The System Order Reduction via Balancing in View of the Method of Singular Perturbation

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This paper presents several techniques for system order reduction, known from literature, all of them based on system balancing by employing the method of singular perturbation. These techniques have the same robustness accuracy evaluated with respect to the H_∞ norm of the reduced-order system like two techniques known as the direct truncation and the balancing residualization method. A modification of these techniques preserves the exact DC gain as the original system, and produces from very good to excellent accuracy at low and medium frequencies. To illustrate the efficiency of the order-reduction techniques here presented, a real simulation example is given.

Keywords: *system balancing, robust order reduction, residualization, singular perturbation, H_∞ method.*

1. INTRODUCTION

In the eighties a technique of robust order reduction was developed for linear, time invariant systems, based on the use of balancing transformation [1-3].

A linear, time invariant system is considered, represented by a state space model:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A\,x(t) + B\,u(t), \qquad x(t_0) = x_0$$
$$y(t) = C\,x(t) + D\,u(t) \tag{1}$$

where $x(t)$ is n – dimensional state vector, $u(t)$ is m – dimensional input vector, and $y(t)$ is p – dimensional output vector. A corresponding transfer function for the open loop system is given by:

$$G(s) = C\,(sI - A)^{-1} B + D\,. \tag{2}$$

It is assumed that the system (1) is asymptotically stable and that $G(s)$ is a minimal realization.

Assumption 1: *A system is asymptotically stable, a pair (A, B) is controllable and a pair (A, C) is observable*.

The controllability and observability gramians of the system (1) satisfy algebraic equations of the Lyapunov as in [2,3]:

$$PA^T + AP + BB^T = 0\,, \tag{3}$$

$$QA + A^T Q + C^T C = 0\,. \tag{4}$$

For a system that is controllable and observable, both gramians, the one of controllability and the observability gramian, are positive definite matrices, i.e. $P > 0$, $Q > 0$.

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2. SYSTEM ORDER REDUCTION THROUGH BALANCING TRANSFORMATION

The balancing transformation is such transformation of the state space vector that makes both the controllability and the observability gramians become identical and diagonal i.e.:

$$x_{\mathrm{b}}(t) = T\,x(t)\,,$$
$$\det(T) \neq 0 \Rightarrow \frac{\mathrm{d}x_{\mathrm{b}}(t)}{\mathrm{d}t} = A_{\mathrm{b}}\,x_{\mathrm{b}}(t) + B_{\mathrm{b}}\,u(t)\,,$$
$$y_{\mathrm{b}}(t) = C_{\mathrm{b}}\,x_{\mathrm{b}}(t) + D_{\mathrm{b}}\,u(t) = y(t)\,, \tag{5}$$

$$A_{\mathrm{b}} = TAT^{-1}, \quad B_{\mathrm{b}} = TB, \quad C_{\mathrm{b}} = CT^{-1}, \quad D_{\mathrm{b}} = D \tag{6}$$

$$P_{\mathrm{b}} = Q_{\mathrm{b}} = \Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}\,,$$
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0\,, \tag{7}$$

where σ_i are Hankel singular values.

Assuming that the original system is controllable and observable, a balanced system will be as well, both controllable and observable, since the balancing transformation preserves controllability and observability of the system [2,3]. Hence, all σ_i are positive. Furthermore, both original and balanced system are of minimal realization.

The transfer function of the balanced system, given by:

$$G_{\mathrm{b}}(s) = C_{\mathrm{b}}(sI - A_{\mathrm{b}})^{-1} B_{\mathrm{b}} + D = G(s)\,, \tag{8}$$

stays unchanged thanks to a coordinate change through a nonsingular transformation. The balanced gramians of controllability and observability are satisfying these algebraic Lyapunov equations:

$$\Sigma A_{\mathrm{b}}^T + A_{\mathrm{b}}\Sigma + B_{\mathrm{b}}B_{\mathrm{b}}^T = 0 \Leftrightarrow$$
$$\Leftrightarrow P_{\mathrm{b}}A_{\mathrm{b}}^T + A_{\mathrm{b}}P_{\mathrm{b}} + B_{\mathrm{b}}B_{\mathrm{b}}^T = 0 \tag{9}$$

$$\Sigma A_{\mathrm{b}} + A_{\mathrm{b}}^T\Sigma + C_{\mathrm{b}}^T C_{\mathrm{b}} = 0 \Leftrightarrow$$
$$\Leftrightarrow Q_{\mathrm{b}}A_{\mathrm{b}} + A_{\mathrm{b}}^T Q_{\mathrm{b}} + C_{\mathrm{b}}^T C_{\mathrm{b}} = 0\,. \tag{10}$$

