Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 763547, 8 pages
http://dx.doi.org/10.1155/2014/763547

Research Article

Symmetric Spaces and Fixed Points of Generalized Contractions

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Received 28 September 2013; Accepted 30 December 2013; Published 5 March 2014

Academic Editor: E. Karapinar

Some fixed point results in semi-metric spaces as well as in symmetric spaces are proved. Applications of our results to probabilistic spaces are also presented.

1. Introduction

There have been a number of generalizations of metric space. Two of them are the notions of symmetric spaces and semi-metric spaces introduced and studied by Wilson [1]. For historical remarks about these spaces see [2]. Fixed point theory of various classes of maps in a metric space and its generalizations has been studied by a number of authors; see, for example, [3–9] and the references cited therein. In 1976, Cicchese proved the first fixed point theorem for contractions in semi-metric spaces. Further fixed point results for this class of spaces were obtained by Jachymski et al. [10], Hicks and Rhoades [11], Aamri and El Moutawakil [12], Aamri et al. [13], Zhu et al. [14], Miheţ [15], Imdad et al. [16], Aliouche [17], and Radenović and Kadelburg [18]. For more information on fixed point theory in symmetric spaces and semi-metric spaces, we refer the reader to [2].

In this paper we prove some fixed point results in semi-metric spaces and symmetric spaces. We also present applications of our results to probabilistic spaces. Our results generalize earlier results obtained by Aranđelović and Kečkić [2], Browder [19], Walter [20], and Maiti et al. [21].

2. Preliminary Notes

A symmetric space is a pair (X, d) consisting of a nonempty set X and a function $d : X \times X \to [0, \infty)$ such that for all x, y in X the following conditions hold:

(W1) $d(x, y) = 0$ if and only if $x = y$;

(W2) $d(x, y) = d(y, x)$.

Let (X, d) be symmetric space. The *open ball* with center $x \in X$ and radius $r > 0$ is defined by

$$B(x, r) = \{y \in X : d(x, y) < r\}. \tag{1}$$

Also if A is a subset of X, then

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\} \tag{2}$$

denotes *the diameter of* A.

Many properties and notions in symmetric spaces are similar to those in metric spaces (but not all, because of the absence of the triangle inequality). For example, a sequence $\{x_n\} \subseteq X$ is said to be d-Cauchy sequence if given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$, for all $m, n \geqslant N$.

In every symmetric space (X, d) one may introduce the topology τ_d by defining the family of closed sets as follows: a set $A \subseteq X$ is closed if and only if for each $x \in X$, $d(x, A) = 0$ implies $x \in A$, where

$$d(x, A) = \inf\{d(x, a) : a \in A\}. \tag{3}$$

The following conditions can be used as partial replacements for the triangle inequality's absence in the symmetric space (X, d) :

(W3) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$;

(W4) $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$;

(W) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$ imply $\lim_{n\to\infty} d(x_n, z_n) = 0$;

(JMS) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$ imply $\lim_{n\to\infty} d(x_n, z_n) \neq \infty$;

(CC) $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$;

(SC) $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\overline{\lim}_{n\to\infty} d(x_n, y) \leq d(x, y)$;

(MT) there exists $s \geqslant 1$ such that for any $x, y, z \in X$

$$d(x,z) \leqslant s\left(d(x,y) + d(y,z)\right). \tag{4}$$

The properties (W3) and (W4) were induced by Wilson [1], (W) by Miheţ [15], (JMS) by Jachymski et al. [10], (CC) by Cho et al. [22] and earlier by Borges [23] (as 1-continuity property), (MT) by Czerwik [24] (see also [25]), and (SC) by Aranđelović and Kečkić [2].

Next statement gives the characterization of symmetric space which satisfies the property (JMS).

Proposition 1 (Jachymski et al. [10]). *Let (X, d) be a symmetric space. Then the following conditions are equivalent.*

(i) *(X, d) satisfies property (JMS).*

(ii) *There exists $\delta, \eta > 0$ such that for any $x, y, z \in X$,*

$$d(x,z) + d(z,y) < \delta \text{ implies that } d(x,y) < \eta. \tag{5}$$

(iii) *There exists $r > 0$ such that*

$$\sup\{\operatorname{diam}(B(x,r)) : x \in X\} < \infty. \tag{6}$$

The convergence of a sequence $\{x_n\}$ in the topology τ_d need not imply $d(x_n, x) \to 0$, although the converse is true (see Proposition 2).

The following two propositions have been well known for a long time, but for the convenience of the reader we will state them without proofs, which can also be found in [2].

