# One View to the Optimization of Thin-Walled Open Sections Subjected to Constrained Torsion

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*One approach to the optimization of thin-walled open section cantilever beams subjected to constrained torsion is considered. The aim of this paper is to determine the minimum mass i.e. minimum cross- sectional area of structural thin-walled I-beam and Channel section beam elements for given loads, material and geometrical characteristics. The area of the cross-section is assumed to be the objective function. The displacement constraints are introduced. The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled beam theory from one side and the basic assumptions of the optimum design from the other. Applying the Lagrange multiplier method, the equations of which the solutions represent the optimal values of the ratios of the parts of the chosen cross sections are derived. The obtained results are used for numerical calculation applying the Finite Element Method.*

***Keywords:*** *optimization, thin-walled beams, optimal dimensions, displacement constraints.*

## 1. INTRODUCTION

In most structures it is possible to find the elements in which, depending on loading cases and the way of their introductions, the effect of constrained torsion is present and its consequences are particularly evident in the case of thin-walled profiles. Thin-walled open section beams are widely applied due to their low weight in many structures. Thin-walled beams have a specific behavior and because of that their optimization represents a particular problem. The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled beam theory from one side and the basic assumptions of the optimum design from the other.

Optimization is a mathematical process through which the set of conditions is obtained giving as the result the maximum or minimum value of a specified function. In the ideal case, one would like to obtain the perfect solution for the considered design situation. But in the reality, one can only achieve the best solution.

The quantities numerically calculated during the process of obtaining the optimal solution are called the design variables and the total region defined by the design variables included in the objective function is called the design space and it is limited by the constraints.

Engineering design is a process of formulating a plan for the satisfaction of human needs through a cycle of steps that include problem definition, conceptualization, embodiment and detailing. Engineers may focus on requirements in engineering aspects, but many of these requirements conflict with each other. Conflicts are ubiquituos in an engineering design process. For instance, improving reliability will increase the cost. Therefore, conflicts always exist in design objectives in any engineering design process, and it is important to resolve the objective conflicts in engineering design.

Many studies have been made on the optimization problems treating the cases where geometric configurations of structures are specified and only the dimensions of members, such as areas of members' cross-sections, are determined in order to attain the minimum structural weight or cost. Many methods have been developed for the determination of the local minimum point for the optimization problem [2, 5, 9, 12].

During the process of dimensioning of a structure, besides requested dimensions which are necessary to permit to the particular part of the structure to support the applied loads, it is also often very important to find the optimal values of the dimensions. Very often used types of cross sections, particularly in steel structures, are the I-section and the Channel section beams.

## 2. DEFINITION OF THE PROBLEM

Open thin-walled steel sections subjected to twisting moments are generally prone to large warping stresses and excessive angles of twist. It is therefore common practice to avoid twisting moments in steel assemblies consisting of steel open sections whenever it is possible. However, in a number of practical applications, twisting cannot be avoided and the designer is compelled to count on the torsional resistance of these members. The classical formulation for open thin-walled sections subjected to torsion was developed by Vlasov [13]. The

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Vlasov formulation is based on two fundamental kinematic assumptions: (a) In-plane deformations of the section are negligible, and (b) shear strains along the section mid-surface are negligible.

The formulation is restricted to the torsional analysis of open section thin-walled beams.

The considered cantilever beam of the length $l$ is subjected to the constrained torsion because of the fact that its one end is fixed and the other free end is loaded by a concentrated torque $M$. The cross section (Fig. 1) is supposed to have flanges of mutually equal widths and thicknesses $b_1 = b_3$, $t_1 = t_3$.

