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# On Learning the Structure of Clusters in Graphs 

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#### Abstract

Graph clustering is a fundamental problem in unsupervised learning, with numerous applications in computer science and in analysing real-world data. In many real-world applications, we find that the clusters have a significant high-level structure. This is often overlooked in the design and analysis of graph clustering algorithms which make strong simplifying assumptions about the structure of the graph. This thesis addresses the natural question of whether the structure of clusters can be learned efficiently and describes four new algorithmic results for learning such structure in graphs and hypergraphs.

The first part of the thesis studies the classical spectral clustering algorithm, and presents a tighter analysis on its performance. This result explains why it works under a much weaker and more natural condition than the ones studied in the literature, and helps to close the gap between the theoretical guarantees of the spectral clustering algorithm and its excellent empirical performance.

The second part of the thesis builds on the theoretical guarantees of the previous part and shows that, when the clusters of the underlying graph have certain structures, spectral clustering with fewer than $k$ eigenvectors is able to produce better output than classical spectral clustering in which $k$ eigenvectors are employed, where $k$ is the number of clusters. This presents the first work that discusses and analyses the performance of spectral clustering with fewer than $k$ eigenvectors, and shows that general structures of clusters can be learned with spectral methods.

The third part of the thesis considers efficient learning of the structure of clusters with local algorithms, whose runtime depends only on the size of the target clusters and is independent of the underlying input graph. While the objective of classical local clustering algorithms is to find a cluster which is sparsely connected to the rest of the graph, this part of the thesis presents a local algorithm that finds a pair of clusters which are densely connected to each other. This result demonstrates that certain structures of clusters can be learned efficiently in the local setting, even in the massive graphs which are ubiquitous in real-world applications.

The final part of the thesis studies the problem of learning densely connected clusters in hypergraphs. The developed algorithm is based on a new heat diffusion process, whose analysis extends a sequence of recent work on the spectral theory of hypergraphs. It allows the structure of clusters to be learned in datasets modelling higher-order relations of objects and can be applied to efficiently analyse many complex datasets occurring in practice.

All of the presented theoretical results are further extensively evaluated on both synthetic


and real-word datasets of different domains, including image classification and segmentation, migration networks, co-authorship networks, and natural language processing. These experimental results demonstrate that the newly developed algorithms are practical, effective, and immediately applicable for learning the structure of clusters in real-world data.

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Thanks to Granny for permission to use her photo in the introduction. I extend to her all my love and congratulations on the occasion of her 90th birthday on 17th August 2022.

## Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

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## Chapter 1



## Introduction

Graphs are a natural tool to represent many datasets and have proved invaluable in many diverse areas of data science such as recommendation systems [ABC+ 08, AM07], image segmentation [SM00, SZ19, ZZL+21], and bioinformatics [HdK21, GPP ${ }^{+}$22]. One of the most fundamental operations in these applications is graph clustering, in which the goal is to partition the vertices of a graph into disjoint clusters. A cluster is usually defined to be a set of vertices in the graph such that there are many edges connecting vertices inside the cluster, while there are few edges connecting the cluster to the rest of the graph. However, this definition fails to tell the whole story. In many real-world applications, we find that the clusters also have a significant high-level structure which is often overlooked in the design and analysis of graph clustering algorithms. Throughout this thesis, we will discuss several methods for learning this high-level structure of clusters, and for exploiting this structure to develop improved algorithms for graph clustering.

In order to better understand the structure of clusters in graphs, let us consider some examples constructed from real-world data. We begin by considering the well-known MNIST dataset which consists of 70,000 images of handwritten digits from 0 to 9 . Each image has $28 \times 28$ greyscale pixels, and each pixel takes a value from 0 to 255 . This dataset is not given to us as a graph, but we can employ a common tool to convert almost any generic dataset into a graph: the $k$ nearest neighbour graph. To do this, we consider each image in the dataset to be a vertex in the graph, and connect each vertex $v$ to the vertices representing the images which are the 'most similar' to the image represented by $v$. This construction is described more formally in Chapter 5. Figure 1.1 illustrates the $k$ nearest neighbour graph for a randomly chosen subset of the data. Notice that in the constructed graph there are 10 clusters, which correspond to the ground truth clusters of images representing each digit. A vertex in a given cluster tends to be more strongly connected with other vertices inside the same cluster than with vertices in other clusters. There is also additional high-level structure


Figure 1.1: (a) Example images from the MNIST dataset. (b) The $k$ nearest neighbour graph of a random subset of the MNIST dataset, for $k=3$. The graph has a cluster corresponding to each digit.
due to the relative similarity of different digits. For example, the cluster representing the digit 4 has more connections with the cluster representing 9 than the one representing 6 . This is the high-level structure which has been overlooked in previous clustering algorithms, and we study such structures in more detail in Chapter 5 .

We will also look at another learning task for image data: image segmentation. In this case, we are given an arbitrary image, and the goal is to partition the pixels into some number of clusters such that each cluster corresponds to a different region of the image. This time, we will construct a graph from a single image in the following way: we first represent each pixel in the image by a point $(r, g, b, x, y)^{\top} \in \mathbb{R}^{5}$ where $r, g, b \in[1,255]$ are the red, green, and blue values of the pixel and $x$ and $y$ are the coordinates of the pixel in the image. Then, we construct the similarity graph by taking each pixel to be a vertex in the graph, and for every pair of pixels $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{5}$, we add an edge with weight $\exp \left(-\|\boldsymbol{u}-\boldsymbol{v}\|^{2} / 2 \sigma^{2}\right)$ for some parameter $\sigma$. Notice that this graph combines the spatial and colour information of each pixel, such that we might expect each region of the image to be a separate cluster in the graph. We can then use a graph clustering algorithm to find these regions and segment the image. Once again, we can reasonably expect that the clusters in the graph constructed from the image will have some high-level structure. For example, a cluster representing a given region of the image will be more closely related to clusters of pixels with similar colours, or clusters representing neighbouring regions. We will see in Chapter 5 that this structure of clusters allows us to develop an improved graph-based image segmentation algorithm than the ones previously studied in the literature, and the results of this algorithm are also shown in Figure 1.2

Finally, we will consider a completely different example dataset in which we are interested in

(a) Original Image

(b) Segmented Image

Figure 1.2: An example of image segmentation using the clustering algorithm in Chapter 5
learning the structure of clusters in a graph. The interstate dispute dataset records every military dispute between pairs of countries during 1816-2010, including the level of hostility resulting from the dispute and the number of casualties if applicable. This dataset already has a natural graph structure: for a given time period, we can construct a graph from the data by representing each country with a vertex and adding an edge between each pair of countries weighted according to the severity of their military disputes. Notice that since the edges in this graph represent a 'negative' relationship between countries, we would expect that a cluster of allied countries would not contain many edges inside the cluster. On the other hand, two different clusters might be densely connected to each other when there are two groups of opposing countries. If we wish to develop an algorithm for finding the clusters in this graph, clearly we must take the high-level structure into account and most existing graph clustering algorithms will not perform well. In Chapter 6, we will develop an algorithm capable of learning the clusters in this dataset which correspond to the underlying geopolitical relationships between countries. Figure 1.3 demonstrates that from this dataset, our algorithm is able to identify changes in geopolitics over time, including the changing roles of Russia, Japan, and Eastern Europe during the 20th century.

Although our highlighted examples come from different domains and the method used to construct a graph is different in each case, their common theme is that the clusters in the generated graphs have a significant high-level structure. In this thesis we will see that these structures can be learned efficiently through our newly developed algorithms and that the cluster structure can be used to significantly improve the performance of spectral clustering. Throughout the thesis, we will touch on a wide variety of techniques from the field of algorithmic spectral graph theory, which studies the matrix representations of graphs and applies the algebraic properties of these matrices to design efficient algorithms. Along the way, we will examine the classical spectral clustering algorithm (Chapters 4 and 5), random


Figure 1.3: Clusters found by the algorithm from Chapter 6 on the interstate dispute dataset. For any given time period and using the USA as a 'seed', the algorithm returns two densely connected clusters representing groups of allied countries. The results are well explained by geopolitics, and capture the changing political picture during the 20th century.
walks and local algorithms (Chapter 6), and the recently emerging spectral hypergraph theory based on non-linear heat diffusion operators (Chapter 7). The newly developed techniques are all extensively evaluated on real-world datasets including those introduced above, and the results demonstrate a wide applicability for learning the structure of clusters in natural datasets. The remainder of the thesis is organised as follows.

- Chapter 2 introduces the basic mathematical concepts and notation which is used throughout the thesis, including a brief introduction to linear algebra, graphs, and hypergraphs. Additionally, this chapter introduces the field of spectral graph theory, on which most of the results in this thesis are based.
- Chapter 3 describes the previous work which is most related to the methods and results presented in later chapters. This includes a detailed description of the history of spectral graph partitioning, techniques for local graph clustering, and spectral hypergraph theory. This chapter forms an important reference in order to understand the technical material of the thesis.
- Chapter 4 studies spectral clustering, and presents a tighter analysis on its performance. This result explains why spectral clustering works under a much weaker and more natural condition than the ones previously studied in the literature, and helps to close the gap between the theoretical guarantees of the spectral clustering algorithm and its excellent empirical performance. This chapter is largely based on Sections 3 and 4 of Macgregor and Sun [MS22].
- Chapter 5 builds on the theoretical guarantees of the previous chapter and shows that, when the clusters of the underlying graph have certain structures, spectral clustering with fewer than $k$ eigenvectors is able to produce better output than classical spectral clustering in which $k$ eigenvectors are employed, where $k$ is the number of clusters. This chapter includes experiments on the MNIST dataset and the BSDS dataset for
image segmentation, and demonstrates improved clustering performance over previous methods. This chapter is based on Sections 5 and 6 of Macgregor and Sun [MS22].
- Chapter 6 considers efficient learning of the structure of clusters with local algorithms, whose runtime depends only on the size of the target clusters and is independent of the underlying input graph. The experimental part of the chapter focuses on graphs like the interstate dispute dataset, in which clusters are densely connected to each other, and demonstrates that certain structures of clusters can be learned efficiently in the local setting. This chapter is based on Macgregor and Sun [MS21b].
- Finally, Chapter 7 studies the problem of learning densely connected clusters in hypergraphs. The developed algorithm is based on a new heat diffusion process, whose analysis extends a sequence of recent work on the spectral theory of hypergraphs. It allows the structure of clusters to be learned in datasets modelling higher-order relations of objects and can be applied to efficiently analyse many complex datasets occurring in practice. This chapter is based on Macgregor and Sun MS21a.