Proposition 2. *If (X, d) is a symmetric space, then the family $\{B(x, r) : r > 0\}$ forms a local basis at x. Also, if $d(x_n, x) \to 0$, then $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) in the topology τ_d.*

Definition 3. A topological space (X, τ) is semimetrizable if there is a symmetric function $d : X \times X \to \mathbf{R}$ such that $\tau_d = \tau$ and that the mapping

$$X \supseteq A \longmapsto c(A) = \{x \in X : d(x, A) = 0\} \tag{7}$$

is the closure operator in τ_d. In terms of d it can be expressed by saying that the operator c is idempotent. In this case we say that (X, d) is *semi-metric space*; d is said to be *semi-metric* function on X (or admissible semi-metric for (X, τ)).

It is worth mentioning that this basis need not consist of open sets. Moreover, in [26], a semimetrizable space (X, τ) was constructed with the property that, for any d that generates τ, there exist $x \in X$ and $r > 0$ such that $B(x, r)$ is not open.

Proposition 4. *Let (X, d) be a symmetric space. Then (X, d) is a semi-metric space if and only if the following conditions hold.*

(1) *(X, τ_d) is first countable.*

(2) *For any sequence $\{x_n\} \subseteq X$, $d(x_n, x) \to 0$ is equivalent to $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) in the topology τ_d.*

Example 5. Let $X = \mathbb{N}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 2^{1/2^{\min\{x,y\}}} - 1 & \text{if } |x-y| = 1 \\ 3 & \text{if } |x-y| \geqslant 2 \\ 0 & \text{if } x = y. \end{cases} \tag{8}$$

Then (X, d) is a d-Cauchy complete semi-metric.

A symmetric space (X, d) is said to be d-Cauchy complete if every d-Cauchy sequence converges to some $x \in X$ in the topology τ_d, and it is said to be *d-weakly complete* if every decreasing sequence $\{F_n\}$ of nonempty closed subsets, such that there exists a sequence $\{x_n\}$, $x_n \in F_n$ with $F_n \subseteq B(x_n, 2^{-n})$ has a nonempty intersection.

Next statement was proved in [27] (see also [28]).

Proposition 6 (Galvin and Shore [27]). *Let (X, d) be a semi-metric space. Then the following are equivalent:*

(1) *(X, τ_d) is d-weakly complete;*

(2) *every d-Cauchy sequence in X has a convergent subsequence;*

(3) *every decreasing sequence $\{F_n\}$ of nonempty closed subsets of X such that $\operatorname{diam}(F_n) \leqslant 2^{-n}$ for each n has a nonempty intersection.*

Let X be a nonempty set and $f : X \to X$. Then $z \in X$ is called a fixed point of f if $z = f(z)$. Let $x \in X$. The sequence $\{x_n\}$ defined by $x_n = f^n(x)$ is called the sequence of *Picard iterates* of f at point x. This sequence $\{x_n\}$ is also called the *orbit* of f at point x. We will denote it by $O(x)$ and use $O(x, y)$ to denote set $O(x) \cup O(y)$.

Let Φ denote the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(a) φ is monotone nondecreasing;

(b) $\lim \varphi^n(t) = 0$ for any $t > 0$.

The function $\varphi \in \Phi$ is known as the comparison function (see [29]). As a consequence of the above properties, we have the following (see [29]).

Lemma 7. *If $\varphi \in \Phi$ then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.*

Definition 8. If (X, d) is a metric space and $f : X \to X$, then f is called a

(1) *contraction* if there exists real number $\alpha \in (0, 1)$ such that

$$d(f(x), f(y)) \leqslant \alpha d(x, y) \quad \forall x, y \in X; \tag{9}$$

(2) *φ-contraction* if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in X$

$$d(f(x), f(y)) \leqslant \varphi(d(x, y)); \tag{10}$$

(3) *generalized φ-contraction* if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that for any $x, y \in X$

$$d(f(x), f(y)) \leqslant \varphi(d_f(x, y)), \tag{11}$$

where

$$d_f(x, y) = \max\{d(x, y), d(x, f(x)), d(y, f(y))\}. \tag{12}$$

Lemma 9 (Aranđelović and Kečkić [2]). *Let X be a nonempty set, let $f : X \to X$, and let n be a fixed positive integer such that the iterate f^n has a unique fixed point z. Then*

(1) *z is a unique fixed point of f;*

(2) *if X is a topological space and any sequence of Picard iterates defined by f^n converges to z, then the sequence of Picard iterates defined by f always converges to z.*

3. Some Topological Results

Proposition 10. *Let (X, d) be a symmetric space satisfying (W). Then it satisfies (W4) and (JMS).*

Proof. The implication (W)⇒(W4) is straightforward (see [15]).