The aim of the paper is to determine the minimal mass of the beam or, in another way, to find the minimal cross-sectional area

$$A = A_{\min} \tag{1}$$

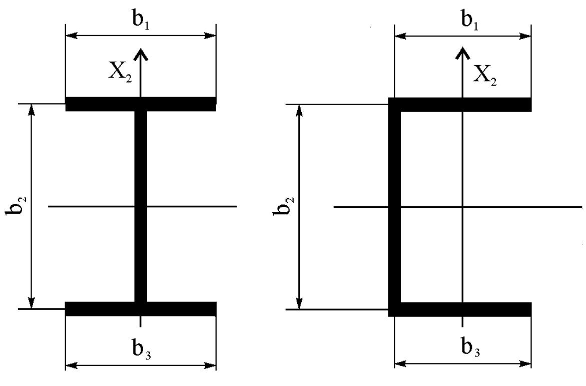
for the given loads and material and geometrical properties of the considered beam.

Formulation of the structural design optimization problem plays an important role in the numerical solution process [6]. A particular choice of the cost function and constraints affect the final solution, and efficiency and robustness of the solution process.

The process of selecting the best solution from various possible solutions must be based on a prescribed criterion known as the objective function. In the considered problem the cross sectional area will be treated as an objective function and it is obvious from the Fig. 1 that

$$A = \sum b_i t_i, \quad i = 1,2,3, \tag{2}$$

where $b_i$ and $t_i$ are widths and thicknesses of the parts of the considered cross sections.




a) b)

**Figure 1. a) I - section; b) Channel-section**

## 3. CONSTRAINTS

Only the displacements will be taken into account in the calculations that follow and the constraints treated in the paper are the displacement constraints.

The considered displacement constraint is allowable angle of twist per unit length, denoted by $\theta_0'$.

The ratio

$$z = b_2 / b_1 \tag{3}$$

will be the optimal relation of the dimensions of the considered cross sections.

The flexural-torsion cross section characteristic [7,11] is given by the expression

$$k = \sqrt{GI_t / EI_\omega}, \tag{4}$$

where:

- $I_t$ – torsion constant,
- $I_\omega$ – sectorial moment of inertia,
- $E$ – modulus of elasticity and
- $G$ – shear modulus.

If the allowable angle of twist per unit length $\theta_0'$ is taken as the constraint, the constrained function $\varphi$ can be written in the form [7,11]

$$\varphi = \theta'_{\max} = \theta'(l) = \frac{M}{GI_t}\left(1 - \frac{1}{\cosh kl}\right) \le \theta_0', \tag{5}$$

or in the form

$$\varphi = \cosh kl \left(1 - \theta_0' \frac{GI_t}{M}\right) - 1 \le 0, \tag{6}$$

where:

- $M$ – torque,
- $l$ – beam length,
- $k$ – flexural-torsion cross section characteristic,
- $\theta'$ – angle of twist per unit length.

## 4. LAGRANGE MULTIPLIER METHOD

Lagrange Multiplier Method [2-5, 9, 12, 14] is the classical approach to the constraint optimization. Lagrange multiplier, which is labeled as $\lambda$, measures the change of the objective function with respect to the constraint.

Applying this method to the vector depending on two parameters $b_i$, ( $i = 1,2$ ), the system of equations ''(7)'' will be obtained

$$\frac{\partial}{\partial b_i}\left(A + \lambda\varphi\right) = 0, \quad i = 1,2, . \tag{7}$$

After the elimination of the multiplier $\lambda$, it will become

$$\frac{\partial A}{\partial b_i}\frac{\partial \varphi}{\partial b_j} = \frac{\partial A}{\partial b_j}\frac{\partial \varphi}{\partial b_i}, \quad (i \ne j,\ i = 1,\ j = 2). \tag{8}$$

## 5. ANALYTICAL APPROACH

The torsion constant for the considered symmetrical open sections [7,11] is given by the expression

$$I_t = \frac{1}{3} b_1 t_1^3 \left(2 + \psi^3 z\right), \tag{9}$$

where:

$$\psi = t_2 / t_1 \quad (10)$$

is the ratio of thicknesses of the parts of the cross section.

