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FURTHER RESULTS ON NON - LYAPUNOV STABILITY OF THE LINEAR NONAUTONOMOUS SYSTEMS WITH DELAYED STATE

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Abstract. *Paper extends some basic results from the area of finite time and practical stability to linear, continuous, time invariant nonautonomous time-delay systems. Sufficient conditions of this kind of stability, for particular class of time-delay systems are derived.*

Keywords: *Linear analysis, Matrix methods, Stability criteria, Time delay*

1. INTRODUCTION

The problem of investigation of time delay system has been exploited over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time delay, regardless if it is present in the control or/and state, may cause undesirable system transient response, or generally, even an instability. Consequently, the problem of stability analysis of this class of systems has been one of the main interest of many researchers. In general, the introduction of time lag factors makes the analysis much more complicated. In the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria

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and generally provides nice algebraic conditions. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method, or on using idea of matrix measure, Lee and Diant (1981), Mori *et al*. (1981), Mori (1985), Hmamed (1986), Lee *et al*. (1986).

In practice one is not only interested in system stability (e.g. in the sense of Lyapunov), but also in bounds of system trajectories. A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined *a priori* in a given problem. Besides that, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval.

These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of system response. Thus, the analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concern.

Motivated by "brief discussion" on practical stability in the monograph of La Salle and Lefschet (1961), Weiss and Infante (1965, 1967) have introduced various notations of stability over finite time interval for continuous-time systems and constant set trajectory bounds. Further development of these results were due to many other authors, Michel (1970), Grujic (1971), Lashirer and Story (1972). Practical stability of simple and interconnected systems with respect to time-varying subsets was considered by Michel (1970) and Grujic (1975.a). A more general type of stability ("practical stability with settling time", practical exponential stability, etc.) which includes many previous definitions of finite stability was introduced and considered by Grujic (1971, 1975.b, 1975.c). Concept of finite-time stability, called "final stability", was introduced by Lashirer and Story (1972) and further development of these results was due to Lam and Weiss (1974).

In the context of practical stability for linear generalized state-space systems, various results were first obtained in Debeljkovic and Owens (1985) and Owens and Debeljkovic (1986). Analysis of nonlinear singular and implicit dynamic systems in terms of the generic qualitative and quantitative concepts, which contain technical and practical stability types as special cases, have been introduced and studied in Bajic (1988, 1992).

In this short overview, the results in the area of finite and practical stability were only concerned for continuous time systems.

Here we examine the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of nonautonomous time-delay systems. To the best knowledge of authors, these problems, using this approach, are not yet analyzed for the time-delay systems and this class of systems.

2. NOTATION AND PRELIMINARIES

A linear, multivariable time-delay system can be represented by differential equation:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau), \tag{1}$$

and with associated function of initial state:

$$\begin{aligned} \mathbf{x}(t) &= \boldsymbol{\psi}_x(t)\ , \\ -\tau &\le t \le 0\ , \\ \mathbf{u}(t) &= \boldsymbol{\psi}_u(t)\ . \end{aligned} \tag{2}$$

Equation (1) is referred to as nonhomogenous or the unforced state equation, $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ control vector), A_0, A_1, B_0 and B_1 are constant system matrices of appropriate dimensions, and τ is pure time delay, $\tau = const.$ ($\tau > 0$).

Dynamical behavior system (1) with initial functions (2) is defined over time interval $J = \{t_0, t_0 + T\}$, where quantity T may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously. It is obvious that $J \in R$.

Time invariant sets, used as a bounds of system trajectories, are assumed to be open, connected and bounded. Let index β stands for the set of all allowable states of system and index α for the set of all initial states of the system, such that the set $S_\alpha \subseteq S_\beta$. In general, one may write:

$$S_\rho = \{\mathbf{x} : \| \mathbf{x} \|_Q^2 < \rho\}\ ,\ \rho \in [\alpha, \beta] \tag{3}$$

where Q will be assumed to be symmetric, positive-definite, real matrix.

S_ε denotes the set of all allowable control actions.

