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Abstract metric spaces and Sehgal–Guseman-type theorems

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ABSTRACT

Recently, Raja and Vaezpour [P. Raja and S.M. Vaezpour, Some extensions of Banach's contraction principle in complete cone metric spaces, Fixed Point Theory Appl. 2008, 11 pages, doi:10.1155/2008/768294. Article ID 768294] proved some results for Sehgal–Guseman-type theorems in the framework of abstract (cone) metric spaces over a normal solid cone. The purpose of this paper is to present this in the framework of symmetric spaces which are associated with abstract (cone) metric spaces introduced by Radenović and Kadelburg [S. Radenović, Z. Kadelburg, Quasi-contractions on symmetric and cone symmetric spaces, ISI J. BJMA (electronic) (in press)]. Our results extend and generalize Sehgal–Guseman-type theorems from metric and abstract metric spaces to some symmetric spaces. Examples are given to illustrate the results.

1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. However, it has been observed [1] that some of the defining properties of the metric are not needed in the proofs of certain metric theorems. Motivated by this fact, Hicks and Rhoades [1] established some common fixed point theorems in symmetric spaces. Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that (i) $d(x, y) = 0$ if and only if $x = y$, and (ii) $d(x, y) = d(y, x)$. Hence, symmetric $d : X \times X \to \mathbb{R}$ has all the properties of a metric except the triangle inequality. This concept was used in some recent papers (see, e.g., [1–6]) to obtain certain fixed point results.

Recently, Raja and Vaezpour [7] proved some fixed point results for Sehgal–Guseman-type theorems in abstract (cone) metric spaces (see the definitions in the next section). Also, Radenović [6] established some coincidence point theorems in symmetric spaces obtained as associated with abstract (cone) metric spaces (see the next section). Theorem 1.2 below is Sehgal's theorem [8] in the setting of abstract (cone) metric spaces with a normal solid cone. Theorems 1.3 and 1.4 below are the corresponding extensions of the Abbas–Jungck fixed point results [9] from abstract (metric) spaces to symmetric spaces.

Lemma 1.1 ([7]). *Let (X, d) be a complete cone metric space, P a normal solid cone with normal constant K, $f : X \to X$ a continuous function, $\beta \in [0, 1)$ such that for every $x \in X$, there is an $n(x) \in \mathbb{N}$ such that*

$$d\left(f^{n(x)}x, f^{n(x)}y\right) \leq \beta d(x, y), \tag{1.1}$$

for every $y \in X$. Then for every $x \in X$, $r(x) = \sup_n \left\| d\left(f^{n(x)}x, x\right)\right\|$ is finite.

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Theorem 1.2 ([7]). *Let (X, d) be a complete cone metric space, P a normal solid cone with normal constant K, $\beta \in [0, 1)$, and $f : X \to X$ a continuous function such that for every $x \in X$, there is an $n(x) \in \mathbb{N}$ such that*

$$d\left(f^{n(x)}x, f^{n(x)}y\right) \preceq \beta d(x, y), \tag{1.2}$$

for every $y \in X$. Then f has a unique fixed point $u \in X$ and $\lim_{n\to\infty} f^n(x_0) = u$ for every $x_0 \in X$.

Theorem 1.3 ([6]). *Let (X, d) be a complete cone metric space, P a normal solid cone with normal constant K. Suppose that the commuting mappings $f, g : X \to X$ are such that for some constant $\lambda \in [0, 1)$ and for every $x, y \in X$,*

$$D(fx, fy) \le \lambda D(gx, gy). \tag{1.3}$$

If the range of g contains the range of f and if g is a continuous, then f and g have a unique common fixed point.