The idea for the order reduction through balancing transformation can be linked with the original paper of Kalman in 1963 on the canonical system decomposition [4]. It was shown there that the system modes that are either uncontrollable or unobservable do not appear in the system transfer function. Therefore, in [2,3] authors deduce that the system modes both weakly controllable and weakly observable have little influence on the system dynamics, so they can be neglected. However, it was noticed that those modes which are weakly controllable and well observable can not be neglected, as well as those which are well controllable and weakly observable modes – they may become particularly important for the closed-loop system performance. Let us assume that the balanced system (5) – (7) is partitioned in such a way that:

$$A_\mathrm{b} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_\mathrm{b} = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix},$$
$$C_\mathrm{b} = \begin{bmatrix} C_{11} & C_{22} \end{bmatrix}, \quad D_\mathrm{b} = D,$$
$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Sigma_1 = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\},$$
$$\Sigma_2 = \mathrm{diag}\{\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n\}, \qquad (11)$$

where A_{11} and A_{22} are matrices of the dimension $r \times r$ and $(n - r) \times (n - r)$, respectively, and other matrices have dimensions that correspond to the system dimensions defined as in (1).

Assuming that $\sigma_r > \sigma_{r+1}$, a balanced truncation produces a system of lower order r, defined by:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = A_{11}\, x_1(t) + B_{11}\, u(t),$$
$$y(t) = C_{11}\, x_1(t) + D\,u(t). \qquad (12)$$

The corresponding transfer function of the reduced order system is:

$$G_{11}(s) = C_{11}(sI - A_{11})^{-1} B_{11} + D. \qquad (13)$$

Reduced order system achieved this way is both controllable and observable since all corresponding Hankel singular values are positive. Furthermore, the reduced order system is balanced and asymptotically stable. It was shown in literature [1] that the H_∞ norm for the reduced order system, obtained through truncation procedure given above, satisfies the condition:

$$\left\| G(s) - G_{11}(s) \right\|_\infty \le 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n). \qquad (14)$$

It was noticed that the reduced order system obtained through balanced truncation procedure gives very good approximation of the original system in the case of the impulse input (good spectra approximation on higher frequencies) but shows considerable steady state error in the case of step input (poor spectra approximation on lower frequencies) [2,3,5]. This error is due to a fact that the original system and the reduced order system have different DC gains. Actually, after the above described truncation through balancing transformation, most of the spectra on lower frequencies are kept and some of the spectra on the higher frequencies also but some of the spectra on lower frequencies are lost as well as most of the spectra on the higher frequencies. By eliminating the part of the spectra on the lower frequencies (the neglected part of the system – state variables $x_2(t)$), we produce gain that differs from the gain of the original system that was balanced. This discrepancy was removed in [2,3] where a technique of balanced residualization was proposed that produces an accurate (exact) DC gain and very good spectra approximation on the lower frequencies and middle frequencies. It should be noticed that in [1] was also used a residualization technique. Improved truncation method that preserves exact DC gain value as in the original system is given by (37).

3. SYSTEM ORDER REDUCTION VIA BALANCED RESIDUALIZATION

Let us describe in short the essence of the system order reduction technique based on the balanced residualization as in [1].

A balanced linear system given by (15) is under consideration:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_{11} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T + D\,u(t), \qquad (15)$$

where A_{11} and A_{22} are matrices of the dimension $r \times r$ and $(n - r) \times (n - r)$, respectively, and other matrices have dimensions that correspond to the system dimensions defined as in (1).