Let $|\mathbf{x}|_{(\cdot)}$ be any vector norm (e.g., $\cdot = 1, 2, \infty$) and $\|(\cdot)\|$ the matrix norm induced by this vector. Here, we use $|\mathbf{x}|_2 \overset{\Delta}{=} (\mathbf{x}^T\mathbf{x})^{1/2}$ and $\|(\cdot)\|_2 = \lambda_{\max}^{1/2}(A^*A)$. Upper indices * and T denote transpose conjugate and transpose, respectively.

Matrix measure has been widely used in the literature when dealing with stability of time delay systems. The matrix measure μ for any matrix $A \in \mathbb{C}^{n\times n}$ is defined as follows

$$\mu(A) \overset{\Delta}{=} \lim_{\varepsilon\to 0} \frac{\| I + \varepsilon A \| - 1}{\varepsilon}\ . \tag{4}$$

The matrix measure defined in (4) can be subdefined in three different ways, depending on the norm utilized in its definitions.

$$\mu_1(A) = \max_k \left(\mathrm{Re}(a_{kk}) + \sum_{\substack{i=1 \\ i\ne k}}^{n} | a_{ik} | \right), \tag{5}$$

$$\mu_2(A) = \frac{1}{2} \max_i \lambda_i (A^* + A)\ , \tag{6}$$

and

$$\mu_\infty(A) = \max_i \left(\mathrm{Re}(a_{ii}) + \sum_{\substack{k=1 \\ k\ne i}}^{n} | a_{ki} | \right), \tag{7}$$

Coppel (1965), or Desoer and Vidysagar (1975).

3. Previous results

Definition 1: System given by (1) satisfying initial condition (2) is finite time stable w.r.t. [$\zeta(t)$, β, τ, T] if and only if:

$$|\boldsymbol{\psi}_x(t)|_2 < \zeta(t), \tag{8}$$

implies:

$$|\mathbf{x}(t)|_2 < \beta\ , \tag{9}$$

$\zeta(t)$ being scalar function with the property $0 < \zeta(t) \leq \alpha$, $-\tau \leq t \leq 0$, where α is a real positive number and $\beta \in R$ and $\beta > \alpha$, with $\boldsymbol{\Psi}_x(t) = \mathbf{x}(t)$, $\forall t \in [-\tau, 0]$.

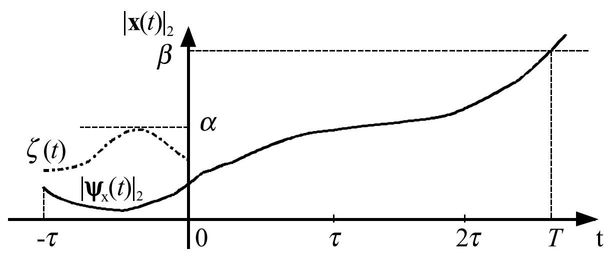

Fig. 1. Illustration of preceding definition.

Theorem 1: The system given by (1) with the initial function (2) is finite time stable with respect to {α, β, τ, T}, if the following condition is satisfied:

$$\|\Phi\|_2 < \frac{\sqrt{\beta/\alpha}}{1+\tau\|A_1\|_2}, \quad \forall t \in [0, T], \tag{10}$$

where $\|(\cdot)\|$ is Euclidean norm and $\Phi(t)$ is fundamental matrix of system (1), Nenadic *et al*. (1997), Debeljkovic *et al*. (1997a).

Theorem 2: The system given by (1) with initial function (2) is finite time stable w.r.t. {α, β, τ, T } if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\sqrt{\beta/\alpha}}{1+\tau\|A_1\|_2}, \quad \forall t \in [0, T] \tag{11}$$

where $\|(\cdot)\|$ denotes Euclidean norm, Debeljkovic *et al*. (1997b).

