Multi-Body Kinematics and Dynamics in Terms of Quaternions: Langrange Formulation in Covariant Form – Rodriguez Approach

Nemanja D. Zoric
Research Assistant
University of Belgrade
Faculty of Mechanical Engineering

Mihailo P. Lazarevic
Full Professor
University of Belgrade
Faculty of Mechanical Engineering

Aleksandar M. Simonovic
Assistant Professor
University of Belgrade
Faculty of Mechanical Engineering

This paper suggests a quaternion approach for the modelling kinematics and dynamics of rigid multi-body systems. Instead of the regular "Newton-Euler" and Lagrange method used in the traditional way, Lagrange's equations of second kind in the covariant form are used by applying Rodriguez approach and quaternion algebra. A model of multi-body system of n rigid bodies in terms of quaternions is obtained, which is useful for studying kinematics, dynamics as well as for research of control system designs.

Keywords: *quaternions, rotation, rigid bodies system.*

1. INTRODUCTION

In general, modelling of kinematics and dynamics of rigid bodies systems has been mostly based on the Euler angles representation of rotation. It is well known that three angles can't afford a regular representation of the rotation, since there are singularities. Euler proposed a solution to circumvent this problem by introducing a set of four quantities, the so-called Euler parameters, based on relations among Euler angles. Later on, Hamilton (1844) invented the quaternions, an extension of complex numbers, and soon afterwards, it was discovered that rotations may be represented by quaternions [1].

In [2] and [3], Lagrange's equations of second kind of rigid bodies system in covariant form were developed using Rodriguez matrix for the representation of orientation of rigid body with respect to the inertial frame. Our goal is to develop the same form of equation, but with the help of quaternions.

Further research will be based on control system designs, because quaternions enable singularity-free mathematical representation of orientations [4,5].

2. MATHEMATICAL BACKGROUND OF THE QUATERNIONS

2.1 Definition

Quaternions are hyper-complex numbers of rank 4 consisting of one real and three imaginary parts. The quaternions were first described by Irish mathematician Sir William Rowan Hamilton in 1844 and applied to mechanics in three-dimensional space. Crucial to this description was his celebrated rule:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} \otimes \mathbf{j} \otimes \mathbf{k} = -1, \tag{1}$$

where $\otimes$ denotes quaternion or the Hamilton product. The quaternion is defined as:

Received: December 2009, Accepted: January 2010
Correspondence to: Nemanja Zoric, M.Sc.
Faculty of Mechanical Engineering,
Kraljice Marije 16, 11120 Belgrade 35, Serbia
E-mail: nzoric@mas.bg.ac.rs

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \tag{2}$$

where q_0 represents real part, and q_1, q_2 and q_3 represent imaginary parts of the quaternion. Pure part of the quaternion (2) is defined as:

$$\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}. \tag{3}$$

2.2 Algebraic properties

Let p and q be two quaternions. The sum of p and q can be written as:

$$p+q = (p_0+q_0)+(p_1+q_1)\mathbf{i}+(p_2+q_2)\mathbf{j}+(p_3+q_3)\mathbf{k}. \tag{4}$$

If c is scalar, then the product of quaternion q and scalar c is given by:

$$cq = cq_0 + cq_1\mathbf{i} + cq_2\mathbf{j} + cq_3\mathbf{k}. \tag{5}$$

For two quaternions p and q, their Hamilton or quaternion product is determined by the product of the basis elements and the distributive law. This gives the following expression [6]:

$$p \otimes q = p_0 q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}. \tag{6}$$

From (6) it can be seen that quaternions form non-commutative under multiplication. Let q be the quaternion. The complex conjugate of quaternion q is defined as:

$$q^* = q_0 - \mathbf{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}. \tag{7}$$

From (6) and (7) it can be concluded the following:

$$(p \otimes q)^* = q^* \otimes p^*. \tag{8}$$

The norm of quaternion q is defined as:

$$N(q) = \sqrt{q^* \otimes q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = |q|, \tag{9}$$

and inverse of quaternion q is:

$$q^{-1} = \frac{q^*}{N^2(q)} = \frac{q^*}{|q|^2}. \tag{10}$$

2.3 Quaternionic representation of rotation of the rigid body

The rigid body (V) rotates about axis 0τ which is represented by unit vector $\mathbf{e}$ (Fig. 1). Reference frame $0xyz$ is inertial, and reference frame $0\xi\eta\zeta$ is body-fixed frame. Unit vectors of axis x, y and z are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and unit vectors of axis ξ, η and ζ are denoted by $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\mathbf{v}$.

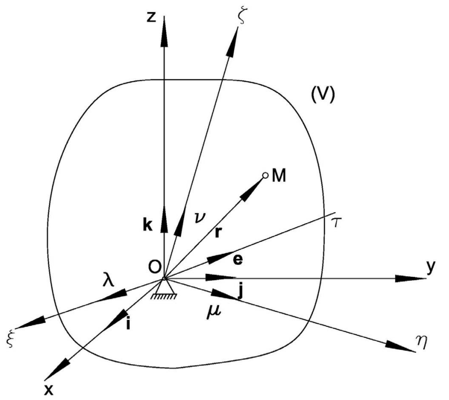

Figure 1. Rotation of the rigid body

In initial time these two reference frames were equivalent. Vector $\mathbf{OM} = \mathbf{r}$ (point M belongs to the body (V)) can be expressed in both reference frames:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\ , \tag{11}$$

and

$$\mathbf{r}^{(1)} = \xi\boldsymbol{\lambda} + \eta\boldsymbol{\mu} + \zeta\mathbf{v}\ . \tag{12}$$

Superscript in (12) denotes body-fixed reference frame in which vector $\mathbf{r}$ is expressed. Vectors (11) and (12) belong to the set of vectors. In order to operate with the quaternion, a vector, which lives in R^3, needs to be treated as a pure quaternion (that is a quaternion which real part is zero) which lives in R^4. The set of all pure quaternions (denoted by Q_0) is the subset of Q, the set of all quaternions. It can be defined as one-to-one correspondence between the set of vectors and the set of pure quaternions, a correspondence in which a vector $\mathbf{r} \in R^3$ corresponds to pure quaternion $r = 0 + \mathbf{r} \in Q_0$, that is:

$$\mathbf{r} \in R^3 \to r = 0 + \mathbf{r} \in Q_0 \subset Q\ . \tag{13}$$

The relation between the vector expressed in inertial frame (11) and the vector expressed in body-fixed frame (12) is given by:

$$\mathbf{r} = q \otimes \mathbf{r}^{(1)} \otimes q^*\ , \tag{14}$$

where q is unit quaternion (a quaternion with norm one) which has the following structure:

$$q = \cos\frac{\theta}{2} + \mathbf{e}\sin\frac{\theta}{2}\ , \tag{15}$$

where θ represents the rotation angle about axis 0τ, and $\mathbf{e}$ is pure quaternion which corresponds to the unit vector axis 0τ. The result of (14) is also pure quaternion which corresponds to the position vector of point M. It can be illustrated in the following figure (Fig. 2):

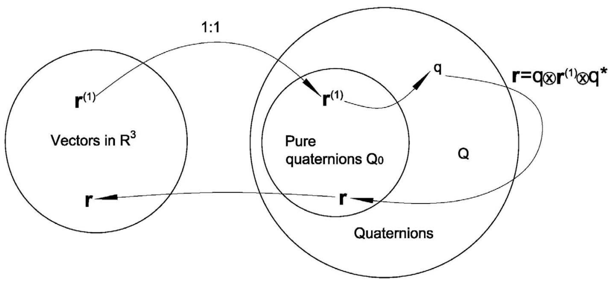

Figure 2. Quaternionic representation of rotation of the rigid body

Vector $\mathbf{e}$ is invariant, and because of that property it is all the same in which the coordinate frame this vector would be expressed.