## Chapter 2

## Preliminaries

This chapter introduces the basic technical concepts which form the prerequisites for understanding the material in this thesis. Although most of the concepts listed here are standard, everything is included so that the thesis is as self-contained as possible. The chapter is divided into six sections. Section 2.1 defines some basic mathematical notation; Section 2.2 gives a brief introduction to concepts from linear algebra; Section 2.3 introduces graphs and spectral graph theory; Section 2.4 defines hypergraphs; Section 2.5 defines asymptotic notation used in the analysis of algorithms; finally, Section 2.6 defines the metrics used to evaluate clustering algorithms.

### 2.1 Basic Notation

Let us first define some of the mathematical notation used throughout the thesis. The set of real numbers is given by $\mathbb{R}$, the set of integers is $\mathbb{Z}$, and the set of complex numbers is $\mathbb{C}$. Occasionally, $\mathbb{R}_{\geq 0}$ or $\mathbb{Z}_{\geq 0}$ are used to restrict these sets to have non-negative values. For any $n \in \mathbb{Z}_{\geq 0}$, the notation $[n]$ is used to mean $\{1,2, \ldots, n\}$. For some logical proposition $X$, the expression $[X]$ is equal to 1 if $X$ is true, or 0 otherwise. For example, we say that $[a=b]$ is equal to 1 if $a=b$, and it is equal to 0 otherwise. For any operators $f, g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, we define $f \circ g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ by $(f \circ g)(v) \triangleq f(g(v))$ for any $v$. Given two sets $A$ and $B$, the set difference is expressed as $A \backslash B$. The symmetric difference between the sets $A$ and $B$ is given by $A \triangle B \triangleq(A \backslash B) \cup(B \backslash A)$. When $A$ is a subset of some universe $U$, the complement of $A$ is given by $\bar{A} \triangleq U \backslash A$. For any set $A$, we use $2^{A}$ to denote the power set of $A$; that is, $2^{A}$ is the set consisting of all possible subsets of $A$.

### 2.2 Linear Algebra

The subject of this thesis lies at the intersection of linear algebra and graph theory. As
such, let us review some basic concepts in linear algebra. A vector $\boldsymbol{x} \in \mathbb{R}^{n}$ is written as $(\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(n))^{\top}$. Notice that we use $\boldsymbol{x}(i)$ to refer to the $i$-th element in the vector $\boldsymbol{x}$. Subscripts, such as $\boldsymbol{x}_{i}$, are used to distinguish between different vectors. For a vector $\boldsymbol{x} \in \mathbb{R}^{n}$ and a set $A \subseteq[n]$, let $\boldsymbol{x}(A)=\sum_{i \in A} \boldsymbol{x}(i)$. For any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n}$, we write $\boldsymbol{x} \preceq \boldsymbol{y}$ if it holds for all $v$ that $\boldsymbol{x}(v) \leq \boldsymbol{y}(v)$. We will use $\mathbf{1}_{n} \in \mathbb{R}^{n}$ to represent the vector with $\mathbf{1}_{n}(i)=1$ for all $i \in[n]$ and $\mathbf{0}_{n}$ to represent the vector with $\mathbf{0}_{n}(i)=0$. For any vector $\boldsymbol{x}$, we define the support of $\boldsymbol{x}$ to be $\operatorname{supp}(\boldsymbol{x})=\{u: \boldsymbol{x}(u) \neq 0\}$. Given any two vectors $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^{n}$, the inner product of $\boldsymbol{v}$ and $\boldsymbol{u}$ is written as $\langle\boldsymbol{v}, \boldsymbol{u}\rangle$ or $\boldsymbol{v}^{\top} \boldsymbol{u}$ and defined to be $\sum_{i=1}^{n} \boldsymbol{v}(i) \boldsymbol{u}(i)$. Throughout the thesis we will use the Euclidean norm, defined for any $\boldsymbol{v} \in \mathbb{R}^{n}$ as

$$
\|\boldsymbol{v}\|=\sqrt{\langle\boldsymbol{v}, \boldsymbol{v}\rangle}=\sqrt{\sum_{i=1}^{n} \boldsymbol{v}(i)^{2}}
$$

Given a set of vectors $\left\{\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{n}\right\}$, their span is denoted by $\operatorname{span}\left(\boldsymbol{v}_{1}, \ldots \boldsymbol{v}_{n}\right)$ and is defined to be the set of vectors $\boldsymbol{x}$ which can be written as a linear combination of $\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{n}$. That is, $\boldsymbol{x} \in \operatorname{span}\left(\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{n}\right)$ if and only if

$$
\boldsymbol{x}=\sum_{i=1}^{n} c_{i} \boldsymbol{v}_{i}
$$

for some numbers $c_{1}, \ldots, c_{n} \in \mathbb{R}$.
Given a matrix $\mathbf{M} \in \mathbb{R}^{n \times m}$, the expression $\mathbf{M}(i, j)$ is the element in the $i$-th row and $j$-th column of the matrix. An eigenvector of a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ is any vector $\boldsymbol{v} \in \mathbb{R}^{n}$ such that $\boldsymbol{v} \neq \mathbf{0}_{n}$ and

$$
\mathbf{M} \boldsymbol{v}=\lambda \boldsymbol{v}
$$

for some $\lambda \in \mathbb{R}$, and $\lambda$ is called an eigenvalue. When discussing some matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$, we often use $\lambda_{1}, \ldots, \lambda_{n}$ to denote the eigenvalues of $\mathbf{M}$ and $f_{1}, \ldots \boldsymbol{f}_{n}$ for the eigenvectors. Occasionally, we consider matrices with complex-valued entries. Recall that for any complex number $x=a+b \cdot i$, the complex conjugate of $x$ is given by $\bar{x}=a-b \cdot i$. For any $\mathbf{M} \in \mathbb{C}^{n \times n}$, the conjugate transpose of $\mathbf{M}$ is written as $\mathbf{M}^{\dagger}$, and is constructed by taking the transpose of $\mathbf{M}$ and replacing each entry with its complex conjugate. A matrix $\mathbf{M}$ is said to be Hermitian if $\mathbf{M}=\mathbf{M}^{\dagger}$. A real matrix $\mathbf{M}$ is positive semi-definite (PSD) if and only if it holds that

$$
x^{\top} \mathrm{M} x \geq 0
$$

for all $\boldsymbol{x} \in \mathbb{R}^{n}$. It is equivalent to say that all of the eigenvalues of M are non-negative.
The following theorem, known as the Courant-Fischer Theorem, characterises the eigenvectors and eigenvalues of a symmetric matrix with respect to an optimisation over $\mathbb{R}^{n}$.

Theorem 2.1 (Courant-Fischer Theorem). Let M be a real symmetric matrix with eigenvalues $\lambda_{1} \leq \ldots \leq \lambda_{n}$. Then

$$
\lambda_{k}=\min _{\substack{S \subset \mathbb{R}^{n} \\ \operatorname{dim}(S)=k}} \max _{\substack{\boldsymbol{x} \in S \\ \boldsymbol{x} \neq \mathbf{0}_{n}}} \frac{\boldsymbol{x}^{\top} \mathbf{M} \boldsymbol{x}}{\boldsymbol{x}^{\top} \boldsymbol{x}}
$$

where the minimisation is taken over all $k$ dimensional subspaces of $\mathbb{R}^{n}$.

### 2.3 Graphs

An undirected graph $G=(V, E, w)$ is defined by a set of vertices $V$, a set of edges $E$, and a weight function $w: V \times V \rightarrow \mathbb{R}_{\geq 0}$. Every edge $e \in E$ is an unordered pair $\{v, u\}$ where $v, u \in V$ and $v \sim u$ is used to mean that $\{v, u\} \in E$. The weight function $w$ is defined such that $w(v, u)>0$ if $\{v, u\} \in E$ and $w(v, u)=0$ otherwise. Sometimes, the weight function $w$ is omitted when discussing a graph and in this case the weight of every edge can be taken to be 1. In this thesis, we consider only graphs with non-negative edge weights. A directed graph is defined similarly to an undirected graph, but in this case each edge $e \in E$ is an ordered pair $(u, v)$. Unless otherwise specified, the graphs defined in this thesis are undirected. Sometimes, the vertex and edge sets of different graphs are distinguished by subscript, such as $G=\left(V_{G}, E_{G}, w_{G}\right)$ and $H=\left(V_{H}, E_{H}, w_{H}\right)$.

Given some graph $G=(V, E, w)$, the number of vertices in a graph is commonly given by $n=|V|$. The degree of any vertex $v \in V$ is defined by $d(v) \triangleq \sum_{u \in V} w(v, u)$, and the set of neighbours of $v$ is $N(v) \triangleq\{u \in V:\{v, u\} \in E\}$. When $G=(V, E)$ is a directed graph, for any $v \in V$ we use $d_{\text {out }}(v)$ and $d_{\text {in }}(v)$ to express the number of edges with $v$ as the tail or the head, respectively. For any $S \subset V$, we define $\operatorname{vol}_{\text {out }}(S)=\sum_{v \in S} d_{\text {out }}(v)$, and $\operatorname{vol}_{\text {in }}(S)=\sum_{v \in S} d_{\text {in }}(v)$.