Theorem 3: The autonomous system given by (1) with the initial function (2) is finite time stable with respect to {α, β, τ, T}, if the following condition is satisfied:

$$e^{\mu_2(A_0)t} < \frac{\beta/\alpha}{1+\mu_2^{-1}(A_0)\|A_1\|_2\,(1-e^{-\mu_2(A_0)\tau})}, \quad \forall t \in [0,T] \quad \text{Debeljkovic } et\ al.\ (1997.c). \tag{12}$$

Theorem 4: The autonomous system given by (1) with the initial function (2) is finite time stable with respect to $\{\sqrt{\alpha}, \sqrt{\beta}, \tau, T, \mu(A_0) \neq 0\}$ if the following condition is satisfied:

$$1 + \|A_1\|_2\ \tau < \beta/\alpha, \tag{13}$$

Debeljkovic *et al*. (1997.c).

Definition 2: System given by (1), with $\mathbf{u}(t-\tau)\equiv 0$, $\forall t$, satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t), \beta, \varepsilon, \tau, J, \mu(A_0)\neq 0\}$ if and only if:

$$\psi_x(t)\in S_\alpha\,,\ \forall t\in[-\tau,0]\,, \tag{14}$$

and

$$\mathbf{u}(t)\in S_\varepsilon\ \ \forall t\in J, \tag{15}$$

imply:

$$\mathbf{x}(t;t_0,\mathbf{x}_0)\in S_\beta\,,\ \forall t\in[0,T] \tag{16}$$

Theorem 5: System given by (1), with initial function (2) is finite time stable w.r.t. $\{\zeta(t), \beta, \varepsilon, \tau, J, \mu(A_0)\neq 0, B_1=0\}$ if the following condition is satisfied:

$$e^{\mu_2(A_0)t}<\frac{\beta/\alpha}{\theta}$$

$$\theta=\mu^{-1}(A_0)(\mu(A_0)+\|A_1\|_2\,(1-e^{-\mu_2(A_0)\tau}))+\mu^{-1}(A_0)\gamma\|B_0\|_2\,(1-e^{-\mu_2(A_0)t}),\quad \forall t\in J. \tag{17}$$

where:

$$\gamma=\varepsilon/\alpha,\quad \mu_2(A_0)=0.5\lambda_{\max}(A_0+A_0^T)\,, \tag{18}$$

Debeljkovic *et al*. (1997.d).

Theorem 6: System given by (1), with $\mathbf{u}(t-\tau)\equiv 0$, $\forall t$, satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t), \beta, \varepsilon, \tau, J, \mu(A_0)\neq 0\}$, if following condition is satisfied:

$$(1+\tau\|A_1\|_2)+\gamma\|B_0\|_2\,t<\beta/\alpha\,,\ \forall t\in J, \tag{19}$$

where γ is given with (18), Debeljkovic *et al*. (1997.d).

Definition 3: System given by (1) satisfying initial condition (2) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_\psi, \varepsilon, \tau, J, \mu(A_0)\neq 0\}$ if and only if

$$\psi_x(t)\in S_\alpha\,,\ \forall t\in[-\tau,0], \tag{20}$$

$$\psi_u(t)\in S_{\varepsilon_\psi}\,,\ \forall t\in[-\tau,0], \tag{21}$$

$$\mathbf{u}(t)\in S_\varepsilon\,,\ \forall t\in J, \tag{22}$$

imply:

$$\mathbf{x}(t,t_0,\mathbf{x}_0,\mathbf{u}(t))\in S_\beta,\ \forall t\in J. \tag{23}$$

Theorem 7: System given by (1), with initial function (2) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_\psi, \varepsilon, \tau, J, \mu(A_0)\neq 0\}$ if the following condition is satisfied:

$$\mu_2^{-1}(A_0)\cdot e^{\mu_2(A_0)t}<\frac{\beta}{\alpha}\cdot\delta^{-1}\,, \tag{24}$$

where:

$$\delta=a_1(\mu_2(A_0)\cdot a_1^{-1}+(1-e^{-\mu_2(A_0)\tau})\cdot C_1+(1-e^{-\mu_2(A_0)t})\cdot C_2) \tag{25}$$

$$c_1=1+b_1(\gamma+\gamma_\psi) \tag{26}$$

$$c_2=\gamma(b_0+b_1) \tag{27}$$

$$a_1=\|A_1\|,\ \ b_1=\frac{\|B_0\|}{a_1},\ \ b_0=\frac{\|B_0\|}{a_1} \tag{28}$$

$$\gamma = \frac{\varepsilon}{\alpha},\ \ \gamma_\psi = \frac{\varepsilon_\psi}{\alpha}, \tag{29}$$

Debeljkovic *et al*. (1998).