The next case is when the rigid body rotates about the moving axis. The rigid body (V) rotates about moving axis $0\tau_2$ represented by unit vector $\mathbf{e}_2$. At the same time, this axis rotates about the axis $0\tau_1$ represented by unit vector $\mathbf{e}_1$. The reference frame $0xyz$ is inertial, $0\xi_2\eta_2\zeta_2$ is body-fixed and $0\xi_1\eta_1\zeta_1$ is fixed on the rotation axis $0\tau_2$ (Fig. 3). At initial time these three frames were equivalent.

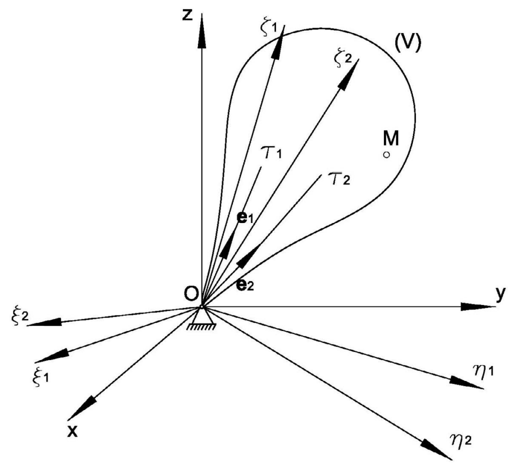

Figure 3. Rotation of the rigid body about moving axis

Vector $\mathbf{OM} = \mathbf{r}$ (point M belongs to the body (V)) expressed in reference frame $0\xi_1\eta_1\zeta_1$ is:

$$\mathbf{r}^{(1)} = q_2 \otimes \mathbf{r}^{(2)} \otimes q_2^*, \tag{16}$$

where

$$q_2 = \cos\frac{\theta_2}{2} + \mathbf{e}_2^{(2)}\sin\frac{\theta_2}{2}\ . \tag{17}$$

It must be mentioned that vector $\mathbf{e}_2$ is invariant in relation to reference frames $0\xi_1\eta_1\zeta_1$ and $0\xi_2\eta_2\zeta_2$, and vector $\mathbf{e}_1$ is invariant in relation to reference frames $0xyz$ and $0\xi_1\eta_1\zeta_1$. Vector $\mathbf{OM} = \mathbf{r}$ expressed in inertial reference frame is:

$$\mathbf{r} = q_1 \otimes \mathbf{r}^{(1)} \otimes q_1^*, \tag{18}$$

where

$$q_1 = \cos\frac{\theta_1}{2} + \mathbf{e}_1\sin\frac{\theta_1}{2}\ . \tag{19}$$

Substituting (16) in (18), one can get following expression:

$$\mathbf{r} = q_1 \otimes q_2 \otimes \mathbf{r}^{(2)} \otimes q_2^* \otimes q_1^* =$$
$$= (q_1 \otimes q_2) \otimes \mathbf{r}^{(2)} \otimes (q_1 \otimes q_2)^* = q \otimes \mathbf{r}^{(2)} \otimes q^*, \quad (20)$$

where

$$q = q_1 \otimes q_2 \quad (21)$$

represents composite quaternion.

3. KINEMATICS OF OPEN CHAIN SYSTEM

3.1 Transformation of coordinates

The open chain system of rigid bodies (V_1), (V_2), ..., (V_n) is shown in Figure 4. The rigid body (V_1) is connected to the fixed stand. Two neighboring bodies, (V_{i-1}) and (V_i) of chain are connected together with joint (i), which allows translation along the axis which is represented by unit vector $\mathbf{e}_i$, or rotation about the same axis body (V_i) in respect to body (V_{i-1}). The values q^i represent generalized coordinates.

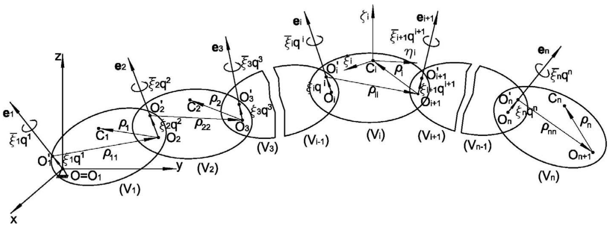

Figure 4. Open chain of the rigid bodies system

The reference frame $0xyz$ is inertial Cartesian frame, and the reference frame $0\xi_i\eta_i\zeta_i$ is local body-frame which is associated to the body (V_i) at the point C_i which represents the centre of inertia of body (V_i). At initial time, corresponding axis of reference frames were parallel. This configuration is called reference configuration and it is denoted by (0). The symbols ξ_i and $\overline{\xi}$ can be introduced, which are defined as:

$$\xi_i = 1, \ \overline{\xi}_i = 0 \quad (22)$$

in the case when bodies (V_{i-1}) and (V_i) are connected with prismatic joint, and

$$\xi_i = 0, \ \overline{\xi}_i = 1 \quad (23)$$

in the case when bodies (V_{i-1}) and (V_i) are connected with cylindrical joint. Arbitrary vector $\boldsymbol{\tau}_j$, associated with the body (V_j) is given (Fig. 5). In reference configuration, this vector is identical in both reference frames

$$\boldsymbol{\tau}_{j(0)}^{(j-1)} = \boldsymbol{\tau}_{j(0)}^{(j)}. \quad (24)$$

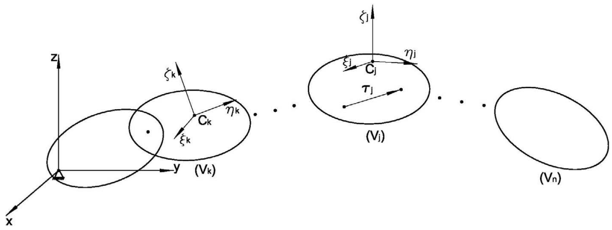

Figure 5. Vector τ_j on rigid body (V_j)

In the case when bodies (V_{j-1}) and (V_j) are connected by cylindrical joint, after rotation of the body (V_j) about the axis $\mathbf{e}_j$ for angle q^j, vector $\boldsymbol{\tau}_j$ in body-fixed reference frame $0\xi_{j-1}\eta_{j-1}\zeta_{j-1}$ has the following value:

$$\boldsymbol{\tau}_j^{(j-1)} = p_j \otimes \boldsymbol{\tau}_j^{(j)} \otimes p_j^*, \quad (25)$$

where p_j represents unit quaternion which is defined as

$$p_j = \cos\frac{q^j}{2} + \mathbf{e}_j^{(j)} \sin\frac{q^j}{2}. \quad (26)$$

If rigid body (V_{j-2}) is connected with (V_{j-1}) by cylindrical joint, then:

$$\boldsymbol{\tau}_j^{(j-2)} = p_{j-2,j} \otimes \boldsymbol{\tau}_j^{(j)} \otimes p_{2-1,j}^*, \quad (27)$$

where

$$p_{j-2,j} = p_{j-1} \otimes p_j, \quad (28)$$

and:

$$p_{j-1} = \cos\frac{q^{j-1}}{2} + \mathbf{e}_{j-1}^{(j-1)} \sin\frac{q^{j-1}}{2}. \quad (29)$$

In the case of prismatic joint, vector $\boldsymbol{\tau}_j$ is the same in both local body-fixed frames:

$$\boldsymbol{\tau}_j^{(j-1)} = \boldsymbol{\tau}_j^{(j)}. \quad (30)$$

In the general case, with the help of symbol $\overline{\xi}$, the quaternion (26) has the following form:

$$p_j = \cos\frac{\overline{\xi}_j q^j}{2} + \mathbf{e}_j^{(j)} \sin\frac{\overline{\xi}_j q^j}{2}. \quad (31)$$

