Journal of Computational and Applied Mathematics 259 (2014) 48–56

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Error bounds of the Micchelli–Sharma quadrature formula for analytic functions☆

Aleksandar V. Pejčev, Miodrag M. Spalević*

Department of Mathematics, University of Beograd, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia

ARTICLE INFO

Article history:
Received 4 October 2012
Received in revised form 16 February 2013

MSC:
primary 65D32
secondary 65D30
41A55

Keywords:
Micchelli–Sharma quadrature formula
Contour integral representation
Error bound

ABSTRACT

Micchelli and Sharma constructed in their paper [On a problem of Turán: multiple node Gaussian quadrature, Rend. Mat. 3 (1983) 529–552] a quadrature formula for the Fourier–Chebyshev coefficients, which has the highest possible precision. For analytic functions the remainder term of this quadrature formula can be represented as a contour integral with a complex kernel. We study the kernel, on elliptic contours with foci at the points ∓ 1 and a sum of semi-axes $\rho > 1$, for the quoted quadrature formula. Starting from the explicit expression of the kernel, we determine the location on the ellipses where the maximum modulus of the kernel is attained, and derive effective error bounds for this quadrature formula. Numerical examples are included.

1. Introduction and preliminary

Let ω be a weight function (integrable, non-negative function on $[a, b]$ that vanishes only at isolated points). Recently, Bojanov and Petrova [1] have considered quadrature formulas of the type

$$\int_a^b \omega(t)P_k(t)f(t)\,dt \approx \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j), \quad a < x_1 < \cdots < x_n < b, \tag{1.1}$$

where ν_j are given natural numbers (multiplicities) and $P_k(t)$ is a monic polynomial of degree k. A number ℓ is the algebraic degree of precision (ADP) of (1.1) if (1.1) is exact for all polynomials of degree ℓ and there is a polynomial of degree $\ell+1$ for which this formula is not exact. By $e(\nu)$ is denoted the smallest non-negative even integer $\geq \nu$ (clearly $e(\nu) = 0$ for $\nu \leq 0$), and by $\sigma(P_k)$ the number of zeros of P_k in (a, b) with odd multiplicities. It is easy to see that the ADP (1.1) does not exceed

$$e(\nu_1-\tau_1)+\cdots+e(\nu_n-\tau_n)+\sigma(P_k)-1,$$

since the formula is not exact for the polynomial

$$(t-x_1)^{e(\nu_1-\tau_1)}\cdots(t-x_n)^{e(\nu_n-\tau_n)}(t-t_1)\cdots(t-t_m),$$

where $m = \sigma(P_k),\ t_1, \ldots, t_m \in (a, b)$, are the zeros of P_k with odd multiplicities, $\tau_i := 1$ if $x_i \in \{t_1, \ldots, t_m\}$ and $\tau_i := 0$ otherwise.

☆ This work was supported in part by the Serbian Ministry of Education, Science and Technological Development (Research Project: "Methods of numerical and nonlinear analysis with applications" (No. #174002)).

* Corresponding author. Tel.: +381 11 3302334; fax: +381 11 3370364.
E-mail addresses: apejcev@mas.bg.ac.rs (A.V. Pejčev), mspalevic@mas.bg.ac.rs (M.M. Spalević).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.03.039

In [1], for the sake of convenience, Bojanov and Petrova defined the formula (1.1) to be Gaussian if it has maximal ADP, that is if

$$\mathrm{ADP}(1.1) = e(\nu_1 - \tau_1) + \cdots + e(\nu_n - \tau_n) + \sigma(P_k) - 1.$$

A complete characterization of the Gaussian formulas of form (1.1) and explicit construction of such formulas in several particular cases are given in [1], and later in [2].

Let

$$\pi_n(\mathbb{R}) := \left\{ P(t) : P(t) = \sum_{k=0}^{n} d_k t^k,\ d_k \in \mathbb{R} \right\}$$

represent the space of all polynomials in one variable of degree at most n. Bojanov and Petrova [1, Section 2] discuss general remarks concerning Gaussian quadrature formulas with multiple nodes, since the study of formulas of type (1.1) for Fourier coefficients can be reduced to the study of standard multiple node quadratures. We repeat the following theorem established by Ghizzetti and Ossicini [3].