The sectorial moments of inertia for the considered sections are given in the following forms:

– I-section

$$I_\omega = \frac{1}{24} b_1^3 b_2^2 t_1 , \quad (11)$$

– Channel-section

$$I_\omega = \frac{1}{12} b_1^3 b_2^2 t_1 \frac{3+2\psi z}{6+\psi z} . \quad (12)$$

Applying the Lagrange multiplier method, after the differentiation of the expression ‘‘(8)’’ with respect to the variables $b_1$ and $b_2$, the expressions ‘‘(6)’’ take the forms ‘‘(13)’’ and ‘‘(15)’’, respectively:

- **I-beam**

In the considered case when the I-section beam is the object of the optimization, the equations ‘‘(8)’’ are reduced to the equation ‘‘(13)’’. The equation of the second order is obtained and its solutions represent the optimal ratios of the cross-sectional dimensions for the chosen shape

$$\sum_{i=0}^{2} c_i z^i = 0 , \quad (13)$$

where the coefficients $c_i$ are given in the form ‘‘(14)’’:

$$c_0 = -8 ,$$

$$c_1 = 2\psi \left[ 2 - \psi^2 + 2 \frac{\psi^2 - 1}{\frac{kl \ \tanh \ kl}{1-\cosh \ kl}} \right] ,$$

$$c_2 = 3\psi^4 . \quad (14)$$

- **Channel-section beam**

In this case, when the Channel-section beam is the object of the optimization, the equations ‘‘(8)’’ are reduced to the equation of the fourth order ‘‘(15)’’

$$\sum_{i=0}^{4} c_i z^i = 0 , \quad (15)$$

where the coefficients $c_i$ are given in the form ‘‘(16)’’:

$$c_0 = -72 ,$$

$$c_1 = -6\psi \left[ 7 + 3\psi^2 - 6 \frac{\psi^2 - 1}{\frac{kl \tanh kl}{1-\cosh kl}} \right] ,$$

$$c_2 = \psi^2 \left[ 13 + 3\psi^2 + 30 \frac{\psi^2 - 1}{\frac{kl \tanh kl}{1-\cosh kl}} \right] ,$$

$$c_3 = 4\psi^3 \left[ 1 + 4\psi^2 + \frac{\psi^2 - 1}{\frac{kl \tanh kl}{1-\cosh kl}} \right] ,$$

$$c_4 = 3\psi^6 . \quad (16)$$

### 5.1 The results obtained by analytical approach

The following expression will be introduced

$$D_1 = \frac{\psi^2 - 1}{\frac{kl \tanh kl}{1-\cosh kl}} . \quad (17)$$

The length of the considered cantilever beam is taken as $25 \le l \le 200$ (cm). The values ($kl$) are calculated using the data for the JUS standard profiles and the ratio ‘‘(12)’’ is taken as $\psi = 0.5;\ 0.75;\ 1$.

The results for the ratios ‘‘(3)’’ $z = b_2 / b_1$ obtained from the equations ‘‘(8)’’ are given in Tables 1 and 2:

**Table 1. Optimal ratios *z* for the I-beam (Fig. 2a)**

| $\psi$ | 1 | 0.75 | | | 0.5 | | |
|---|---|---|---|---|---|---|---|
| $D_l$ | 0 | 0.22 | 0.58 | 437.5 | 0.38 | 1 | 750 |
| $z$ | 1.33 | 1.78 | 1.50 | 0 | 2.67 | 1.94 | 0 |

**Table 2. Optimal ratios *z* for the Channel-section (Fig. 2b)**

| $\psi$ | 1 | 0.75 | | | 0.5 | | |
|---|---|---|---|---|---|---|---|
| $D_l$ | 0 | 0.22 | 0.58 | 2.88 | 0.38 | 1 | 4.93 |
| $z$ | 1.72 | 2.28 | 1.90 | 0.79 | 3.43 | 2.39 | 0.74 |

The results are also presented graphically in Fig. 2a and 2b.