Vector $\boldsymbol{\tau}_j$ in inertial reference frame $0xyz$ has the following value:

$$\boldsymbol{\tau}_j = p_{0,j} \otimes \boldsymbol{\tau}_j^{(j)} p_{0,j}^*, \quad (32)$$

where

$$p_{0,j} = p_1 \otimes p_2 \otimes ... \otimes p_j, \quad (33)$$

and

$$p_{j,j} = 1. \quad (34)$$

3.2 Velocity of inertia centre of the rigid body (V_i)

The position vector of inertia centre C_i of the rigid body (V_i) has the following value in inertial reference frame (Fig. 4):

$$\mathbf{OC}_i = \mathbf{r}_i = \sum_{k=1}^{i} \left(\boldsymbol{\rho}_{kk} + \xi_k \mathbf{e}_k q^k \right) + \boldsymbol{\rho}_i, \quad (35)$$

where

$$\boldsymbol{\rho}_{kk} = p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes p_{0,k}^*, \quad (36)$$

$$\mathbf{e}_k = p_{0,k-1} \otimes \mathbf{e}_k^{(k)} \otimes p_{0,k-1}^*, \quad (37)$$

$$\boldsymbol{\rho}_i = p_{0,i} \otimes \boldsymbol{\rho}_i^{(i)} \otimes p_{0,i}^*. \quad (38)$$

Velocity of the centre of inertia is:

$$\mathbf{V}_i = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_{\alpha=1}^{i} \frac{\partial \mathbf{r}_i}{\partial q^\alpha} \dot{q}^\alpha = \sum_{\alpha=1}^{i} \mathbf{T}_{\alpha(i)} \dot{q}^\alpha , \tag{39}$$

where

$$\mathbf{T}_{\alpha(i)} = \frac{\partial \mathbf{r}_i}{\partial q^\alpha} , \tag{40}$$

which is called the quasi-basic vector. Generalized coordinate partial derivatives of the quaternion are:

$$\frac{\partial p_k}{\partial q^\alpha} = 0 , \ k \neq \alpha , \tag{41}$$

and

$$\frac{\partial p_\alpha}{\partial q^\alpha} = -\frac{1}{2}\bar{\xi}_\alpha \sin\frac{\bar{\xi}_\alpha q^\alpha}{2} + \frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha^{(\alpha)} \cos\frac{\bar{\xi}_\alpha q^\alpha}{2} =$$

$$= \frac{1}{2}\bar{\xi}_\alpha \left(-\sin\frac{\bar{\xi}_\alpha q^\alpha}{2} + \mathbf{e}_\alpha^{(\alpha)} \cos\frac{\bar{\xi}_\alpha q^\alpha}{2} \right) . \tag{42}$$

Since

$$-\mathbf{e}_\alpha^{(\alpha)} \otimes \mathbf{e}_\alpha^{(\alpha)} = 1 \tag{43}$$

then, multiplication of (42) with (43) on the left side, (43) becomes:

$$\frac{\partial p_\alpha}{\partial q^\alpha} = \frac{1}{2}\bar{\xi}_\alpha \left(-\mathbf{e}_\alpha^{(\alpha)} \right) \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes$$

$$\otimes \left(-\sin\frac{\bar{\xi}_\alpha q^\alpha}{2} + \mathbf{e}_\alpha^{(\alpha)} \cos\frac{\bar{\xi}_\alpha q^\alpha}{2} \right) =$$

$$= -\frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha^{(\alpha)} \otimes \left(-\mathbf{e}_\alpha^{(\alpha)} \sin\frac{\bar{\xi}_\alpha q^\alpha}{2} - \cos\frac{\bar{\xi}_\alpha q^\alpha}{2} \right) =$$

$$= \frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha^{(\alpha)} \otimes \left(\mathbf{e}_\alpha^{(\alpha)} \sin\frac{\bar{\xi}_\alpha q^\alpha}{2} + \cos\frac{\bar{\xi}_\alpha q^\alpha}{2} \right) =$$

$$= \frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha^{(\alpha)} \otimes p_\alpha . \tag{44}$$

Also, taking into account:

$$\overset{*}{p} \otimes p = 1 , \tag{45}$$

then

$$\frac{\partial}{\partial q^\alpha}\left(\overset{*}{p}_\alpha \otimes p_\alpha \right) = \frac{\partial \overset{*}{p}_\alpha}{\partial q^\alpha} \otimes p_\alpha + \overset{*}{p}_\alpha \otimes \frac{\partial p_\alpha}{\partial q^\alpha} = 0 , \tag{46}$$

and, from the previous expression:

$$\frac{\partial \overset{*}{p}_\alpha}{\partial q^\alpha} \otimes p_\alpha = -\overset{*}{p}_\alpha \otimes \frac{\partial p_\alpha}{\partial q^\alpha} = -\frac{1}{2}\overset{*}{p}_\alpha \otimes \bar{\xi}_\alpha \mathbf{e}_\alpha^{(\alpha)} \otimes p_\alpha . \tag{47}$$

Multiplication of the previous expression with $\overset{*}{p}_\alpha$ on the right side gives:

$$\frac{\partial \overset{*}{p}_\alpha}{\partial q^\alpha} = -\frac{1}{2}\bar{\xi}_\alpha \overset{*}{p}_\alpha \otimes \mathbf{e}_\alpha^{(\alpha)} . \tag{48}$$

In the case $\alpha \le k$, generalized coordinate partial derivative of vector $\boldsymbol{\rho}_{kk}$ is:

$$\frac{\partial \boldsymbol{\rho}_{kk}}{\partial q^\alpha} = \frac{\partial \left(p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \overset{*}{p}_{0,k} \right)}{\partial q^\alpha} =$$

$$= \frac{\partial p_{0,k}}{\partial q^\alpha} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \overset{*}{p}_{0,k} + p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \frac{\partial \overset{*}{p}_{0,k}}{\partial q^\alpha} , \tag{49}$$

where

$$\frac{\partial p_{0,k}}{\partial q^\alpha} = p_1 \otimes p_2 \otimes \ldots \otimes \frac{\partial p_\alpha}{\partial q^\alpha} \otimes \ldots \otimes p_k =$$

$$= \frac{1}{2}\bar{\xi}_\alpha p_1 \otimes \ldots \otimes p_{\alpha-1} \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes p_\alpha \otimes \ldots \otimes p_k =$$

$$= \frac{1}{2}\bar{\xi}_\alpha p_{0,\alpha-1} \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes p_{\alpha-1,k} , \tag{50}$$

and

$$\frac{\partial \overset{*}{p}_{0,k}}{\partial q^\alpha} = \overset{*}{p}_k \otimes \ldots \otimes \frac{\partial \overset{*}{p}_\alpha}{\partial q^\alpha} \otimes \ldots \otimes \overset{*}{q}_1 =$$

$$= -\frac{1}{2}\bar{\xi}_\alpha \overset{*}{p}_k \otimes \ldots \otimes \overset{*}{p}_\alpha \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes \overset{*}{p}_{\alpha-1} \otimes \ldots \otimes \overset{*}{p}_1 =$$

$$= -\frac{1}{2}\bar{\xi}_\alpha \overset{*}{p}_{\alpha-1,k} \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes \overset{*}{p}_{0,\alpha-1} . \tag{51}$$