Now let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X such that

$$\lim_{n\to\infty} d(x_n, y_n) = 0, \qquad \lim_{n\to\infty} d(y_n, z_n) = 0. \tag{13}$$

From (W) it follows that

$$\lim_{n\to\infty} d(x_n, z_n) \neq \infty. \tag{14}$$

So, (X, d) satisfies (JMS). □

A semi-metric space in which all balls $B(x, r)$, $x \in X$ and $r > 0$, are open will be called a semi-metric space with open balls.

Proposition 11. *Let (X, d) be a compact semi-metric space with open balls and $K \subseteq X$ a nonempty compact set. Then K is bounded.*

Proof. (X, τ_d) is first countable [2, Proposition 3] and T_1-space [2, page 5161]. Also, (X, d) satisfies the property (SC) because all $B(x, r)$ are open sets [2, Theorem 1]. K is countably compact because it is compact [30, Theorem 11.9]. It is sequentially compact, as a first countable countably compact set [30, Problem 10.7].

Suppose that K is not bounded. Let $x_0 \in K$. For each positive integer n there exists $x_n \in K$ such that $d(x_n, x_0) > n$. Then there exists $x_* \in K$ and an increasing sequence of positive integers n_k such that, in the topology τ_d,

$$\lim_{k\to\infty} x_{n_k} = x_*, \tag{15}$$

because K is sequentially compact. So, we get that

$$\overline{\lim_{k\to\infty}} d(x_{n_k}, x_0) \leqslant d(x_*, x_0) \tag{16}$$

which is a contradiction because

$$\lim_{k\to\infty} d(x_{n_k}, x_0) = \infty. \tag{17}$$

□

4. Bounded Semi-Metric Spaces: Fixed Point Results

In this section, we obtain generalizations of fixed point results of Browder [19] and Walter [20] (see also [7]).

Theorem 12. *Let (X, d) be a bounded semi-metric and d-Cauchy complete space satisfying (W4). Suppose $f : X \to X$ satisfies that, for $x \in X$, there exists $\nu(x) \in \mathbb{N}$ such that, for any $\nu \geqslant \nu(x)$ and $y \in X$,*

$$d(f^\nu(x), f^\nu(y)) \leqslant \varphi(\operatorname{diam}(O(x, y))) \tag{18}$$

with $\varphi \in \Phi$. Then there exists $z \in X$ such that $\lim_{p\to\infty} f^p(x) = z$ in the topology τ_d (or equivalently $d(f^p(x), z) \to 0$ as $p \to \infty$), $\forall x \in X$.

Proof. Let $\mu = \max\{\nu(x), \nu(y)\}$ and $\nu \geqslant \mu$.

Each element $\alpha \in O(f^\mu(x), f^\mu(y))$ has one of forms $f^{\nu+\rho}(x)$ or $f^{\nu+\rho}(y)$, with $\rho \geqslant 0$. Now let $\alpha, \beta \in O(f^\mu(x), f^\mu(y))$ and consider the case with $\alpha = f^{\nu+\rho}(x)$ and $\beta = f^{\nu+\rho}(y)$; then

$$\begin{aligned} d(\alpha, \beta) &= d(f^{\nu+\rho}(x), f^{\nu+\rho}(y)) \\ &\leqslant \varphi(\operatorname{diam}(O(f^\rho(x), f^\rho(y)))) \\ &\leqslant \varphi(\operatorname{diam}(O(x, y))). \end{aligned} \tag{19}$$

Other cases will lead to the same inequality:

$$\operatorname{diam}(O(f^\mu(x), f^\mu(y))) \leqslant \varphi(\operatorname{diam}(O(x, y))). \tag{20}$$

Now, define the sequences $\{E_n\}_{n=0}^{\infty} \subset X$ and $\{p(n)\}_{n=0}^{\infty}$ by $p(0) = 0$, $p(n+1) = p(n) + \max\{\nu(f^{p(n)}(x)), \nu(f^{p(n)}(y))\}$, and $E_n = O(f^{p(n)}(x), f^{p(n)}(y))$, $n = 0, 1, 2, \ldots$.

We want to prove that

$$\operatorname{diam}(E_{n+1}) \leqslant \varphi(\operatorname{diam}(E_n)), \quad n = 0, 1, 2, \ldots. \tag{21}$$

By (20), we get that (21) is valid for $n = 0$.

