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On the instability of equilibrium of nonholonomic systems with nonhomogeneous constraints

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ABSTRACT

The first Lyapunov method, extended by V. Kozlov to nonlinear mechanical systems, is applied to the study of the instability of the equilibrium position of a mechanical system moving in the field of potential and dissipative forces. The motion of the system is subject to the action of the ideal linear nonholonomic nonhomogeneous constraints. Five theorems on the instability of the equilibrium position of the above mentioned system are formulated. The theorem formulated in [V.V. Kozlov, On the asymptotic motions of systems with dissipation, J. Appl. Math. Mech. 58 (5) (1994) 787–792], which refers to the instability of the equilibrium position of the holonomic scleronomic mechanical system in the field of potential and dissipative forces, is generalized to the case of nonholonomic systems with linear nonhomogeneous constraints. In other theorems the algebraic criteria of the Kozlov type are transformed into a group of equations required only to have real solutions. The existence of such solutions enables the fulfillment of all conditions related to the initial algebraic criteria. Lastly, a theorem on instability has also been formulated in the case where the matrix of the dissipative function coefficients is singular in the equilibrium position. The results are illustrated by an example.

1. Introduction

This paper deals with the motion of a scleronomic mechanical system with m degrees of freedom in a stationary field of viscous forces and a stationary field of potential forces, this motion being subject to the action of l $(l < n)$ ideal, linear, nonhomogeneous nonholonomic mutually independent constraints. Let the configuration of the system be determined by $n = m + l$ generalized coordinates $\mathbf{q} = (q^1, \ldots, q^n)$, with a corresponding vector of generalized velocities $\dot{\mathbf{q}} = \left(\dot{\mathbf{q}}', \dot{\mathbf{q}}''\right)$, where $\dot{\mathbf{q}}' = \left(\dot{q}^1, \ldots, \dot{q}^m\right)$, $\dot{\mathbf{q}}'' = \left(\dot{q}^{m+1}, \ldots, \dot{q}^n\right)$. The equations of the constraints, the kinetic energy, potential energy and the Rayleigh dissipative function have the form, respectively,[1]

$$B_i^\nu(\mathbf{q})\,\dot{q}^i + B^\nu(\mathbf{q}) = 0, \tag{1}$$

$$T = \frac{1}{2}a_{ij}(\mathbf{q})\,\dot{q}^i\dot{q}^j, \tag{2}$$

$$\Pi = \Pi(\mathbf{q}), \tag{3}$$

$$\Phi = \frac{1}{2}d_{ij}(\mathbf{q})\,\dot{q}^i\dot{q}^j. \tag{4}$$

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[1] In this paper indices take the following values: $\alpha, \beta, \gamma, \delta = 1, \ldots, m$; $\nu, \rho, \theta = m+1, \ldots, n$; $i, j = 1, \ldots, n$. Further: $\delta_i^j = 1 \forall i = j$, $\delta_i^j = 0 \forall i \neq j$.

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The kinetic energy T is positively definite, while the Rayleigh dissipative function Φ is in general semi-definite in $\dot{q}^i$, for all values of q^i.

Motions of the given system are solutions of the differential equations (1) which, as the constraints are mutually independent ($rank\left[B_i^\nu(\mathbf{x})\right] = l$), can be presented in the form:

$$\dot{q}^\nu = b_\alpha^\nu(\mathbf{q})\,\dot{q}^\alpha + b^\nu(\mathbf{q})\,, \tag{5}$$

and in the form of the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} + \frac{\partial \Phi}{\partial \dot{q}^\alpha} + \frac{\partial \Pi}{\partial q^\alpha} + b_\alpha^\nu\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^\nu}\right) - \frac{\partial T}{\partial q^\nu} + \frac{\partial \Phi}{\partial \dot{q}^\nu} + \frac{\partial \Pi}{\partial q^\nu}\right] = 0, \tag{6}$$

or, in explicit form

$$a_{\alpha j}\ddot{q}^j + \Gamma_{jk,\alpha}\dot{q}^j\dot{q}^k + d_{\alpha j}\dot{q}^j + \partial\Pi/\partial q^\alpha + b_\alpha^\nu\left(a_{\nu j}\ddot{q}^j + \Gamma_{jk,\nu}\dot{q}^j\dot{q}^k + d_{\nu j}\dot{q}^j + \partial\Pi/\partial q^\nu\right) = 0, \tag{7}$$

where $\Gamma_{jk,i}$ are Christoffel's symbols of the first kind with respect to the metrics $d\sigma^2 = a_{jk}dq^jdq^k/2$.

One assumes that the functions $\Pi(\mathbf{q})$, $a_{ij}(\mathbf{q})$, $d_{ij}(\mathbf{q})$, $b_\alpha^\nu(\mathbf{q})$ and $b^\nu(\mathbf{q})$ are infinitely differentiable.

If the conditions $\left(\partial\Pi/\partial q^i\right)(\mathbf{q}_o) = 0$ and $b^\nu(\mathbf{q}_o) = 0$ are satisfied at some point $\mathbf{q} = \mathbf{q}_o$, then $q^i = q_o^i$ is the solution (for $t \geq t_o$) of differential equations (5) and (7). In that case the point $\mathbf{q} = \mathbf{q}_o$ is a position of equilibrium of the second (**II**) kind. Further, without loss of generality, it can be accepted that $\mathbf{q_o} = 0$.