Theorem 1.1. *For any given set of odd multiplicities* $\nu_1, \ldots, \nu_n (\nu_j = 2s_j + 1,\ s_j \in \mathbb{N}_0,\ j = 1, \ldots, n)$, *there exists a unique quadrature formula of the form*

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \quad a \le x_1 < \cdots < x_n \le b, \tag{1.2}$$

of $\mathrm{ADP} = \nu_1 + \cdots + \nu_n + n - 1$, *which is well known as the Chakalov–Popoviciu quadrature formula (see* [4,5]*). The nodes* $x_1, \ldots, x_n$ *of this quadrature are determined uniquely by the orthogonal property*

$$\int_a^b \omega(t) \prod_{k=1}^{n} (t - x_k)^{\nu_k} Q(t)\, dt = 0, \quad \forall Q \in \pi_{n-1}(\mathbb{R}).$$

The corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is known as the σ-orthogonal polynomial, with $\sigma = (s_1, \ldots, s_n)$.

Quadratures of type (1.2) with equal multiplicities $\nu_1 = \cdots = \nu_n = \nu$, with ν being an odd number ($\nu = 2s + 1,\ s \in \mathbb{N}$), have been studied by Turán [6]. In this case, the Gaussian quadrature is called the Gauss–Turán quadrature of type ν ($= 2s + 1$), and the corresponding (monic) orthogonal polynomial $\prod_{k=1}^{n}(t - x_k)$ is called the s-orthogonal polynomial.

Bojanov and Petrova [1] describe the connection between quadratures with multiple nodes and formulas of type (1.1). For the system of nodes $\mathbf{x} := (x_1, \ldots, x_n)$ with corresponding multiplicities $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, they define the polynomials

$$\Lambda(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m), \qquad \Lambda_j(t; \mathbf{x}) := \frac{\Lambda(t; \mathbf{x})}{t - x_j}, \qquad \Lambda^{\bar{\nu}}(t; \mathbf{x}) := \prod_{m=1}^{n} (t - x_m)^{\nu_m},$$

set $x_j^{\nu_j} := (x_j, \ldots, x_j)$ [x_j repeats ν_j times], $j = 1, \ldots, n$, by $g[x_1^{\nu_1}, \ldots, x_m^{\nu_m}]$ denote the divided difference of g at the points $x_1^{\nu_1}, \ldots, x_m^{\nu_m}$, and state and prove the following important theorem which reveals the relation between the standard quadratures and the quadratures for Fourier coefficients.

Theorem 1.2. *For any given sets of multiplicities* $\bar{\mu} := (\mu_1, \ldots, \mu_k)$ *and* $\bar{\nu} := (\nu_1, \ldots, \nu_n)$, *and nodes* $y_1 < \cdots < y_k$, $x_1 < \cdots < x_n$, *there exists a quadrature formula of the form*

$$\int_a^b \omega(t) \Lambda^{\bar{\mu}}(t; \mathbf{y}) f(t)\, dt \approx \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} c_{ji} f^{(i)}(x_j), \tag{1.3}$$

with $\mathrm{ADP} = N$ *if and only if there exists a quadrature formula of the form*

$$\int_a^b \omega(t) f(t)\, dt \approx \sum_{m=1}^{k} \sum_{\lambda=0}^{\mu_m - 1} b_{m\lambda} f^{(\lambda)}(y_m) + \sum_{j=1}^{n} \sum_{i=0}^{\nu_j - 1} a_{ji} f^{(i)}(x_j), \tag{1.4}$$

which has degree of precision $N + \mu_1 + \cdots + \mu_k$. *In the case* $y_m = x_j$ *for some* m *and* j, *the corresponding terms in both sums combine in one term of the form*