Now, let n be arbitrary and set $\gamma = f^{p(n)}(x), \xi = f^{p(n)}(y)$, and $\eta = \max\{\nu(f^{p(n)}(x)), \nu(f^{p(n)}(y))\}$; then

$$\begin{aligned}\operatorname{diam}&(O(f^{\eta}(x), f^{\eta}(y)))\\ &\leqslant \varphi(\operatorname{diam}(O(\gamma, \xi)))\\ &= \varphi\left(\operatorname{diam}\left(O\left(f^{p(n)}(x), f^{p(n)}(y)\right)\right)\right)\\ &= \varphi(\operatorname{diam}(E_n)).\end{aligned} \tag{22}$$

But $f^{\eta}(\gamma) = f^{p(n)+\eta}(x) = f^{p(n+1)}(x)$ and $f^{\eta}(\xi) = f^{p(n)+\eta}(y) = f^{p(n+1)}(y)$. Thus $\operatorname{diam}(O(f^{\eta}(\gamma), f^{\eta}(\xi))) = \operatorname{diam}(O(f^{p(n+1)}(x), f^{p(n+1)}(y))) = \operatorname{diam}(E_{n+1})$.

Therefore (21) holds for all $n = 0, 1, 2, \ldots$.

Now, by (21) and the monotonicity of φ, we get that $\operatorname{diam}(E_{n+1}) \leqslant \varphi^{(n+1)}(\operatorname{diam}(E_0)) = \varphi^{(n+1)}(\operatorname{diam}(O(x, y))) \to 0$ as $n \to \infty$ that is, $\lim_{n\to\infty} \operatorname{diam}(E_n) = \lim_{n\to\infty} \operatorname{diam}(O(f^{p(n)}(x), f^{p(n)}(y))) = 0$ which is equivalent to $\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x), f^{p}(y))) = 0$. But

$$\begin{aligned}\lim_{p\to\infty} &\operatorname{diam}(O(f^{p}(x)))\\ &\leqslant \lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x), f^{p}(y))) = 0,\end{aligned} \tag{23}$$

which implies

$$\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(x))) = 0. \tag{24}$$

Similarly

$$\lim_{p\to\infty} \operatorname{diam}(O(f^{p}(y))) = 0. \tag{25}$$

Hence, $\{f^{p}(x)\}$ and $\{f^{p}(y)\}$ are d-Cauchy sequences, and by the d-completeness of X, there exists $z, w \in X$ such that $\lim_{p\to\infty} f^{p}(x) = z$ and $\lim_{p\to\infty} f^{p}(y) = w$ in the topology τ_d.

Since $\lim_{p\to\infty} d(f^{p}(x), z) = 0$ and $\lim_{p\to\infty} d(f^{p}(x), f^{p}(y)) = 0$, (W4) implies that $\lim_{p\to\infty} d(f^{p}(y), z) = 0$. But $\lim_{p\to\infty} f^{p}(y) = w$ in the topology τ_d and so $\lim_{p\to\infty} d(f^{p}(y), w) = 0$. Since (W4) implies (W3), we have $z = w$. Since y is arbitrary in X, $\lim_{p\to\infty} f^{p}(x) = z$ in the topology τ_d, $\forall x \in X$. □

Corollary 13. *If, in addition to the hypothesis of Theorem 12, one assumes that f is τ_d-continuous (i.e, $x_n \to x$ in the topology τ_d implies $f(x_n) \to f(x)$ in the topology τ_d) then f has a fixed point.*

Proof. Since $\lim_{p\to\infty} f^{p}(x) = z$ in the topology τ_d, by the τ_d-continuity of f, $\lim_{p\to\infty} f^{p+1}(x) = f(z)$ in the topology τ_d. Therefore, since (W4) implies (W3), $f(z) = z$. Hence, $z \in X$ is a fixed point. □

Theorem 14. *Let (X, d) be a bounded semi-metric and d-Cauchy complete space satisfying (W4), (CC) and (JMS). Suppose that f is a self-map on X, and for $x, y \in X$.*

$$d(f(x), f(y)) \leqslant \varphi(\operatorname{diam}(O(x, y))). \tag{26}$$

Then f has a unique fixed point and $\lim_{p\to\infty} f^{p}(x) = z$ in the topology τ_d (or equivalently $d(f^{p}(x), z) \to 0$ as $p \to \infty$), $\forall x \in X$.

Proof. By Theorem 12, there exists $z \in X$ such that $\lim_{p\to\infty} d(f^{p}(x), z) = 0$ for all $x \in X$.

Next, assume that $z \neq f(z)$; that is, $\operatorname{diam}(O(z)) = \beta > 0$.

Thus, it is possible to choose two sequences $\{i(p)\}, \{j(p)\}$ such that

$$\lim_{p\to\infty} d\left(f^{i(p)}(z), f^{j(p)}(z)\right) = \beta. \tag{27}$$

So one can pick $\delta > 0$ with a corresponding $\eta > 0$, such that $\eta \leqslant \beta/2$.