Theorem 1.4 ([6]). *Let (X, d) be a complete cone metric space, P a normal solid cone with normal constant K. Suppose mappings $f, g : X \to X$ satisfy*

$$D(fx, fy) \le \lambda D(gx, gy) \quad \text{for all } x, y \in X, \tag{1.4}$$

where $\lambda \in [0, 1)$,

$$\text{or } D(fx, fy) \le \lambda\left(D(fx, gx) + D(fy, gy)\right) \quad \text{for all } x, y \in X, \tag{1.5}$$

where $\lambda \in [0, \frac{1}{2K})$,

$$\text{or } D(fx, fy) \le \lambda\left(D(fx, gy) + D(fy, gx)\right) \quad \text{for all } x, y \in X, \tag{1.6}$$

where $\lambda \in [0, \frac{1}{2K})$. If the range of g contains the range of f and $g(X)$ is complete subspace of X, then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

In this paper we show that the continuity condition for function f in Theorem 1.2 may be dropped. Also, these results are extended using the concept of the associated symmetric space to the cone metric space introduced in [10]. From the proof of Lemma 1.1 in [7] it follows that the completeness of the space (X, d) and continuity of the function f are superfluous.

2. Preliminaries

Abstract (cone) metric spaces were introduced in [11]. The authors there described convergence in cone metric spaces and introduced completeness. Then they proved some fixed point theorems for contractive mappings on cone metric spaces. Recently, in [6,7,9,12–14], some common fixed point theorems were proved for maps on cone metric spaces. Actually, the study of fixed points in (cone) abstract spaces started a long time before the work of Huang and Zhang (2007), at least with the paper of Zabreiko [15].

Consistent with [11,16] (see also [15,17]), the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int\, P$ (interior of P).

There exist two kinds of cones (see [16]): normal with normal constant $K \geq 1$ and non-normal cones.

Let E be a real Banach space, $P \subset E$ a cone and $\preceq$ the partial ordering defined by P. Then P is called normal if

$$\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0, \tag{2.1}$$

or equivalently, there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \preceq x \preceq y \quad \text{imply} \quad \|x\| \leq K\|y\|, \tag{2.2}$$

or equivalently, if $(\forall n)\, x_n \preceq y_n \preceq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{2.3}$$

The least positive number satisfying (2.2) is the normal constant of P. It is clear that $K \geq 1$. From details see [16]

From ([16], Example 19.1.(ii)) we know that there exists an ordered Banach space E with cone P which is not normal but is solid, i.e., $int\, P \neq \emptyset$. Hence, a cone P is called a solid if $int\, P \neq \emptyset$.

Definition 2.1 ([11]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called an abstract (cone) metric space. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Definition 2.2 (*[11]*). Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;
(f) a convergent sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

In the following we always suppose that E is a Banach space, P is a normal cone in E and $\preceq$ is the partial ordering with respect to P.

In the sequel, we shall need some definitions and results. For a given cone metric space (X, d) one can construct a symmetric space (X, D) where "symmetric" $D : X \times X \to \mathbb{R}$ is given by $D(x, y) = \|d(x, y)\|$ (see [10]).

Definition 2.3 (*[10]*). The space (X, D) is called the symmetric space associated with the cone symmetric space (X, d).

In the case of cone metric spaces with normal cones, the triangle inequality

$$d(x, y) \preceq d(x, z) + d(z, y)$$

for each $x, y, z \in X$ implies that the symmetric D satisfies the condition

$$\begin{aligned} D(x, y) &= \|d(x, y)\| \leq K \|d(x, z) + d(z, y)\| \\ &\leq K \cdot D(x, z) + K \cdot D(z, y), \end{aligned}$$

where $K \geq 1$ is the coefficient of normality for the cone P. So, the symmetric D satisfies: for each $x, y, z \in X$

$$D(x, y) \leq K \cdot D(x, z) + K \cdot D(z, y). \tag{2.4}$$

Hence, in this case the symmetric space (X, D) is "almost" a metric space. It is a metric space if the normal constant $K = 1$.

Now, for $x \in X$ and $\varepsilon > 0$ let $B_\varepsilon(x) = \{y \in X : D(y, x) < \varepsilon\}$. Let t_D be the topology on X generated by the balls of the form $B_\varepsilon(x)$, $x \in X$, $\varepsilon > 0$.