Since

$$\mathbf{e}_\alpha^{(\alpha)} = \overset{*}{p}_{0,\alpha-1} \otimes \mathbf{e}_\alpha \otimes p_{0,\alpha-1} , \tag{52}$$

then (50) and (51) become:

$$\frac{\partial p_{0,k}}{\partial q^\alpha} = \frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha \otimes p_{0,k} , \tag{53}$$

and

$$\frac{\partial \overset{*}{p}_{0,k}}{\partial q^\alpha} = -\frac{1}{2}\bar{\xi}_\alpha \overset{*}{p}_{0,k} \otimes \mathbf{e}_\alpha . \tag{54}$$

Substituting (53) and (54) in (49), one can get following expression:

$$\frac{\partial \boldsymbol{\rho}_{kk}}{\partial q^\alpha} = \frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha \otimes p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \overset{*}{p}_{0,k} -$$

$$- \frac{1}{2}\bar{\xi}_\alpha p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \overset{*}{p}_{0,k} \otimes \mathbf{e}_\alpha =$$

$$= \frac{1}{2}\bar{\xi}_\alpha \mathbf{e}_\alpha \otimes \boldsymbol{\rho}_{kk} - \frac{1}{2}\bar{\xi}_\alpha \boldsymbol{\rho}_{kk} \otimes \mathbf{e}_\alpha =$$

$$= \frac{1}{2}\bar{\xi}_\alpha \left(-\mathbf{e}_\alpha \cdot \boldsymbol{\rho}_{kk} + \mathbf{e}_\alpha \times \boldsymbol{\rho}_{kk} + \boldsymbol{\rho}_{kk} \cdot \mathbf{e}_\alpha - \boldsymbol{\rho}_{kk} \times \mathbf{e}_\alpha \right) =$$

$$= \frac{1}{2}\bar{\xi}_\alpha \left(-\mathbf{e}_\alpha \cdot \boldsymbol{\rho}_{kk} + \mathbf{e}_\alpha \times \boldsymbol{\rho}_{kk} + \mathbf{e}_\alpha \cdot \boldsymbol{\rho}_{kk} + \mathbf{e}_\alpha \times \boldsymbol{\rho}_{kk} \right) =$$

$$= \bar{\xi}_\alpha \mathbf{e}_\alpha \times \boldsymbol{\rho}_{kk} . \tag{55}$$

In the case $\alpha > k$, there follows:

$$\frac{\partial \boldsymbol{\rho}_{kk}}{\partial q^{\alpha}} = 0 \,. \tag{56}$$

Similar to the previous expressions, it can be written the following:

$$\frac{\partial \mathbf{e}_k}{\partial q^{\alpha}} = \bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \mathbf{e}_k \,, \tag{57}$$

when $\alpha \le k$,

$$\frac{\partial \mathbf{e}_k}{\partial q^{\alpha}} = 0 \,, \tag{58}$$

when $\alpha > k$,

$$\frac{\partial \boldsymbol{\rho}_i}{\partial q^{\alpha}} = \bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \boldsymbol{\rho}_i \,. \tag{59}$$

Also, when $\alpha \le i$, and

$$\frac{\partial \boldsymbol{\rho}_i}{\partial q^{\alpha}} = 0 \,, \tag{60}$$

when $\alpha > i$. According to the previous expressions, (40) becomes:

$$\begin{aligned}\mathbf{T}_{\alpha(i)} = \frac{\partial \mathbf{r}_i}{\partial q^{\alpha}} = \sum_{k=\alpha}^{i} \left(\bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \boldsymbol{\rho}_{kk} + \xi_k \bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \mathbf{e}_k q^k \right) + \\ + \xi_{\alpha} \mathbf{e}_{\alpha} + \bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \boldsymbol{\rho}_i = \\ = \bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \left[\sum_{k=\alpha}^{i} \left(\boldsymbol{\rho}_{kk} + \xi_k \mathbf{e}_k q^k \right) + \boldsymbol{\rho}_i \right] + \xi_{\alpha} \mathbf{e}_{\alpha}\end{aligned} \tag{61}$$

when $\forall \alpha \le i$, and:

$$\mathbf{T}_{\alpha(i)} = 0 \tag{62}$$

when $\forall \alpha > i$. According to (61) and (62), the expression for velocity of the centre inertia of the rigid body (V_i) is the following:

$$\mathbf{V}_i = \sum_{\alpha=1}^{n} \mathbf{T}_{\alpha(i)} \dot{q}^{\alpha} \,. \tag{63}$$

If vectors in (61) are expressed in local body-fixed coordinate frames, then quasibasic vectors become:

$$\begin{aligned}\mathbf{T}_{\alpha(i)} = \bar{\xi}_{\alpha} \left(p_{0,\alpha-1} \otimes \mathbf{e}_{\alpha}^{(\alpha)} \otimes \overset{*}{p}_{0,\alpha-1} \right) \times \\ \times \left[\sum_{k=\alpha}^{i} \left(p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \overset{*}{p}_{0,k} + \xi_k \left(p_{0,k-1} \otimes \mathbf{e}_k^{(k)} \otimes \overset{*}{p}_{0,k-1} \right) q^k \right) + \right. \\ \left. + p_{0,i} \otimes \boldsymbol{\rho}_i^{(i)} \otimes \overset{*}{p}_{0,i} \right] + \\ + \xi_{\alpha} \left(p_{0,\alpha-1} \otimes \mathbf{e}_{\alpha}^{(\alpha)} \otimes \overset{*}{p}_{0,\alpha-1} \right)\end{aligned} \tag{64}$$

when $\forall \alpha \le i$.

3.3 Acceleration of the inertia centre of the rigid body (V_i)

Acceleration of the inertia centre of the rigid body (V_i) is time derivative of (39):

$$\mathbf{a}_{C_i} = \mathbf{a}_i = \sum_{\alpha=1}^{i} \mathbf{T}_{\alpha(i)} \ddot{q}^{\alpha} + \sum_{\alpha=1}^{i} \frac{\mathrm{d}\mathbf{T}_{\alpha(i)}}{\mathrm{d}t} \dot{q}^{\alpha} \,. \tag{65}$$

The second part of the previous expression can be written as:

$$\frac{\mathrm{d}\mathbf{T}_{\alpha(i)}}{\mathrm{d}t} = \sum_{\beta=1}^{i} \frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^{\beta}} \dot{q}^{\beta} \,. \tag{66}$$

According to (40), it can be written the following:

$$\frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^{\beta}} = \frac{\partial^2 \mathbf{r}_i}{\partial q^{\alpha} \partial q^{\beta}} \Rightarrow \frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^{\beta}} = \frac{\partial \mathbf{T}_{\beta(i)}}{\partial q^{\alpha}} \,, \tag{67}$$

then (65) becomes:

$$\mathbf{a}_i = \sum_{\alpha=1}^{i} \mathbf{T}_{\alpha(i)} \ddot{q}^{\alpha} + \sum_{\alpha=1}^{i} \sum_{\beta=1}^{i} \boldsymbol{\Gamma}_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta} \,, \tag{68}$$

where

$$\boldsymbol{\Gamma}_{\alpha\beta(i)} = \frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^{\beta}} \,, \tag{69}$$

and

$$\boldsymbol{\Gamma}_{\alpha\beta(i)} = \boldsymbol{\Gamma}_{\beta\alpha(i)} \,. \tag{70}$$