Since there exists $p_0 \in \mathbb{N}$ such that

$$d(f^{n}(z), z) \leqslant \frac{\delta}{2}, \quad d(f^{m}(z), (z)) \leqslant \frac{\delta}{2}, \quad \forall n, m \geqslant p_0, \tag{28}$$

therefore $d(f^{n}(z), z) + d(f^{m}(z), z) \leqslant \delta$, by Proposition 1(ii)

$$d(f^{n}(z), f^{m}(z)) \leqslant \eta \leqslant \frac{\beta}{2}, \quad \forall n, m \geqslant p_0. \tag{29}$$

So, $i(p) \equiv i$ for infinitely many p, with $0 \leqslant i \leqslant p_0$; thus there exists a sequence $\{r(p)\} \subseteq \{j(p)\}$ such that $\lim_{p\to\infty} d(f^{i}(z), f^{r(p)}(z)) = \beta$. So, either $r(p) \equiv j$ for infinitely many p (i.e., $d(f^{i}(z), f^{j}(z)) = \beta$) or there exists a sequence $\{s(p)\} \subseteq \{r(p)\}$ with $s(p) \to \infty$ as $p \to \infty$ which implies $d(f^{i}(z), z) = \beta$.

In both cases, one can conclude that there exists $i, j \geqslant 0$ such that $d(f^{i}(z), f^{j}(z)) = \beta$.

If $d(z, f^{j}(z)) = \beta$, since $\lim_{p\to\infty} d(f^{p}(x), z) = 0$ and by (CC) of d, we get

$$\begin{aligned}\beta &= d\left(z, f^{j}(z)\right)\\ &= \lim_{p\to\infty} d\left(f^{p}(z), f^{j}(z)\right)\\ &\leqslant \lim_{p\to\infty} \varphi\left(\operatorname{diam}\left(O\left(f^{p-1}(z), f^{j-1}(z)\right)\right)\right)\\ &\leqslant \lim_{p\to\infty} \varphi(\operatorname{diam}(O(z))) = \varphi(\beta),\end{aligned} \tag{30}$$

which is a contradiction with $\varphi(\beta) < \beta$, for $\beta > 0$.

On the other hand, if $i, j \geqslant 1$, by (26),

$$\begin{aligned}\beta &= d\left(f^{i}(z), f^{j}(z)\right)\\ &\leqslant \varphi\left(\operatorname{diam}\left(O\left(f^{i-1}(z), f^{j-1}(z)\right)\right)\right)\\ &\leqslant \varphi(\beta)\end{aligned} \tag{31}$$

which is also a contradiction. Hence $\beta = 0$; that is, $f(z) = z$. □

Corollary 15. *Let (X, d) be a bounded semi-metric and d-Cauchy complete space satisfying (W) and (CC). Suppose that f is a self-map on X, and for $x, y \in X$*

$$d(f(x), f(y)) \leqslant \varphi(\operatorname{diam}(O(x, y))). \quad (32)$$

Then f has a unique fixed point and $\lim_{p\to\infty} f^p(x) = z$ in the topology τ_d (or equivalently $d(f^p(x), z) \to 0$ as $p \to \infty$), for all $x \in X$.

5. Symmetric Spaces: Fixed Point Results

In this section, we extend results attributed to Maiti et al. [21, Theorem 4] and Aranđelović and Kečkić [2, Theorem 3].

Theorem 16. *Let (X, d) be a d-Cauchy complete symmetric space satisfying (W3) and (JMS). Let $f : X \to X$ be a τ_d-continuous map such that*

$$d(f(x), f(y)) \leqslant \varphi(d_f(x, y)), \quad (33)$$

for all $x, y \in X$, and $\varphi \in \Phi$. Then f has a unique fixed point $z \in X$ and for each $x \in X$, the sequence of Picard iterates defined by f at x converges to z in the topology τ_d.

Proof. Define $d^* : X \times X$ as follows: $d^*(x, y) = 0$ for $x = y$ and $d^*(x, y) = d_f(x, y)$ otherwise. Then the space (X, d^*) is a symmetric space. Also, we have $d(x, y) \leqslant d^*(x, y)$ for any $x, y \in X$. So, if $\{x_n\} \subseteq X$ is an arbitrary d^*-Cauchy sequence in (X, d^*), then $\{x_n\}$ is a d-Cauchy sequence in (X, d).

Let $x, y \in X$. From

$$\begin{gathered} d(f^2(x), f(x)) \leqslant \varphi(d(x, f(x))), \\ d(f^2(y), f(y)) \leqslant \varphi(d(y, f(y))), \\ d(f(x), f(y)) \leqslant \varphi(d_f(x, y)), \end{gathered} \quad (34)$$

it follows that

$$d^*(f(x), f(y)) \leqslant \varphi(d^*(x, y)). \quad (35)$$

So f is a φ-contraction on (X, d^*).