$$\sum_{\lambda=0}^{\mu_m + \nu_j - 1} d_{m\lambda} f^{(\lambda)}(y_m).$$

1.1. Numerical construction

Let us suppose that the coefficients a_{ji} $(j = 1, \ldots, n;\ i = 0, \ldots, \nu_j - 1)$ in (1.4) are known. By acting as in the first part of the proof of Theorem 2.1 in [1] we can determine the coefficients c_{ji} $(j = 1, \ldots, n;\ i = 0, \ldots, \nu_j - 1)$ in (1.3). Namely, applying (1.4) to the polynomial $\Lambda^{\bar{\mu}}(\cdot; \mathbf{y})f$, where $f \in \pi_N(\mathbb{R})$, the first sum in (1.4) vanishes and we can obtain (see [1, Eq. (2.4)])

$$\int_a^b \omega(t)\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\,dt = \sum_{j=1}^{n}\left(\sum_{i=0}^{\nu_j-1} a_{ji}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})f(t)\right]^{(i)}\Big|_{t=x_j}\right) = \sum_{j=1}^{n}\sum_{i=0}^{\nu_j-1} c_{ji}f^{(i)}(x_j),$$

where

$$c_{ji} = \sum_{s=i}^{\nu_j-1} a_{js}\binom{s}{i}\left[\Lambda^{\bar{\mu}}(t;\mathbf{y})\right]^{(s-i)}\Big|_{t=x_j} \quad (j = 1, 2, \ldots, n;\ i = 0, 1, \ldots, \nu_j - 1). \tag{1.5}$$

2. On the Micchelli–Sharma quadrature formula

In [7], for every $s > 0$, Micchelli and Sharma constructed a multiple node formula for the Fourier–Chebyshev coefficients of a function f of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t)f(t)\,dt \approx \sum_{j=0}^{s}\left[A_j f^{(j)}(-1) + B_j f^{(j)}(1)\right] + \sum_{j=1}^{n-1}\sum_{i=0}^{2s} a_{ji}f^{(i)}(x_j), \tag{2.1}$$

with ADP (2.1) $= (2s + 3)n - 1$, which has the highest possible precision. The nodes of their formula are located at the extremal points $-1, \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1}, 1$ of the Chebyshev polynomial of first kind T_n. Note that $\{\tilde{\eta}_j\}_{j=1}^{n-1}$ are also the zeros of the Chebyshev polynomial of second kind U_{n-1}. The uniqueness of the formula (2.1) has been proved by Bojanov and Petrova (see [1, Th. 2.6]).

The Micchelli–Sharma quadrature formula (2.1) can be represented in the form (see [7,1])

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t)f(t)\,dt \approx \frac{\pi}{2^n}\left\{\mathcal{M}_1[f] + 2\sum_{j=1}^{s}\frac{(-1)^j j a_j}{j+1}\mathcal{M}_{2j+1}[f]\right\}, \tag{2.2}$$

where a_j are defined by their generating function

$$\sum_{j=0}^{\infty} j a_j t^j = \frac{1}{2}\left[\left(1 - 4^{-n+1}t\right)^{-1/2} - 1\right],$$

and

$$\mathcal{M}_1[f] = f\left[-1, \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1}, 1\right],$$
$$\mathcal{M}_{2j+1}[f] = f\left[(-1)^{j+1}, \tilde{\eta}_1^j, \ldots, \tilde{\eta}_{n-1}^j, 1^{j+1}\right], \quad j = 1, \ldots, s.$$

By using the proposed numerical method in Section 1.1 the Micchelli–Sharma quadrature formula (2.1) can be calculated from the quadrature formula (see [1, proof of Th. 2.6])

$$\int_{-1}^{1} (1-t^2)^{1/2+s} f(t)\,dt \approx \sum_{j=1}^{n-1} \alpha_{ji}f^{(i)}(x_j) + \sum_{j=1}^{n}\alpha_j f(\xi_j).$$

Here, we use the form (2.2) of the Micchelli–Sharma quadrature formula and calculate it as follows.