Let us assume that corresponding Hankel singular values are satisfying $\sigma_r > \sigma_{r+1}$. Assuming that the state space variable x_2 has reached its quasi-steady state value (it should be noticed that x_2 is asymptotically stable fast space variable) we can place zero instead of corresponding time derivative, which leads to the next approximation of the reduced order system:

$$\begin{bmatrix} \dot{x}_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_{11} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T + D\,u(t). \qquad (16)$$

A matrix A_{22} is asymptotically stable – it has been shown in [1] that this matrix has all eigenvalues in the closed left half of the complex plane which makes it invertible matrix. Hence, from the second line of the (16) we can find $x_2(t)$ expressed as a function of $x_1(t)$ and $u(t)$ as in:

$$x_2(t) = -A_{22}^{-1}\left(A_{21}\, x_1(t) + B_{22}\, u(t)\right), \qquad (17)$$

which leads to the form of the residualized system of the lower order:

$$\dot{x}_1(t) = A_\mathrm{r}\, x_1(t) + B_\mathrm{r}\, u(t),$$
$$y(t) = C_\mathrm{r}\, x_1(t) + D_\mathrm{r}\, u(t). \qquad (18)$$

$$A_\mathrm{r} = A_{11} - A_{12} A_{22}^{-1} A_{21},$$
$$B_\mathrm{r} = B_{11} - A_{12} A_{22}^{-1} B_{22},$$
$$C_\mathrm{r} = C_{11} - C_{22} A_{22}^{-1} A_{21},$$
$$D_\mathrm{r} = D_{11} - C_{22} A_{22}^{-1} B_{22}. \qquad (19)$$

According to the theory of singular perturbation [6], this system represents zero approximation of the original system defined by (15). It is mentioned in [2,3] that the transfer function of the residualized system:

$$G_{\mathrm{r}}(s) = C_{\mathrm{r}}(sI - A_{\mathrm{r}})^{-1} B_{\mathrm{r}} + D_{\mathrm{r}}, \qquad (20)$$

apart from having the same property as the one for the reduced order system achieved through truncation, that is:

$$\|G(s) - G_{\mathrm{r}}(s)\|_{\infty} \le 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n), \quad (21)$$

also keeps true value of the DC gain as in the case of the original system. It can be noticed from (13) that the DC gain of the reduced order system is not the same as the DC gain of the original system, i.e.:

$$G_{11}(0) = -C_{11} A_{11}^{-1} B_{11} + D \neq G(0)$$
$$G(0) = -CA^{-1}B + D = G_{\mathrm{b}}(0),$$
$$G_{\mathrm{b}}(0) = -\begin{bmatrix} C_{11} & C_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} + D. \quad (22)$$

On the other hand, it was shown in control literature, through several matrix algebraic operations, that the DC gain of the residualized system is identical to the DC gain of the original system, i.e.:

$$G_{\mathrm{r}}(0) = -C_{\mathrm{r}} A_{\mathrm{r}}^{-\mathbf{1}} B_{\mathrm{r}} + D_{\mathrm{r}} = -\left(C_{11} - C_{22} A_{22}^{-1} A_{21}\right) \times$$
$$\times \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} \left(B_{11} - A_{12} A_{22}^{-1} B_{22}\right) +$$
$$+ \left(D - C_{22} A_{22}^{-1} B_{22}\right) =$$
$$= G(0) = -\begin{bmatrix} C_{11} & C_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} + D. \quad (23)$$

It was noticed that the residualized system of the reduced order gives well the approximation of the system spectrum on lower and middle frequencies. Hence, step responses of the residualized systems of reduced order are good approximation of the corresponding step responses of the original system. It is interesting that the reduced order system obtained through truncation have better spectra approximation on higher frequencies than the reduced order system obtained using balanced residualization.

In the next part a further generalization of the results displayed above will be shown, as it was given in [2,3], that will lead to alternative techniques development. Techniques obtained will be based on the transformation known from the theory of singular perturbation by a name *the Chang transformation*.

4. GENERALIZED BALANCED RESIDUALIZATION IN ORDER REDUCTION

The order reduction technique explained in previous section actually represents the zero order approximation obtained by using the theory of singular perturbation [6]. It should be noticed that the theory of singular perturbation was used for the order reduction through balancing in a wide variety of papers – see the reference list in [2,3].

A singularly perturbed control linear system has model of the form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \mu \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t), \qquad (24)$$

where μ is small positive parameter of the singular perturbation that exhibits differing of the state space variables in two groups – slowly varying variables $x_1(t)$ and fast varying variables $x_2(t)$.