4. Main results

Equation (2) can be rewritten in it's general form as:

$$\mathbf{x}(t_0+\theta) = \boldsymbol{\psi}_x(t_0+\theta), \quad t_0-\tau \le \theta \le 0, \tag{30}$$

$$\mathbf{u}(t_0+\theta) = \boldsymbol{\psi}_u(t_0+\theta), \quad t_0-\tau \le \theta \le 0, \tag{31}$$

with:

$$\boldsymbol{\psi}_x(t_0+\theta) \in C[t_0-\tau, 0], \tag{32}$$

$$\boldsymbol{\psi}_u(t_0+\theta) \in C[t_0-\tau, 0], \tag{33}$$

where t_0 is observation initial time of the system, given by (1), and $C[(\cdot)]$ is a Banach space of continuous functions over a time interval $[t-\tau, t]$ into R^n with the norm defined in the following manner:

$$\| \boldsymbol{\psi} \|_C = \max_{t_0-\tau\le\theta\le t_0} \| \boldsymbol{\psi}(t_0+\theta) \|. \tag{34}$$

It is assumed that the usual smoothness condition are present so there is no difficulty with questions of existence, uniqueness and continuity of solutions with respect to initial data.

Definition 4: System given by (1) satisfying initial conditions (30) and (31) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_0, \varepsilon, t_0, J\}$, $\alpha < \beta$, if and only if:

$$\| \boldsymbol{\psi}_x \|_C < \alpha,\ \| \boldsymbol{\psi}_u \|_C < \varepsilon_0, \tag{35}$$

$$\| \mathbf{u}(t) \| < \varepsilon, \quad \forall t \in J, \tag{36}$$

imply:

$$\| \mathbf{x}(t) \| < \beta, \quad \forall t \in J. \tag{37}$$

Theorem 8: System given by (1) satisfying initial conditions (30) and (31) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_0, \varepsilon, t_0, J\}$, $\alpha < \beta$, if the following condition is satisfied:

$$(1+(t-t_0)\sigma_{\max}^A)\, e^{\sigma_{\max}^A (t-t_0)} + \gamma_1^*(t-t_0) + \gamma_0^*\tau \le \beta/\alpha, \quad \forall t \in J. \tag{38}$$

where:

$$\gamma_1^* = \gamma_1/\alpha,\ \ \gamma_0^* = \gamma_0/\alpha,\ \ \gamma_1 = \frac{(b_1+b_0)}{\alpha}$$

$$\| B_0 \| = b_0,\ \ \| B_1 \| = b_1,\ \ \gamma_0 = \frac{\varepsilon_0 - \varepsilon}{\alpha} \cdot b_1$$

$$\sigma_{\max}^A = \sigma_{\max}(A_0) + \sigma_{\max}(A_1) \tag{39}$$

Proof: In accordance with the property of the norm, one can immediately write:

$$\begin{aligned}\|\dot{\mathbf{x}}(t)\| &= \|A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)+B_0\mathbf{u}(t)+B_1\mathbf{u}(t-\tau)\| \\ &\leq \|A_0\|\|\mathbf{x}(t)\|+\|A_1\|\|\mathbf{x}(t-\tau)\| \\ &\quad +\|B_0\|\|\mathbf{u}(t)\|+\|B_1\|\|\mathbf{u}(t-\tau)\|\end{aligned} \tag{40}$$