Let

$$x_0 := -1, \qquad x_j := \tilde{\eta}_j, \quad j = 1, \ldots, n-1,\ x_n := 1,$$

then we have (see [8])

$$\mathcal{M}_1[f] = f[x_0, x_1, \ldots, x_{n-1}, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{u'(x_k)},$$

where $u(t) = (t - x_0)(t - x_1)\cdots(t - x_n)$, and

$$\mathcal{M}_{2j+1}[f] = f\left[x_0^{j+1}, x_1^j, \ldots, x_{n-1}^j, x_n^{j+1}\right] = \frac{(Wf)\left(x_0^{j+1}, x_1^j, \ldots, x_{n-1}^j, x_n^{j+1}\right)}{V\left(x_0^{j+1}, x_1^j, \ldots, x_{n-1}^j, x_n^{j+1}\right)}, \quad j = 1, \ldots, s,$$

where $(M := (n+1)j)$

$$
\begin{aligned}
&(Wf)\left(x_0^{j+1}, x_1^j, \dots, x_{n-1}^j, x_n^{j+1}\right)\\
&= \begin{vmatrix}
1 & x_0 & \dots & x_0^j & \dots & x_0^{M-1} & f(x_0)\\
0 & 1 & \dots & jx_0^{j-1} & \dots & (M-1)x_0^{M-2} & f'(x_0)\\
 & & & & \vdots & & \\
0 & 0 & \dots & j! & \dots & (M-1)(M-2)\cdots(M-j)x_0^{M-j-1} & f^{(j)}(x_0)\\
1 & x_1 & \dots & x_1^{j-1} & \dots & x_1^{M-1} & f(x_1)\\
0 & 1 & \dots & (j-1)x_1^{j-2} & \dots & (M-1)x_1^{M-2} & f'(x_1)\\
 & & & & \vdots & & \\
0 & 0 & \dots & (j-1)! & \dots & (M-1)(M-2)\cdots(M-j+1)x_1^{M-j} & f^{(j-1)}(x_1)\\
 & & & & \vdots & & \\
1 & x_n & \dots & x_n^j & \dots & x_n^{M-1} & f(x_n)\\
0 & 1 & \dots & jx_n^{j-1} & \dots & (M-1)x_n^{M-2} & f'(x_n)\\
 & & & & \vdots & & \\
0 & 0 & \dots & j! & \dots & (M-1)(M-2)\cdots(M-j)x_n^{M-j-1} & f^{(j)}(x_n)
\end{vmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
&V\left(x_0^{j+1}, x_1^j, \dots, x_{n-1}^j, x_n^{j+1}\right)\\
&= \begin{vmatrix}
1 & x_0 & \dots & x_0^j & \dots & x_0^{M}\\
0 & 1 & \dots & jx_0^{j-1} & \dots & Mx_0^{M-1}\\
 & & & & \vdots & \\
0 & 0 & \dots & j! & \dots & M(M-1)\cdots(M-j+1)x_0^{M-j}\\
1 & x_1 & \dots & x_1^{j-1} & \dots & x_1^{M}\\
0 & 1 & \dots & (j-1)x_1^{j-2} & \dots & Mx_1^{M-1}\\
 & & & & \vdots & \\
0 & 0 & \dots & (j-1)! & \dots & M(M-1)\cdots(M-j+2)x_1^{M-j+1}\\
 & & & & \vdots & \\
1 & x_n & \dots & x_n^j & \dots & x_n^{M}\\
0 & 1 & \dots & jx_n^{j-1} & \dots & Mx_n^{M-1}\\
 & & & & \vdots & \\
0 & 0 & \dots & j! & \dots & M(M-1)\cdots(M-j+1)x_n^{M-j}
\end{vmatrix}.
\end{aligned}
$$