In the case $\alpha \le \beta$ then (see (55)):

$$\boldsymbol{\Gamma}_{\alpha\beta(i)} = \frac{\partial \mathbf{T}_{\beta(i)}}{\partial q^{\alpha}} = \bar{\xi}_{\alpha} \mathbf{e}_{\alpha} \times \mathbf{T}_{\beta(i)} \,, \tag{71}$$

and, in the case $\alpha > \beta$:

$$\boldsymbol{\Gamma}_{\alpha\beta(i)} = \frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^{\beta}} = \bar{\xi}_{\beta} \mathbf{e}_{\beta} \times \mathbf{T}_{\alpha(i)} \,. \tag{72}$$

The expressions (71) and (72) can be written in the unique form:

$$\boldsymbol{\Gamma}_{\alpha\beta(i)} = \bar{\xi}_{\inf(\alpha,\beta)} \mathbf{e}_{\inf(\alpha,\beta)} \times \mathbf{T}_{\sup(\alpha,\beta)(i)} \,. \tag{73}$$

The expression (65) can be written in the following way:

$$\mathbf{a}_i = \sum_{\alpha=1}^{n} \mathbf{T}_{\alpha(i)} \ddot{q}^{\alpha} + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \boldsymbol{\Gamma}_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta} \,. \tag{74}$$

The vectors in (73), expressed in local body-fixed coordinate frames have the following form:

$$\begin{aligned}\mathbf{e}_{\inf(\alpha,\beta)} = \\ = \left(p_{0,\inf(\alpha,\beta)-1} \otimes \mathbf{e}_{\inf(\alpha,\beta)}^{(\inf(\alpha,\beta))} \otimes \overset{*}{p}_{0,\inf(\alpha,\beta)-1} \right),\end{aligned} \tag{75}$$

$$\begin{aligned}\mathbf{T}_{\sup(\alpha,\beta)(i)} = \bar{\xi}_{\sup(\alpha,\beta)} \left(p_{0,\alpha-1} \otimes \mathbf{e}_{\alpha}^{(\alpha)} \otimes \overset{*}{p}_{0,\alpha-1} \right) \times \\ \times \left[\sum_{k=\alpha}^{i} \left(p_{0,k} \otimes \boldsymbol{\rho}_{kk}^{(k)} \otimes \overset{*}{p}_{0,k} + \xi_k \left(p_{0,k-1} \otimes \mathbf{e}_k^{(k)} \otimes \overset{*}{p}_{0,k-1} \right) q^k \right) + \right. \\ \left. + p_{0,i} \otimes \boldsymbol{\rho}_i^{(i)} \otimes \overset{*}{p}_{0,i} \right] + \\ + \xi_{\alpha} \left(p_{0,\alpha-1} \otimes \mathbf{e}_{\alpha}^{(\alpha)} \otimes \overset{*}{p}_{0,\alpha-1} \right).\end{aligned} \tag{76}$$

3.4 Angular velocity of the rigid body (V_i)

Angular velocity of the rigid body (V_i) can be obtained from the following expression:

$$\boldsymbol{\omega}_i = \sum_{\alpha=1}^{i} \overline{\xi}_\alpha \mathbf{e}_\alpha \dot{q}^\alpha , \tag{77}$$

or

$$\boldsymbol{\omega}_i = \sum_{\alpha=1}^{n} \boldsymbol{\Omega}_{\alpha(i)} \dot{q}^\alpha , \; \boldsymbol{\Omega}_{\alpha(i)} = \begin{cases} \overline{\xi}_\alpha \mathbf{e}_\alpha , & \forall \alpha \le i \\ 0, & \forall \alpha > i \end{cases} . \tag{78}$$

If unit vector of axis in (77) is expressed in the local body-fixed reference frames, then (77) has the following form:

$$\boldsymbol{\omega}_i = \sum_{\alpha=1}^{i} \overline{\xi}_\alpha \left(p_{0,\alpha-1} \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes \overset{*}{p}_{0,\alpha-1} \right) \dot{q}^\alpha . \tag{79}$$

4. KINETIC ENERGY OF THE RIGID BODIES SYSTEM

Consider an open chain of the rigid bodies system (V_1), (V_2), ..., (V_n). Differential of kinetic energy of the body (V_i) (Fig. 6) is:

$$\mathrm{d}E_{k(i)} = \frac{1}{2} \mathrm{d}m_i \mathbf{V}_{M_i}^2 , \tag{80}$$

where C_i represents inertia centre, $\mathrm{d}V_i$ is infinitesimal volume of the body (V_i) which corresponds to infinitesimal mass $\mathrm{d}m_i$.

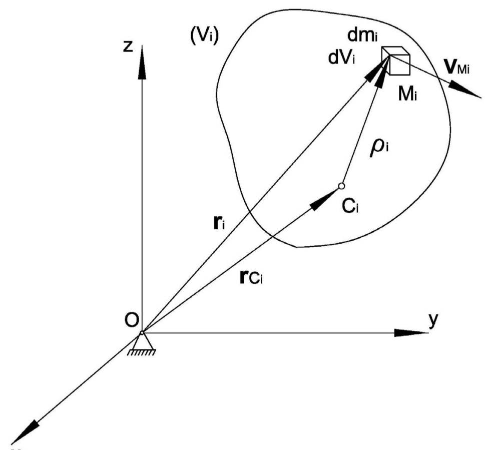

Figure 6. Characteristic vectors of the rigid body (V_i)

Velocity of the point M_i which belongs to the body (V_i) is:

$$\mathbf{V}_{M_i} = \mathbf{V}_i = \mathbf{V}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i . \tag{81}$$

Kinetic energy of the body (V_i) is:

$$E_{k(i)} = \frac{1}{2} \int_{(V_i)} \left(\mathbf{V}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \cdot \left(\mathbf{V}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \mathrm{d}m_i . \tag{82}$$

Due to (see[3]):

$$\int_{(V_i)} \boldsymbol{\rho}_i dm_i = m_i \boldsymbol{\rho}_{C_i} = 0 , \tag{83}$$

the expression (82) becomes:

$$E_{k(i)} = \frac{1}{2} m_i \mathbf{V}_{C_i} \cdot \mathbf{V}_{C_i} + \\ + \frac{1}{2} \int_{(V_i)} \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \cdot \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \mathrm{d}m_i . \tag{84}$$

Kinetic energy of the system of the rigid bodies is equal to the sum of kinetic energies of each body:

$$E_k = \sum_{i=1}^{n} E_{k(i)} = \frac{1}{2} \sum_{i=1}^{n} m_i \mathbf{V}_{C_i} \cdot \mathbf{V}_{C_i} + \\ + \frac{1}{2} \sum_{i=1}^{n} \int_{(V_i)} \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \cdot \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \mathrm{d}m_i . \tag{85}$$

Since

$$\mathbf{V}_{C_i} = \sum_{\alpha=1}^{n} \frac{\partial \mathbf{r}_{C_i}}{\partial q^\alpha} \dot{q}^\alpha \tag{86}$$

then

$$\mathbf{V}_{C_i} \cdot \mathbf{V}_{C_i} = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \frac{\partial \mathbf{r}_{C_i}}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_{C_i}}{\partial q^\beta} \dot{q}^\alpha \dot{q}^\beta . \tag{87}$$

The second part of (85), using (78), can be transformed as follows:

$$\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i = \sum_{\alpha=1}^{\mathbf{n}} \left(\boldsymbol{\Omega}_{\alpha(i)} \times \boldsymbol{\rho}_i \right) \dot{q}^\alpha = \sum_{\alpha=1}^{n} \frac{\partial \boldsymbol{\rho}_i}{\partial q^\alpha} \dot{q}^\alpha \tag{88}$$

then, it becomes:

$$\sum_{i=1}^{n} \int_{(V_i)} \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \cdot \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_i \right) \mathrm{d}m_i = \\ = \sum_{i=1}^{n} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \int_{(V_i)} \frac{\partial \boldsymbol{\rho}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{\rho}_i}{\partial q^\beta} \dot{q}^\alpha \dot{q}^\beta \mathrm{d}m_i , \tag{89}$$