Employing the following inequality:

$$\|\mathbf{x}(t-\tau)\| \leq \sup_{t-\tau\leq t^*\leq t}\|\mathbf{x}(t^*)\|, \tag{41}$$

equation (40) can be written in the following manner:

$$\begin{aligned}\|\dot{\mathbf{x}}(t)\| &\leq \sigma_{\max}(A_0)\|\mathbf{x}(t)\|+\sigma_{\max}(A_1)\sup_{t-\tau\leq t^*\leq t}\|\mathbf{x}(t^*)\| \\ &\quad +\|B_0\|\cdot\|\mathbf{u}(t)\|+\|B_1\|\cdot\|\mathbf{u}(t-\tau)\|, \\ &\quad t>t_0+\tau\end{aligned} \tag{42}$$

or:

$$\begin{aligned}\|\dot{\mathbf{x}}(t)\| &\leq \sigma_{\max}^{A}\Big(\sup_{t-\tau\leq t^*\leq t}\|\mathbf{x}(t^*)\|+\|\boldsymbol{\psi}_x\|_C\Big) \\ &\quad +\|B_0\|\cdot\|\mathbf{u}(t)\|+\|B_1\|\cdot\|\mathbf{u}(t-\tau)\|, \\ &\quad t>t_0,\end{aligned} \tag{43}$$

where $\sigma_{\max}(A_i)$ denotes maximal sigular value of matrix A_i, $i = 1, 2$.

To obtain the final result, one has to integrate equation (1), so:

$$\int_{t_0}^{t}\dot{\mathbf{x}}(t)dt=\int_{t_0}^{t}\big(A_0\mathbf{x}(s)+A_1\mathbf{x}(s-\tau)\big)ds+\int_{t_0}^{t}\big(B_0\mathbf{u}(s)+B_1\mathbf{u}(s-\tau)\big)ds \tag{44}$$

and combining with (43), it is obvious that:

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq \|\mathbf{x}(t_0)\| \\ &\quad +\int_{t_0}^{t}\|A_0\|\cdot\|\mathbf{x}(s)\|ds+\int_{t_0}^{t}\|A_1\|\cdot\|\mathbf{x}(s-\tau)\|ds \\ &\quad +\int_{t_0}^{t}\|B_0\|\cdot\|\mathbf{u}(s)\|ds+\int_{t_0}^{t}\|B_1\|\cdot\|\mathbf{u}(s-\tau)\|ds\end{aligned} \tag{45}$$

and:

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq \|\mathbf{x}(t_0)\|+\sigma_{\max}^{A}\int_{t_o}^{t}\Big(\sup_{t-\tau\leq t^*\leq t}\|\mathbf{x}(t^*)\|+\|\boldsymbol{\psi}_x\|_C\Big)ds \\ &\quad +\|B_0\|\int_{t_0}^{t}\|\mathbf{u}(s)\|ds+\|B_1\|\int_{t_0-\tau}^{t_0}\|\boldsymbol{\psi}_u\|_C\,ds \\ &\quad +\|B_1\|\int_{t_0}^{t-\tau}\|\mathbf{u}(s)\|ds,\end{aligned} \tag{46}$$

or

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_0)\| + \sigma_{\max}^{A} \int_{t_o}^{t} \Big(\sup_{t-\tau \leq t^* \leq t} \|\mathbf{x}(t^*)\| + \|\boldsymbol{\psi}_x\|_C \Big) ds$$
$$+ (\|B_0\| + \|B_1\|)\varepsilon \cdot (t - t_0) + (\varepsilon_0 - \varepsilon)\|B_1\| \cdot \tau. \tag{47}$$

Since:

$$\|\mathbf{x}(t_0)\| = \|\boldsymbol{\psi}_x(0)\| \leq \|\boldsymbol{\psi}_x\|_C, \tag{48}$$

it yields:

$$\|\mathbf{x}(t)\| \leq \|\boldsymbol{\psi}_x\|_C \cdot (1 + \sigma_{\max}^{A}(t - t_0))$$
$$+ \sigma_{\max}^{A} \int_{t_0}^{t} \sup_{s-\tau \leq t^* \leq s} \|\mathbf{x}(t^*)\| ds + (b_1 + b_0)\varepsilon(t - t_0) + (\varepsilon_0 - \varepsilon) b_1 \tau. \tag{49}$$

Obviously:

$$\rho(t) = \|\boldsymbol{\psi}_x\|_C \cdot (1 + \sigma_{\max}^{A}(t - t_0)) \tag{50}$$

is nondecreasing function, so one can write:

$$\sup_{s-\tau \leq t^* \leq s} \|\mathbf{x}(t^*)\| \leq \rho(t) + \sigma_{\max}^{A} \int_{t_0}^{t} \sup_{s-\tau \leq t^* \leq s} \|\mathbf{x}(t^*)\| ds \tag{51}$$

