp(d(T(u),T(v))) \preceq \exp(A(u,v)), \tag{23}$$

$$D_3 - \frac{1}{d(T(u),T(v))} \preceq -\frac{1}{A(u,v)}, \tag{24}$$

$$D_4 - \frac{1}{d(T(u),T(v))} + d(T(u),T(v)) \preceq -\frac{1}{A(u,v)} + A(u,v), \tag{25}$$

$$D_5 + \frac{1}{1-\exp(d(T(u),T(v)))} \preceq \frac{1}{1-\exp(A(u,v))}, \tag{26}$$

$$D_6 + \frac{1}{\exp(-d(T(u),T(v))) - \exp(d(T(u),T(v)))} \preceq \frac{1}{\exp(-A(u,v)) - \exp(A(u,v))}, \tag{27}$$

where $A(u,v) = a \cdot d(u,v) + b \cdot d(u,T(u)) + c \cdot d(v,T(v)) + \delta \cdot d(u,T(v)) + e \cdot d(v,T(u))$, while a,b,c,δ,e are non-negative numbers: $\delta < \frac{1}{2}, c < 1, a+b+c+2\delta+e = 1, 0 < a+\delta+e \leq 1$. Then, in each of these cases, there exists $u^* \in X$ such that $Tu^* = u^*$ and, for every $u \in X$, the sequence $\{T^n(u)\}_{n\in\mathbb{N}}$ converges to u^*.

Proof. Take in Theorem 2. $F(\xi) = \xi, F(\xi) = \exp \xi, F(\xi) = -\frac{1}{\xi}$, $F(\xi) = -\frac{1}{\xi} + \xi, F(\xi) = \frac{1}{1-\exp\xi}, F(\xi) = \frac{1}{\exp(-\xi)-\exp\xi}$, respectively. Because every F is strictly increasing on $(0,+\infty)$, the result follows according to Theorem 2. □

In the sequel, we present an example, supporting our results. This example shows that Theorem 3 from [2] cannot be applied. In addition, the mapping T in this example is not a Perov type contraction.

Example 1. *Let (X,d) be a complete vector-valued metric space where $X = \left\{u_n = \frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $d : X \times X \to \mathbb{R}^2$ is given by:*

$$d(u,v) = (|u-v|, |u-v|). \tag{28}$$

Define $T : X \to X$ as

$$T(u) = \begin{cases} 0, u = 0 \\ u_{n+1}, u = u_n \end{cases} \tag{29}$$

and $F : (0,+\infty)^2 \to (-\infty,+\infty)^2$ by:

$$F(r_1,r_2) = \left(-\frac{1}{r_1}, -\frac{1}{r_2}\right) = (F_1(r_1), F_2(r_2)). \tag{30}$$

It is evident that F is strictly increasing mapping, but $F \notin \mathcal{F}^2$ in the sense of [2]. Now, we claim that T is a Perov type $F-$contraction with $\tau = (1,1) \in \mathbb{R}^2$. For this, we have to show that

$$\tau + F(d(T(u),T(v))) \preceq F(d(u,v)) \tag{31}$$

for all $u,v \in X$. Two cases are possible:

Case 1. $u = 0, v = u_n$. Then, (31) became:

$$(1,1) + \left(-\frac{1}{|T(0)-T(u_n)|}, -\frac{1}{|T(0)-T(u_n)|}\right) \preceq \left(-\frac{1}{|0-u_n|}, -\frac{1}{|0-u_n|}\right), \tag{32}$$

or equivalently
$$1+\left(-\frac{1}{u_{n+1}}\right)\leq\left(-\frac{1}{u_n}\right), \tag{33}$$
i.e., $1-(n+1)\leq -n$, which holds true.

Case 2. $u=u_n, v=u_m, n<m$. In this case, (31) became:
$$(1,1)+\left(-\frac{1}{|u_{n+1}-u_{m+1}|},-\frac{1}{|u_{n+1}-u_{m+1}|}\right)\preceq\left(-\frac{1}{(|u_n-u_m|,|u_n-u_m|)}\right), \tag{34}$$
or equivalently
$$1+\left(-\frac{1}{\frac{1}{n+1}-\frac{1}{m+1}}\right)\leq\left(-\frac{1}{\frac{1}{n}-\frac{1}{m}}\right), \tag{35}$$
that is,
$$1\leq\frac{(n+1)(m+1)}{m-n}-\frac{mn}{m-n}=\frac{n+m+1}{m-n}. \tag{36}$$
The last condition is evidently true for all $n,m\in\mathbb{N}, n<m$.