Substituting (87) and (89) in (85), it can be obtained:

$$E_k = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \left[\sum_{i=1}^{n} m_i \frac{\partial \mathbf{r}_{C_i}}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_{C_i}}{\partial q^\beta} + \right. \\ \left. + \sum_{i=1}^{n} \int_{(V_i)} \frac{\partial \boldsymbol{\rho}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{\rho}_i}{\partial q^\beta} \mathrm{d}m_i \right] \dot{q}^\alpha \dot{q}^\beta = \\ = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta , \tag{90}$$

where

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \frac{\partial \mathbf{r}_{C_i}}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_{C_i}}{\partial q^\beta} + \\ + \sum_{i=1}^{n} \int_{(V_i)} \frac{\partial \boldsymbol{\rho}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{\rho}_i}{\partial q^\beta} \mathrm{d}m_i . \tag{91}$$

Coefficients $a_{\alpha\beta}$ are called the covariant coordinates of the basic metric tensor, and matrix $\left[a_{\alpha\beta} \right] \in R^{n \times n}$ are called basic metric tensor. According to (40), the first part of (91) becomes:

$$\sum_{i=1}^{n} m_i \frac{\partial \mathbf{r}_{C_i}}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_{C_i}}{\partial q^\beta} = \sum_{i=1}^{n} m_i \mathbf{T}_{\alpha(i)} \cdot \mathbf{T}_{\beta(i)} =$$

$$= \sum_{i=1}^{n} m_i \left(\mathbf{T}_{\alpha(i)}\right)\left\{\mathbf{T}_{\beta(i)}\right\}. \qquad (92)$$

The second part of the right side of (91) can be transformed in the following way:

$$\sum_{i=1}^{n} \int_{(V_i)} \frac{\partial \boldsymbol{\rho}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{\rho}_i}{\partial q^\beta} \mathrm{d}m_i =$$

$$= \sum_{i=1}^{n} \int_{(V_i)} \bar{\xi}_\alpha \bar{\xi}_\beta \left(\mathbf{e}_\alpha \times \boldsymbol{\rho}_i\right) \cdot \left(\mathbf{e}_\beta \times \boldsymbol{\rho}_i\right) \mathrm{d}m_i . \qquad (93)$$

Since (see[3]):

$$\left(\mathbf{e}_\alpha \times \boldsymbol{\rho}_i\right)\left\{\mathbf{e}_\beta \times \boldsymbol{\rho}_i\right\} = -\left(\mathbf{e}_\alpha\right)\left[\rho_i^d\right]^2 \left\{\mathbf{e}_\alpha\right\}, \qquad (94)$$

then (93) becomes:

$$\sum_{i=1}^{n} \int_{(V_i)} \frac{\partial \boldsymbol{\rho}_i}{\partial q^\alpha} \cdot \frac{\partial \boldsymbol{\rho}_i}{\partial q^\beta} \mathrm{d}m_i =$$

$$= \sum_{i=1}^{n} \bar{\xi}_\alpha \bar{\xi}_\beta \left(\mathbf{e}_\alpha\right) \left(\int_{(V_i)} \left[\rho_i^d\right]^2 \mathrm{d}m_i \right) \left\{\mathbf{e}_\beta\right\} =$$

$$= \sum_{i=1}^{n} \bar{\xi}_\alpha \bar{\xi}_\beta \left(\mathbf{e}_\alpha\right) \left[J_{C_i}\right] \left\{\mathbf{e}_\beta\right\}, \qquad (95)$$

where

$$\left[J_{C_i}\right] = \int_{(V_i)} \left[\rho_i^d\right]^2 \mathrm{d}m_i =$$

$$= \int_{(V_i)} \begin{bmatrix} \eta_i^2 + \zeta_i^2 & -\xi_i \eta_i & -\xi_i \zeta_i \\ -\eta_i \xi_i & \zeta_i^2 + \xi_i^2 & -\eta_i \zeta_i \\ -\zeta_i \xi_i & -\zeta_i \eta_i & \eta_i^2 + \xi_i^2 \end{bmatrix} \mathrm{d}m_i \qquad (96)$$

denotes inertia tensor of the rigid body (V_i). It is most convenient for the inertia tensor of the body (V_i) to be expressed in local body-fixed reference frame $0\xi_i\eta_i\zeta_i$, because, in this case, the inertia tensor is constant. After that, covariant coordinates have the following form:

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \left(\mathbf{T}_{\alpha(i)}\right)\left\{\mathbf{T}_{\beta(i)}\right\} +$$

$$+ \sum_{i=1}^{n} \bar{\xi}_\alpha \bar{\xi}_\beta \left(\mathbf{e}_\alpha^{(i)}\right) \left[J_{C_i}\right] \left\{\mathbf{e}_\beta^{(i)}\right\}, \qquad (97)$$

or

$$a_{\alpha\beta} = \sum_{i=\sup(\alpha,\beta)}^{n} m_i \left(\mathbf{T}_{\alpha(i)}\right)\left\{\mathbf{T}_{\beta(i)}\right\} +$$

$$+ \sum_{i=\sup(\alpha,\beta)}^{n} \bar{\xi}_\alpha \bar{\xi}_\beta \left(\mathbf{e}_\alpha^{(i)}\right) \left[J_{C_i}\right] \left\{\mathbf{e}_\beta^{(i)}\right\} = a_{\alpha\beta}^{tr} + a_{\alpha\beta}^{rot}, \qquad (98)$$

where $a_{\alpha\beta}^{tr}$ and $a_{\alpha\beta}^{rot}$ denote translational and rotational component of covariant coordinates:

$$a_{\alpha\beta}^{tr} = \sum_{i=\sup(\alpha,\beta)}^{n} m_i \left(\mathbf{T}_{\alpha(i)}\right)\left\{\mathbf{T}_{\beta(i)}\right\}, \qquad (99)$$

and

$$a_{\alpha\beta}^{rot} = \sum_{i=\sup(\alpha,\beta)}^{n} \bar{\xi}_\alpha \bar{\xi}_\beta \left(\mathbf{e}_\alpha^{(i)}\right) \left[J_{C_i}\right] \left\{\mathbf{e}_\beta^{(i)}\right\}. \qquad (100)$$

From (98) it can be concluded that coefficients $a_{\alpha\beta}$ have the following property:

$$a_{\alpha\beta}\left(q^1, ..., q^n\right) = a_{\beta\alpha}\left(q^1, ..., q^n\right). \qquad (101)$$

Unit vectors in (100), expressed in body-fixed coordinate frame have the following form:

$$\mathbf{e}_\alpha^{(i)} = \overset{*}{p}_{\alpha-1,i} \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes p_{\alpha-1,i} =$$

$$= \overset{*}{p}_{\alpha,i} \otimes \mathbf{e}_\alpha^{(\alpha)} \otimes p_{\alpha,i}, \qquad (102)$$

and

$$\mathbf{e}_\beta^{(i)} = \overset{*}{p}_{\beta-1,i} \otimes \mathbf{e}_\beta^{(\beta)} \otimes p_{\beta-1,i} =$$

$$= \overset{*}{p}_{\beta,i} \otimes \mathbf{e}_\beta^{(\beta)} \otimes p_{\beta,i}. \qquad (103)$$