Theorem 2.4 (*[10]*). *Let (X, d) be a cone metric space with a normal cone P and let D be the associated symmetric. Then $t_d = t_D$; moreover, d is a cone semi-metric and D is a semi-metric.*

In other words, spaces (X, d) and (X, D) have the same collections of open, closed and compact sets, and also the same convergent and Cauchy sequences and the same continuous functions.

3. Main results

First, we shall show the cone version of Guseman's theorem ([18], 1970), i.e., Theorem 1.2 in which the continuity condition is dropped.

Theorem 3.1. *Let (X, d) be a complete cone metric space, P be a normal solid cone with normal constant K and $f : X \to X$ be a mapping such that for some $\lambda \in [0, 1)$ and for every $x \in X$, there is an $n(x) \in \mathbb{N}$ such that*

$$d\left(f^{n(x)}x, f^{n(x)}y\right) \preceq \beta d(x, y), \tag{3.1}$$

for every $y \in X$. Then f has a unique fixed point $u \in X$ and $\lim_{n\to\infty} f^n(x_0) = u$ for every $x_0 \in X$.

Proof. Let $x \in X$ be given. Consider the following sequence:

$$x_0 = x, x_1 = f^{k(x_0)}x_0, x_2 = f^{k(x_1)}x_1, \ldots, x_{n+1} = f^{k(x_n)}x_n, \ldots.$$

We show that $\{x_n\}$ is a Cauchy sequence. According to (3.1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d\left(x_n, f^{k(x_n)}x_n\right) \preceq d(f^{k(x_{n-1})}x_{n-1}, f^{k(x_n)}f^{k(x_{n-1})}x_{n-1}) \\ &= d(f^{k(x_{n-1})}x_{n-1}, f^{k(x_{n-1})}f^{k(x_n)}x_{n-1}) \preceq \lambda d\left(x_{n-1}, f^{k(x_n)}x_{n-1}\right). \end{aligned}$$

Repeating this argument n times we obtain

$$d(x_n, x_{n+1}) \preceq \lambda^n d\left(x_0, f^{k(x_n)}x_0\right). \tag{3.2}$$

Thus, by the triangle inequality, for $m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq \lambda^n d\left(x_0, f^{k(x_n)}x_0\right) + \lambda^{n+1} d\left(x_0, f^{k(x_{n+1})}x_0\right) + \cdots + \lambda^{m-1} d\left(x_0, f^{k(x_{m-1})}x_0\right) \\ &= \lambda^n \left(d\left(x_0, f^{k(x_n)}x_0\right) + \lambda d\left(x_0, f^{k(x_{n+1})}x_0\right) + \cdots + \lambda^{m-n-1} d\left(x_0, f^{k(x_{m-1})}x_0\right)\right) \\ &\to \theta \quad \text{in the Banach space } E, \text{ as } n \to \infty. \end{aligned}$$

Indeed, according to Lemma 1.1 we have

$$\begin{aligned} &\left\| d\left(x_0, f^{k(x_n)}x_0\right) + \lambda d\left(x_0, f^{k(x_{n+1})}x_0\right) + \cdots + \lambda^{m-n-1} d\left(x_0, f^{k(x_{m-1})}x_0\right)\right\| \\ &\leq \left\| d\left(x_0, f^{k(x_n)}x_0\right)\right\| + \left\|\lambda d\left(x_0, f^{k(x_{n+1})}x_0\right)\right\| + \cdots + \left\|\lambda^{m-n-1} d\left(x_0, f^{k(x_{m-1})}x_0\right)\right\| \\ &\leq (1 + \lambda + \cdots + \lambda) \cdot r(x_0) \leq \frac{1}{1-\lambda} r(x_0) < +\infty. \end{aligned}$$

Now, (2.3) implies that $d(x_n, x_m) \to \theta$, as $n \to \infty$, that is $\{x_n\}$ is a Cauchy sequence. From the completeness of (X, d) we have $x_n \to u$, for some $u \in X$. Now, we shall show that $fu = u$. For this u there is $k(u) \in \mathbb{N}$ such that and from

$$d\left(f^{k(u)}u, f^{k(u)}x_n\right) \preceq \lambda d(u, x_n) \to \theta, \quad \text{as } n \to \infty.$$

