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AN INEQUALITY FOR THE LEBESGUE MEASURE

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In this paper we present a new inequality for the Lebesgue measure and give some of its applications.

If λ is the Lebesgue measure on the set of real numbers $\mathbb{R}$ and $\{A_n\}$ sequences of Lebesgue measurable sets in $\mathbb{R}$, then we have the following inequality: $\lambda\left(\underline{\lim}\, A_n\right) \le \underline{\lim}\, \lambda\left(A_n\right)$. But for the inequality $\overline{\lim}\, \lambda\left(A_n\right) \le \lambda\left(\overline{\lim}\, A_n\right)$ we must suppose that $\lambda(\cup_{i=n}^{\infty} A_n) < \infty$ for at least one value of n (see [**1**], p 40.). Example: for a family of intervals $I_n = [n, n+1)$ $n = 0, 1, \ldots$ we have: $\overline{\lim}\, \lambda\left(A_n\right) = 1$ and $\lambda\left(\overline{\lim}\, A_n\right) = 0$.

But in the general case we have the following inequality:

Proposition. *Let A be a measurable set of a positive measure and $\{x_n\}$ a bounded sequence of real numbers. Then*

$$\lambda(A) \le \lambda\left(\overline{\lim}\,(x_n + A)\right). \tag{1}$$

Proof. Let $K \subseteq A$ be a compact set. From $\overline{\lim}\,(x_n + K) \subseteq \overline{\lim}\,(x_n + A)$ follows $\lambda\left(\overline{\lim}\,(x_n + K)\right) \le \lambda\left(\overline{\lim}\,(x_n + A)\right)$. Now we have

$$\lambda(K) = \overline{\lim}\, \lambda\left((x_n + K)\right) \le \lambda\left(\overline{\lim}\,(x_n + K)\right) \le \lambda\left(\overline{\lim}\,(x_n + A)\right) \tag{2}$$

Since $\lambda(A) = \sup\{\lambda(K) \ : \ K \text{ is the compact subset of } A\}$, (2) implies (1).

Corollary 1. (H. Steinhaus [**2**]) *Let A be a Lebesgue measurable set of a positive measure. Then in A exist at least two points such that distance between them is a rational number.*

Proof. Let (q_n) be arbitrary bounded sequence of rational numbers, such that their member are different. From

$$0 < \lambda(A) \le \lambda\left(\overline{\lim}\,(A + q_n)\right)$$

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it follows that the set $\overline{\lim}\,(A+q_n)$ is nonempty. So, there exists $p_1, p_2 \in A$ and i, j such that $p_1 + q_i = p_2 + q_j$, which implies $|p_1 - p_2| = |q_i - q_j|$.

As a further practical application of the inequality (1) we give the following short and simple proof of the famous Steinhaus' result.

Corollary 2. (H. Steinhaus [**2**]) *Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set of a positive measure. Then its difference set $A - A = \{x \mid x = a_1 - a_2,\ a_1, a_2 \in A\}$ contains a neighborhood of zero.*

Proof. Assume that the statement is wrong. Then there exists compact set of positive measure $K \subseteq A$ such that the difference set $K - K$ does not contain a neighborhood of zero. It follows that there exists convergent sequence $\{x_n\} \subseteq \mathbb{R}$ such that $\lim x_n = 0$ and $\{x_n\} \cap (K - K) = \emptyset$. From

$$0 < \lambda(K) \leq \lambda\left(\overline{\lim}\,(K - x_n)\right)$$

it follows that the set $\overline{\lim}\,(K - x_n)$ is nonempty which implies that there exists $t \in \mathbb{R}$ such that $\{x_n + t\} \in K$ for infinitely many values of n. From $\lim x_n = 0$ it follows that $t \in K$, as K is a closed set. Thus we have that there exists infinite sequences $\{a_j\} \subseteq K$ and $\{x_{n_j}\} \subseteq \{x_n\}$ such that $x_{n_j} = a_j - t \in K - K$, which is a contradiction.

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