3. The remainder term of Micchelli–Sharma quadrature formulas for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and $\mathcal{D}$ be its interior. Suppose f is an analytic function in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$. If we know values of the function f and of the first derivative f' of f in the nodes $-1, x_1, x_2, \dots, x_{n-1}, 1$ of the interval $[-1, 1]$, then the residue of Hermite interpolation of the function f can be written in the form (see Gončarov [9])

$$
\begin{aligned}
r_{n,s}(f;t) &= f(t) - \sum_{j=0}^{s}\left[a_j(t)f^{(j)}(-1) + b_j(t)f^{(j)}(1)\right] - \sum_{\nu=1}^{n-1}\sum_{i=0}^{2s}\ell_{i,\nu}(t)f^{(i)}(x_\nu)\\
&= \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)\Omega_{n,s}(t)}{(z-t)\Omega_{n,s}(z)}\,dz, \qquad (3.1)
\end{aligned}
$$

where $a_j, b_j, \ell_{i,\nu}$ are the fundamental functions of Hermite interpolation and

$$
\Omega_{n,s}(z) = (1-t^2)^{s+1}\prod_{\nu=1}^{n-1}(z-x_\nu)^{2s+1} = (1-t^2)^{s+1}\left(U_{n-1}(z)\right)^{2s+1}.
$$

If we choose x_ν to be the zeros of the Chebyshev polynomial of the second kind, i. e., $x_\nu = \xi_\nu$, by multiplying (3.1) with $\omega(t)T_n(t)$, where $\omega(t) = 1/\sqrt{1-t^2}$, and integrating in t over $(-1, 1)$, we get a contour integral representation of the remainder term in (2.1), i. e., (2.2),

$$R_{n,s}(fT_n) = \int_{-1}^{1} r_{n,s}(f;t)\omega(t)T_n(t)dt = \int_{-1}^{1} f(t)\omega(t)T_n(t)dt - \sum_{j=0}^{s}\left[A_j f^{(j)}(-1) + B_j f^{(j)}(1)\right] - \sum_{\nu=1}^{n-1}\sum_{i=0}^{2s} a_{i,\nu} f^{(i)}(x_\nu),$$

where

$$A_j = \int_{-1}^{1} a_j(t)T_n(t)\omega(t)dt, \qquad B_j = \int_{-1}^{1} b_j(t)T_n(t)\omega(t)dt, \qquad a_{i,\nu} = \int_{-1}^{1} \ell_{i,\nu}(t)T_n(t)\omega(t)dt.$$

Finally, we get the representation

$$R_{n,s}(fT_n) = \frac{1}{2\pi i}\oint_{\Gamma} K_{n,s}(z)f(z)dz, \tag{3.2}$$

where the *kernel* is given by

$$K_{n,s}(z) = \frac{\rho_{n,s}(z)}{(1-z^2)^{s+1}U_{n-1}^{2s+1}(z)}, \tag{3.3}$$

and

$$\rho_{n,s}(z) = \int_{-1}^{1} \frac{(1-t^2)^{s+1/2}}{z-t}\, U_{n-1}^{2s+1}(t)T_n(t)dt. \tag{3.4}$$

The integral representation (3.2) leads to the error bound

$$\left|R_{n,s}(fT_n)\right| \le \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{3.5}$$

where $\ell(\Gamma)$ is the length of the contour Γ.

In many papers error bounds $|R(f)|$ of interpolatory quadrature formulas, where f is an analytic function, are considered. Two choices of the contour Γ have been widely used:

- a circle C_r with a center at the origin and a radius $r\ (> 1)$, i. e., $C_r = \{z|\ |z| = r\}$, $r > 1$, and
- an ellipse $\mathcal{E}_\rho$ with foci at the points ∓ 1 and a sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}\left(u + u^{-1}\right),\ 0 \le \theta \le 2\pi \right\}, \quad u = \rho\, e^{i\theta}. \tag{3.6}$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1, 1]$, while with increasing ρ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1.

In this paper we take $\Gamma = \mathcal{E}_\rho$.