Let δ, η be defined as in (ii) of Proposition 1. Then there exists the least positive integer $j \geqslant 1$ such that $\varphi^j(\eta) \leqslant \delta/2$.

Let $g = f^j$. We have that g is continuous (in τ_d). Then

$$\begin{aligned} d^*(g(x), g(y)) &= d^*(f(f^{j-1}(x)), f(f^{j-1}(y))) \\ &\leq \varphi(d^*(f^{j-1}(x), f^{j-1}(y))) \\ &\leqslant \varphi^j(d^*(x, y)). \end{aligned} \quad (36)$$

So g is a φ^j-contraction on (X, d^*).

Let $x \in X$ and $\psi = \varphi^j$. Then $\psi \in \Phi$ and

$$d^*(g^{m+n}(x), g^n(x)) \leqslant \psi^n(d^*(x, g^m(x))) \quad \text{for any } m, n \in \mathbb{N}. \quad (37)$$

So

$$d^*(g^{n+1}(x), g^n(x)) \leqslant \psi^n(d^*(x, g(x))) \quad (38)$$

which implies that

$$d^*(g^{n+1}(x), g^n(x)) \longrightarrow 0. \quad (39)$$

Then there exists $k \in \mathbb{N}$ such that

$$d^*(g^k(x), g^{k+1}(x)) \leqslant \min\left\{\frac{\delta}{2}, \eta\right\}. \quad (40)$$

We will prove that, for all $n \in \mathbb{N}$,

$$d^*(g^k(x), g^{k+n}(x)) \leqslant \eta. \quad (41)$$

By definition of k, we get that (41) is valid for $n = 1$. Now, assume that (41) is satisfied for some $n \in \mathbb{N}$. From

$$\begin{aligned} & d^*(g^k(x), g^{k+1}(x)) \leqslant \frac{\delta}{2}, \\ & d^*(g^{k+1}(x), g^{k+n+1}(x)) \\ & \quad \leqslant \psi(d^*(g^k(x), g^{k+n}(x))) \\ & \quad \leqslant \psi(\eta) \leqslant \frac{\delta}{2}, \end{aligned} \quad (42)$$

it follows that

$$d^*(g^k(x), g^{k+1}(x)) + d^*(g^{k+1}(x), g^{k+n+1}(x)) \leqslant \delta, \quad (43)$$

which by Proposition 1 implies that

$$d^*(g^k(x), g^{k+n+1}(x)) \leqslant \eta. \quad (44)$$

So, by induction we get that (41) is satisfied for any $n \geqslant 1$. Thus

$$d^*(g^{k+n}(x), g^{k+n+m}(x)) \leqslant \psi^n(\eta), \quad \text{for any } m, n \in \mathbb{N}. \quad (45)$$

Hence $\{g^n(x)\}$ is a d^*-Cauchy sequence in (X, d^*), which implies that $\{g^n(x)\}$ is a d-Cauchy sequence in (X, d). It follows that there exists $z \in X$ such that $\lim_{n\to\infty} g^n(x) = z$ (in the topology τ_d) because (X, d) is d-Cauchy complete. Then $\lim_{n\to\infty} g^{n+1}(x) = g(z)$ (in the topology τ_d) because g is τ_d-continuous. Now we get that $g(z) = z$ because (X, d) satisfies (W3).

If y is another fixed point of f, then for all n we have

$$d^*(y, z) = d^*(g^n(y), g^n(z)) \leqslant \psi^n(d^*(y, z)) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (46)$$

So z is a unique fixed point of g. By Lemma 9 we get that z is a unique fixed point of f.

From

$$d^*(z, g^{n+1}(x)) \leqslant \varphi(\max\{d(z, g(z)), d(z, g^n(x)), d(g^n(x), g^{n+1}(x))\}) \quad (47)$$

it follows that for each $x \in X$ the sequence of Picard iterates defined by $g = f^j$ at x converges, in the topology τ_{d^*}, to z, which implies their convergence in the topology τ_d. So, by Lemma 9, we obtain that for each $x \in X$ the sequence of Picard iterates defined by f at x converges, in the topology τ_d, to z. □

Remark 17. The next example of [10] illustrates that the continuity of f in Theorem 16 can not be omitted.

Example 18. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and let d be defined as follows:

$d(0,1) = d(1,0) = 1$;

$d(1,1/n) = d(1/n,1) = 2/3$ for $n \geqslant 2$;

$d(1,1) = 0$; otherwise $d(x,y) = |x-y|$.

Let $f : X \to X$ given by

$$f(x) = \begin{cases} \dfrac{x}{4}, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases} \tag{48}$$

Then (X,d) is a bounded d-Cauchy complete semi-metric space and

$$d(f(x), f(y)) \leqslant \varphi(d_f(x,y)) \tag{49}$$

for all $x, y \in X$, (see [10, Example 3]). (X,d) satisfies (W3) and (JMS).