Remark 3. *The function* $F=\left(-\frac{1}{r_1},-\frac{1}{r_2}\right)=(F_1,F_2)$ *does not satisfy (F3) from ([2], p. 3). Indeed,* $\lim_{r_i\to 0^+} r_i^k\cdot\left(-\frac{1}{r_i}\right)=-\lim_{r_i\to 0^+} r_i^{k-1}\neq 0$ *for all* $k\in(0,1)$ *and* $i=1,2$. *This shows that our result is a genuine generalization of Theorem 3 from [2]. In addition, it is easy to check that the mapping* T *in the above example is not a Perov contraction. Moreover, it is not also a Banach contraction in the sense of [15]. On the other hand, each result in the setting of vector-valued metric space is equivalent to the corresponding one in ordinary metric space.*

The next remark is maybe the most significant for results in the vector-valued metric spaces. Namely, in fact, results are the same for each $m\geq 2$. This means that we can suppose that $m=2$. In this case, $F-$contraction $T:X\to X$ in Perov's sense has the following form: there exists $\tau=(\tau_1,\tau_2)\succ\theta=(0,0)$ such that, for $F\in\mathcal{F}^2$,
$$\tau+F(d(T(u),T(v)))\preceq F(d(u,v)), \tag{37}$$
or equivalently
$$\tau_i+F_i(d_i(T(u),T(v)))\leq F_i(d_i(u,v)), \tag{38}$$
is yielded, where $i\in\{1,2\}$.

If $(\mathcal{X},d)$ is a generalized metric space (or vector-valued metric space), then one could join to the same group the following new classes of generalized metric spaces as well:

- generalized partial metric space;
- generalized metric like space;
- generalized b-metric space;
- generalized partial b-metric space;
- generalized b-metric like space.

How do these arise? Take a natural number $m\geq 2$, and some of the following spaces:

- partial metric space;
- metric like space;
- b-metric space;
- partial b-metric space;
- b-metric like space.

if we denote with $\overline{d}$ the distance of any of above spaces and with $\mathcal{D}$:
$$\mathcal{D}(x,y)=\left(\alpha_1\overline{d}(x,y),\alpha_2\overline{d}(x,y),\ldots,\alpha_m\overline{d}(x,y)\right)$$
where $\alpha_i\geq 0, i=1,2,\ldots,m$ is such that at least one of them is different from 0.

Then, $\mathcal{D}(x,y)$ is called generalized partial metric (resp. generalized metric like generalized b-metric, generalized partial b-metric, and, eventually, generalized b-metric like).

Example 2. *Let* $\left(\mathcal{X},\overline{d}^{ml}\right)$*, where* $\mathcal{X}=(C[0,1],\mathbb{R})$ *is the set of real continuous functions on* $[0,1]$ *and* $\overline{d}^{ml}(u,v)=\sup_{t\in[0,1]}(|u(t)|+|v(t)|)$ *for all* $u,v\in(C[0,1],\mathbb{R})$*. This is an example of metric-like space that is not a partial metric space.*

Let $m=3$*. Then,* $(\mathcal{X},\mathcal{D})$ *where* $\mathcal{D}(u,v)=\left(2\overline{d}^{ml}(u,v),\frac{1}{10}\overline{d}^{ml}(u,v),3\overline{d}^{ml}(u,v)\right)$ *is an example of generalized metric like space (with* $m=3$*) and* $\alpha_1=2,\alpha_2=\frac{1}{10},\alpha_3=3$*.*

In order to show some application of a generalized metric space, we actually have to do the following: on some given generalized metric space, we should formulate (or find) a theorem, then use it to determine whether, for instance, a fractional differential equation or an ordinary algebraic equation or similar have solutions.