This system is dual to the system (15), literally – they are identical for $\mu = 1$. Actually, the balanced system too has slow and fast modes: fast modes are those corresponding to small Hankel singular values and slow modes are those corresponding to relatively large Hankel singular values. Hence, it is possible to express the system (15) in a singularly perturbed form, e.g. assuming that $\mu = \sigma_{r+1} / \sigma_r$ or even $\mu = \sigma_{r+1} / \sigma_1$ and then multiplying by μ the second line in (15), the one corresponding to fast state space variables $x_2(t)$. This procedure will scale appropriately the corresponding matrices making their elements of the same size order as elements in matrices corresponding to slow variables $x_1(t)$. Matrices from (15) and (24) are satisfying:

$$A_1 = A_{11},\ A_2 = A_{12},\ A_3 = \mu A_{21},\ A_4 = \mu A_{22},$$
$$B_1 = B_{11},\ B_2 = \mu B_{22},\quad C_1 = C_{11}, C_2 = C_{22}. \quad (25)$$

By using Chang transformation a singularly perturbed control system (24) can be partitioned (decomposed) in two independent subsystems, slow and fast:

$$\begin{bmatrix} \dot{z}_1(t) \\ \mu \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{s}} & 0 \\ 0 & A_{\mathrm{f}} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} B_{\mathrm{s}} \\ B_{\mathrm{f}} \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} C_{\mathrm{s}} & C_{\mathrm{f}} \end{bmatrix} \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T + Du(t), \quad (26)$$

where notation was introduced as in:

$$A_{\mathrm{s}} = A_1 - A_2 L,\quad A_{\mathrm{f}} = A_4 + \mu L A_2,$$
$$B_{\mathrm{s}} = B_1 - MB_2 - \mu MLB_1,\quad B_{\mathrm{f}} = B_2 + \mu LB_1,$$
$$C_{\mathrm{s}} = C_1 - C_2 L,\quad C_{\mathrm{f}} = C_2 - \mu C_2 LM + \mu C_1 M. \ (27)$$

L and M are matrices satisfying algebraic equation:

$$A_4 L - A_3 - \mu L\left(A_1 - A_2 L\right) = 0,$$
$$MA_4 - A_2 + \mu\left[MLA_2 - \left(A_1 - A_2 L\right)M\right] = 0. \quad (28)$$

These equations could be successfully solved for small values of μ, using either the fixed point method or Newton's method. For relatively large values of μ a method of eigenvectors can be used. Numerical methods mentioned above could be found in literature – see papers referenced in [2,3]. It should be noticed that in the case of small μ values an approximation of zero-order for solution of the (28) is given by:

$$L^{(0)} = A_4^{-1} A_3,\quad M^{(0)} = A_2 A_4^{-1}. \qquad (29)$$

Output of the system (26) could be represented as:

$$y(t) = \begin{bmatrix} C_{\mathrm{s}} & C_{\mathrm{f}} \end{bmatrix} \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T + Du(t) =$$
$$= y_{\mathrm{s}}(t) + y_{\mathrm{f}}(t), \qquad (30)$$

where outputs of the slow and fast subsystem are defined respectively as:

$$y_{\mathrm{s}}(t) = C_{\mathrm{s}} z_1(t) + D u(t) ,$$
$$y_{\mathrm{f}}(t) = C_{\mathrm{f}} z_2(t) . \qquad (31)$$

It is not important where the feed-forward loop (direct output control branch) is included, whether it is associated with fast or with slow subsystem, which depends primarily upon the nature of the input signal of the system. Respectively, transfer function for the fast and the slow subsystem obtained from partitioning of the system given above are:

$$G_{\mathrm{s}}(s) = C_{\mathrm{s}}(sI - A_{\mathrm{s}})^{-1} B_{\mathrm{s}} + D ,$$
$$G_{\mathrm{f}}(s) = C_{\mathrm{f}}(sI - \frac{1}{\mu} A_{\mathrm{f}})^{-1} \frac{1}{\mu} B_{\mathrm{f}} =$$
$$= C_{\mathrm{f}}(\mu sI - A_{\mathrm{f}})^{-1} B_{\mathrm{f}} . \qquad (32)$$

In order to obtain exact partitioning (30) of the system in slow and fast subsystem, it is necessary to solve *L-M* equations. For the small values of the μ parameter this could be easily achieved using, for example, the fixed point algorithm. Several algorithms for solving *L-M* equations are suggested in [2,3].