Let a regular transformation of coordinates

$$q^\alpha = \xi^\alpha, \qquad q^\nu = \xi^\nu + b_{\alpha o}^\nu\xi^\alpha, \qquad b_{\alpha o}^\nu = b_{\alpha(\mathbf{q}=\mathbf{0})}^\nu, \tag{8}$$

be performed, after which the equations of nonholonomic constraints (5) take the form

$$\dot{\xi}^\nu = \bar{b}_\beta^\nu\dot{\xi}^\beta + \bar{b}^\nu, \qquad \bar{b}_\alpha^\nu = b_{\alpha(q^\alpha=\xi^\alpha,q^\nu=\xi^\nu+b_{\alpha o}^\nu\xi^\alpha)}^\nu - b_{\alpha o}^\nu, \qquad \bar{b}^\nu = b_{(q^\alpha=\xi^\alpha,q^\nu=\xi^\nu+b_{\alpha o}^\nu\xi^\alpha)}^\nu$$

with

$$\bar{b}_{\alpha o}^\nu = \bar{b}_{\alpha(\boldsymbol{\xi}=\mathbf{0})}^\nu = 0, \tag{9}$$

where $\boldsymbol{\xi} = \left(\xi^1, \ldots, \xi^n\right)$. It will be assumed further that the transformation (8) is performed, and that the previous notations are preserved for all quantities, including the generalized coordinates. In addition to this, it will be considered that a part of the dissipative function

$$\Phi^* = \frac{1}{2}d_{\alpha\beta}(\mathbf{q})\,\dot{q}^\alpha\dot{q}^\beta \tag{10}$$

is positively defined in $\dot{\mathbf{q}}' = (\dot{q}^1, \ldots, \dot{q}^m)$.

Let

$$\Pi(\mathbf{q}) = \Pi^{(r+1)}(\mathbf{q}) + \Pi^{(r+2)}(\mathbf{q}) + \cdots \tag{11}$$

be Maclaurin's series for potential energy. Furthermore, let

$$b^\nu(\mathbf{q}) = b_{(s)}^\nu(\mathbf{q}) + b_{(s+1)}^\nu(\mathbf{q}) + \cdots \tag{12}$$

be Maclaurin's series for the function $b^\nu(\mathbf{q})$ appearing in (5), where $b^\nu(\mathbf{0}) = 0$ is taken into account. In previous relations $(\cdot)_{(p)}(\mathbf{q})$, $(\cdot)^{(p)}(\mathbf{q})$ denote the corresponding homogeneous forms of degree p.

Forms linear with respect to generalized velocities are also presented in the explicit expressions of Eqs. (5) and (7). The solution of these equations with respect to forms quoted is

$$b_{ij}(\mathbf{q})\dot{q}^j = F_i(\mathbf{q}) + G_{i,j,k}(\mathbf{q})\dot{q}^j\dot{q}^k + G_{ij}(\mathbf{q})\ddot{q}^j, \tag{13}$$

with

$$\begin{aligned}
&b_{\alpha j}(\mathbf{q}) = d_{\alpha j}(\mathbf{q}) + b_\alpha^\nu(\mathbf{q})d_{\nu j}(\mathbf{q}), \qquad b_{\nu\alpha}(\mathbf{q}) = -\delta_{\nu\rho}b_\alpha^\rho(\mathbf{q}), \qquad b_{\nu\rho}(\mathbf{q}) = \delta_{\nu\rho}\\
&F_\alpha(\mathbf{q}) = -\partial\Pi/\partial q^\alpha - b_\alpha^\nu\partial\Pi/\partial q^\nu, F_\nu(\mathbf{q}) = \delta_{\nu\rho}b^\rho(\mathbf{q}), \qquad G_{\alpha j}(\mathbf{q}) = -a_{\alpha j} - b_\alpha^\nu a_{\nu j}\\
&G_{\nu j}(\mathbf{q}) \equiv 0, \qquad G_{i,j,\alpha}(\mathbf{q}) = -\Gamma_{jk,\alpha} - b_\alpha^\nu\Gamma_{jk,\nu}, G_{ij,\nu}(\mathbf{q}) \equiv 0.
\end{aligned}$$

wherefrom, taking into account that $\det[b_{ij}(\mathbf{q}=\mathbf{0})] = \det[d_{\alpha\beta}(\mathbf{q}=\mathbf{0})] \neq 0$ holds, one can form the so-called truncated equations (cf. (11) and (12))

$$\begin{aligned}
&\dot{q}^\nu = 0,\\
&d_{\alpha i}(\mathbf{0})\dot{q}^i = -\partial\Pi^{(r+1)}(\mathbf{q})/\partial q^\alpha,
\end{aligned} \tag{14}$$

in the case of $r < s$ (case C_1) or

$$\dot{q}^\nu = b^\nu_{(r)}(\mathbf{q}),$$
$$d_{\alpha i}(\mathbf{0})\dot{q}^i = -\partial \Pi^{(r+1)}(\mathbf{q})/\partial q^\alpha, \tag{15}$$

in the case of $s = r$ (case C_2) or, finally,

$$\dot{q}^\nu = b^\nu_{(s)}(\mathbf{q}),$$
$$d_{\alpha i}(\mathbf{0})\dot{q}^i = 0, \tag{16}$$

in the case of $r > s$ (case C_3).

In accordance with the main results of papers [1–4] relating to the generalization of the first Lyapunov method to the strictly nonlinear systems of differential equations, further studies are based on the following statement: if differential equations (13) allow for the existence of the solution $\mathbf{q} = \mathbf{q}(t)$ with the characteristic $\mathbf{q}(t) \to 0$ as $t \to -\infty$ then the equilibrium state $\mathbf{q} = \mathbf{0}, \dot{\mathbf{q}} = \mathbf{0}$ of the discussed mechanical system will be unstable. In the case of $r > 1$ (cf. (14)),the mentioned solution is sought in the form of the infinite series (possibly also divergent):

$$\sum_{j=1}^{\infty} \mathbf{a}_j \left(\ln(-t)\right)(-t)^{-j\mu}, \tag{17}$$

where (cf. [1,2]) $\mathbf{a}_1 = const.$, whereas $\mathbf{a}_2, \mathbf{a}_3 \ldots$ are vector polynomials in $\ln(-t)$ and $\mu > 0$. If the above series exists and if it is convergent, it represents the solution $\mathbf{q} = \mathbf{q}(t)$ of the Eq. (13) having the property $\mathbf{q}(t) \to \mathbf{0}$ as $t \to -\infty$. If this series exists and if it is divergent, then, as shown in [5], there exists the solution $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(\mathbf{t})$ of the Eq. (13) for which the series (17) represents an asymptotic presentation. The conclusion is that the existence of the series (17) has as a consequence the instability of the equilibrium state $\mathbf{q} = \mathbf{0}, \dot{\mathbf{q}} = \mathbf{0}$ of the system whose movement is described by the differential equations (13).