Definition 3. *Let* $\mathbf{T}$ *be a map of b-metric like space* $\left(\mathcal{X},d^{bml}\right)$ *onto itself. Then, it is called generalized* $(s,q)-$*Jaggi* $\mathcal{F}-$*contraction if there exists a strictly increasing map* $\mathcal{F}:(0,+\infty)\to(-\infty,+\infty)$ *and* $\tau>0$ *such that, for all* $x,y\in\mathcal{X}$ *with* $d^{bml}(\mathcal{T}x,\mathcal{T}y)>0$ *and* $d^{bml}(x,y)>0$ *holds*

$$\tau+\mathcal{F}\left(s^q\cdot\overline{d}^{bml}(\mathcal{T}x,\mathcal{T}y)\right)\leq\mathcal{F}\left(\mathcal{M}_{bml}^{\alpha,\beta,\gamma}(x,y)\right),$$

where

$$\mathcal{M}_{bml}^{\alpha,\beta,\gamma}(x,y)=\overline{A}\cdot\frac{\overline{d}^{bml}(x,\mathcal{T}x)\cdot\overline{d}^{bml}(y,\mathcal{T}y)}{\overline{d}^{bml}(x,y)}+\overline{B}\cdot\overline{d}^{bml}(x,y)+\overline{C}\cdot\overline{d}^{bml}(x,\mathcal{T}y)$$

with $\overline{A}+\overline{B}+2\overline{C}s<1$*, and* $q>1$*.*

Now, we have obtained a positive result whose proof is performed routinely.

Theorem 4. *Let* $\left(\mathcal{X},\overline{d}^{bml}\right)$ *be a 0-complete b-metric like space and* $\mathcal{T}:X\to X$ *a generalized Jaggi* $\mathcal{F}$*-contraction map. Then,* $\mathcal{T}$ *has a unique fixed point, say* $\sigma\in\mathcal{X}$*, and, if* $\mathcal{T}$ *is a* $\overline{d}^{bml}$*-continuous map, then, for any* $\kappa\in\mathcal{X}$*, the sequence* $\mathcal{T}^n\kappa$ *converges to* σ*.*

With the help of the previous definition and its corresponding theorem, we will show the application of a generalized metric space to the solution of some fractional differential equations.

3. Applications in Nonlinear Fractional Differential Equations

Given the function $h:[0,+\infty)\to\mathbb{R}$, we say that its Caputo derivative (see [34–36]) ${}^{C}D^{\beta}(h(t))$ of order $\beta>0$ is defined as

$${}^{C}D^{\beta}(h(t))=\frac{1}{\Gamma(n-\beta)}\int_0^t(t-s)^{n-\beta-1}h^{(n)}(s)ds\ (n-1<\beta<n,n=[\beta]+1),$$

where $[\beta]$ denotes the integer part of the positive real number β, and Γ is the gamma function. For recent examples of fractional order differential equations involving Caputo derivatives, see [37,38].

The scope of this section is to apply the previous theorem to prove the existence of solutions for nonlinear fractional differential equations of the form

$${}^{C}D(\nu(t)+g(t,\nu(t)))=0\ (0\leq t\leq 1,\beta<1)$$

with boundary conditions $\nu(0) = 0 = \nu(1)$, where $\nu \in C([0,1],\mathbb{R})$ and $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the set of all continuous functions on $[0,1]$ in $\mathbb{R}$, while $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The associated Green's function to ([34–36,39]) is given by

$$\mathcal{G}(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, \text{ if } 0 \le s \le t \le 1 \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, \text{ if } 0 \le t \le s \le 1. \end{cases}$$

We shall now state and prove the main result of this section.

Theorem 5. *Consider the nonlinear fractional differential equation ([34–36,39]). Let $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given map and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ a continuous function.*

Suppose also that all the following statements hold true:

(i) There exists $\nu_0 \in (C[0,1],\mathbb{R})$ such that $\xi\left(\nu_0(t), \int_0^t \mathcal{T}\nu_0(t)\right) \ge 0$ for all $t \in [0,1]$, where the map $\mathcal{T} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is defined as:

$$\mathcal{T}\nu(t) = \int_0^t \mathcal{G}(t,s)g(s,\nu(s))ds.$$

(ii) There exists $\tau > 0$ such that for all $\mu, \nu \in \mathcal{X}$

$$\overline{d}^{bml}(\mathcal{T}\mu, \mathcal{T}\nu) > 0 \text{ and } \overline{d}^{bml}(\mu,\nu) > 0 \text{ yields}$$

$$|g(t,a) + g(t,b)| \le \frac{1}{s^{\frac{q}{2}}} \mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\mu,\nu)e^{-\tau},$$

for all $t \in [0,1]$ and $a, b \in \mathbb{R}$ with $\nu(a,b) > 0$, where

$$M_{bml}^{\alpha,\beta,\gamma}(\mu,\nu) = \overline{A} \cdot \frac{\overline{d}^{bml}(\mu, g\mu) \cdot \overline{d}^{bml}(\nu, g\nu)}{\overline{d}^{bml}(\mu,\nu)} + \overline{B} \cdot \overline{d}^{bml}(\mu,\nu) + \overline{C} \cdot \overline{d}^{bml}(\nu, g\mu)$$

$\overline{A}, \overline{B}, \overline{C} \ge 0$ with $\overline{A} + \overline{B} + 2\overline{C}s < 1$ and $q > 1$.