It is possible to conclude (Tables 2 and 3 and Figure 2) that when $\psi$ is decreasing (i.e. if $D_1$ is increasing) the value for $z$ will be decreasing. Based on the performed calculations, the regions of optimal dimension values of the considered cantilever beams are defined in the following way:

- I-section beam:

– $\psi = 1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.33$,

– $\psi = 0.75 \Rightarrow 0.22 \le D_1 \le 437.5 \Rightarrow 1.78 \ge z \ge 0$,

– $\psi = 0.5 \Rightarrow 0.38 \le D_1 \le 750 \Rightarrow 2.67 \ge z \ge 0$.

The calculations show that the optimal values of $z$ for the I-section beam are very small for the lengths $l > 100$ cm. Because of that it is possible to say that the application of this criterion makes sense for the following lengths:

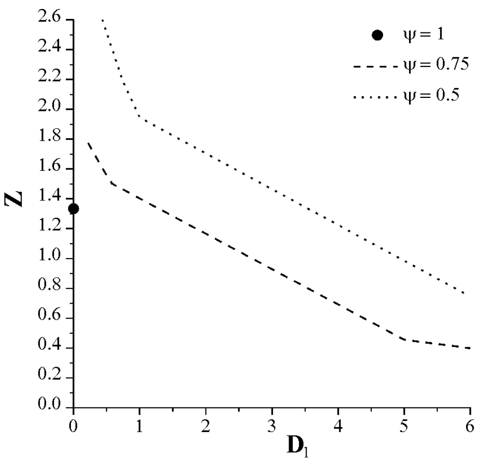
- for $\psi = 0.75$ : $l \approx 95$ cm $\Rightarrow z \geq 0.45$ and
- for $\psi = 0.5$ : $l \approx 90$ cm $\Rightarrow z \geq 0.51$ .

- Channel-section beam:

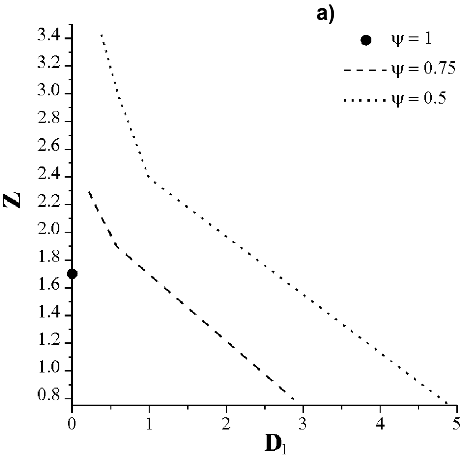
$- \psi = 1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.72$ ,

$- \psi = 0.75 \Rightarrow 0.22 \leq D_1 \leq 2.88 \Rightarrow 2.88 \geq z \geq 0.79$ ,

$- \psi = 0.5 \Rightarrow 0.38 \leq D_1 \leq 4.93 \Rightarrow 3.43 \geq z \geq 0.74$ .




a)



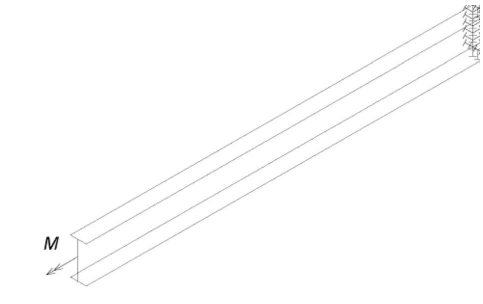

b)

**Figure 2. The optimal ratios *z***
**a) I-beam, b) Channel-section beam**

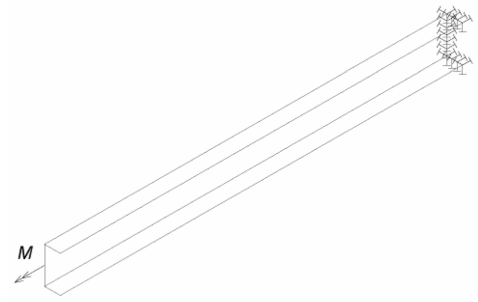
## 6. APPLICATION OF THE FINITE ELEMENT METHOD

Using the optimal values of *z*, obtained in the previous chapter by the $\theta_0^{`}$ criterion, the numerical calculation applying the Finite Element Method (FEM) was done.