From now on $r > 1$ will be valid. The case $r = 1$ has been solved in [6] and will not be discussed here.

2. The instability of equilibrium in the field of potential and dissipative forces — case C_1

The theorem on the instability of equilibrium of the holonomic scleronomic mechanical system, moving in a field of both conservative and dissipative forces, formulated in [1] will be further generalized to the case when linear nonhomogeneous nonholonomic constraints are imposed to the system, by the following

Theorem 1. *Let $r < s$, and let the function $\tilde{\Pi}^{(r+1)} = \Pi^{(r+1)}(\mathbf{q}', \mathbf{q}'' = \mathbf{0})$ has no minimum in the point $\mathbf{q}' = \mathbf{0}$. Under these conditions the equilibrium state $\mathbf{q} = \mathbf{0}, \dot{\mathbf{q}} = \mathbf{0}$ is unstable.*

Proof. In order to find conditions for the existence of series (17) under the conditions of Theorem 1, it is assumed that (cf. [1]): $\mu = 1/(r-1)$, $\mathbf{a}_1 = \lambda\mathbf{e}$, where $\lambda > 0$, $e = (e^1, \ldots, e^n)$, $\sqrt{d_{ij}(\mathbf{0})\, e^i e^j} - 1 = 0$. The series (17), included in the differential equations (5) and (7), gives the following relations (shown are only terms relevant to defining vector $\mathbf{a}_1$):

$$\frac{1}{r-1}\lambda e^\nu (-t)^{-r/(r-1)} + \cdots = 0, \tag{18}$$
$$\left(\frac{1}{r-1}\lambda e^i d_{\alpha i}(\mathbf{0}) + \lambda^r \frac{\partial \Pi^{(r+1)}}{\partial q^\alpha}(\mathbf{e})\right)(-t)^{-r/(r-1)} + \cdots = 0, \tag{19}$$

or

$$e^\nu = 0, \tag{20}$$
$$\kappa d_{\alpha\beta}(\mathbf{0}) e^\beta = -\frac{\partial \tilde{\Pi}^{(r+1)}}{\partial q^\alpha}(\mathbf{e}') \tag{21}$$

where

$$\mathbf{e}' = (e^1, \ldots, e^m), \tag{22}$$
$$\kappa = \frac{1}{\lambda^{r-1}(r-1)}. \tag{23}$$

The condition $\lambda > 0$ will be fulfilled if $\kappa > 0$. As from (21), with the condition

$$\sqrt{d_{ij}(\mathbf{0})\, e^i e^j} - 1 = 0, \tag{24}$$

or (cf. (20))

$$\sqrt{d_{\alpha\beta}\,(\mathbf{0})\,e^{\alpha}e^{\beta}}-1=0, \tag{25}$$

one obtains

$$\kappa=-\,(r+1)\,\tilde{\Pi}^{(r+1)}(\mathbf{e}'),$$

it follows that $\kappa>0$ will exist if the vector $\mathbf{e}'=\mathbf{e}'^{*}$ representing the solution of the Eqs. (21) and (25) satisfies the condition $\tilde{\Pi}^{(r+1)}(\mathbf{e}'^{*})<0$, too. In accordance with the conditions of the theorem it is fulfilled, which can be proved, as in [1,2], using the conditions of the minimum of the function $\tilde{\Pi}^{(r+1)}=\tilde{\Pi}^{(r+1)}(\mathbf{q}')$ on the ellipsoid (25). This minimum exists under the conditions of Theorem 1, and is determined just by the Eqs. (21) and (25).

In this way the existence of the vector $\mathbf{e}$, such that $\mathbf{a}_1=\lambda\mathbf{e}$, $\lambda>0$, is proved. It follows that series (17) exists, as the existence of its first term with the properties described leads to the existence of the remaining terms of the series through the chain of linear nonhomogeneous differential equations with constant coefficients. Functions which lead to the nonhomogeneity of these equations represent known polynomials of the variable $\ln(-t)$. This establishes the existence of the solution demanded with asymptotic behavior, and there from also the instability of the equilibrium state $\mathbf{q}=\mathbf{0}$, $\dot{\mathbf{q}}=\mathbf{0}$ of the system whose motion is described by the differential equation (13), is established by this. Theorem 1 is proved. □

Note 1. The case of absence of dissipation, if $b^{\nu}\equiv 0$, is solved in [7,8].

Note 2. As (cf. (4)) can be a semi-definite function of variables $\dot{\mathbf{q}}=(\dot{q}^1,\ldots,\dot{q}^n)$, in the following consideration instead of (24), the norm

$$\sqrt{a_{ij}\,(\mathbf{0})\,e^{i}e^{j}}-1=0. \tag{26}$$

is accepted. The case with dissipation, if $b^{\nu}\equiv 0$, is solved in [12].

3. The instability of equilibrium in the field of potential and dissipative forces — case C_2

The series (17), for $\mu=1/(r-1)$, included in the differential equations (5) and (7), gives the following relations (shown are only terms relevant to defining vector $\mathbf{a}_1$):

$$\left(\frac{1}{r-1}\lambda e^{\nu}-\lambda^{r}b^{\nu}_{(r)}(\mathbf{e})\right)(-t)^{-r/(r-1)}+\cdots=0, \tag{27}$$

$$\left(\frac{1}{r-1}\lambda e^{i}d_{\alpha i}(\mathbf{0})+\lambda^{r}\frac{\partial\Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e})\right)(-t)^{-r/(r-1)}+\cdots=0, \tag{28}$$

which result in

$$\kappa e^{\nu}-b^{\nu}_{(r)}(\mathbf{e})=0, \tag{29}$$

$$\kappa d_{\alpha i}(\mathbf{0})e^{i}+\frac{\partial\Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e})=0, \tag{30}$$

where κ is defined by relation (23).