Using term $O(\mu)$, from (28) and (29) we can deduce:

$$L = L^{(0)} + O(\mu) = A_4^{-1} A_3 + O(\mu) ,$$
$$M = M^{(0)} + O(\mu) = A_2 A_4^{-1} + O(\mu) . \qquad (33)$$

These observations are suggesting that matrices obtained through exact system partitioning to slow and fast subsystem can be computed from matrices used in residualization approximation via following relations:

$$A_{\mathrm{s}} = A_1 - A_2 A_4^{-1} A_3 + O(\mu) = A_{\mathrm{r}} + O(\mu) ,$$
$$B_{\mathrm{s}} = B_1 - A_2 A_4^{-1} B_2 + O(\mu) = B_{\mathrm{r}} + O(\mu) ,$$
$$C_{\mathrm{s}} = C_1 - C_2 L = C_1 - C_2 A_4^{-1} A_3 + O(\mu) ,$$
$$A_{\mathrm{f}} = A_4 + \mu L A_2 = A_4 + O(\mu) ,$$
$$B_{\mathrm{f}} = B_2 + \mu L B_1 = B_2 + O(\mu) ,$$
$$C_{\mathrm{f}} = C_2 - \mu C_2 L M + \mu C_1 M = C_2 + O(\mu) . \qquad (34)$$

Hence, the results known from the literature could also be obtained by perturbing matrices from (33) and (34) by adding the term $O(\mu)$ and placing $\mu = 0$ in the transfer function of the fast subsystem that was defined in (32), which can be fair enough approximation on lower and middle frequencies (under assumption that μ is sufficiently small).

A generalization of the residualization approximation can be obtained using slow subsystem with approximation of the fast one with its DC gain which equals $-C_{\mathrm{f}} A_{\mathrm{f}}^{-1} B_{\mathrm{f}}$. Transfer function approximated with reduced order model is:

$$G(s) \approx C_{\mathrm{s}}(sI - A_{\mathrm{s}})^{-1} B_{\mathrm{s}} + D - C_{\mathrm{f}} A_{\mathrm{f}}^{-1} B_{\mathrm{f}} . \qquad (35)$$

The DC gain in generalized residualization satisfies following lemma given and proven in [2,3]. Here the proof will be omitted.

Lemma: The procedure of generalized residualization preserves original value of the DC gain, i.e.:

$$-CA^{-1}B + D = -C_{\mathrm{s}} A_{\mathrm{s}}^{-1} B_{\mathrm{s}} + D - C_{\mathrm{f}} A_{\mathrm{f}}^{-1} B_{\mathrm{f}} . \qquad (36)$$

Finally, in [1,2] it was emphasized that the DC gain for the model approximation using technique of balanced order reduction based on truncation could be improved if the transfer function of the truncated i.e. selected slow subsystem is chosen in the modified form as in:

$$G_{\mathrm{s,trunc}}^{\mathrm{corr}}(s) = C_{11}(sI - A_{11})^{-1} B_{11} +$$
$$+ C_{11} A_{11}^{-1} B_{11} - CA^{-1} B , \qquad (37)$$

which itself represents a correction of the truncation method.

Simulations carried out for several examples have shown that such approximation can become closer to the approximation using balanced residualization.

Table 1. System matrices for the model of the power system consisting of two machines

$A = \begin{bmatrix} 0 & -314.1590 & 314.1590 & 0 & 0 & 0 & 0 \\ 0.0030 & -0.1310 & -0.0120 & -0.1410 & -0.0060 & 0 & 0 \\ -0.2710 & -0.3520 & -2.7630 & -0.1820 & -0.3710 & 0 & 0 \\ 0.0050 & -0.2900 & -0.0080 & -0.3730 & -0.0050 & 314.1590 & 0 \\ -0.2900 & -0.1270 & -0.7240 & 0.0250 & -1.2610 & 0 & 314.1500 \\ 0 & 0 & 0 & 0 & 0 & -33.3300 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -33.3300 \end{bmatrix}$
$B^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.0620 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2010 \end{bmatrix}$
$C = \begin{bmatrix} 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0.0010 & 0.5070 & 0.0270 & 0.4720 & 0.0350 & 0 & 0 \\ -0.1250 & 0.2250 & 0.1980 & 0.2580 & 0.1350 & 0 & 0 \end{bmatrix}$
$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. EXAMPLE

Methods displayed here for the order reduction were tested on several examples known from literature and taken from [7,8]. A multivariable model of two machine power system was simulated using Matlab. The example contains the model of the system having two inputs and three outputs. The value for the reduced order was determined from Hankel singular values. Several reduced order models were produced, using methods described above. The efficiency of these approximations was compared for typical input functions: impulse, step, ramp and sine. In the open loop case a comparison with the original model was made, for all frequency characteristics (both magnitude and phase spectra) of all available transfer functions.