Further, according to (2.3), $f^{k(u)}x_n \to f^{k(u)}u$, as $n \to \infty$. Thus, by ([11], Lemma 5) it follows that

$$d\left(f^{k(u)}x_n, x_n\right) \to d(f^{k(u)}u, u).$$

Now, from (3.1) we have

$$d\left(f^{k(u)}x_n, x_n\right) = d\left(f^{k(x_{n-1})}f^{k(u)}x_{n-1}, f^{k(x_{n-1})}x_{n-1}\right) \preceq \lambda d\left(f^{k(u)}x_{n-1}, x_{n-1}\right).$$

Repeating this argument n times we obtain

$$d\left(f^{k(u)}x_n, x_n\right) \preceq \lambda^n d\left(f^{k(u)}x_0, x_0\right).$$

Since $\left\|d\left(f^{k(u)}x_0, x_0\right)\right\| \leq r(x_0)$ it follows that $\lambda^n d\left(f^{k(u)}x_0, x_0\right) \to \theta$ in the Banach space E, as $n \to \infty$. Hence, according to (2.3) we have $d\left(f^{k(u)}x_n, x_n\right) \to \theta$. Therefore, $d(f^{k(u)}u, u) = \theta$, that is, $f^{k(u)}u = u$. From (3.1) it follows that u is the unique fixed point of $f^{k(u)}$. But for fu we have

$$f^{k(u)}fu = ff^{k(u)}u = fu,$$

i.e., fu is also a fixed point of $f^{k(u)}$. Thus, $fu = u$.

Now we show that $\lim_{n\to\infty} f^n x = u$. Let n be sufficiently large. Then we have $n = p \cdot k(u) + q$ with $p > 0$ and $0 \leq q < k(u)$. From (3.1) we obtain

$$\begin{aligned} d\left(u, f^n x\right) &= d\left(f^{k(u)}u, f^{k(u)}f^{(p-1)k(u)+q}x\right) \preceq \lambda d\left(u, f^{(p-1)k(u)+q}x\right) \\ &\preceq \cdots \preceq \lambda^p d\left(u, f^q x\right) \preceq \lambda^p \left(d(u, x) + d\left(x, f^q x\right)\right) \to \theta, \quad \text{as } p \to \infty, \end{aligned}$$

because $\|d(u, x) + d(x, f^q x)\| \leq K \cdot (\|d(u, x)\| + \|d(x, f^q x)\|) < +\infty$. Since $p \to \infty$ as $n \to \infty$ and as $\lambda \in [0, 1)$, (2.3) implies $\lim_{n\to\infty} f^n x = u$. □

Corollary 3.2. *In Theorem 3.1 by setting $E = \mathbb{R}$, $P = [0, +\infty)$, $\|x\| = |x|$, $x \in E$, we get the well known Guseman result from ([18], 1970).*

The following theorem gives a proper generalization of a results from [7,8,18], since Example 4.3 given below (Section 4) shows that there exists a cone metric space with a non-normal cone and a point in it with a bounded orbit. The orbit of a map $f : X \to X$ at point $x \in X$ will be denoted as $O_f(x; \infty) = \left\{f^k x : k = 0, 1, 2, \ldots\right\}$.

The following remark will be useful in the proof of the next theorem:

Remark 3.3. (1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(2) If $\theta \preceq u \ll c$ for each $c \in int\,P$, then $u = \theta$.
(3) If $c \in int\,P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer n_0 such that for all $n > n_0$ we have $a_n \ll c$.

Theorem 3.4. *Let (X, d) be a complete cone metric space, P be a solid cone and $f : X \to X$ be a mapping such that for some $\lambda \in [0, 1)$ and for every $x \in X$, there is an $n(x) \in \mathbb{N}$ such that*

$$d\left(f^{n(x)}x, f^{n(x)}y\right) \preceq \lambda d(x, y), \tag{3.3}$$

for every $y \in X$. If there exists a point $x \in X$ having a bounded orbit $O_f(x; \infty)$, then f has a unique fixed point $u \in X$.