4. Error bounds based on the analysis of maximum of the modulus of the kernel of the Micchelli–Sharma quadrature formula

We have from (3.4), by substitution $t = \cos\theta$,

$$\rho_{n,s}(z) = \int_{0}^{\pi} \frac{\left[\sin^{2s+1} n\theta \cos n\theta \sin\theta\right]}{z - \cos\theta} d\theta.$$

On the basis of the formula from [10], we deduce

$$\begin{aligned}\rho_{n,s}(z) &= \frac{1}{2^{2s}}\int_{0}^{\pi} \frac{\sum_{k=0}^{s}(-1)^{s-k}\binom{2s+1}{k}\sin(2s+1-k)n\theta}{z-\cos\theta}\cos n\theta\sin\theta d\theta \\ &= \frac{1}{2^{2s+1}}\int_{0}^{\pi} \frac{\sum_{k=0}^{s}(-1)^{s-k}\binom{2s+1}{k}(\sin(2s-2k)n\theta + \sin(2s-2k+2)n\theta)}{z-\cos\theta}\sin\theta d\theta\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{2s+1}} \int_0^\pi \frac{\sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) \sin(2s-2k)n\theta}{z - \cos\theta} \sin\theta \, d\theta \\
&= \frac{1}{2^{2s+2}} \int_0^\pi \frac{1}{z - \cos\theta} \sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) \\
&\quad \times \left(\cos((2s-2k)n-1)\theta - \cos((2s-2k)n+1)\theta\right) d\theta \\
&= \frac{\pi}{\sqrt{z^2-1}} \frac{1}{2^{2s+2}} \sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) \\
&\quad \times \left(\left(z - \sqrt{z^2-1}\right)^{(2s-2k)n-1} - \left(z - \sqrt{z^2-1}\right)^{(2s-2k)n+1}\right),
\end{aligned}
$$

where we have used (see, for example, [11])

$$
\int_0^\pi \frac{\cos m\theta}{z - \cos\theta} d\theta = \frac{\pi}{\sqrt{z^2-1}} \left(z - \sqrt{z^2-1}\right)^m, \quad m \in \mathbb{N}_0.
$$

Substituting $z = \frac{1}{2}(u + u^{-1})$ ($u = z + \sqrt{z^2-1}$), and using

$$
T_n(z) = \left(u^n + u^{-n}\right)/2, \qquad U_{n-1}(z) = \frac{u^n - u^{-n}}{u - u^{-1}}, \tag{4.1}
$$

we get

$$
\begin{aligned}
K_{n,s}(z) &= \frac{\frac{\pi}{\frac{1}{2}(u-u^{-1})} \frac{1}{2^{2s+2}} \sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) \left(u^{-(2s-2k)n+1} - u^{-(2s-2k)n-1}\right)}{\left(\frac{1}{2}(u-u^{-1})\right)^{2s+2} \left(\frac{u^n-u^{-n}}{u-u^{-1}}\right)^{2s+1}} \\
&= \frac{2\pi \sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) u^{-(2s-2k)n}}{\left(u - u^{-1}\right)\left(u^n - u^{-n}\right)^{2s+1}}.
\end{aligned}
$$

If we use the usual notation (see [11])

$$
a_j = a_j(\rho) = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j \in \mathbb{N} \ (\rho > 1),
$$

when $u = \rho e^{i\theta}$, we have

$$
\begin{aligned}
&\left| \sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) u^{-(2s-2k)n} \right|^2 \\
&\quad = \left(\sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) \rho^{-(2s-2k)n} \cos((2s-2k)n\theta) \right)^2 \\
&\qquad + \left(\sum_{k=-1}^{s-1} (-1)^{s-k} \left(\binom{2s+1}{k} - \binom{2s+1}{k+1}\right) \rho^{-(2s-2k)n} \sin((2s-2k)n\theta) \right)^2 = a, \\
&\left|u - u^{-1}\right|^2 = 2b, \qquad \left|u^n - u^{-n}\right|^2 = 2c,
\end{aligned}
$$

where

$$
b = a_2 - \cos 2\theta, \qquad c = a_{2n} - \cos 2n\theta,
$$

and

$$
|K_n(z)| = \frac{\pi}{2^s} \cdot \sqrt{\frac{a}{bc^{2s+1}}}. \tag{4.2}
$$

Now we can formulate the main statement.