A mathematical model of a part of Serbian power system working in isolated regime is chosen for simulation and taken from [7]. It is the two machine power system having the system matrices given in the Table 1.

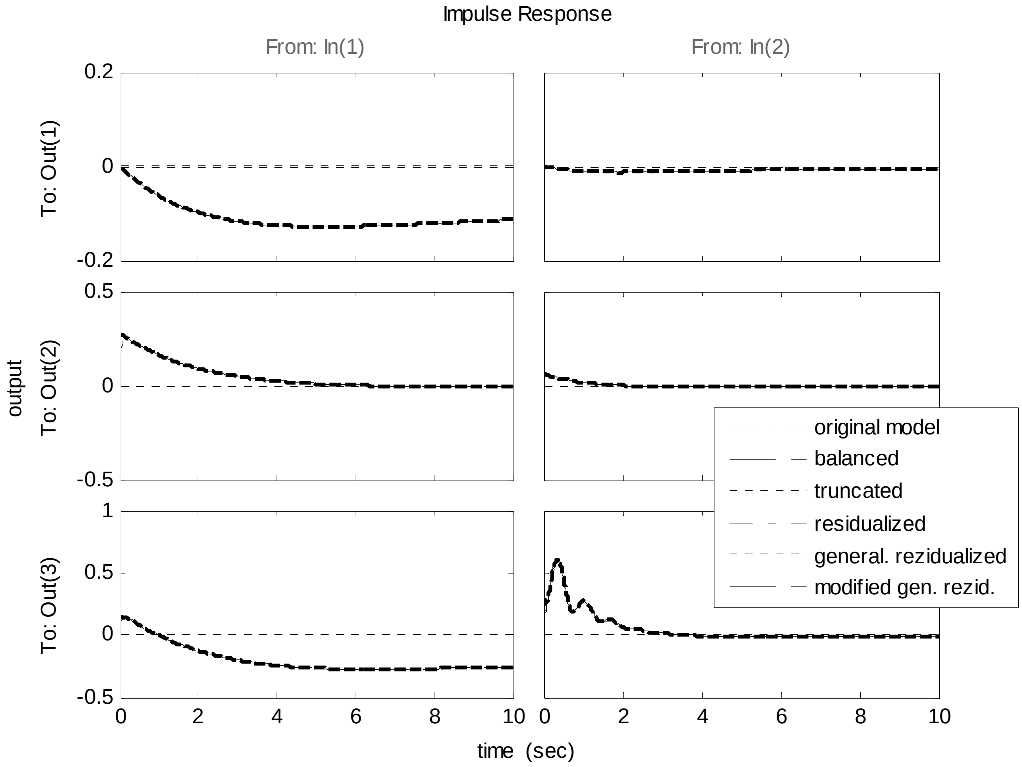

Figure 1. Impulse responses: original and balanced systems are compared with four different reduced order approximations