Proof. Let $x \in X$ be the point with a bounded orbit, i.e., the set $O_f\,(x;\infty)$ has a finite diameter M. This means that

$$\sup_{0\le i,j<+\infty}\left\{\left\|d\left(f^ix,f^jx\right)\right\|\right\}=M<+\infty.$$

Further, as in the proof of Theorem 3.1, for the sequence

$$x_0=x,x_1=f^{k(x_0)}x_0,x_2=f^{k(x_1)}x_1,\dots,x_{n+1}=f^{k(x_n)}x_n,\dots.$$

we have $d\,(x_n,x_{n+1})\preceq\lambda^n d\left(x_0,f^{k_n(x_n)}x_0\right)$, and for $m>n$

$$\begin{aligned}d\,(x_n,x_m)&\preceq\lambda^n\left(d\left(x_0,f^{k(x_n)}x_0\right)+\lambda d\left(x_0,f^{k\left(x_{n+1}\right)}x_0\right)+\cdots+\lambda^{m-n-1}d\left(x_0,f^{k\left(x_{m-1}\right)}x_0\right)\right)\\&=\lambda^n u_{m,n}\,(\lambda,x_0)\,.\end{aligned}$$

We have that sequence $u_{m,n}\,(\lambda,x_0)$ satisfies

$$\left\|u_{m,n}\,(\lambda,x_0)\right\|\le(1+\lambda+\cdots+\lambda)\cdot diam\,O_f\,(x_0;\infty)<\frac{M}{1-\lambda}.$$

Since the sequence $u_{m,n}\,(\lambda,x_0)$ is norm bounded with $\frac{M}{1-\lambda}$, it follows that the vector sequence $\lambda^n u_{m,n}\,(\lambda,x_0)$ converges to θ in the norm of the space E, as $n\to\infty$. Hence, using Remark 3.3(3) and (1), $d\,(x_n,x_m)\ll c$ as $m,n\to\infty$. This means, by definition, that the sequence x_n is a Cauchy sequence, and hence convergent to a certain point $u\in X$.

Now, we shall show that $fu=u$. For $u\in X$ there is a $k\,(u)\in\mathbb{N}$ such that by (3.3)

$$d\left(f^{k(u)}u,f^{k(u)}x_n\right)\preceq\lambda d\,(u,x_n)\,.$$

Let $\theta\ll c$ be given. Choose a natural number n_0 such that for all $n\ge n_0$ we have $d\,(u,x_n)\ll\frac{c}{3}$. According to Remark 3.3(1), $d\left(f^{k(u)}u,f^{k(u)}x_n\right)\ll\frac{c}{3}$. This means that $f^{k(u)}x_n\to df^{k(u)}u$, as $n\to\infty$.

Further, from the inequality $d\left(f^{k(u)}x_n,x_n\right)\preceq\lambda^n d\left(f^{k(u)}x_0,x_0\right)$ it follows that there is a natural number n_1 such that for all $n\ge n_1$ we have $d\left(f^{k(u)}x_n,x_n\right)\ll\frac{c}{3}$.

Finally, it clear that there is a natural number n_2 such that for all $n\ge n_2$ we have

$$\begin{aligned}d\left(f^{k(u)}u,u\right)&\preceq d\left(f^{k(u)}u,f^{k(u)}x_n\right)+d\left(f^{k(u)}x_n,x_n\right)+d\,(x_n,u)\\&\ll\frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c.\end{aligned}$$

Now according to Remark 3.3(1), $d\left(f^{k(u)}u,u\right)\ll c$ for each $c\in int\,P$. This means, by Remark 3.3(2), $d\left(f^{k(u)}u,u\right)=\theta$, i.e., $f^{k(u)}u=u$. The rest of the proof is the same as in Theorem 3.1. □

The following theorem can be compared with Theorem 1.4 in the case where $g=I_X$ in (2.1).