(iii) For each $t \in [0,1]$ and $\mu, \nu \in C([0,1],\mathbb{R}), \xi(\nu(t), \mu(t)) \ge 0$ yields $\xi(\mathcal{T}\mu(t), \mathcal{T}(\nu)) \ge 0$;

(iv) For each $t \in [0,1]$, if $\{\nu_n\}$ is a sequence in $C([0,1],\mathbb{R})$ such that $\nu_n \to \nu$ in $C([0,1],\mathbb{R})$ and $\xi(\nu_n(t), \nu_{n-1}(t)) \ge 0$ for all $n \in \mathbb{N}$, then $\xi(\nu_n(t), \nu(t)) \ge 0$ for all $n \in \mathbb{N}$.

Then, problem ([34–36,39]) has at least one solution.

Proof. Let $\left(\Xi, \overline{d}_{bml}, s \ge 1\right) = C([0,1],\mathbb{R})$ be endowed with a b-metric like

$$\overline{d}_{bml}(\nu,\mu) = \sup_{t\in[0,1]} (|\nu(t)| + |\mu(t)|)^2,$$

for all $\nu, \mu \in \Xi$.

Now, define a generalized b-metric like on Ξ with

$$\mathcal{D}_{bml}(\nu,\mu) =: \left(\underbrace{\overline{d}_{bml}(\nu,\mu), \ldots, \overline{d}_{bml}(\nu,\mu)}_{m}\right),$$

where m is a given natural number larger than or equal to 2.

It is easy to show that $(\Xi, \mathcal{D}_{bml}, s \ge 1)$ is a 0-complete generalized b-metric like space with parameter $s = 2$.

It is obvious that $\nu^* \in \Xi$ is a solution of ([34–36,39]) if and only if $\nu^* \in \Xi$ is a solution of $\nu(t) = \int_0^t \mathcal{G}(t,s)g(s,\nu(s))ds$ for all $t \in [0,1]$. Therefore, the problem ([34–36,39]) can be considered as the problem of finding an element $\nu^* \in \Xi$ that is the fixed point of operator (map $\mathcal{T}$). To that end, let $\nu, \mu \in \Xi$ such that $\xi(\nu(t), \mu(t)) \geq 0$ for all $t \in [0,1]$. According to (iii), we have that $\xi(\mathcal{T}\nu, \mathcal{T}\mu) > 0$. Then, using hypotheses (i) and (ii), we obtain the following inequalities:

$$
\begin{aligned}
|\mathcal{T}\nu(t) + \mathcal{T}\mu(t)| &= \left| \int_0^t \mathcal{G}(t,s)g(s,\nu(s))ds + \int_0^t \mathcal{G}(t,s)g(s,\mu(s))ds \right| \\
&\leq \left| \int_0^t \mathcal{G}(t,s)|g(s,\nu(s))|ds \right| + \left| \int_0^t \mathcal{G}(t,s)|g(s,\mu(s))|ds \right| \\
&\leq \sup_{t\in[0,1]} |g(s,\nu(s)) + g(s,\mu(s))| \int_0^t \mathcal{G}(t,s)ds \\
&\leq \frac{1}{\sqrt{m s^{\frac{q}{2}}}} \sup_{t\in[0,1]} \sqrt{\mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)e^{-\tau}} \cdot \sup_{t\in[0,1]} \int_0^t \mathcal{G}(t,s)ds \\
&\leq \frac{1}{\sqrt{m s^{\frac{q}{2}}}} \sup_{t\in[0,1]} \sqrt{\mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)e^{-\tau}}.
\end{aligned}
$$

This means that

$$
|\mathcal{T}\nu(t) + \mathcal{T}\mu(t)|^2 \leq \frac{1}{m \cdot s^q} \mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)e^{-\tau}.
$$

Then, we have

$$
s^q \overline{d}_{bml}(\nu,\mu) \leq \mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)e^{-\tau}.
$$