As the numerical example, considered cantilever beam having the lengths $l = 100$ cm, fixed at one end and subjected to the concentrated torque $M = 10$ kNcm at its free end (Fig. 3), will be considered by the FEM using the software programme KOMIPS [8].




a)




b)

**Figure 3. Middle surface, load, supports**
**a) I- beam; b) Channel-section beam**

### 6.1 The results obtained by FEM

As the example for the numerical calculation, one I-beam section (I 10 - JUS C.B3.131) and one Channel section (U 10 - JUS C.B3.141) are considered. The problem is analyzed in three different ways [1]:

a) Taking into account the initial dimensions that represent the initial model the optimal relations $z_{\text{optimal}}$ are obtained from the expressions derived in this paper.

- I-beam section (I 10): For $b_{1\text{initial}} = b_{3\text{initial}} = 5$ cm, $b_{2\text{initial}} = 9.32$ cm , $t_1 = 0.68$ cm , $t_2 = 0.45$ cm, the initial ratio is $z_{\text{initial}} = 1.86$ . For the initial values $t_1$ and $t_2$ the optimal relation $z_{\text{optimal}} = 1.65$ is obtained.
- Channel section (U 10): For $b_{1\text{initial}} = 4.7$ cm, $b_{2\text{initial}} = 9.15$ cm , $t_1 = 0.85$ cm , $t_2 = 0.60$ cm , the initial ratio is $z_{\text{initial}} = 1.95$ . For the initial values $t_1$ and $t_2$ the optimal relation $z_{\text{optimal}} = 2.34$ is obtained.

b) The optimal dimensions of the cross section $b_{1\text{optimal}}$ and $b_{2\text{optimal}}$ are obtained by equalizing initial and optimal areas ( $A_{\text{initial}} = A_{\text{optimal}}$ ) and by using the calculated optimal relation $z_{\text{optimal}}$ (it represents the optimal model no. 1)

c) The optimal dimensions of the cross section $b_{1\text{optimal}}$ and $b_{2\text{optimal}}$ are obtained from the assumption $b_{2\text{optimal}} = b_{2\text{initial}}$ and by using the calculated optimal ratio $z_{\text{optimal}}$ (it represents the optimal model no. 2).

For each model the optimal values $z$ are calculated and the results are given in Table 3:

**Table 3. Optimal ratios z**

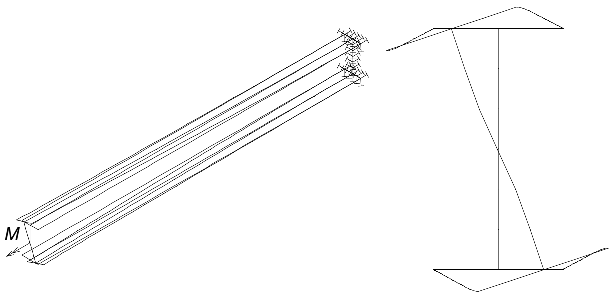
| Model | | z | |
|---|---|---|---|
| | | I-beam | U-beam |
| 1 | Initial | 1.86 | 1.95 |
| 2 | Optimal no.1 | 1.65 | 2.34 |
| 3 | Optimal no.2 | 1.65 | 2.34 |

The cross-sectional areas are also calculated and the results are given in Table 4:

**Table 4. Cross-sectional areas and angles of twist per unit length $\theta_0^{`}$**

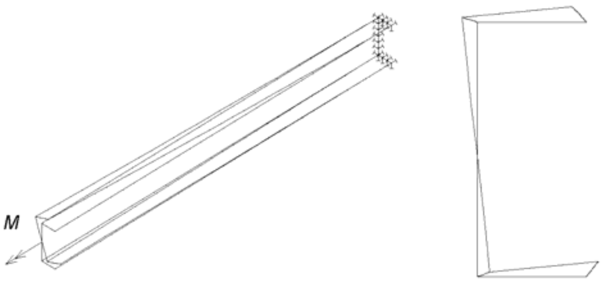
| Model | $A$ (cm$^2$) | | $\theta_0^{`}$ (°/m) | |
|---|---|---|---|---|
| | I-beam | U-beam | I-beam | U-beam |
| 1 | 10.99 | 13.49 | 5.05 | 2.68 |
| 2 | 10.99 | 13.49 | 4.79 | 2.25 |
| 3 | 11.88 | 12.14 | 3.64 | 2.68 |