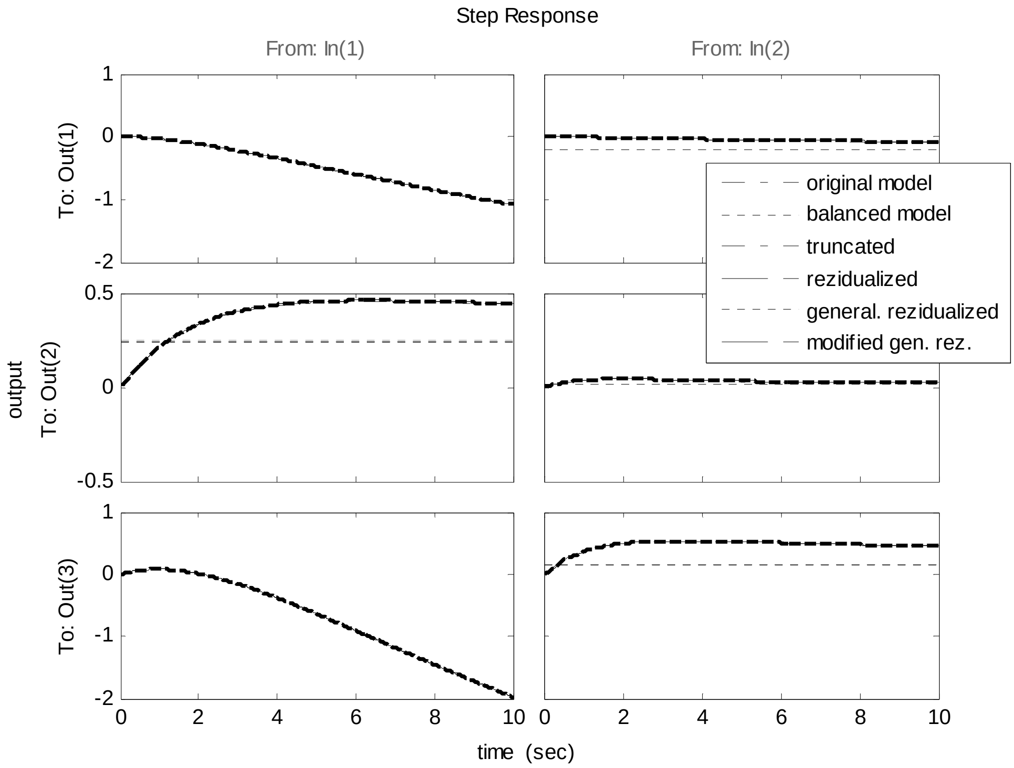

Figure 2. Step responses: original and balanced systems are compared with four different reduced order approximations