Theorem 3.5. *Let (X,d) be a complete cone metric space, P a normal solid cone with normal constant K, and $f:X\to X$ a mapping such that for some $\lambda\in[0,1)$, $\lambda K<1$ and for every $x\in X$, there is an $n\,(x)\in\mathbb{N}$ such that*

$$D\left(f^{n(x)}x,f^{n(x)}y\right)\le\lambda D\,(x,y)\,,\tag{3.4}$$

for every $y\in X$. Then f has a unique fixed point $u\in X$, and $\lim_{n\to\infty}f^nx=u$, for every $x\in X$.

In order to prove Theorem 3.5 we shall need the following lemma.

Lemma 3.6. *Let (X,d) be a cone metric space, P a normal solid cone with normal constant K, and $f:X\to X$ a mapping such that some $\lambda\in[0,1)$, $\lambda K<1$ and for every $x\in X$, there is an $n\,(x)\in\mathbb{N}$ such that*

$$D\left(f^{n(x)}x,f^{n(x)}y\right)\le\lambda D\,(x,y)\,,$$

for every $y\in X$. Then for every $x\in X$, $r\,(x)=\sup_n D\left(f^{n(x)}x,x\right)$ is finite.

Proof. Let $x\in X$ be arbitrary. We show that for any fixed $m\in\mathbb{N}$,

$$\max_{1\le n\le m}D\left(x,f^nx\right)\le\frac{K}{1-\lambda K}\max\left\{D\left(x,f^ix\right):1\le i\le k\,(x)\right\}.\tag{3.5}$$

Denote by q a positive integer such that

$$D\left(x,f^qx\right)=\max_{1\le n\le m}D\left(x,f^nx\right).\tag{3.6}$$

If $q \le k(x)$, then it is easy to see that (3.5) holds. If $q > k(x)$, then by the triangle inequality, we have

$$\begin{aligned} d\left(x, f^{q}x\right) &\preceq d\left(x, f^{k(x)}x\right) + d\left(f^{k(x)}x, f^{q}x\right) \\ &= d\left(x, f^{k(x)}x\right) + d\left(f^{k(x)}x, f^{k(x)}f^{q-k(x)}x\right). \end{aligned}$$

Now according to (2.4) and (3.6) it follows that

$$\begin{aligned} D\left(x, f^{q}x\right) &\le K\left(D\left(x, f^{k(x)}x\right) + D\left(f^{k(x)}x, f^{k(x)}f^{q-k(x)}x\right)\right) \\ &\le KD\left(x, f^{k(x)}x\right) + K\lambda D\left(x, f^{q-k(x)}x\right) \\ &\le KD\left(x, f^{k(x)}x\right) + K\lambda D\left(x, f^{q}x\right). \end{aligned}$$

Hence, $D(x, f^{q}x) \le \frac{K}{1-\lambda K} D\left(x, f^{k(x)}x\right)$. From this it follows that

$$r(x) = \sup_{n} D\left(f^{n(x)}x, x\right) \le \frac{K}{1-\lambda K} \max\left\{D\left(x, f^{i}x\right) : 1 \le i \le k(x)\right\}.$$

So, we have proved (3.5). Thus, for any $x \in X$, $r(x)$ is finite real number. $\square$

Proof of the Theorem 3.5. Let $x \in X$ be given. Consider the following sequence:

$$x_0 = x, x_1 = f^{k(x_0)}x_0, x_2 = f^{k(x_1)}x_1, \ldots, x_{n+1} = f^{k(x_n)}x_n, \ldots.$$

We show that $\{x_n\}$ is a Cauchy sequence. According to (3.4) we have

$$\begin{aligned} D(x_n, x_{n+1}) &= D\left(x_n, f^{k(x_n)}x_n\right) = D\left(f^{k(x_{n-1})}x_{n-1}, f^{k(x_n)}f^{k(x_{n-1})}x_{n-1}\right) \\ &= D\left(f^{k(x_{n-1})}x_{n-1}, f^{k(x_{n-1})}f^{k(x_n)}x_{n-1}\right) \le \lambda D\left(x_{n-1}, f^{k(x_n)}x_{n-1}\right). \end{aligned}$$