For an undirected graph $G=(V, E)$ and any $S \subseteq V$, the volume of $S$ is defined by $\operatorname{vol}(S) \triangleq \sum_{v \in S} d(v)$, and for two sets $S_{1}, S_{2} \subseteq V$ let us define the weight between $S_{1}$ and $S_{2}$ by

$$
w\left(S_{1}, S_{2}\right) \triangleq \sum_{v \in S_{1}} \sum_{u \in S_{2}} w(v, u)
$$

The conductance of $S \subset V$ is given by

$$
\Phi(S) \triangleq \frac{w(S, \bar{S})}{\min \{\operatorname{vol}(S), \operatorname{vol}(\bar{S})\}}
$$

Notice that if $\Phi(S)=0$, then $S$ is disconnected from the rest of the graph. Loosely, if $\Phi(S)$ is low, then there are only a small number of edges connecting $S$ to the rest of the graph and we can say that $S$ forms a cluster in the graph. The conductance of the graph $G=(V, E)$ is given by

$$
\Phi_{G} \triangleq \min _{\substack{S \subseteq V \\ S \neq \emptyset}} \Phi(S)
$$

The boundary of $S \subset V$ is given by

$$
\partial(S) \triangleq\{\{u, v\} \in E: u \in S, v \notin S\} .
$$

The indicator vector of a set $S \subseteq V$ is given by $\chi_{S} \in \mathbb{R}^{n}$, where

$$
\chi_{S}(v)= \begin{cases}1 & \text { if } v \in S, \\ 0 & \text { otherwise }\end{cases}
$$

If $S$ contains only a single vertex $v$, we can write $\chi_{v}$ instead of $\chi_{\{v\}}$.
A graph $G=(V, E)$ is said to be bipartite if the vertex set can be decomposed into disjoint sets $L$ and $R$ such that every edge $e \in E$ connects a vertex in $L$ with a vertex in $R$. That is, for all $e \in E$, it holds that $e \cap L \neq \emptyset$ and $e \cap R \neq \emptyset$. A connected component of $G$ is any maximal set of vertices $S \subseteq V$ such that there is a path from any vertex in $S$ to every other vertex in $S$.

### 2.3.1 Spectral Graph Theory

Having introduced linear algebra and graph theory, let us now take their intersection. Spectral graph theory is the study of the eigenvalues and eigenvectors of various matrices associated with a graph. This section gives a brief introduction to the key concepts in spectral graph theory, and Chapter 3 includes several specific results which are used in this thesis. For a comprehensive overview of the field, we refer the reader to Chung's excellent book [Chu97].

Given a graph $G=(V, E, w)$ with $n$ vertices, let us label the vertices as $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. Then, the adjacency matrix of $G$ is given by $\mathbf{A} \in \mathbb{R}^{n \times n}$ where $\mathbf{A}(i, j)=w\left(v_{i}, v_{j}\right)$. The degree matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$ is a diagonal matrix whose non-zero entries are $\mathbf{D}(i, i)=d\left(v_{i}\right)$ for $i \in[n]$. A key object of study in spectral graph theory is the Laplacian matrix of $G$ which is defined to be

$$
\mathbf{L}=\mathbf{D}-\mathbf{A}
$$

The most useful property of this operator is the expression given by its quadratic form. Given some vector $\boldsymbol{x} \in \mathbb{R}^{n}$, the Laplacian quadratic form is given by

$$
\boldsymbol{x}^{\top} \mathbf{L} \boldsymbol{x}=\sum_{\{u, v\} \in E} w(u, v)(\boldsymbol{x}(u)-\boldsymbol{x}(v))^{2} .
$$

It is also common to instead consider the normalised Laplacian matrix, which is defined as

$$
\mathbf{N}=\mathbf{D}^{-\frac{1}{2}} \mathbf{L} \mathbf{D}^{-\frac{1}{2}}=\mathbf{I}-\mathbf{D}^{-\frac{1}{2}} \mathbf{A} \mathbf{D}^{-\frac{1}{2}}
$$

We also use the signless Laplacian operator defined as

$$
\mathbf{J} \triangleq \mathbf{D}+\mathbf{A},
$$



Figure 2.1: A random graph sampled from $\operatorname{SBM}(60,3,0.9,0.05)$.
and its normalised counterpart

$$
\mathbf{Z} \triangleq \mathbf{D}^{-\frac{1}{2}} \mathbf{J} \mathbf{D}^{-\frac{1}{2}}
$$

Many results in spectral graph theory focus on the eigenvectors and eigenvalues of the normalised Laplacian matrix. The $n$ eigenvalues are often given in order of increasing value, and are represented by $\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$. The corresponding eigenvectors are often denoted by $f_{1}, \ldots, \boldsymbol{f}_{n}$. Notice that the Laplacian matrix uniquely defines the original graph, and so the eigenvectors and eigenvalues of the Laplacian fully describe the graph. A recurring theme in spectral graph theory is that, in many cases, we can tell a lot about a graph from only a small number of eigenvalues and eigenvectors. The following basic results in spectral graph theory are not difficult to prove, and can be found in [Chu97].

1. The eigenvalues of $\mathbf{N}$ all lie between 0 and 2 .
2. The smallest eigenvalue of $\mathbf{N}$ is always equal to 0 .
3. If the graph has $k$ connected components, then the smallest eigenvalue of $\mathbf{N}$ has multiplicity $k$.
4. The largest eigenvalue of $\mathbf{N}$ is equal to 2 if and only if the graph has a bipartite connected component.

### 2.3.2 The Stochastic Block Model

It is useful to evaluate the performance of graph clustering algorithms on graphs with a known ground-truth clustering. A popular model for generating such random graphs is the stochastic block model (SBM), and we use several variants of this model throughout the thesis. For $n, k \in \mathbb{Z}_{\geq 0}$ and $p, q \in[0,1]$ the $\operatorname{model} \operatorname{SBM}(n, k, p, q)$ defines a random graph


Figure 2.2: An example of a 3-uniform hypergraph with 6 vertices and 4 edges
on $n$ vertices in which the vertices are split evenly between $k$ clusters $S_{1}, \ldots, S_{k}$. Every pair of vertices $u$ and $v$ are connected with probability

- $p$ if they are in the same cluster, and
- $q$ otherwise.

Figure 2.1 shows an example graph drawn from the stochastic block model. When $p>q$, the stochastic block model typically generates graphs with a ground-truth cluster structure which makes it a useful model for evaluating the performance of graph clustering algorithms. For the interested reader, there is an extensive literature on several variants of the stochastic block model [Abb17, KMS16].

### 2.4 Hypergraphs

A hypergraph $H=(V, E, w)$ is defined by a vertex set $V$, a set of edges $E \subseteq 2^{V}$ and a weight function $w: E \rightarrow \mathbb{R}_{\geq 0}$. The difference between a graph and a hypergraph is that each edge in a graph connects exactly two vertices while edges in a hypergraph can contain any number of vertices. Note that this means a graph is a special case of a hypergraph. Figure 2.2 shows an example hypergraph. Let $H=(V, E, w)$ be a hypergraph with $n=|V|$ vertices. For any vertex $v \in V$, the degree of $v$ is defined by $d(v) \triangleq \sum_{e \in E} w(e) \cdot[v \in e]$, recalling that $[X]=1$ if event $X$ holds and $[X]=0$ otherwise. The rank of edge $e \in E$ is defined to be the total number of vertices in $e$ and is written as $\operatorname{rank}(e)$. A hypergraph is said to be $\boldsymbol{r}$-uniform if every edge $e \in E$ has rank $r$.

### 2.5 Asymptotic Notation

When studying the running time of algorithms, we are often interested in how the running time grows as a function of the size of the algorithm's input. A useful tool to this end is
asymptotic notation. Suppose we are given two functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$. Then, we say that

$$
f(x)=O(g(x))
$$

if there exist constants $c, t \in \mathbb{R}$ such that for all $x \geq t$, it holds that $f(x) \leq c \cdot g(x)$. For example, if $f(n)$ gives the running time of some algorithm for an input of size $n$, then $f(n)=O(n)$ tells us that for sufficiently large inputs, the running time of the algorithm is at most linear in $n$. On the other hand,

$$
f(x)=\Omega(g(x))
$$

tells us that there exist constants $c, t \in \mathbb{R}$ such that for all $x \geq t$, it holds that $f(x) \geq c \cdot g(x)$. In other words, $f(x)=\Omega(g(x))$ if and only if $g(x)=O(f(x))$. Finally, we can say that

$$
f(x)=\Theta(g(x))
$$

if $f(x)=O(g(x))$ and $f(x)=\Omega(g(x))$. When discussing the running time of algorithms, it is common to use poly $(n)$ to represent some polynomial in $n$ and $\operatorname{polylog}(n)$ for some polynomial in $\log (n)$. Intuitively, $O(\cdot)$ gives an upper-bound on some function, and hides a constant factor. We can take this one step further and use $\widetilde{O}(\cdot)$ to hide constants and logarithmic factors. For example, suppose $f(x)=100 \cdot x \log (x)$. Then, we could say that $f(x)=O(x \log (x))$ or $f(x)=\widetilde{O}(x)$.

Occasionally, we will use asymptotic notation in order to hide a fixed constant when we are not discussing the asymptotic behaviour of a function. For example, $x=\Omega(y)$ means that there exists a universal constant $c$ such that $x \geq c y$. This use of asymptotic notation is common throughout theoretical computer science [LOGT14, PSZ17].