5. DIFFERENTIAL EQUATIONS OF MOTIONS OF THE RIGID BODY SYSTEM

In this section differential equations of motion of rigid bodies system in covariant form using quaternion algebra will be derived. Consider open chain system of rigid bodies (V_1), (V_2), ..., (V_n). It is assumed that constraints are holonomic, scleronomic and ideal. A system of independent coordinates (q^1, q^2,…,q^n) can be chosen, which allows that kinetic energy to be written as the function of these coordinates and their time derivatives. In this case, differential equations of motion can be represented in the form of Lagrange's equation expressed only as the function of generalized coordinates (q^1, q^2,…,q^n) and their time derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) - \frac{\partial E_k}{\partial q^\gamma} = Q_\gamma, \qquad (104)$$

where Q_γ denotes generalized force of active forces system which act on the rigid body system, which corresponds to generalized coordinate q^γ. Generalized velocity partial derivatives of kinetic energy are:

$$\frac{\partial E_k}{\dot{q}^\gamma} = \frac{1}{2} \sum_{\beta=1}^{n} a_{\gamma\beta} \dot{q}^\beta + \frac{1}{2} \sum_{\alpha=1}^{n} a_{\alpha\gamma} \dot{q}^\alpha . \qquad (105)$$

According to (98), (105) can be written as:

$$\frac{\partial E_k}{\dot{q}^\gamma} = \sum_{\alpha=1}^{n} a_{\alpha\gamma} \dot{q}^\alpha . \qquad (106)$$

Time derivative of (106) is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\dot{q}^\gamma}\right)=\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + $$
$$+\frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\left(\frac{\partial a_{\alpha\gamma}}{\partial q^\beta}+\frac{\partial a_{\beta\gamma}}{\partial q^\alpha}\right)\dot{q}^\alpha\dot{q}^\beta . \qquad (107)$$

Generalized coordinate partial derivatives of kinetic energy are:

$$\frac{\partial E_k}{\partial q^\gamma}=\frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\dot{q}^\alpha\dot{q}^\beta . \qquad (108)$$

Substituting (107) and (108) in (104), the Lagrange equations have the following form [3]:

$$\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha+\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}^\alpha\dot{q}^\beta=Q_\gamma , \qquad (109)$$

where

$$\Gamma_{\alpha\beta,\gamma}=\frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^\alpha}+\frac{\partial a_{\gamma\alpha}}{\partial q^\beta}-\frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\right) \qquad (110)$$

denotes Christoffel symbols of the first kind. This form of equations is called covariant. According to (98), it can be written:

$$\Gamma_{\alpha\beta,\gamma}=\Gamma^{tr}_{\alpha\beta,\gamma}+\Gamma^{rot}_{\alpha\beta,\gamma} , \qquad (111)$$

where

$$\Gamma^{tr}_{\alpha\beta,\gamma}=\frac{1}{2}\left(\frac{\partial a^{tr}_{\beta\gamma}}{\partial q^\alpha}+\frac{\partial a^{tr}_{\gamma\alpha}}{\partial q^\beta}-\frac{\partial a^{tr}_{\alpha\beta}}{\partial q^\gamma}\right) \qquad (112)$$

denotes the translational components, and

$$\Gamma^{rot}_{\alpha\beta,\gamma}=\frac{1}{2}\left(\frac{\partial a^{rot}_{\beta\gamma}}{\partial q^\alpha}+\frac{\partial a^{rot}_{\gamma\alpha}}{\partial q^\beta}-\frac{\partial a^{rot}_{\alpha\beta}}{\partial q^\gamma}\right) \qquad (113)$$

denotes rotational components of Christoffel symbols. Deriving (99), applying the following properties:

$$\frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^\beta}=\frac{\partial \mathbf{r}_i}{\partial q^\alpha\partial q^\beta}=\frac{\partial \mathbf{T}_{\beta(i)}}{\partial q^\alpha} , \qquad (114)$$

(71), (72) and (73), and substituting it in (112), the translational components of the Christoffel symbols become:

$$\Gamma^{tr}_{\alpha\beta,\gamma}=\sum_{i=\sup(\alpha,\beta,\gamma)}^{n} m_i\frac{\partial \mathbf{T}_{\alpha(i)}}{\partial q^\beta}\cdot\mathbf{T}_{\gamma(i)}=$$
$$=\sum_{i=\sup(\alpha,\beta,\gamma)}^{n} m_i\bar{\xi}_{\inf(\alpha,\beta)}\Big(\mathbf{e}_{\inf(\alpha,\beta)}\times$$
$$\times\mathbf{T}_{\sup(\alpha,\beta)(i)}\Big)\left\{\mathbf{T}_{\gamma(i)}\right\} , \qquad (115)$$

where

$$\mathbf{e}_{\inf(\alpha,\beta)}=$$
$$=p_{0,\inf(\alpha,\beta)-1}\otimes\mathbf{e}^{(\inf(\alpha,\beta))}_{\inf(\alpha,\beta)}\otimes p^*_{0,\inf(\alpha,\beta)-1} . \qquad (116)$$

The rotational components of the Christoffel symbols are as follows:

$$\frac{\partial a^{rot}_{\beta\gamma}}{\partial q^\alpha}=\sum_{i=\sup(\beta,\gamma)}^{n}\bar{\xi}_\beta\bar{\xi}_\gamma\left(\frac{\partial \mathbf{e}^{(i)}_\beta}{\partial q^\alpha}\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\gamma\right\}+$$
$$+\sum_{i=\sup(\beta,\gamma)}^{n}\bar{\xi}_\beta\bar{\xi}_\gamma\left(\mathbf{e}^{(i)}_\beta\right)\left[J_{C_i}\right]\left\{\frac{\partial \mathbf{e}^{(i)}_\gamma}{\partial q^\alpha}\right\} . \qquad (117)$$

According to (102) and (103), and doing the same as in Section 3.2, partial derivatives of unit vectors of the axis in the case $\alpha \geq \beta,\gamma$, are:

$$\frac{\partial \mathbf{e}^{(i)}_\beta}{\partial q^\alpha}=\bar{\xi}_\alpha\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_\alpha , \qquad (118)$$

and

$$\frac{\partial \mathbf{e}^{(i)}_\gamma}{\partial q^\alpha}=\bar{\xi}_\alpha\mathbf{e}^{(i)}_\gamma\times\mathbf{e}^{(i)}_\alpha . \qquad (119)$$

Substituting (118) and (119) into (117), it can be obtained:

$$\frac{\partial a^{rot}_{\beta\gamma}}{\partial q^\alpha}=\sum_{i=\sup(\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma\left(\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\gamma\right\}+$$
$$+\sum_{i=\sup(\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma\left(\mathbf{e}^{(i)}_\beta\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\gamma\times\mathbf{e}^{(i)}_{\sup(\alpha,\gamma)}\right\} . \qquad (120)$$

Doing similarly with other components as in (113), and substituting it into (126), the rotational components of the Christoffel symbols become:

$$\Gamma^{rot}_{\alpha\beta,\gamma}=\frac{1}{2}\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma\Big(\left(\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\gamma\right\}+$$
$$+\left(\mathbf{e}^{(i)}_\beta\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\gamma\times\mathbf{e}^{(i)}_{\sup(\alpha,\gamma)}\right\}+\left(\mathbf{e}^{(i)}_\gamma\times\mathbf{e}^{(i)}_{\sup(\beta,\gamma)}\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\alpha\right\}+$$
$$+\left(\mathbf{e}^{(i)}_\gamma\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\alpha\times\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right\}-\left(\mathbf{e}^{(i)}_\alpha\times\mathbf{e}^{(i)}_{\sup(\alpha,\gamma)}\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\beta\right\}-$$
$$-\left(\mathbf{e}^{(i)}_\alpha\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_{\sup(\beta,\gamma)}\right\}\Big) . \qquad (121)$$