Theorem 4.1. *For each fixed $n, s \in \mathbb{N}$ there exists $\rho_0 = \rho_0(n, s)$ such that*

$$\max_{z\in\mathcal{E}_\rho} \left|K_{n,s}(z)\right| = \left|K_{n,s}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|,$$

for each $\rho > \rho_0$.

Proof. Using (4.2), we need to show the inequality

$$\frac{a}{bc^{2s+1}} \le \frac{A}{BC^{2s+1}},$$

i. e.,

$$I = aBC^{2s+1} - Abc^{2s+1} \le 0$$

for each ρ greater than some $\rho_0 = \rho_0(n, s)$, where A, B, C are the values of a, b, c at $\theta = 0$. The expression $I = I(\rho)$ is a rational expression of ρ and it will be negative for enough large ρ if and only if its coefficient which multiplies the highest degree of ρ is negative. This coefficient is equal to

$$\frac{1}{2^{2s+1}}\left(\binom{2s+1}{s} - \binom{2s+1}{s-1}\right)^2 (\cos 2\theta - 1)$$

and it is obvious negative for $\theta \in (0, \pi)$. $\square$

We are particularly interested in the cases when ρ_0 is close to its actual value, which is close to 1. Numerical experiments show that for all n, s the corresponding values of $\rho_0(n, s)$ are very close to 1, in the most cases they are less than 1.001.

In the expression $I(\rho)$, the lowest degree of ρ, which it contains, is equal to $-(4s+3)n-2$. This means that $J(\rho) = \rho^{(4s+3)n+2} I(\rho)$ is a polynomial of ρ, i.e.

$$J = J(\rho) = \sum_{i=0}^{d} a_i(\theta)\rho^i, \tag{4.3}$$

where $d = \deg(J) = (12s+4)n+2$.

From the practical point of view we are interested in the precise determination of ρ_0. If we want to get that J is non-positive for each ρ greater than $\rho^* = \rho^*(n, s)$, we can rewrite this polynomial as a polynomial of $\rho - \rho^*$, i.e.

$$J(\rho) = \sum_{i=0}^{d} b_i(\theta)(\rho-\rho^*)^i, \tag{4.4}$$

for some other coefficients $b_0, b_1, \dots, b_d$ which are trigonometric functions of θ again, i. e., $b_i = b_i(\theta),\ i = 0, 1, \dots, d$. It is enough to show that they are non-positive whenever θ belongs to $[0, 2\pi]$. It is obvious that those functions are π-periodic because they are constructed by applying elementary arithmetic operations on functions of the form $\cos 2k\theta$, where k is an integer. Moreover, the graphs of those functions are symmetric with respect to the line $x = \pi/2$. Because of that, it is enough to consider the corresponding graphs on the interval $(0, \pi/2)$.

In general, it is very complicate to express explicitly the coefficients $a_i(\theta),\ i = 0, 1, \dots, d$, from (4.3) (their expressions are very long), especially the coefficients $b_i(\theta),\ i = 0, 1, \dots, d$, from (4.4). We can get the formulas for explicitly expressing the coefficients $b_i(\theta),\ i = 0, 1, \dots, d$, in functions of the coefficients $a_j(\theta),\ j = 0, 1, \dots, d$, by using the binomial formula, but for numerical calculation it is more practical to do it step by step by using the well known Horner scheme.

Using this method, we get that for all n, s all the coefficients $b_i(\theta),\ i = 0, 1, \dots, d$ become non-positive for each $\rho^* > 1$. We have been calculating ρ_0 with two significant decimal digits, so $\rho_0 = 1.01$.