### 2.6 Clustering Metrics

When evaluating the performance of a clustering algorithm, we would like to measure the similarity between the ground truth clusters and the clusters returned by the algorithm. The Rand Index [Ran71], also referred to as the clustering accuracy, is a common metric used for this purpose.

Suppose we are given a dataset with $n$ data points and $k$ clusters. The ground truth clusters are given by $C_{1}, \ldots, C_{k}$ and the clusters returned by our algorithm are $S_{1}, \ldots, S_{k}$. Let $\sigma_{\mathrm{C}}:[n] \rightarrow[k]$ be a function which maps each data point to its ground truth cluster. That is, if $j \in C_{i}$ then $\sigma_{\mathrm{C}}(j)=i$. Let $\sigma_{\mathrm{S}}:[n] \rightarrow[k]$ be the equivalent function for the clusters $S_{1}, \ldots, S_{k}$. Then, let

$$
\mathrm{TP} \triangleq\left|\left\{(i, j) \in[n] \times[n]: i \neq j \wedge \sigma_{\mathrm{C}}(i)=\sigma_{\mathrm{C}}(j) \wedge \sigma_{\mathrm{S}}(i)=\sigma_{\mathrm{S}}(j)\right\}\right|
$$

be the number of pairs of data points which are in the same ground truth cluster and the same cluster returned by the algorithm. Similarly, let

$$
\mathrm{TN} \triangleq\left|\left\{(i, j) \in[n] \times[n]: i \neq j \wedge \sigma_{\mathrm{C}}(i) \neq \sigma_{\mathrm{C}}(j) \wedge \sigma_{\mathrm{S}}(i) \neq \sigma_{\mathrm{S}}(j)\right\}\right|
$$

be the number of pairs of data points which the algorithm correctly identifies as belonging to different clusters. Then, the Rand Index is

$$
\mathrm{RI} \triangleq \frac{\mathrm{TP}+\mathrm{TN}}{\binom{n}{2}}
$$

That is, the Rand Index measures the proportion of pairs of data points which the algorithm correctly classifies as belonging to the same cluster or different clusters. While this is a natural definition, notice that even a baseline algorithm which assigns data points to clusters at random will achieve a non-zero Rand Index. To correct for this, the Adjusted Rand Index [GA17] normalises the Rand Index by the expected value for a random assignment. Further defining

$$
\mathrm{FP} \triangleq\left|\left\{(i, j) \in[n] \times[n]: i \neq j \wedge \sigma_{\mathrm{C}}(i) \neq \sigma_{\mathrm{C}}(j) \wedge \sigma_{\mathrm{S}}(i)=\sigma_{\mathrm{S}}(j)\right\}\right|
$$

and

$$
\mathrm{FN} \triangleq\left|\left\{(i, j) \in[n] \times[n]: i \neq j \wedge \sigma_{\mathrm{C}}(i)=\sigma_{\mathrm{C}}(j) \wedge \sigma_{\mathrm{S}}(i) \neq \sigma_{\mathrm{S}}(j)\right\}\right|,
$$

the Adjusted Rand Index is given by

$$
\mathrm{ARI} \triangleq \frac{2(\mathrm{TP} \cdot \mathrm{TN}-\mathrm{FP} \cdot \mathrm{FN})}{(\mathrm{TP}+\mathrm{FN})(\mathrm{TN}+\mathrm{FN})+(\mathrm{TN}+\mathrm{FP})(\mathrm{TP}+\mathrm{FP})}
$$

In contrast with the Rand Index, which gives a value between 0 and 1, the Adjusted Rand Index can give negative values. A random assignment of data points to clusters will give an expected Adjusted Rand Index of 0 .

## Chapter 3

## -

## Related Work

The goal of this chapter is to provide a broad overview of the techniques related to the algorithms developed later in the thesis, and to serve as an introduction to spectral graph partitioning, local graph clustering, and spectral hypergraph theory. As well as placing the novel work of this thesis in the context of the related work in the field, this chapter also introduces several technical concepts which are built on in the later chapters. The structure of the chapter is given below.

- Section 3.1 gives an introduction to spectral graph partitioning, and describes several key points in the development of the field which underpins this thesis.
- Section 3.2 describes some recently developed techniques for finding clusters in graphs with specific structures of clusters.
- Section 3.3 introduces local graph clustering and describes two key algorithms based on the personalised Pagerank and evolving set process, respectively.
- Section 3.4 gives a brief overview of spectral hypergraph theory, with a focus on the hypergraph Laplacian operator recently developed by Chan et al. [CLTZ18].


### 3.1 Spectral Graph Partitioning

Spectral graph partitioning is the use of the eigenvectors of graph matrices to help with finding partitions (or clusters) in graphs. The roots of this idea stretch back to the 1970s, when Donath and Hoffman [DH72] first proposed to use the eigenvectors of the adjacency matrix for partitioning, although they gave no theoretical justification for this choice.

### 3.1.1 Fiedler Vector

In 1973, Fiedler [Fie73] proved a number of results relating the second smallest eigenvalue of the graph Laplacian to the combinatorial connectivity of the graph. This gave the first theoretical justification for using eigenvectors in graph partitioning. Many authors refer to the eigenvector corresponding to the second smallest eigenvalue of the Laplacian as the Fiedler vector of the graph. The corresponding eigenvalue is sometimes referred to as the algebraic connectivity of the graph.

### 3.1.2 Cheeger Inequality

In parallel to this, Cheeger [Che70] studied the connectivity of Riemannian manifolds, and proved an inequality relating the second smallest eigenvalue of the Laplacian operator to the connectivity of the manifold. The significance of Cheeger's result for graph partitioning was demonstrated several years later, when Dodziuk [Dod84] and Alon [Alo86] independently proved (slightly weaker) versions of Cheeger's inequality in the graph setting. Chung [Chu96] later tightened the result and proved the following theorem, which is commonly referred to as the Cheeger inequality in the context of spectral graph theory.

Theorem 3.1 (Cheeger Inequality). Let $G$ be a graph and $\mathbf{N}_{G}$ be its normalised Laplacian matrix with eigenvalues $0=\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$. Then,

$$
\frac{\lambda_{2}}{2} \leq \Phi_{G} \leq \sqrt{2 \lambda_{2}}
$$

where $\Phi_{G}$ is the conductance of $G$.

The theorem relates the second smallest eigenvalue of the graph Laplacian to the conductance of the graph. While the theorem itself already provides significant evidence that the Laplacian spectrum may be helpful when partitioning graphs, its proof gives even more. The proof of the upper bound of Theorem 3.1 is constructive: it presents an algorithm which, given a graph $G=(V, E)$, finds a set $S \subset V$ such that $\Phi(S) \leq \sqrt{2 \lambda_{2}}$. The algorithm is formally described in Algorithm 3.1, and for any given graph the partition returned by this algorithm is often called the Cheeger cut of the given graph.

```
Algorithm 3.1: SWEEPSET( \(G\) )
    1 Compute the second smallest eigenvalue \(\lambda_{2}\) of \(\mathbf{N}_{G}\), and its eigenvector \(\boldsymbol{f}_{2}\)
    2 Sort the vertices \(v_{1}, \ldots, v_{n} \in V_{G}\) such that \(\boldsymbol{f}_{2}\left(v_{1}\right) / \sqrt{d\left(v_{1}\right)} \leq \ldots \leq \boldsymbol{f}_{2}\left(v_{n}\right) / \sqrt{d\left(v_{n}\right)}\)
    3 Let \(i=\arg \min _{i \in[n-1]} \Phi_{G}\left(\left\{v_{1}, \ldots, v_{i}\right\}\right)\)
    4 Return \(\left\{v_{1}, \ldots, v_{i}\right\}\)
```

It is worth noting that both sides of the Cheeger inequality are tight up to a constant, as demonstrated in Examples 3.1 and 3.2 .

Example 3.1 (Tightness of Cheeger Inequality lower bound).
Let $G$ be the d-dimensional hypercube graph. Then,

$$
\lambda_{2}=\frac{2}{d}
$$

and

$$
\Phi(G)=\frac{2^{d-1}}{d \cdot 2^{d-1}}=\frac{1}{d}=\frac{\lambda_{2}}{2}
$$

which demonstrates that the lower bound is tight.


Figure 3.1: A hypercube graph and the cut with minimum conductance.

Example 3.2 (Tightness of Cheeger Inequality upper bound).
Let $G$ be the cycle graph on $n$ vertices. Then,

$$
\lambda_{2}=2-2 \cos \left(\frac{2 \pi}{n}\right)=O\left(\frac{1}{n^{2}}\right)
$$

using the fact that $1-\cos (x)=2 \sin ^{2}(x / 2)$ and $\sin (x) \leq x$.


Moreover, we have that

$$
\Phi_{G}=\frac{2}{n}=\Omega\left(\sqrt{\lambda_{2}}\right)
$$

Figure 3.2: A cycle graph and the cut with minimum conductance.
which shows that the Cheeger inequality is tight up to a constant.

Finally, Kwok et al. $\left[K_{L L}+13\right]$ gave the following improved version of the Cheeger inequality, which shows that $\lambda_{2}$ better approximates the conductance $\Phi_{G}$ when there is a large gap between $\lambda_{2}$ and $\lambda_{3}$.

Theorem 3.2 (Improved Cheeger Inequality). Let $G$ be a graph, and $\mathbf{N}_{G}$ be its normalised Laplacian matrix with eigenvalues $0=\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$. Then, it holds for any $2 \leq k \leq n$ that

$$
\Phi_{G} \leq O(k) \frac{\lambda_{2}}{\sqrt{\lambda_{k}}}
$$

where $\Phi_{G}$ is the conductance of $G$.