As stated in the previous section, the existence of series (17) is ensured by the real solutions $\mathbf{e}=\mathbf{e}^{*}\neq\mathbf{0}$, $\kappa>0$ of Eqs. (29), (30) and (26). Further study will bring these equations to a form much more simple than the original one. The discussion that ensues can be applied also to all analogous algebraic criteria present in the papers [1–4,9].

Let the Eqs. (26), (29) and (30) have the real solutions

$$e^{i}=e^{*i},\qquad \kappa=\kappa^{*},\qquad \kappa^{*}>0, \tag{31}$$

and let, further, the transformation

$$\tilde{e}^{i}=\rho e^{*i},\quad \rho=const.,\,\rho>0, \tag{32}$$

be performed.

With this remark the Eqs. (29) and (30) get the form, respectively,

$$\kappa^{*}\tilde{e}^{\nu}\rho^{r-1}-b^{\nu}_{(r)}(\tilde{\mathbf{e}})=0 \tag{33}$$

$$\kappa^{*}d_{\alpha i}(\mathbf{0})\tilde{e}^{i}\rho^{r-1}+\frac{\partial\Pi^{(r+1)}}{\partial q^{\alpha}}(\tilde{\mathbf{e}})=0, \tag{34}$$

where $\tilde{\mathbf{e}}=\left(\tilde{e}^{1},\ldots,\tilde{e}^{n}\right)$.

If choosing a positive scalar ρ such that

$$\kappa^* \rho^{r-1} = 1 \tag{35}$$

the vector $\mathbf{e} = \tilde{\mathbf{e}}$ will satisfy the equations (cf. (33) and (34))

$$\tilde{e}^\nu - b^\nu_{(r)}(\tilde{\mathbf{e}}) = 0, \tag{36}$$

$$d_{\alpha i}(\mathbf{0})\tilde{e}^i + \frac{\partial \Pi^{(r+1)}}{\partial q^\alpha}(\tilde{\mathbf{e}}) = 0. \tag{37}$$

Hence follows that if the equations

$$e^\nu - b^\nu_{(r)}(\mathbf{e}) = 0, \tag{38}$$

$$d_{\alpha i}(\mathbf{0})e^i + \frac{\partial \Pi^{(r+1)}}{\partial q^\alpha}(\mathbf{e}) = 0. \tag{39}$$

have the real solution $\mathbf{e} = \tilde{\mathbf{e}}$, then Eqs. (29), (30) and (26) also have the real solution $\mathbf{e} = \mathbf{e}^* = \tilde{\mathbf{e}}/\rho$, $\kappa^* = 1/\rho^{r-1}$. Then it is obvious that

$$\rho = \sqrt{a_{ij}(\mathbf{0})\,\tilde{e}^i\tilde{e}^j}. \tag{40}$$

It is possible now to formulate the following

Theorem 2. *Let $r = s$, and let the equations*

$$d_{\alpha i}(\mathbf{0})e^i + \frac{\partial \Pi^{(r+1)}}{\partial q^\alpha}(\mathbf{e}) = 0, \qquad e^\nu - b^\nu_{(r)}(\mathbf{e}) = 0,$$

have real nontrivial solution $\mathbf{e} = \tilde{\mathbf{e}}$. Under these conditions the equilibrium state $\mathbf{q} = \mathbf{0}$, $\dot{\mathbf{q}} = \mathbf{0}$ is unstable.

Proof. If the mentioned algebraic equations have the real solution $\mathbf{e} = \tilde{\mathbf{e}}$ it follows, according to the previous discussions, that the Eqs. (29) and (30) as well have the solutions $\mathbf{e} = \mathbf{e}^* = \tilde{\mathbf{e}}/\rho$, $\kappa = \kappa^* = 1/\rho^{r-1}$, where $\rho = \left(a_{ij}(\mathbf{0})\,\tilde{e}^i\tilde{e}^j\right)^{1/2}$. It follows therefore that the series (17) exists and, especially, that the equilibrium state $\mathbf{q} = \mathbf{0}$, $\dot{\mathbf{q}} = \mathbf{0}$ of the system (13) is unstable. □

4. The instability of equilibrium in the field of potential and dissipative forces — case C_3

The case discussed next is $r > s$, $s > 1$. In this case the series (17), for $\mu = 1/(s-1)$, included in the differential equations (5) and (7), gives the following relations (shown are only terms relevant to defining vector $\mathbf{a}_1$):

$$\left(\frac{1}{s-1}\lambda e^\nu - \lambda^s b^\nu_{(s+1)}(\mathbf{e})\right)(-t)^{-s/(s-1)} + \cdots = 0, \tag{41}$$

$$\frac{1}{s-1}\lambda e^i d_{\alpha i}(\mathbf{0})(-t)^{-s/(s-1)} + \cdots = 0, \tag{42}$$

which gives

$$\kappa_1 e^\nu - b^\nu_{(s)}(\mathbf{e}) = 0, \tag{43}$$

$$d_{\alpha i}(\mathbf{0})e^i = 0, \tag{44}$$

where

$$\kappa_1 = \frac{1}{\lambda^{s-1}(s-1)}.$$

Eq. (44) can be presented in the form shown below

$$e^\alpha = -r^{\alpha\beta} d_{\beta\nu}(\mathbf{0})e^\nu \tag{45}$$

where $d_{\alpha\beta}(\mathbf{0})r^{\beta\gamma} = \delta^\gamma_\alpha$.

If the function $\tilde{b}^\nu_{(s)} = \tilde{b}^\nu_{(s)}(\mathbf{q}'')$ is introduced in the following way

$$\tilde{b}^\nu_{(s)}(\mathbf{q}'') = \left(b^\nu_{(s)}(\mathbf{q})\right)_{\left(q^\alpha = -r^{\alpha\beta} d_{\beta\nu} q^\nu\right)}, \tag{46}$$

the Eq. (43) becomes

$$\kappa_1 e^\nu - \tilde{b}^\nu_{(s)}(\mathbf{e}'') = 0, \tag{47}$$

where $\mathbf{e}'' = (e^{m+1}, \ldots, e^n)$.