But f does not have a fixed point in X. Note that f is not continuous.

6. Applications

We now present applications of our results to probabilistic spaces. We begin with some essential definitions.

Definition 19. Let X be a set and $\mathcal{F}$ a mapping of $X \times X$ into a collection $\mathcal{L}$ of all distribution functions F (a distribution function F is a nondecreasing and left continuous mapping of reals into $[0,1]$ with $\inf\{F(x)\} = 0$ and $\sup\{F(x)\} = 1$). Consider the following conditions:

(I) $F_{x,y}(0) = 0$ for all $x, y \in X$, where $F_{x,y}$ denotes the value of $\mathcal{F}$ at $(x,y) \in X \times X$.

(II) $F_{x,y} = H$ if and only if $x = y$, where H denotes the distribution function defined by $H(x) = 0$ if $x \leqslant 0$ and $H(x) = 1$ if $x > 0$.

(III) $F_{x,y} = F_{y,x}$.

(IV) If $F_{x,y}(\varepsilon) = 1$ and $F_{y,z}(\delta) = 1$, then $F_{x,z}(\varepsilon + \delta) = 1$.

If $\mathcal{F}$ satisfies (I) and (II), then it is called a *PPM-structure* on X and the pair $(X, \mathfrak{F})$ is called a *PPM-space*. $\mathfrak{F}$ satisfying (III) is said to be symmetric. A symmetric PPM-space satisfying (IV) is a probabilistic metric space (or briefly *PM-space*).

The topology $\tau_{\mathfrak{F}}$ in $(X, \mathfrak{F})$ is generated by the family

$$\mathcal{U} = \{U_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda > 0\}, \tag{50}$$

where the set

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\} \tag{51}$$

is called (ε, λ)-*neighborhood* of $x \in X$. A sequence $\{x_n\}$ is said to be a Cauchy sequence if, for every given $\varepsilon, \lambda > 0$, there exists a positive integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ for all $m, n \geqslant n_0$. A T_1 topology $\tau_{\mathfrak{F}}$ on X is defined as follows: $U \in \tau_{\mathfrak{F}}$ if, for any $x \in U$, there exists $\varepsilon > 0$ such that $U_x(\varepsilon, \varepsilon) \subset U$. If $U_x(\varepsilon, \varepsilon) \in \tau_{\mathfrak{F}}$, then $\tau_{\mathfrak{F}}$ is said to be *topological*.

The space $(X, \mathfrak{F})$ is called $\mathfrak{F}$-complete if for every Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that $\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1$ for all $\varepsilon > 0$.

Remark 20. (1) The condition (W) is equivalent to

$$\lim_{n\to\infty} F_{x_n,y_n}(\varepsilon) = 1, \qquad \lim_{n\to\infty} F_{y_n,z_n}(\varepsilon) = 1, \qquad \text{imply } \lim_{n\to\infty} F_{x_n,z_n}(\varepsilon) = 1. \tag{P}$$

(2) The condition (W4) is equivalent to

$$\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1, \qquad \lim_{n\to\infty} F_{x_n,y_n}(\varepsilon) = 1, \qquad \text{imply } \lim_{n\to\infty} F_{y_n,x}(\varepsilon) = 1. \tag{P4}$$

The following lemma was proved in [11].

Lemma 21 (Hicks and Rhoades [11]). *Let $(X, \mathfrak{F})$ be a symmetric PPM-space. Set*

$$d(x,y) = \begin{cases} 0, & \text{if } y \notin U_x(\varepsilon,\varepsilon),\ \forall \varepsilon > 0 \\ \sup\{\varepsilon : y \notin U_x(\varepsilon,\varepsilon), \ \varepsilon > 0\}, & \text{otherwise.} \end{cases} \tag{52}$$

Then d is a bounded compatible symmetric for X.

Lemma 22 (Hicks and Rhoades [11]). *Let $(X, \mathfrak{F})$ be a symmetric PPM-space. Define d as in (52). Then*

(1) *$d(x,y) < t$ if and only if $F_{x,y}(t) > 1 - t$;*

(2) *d is compatible symmetric for $\tau_{\mathfrak{F}}$;*

(3) *$(X, \mathfrak{F})$ is complete if and only if (X,d) is d-Cauchy complete symmetric space;*

(4) *if $\tau_{\mathfrak{F}}$ is topological, d is semi-metric.*

$f : (X, \mathcal{F}) \to (X, \mathcal{F})$ is $\mathcal{F}$-continuous if $F_{x_n,x}(t) \to 1$ for all $t > 0$ implies $F_{f(x_n),f(x)}(t) \to 1$. This is equivalent to the continuity of $f : (X,d) \to (X,d)$, where d is as in Lemma 21.