### 3.1.3 Higher-order Cheeger Inequality

The Cheeger inequality clearly justifies the use of the second eigenvector of the graph Laplacian for partitioning a graph into two parts. This raises a natural question of whether the other eigenvalues give any information for partitioning the graph into more than two
parts. Lee et al. [LOGT14] considered the $\boldsymbol{k}$-way expansion constant of a graph, which generalises the conductance of a graph to a partitioning with more than two clusters.

Definition 3.1 ( $k$-way expansion). Given a graph $G$, the $k$-way expansion is defined as

$$
\rho_{G}(k) \triangleq \min _{\text {partition } S_{1}, \ldots, S_{k}} \max _{i \in[k]} \Phi_{G}\left(S_{i}\right),
$$

where the minimum is taken over all sets $S_{1}, \ldots, S_{k} \subset V_{G}$ where $S_{i} \cap S_{j}=\emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{k} S_{i}=V_{G}$.

It is clear that $\rho_{G}(2)=\Phi_{G}$, and so this generalises the definition of graph conductance. Lee et al. [LOGT14] proved the following theorem, which relates the $k$-way expansion of a graph to the $k$-th smallest eigenvalue of the normalised Laplacian.

Theorem 3.3 (Higher-order Cheeger Inequality). Let $G=(V, E)$ be a graph, and $\mathbf{N}_{G}$ be its normalised Laplacian matrix with eigenvalues $0=\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$. Then,

$$
\frac{\lambda_{k}}{2} \leq \rho_{G}(k) \leq O\left(k^{3}\right) \sqrt{\lambda_{k}}
$$

where $\rho_{G}(k)$ is the $k$-way expansion constant of $G$.
Once again, the proof of this theorem is constructive, and gives an algorithm for finding clusters $S_{1}, \ldots, S_{k}$ satisfying the inequalities. The dependence on $k$ in the upper bound of Theorem 3.3 is needed, but it is not known if the inequality is tight [LOGT14].

### 3.1.4 Spectral Clustering

The spectral clustering algorithm was proposed over a decade before the higher-order Cheeger inequality [NJW01], and is widely used in practice due to its simplicity and empirical effectiveness. There are several variants of this algorithm since it is possible to use the adjacency matrix, Laplacian matrix, or normalised Laplacian matrix for constructing the spectral embedding [VL07]. However, they are all composed of the same three steps:

1. Compute $k$ eigenvectors of some graph matrix;
2. Embed the vertices of the graph into $\mathbb{R}^{k}$ according to the values of the eigenvectors;
3. Apply a simple clustering algorithm, such as $k$-means, to obtain the final clusters.

See Algorithm 3.2 for the formal description of spectral clustering.
Although spectral clustering has been used in practice for several decades, the theoretical justification for its good performance has seen some recent progress [PSZ17] and deserves further study. There are two technical approaches which are common when analysing spectral clustering. The first is to see that the algorithm solves a relaxation of a natural optimisation


Figure 3.3: An illustration of the intuition behind spectral clustering. If the clusters are disconnected, then the Laplacian is block diagonal and the spectral embedding perfectly separates the clusters. If a small number of edges are added between the clusters, then the embedding doesn't change 'too much'.
problem. For more details on this perspective, the reader can refer to the excellent tutorial of Von Luxburg [vL07]. The second comes from considering the structure of clusters in the eigenspace of the graph matrices. Ng et al. [NJW01] introduced this idea with the following observations, which are illustrated in Figure 3.3.

1. If the clusters in the input graph $G$ are disconnected, then the normalised Laplacian matrix $\mathbf{N}_{G}$ is block diagonal. Therefore, the first $k$ eigenvectors of $\mathbf{N}_{G}$ are the indicator vectors of the connected components.
2. If the clusters are then connected by a small number of edges, then the Laplacian matrix doesn't change much, and accordingly the eigenvalues and eigenvectors of the Laplacian do not change too much.
3. As such, if a graph possesses a clear cluster structure, then the clusters are well separated in the spectral embedding.

Peng et al. [PSZ17] built on this intuition and proved that, not only are the clusters well

```
Algorithm 3.2: \(\operatorname{SpectralCluster}(G, k)\)
    Construct the normalised graph Laplacian matrix \(\mathbf{N}_{G}\)
    2 Compute eigenvectors \(\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{k}\) of \(\mathbf{N}_{G}\) corresponding to the \(k\) smallest eigenvalues
    3 Embed each \(v \in V_{G}\) to the point \(\left(\boldsymbol{f}_{1}(v), \ldots, \boldsymbol{f}_{k}(v)\right)^{\top}\)
    4 Apply \(k\)-means to the embedded vertices; partition \(V_{G}\) based on the output
```

separated in the spectral embedding, but the output of the $k$-means algorithm is guaranteed to correctly cluster most of the vertices. To quantify the 'clusterability' of a given graph $G$, they introduced the parameter $\Upsilon(k)$ defined as follows:

Definition 3.2. For a graph $G$, let $\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$ be the eigenvalues of the normalised Laplacian matrix $\mathbf{N}_{G}$. Then,

$$
\Upsilon(k) \triangleq \frac{\lambda_{k+1}}{\rho(k)},
$$

where $\rho(k)$ is the $k$-way expansion constant.
To understand the motivation behind this definition, suppose that a graph $G=(V, E)$ contains exactly $k$ clusters with low conductance. That is,

1. there is a partition of $V$ into $k$ sets $S_{1}, \ldots, S_{k}$ such that $\max _{i \in[k]} \Phi\left(S_{i}\right)$ is low, and
2. for any partition of $V$ into $k+1$ sets $S_{1}, \ldots, S_{k+1}$, there is some $i \in[k+1]$ such that $\Phi\left(S_{i}\right)$ is high.

Then, by the definition of $\rho(k)$, the denominator of $\Upsilon(k)$ is low, and by the higher-order Cheeger inequality (Theorem 3.3), the numerator is high. Thus, a large value of $\Upsilon(k)$ indicates that there are exactly $k$ low-conductance clusters in the graph. The definition of $\Upsilon(k)$ also closely resembles the widely-used eigen-gap heuristic for spectral clustering [NJW01, vL07], which suggests that spectral clustering works when the value of $\left|\lambda_{k+1}-\lambda_{k}\right|$ is much larger than $\left|\lambda_{k}-\lambda_{k-1}\right|$.

Peng et al. [PSZ17] prove that for graphs with a high value of $\Upsilon(k)$, the eigenvectors of the Laplacian are close to the indicator vectors of the optimal clusters. Let $S_{1}, \ldots, S_{k} \subset V$ be the optimal clusters which achieve the $k$-way expansion $\rho(k)$, then the normalised indicator vector of each cluster $S_{i}$ is defined as

$$
\overline{\boldsymbol{g}}_{i}=\frac{\mathbf{D}^{\frac{1}{2}} \boldsymbol{\chi}_{S_{i}}}{\left\|\mathbf{D}^{\frac{1}{2}} \boldsymbol{\chi}_{S_{i}}\right\|}
$$

Their result is given in the following theorem.
Theorem 3.4 ([PSZ17], Theorem 1.1). Let $G$ be a graph with $\Upsilon(k)=\Omega\left(k^{2}\right)$, and let $f_{1}, \ldots, f_{n}$ be the eigenvectors of $\mathrm{N}_{G}$ corresponding to eigenvalues $\lambda_{1} \leq \ldots \leq \lambda_{n}$. Then,

1. for every $\overline{\boldsymbol{g}}_{i}$, there is a linear combination of $\left\{\boldsymbol{f}_{i}\right\}_{i=1}^{k}$ called $\widehat{\boldsymbol{f}}_{i}$ such that $\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\| \leq$ $1 / \Upsilon(k) ;$
2. for every $\boldsymbol{f}_{i}$, there is a linear combination of $\left\{\overline{\boldsymbol{g}}_{i}\right\}_{i=1}^{k}$, called $\widehat{\boldsymbol{g}}_{i}$, such that $\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq$ $1.1 k / \Upsilon(k)$.

Peng et al. [PSZ17] used this to prove an upper bound on the number of vertices misclassified by the $k$-means clustering step of the spectral clustering algorithm.

Theorem 3.5 ([PSZ17], Theorem 1.2). Let $G$ be a graph with $\Upsilon(k)=\Omega\left(k^{3}\right)$. Let $A_{1}, \ldots, A_{k}$ be the clusters found by spectral clustering, and let the optimal correspondent of $A_{i}$ be $S_{i}$. Then, it holds for any $i \in[k]$ that

$$
\frac{\operatorname{vol}\left(A_{i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}=O\left(k^{3} / \Upsilon(k)\right)
$$

Kolev and Melhorn [KM16] and Mizutani [Miz21] developed similar results under the weaker assumptions of $\Upsilon(k)=\Omega\left(k^{2}\right)$ and $\Upsilon(k)=\Omega(k)$ respectively, although their results are not directly comparable since they use a slightly different definition of the gap $\Upsilon(k)$.

### 3.2 Clustering Structured Graphs

Spectral clustering gives an excellent starting point for the subject of graph partitioning; however, the classical analysis is usually based only on the assumption that the clusters are sparsely connected to each other and does not take into account the high-level structure of clusters. As demonstrated in Chapter 1, graphs created from real-world data often exhibit a high-level structure, and this section describes techniques for learning clusters in such graphs.

### 3.2.1 Partitioning Almost-Bipartite Graphs

Let us start by considering the case of graphs exhibiting heterophily. That is, graphs in which vertices have a tendency to share connections with those in a different cluster to their own $1^{1}$ An example of such a graph is the inter-state dispute graph introduced in Chapter 1 . More generally, graphs in which edges represent 'negative' relationships tend to exhibit heterophily.