Study analogous to that from the previous section results in the conclusion that if equations

$$e^{\nu} - \tilde{b}^{\nu}_{(s)}(\mathbf{e}'') = 0$$

have real nontrivial solution $\mathbf{e}'' = \tilde{\mathbf{e}}''$ then (47) will also have the solution $\mathbf{e}'' = \mathbf{e}''^{*} = \tilde{\mathbf{e}}''/\rho, \kappa_1^* = 1/\rho^{(s-1)}$, where $\rho = \sqrt{a_{ij}(\mathbf{0})\,\tilde{e}^i\tilde{e}^j}$. It is obvious that (cf. (45)) $\tilde{e}^{\alpha} = -r^{\alpha\beta}d_{\beta\nu}(0)\tilde{e}^{\nu}$ and $e^{*\alpha} = -r^{\alpha\beta}d_{\beta\nu}(0)e^{*\nu}$ are valid.

Now it is possible to formulate the next

Theorem 3. *Let $r > s, s > 1$ hold, and let the equations*

$$e^{\nu} - \tilde{b}^{\nu}_{(s)}(\mathbf{e}'') = 0$$

have real nontrivial solution $\mathbf{e}'' = \tilde{e}''$. Under these conditions the equilibrium state $\mathbf{q} = \mathbf{0}, \dot{\mathbf{q}} = \mathbf{0}$ is unstable.

Proof. If the algebraic equations mentioned above have a real solution $\mathbf{e} = \tilde{\mathbf{e}}$ it follows, according to previous analysis, that the Eqs. (29) and (30) have nontrivial solutions $\mathbf{e} = \mathbf{e}^* = \tilde{\mathbf{e}}/\rho,\ \kappa_1 = \kappa_1^* = 1/\rho^{s-1}$, where $\rho = \left(a_{ij}(\mathbf{0})\,\tilde{e}^i\tilde{e}^j\right)^{1/2}$, too. From here it follows that the series (17) exists and, especially, that the equilibrium state $\mathbf{q} = \mathbf{0}, \dot{\mathbf{q}} = \mathbf{0}$ of system (13) is unstable. □

5. The singular case

The case under consideration is the one when Maclaurin's series for dissipative function coefficients $d_{ij} = d_{ij}(\mathbf{q})$ has the form

$$d_{ij} = d^{(l)}_{ij}(\mathbf{q}) + d^{(l+1)}_{ij}(\mathbf{q}) + \cdots, \tag{48}$$

where $d^{(l)}_{ij} = d^{(l)}_{ij}(\mathbf{q})$ is the homogeneous forms of degree l. It is obvious that in this case, which will be called singular, the below relation is valid

$$\det[d_{ij}(\mathbf{0})] = 0, \tag{49}$$

the implication being that differential equations (13) cannot be solved explicitly by generalized velocities. In order to overcome this problem, equations of nonholonomic constraints (5) are being differentiated by the time:

$$\ddot{q}^{\nu} = b^{\nu}_{\alpha}(\mathbf{q})\,\ddot{q}^{\alpha} + \frac{\partial b^{\nu}_{\alpha}(\mathbf{q})}{\partial q^i}\dot{q}^{\alpha}\dot{q}^i + \frac{\partial b^{\nu}(\mathbf{q})}{\partial q^i}\dot{q}^i. \tag{50}$$

The next case under discussion is $l > (r-1)/2$ and $s > (r+1)/2$. When series (17), with $\mu = 2/(r-1)$ gets inserted into differential equations (7) and (50) by standard procedure the below algebraic criteria is achieved

$$\kappa_2 a_{\alpha j}(\mathbf{0})e^j + \frac{\partial \Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e}) = 0, \tag{51}$$

$$e^{\nu} = 0, \tag{52}$$

where

$$\kappa_2 = 2\frac{(r+1)}{(r-1)^2\,\lambda^{r-1}}. \tag{53}$$

It is clear that (51), in view of (52), can also be written in the form

$$\kappa_2 a_{\alpha\beta}(\mathbf{0})e^{\beta} + \frac{\partial \tilde{\Pi}^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e}') = 0, \tag{54}$$

where $\tilde{\Pi}^{(r+1)} = \Pi^{(r+1)}(\mathbf{q}', \mathbf{q}'' = \mathbf{0})$.

The theorem on instability of equilibrium of the holonomic scleronomic mechanical system, moving in a field of both conservative and dissipative forces, formulated in [1], will be further generalized to the singular case when linear nonhomogeneous nonholonomic constraints are imposed on the system, by the following

Theorem 4. *Let $l > (r-1)/2 \wedge s > (r+1)/2$, and let the function $\tilde{\Pi}^{(r+1)} = \Pi^{(r+1)}(\mathbf{q}', \mathbf{q}'' = \mathbf{0})$ has no minimum in the point $\mathbf{q}' = \mathbf{0}$. Under these conditions the equilibrium state $\mathbf{q} = \mathbf{0}, \dot{\mathbf{q}} = \mathbf{0}$ of the system (7) and (50) is unstable.*