If we now take $\mathcal{F}(\omega) = \ln \omega$ for any $\omega > 0$, then $\mathcal{F}$ satisfies all conditions of the theorem, and we obtain

$$
\ln\Big(s^q \overline{d}_{bml}(\nu,\mu)\Big) \leq \ln\Big(\mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)e^{-\tau}\Big),
$$

that is,

$$
\tau + \ln\Big(s^q \overline{d}_{bml}(\nu,\mu)\Big) \leq \ln\Big(\mathcal{M}_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\Big).
$$

The above is clearly equivalent to

$$
\tau + \ln\Big(s^q \overline{d}_{bml}(\nu,\mu)\Big) \leq \mathcal{F}\left(\overline{A} \cdot \frac{\overline{d}^{bml}(\mu,g\mu) \cdot \overline{d}^{bml}(\nu,g\nu)}{\overline{d}^{bml}(\mu,\nu)} + \overline{B} \cdot \overline{d}^{bml}(\mu,\nu) + \overline{C} \cdot \overline{d}^{bml}(\nu,g\mu)\right),
$$

where $\overline{A}, \overline{B}, \overline{C} \geq 0$ with $\overline{A} + \overline{B} + 2\overline{C}s < 1$ and $q > 1$.

Applying the above theorem now with $q = 2$, we conclude that map $\mathcal{T}$ has a fixed point, which in turn shows that problem ([34–36,39]) has at least one solution. □

4. Conclusions

In this article, we considered F-contraction mappings in the setting of Perov type cone metric space. Among other things, we have proved that recently established results for F-contractions of Perov's type are in fact equivalent to the corresponding ones in ordinary metric spaces. We have also provided an example application of proposed results for fractional differential equations in generalized b-metric-like spaces.

Author Contributions: Conceptualization N.M. and S.R. (Stojan Radenović); methodology, N.M. and S.R. (Stojan Radenović); formal analysis, N.M. and S.R. (Slobodan Radojević); writing—original draft preparation, N.M. and S.R. (Slobodan Radojević); supervision, S.R. (Stojan Radenović). All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: N.M.'s research was funded by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Altun, I.; Olgun, M. Fixed point results for Perov type F-contractions and an application. *J. Fixed Point Theory Appl.* **2020**, *22*, 46. [CrossRef]
3. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
4. Kurepa, Đ.R. Tableaux ramifies d'ensambles. *C. R. Acad. Sci. Paris* **1934** *198*, 1563–1565.
5. Perov, A.I. On Cauchy problem for a system of ordinary differential equations. *Priblizhen. Met. Reshen. Diff. Uravn.* **1964**, *2*, 115–134.
6. Abbas, M.; Jungck, G. Common fixed point results for noncommuting mappings without continuity in cone metric space. *J. Math. Anal. Appl.* **2008**, *341*, 416–420. [CrossRef]
7. Abbas, M.; Nazir, T.; Rakočević, V. Common fixed points results of multivalued Perov type contractions on cone metric spaces with a directed graph. *Bull. Belg. Math. Soc. Simon Stevin* **2018**, *25*, 331–354. [CrossRef]
8. Abbas, M.; Rakočević, V.; Hussain, A. Best proximity point of Zamfirescu contractions of Perov type on regular cone metric spaces. *Fixed Point Theory* **2020**, *21*, 3–18. [CrossRef]
9. Altun, I.; Hussain, N.; Qasim, M.; Alsulami, H.H. A new fixed point result of Perov type and its application to a semilinear operator system. *Mathematics* **2019**, *7*, 1019. [CrossRef]
10. Aleksić, S.; Paunović, L.; Radenović, S.; Vetro, F. Some critical remarks on the paper 'A note on the metrizability of TVS-cone metric spaces'. *Vojn. Teh. Glas. Tech. Cour.* **2017**, *65*, 1–8. [CrossRef]
11. Karapinar, E. Couple fixed point theorems for nonlinear contractions in cone metric spaces. *Comput. Math. Appl.* **2010**, *59*, 3656–3668. [CrossRef]
12. Aleksić, S.; Kadelburg, Z.; Mitrović, Z.D.; Radenović, S. A new survey: Cone metric spaces. *J. Int. Math. Virtual Inst.* **2019**, *9*, 93–121.
13. Cvetković, M. On the equivalence between Perov fixed point theorems and Banach contraction principle. *Filomat* **2017** *31*, 3137–3146. [CrossRef]
14. Hardy, G.E.; Rogers, T.D. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **1973**, *16*, 201–206. [CrossRef]
15. Huang, L.G.; Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **2007**, *332*, 1468–1476. [CrossRef]
16. Jachymski, J.; Klima, J. Around Perov's fixed point theorem for mappings on generalized metric spaces. *Fixed Point Theory* **2016**, *17*, 367–380.
17. Janković, S.; Kadelburg, Z.; Radenović, S. On cone metric spaces: A survey. *Nonlinear Anal.* **2011**, *74*, 2591–2601. [CrossRef]
18. Popescu, O.; Stan, G. Two fixed point theorems concerning F-contraction in complete metric spaces. *Symmetry* **2020**, *12*, 58. [CrossRef]
19. Radenović, S.; Vetro, F.; Xu, S. Some results of Perov type mappings. *J. Adv. Math. Stud.* **2017**, *10*, 396–409.
20. Radenović, S.; Vetro, F. Some remarks on Perov type mappings in cone metric spaces. *Mediterr. J. Math.* **2017**, *14*, 240. [CrossRef]
21. Vetro,F.; Radenović, S. Some results of Perov type on rectangular cone metric spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 1–16. [CrossRef]
22. Vujaković, J.; Radenović, S. On some F-contraction of Piri-Kumam-Dung-type mappings in metric spaces. *Vojno Tehnički Glasnik* **2020**, *68*, 697–714. [CrossRef]
23. Wardowski, D.; Van Dung, N. Fixed points of F-weak contractions on complete metric spaces. *Demonstr. Math.* **2014**, *47*, 146–155. [CrossRef]
24. Wardowski, D. Solving existence problems via F-contractions. *Proc. Am. Math. Soc.* **2018**, *146*, 1585–1598. [CrossRef]
25. Xu, S.; Dolićanin, Ć.; Radenović, S. Some remarks on results of Perov type. *J. Adv. Math. Stud.* **2016**, *9*, 361–369.
26. Zabrejko, P.P. K-metric and K-normed linear spaces; survey. *Collect. Math.* **1997**, *48*, 825–859.