According to the certain properties of members in the previous expression, (121) becomes:

$$\Gamma^{rot}_{\alpha\beta,\gamma}=$$
$$=\frac{1}{2}\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma\Big(\left(\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\times\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\gamma\right\}-$$
$$-\left(\mathbf{e}^{(i)}_\alpha\times\mathbf{e}^{(i)}_\gamma\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\beta\right\}-\left(\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_\gamma\right)\left[J_{C_i}\right]\left\{\mathbf{e}^{(i)}_\alpha\right\}\Big) . \qquad (122)$$

Due to the property of dual object, the inertia tensor can be written in the following way [3]:

$$\left[J_{C_i}\right]=-\int_{(V_i)}\left[\rho_i^d\right]^2 \mathrm{d}m_i=\int_{(V_i)}\left\{\rho_i^2[I]-\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\right\}\mathrm{d}m_i\,,\tag{123}$$

and substituting (123) in (122), the Christoffel symbols become:

$$\Gamma^{rot}_{\alpha\beta,\gamma}=$$
$$=\frac{1}{2}\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma$$
$$\int_{(V_i)}\Bigg(-\left(\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\times\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right)\left[\rho_i^d\right]^2\left\{\mathbf{e}^{(i)}_\gamma\right\}+$$
$$+\left(\mathbf{e}^{(i)}_\alpha\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_\beta\right\}+$$
$$+\left(\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_\alpha\right\}\Bigg)\mathrm{d}m_i\,.\tag{124}$$

The last two terms of subintegral function in previous expression can be written as follows:

$$\left(\mathbf{e}^{(i)}_\alpha\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_\beta\right\}+\left(\mathbf{e}^{(i)}_\beta\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_\alpha\right\}=$$
$$=\left(\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right\}+$$
$$+\left(\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\right\}.\tag{125}$$

Subintegral function can be easily transformed in the following expression:

$$-\left(\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\times\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right)\left[\rho_i^d\right]\left[\rho_i^d\right]\left\{\mathbf{e}^{(i)}_\gamma\right\}+$$
$$+\left(\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\right\}=$$
$$=\left(\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\right\},\tag{126}$$

and, the rotational components of the Christoffel symbols become:

$$\Gamma^{rot}_{\alpha\beta,\gamma}=$$
$$=\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma$$
$$\int_{(V_i)}\left\{\left(\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\times\mathbf{e}^{(i)}_\gamma\right)\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\left\{\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\right\}\right\}\mathrm{d}m_i=$$
$$=\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma\left(\mathbf{e}^{(i)}_{\sup(\alpha,\beta)}\times\mathbf{e}^{(i)}_\gamma\right)[\Pi_i]\left\{\mathbf{e}^{(i)}_{\inf(\alpha,\beta)}\right\},\tag{127}$$

where

$$[\Pi_i]=\int_{(V_i)}\{\boldsymbol{\rho}_i\}(\boldsymbol{\rho}_i)\,\mathrm{d}m_i\tag{128}$$

denotes the planar moment inertia [3]. Unit vectors in (127) can be expressed in local-body fixed coordinate frame like in (102), so the Christoffel symbols finally become:

$$\Gamma_{\alpha\beta,\gamma}=\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}m_i\bar{\xi}_{\inf(\alpha,\beta)}\Bigg(\Bigg(p_{0,\inf(\alpha,\beta)-1}\otimes$$
$$\otimes\mathbf{e}^{(\inf(\alpha,\beta))}_{\inf(\alpha,\beta)}\otimes\overset{*}{p}_{0,\inf(\alpha,\beta)-1}\Bigg)\times\mathbf{T}_{\sup(\alpha,\beta)(i)}\Bigg)\left\{\mathbf{T}_{\gamma(i)}\right\}+$$
$$+\sum_{i=\sup(\alpha,\beta,\gamma)}^{n}\bar{\xi}_\alpha\bar{\xi}_\beta\bar{\xi}_\gamma\Bigg(\Bigg(\overset{*}{p}_{\sup(\alpha,\beta),i}\otimes\mathbf{e}^{(\sup(\alpha,\beta))}_{\sup(\alpha,\beta)}\otimes$$
$$\otimes p_{\sup(\alpha,\beta),i}\Bigg)\times\left(\overset{*}{p}_{\gamma,i}\otimes\mathbf{e}^{(\gamma)}_\gamma\otimes p_{\gamma,i}\right)\Bigg)[\Pi_i]$$
$$\left\{\overset{*}{p}_{\inf(\alpha,\beta),i}\otimes\mathbf{e}^{(\inf(\alpha,\beta))}_{\inf(\alpha,\beta)}\otimes p_{\inf(\alpha,\beta),i}\right\}.\tag{129}$$

6. CONCLUSION

This paper has shown the development of Lagrange's equations of the second kind of the rigid bodies system in the covariant form using the quaternion algebra. It can be concluded that every vector which belongs to the arbitrary body of the rigid bodies system can be easily expressed in the body-fixed reference frame of another body making composite quaternion, which consists of Hamiltonian product of quaternions representing the rotation neighbouring bodies, avoiding trigonometric functions characteristic of Euler's angles. Also, it is easy to find generalized coordinate partial derivatives of that vector.

Unlike the existing results, where quaternionic approach has been applied only for the case of rotation of one or two bodies, it is here presented the procedure of obtaining the model of multi-body system of *n* rigid bodies in terms of quaternions, which is useful for studying kinematics, dynamics as well as for research of control system designs.

ACKNOWLEDGMENT

This work is partially funded by the Ministry of Science and Environmental Protection of Republic of Serbia as T.R.P. No. 20152.

REFERENCES

[1] Hamilton, W.R.: On Quaternions, or a new system of imaginaries in algebra, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Vol. 25-36, 1844-1850.

[2] Markovic, S.: *Rapid Establishment of Differential Equations of Motion of Rigid Bodies System in an Analytical Form*, MSc thesis, Faculty of Mechanical Engineering, University of Belgrade, Belgrade, 1992, (in Serbian).

[3] Covic, V. and Lazarevic M.: *Mechanics of Robots*, Faculty of Mechanical Engineering, University of Belgrade, Belgrade, 2009, (in Serbian).

[4] Isenberg, D.R. and Kakad, Y.P.: Computed torque control of a quaternion based space robot, in: *Proceedings of the 19th International Conference*

on System Engineering, 19-21.08.2008, Las Vegas, USA, pp. 59-64.

[5] Arribas, M., Elipe, A. and Palacios, M.: Quaternions and the rotation of a rigid body, Celestial Mechanics and Dynamical Astronomy, Vol. 96, No. 3-4, pp. 239-251, 2006.

[6] Kuipers, J.B.: *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality*, Princeton University Press, New Jersey, 1999.

КИНЕМАТИКА И ДИНАМИКА СИСТЕМА КРУТИХ ТЕЛА У КВАТЕРНИОНСКОЈ ФОРМИ: ЛАГРАНЖЕВА ФОРМУЛАЦИЈА У КОВАРИЈАНТНОМ ОБЛИКУ – РОДРИГОВ ПРИСТУП

Немања Д. Зорић, Михаило П. Лазаревић, Александар М. Симоновић

У овом раду се предлаже кватернионски приступ за моделирање кинематике и динамике система крутих тела. Уместо регуларног „Њутн-Ојлеровог“ и Лагранжевог метода коришћеног на традиционалан начин, употребљавају се Лагранжеве једначине друге врсте у коваријантном облику применом Родриговог приступа и кватернионске алгебре. Добијен је модел система од *n* крутих тела у кватернионској форми који је користан за проучавање кинематике, динамике система за општи случај кретања, као и за синтезу система управљања.