Proof. The first proof phase is identical as in the case of Theorem 1. That phase proves the existence of series (17) for $\mu = 2/(r-1)$. It follows that $\kappa_2 > 0$ will exist if the vector $\mathbf{e}' = \mathbf{e}'^{*}$ representing the solution of the Eqs. (26) and (54) satisfies the condition $\tilde{\Pi}^{(r+1)}(\mathbf{e}'^{*}) < 0$, as well. In accordance with the conditions of the theorem it has been fulfilled, which can be proved, as in Theorem 1, using the conditions of the minimum of the function $\tilde{\Pi}^{(r+1)} = \tilde{\Pi}^{(r+1)}(\mathbf{q}')$ on the ellipsoid

$$\sqrt{a_{\alpha\beta}(\mathbf{0})\,e^{\alpha}e^{\beta}} - 1 = 0. \tag{55}$$

This minimum exists under the conditions of Theorem 4, and is determined just by the Eqs. (54) and (55). It remains to prove that conclusions relating to Eq. (50) are also extended to (5). That is, differential equation (50) have the first integrals of the form

$$\dot{q}^{\nu} = b^{\nu}_{\alpha}\left(\mathbf{q}\right)\dot{q}^{\alpha} + b^{\nu}\left(\mathbf{q}\right) + c^{\nu}, \quad c^{\nu} = const., \tag{56}$$

which correspond with the equations of nonholonomic constraints only if $c^{\nu} = 0$. Taking into account that $\mathbf{q}(t) \to 0 \wedge \dot{\mathbf{q}}(t) \to 0$ as $t \to -\infty$ it follows that $b^{\nu}\left(\mathbf{q}\right) \to 0$ as $t \to -\infty$ wherefrom it can be concluded that $c^{\nu} = 0$. Theorem 4 is proved. □

In the singular case where it holds that $s = l+1 \wedge l < (r-1)/2$ the series (17), for $\mu = 1/(r-l-1)$, included in differential equations (7) and (50), gives the following relations, respectively (shown are only terms relevant to defining vector $\mathbf{a}_1$):

$$\begin{aligned}&(-t)^{-(2r-2l-1)/(r-l-1)}\lambda(r-l)a^{*}_{\alpha i}(\mathbf{0})e^{i}/(r-l-1)^{2} + b^{(l)}_{\alpha i}(\mathbf{e}')\lambda^{l+1}e^{i}\left(-t\right)^{-r/(r-l-1)}/(r-l-1)^{2}\\ &\quad+\lambda^{r}\frac{\partial \Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e})\left(-t\right)^{-r/(r-l-1)} + \cdots = 0,\end{aligned} \tag{57}$$

$$(-t)^{-(2r-2l-1)/(r-l-1)}\lambda(r-l)e^{\nu}/(r-l-1)^{2} - \frac{\partial b^{\nu}_{(l)}\left(\mathbf{q}\right)}{\partial q^{i}}(\mathbf{e})\lambda^{l+1}e^{i}\left(-t\right)^{-r/(r-l-1)}/(r-l-1)^{2} = 0. \tag{58}$$

Taking into account that $l < (r-1)/2$ holds, one concludes that $r-l-1 > l \to r-l-1 > 0$ and $2r-2l-1 > r$. Wherefrom, with respect to (57) and (58), it follows

$$\kappa_3 d^{(l)}_{\alpha i}(\mathbf{e})e^{i} + \frac{\partial \Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e}) = 0, \tag{59}$$

$$b^{\nu}_{(l+1)}\left(\mathbf{e}\right) = 0, \tag{60}$$

where

$$\kappa_3 = \frac{1}{\lambda^{r-l-1}(r-l-1)^{2}}. \tag{61}$$

The truncated differential equations corresponding to the algebraic criteria (59) and (60) have the form

$$r^{(l)}_{\alpha i}(\mathbf{q})\dot{q}^{i} + \frac{\partial \Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{q}) = 0, \tag{62}$$

$$r^{(l)}_{\nu i}(\mathbf{q})\dot{q}^{i} = 0, \tag{63}$$

where $r^{(l)}_{\alpha i}(\mathbf{q}) = d^{(l)}_{\alpha i}(\mathbf{q})$, $r^{(l)}_{\nu i}(\mathbf{q}) = \partial b^{\nu}_{(l+1)}\left(\mathbf{q}\right)/\partial q^{i}$.

The differential equations (62) and (63) cannot be solved with respect to the generalized velocities, as $\det[r^{(l)}_{ij}(0)] = 0$. Despite this fact (cf. [10]), if there exists real solution $\mathbf{e} = \tilde{\mathbf{e}}$ for algebraic equations

$$d^{(l)}_{\alpha i}(\mathbf{e})e^{i} + \frac{\partial \Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e}) = 0, \tag{64}$$

$$b^{\nu}_{(l+1)}\left(\mathbf{e}\right) = 0, \tag{65}$$

and the condition $\det[r^{(l)}_{ij}(\tilde{\mathbf{e}})] \neq 0$ is fulfilled, the existence of the series will be provided (17).

Now it is possible to formulate the next

Theorem 5. *Let $s = l+1 \wedge l < (r-1)/2$ holds, and let the condition $\det[d^{(s)}_{ij}(\tilde{\mathbf{e}})] \neq 0$ is fulfilled, where $\mathbf{e} = \tilde{\mathbf{e}}$ is a real zero of the vector fields $b^{(s)}_{\alpha i}(\mathbf{e})e^{i} + \frac{\partial \Pi^{(r+1)}}{\partial q^{\alpha}}(\mathbf{e})$ and $b^{\nu}_{(s)}\left(\mathbf{e}\right)$.*

Under these conditions the equilibrium state $\mathbf{q} = \mathbf{0}$, $\dot{\mathbf{q}} = 0$ of the system (7) and (50) is unstable.

The proof for Theorem 5 follows up the scheme as given in proof for Theorem 4 from [10].

6. Example

A. Systems with cyclic coordinates. One considers a holonomic mechanical system moving in a field of potential and dissipative forces. The configuration of the system is determined by a set of generalized coordinates $\mathbf{q} = (q^{1}, \ldots, q^{m}, \xi^{m+1}, \ldots, \xi^{n})$. Let the kinetic and potential energy of the system have the form

$$T = \frac{1}{2}\left(a_{\alpha\beta}\left(\mathbf{q}'\right)\dot{q}^{\alpha}\dot{q}^{\beta} + a_{\alpha\nu}\dot{q}^{\alpha}\dot{\xi}^{\nu} + a_{\nu\alpha}\dot{\xi}^{\nu}\dot{q}^{\alpha} + a_{\nu\rho}\dot{\xi}^{\nu}\dot{\xi}^{\rho}\right), \tag{66}$$

$$\Pi = \Pi(\mathbf{q}'), \tag{67}$$

where $\mathbf{q}' = \left(q^{1}, \ldots, q^{m}\right)$, $m = n-l$, $l > 0$.