A bipartite graph is a 'perfect' example of a two-cluster graph with heterophily. Of course, it is not difficult to separate a bipartite graph into its two clusters, but what if the graph is only almost bipartite? To answer this, Bauer and Jost [BJ13] and Trevisan [Tre12] introduced the bipartiteness of two sets $L, R \subset V_{G}$, which measures how close the sets $L$ and $R$ are to forming a bipartite connected component.

Definition 3.3 (Bipartiteness). Given a graph $G=(V, E)$ and two clusters $L, R \subset V$, the bipartiteness of $L$ and $R$ is

$$
\beta(L, R) \triangleq 1-\frac{2 w(L, R)}{\operatorname{vol}(L \cup R)}
$$

The bipartiteness of the graph $G$ is given by

$$
\beta_{G} \triangleq \min _{\substack{L, R \subset V_{G} \\ L \cap R=\emptyset}} \beta(L, R) .
$$

[^0]Notice that $\beta(L, R)=0$ if $L$ and $R$ form a bipartite and connected component of $G$. Given a graph $G$, let $0=\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n} \leq 2$ be the eigenvalues of the normalised Laplacian $\mathbf{N}_{G}$. It is a classical result in spectral graph theory that $\lambda_{n}=2$ if and only if $\beta_{G}=0$, and so a natural question is whether there is an analogue of the Cheeger inequality which connects the bipartiteness of a graph with the largest eigenvalue $\lambda_{n}$. Bauer and Jost [BJ13] and Trevisan [Tre12] independently showed that this is indeed the case with the following theorem.

Theorem 3.6. Given a graph $G$, let $\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$ be the eigenvalues of the normalised Laplacian $\mathbf{N}_{G}$. Then,

$$
\frac{2-\lambda_{n}}{2} \leq \beta_{G} \leq \sqrt{2\left(2-\lambda_{n}\right)}
$$

where $\beta_{G}$ is the bipartiteness of the graph.
Trevisan used this result to develop a novel approximation algorithm for the MAX-CUT problem. Soto [Sot15] improved Trevisan's analysis and showed that the algorithm achieves an approximation ratio of 0.614247 . Liu [Liu15] generalised Theorem 3.6 and proved a higher-order version of this inequality which links the top eigenvalues of a graph Laplacian to the bipartiteness of multiple clusters in the graph.

The problem of learning the clusters in graphs exhibiting heterophily has also been studied from a supervised and semi-supervised learning perspective, such as in $\mathrm{MYZ}^{+} 11, \mathrm{PWC}^{+} 19$, $\left[\mathrm{ZYZ}^{+} 20\right]$. These techniques are not covered in detail here since the focus of this thesis is on algorithms for unsupervised learning.

### 3.2.2 Learning Clusters from Edge Directions

In the preceding sections, we considered algorithms for learning clusters which are defined by sparse (or dense) connections. In this section, we consider directed graphs in which the clusters are defined by the directions of their connecting edges.

Cucuringu et al. [CLSZ20] studied this problem and introduced the so-called cut imbalance between two clusters $S_{1}, S_{2} \subset V$, which measures the imbalance of edge directions between the clusters. They proposed to use the Hermitian adjacency matrix of a directed graph to find clusters with a large cut imbalance. The Hermitian adjacency matrix of a directed graph $G$ is the matrix $\mathbf{A}_{h} \in \mathbb{C}^{n \times n}$ where $\mathbf{A}_{h}(u, v)=\overline{\mathbf{A}_{h}(v, u)}=i$ if $G$ contains an edge from $u$ to $v$ and $\mathbf{A}_{h}(u, v)=\mathbf{A}_{h}(v, u)=0$ otherwise. They develop a spectral clustering algorithm based on the complex-valued eigenvectors of $\mathbf{A}_{h}$. For graphs generated from a directed stochastic block model, their algorithm provably recovers the ground truth clusters when the cut imbalance between clusters is sufficiently large.

Laenen and Sun [LS20] further studied spectral clustering with the Hermitian adjacency


Figure 3.4: An example random walk starting with $x_{1}=v_{1}$. At each time step, the current position is shown by the red vertex, and the next vertex of the walk is chosen randomly from the neighbours highlighted in blue.
matrix, and proved that if the clusters form a directed path structure, then they can be recovered using a single complex-valued eigenvector. In Chapter 5 , we see that this can be generalised to undirected graphs and arbitrary structures of clusters. Moreover, we prove that when the clusters have a clearly defined structure, spectral clustering with fewer than $k$ eigenvectors performs better than spectral clustering with $k$ eigenvectors.

### 3.3 Local Graph Clustering

Let us now turn our attention to the problem of local graph clustering, which was first introduced by Spielman and Teng [ST13]. In the classical local clustering problem, we are given as input a large graph $G=(V, E)$ and some starting vertex $v \in V$. Our aim is to find a low-conductance cluster $S \subset V$ which contains the vertex $v$. Moreover, the algorithm should have running time which depends on $\operatorname{vol}(S)$ and is independent of $\operatorname{vol}(V)$. This sublinear running time makes local clustering a crucially important technique for studying massive datasets occurring in real-world applications. This section introduces two techniques which form the basis for several local clustering algorithms.

### 3.3.1 Pagerank Based Algorithm

The first key local clustering technique that we consider is known as the personalised Pagerank. With its roots in the well-known Pagerank algorithm for ranking web pages [PBMW99], personalised Pagerank can be used to find vertices in a graph close to some starting vertex. This technique has proved useful for many graph learning applications ACL06, ACL07, AHN19, YBLG17, TMIY20.

Random walks. Before coming to the personalised Pagerank, we first discuss random walks on graphs. In a graph $G=(V, E)$, a random walk is a sequence of vertices $x_{1}, x_{2}, \ldots, x_{n} \in V$ where each $x_{t}$ is chosen uniformly at random from the neighbours of $x_{t-1}$. See Figure 3.4 for an illustration.

We can represent this process algebraically by defining the random walk matrix to be

$$
\mathbf{X} \triangleq \mathbf{A D}^{-1}
$$

Let $\boldsymbol{p}_{t} \in \mathbb{R}^{n}$ be the probability distribution at time $t$, where $\boldsymbol{p}_{t}(v)$ is the probability that $x_{t}=v$. Define $\boldsymbol{p}_{0} \in \mathbb{R}^{n}$ to be some starting distribution such that $\sum_{i=1}^{n} \boldsymbol{p}(i)=1$. It is common to use $\boldsymbol{p}_{0}=\boldsymbol{\chi}_{v}$ for some vertex $v \in V$. Then, we can write $\boldsymbol{p}_{t+1}$ as

$$
\boldsymbol{p}_{t+1}=\mathbf{X} \boldsymbol{p}_{t}
$$

In order to ensure that the random walk converges, we instead use the lazy random walk defined by the matrix

$$
\mathbf{W} \triangleq \frac{1}{2}(\mathbf{I}+\mathbf{X}) .
$$

This corresponds to a random walk process in which with probability $1 / 2$, the next vertex $x_{t+1}$ is equal to $x_{t}$, and with probability $1 / 2$ the next vertex $x_{t+1}$ is chosen uniformly at random from the neighbours of $x_{t}$. If the graph is connected, this random walk process always converges to the stationary distribution denoted by $\pi$. 2 In particular, by setting

$$
\boldsymbol{\pi}(v)=\frac{d(v)}{\operatorname{vol}(V)}
$$

for every $v \in V$, it is easy to verify that

$$
\mathbf{W} \pi=\pi .
$$

Many algorithms for local graph clustering are based on the fact that a random walk starting inside some low conductance cluster is likely to stay inside the cluster. We can study this through the escape probability, which is the probability that a lazy random walk starting at $v$ leaves a set $A$ within the first $T$ steps and is defined as

$$
\operatorname{esc}(v, T, A) \triangleq \mathbb{P}\left[\bigcup_{j=0}^{T} x_{j} \notin A \mid x_{0}=v\right] .
$$

Spielman and Teng [ST13] gave the following result which connects the escape probability to the conductance of the set $S$.

Proposition 3.1 ([ST13], Proposition 2.5). Let $G=(V, E)$ be a graph, and $A \subset V$. Then, there is a set $A_{T} \subseteq A$ with $\operatorname{vol}\left(A_{T}\right) \geq \frac{1}{2} \operatorname{vol}(A)$, such that it holds for any $x \in A_{T}$ that

$$
\operatorname{esc}(x, T, A) \leq T \Phi(A)
$$

[^1]Personalised Pagerank. In the local clustering problem, we are interested in finding a cluster in the graph $G$ which is close to some starting vertex $v$. In order to find such a cluster, let us consider a new random walk process called the teleporting random walk. We first define some starting distribution $s \in \mathbb{R}^{n}$, and a teleport probability $\alpha$. Then, at time $t$,

1. with probability $\alpha$, the next vertex $x_{t+1}$ is chosen according to the distribution $s$;
2. with probability $1-\alpha$, the vertex $x_{t+1}$ is chosen according to the lazy random walk from $x_{t}$.

If we take $s$ to be the indicator vector of some vertex $v$, then at each step the random walk 'teleports' to $v$ with probability $\alpha$. We can represent this walk with the following update rule:

$$
\boldsymbol{p}_{t+1}=\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \boldsymbol{p}_{t} .
$$

The personalised Pagerank vector $\operatorname{ppr}(\alpha, s)$ is the stationary distribution of this random walk for some specified $\alpha$ and $s$. That is, one can view $\mathrm{ppr}: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ as a function such that, for any $\alpha$ and $s, \operatorname{prr}(\alpha, s)$ is the unique solution to the equation

$$
\operatorname{ppr}(\alpha, s)=\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{ppr}(\alpha, s) .
$$

Andersen et al. [ACL06] showed that the personalised Pagerank vector can also be written as

$$
\operatorname{ppr}(\alpha, \boldsymbol{s})=\alpha \sum_{t=0}^{\infty}(1-\alpha)^{t} \mathbf{W}^{t} \boldsymbol{s} .
$$

Therefore, we can also study $\operatorname{ppr}(\alpha, s)$ through the following random process: pick some integer $t \in \mathbb{Z}_{\geq 0}$ with probability $\alpha(1-\alpha)^{t}$, and perform a $t$-step lazy random walk, where the starting vertex of the random walk is picked according to $s$. Then, $\operatorname{ppr}(\alpha, s)$ describes the probability of reaching each vertex in this process. Intuitively, we would expect the vertices which are 'close' to the vertices in the starting distribution $s$ to have a higher value in the $\operatorname{ppr}(\alpha, s)$ vector.