Repeating this argument n times we obtain

$$D(x_n, x_{n+1}) \le \lambda^{n} D\left(x_0, f^{k(x_n)}x_0\right) \le \lambda^{n} r(x_0).$$

Thus, by the triangle inequality, for $m > n$ we have

$$d(x_n, x_m) \preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m).$$

Hence, as P is a normal cone, by (2.4) it follows that

$$\begin{aligned} D(x_n, x_m) &\le K\left(D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + \cdots + D(x_{m-1}, x_m)\right) \\ &\le K\left(\lambda^{n} + \lambda^{n+1} + \cdots + \lambda^{m-1}\right) \le \frac{K\lambda^{n}}{1-\lambda} r(x_0) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, $\{x_n\}$ is a D-Cauchy (and also d-Cauchy) sequence. Let $\lim_{n\to\infty} x_n = u \in X$. Then we have

$$D\left(f^{k(u)}u, f^{k(u)}x_n\right) \le \lambda_1 D(u, x_n) \to 0, \quad \text{as } n \to \infty.$$

Hence, $\lim_{n\to\infty} f^{k(u)} = f^{k(u)}u$. Thus,

$$D\left(f^{k(u)}u, u\right) = \lim_{n\to\infty} D\left(f^{k(u)}x_n, x_n\right).$$

From (3.4) we have

$$D\left(f^{k(u)}x_n, x_n\right) = D\left(f^{k(x_{n-1})}f^{k(u)}x_{n-1}, f^{k(x_{n-1})}x_{n-1}\right) \le \lambda D\left(f^{k(u)}x_{n-1}, x_{n-1}\right).$$

Repeating this argument n times we obtain

$$D\left(f^{k(u)}x_n, x_n\right) \le \lambda^{n} D\left(f^{k(u)}x_0, x_0\right) \le \lambda^{n} r(x_0) \to 0, \quad \text{as } n \to \infty.$$

Therefore, $D\left(f^{k(u)}u, u\right) = 0$. Hence $f^{k(u)}u = u$. From (3.4) it follows that u is the unique fixed point of $f^{k(u)}$. But for fu we have

$$f^{k(u)}fu = ff^{k(u)}u = fu,$$

i.e., fu is also a fixed point of f^{k}. Thus, $fu = u$.

Now we show that $\lim_{n\to\infty} f^{n}x = u$. Let n be sufficiently large. Then we have $n = p \cdot k(u) + q$ with $p > 0$ and $0 \le q < k(u)$. From (3.4) we obtain

$$\begin{aligned} D\left(u, f^{n}x\right) &= D\left(f^{k(u)}u, f^{k(u)}f^{(p-1)k(u)+q}x\right) \le \lambda D\left(u, f^{(p-1)k(u)+q}x\right) \\ &\le \cdots \le \lambda^{p} D\left(u, f^{q}x\right) \le \lambda^{p}\left(D(u, x) + r(x)\right). \end{aligned}$$

Since $p \to \infty$ as $n \to \infty$ and as $\lambda \in [0, 1)$, it follows that $\lim_{n\to\infty} f^{n}x = u$. $\square$

Corollary 3.7. *In Theorem 3.5, by setting $E = \mathbb{R}$, $P = [0, +\infty)$, $\|x\| = |x|$, $x \in E$, we get again Guseman's theorem from ([18], 1970).*

Remark 3.8. Note that condition (3.4) of Theorem 3.5 is in the case $K = 1$ weaker than the respective condition (3.1) of Theorem 3.1. In the case $K > 1$, the first condition is not stronger than the latter. Note also that for old and open problems in fixed point theory reader the can see [19,20].

4. Examples

Examples are given to show that our results are distinct from existing ones.

The next example shows that the fixed point problem cannot be solved in symmetric spaces as in the metric setting.