Let the dissipation be incomplete, with the dissipation function

$$\Phi^* = \frac{1}{2} d_{\alpha\beta}\left(\mathbf{q}'\right)\dot{q}^\alpha \dot{q}^\beta, \tag{68}$$

positively definite in $\dot{\mathbf{q}}' = \left(\dot{q}^1, \ldots, \dot{q}^m\right)$.

It is obvious that the differential equations of motion of the considered system have l cyclic integrals in the form

$$\frac{\partial T}{\partial \dot{q}^\nu} = c_\nu \to a_{\nu\alpha}\dot{q}^\alpha + a_{\nu\rho}\dot{\xi}^\rho = c_\nu, \quad c_\nu = const.,$$

whose solution with respect to cyclic generalized velocities $\dot{\boldsymbol{\xi}}'' = \left(\dot{\xi}^{m+1}, \ldots, \dot{\xi}^n\right)$ has the form

$$\dot{\xi}^\nu = \bar{b}^\nu_\alpha\left(\mathbf{q}'\right)\dot{q}^\alpha + \bar{b}^\nu\left(\mathbf{q}'\right), \tag{69}$$

where

$$\bar{b}^\nu_\alpha\left(\mathbf{q}'\right) = -e^{\nu\rho}\left(\mathbf{q}'\right) a_{\rho\alpha}\left(\mathbf{q}'\right), \qquad \bar{b}^\nu\left(\mathbf{q}'\right) = e^{\nu\rho}\left(\mathbf{q}'\right) c_\rho. \tag{70}$$

The above expression includes the coordinates of the matrix $[e^{\nu\rho}]$, which is inverse to the matrix $[a_{\nu\rho}] : e^{\nu\rho} a_{\theta\rho} = \delta^\nu_\theta$.

Relation (69) is being transformed into the form

$$\dot{q}^\nu = b^\nu_\alpha\left(\mathbf{q}'\right)\dot{q}^\alpha + b^\nu\left(\mathbf{q}'\right),$$

where

$$q^\nu = \xi^\nu - \bar{b}^\nu_\alpha(\mathbf{0})\, q^\alpha - \bar{b}^\nu(\mathbf{0})\, t,$$
$$b^\nu_\alpha\left(\mathbf{q}'\right) = \bar{b}^\nu_\alpha\left(\mathbf{q}'\right) - \bar{b}^\nu_\alpha(\mathbf{0}),$$
$$b^\nu\left(\mathbf{q}'\right) = \bar{b}^\nu\left(\mathbf{q}'\right) - \bar{b}^\nu(\mathbf{0}).$$

It is well known that the system given with cyclic coordinates can be considered as a system with nonholonomic linear nonhomogeneous constraints, which represent the cyclic integrals (69). Obviously, the quoted nonholonomic system is of the Chaplygin type. Its differential equations are given by (69) and by the equations and by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial \dot{q}^\alpha} - \frac{\partial T^*}{\partial q^\alpha} + c_\nu\left(\gamma^\nu_{\alpha\beta}\dot{q}^\beta + \gamma^\nu_\alpha\right) + \frac{\partial \Pi}{\partial q^\alpha} + \frac{\partial \Phi}{\partial \dot{q}^\alpha} = 0, \tag{71}$$

where

$$T^* = T_2^* + \frac{1}{2} e^{\nu\rho}\left(\mathbf{q}'\right) c_\nu c_\rho.$$

A particular example of a system with cyclic coordinates is presented below.

It is obvious that differential equations (71) can also be presented in the form (6).

B. Example. The rod 4, orthogonally bent in C (Fig. 1), can slide without friction along the symmetric guide 7, rigidly connected to the telescopic rod 1 of negligible mass. The axis of the guide is orthogonal to the axis of rod 1. Another end of the telescopic rod 1 is connected to joint A, having the vertical axis which contains the centre of mass of guide 2. The moment of inertia of guide 7 with respect to the fixed central vertical axis with which it is hinged, is $J = ml^2$. Telescopic rod, of negligible mass, is hinged by one end B to end of rod 4 while its other end is rigidly attached to the symmetric homogeneous guide 2 having the axis perpendicular to the axis of the telescopic rod 3. Rod 5 can slide without friction in guide 2. Rod 5 is attached by one end to a joint on whose vertical axis is point C. The mass of guide 3 is $2m$, its moment of inertia with respect to the central vertical axis, which cuts the axis of telescopic rod 3, is $J = ml^2$. The configuration of the system is determined by generalized coordinates (φ, θ). The forces acting in points A and B are $\vec{F} = F(\varphi)\,\vec{e}$ and $\vec{F}' = -\vec{F}$, respectively, where $\vec{e} = \overrightarrow{BA}/\overline{BA}$, $F(\varphi = 0) = 0$, $F'(\varphi = 0) \neq 0$. During the motion there appear, between the guide 2 and the rod 5, forces of viscous friction $\vec{F}_w = -\beta\vec{v}_r$ and $\vec{F}'_w = -\beta\vec{v}'_r$ acting on guide 2 and rod 5, respectively, where $\vec{v}_r$ is the relative velocity of guide 2 with respect to rod 5, $\vec{v}'_r = -\vec{v}_r$, $\beta = const.$, $\beta > 0$. Prove that there exists a stationary motion of the system $\varphi = 0, \theta = \dot{\theta}_0 t$, and verify its stability. Neglect the mass of rods 4 and 5. $\overline{CB} = l$. Function $F(\varphi)$ is infinitely differentiable. The system moves in a horizontal plane.