Applying the FEM, the angle of twist per unit length $\theta_0^{`}$ is calculated for each model. The results obtained for the cantilever beam of the length $l = 100$ cm (Figs. 4 and 5), are also presented in Table 4.




a) b)

**Figure 4. I-beam deformations ($f_{max}$=0.4cm): (a) Isometric view, (b) xy – plane**




a) b)

**Figure 5. Channel-section beam deformations ($f_{max}$=0.334cm): (a) Isometric view, (b) xy – plane**

Results obtained by applying KOMIPS program (Table 4) correspond to the analytically obtained values for the initial model of the length $l$ = 100 cm:

- I-section beam:

$$\theta_{0\text{analytical}}^{`} = 5.15 \text{ °/m} \quad \text{and} \quad \theta_{0\text{KOMIPS}}^{`} = 5.05 \text{ °/m}.$$

- Channel section beam:

$$\theta_{0\text{analytical}}^{`} = 2.64 \text{ °/m} \quad \text{and} \quad \theta_{0\text{KOMIPS}}^{`} = 2.68 \text{ °/m}.$$

## 7. CONCLUSION

In this paper, one approach to the optimization of the thin-walled open-section beams, using the Lagrange multiplier method is presented. Accepting the cross-sectional area for the objective function and displacement constrains for the constrained functions, it is possible to find the optimal relation between the dimensions of the web and the flanges of the considered cross-section. First, the analytical calculations were done, and then the obtained results were used for the calculations applying the Finite Element Method.

Results obtained by the Finite Element Method show:

- I-section beam:

The initial and optimal model no. 1 (Table 4) have the same mass, but the optimal model no. 1 has lower angle of twist per unit length $\theta_0^{`}$. Optimal model no. 2 has the lowest value of the $\theta_0^{`}$, but this is the optimum model with the highest mass. This model is the best regarding the displacement constraints, but it is also the heaviest one.

- Channel-section beam:

The initial and optimal model no. 2, which have the minimum mass (Table 4), have the same angle of twist per unit length $\theta_0^{`}$. Initial and optimal model no. 1 have the same mass, but the optimal model no. 1 has the lower values of the angle of twist per unit length $\theta_0^{`}$.

As a conclusion, it is possible to say that all optimal models are better than the initial one. On the basis of the proposed optimization procedure, it is possible to calculate the optimal ratios between the parts of the considered thin-walled profiles in a simple way.

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## ЈЕДАН ПРИСТУП ОПТИМИЗАЦИЈИ ТАНКОЗИДИХ ОТВОРЕНИХ ПОПРЕЧНИХ ПРЕСЕКА ИЗЛОЖЕНИХ ОГРАНИЧЕНОЈ ТОРЗИЈИ

**Нина Анђелић**

У овом раду је разматрана оптимизација танкозидих конзолних конструкционих елемената отворених попречних пресека изложених ограниченој торзији. Циљ рада је одређивање минималне масе, тј, одређивање минималне површине попречног пресека танкозидих конструкционих елемената облика I и U-профила за задата оптерећења, материјал и геометријске карактеристике. Због тога је за функцију циља одабрана површина попречног пресека носача. За критеријум ограничења одабран је критеријум ограничења деформација. При формирању основног математичког модела пошло се од претпоставки теорије танкозидих штапова са једне стране и основних претпоставки проблема оптималног пројектовања са друге. Применом методе Лагранжовог множитеља изведене су једначине чија решења представљају оптималне односе димензија попречног пресека изабраног облика. Добијени резултати су искоришћени при нумеричком прорачуну применом Методе коначних елемената.