If one applies very well known Bellman-Gronwall lemma, Hale (1971), it is easy to show that:

$$\|\mathbf{x}(t)\| \leq \sup_{s-\tau \leq t^* \leq s} \|\mathbf{x}(t)\| \leq \rho(t) e^{\sigma_{\max}^{A}(t - t_0)} \tag{52}$$

and:

$$\|\mathbf{x}(t)\| \leq \alpha(1 + (t - t_0)\sigma_{\max}^{A}) e^{\sigma_{\max}^{A}(t - t_0)} + (b_1 + b_0)\varepsilon(t - t_0) + (\varepsilon_0 - \varepsilon) b_1 \tau. \tag{53}$$

Finally, if one use the basic condition of Theorem 8, namely equation (38), it yields:

$$\|\mathbf{x}(t)\| < \beta, \quad \forall t \in J, \tag{54}$$

what had to be proved.

Prior to our another main result, we present the following lemma.

Lemma 1. Let $Q(t)$ be a $n \times n$ characteristic matrix for the system given by (1) with initial function (2), also continuous and differentiable in $[0, \tau]$ and zero elsewhere. Define the following vector:

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_{0}^{\tau} Q(\theta)\mathbf{x}(t - \theta) d\theta, \tag{55}$$

where matrix $Q(t)$ satisfy the following matrix equation:

$$\dot{Q}(\theta) = (A_0 + Q(0)) \cdot Q(\theta), \quad \theta \in [0, \tau], \tag{56}$$

with boundary value:

$$Q(\tau) = A_1. \tag{57}$$

If:

$$V(\mathbf{y}(t)) = \mathbf{y}^T(t)\mathbf{y}(t) \tag{58}$$

is aggregation function for system given by (1), then:

$$\dot{V}(\mathbf{y}(t)) = \mathbf{y}^T(t)(-R)\mathbf{y}(t)\,, \tag{59}$$

where:

$$-R = (A_0 + Q(0))^T + (A_0 + Q(0))\,, \tag{60}$$

The proof is omitted, for the sake of brevity and can be found in Lee and Diant (1981).

Now, we can state our another main result.

Theorem 8. If λ_M is maximal eigenvalue of matrix $(-R)$ being defined by (60), then:

$$\int_0^{\tau} \| Q(\theta)\mathbf{x}(t-\theta)\| \, d\theta \le \| Q(0)\| \cdot \int_0^{\tau} e^{\frac{\lambda_M}{2}\theta} \, \| \mathbf{x}(t-\theta)\| \, d\theta \tag{61}$$

Proof. If we introduce $p = t - \theta$, then:

$$\int_0^{\tau} \| Q(\theta)\mathbf{x}(t-\theta)\| \, d\theta = \int_t^{t-\tau} \| Q(t-p)\mathbf{x}(p)\| \, (-dp) = \int_{t-\tau}^{t} \| Q(t-p)\mathbf{x}(p)\| \, dp. \tag{62}$$

Let:

$$\mathbf{q}(t,p) = Q(t-p)\mathbf{x}(p) = Q(t,p)\mathbf{x}(p)\,. \tag{63}$$

Then:

$$\frac{\partial}{\partial t}[\mathbf{q}(t,p)] = \frac{\partial}{\partial t}[Q(t,p)]\mathbf{x}(p) = [A_0 + Q(0)]Q(t,p)\mathbf{x}(p) = [A_0 + Q(0)]\mathbf{q}(t,p)\,, \tag{64}$$

$$\frac{\partial}{\partial t}[\mathbf{q}^T(t,p)\mathbf{q}(t,p)] = \mathbf{q}^T(t,p)[A_0 + Q(0)]^T\mathbf{q}(t,p) + \mathbf{q}^T(t,p)[A_0 + Q(0)]\mathbf{q}(t,p), \tag{65}$$