27. Karapinar, E.; Fulga, A.; Agarwal, R.P. A survey: F-contractions with related fixed point results. *J. Fixed Point Theory Appl.* **2020**, *22*, 69. [CrossRef]
28. Agarwal, R.P.; Aksoy, U.; Karapinar, E.; Erhan, I.M. F-contraction mappings on metric-like spaces in connection with integral equations on time scales. *RACSAM* **2020**, *114*, 147. [CrossRef]
29. Aydi, H.; Karapinar, E.; Yazidi, H. Modified F-Contractions via α-Admissible Mappings and Application to Integral Equations. *Filomat* **2017**, *31*, 1141–1148. [CrossRef]
30. Alsulami, H.H; Karapinar, E.; Piri, H. Fixed Points of Modified F-Contractive Mappings in Complete Metric-Like Space Cone metric spaces. *J. Funct. Spaces* **2015**, *2015*, 270971. [CrossRef]
31. Ćirić, L.B. *Some Recent Results in Metrical Fixed Point Theory*; University of Belgrade: Belgrade, Serbia, 2003.
32. Collaco, P.; Silva, J.C. A complete comparison of 23 contraction conditions. *Nonlinear Anal. TMA* **1997**, *30*, 471–476. [CrossRef]
33. Rhoades, B.E. A comparison of various definitions of contractive mappings. *Trans. Am. Math. Soc.* **1997**, *226*, 257–290. [CrossRef]
34. Caputo, M. Linear model of dissipation whose Q is almost frequency independent—II. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
35. Kilbas, A.A.; Srivastava, H.H.; Trujillo, J.J. *Theory and Appolications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherland, 2006; Volume 204.
36. Samko, S.G.; Kilbas, A.A.; Marichey, O.I. *Fractional Integral and Derivative*; Gordon and Breach: London, UK, 1998.
37. Tuan, N.H.; Mohammadi, H.; Rezapour, S. A mathematical model for COVID-19 transmission by using the Caputo fractional derivative. *Chaos Solitons Fractals* **2020**, *140*, 110107. [CrossRef]
38. Rezapour, S.; Mohammadi, H.; Jajarmi, A. A new mathematical model for Zika virus transmission. *Adv. Differ. Equ.* **2020**, *2020*, 589. [CrossRef]
39. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.