Approximate personalised Pagerank. Our goal is to use the personalised Pagerank vector as the basis of local graph clustering algorithms. However, computing a personalised Pagerank vector $\operatorname{ppr}(\alpha, s)$ exactly is equivalent to computing the stationary distribution of a Markov chain on the vertex set $V$, which has a time complexity of $\Omega(n)$. This is too slow for any local clustering algorithm, whose running time is independent of the size of the input graph.

The good news is that, since the probability mass of a personalised Pagerank vector is concentrated around some starting vertex, it is possible to compute a good approximation of
the Pagerank vector in a local way. Andersen et al. $\mathrm{ACLO6}$ introduced the approximate Pagerank vector for this purpose.

Definition 3.4 (Approximate Pagerank). A vector $\boldsymbol{p}=\operatorname{apr}(\alpha, s, \boldsymbol{r})$ is called an approximate Pagerank vector if $\boldsymbol{p}+\operatorname{ppr}(\alpha, \boldsymbol{r})=\operatorname{ppr}(\alpha, \boldsymbol{s})$. The vector $\boldsymbol{r}$ is called the residual vector.

The Lovász-Simonovits curve. The Lovász-Simonovits curve was originally introduced by Lovász and Simonovits [LS90] to reason about the mixing rate of Markov chains, and has been used extensively in the analysis of local clustering algorithms based on the personalised Pagerank [ACL06, ALM13]. For any vector $\boldsymbol{p} \in \mathbb{R}_{\geq 0}^{n}$, order the vertices such that

$$
\frac{\boldsymbol{p}\left(v_{1}\right)}{d\left(v_{1}\right)} \geq \frac{\boldsymbol{p}\left(v_{2}\right)}{d\left(v_{2}\right)} \geq \ldots \geq \frac{\boldsymbol{p}\left(v_{n}\right)}{d\left(v_{n}\right)},
$$

and define the sweep sets of $\boldsymbol{p}$ to be $S_{j}^{\boldsymbol{p}}=\left\{v_{1}, \ldots, v_{j}\right\}$ for $1 \leq j \leq n$. Then, the Lovász-Simonovits curve, denoted by $\boldsymbol{p}[x]$ for $x \in[0, \operatorname{vol}(V)]$, is defined by the points

$$
\boldsymbol{p}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right] \triangleq \boldsymbol{p}\left(S_{j}^{\boldsymbol{p}}\right),
$$

and is linear between those points for consecutive $j$. Lovász and Simonovits also showed that

$$
\begin{equation*}
\boldsymbol{p}[x]=\max _{\substack{\boldsymbol{w} \in[0,1]^{n} \\ \sum_{u \in V} \boldsymbol{w}(u) d(u)=x}} \sum_{u \in V} \boldsymbol{w}(u) \boldsymbol{p}(u), \tag{3.1}
\end{equation*}
$$

which leads immediately to the following proposition.
Proposition 3.2. For any $\boldsymbol{p} \in \mathbb{R}_{\geq 0}^{n}$ and any $S \subseteq V$, it holds that $\boldsymbol{p}(S) \leq \boldsymbol{p}[\operatorname{vol}(S)]$.
Proof. We have that $\boldsymbol{p}(S)=\sum_{u \in V} \chi_{S}(u) \boldsymbol{p}(u)$. Hence, the statement follows by (3.1).
It is worth taking time to understand this definition and gain some of the intuition behind the Lovász-Simonovits curve. First of all, we have that $\boldsymbol{p}[0]=0$ and $\boldsymbol{p}[\operatorname{vol}(V)]=1$. Then, for some $x \in[0, \operatorname{vol}(V)]$, the value of $\boldsymbol{p}[x]$ is the maximum probability mass in $\boldsymbol{p}$ on any vertex set with volume $x$. Notice that for the stationary distribution $\pi$, we have

$$
\boldsymbol{\pi}[x]=\frac{x}{\operatorname{vol}(V)}
$$

Andersen et al. [ACL06] used the Lovász-Simonovits curve to prove that the sweep sets of an approximate Pagerank vector can be used to find a low-conductance cluster close to some starting vertex. Their proof is built around the following two results.

1. If there is a set $S \subset V$ with low conductance, then for most of the vertices $v \in S$, the Lovász-Simonovits curve of the approximate Pagerank vector $\boldsymbol{p}=\operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)$ deviates from that of the stationary distribution. That is, $\boldsymbol{p}[x]-\boldsymbol{\pi}[x]$ is large for some $x \in[0, \mathrm{vol}(V)]$. This corresponds to our intuition that a random walk is likely to stay inside a cluster with low conductance.
2. If $\boldsymbol{p}[x]-\boldsymbol{\pi}[x]$ is large for some $x \in[0, \operatorname{vol}(V)]$, then there is a sweep set of $\boldsymbol{p}$ with low conductance.

These two results lead to the following simple local graph clustering algorithm, which takes as input a graph $G$, a seed vertex $v \in V$, and a parameter $\alpha$.

1. Compute vectors $\boldsymbol{p}$ and $\boldsymbol{r}$ satisfying $\boldsymbol{p}=\operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)$.
2. Compute $j=\arg \min _{j \in[|\operatorname{supp}(p)|]} \Phi\left(S_{j}^{\boldsymbol{p}}\right)$ and return $S_{j}^{\boldsymbol{p}}$.

Andersen et al. [ACL06] also provided an algorithm for computing the approximate Pagerank locally, which is used in the first step of the clustering algorithm.

### 3.3.2 Evolving Set Process

The evolving set process (ESP) is another key technique for designing local clustering algorithms [AC09, AOGPT16]. Given a graph $G=(V, E)$, the ESP is a Markov chain whose states are sets of vertices $S_{i} \subseteq V$. Given a state $S_{i}$, the next state $S_{i+1}$ is determined by the following process: (1) choose $t \in[0,1]$ uniformly at random; (2) let $S_{i+1}=\{v \in$ $\left.V \mid \boldsymbol{\chi}_{S_{i}}^{\top} \mathbf{W} \boldsymbol{\chi}_{v} \geq t\right\}$, where $\boldsymbol{\chi}_{S}$ is the indicator vector defined in Section 2.3. To understand this update process, notice that $\chi_{S_{i}}^{\top} \mathbf{W} \chi_{v}$ is the probability that a one-step random walk starting at $v$ ends inside $S_{i}$. Since the ESP is based on a lazy random walk, we have $S_{i} \subseteq S_{i+1}$ if $t \leq 1 / 2$, and $S_{i+1} \subseteq S_{i}$ otherwise.

If the graph $G$ is connected, then the ESP has two absorbing states: $\emptyset$ and $V$. Given that the transition kernel of the evolving set process is given by $\mathbf{K}\left(S, S^{\prime}\right)=\mathbb{P}\left[S_{i+1}=S^{\prime} \mid S_{i}=S\right]$, the volume-biased ESP is defined by the transition kernel

$$
\widehat{\mathbf{K}}\left(S, S^{\prime}\right)=\frac{\operatorname{vol}\left(S^{\prime}\right)}{\operatorname{vol}(S)} \mathbf{K}\left(S, S^{\prime}\right)
$$

and is guaranteed to absorb in the state $V$ rather than $\emptyset$. Andersen and Peres [AP09] gave a local algorithm for undirected graph clustering using the volume-biased ESP. As part of this, they described an algorithm, $\operatorname{GenerateSample}(G, u, T)$, which samples $S_{T}$ from the volume-biased ESP with $S_{0}=\{u\}$ and has running time proportional to the volume of the output set.

Andersen and Peres [AP09] gave two further results which are useful in the analysis of the evolving set process. The first tells us that an evolving set process running for sufficiently many steps is very likely to have at least one state with low conductance ${ }^{3}$

[^2]Proposition 3.3 ([AP09], Corollary 1). Let $S_{0}, S_{1}, \ldots, S_{T}$ be sets drawn from the volumebiased evolving set process beginning at $S_{0}$. Then, it holds for any constant $c \geq 0$ that

$$
\mathbb{P}\left[\min _{j \leq T} \Phi\left(S_{j}\right) \leq \sqrt{\frac{4 c}{T} \ln (\operatorname{vol}(V))}\right] \geq 1-\frac{1}{c}
$$

Secondly, they showed that the overlap between the sets generated by the evolving set process and some target cluster $A$ is closely linked to the escape probability of the lazy random walk.

Proposition 3.4 ( $\boxed{A P 09], ~ L e m m a ~ 2) . ~ F o r ~ a n y ~ v e r t e x ~} x$, let $S_{0}, S_{1}, \ldots, S_{T}$ be sampled from a volume-biased ESP starting from $S_{0}=\{x\}$. Then, it holds for any set $A \subseteq V$ and $\lambda>0$ that

$$
\mathbb{P}\left[\max _{t \leq T} \frac{\operatorname{vol}\left(S_{t} \backslash A\right)}{\operatorname{vol}\left(S_{t}\right)}>\lambda \operatorname{esc}(x, T, A)\right]<\frac{1}{\lambda} .
$$

### 3.3.3 Other Local Clustering Techniques

Chung [Chu09] and Kloster and Gleich [KG14] developed local clustering algorithms based on the heat-kernel diffusion process. Orecchia and Allen-Zhu [OA14] developed an algorithm based on network flows which can be used to find a low-conductance cluster close to some seed set. Li and Peng [LP13] developed a local algorithm for finding densely connected clusters in graphs, based on truncated random walks.