$$\begin{aligned}\frac{\partial}{\partial t}[\mathbf{q}^T(t,p)\mathbf{q}(t,p)] &= \mathbf{q}^T(t,p)\{[A_0 + Q(0)]^T \\ &+ [A_0 + Q(0)]\}\mathbf{q}(t,p) = \mathbf{q}^T(t,p)(-R)\mathbf{q}(t,p),\end{aligned} \tag{66}$$

$$\frac{\partial}{\partial t}[\mathbf{q}^T(t,p)\mathbf{q}(t,p)] \le \lambda_M \mathbf{q}^T(t,p)\mathbf{q}(t,p)\,, \tag{61*}$$

$$\frac{d[\mathbf{q}^T(t,p)\mathbf{q}(t,p)]}{\mathbf{q}^T(t,p)\mathbf{q}(t,p)} \le \lambda_M \, dt\,. \tag{62*}$$

If one integrate the previous inequality we get:

$$\mathbf{q}^T(t,p)\mathbf{q}(t,p) \le \mathbf{q}^T(p,p)\mathbf{q}(p,p)e^{\lambda_M(t-p)}\,. \tag{63*}$$

It follows:

$$\| \mathbf{q}(t,p)\|^2 \le \| \mathbf{q}(p,p)\|^2 \, e^{\lambda_M(t-p)}\,. \tag{64*}$$

$$\mathbf{q}(p,p) = Q(p,p)\mathbf{x}(p) = Q(p-p)\mathbf{x}(p) = Q(0)\mathbf{x}(p)\,. \tag{65*}$$

Finally if one make square root on both sides of eq. (44), one can get:

$$\| \mathbf{q}(t,p)\| \le \| Q(0)\mathbf{x}(p)\| \, e^{\frac{\lambda_M}{2}(t-p)} \le \| Q(0)\| \| \mathbf{x}(p)\| \, e^{\frac{\lambda_M}{2}(t-p)} \tag{66*}$$

$$\int_{t-\tau}^{t} \| Q(t-p)\mathbf{x}(p) \| \, dp \leq \| Q(0) \| \int_{t-\tau}^{t} \| \mathbf{x}(p) \| \, e^{\frac{\lambda_M}{2}(t-p)} = \| Q(0) \| \int_{\tau}^{0} \| \mathbf{x}(t-\theta) \| \, e^{\frac{\lambda_M}{2}\theta} (-d\theta), \quad (67)$$

and:

$$\int_{0}^{\tau} \| Q(\theta)\mathbf{x}(t-\theta) d\theta \leq \| Q(0) \| \int_{0}^{\tau} \| \mathbf{x}(t-\theta) \| \, e^{\frac{\lambda_M}{2}\theta} d\theta , \quad (68)$$

what ends the proof.

5. CONCLUSION

The matrix measure has been widely used in the literature dealing with stability and asymptotic stability of time-delay systems. This approach and Bellman-Gronwall lemma have been used here in order to develop some results which have an evident advantage to those derived earlier, since they overcome need of computing fundamental matrix for time-delay systems. In that sense, delay dependent criteria expressed by a simple inequalities, have been derived yielding sufficient conditions of non-Lyapunov stability of system considered. These results have been obtained using quite different techniques from approaches used by some other authors working in the area of finite and practical stability.These results can be directly applied to all chemical processes that posseses pure time delay, nuber of which is realy graet.

To the best knowledge of authors, these problems have not yet been analyzed for this class of nonautonomous linear time-delay systems, and represent natural extension of results presented earlier in Lazarevic *et al*. (2000). Moreover, a new theorem has been proved that enables one to apply very well Bellman-Gronwall lemma for the time delay systems.

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NOVI REZULTATI O NE-LJAPUNOVSKOJ STABILNOSTI LINEARNIH NEAUTONOMNIH SISTEMA SA KAŠNJENJEM

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Ovaj rad predstavlja dalji rad na osnovnim rezultatima u oblasti ograničenog vremena i praktične stabilnosti linearnih, kontinualnih stacionarnih, neautonomnih sistema sa kašnjenjem. Izdvojeni su dovoljni uslovi ovog tipa stabilnosti za određenju klasu sistema sa kašnjenjem.