Example 4.1. Let $X = [1, +\infty)$ and $d(x, y) = (x - y)^2$. Obviously, (X, d) is a symmetric space. The mapping $fx = \frac{1}{2}x$, $x \in X$ is a contraction in the Banach sense with $\lambda \in [\frac{1}{4}, 1)$, because

$$d(fx, fy) = (fx - fy)^2 = \frac{1}{4}(x - y)^2 = \frac{1}{4}d(x, y) \leq \lambda d(x, y),$$

for $\lambda \in [\frac{1}{4}, 1)$. However, f has no fixed points.

If in Theorem 3.1 or 3.4 the order k of the iterate of a mapping f depends of x and y, then f need not have a fixed point. The following example is one such case.

Example 4.2. Let $X = \{\ln 2, \ln 3, \ldots, \ln n, \ldots\}$, $E = \mathbb{R}^2$, $P = \{(a, b) \in E : a, b \geq 0\}$, $d : X \times X \to E$,

$d(x, y) = (|x - y|, \alpha |x - y|)$, $\alpha \geq 0$. Then, (X, d) is a complete cone metric space, P is a normal solid cone, $K = 1$. Let $f : X \to X$ be defined by $f(\ln n) = \ln(1 + n)$. Then

$$f^2(\ln n) = f(\ln(n + 1)) = \ln(2 + n), \ldots, f^k(\ln n) = \ln(k + n).$$

For fixed $x = \ln n$ and $y = \ln m$ we have

$$d\left(f^k x, f^k y\right) = \left(\ln \frac{k + n}{k + m}, \alpha \ln \frac{k + n}{k + m}\right) \to (0, 0), \quad \text{as } k \to +\infty.$$

Hence, for any $\lambda \in [0, 1)$ and every $x, y \in X$ there exists $k(x, y)$ such that

$$d\left(f^{k(x,y)}x, f^{k(x,y)}y\right) \preceq \lambda d(x, y),$$

holds. But, $fx \neq x$ for each $x \in X$.

The following example verifies that Theorem 3.5 is a proper generalization of results from ([7], Theorem 3.9, [8,18]).

Example 4.3. Let $X = [0, 1]$, $E = C^1_{\mathbb{R}}[0, 1]$, $P = \{f \in E : f \geq 0\}$ as in ([16], Example 19.1.(ii)). Define $d : X \times X \to E$ by $d(x, y) = |x - y|\varphi$ where $\varphi : [0, 1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. It is easy to see that d is a cone metric on X. Consider the mapping $f : X \to X$ defined by $fx = \alpha x$, $\alpha \in (0, 1)$. The mapping f has a bounded orbit at any $x \in X$. Really, $O_f(x; \infty) = \left\{x, \alpha x, \alpha^2 x, \ldots\right\}$ and so for $i \geq j$,

$$\begin{aligned}\left\|d\left(f^i x, f^j x\right)\right\|_E &= \max_{0 \leq t \leq 1}\left|\alpha^i - \alpha^j\right| e^t + \max_{0 \leq t \leq 1}\left|\alpha^i - \alpha^j\right| e^t \\ &= 2e|x|\left|\alpha^i - \alpha^j\right| < 4e|x| < +\infty,\end{aligned}$$

i.e., the diameter of the orbit $O_f(x; \infty)$ is finite.

Further, for fixed $x \in X$ and any $y \in X$, we have

$$d\left(f^k x, f^k y\right)(t) = \left|f^k x - f^k y\right| e^t = \alpha^k |x - y| e^t \to \theta, \quad \text{as } k \to +\infty,$$

and $d(x, y)(t) = |x - y| e^t$.

Hence, for any fixed $\lambda \in [0, 1)$ and every $x \in X$ there exists $k(x) \in \mathbb{N}$ such that for every $y \in X$

$$d\left(f^{k(x)}x, f^{k(x)}y\right) \preceq \lambda d(x, y),$$

holds. This means that f satisfies the conditions of Theorem 3.5 and has a fixed point $x = 0$.

This is an example where the mapping f has a bounded orbit $O_f(x; \infty)$ at each point $x \in X$ (which is more than necessary for the conclusion in Theorem 3.5). On the other hand, it is known that the cone P is non-normal. So, in this case it is not possible to apply Theorem 3.9 from [7] and results from [8,18].

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