5. Numerical results

The length of the ellipse (3.6) can be estimated by (see [12, Eq. (2.2)],

$$\ell(\mathcal{E}_\rho) \le 2\pi a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\right), \tag{5.1}$$

where $a_1 = (\rho+\rho^{-1})/2$ and then (3.5) gets the form

$$\left|R_{n,s}(fT_n)\right| \le \frac{\pi}{2^s} \cdot \sqrt{\frac{A}{BC^{2s+1}}} \cdot a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\right)\left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right). \tag{5.2}$$

Table 5.1
The values of $e_{3,s}(f_1T_3)$, $e_{3,s}(f_2T_3)$ for $s = 1, 2, 3$.

s	Error	$e_{3,s}(f_1T_3)$	Error	$e_{3,s}(f_2T_3)$
1	6.562(−1)	7.715(+0)	1.489(−16)	1.458(−15)
2	4.449(−4)	5.700(−3)	1.483(−25)	1.712(−24)
3	7.253(−8)	1.011(−6)	3.036(−35)	3.970(−34)

Table 5.2
The values of $e_{7,s}(f_0T_7)$, $e_{7,s}(f_1T_7)$, $e_{7,s}(f_1T_7)$ for $s = 1, 2$.

s	Error	$e_{7,s}(f_0T_7)$	Error	$e_{7,s}(f_1T_7)$	Error	$e_{7,s}(f_1T_7)$
1	8.891(−4)	1.530(−2)	3.521(−15)	5.443(−14)	1.782(−50)	2.650(−49)
2	1.839(−12)	3.491(−11)	7.545(−28)	1.354(−26)	4.610(−77)	8.105(−76)

Since $\mathcal{E}_\rho$ ($\rho > 1$) shrinks to the interval $[-1, 1]$ as ρ tends to $1+$, there exists a maximal parameter $\rho_{\max}$ such that f is analytic inside $\mathcal{E}_\rho$ for $\rho_0 < \rho < \rho_{\max}$. Now, from (5.2) one can obtain the error bound

$$\left|R_{n,s}(fT_n)\right| \le e_{n,s}(fT_n), \tag{5.3}$$

where

$$e_{n,s}(fT_n) = \inf_{\rho_0<\rho<\rho_{\max}} \left[\frac{\pi}{2^s} \cdot \sqrt{\frac{A}{BC^{2s+1}}} \cdot a_1 \left(1 - \frac{1}{4}a_1^{-2} - \frac{3}{64}a_1^{-4} - \frac{5}{256}a_1^{-6}\right) \left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right)\right].$$

Example 1. Let us consider the integral

$$I(f) = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, T_3(t)\, dt,$$

in which we have fixed $n = 3$, where $T_3(t) = 4t^3 - 3t$.

In the case when $f(t) = f_1(t) = e^{10t}$ its exact value is equal to

$$I(f_1) = 5524.115941518612650\ldots(+0),$$

and in the case when $f(t) = f_2(t) = e^t$ its exact value is equal to

$$I(f_2) = 0.696441608839379728807495043398641536635 3\ldots(-1).$$

By “Error” we denote the actual (sharp) error of calculating the corresponding integral by the quadrature formula (2.2). In Table 5.1 the error bounds of type (5.3), when $s = 1, 2, 3$, are displayed.

Example 2. Let us consider the integral

$$I(f) = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, T_7(t)\, dt,$$

in which we have fixed $n = 7$, where $T_7(t) = 64t^7 - 112t^5 + 56t^3 - 7t$.

In the case when $f(t) = f_0(t) = e^{20t}$ its exact value is equal to

$$I(f_0) = 39467431.6804759993964555\ldots(+0),$$

in the case when $f(t) = f_1(t) = e^{10t}$ its exact value is equal to

$$I(f_1) = 747.779428498011246752868042035241 1\ldots(+0),$$

and in the case when $f(t) = f_2(t) = e^t$ its exact value is equal to

$$I(f_2) = 0.502409224662791015248738 37\ldots(-5).$$

In Table 5.2 the error bounds of type (5.3), when $s = 1, 2$, are displayed.

In the examples we have done numerical experiments with increased accuracy. This includes symbolic versions of OPQ routines currently available in the toolbox from Gautschi [13] (see also [14]). The exact values of $I(f)$ were evaluated by the Gauss–Chebyshev quadrature formula of the first kind for the integrands fT_3 and fT_7.

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