Solution. The kinetic energy of the system is

$$T = \frac{1}{2} ml^2 (2 - \cos 4\varphi)\,\dot{\varphi}^2 + \frac{1}{2} ml^2\left(2 + 2\sin^4\varphi\right)\dot{\theta}^2 - ml^2\dot{\varphi}\dot{\theta}, \tag{72}$$

and the generalized forces of the system $\left(\vec{F}, \vec{F}'\right)$ are

$$Q_\varphi = -lF(\varphi)\sin\varphi, \qquad Q_\theta \equiv 0, \tag{73}$$

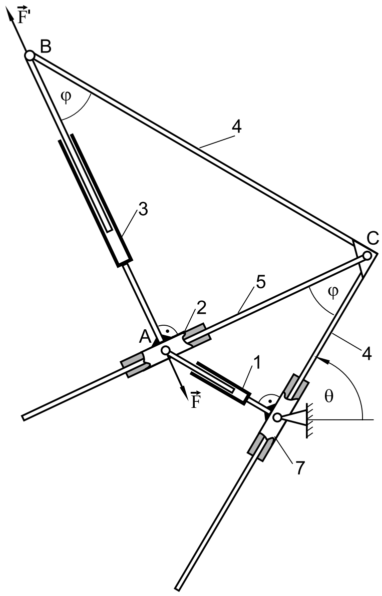

Fig. 1.

wherefrom it follows that the potential energy of this force has the form

$$\Pi_1 = l\int_0^{\varphi} F(\varphi)\sin\varphi \mathrm{d}\varphi \tag{74}$$

while its Maclaurin's series is

$$\Pi_1 = \frac{l}{3}F'_{(0)}\varphi^3 + \frac{l}{8}F''_{(0)}\varphi^4 + \cdots \tag{75}$$

The dissipative function has the form

$$\Phi = \frac{\beta l^2}{2}\cos^2\varphi\dot{\varphi}^2. \tag{76}$$

It is obvious that θ is a cyclic coordinate. The corresponding cyclic integral reads

$$ml^2\left(2+2\sin^4\varphi\right)\dot{\theta} - ml^2\dot{\varphi} = c, \quad c = const., \tag{77}$$

or, in the form of a linear nonholonomic nonhomogeneous constraint

$$\dot{\theta} = \frac{1}{2\left(1+\sin^4\varphi\right)}\dot{\varphi} + \frac{c}{2ml^2\left(1+\sin^4\varphi\right)}, \tag{78}$$

or

$$\dot{\theta} - \frac{1}{2}\dot{\varphi} - \frac{c}{2ml^2} = \left[\frac{1}{2\left(1+\sin^4\varphi\right)} - \frac{1}{2}\right]\dot{\varphi} + \left[\frac{c}{2ml^2\left(1+\sin^4\varphi\right)} - \frac{c}{2ml^2}\right]. \tag{79}$$

By transformation

$$\theta = \epsilon + \frac{1}{2}\varphi + \frac{c}{2ml^2}t, \tag{80}$$

Eq. (79) obtains the form

$$\dot{\epsilon} = \left[\frac{1}{2\left(1+\sin^4\varphi\right)} - \frac{1}{2}\right]\dot{\varphi} + \left[\frac{c}{2ml^2\left(1+\sin^4\varphi\right)} - \frac{c}{2ml^2}\right], \tag{81}$$

and time t does not figure explicitly in the equation for kinetic energy which now looks like

$$\tilde{T} = \frac{1}{2}ml^2(2-\cos 4\varphi)\dot{\varphi}^2 + \frac{1}{2}ml^2\left(2+2\sin^4\varphi\right)\left(\dot{\epsilon}+\frac{1}{2}\dot{\varphi}+\frac{c}{2ml^2}\right)^2 - ml^2\dot{\varphi}\left(\dot{\epsilon}+\frac{1}{2}\dot{\varphi}+\frac{c}{2ml^2}\right). \tag{82}$$

The differential equation of motion in regards to the coordinate φ is

$$-\frac{1}{16}ml^2\left(-27+4\cos 2\varphi+15\cos 4\varphi\right)\ddot{\varphi}+\left(ml^2\sin^4\varphi\right)\ddot{\epsilon}+ml^2\left(\cos\varphi\sin^3\varphi+2\sin 4\varphi\right)\dot{\varphi}^2+\left(\beta l^2\cos^2\varphi\right)\dot{\varphi}$$
$$-\left(4ml^2\cos\varphi\sin^3\varphi\right)\dot{\epsilon}^2-\left(4c\cos\varphi\sin^3\varphi\right)\dot{\epsilon}+F(\varphi)l\sin\varphi-\frac{c^2\cos\varphi\sin^3\varphi}{ml^2}=0. \tag{83}$$

The reduced potential energy has the form

$$\Pi=l\int_0^{\varphi}\left[F\left(\varphi\right)\sin\varphi-\frac{c^2\cos\varphi\sin^3\varphi}{ml^2}\right]\mathrm{d}\varphi \tag{84}$$

and its Maclaurin's series is

$$\Pi=\frac{l}{3}F'_{(0)}\varphi^3+\frac{1}{4}\left(\frac{l}{2}F''_{(0)}-\frac{c^2}{ml^2}\right)\varphi^4+\frac{l}{30}\left(F'''_{(0)}-F'_{(0)}\right)\varphi^5+\cdots \tag{85}$$

It is obvious that differential equations (81) and (83) allow the solution

$$\varphi=0,\qquad \epsilon=0,\qquad t\in[t_o,\infty) \tag{86}$$

finding suitable the (cf. (80)) steady motion

$$\varphi=0,\qquad \theta=\frac{c}{2ml^2}t,\quad t\in[t_o,\infty). \tag{87}$$

Maclaurin's series (12) (cf. (81)) is

$$b^2=-\frac{c}{2ml^2}\varphi^4+\frac{c}{32ml^2}\varphi^6+\cdots \tag{88}$$

while the first nontrivial form in (85) is of an odd degree. Therefrom, in accordance with Theorem 1, the equilibrium position (86) is unstable. The same conclusion applies to steady motion (87).

Note 3. Because of the presence of dissipative forces the previous problem cannot be solved by applying (see: [10,11]) Hagedorn's variational approach to the stability of motion.

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