Let Φ' denote the set of all functions $\varphi \in \Phi$ satisfying

$$\lim_{\epsilon\to 0} \varphi(t + \epsilon) = \varphi(t) \tag{53}$$

for all $t > 0$.

Theorem 23. *Let $(X, \mathfrak{F})$ be a complete symmetric PPM-space that satisfies (P4), where $\tau_{\mathfrak{F}}$ is a topological. Suppose $f : X \to$*

X is $\mathcal{F}$-continuous and satisfies that for $x \in X$ there exists $\nu(x) \in \mathbb{N}$ such that for any $\nu \geqslant \nu(x)$ and $y \in X$

$$F_{u,v}(t) > 1 - t \quad \text{implies } F_{f^{\nu}(x), f^{\nu}(y)}(\varphi(t)) > 1 - \varphi(t), \quad u, v \in O(x, y) \tag{54}$$

for every $t > 0$ with $\varphi \in \Phi'$. Then f has a fixed point.

Proof. Define d as in (52). According to Lemmas 21 and 22, (X, d) is a bounded d-Cauchy complete semi-metric space satisfying (W4). Now assume that (54) is satisfied. Let $\varepsilon > 0$ be given and let $t = d(u, v) + \varepsilon$. Then $d(u, v) < t$ gives $F_{u,v}(t) > 1 - t$ and so $F_{f^{\nu}(x), f^{\nu}(y)}(\varphi(t)) > 1 - \varphi(t)$ and so $d(f^{\nu}(x), f^{\nu}(y)) < \varphi(t) = \varphi(d(u, v) + \varepsilon)$. Since ε was arbitrary, we have that

$$\begin{aligned} d(f^{\nu}(x), f^{\nu}(y)) &\leqslant \varphi(d(u, v)) \\ &\leqslant \varphi\left(\sup_{u,v \in O(x,y)} d(u, v)\right) \\ &= \varphi(\operatorname{diam}(O(x, y))). \end{aligned} \tag{55}$$

By Corollary 13, f has a fixed point. □

Theorem 24. *Let $(X, \mathfrak{F})$ be a complete symmetric PPM-space that satisfies (P). Let $f : X \to X$ be $\mathcal{F}$-continuous such that*

$$F_{x,y}(t) > 1 - t \quad \text{implies } F_{f(x), f(y)}(\varphi(t)) > 1 - \varphi(t) \tag{56}$$

for all $x, y \in X$, and $\varphi \in \Phi'$. Then f has a unique fixed point.

Proof. Define d as in (52). According to Lemma 21, d is a bounded compatible symmetric for $\tau_{\mathfrak{F}}$ and (X, d) is d-Cauchy complete symmetric space satisfying (W3) and (JMS). Suppose (56) is satisfied and let $\varepsilon > 0$ be given. Let $t = \max\{d(x, y) + \varepsilon, d(x, f(x)), d(y, f(y))\}$. Then $d(x, y) + \varepsilon \leqslant t$ and so $d(x, y) \leqslant t - \varepsilon < t$, which implies $F_{x,y}(t) > 1 - t$. This further implies that $F_{f(x), f(y)}(\varphi(t)) > 1 - \varphi(t)$ and so

$$\begin{aligned} & d(f(x), f(y)) \\ &\quad < \varphi(t) = \varphi(\max\{d(x, y) + \varepsilon, d(x, f(x)), d(y, f(y))\}) \\ &\quad = \max\{\varphi(d(x, y) + \varepsilon), \varphi(d(x, f(x))), \\ &\qquad \varphi(d(y, f(y)))\}. \end{aligned} \tag{57}$$

Since ε was arbitrary, we have that

$$\begin{aligned} & d(f(x), f(y)) \\ &\quad \leqslant \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y)))\} \\ &\quad = \varphi(\max\{d(x, y), d(x, f(x)), d(y, f(y))\}). \end{aligned} \tag{58}$$

Now Theorem 16 guarantees that f has a unique fixed point $z \in X$. □

7. Some Open Problems

Problem 25 (see [2]). Let (X, d) be a symmetric space which satisfies the property (MT). Is it a semi-metric space (not necessarily with open balls)?

Problem 26. Does Theorem 16 hold if $d_f(x, y)$ is replaced with

$$\begin{aligned} D_f(x, y) = \max\{&d(x, y), d(x, f(x)), d(y, f(y)), \\ &d(x, f(y)), d(f(x), y)\}? \end{aligned} \tag{59}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the reviewers for their valuable suggestions. This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The first and third authors acknowledge with thanks DSR for the financial support. The second author was supported by the Ministry of Education, Science and Technological Development of Serbia, Grant no. 174002.

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