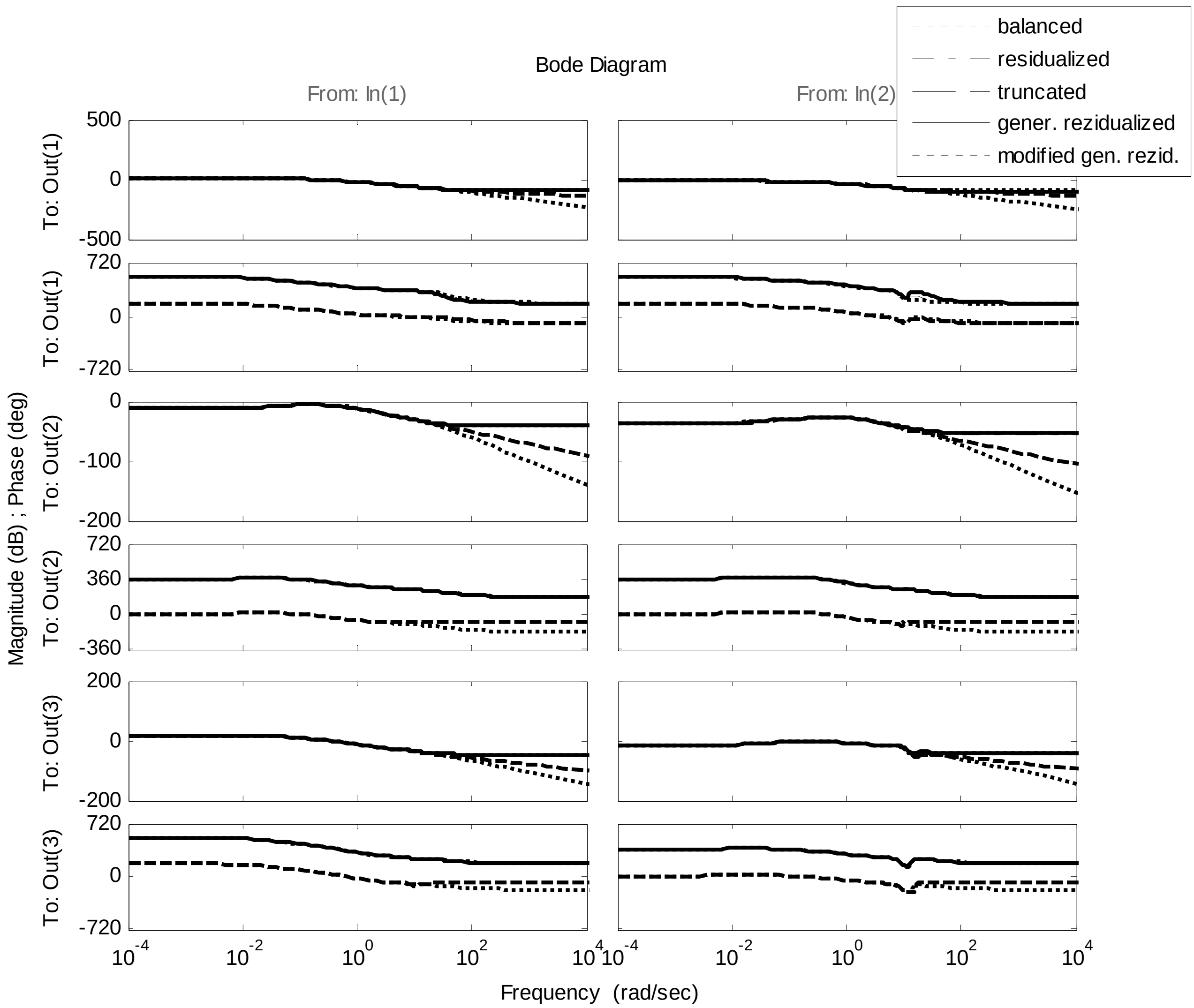

Figure 3. Magnitude and phase spectra: original and balanced systems are compared with four different reduced order approximations

From matrix A dimensions (Table 1) it is clear that the system has order n = 7, and from Hankel singular values it was determined that the reduced order could be r = 5. System has two inputs and three outputs.

Each of the model approximations had reduced order r = 5, and each of them was tested, as well as the original and balanced full order model, for typical input functions: impulse, step, ramp and sine. In time domain all of the approximations produce similar and good performance.

From Figure 1 (at the end of the text) it could be seen that impulse responses for the original system, balanced model of order n and four reduced order models of order r are quite close. All six models exhibit similar performances in the open loop.

From Figure 2 it is obvious that step responses for the original system, balanced model of order n and four reduced order models of order r are similar. All six models exhibit similar performances in the open loop.

Frequency responses were compared for all available six transfer functions: from input 1 to all of three outputs and from input 2 to all of three outputs. For all six transfer functions, a magnitude and phase spectra were shown for all reduced order approximations, and for the original and balanced model as well. A μ value was 0.1173.

Figure 3 shows frequency responses – both magnitude and phase spectra, of the original system, the balanced system of order n and four reduced order systems, all off order r = 5, obtained in different manners described above.

The results show rather well behaviour of all approximations. Substantial accuracy can be achieved, better on lower frequency and worse on higher frequencies. It was chosen that the upper frequency bound for the open loop modelling is 34 rad/s.

6. CONCLUSION

It could be concluded that the generalized residualization method as well as its versions is very appropriate for the use on lower and middle frequencies. In several papers, e.g. [2,3,9,10] was noticed that in the closed loop controller design it is very important to take into account the high frequency dynamics. Several ways to overcome this problem are suggested in [2,3]. In the process of the order reduction it stays unclear where the boundary lies between linear systems with oscillatory modes and linear systems with highly oscillatory modes.

Hence, it should be tried out both order reduction based on slow subsystem and order reduction based on fast subsystem, and depending on the results achieved for the system taken a decision should be made which technique for the order reduction gives better approximation. However, engineering experience will be an advantage in the method application.

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РЕДУКЦИЈА РЕДА СИСТЕМА КРОЗ УРАВНОТЕЖЕЊЕ СА ГЛЕДИШТА МЕТОДЕ СИНГУЛАРНИХ ПЕРТУРБАЦИЈА

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У раду је представљено неколико техника за редукцију реда система, познатих из литературе, које су све засноване на уравнотежењу система уз примену методе сингуларних пертурбација. Ове технике имају исту робусност тачности израчунату у складу са H_{∞} нормом система редукованог реда као и две технике познате под називом директно одсецање и метод балансиране резидуализације. Модификација ових техника задржава тачну вредност појачања једносмерног сигнала каква је код оригиналног система и даје апроксимацију од веома добре до одличне тачности на нижим и средњим учестаностима. Ефикасност приказаних техника за редукцију реда модела дата је на реалном примеру.