### 3.4 Spectral Hypergraph Theory

Given the usefulness of hypergraphs for modelling higher-order relationships between data and the effectiveness of spectral graph theory for learning the structure of graphs, it is natural to look for an extension of these techniques to hypergraphs. Several authors have proposed different ways to generalise the Laplacian matrix from graphs to hypergraphs Chu93, HSJR13, JM19, LM18]. This thesis focuses on the Laplacian operator proposed by Chan et al. [CLTZ18] which has been applied for learning clusters in hypergraphs [TMIY20].

Given some hypergraph $H$ on $n$ vertices, Chan et al. [CLTZ18] defined a new non-linear operator $\mathbf{L}_{H}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ which describes a heat diffusion process on hypergraphs: for any vector $f_{t} \in \mathbb{R}^{n}$, the diffusion proceeds according to the equation

$$
\frac{\mathrm{d} \boldsymbol{f}_{t}}{\mathrm{~d} t}=-\mathbf{L}_{H} \boldsymbol{f}_{t}
$$

Among many properties of $\mathbf{L}_{H}$, they proved that for any vector $\boldsymbol{f}$, the value of $\mathbf{L}_{H} \boldsymbol{f}$ can be computed in polynomial time. Moreover, the heat diffusion process is guaranteed to converge to an eigenvector of $\mathbf{L}_{H}$, and the corresponding eigenvalue satisfies an analogue of the Cheeger inequality for the hypergraph conductance.

Definition 3.5 (Hypergraph conductance). Given a hypergraph $H=(V, E, w)$ and any set $S \subset V$, the conductance of $S$ is

$$
\Phi_{H}(S)=\frac{w(S, \bar{S})}{\operatorname{vol}(S)}
$$

where $w(S, \bar{S})=\sum_{e \in E} w(e) \cdot[e \cap S \neq \emptyset \wedge e \cap \bar{S} \neq \emptyset]$ and $\operatorname{vol}(S)=\sum_{v \in S} d(v)$.
Theorem 3.7 ([CLTZ18], Theorem 3.3). For any hypergraph $H$, let $\lambda_{2}$ be the second smallest eigenvalue of $\mathbf{L}_{H}$. Then,

$$
\frac{\lambda_{2}}{2} \leq \Phi_{H} \leq 2 \sqrt{\lambda_{2}}
$$

where $\Phi_{H}$ is the hypergraph conductance.
Chan and Liang [CL20] generalised this result by considering a slightly different diffusion process, and Chan et al. [CTWZ19] generalised the hypergraph Laplacian operator to directed hypergraphs. Takai et al. [TMIY20] showed that the hypergraph Laplacian operator can be used to compute the personalised Pagerank vector in a hypergraph. They further used this to develop a local clustering algorithm for hypergraphs. Finally, Yoshida [Yos19] described a Laplacian operator for hypergraphs with submodular cut functions, and posed the question of whether there is a hypergraph operator which satisfies an inequality like Trevisan's bipartiteness inequality given in Theorem 3.6. We give an affirmative answer to Yoshida's question in Chapter 7 and demonstrate that the new hypergraph signless Laplacian operator can be used to find densely connected clusters in hypergraphs.

## Chapter 4

## A Tighter Analysis of Spectral Clustering

Spectral clustering, as described in Algorithm 3.2, has been widely applied over the past two decades due to its simplicity and excellent empirical performance [NJW01]. In this chapter we theoretically analyse the spectral clustering algorithm, and develop a tighter guarantee for well-clustered graphs than the ones previously given in the literature. Informally, we analyse the performance guarantee of spectral clustering under a simple assumption on the input graph. While all the previous work (e.g., [LOGT14, KM16, Miz21, [PSZ17]) on the same problem suggests that the assumption on the input graph must depend on $k$, we show that the performance of spectral clustering can be rigorously analysed under a general condition independent of $k$. This is the first result of its kind, and could have further applications in graph clustering.

Given a graph $G=(V, E)$, the goal of graph clustering is to partition $V$ into $k$ clusters $S_{1}, \ldots, S_{k}$ according to some objective function. Recall that we can measure the quality of a cluster $S_{i}$ by its conductance

$$
\Phi\left(S_{i}\right)=\frac{w\left(S_{i}, \overline{S_{i}}\right)}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(\overline{S_{i}}\right)\right\}},
$$

and that the $k$-way expansion of $G$ is a measure of the quality of the optimal partitioning:

$$
\rho(k)=\min _{\text {partition } S_{1}, \ldots, S_{k}} \max _{i \in[k]} \Phi\left(S_{i}\right) .
$$

Let $\mathbf{N}$ be the normalised Laplacian matrix of $G$, and $\lambda_{1} \leq \lambda_{2}, \ldots, \leq \lambda_{n}$ be the eigenvalues of $\mathbf{N}$. In this chapter, we study spectral clustering on $G$ in terms of the function $\Upsilon(k)$ defined by

$$
\begin{equation*}
\Upsilon(k) \triangleq \frac{\lambda_{k+1}}{\rho(k)}, \tag{4.1}
\end{equation*}
$$

which was introduced in Section 3.1.4. The first result in this chapter improves on the structure theorem of Peng et al. [PSZ17] given in Theorem 3.4.

Main Result 1 (See Theorem 4.1 for the formal statement). Given any graph $G$, let $S_{1}, \ldots, S_{k}$ be the clusters which achieve $\rho(k)$. The normalised indicator vectors of $S_{1}, \ldots, S_{k}$ are well-approximated by the $k$ eigenvectors of the graph Laplacian $\mathbf{N}$ corresponding to $\lambda_{1}, \ldots, \lambda_{k}$. The approximation guarantee is parameterised by $\Upsilon(k)$, and is stronger for larger values of $\Upsilon(k)$.

In the remainder of the chapter, we consider spectral clustering for graphs with clusters of almost-balanced size.

Definition 4.1. (Almost-balanced) Let $G$ be a graph with $k$ clusters $S_{1}, \ldots, S_{k}$. We say that the clusters are almost-balanced if

$$
(1 / 2) \cdot \operatorname{vol}\left(V_{G}\right) / k \leq \operatorname{vol}\left(S_{i}\right) \leq 2 \cdot \operatorname{vol}\left(V_{G}\right) / k
$$

for all $i \in[k]$.
The second result of this chapter is a guarantee on the performance of spectral clustering under the assumption that $\Upsilon(k) \geq C$ for some constant $C$.

Main Result 2 (See Theorem 4.2 for the formal statement). Let $G=(V, E)$ be a graph, and $S_{1}, \ldots, S_{k}$ the $k$ clusters of almost-balanced size that achieves $\rho(k)$. Given $G$ as the input, let the output of spectral clustering be $A_{1}, \ldots, A_{k}$ and the 'optimal' correspondent of $A_{i}$ be $S_{i}$ for any $i \in[k]$. Then, $\sum_{i=1}^{k} \operatorname{vol}\left(A_{i} \triangle S_{i}\right) \leq C \cdot \operatorname{vol}(V) / \Upsilon(k)$ for some constant $C$.

Notice that some condition on $\Upsilon(k)$ is needed to ensure that an input graph $G$ has $k$ well-defined clusters, so that misclassified vertices can be formally defined. Taking this into account, our result is non-trivial as long as $\Upsilon(k)$ is lower bounded by some constant ${ }^{1]}$. This significantly improves most of the previous analysis of graph clustering algorithms, which make stronger assumptions on the input graphs. For example, Peng et al. [PSZ17] assumes that $\Upsilon(k)=\Omega\left(k^{3}\right)$, Mizutani [Miz21] assumes that $\Upsilon(k)=\Omega(k)$, the algorithm presented by Oveis Gharan and Trevisan [OGT14] assumes that $\lambda_{k+1}=\Omega\left(\operatorname{poly}(k) \lambda_{k}^{1 / 4}\right)$, and the one presented by Dey et al. [DPRS19] further assumes some condition with respect to $k, \lambda_{k}$, and the maximum degree of $G$. While these assumptions require at least a linear dependency on $k$, making it difficult for the instances with a large value of $k$ to satisfy, the result in this chapter suggests that the performance of spectral clustering can be rigorously analysed for these graphs. In particular, compared with previous work, our result better justifies the widely used eigen-gap heuristic for spectral clustering [NJW01, vL07]. This heuristic suggests that spectral clustering works when the value of $\left|\lambda_{k+1}-\lambda_{k}\right|$ is much larger than $\left|\lambda_{k}-\lambda_{k-1}\right|$, and in practice the ratio between the two gaps is usually a constant rather than some function of $k$.

[^3]
### 4.1 Stronger Structure Theorem

Let $\left\{S_{i}\right\}_{i=1}^{k}$ be any optimal partition that achieves $\rho(k)$. Recall that $\chi_{S_{i}}$ is the indicator vector of $S_{i}$, and that $\overline{\boldsymbol{g}}_{i}$ is the corresponding normalised indicator vector, which is formally defined in Section 3.1.4. Additionally, let $f_{1}, \ldots \boldsymbol{f}_{n}$ be the eigenvectors of $\mathbf{N}$ corresponding to the eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$. One of the basic results in spectral graph theory states that $G$ consists of at least $k$ connected components if and only if $\lambda_{i}=0$ for any $i \in[k]$, and $\operatorname{span}\left(\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{k}\right)=\operatorname{span}\left(\overline{\boldsymbol{g}}_{1}, \ldots, \overline{\boldsymbol{g}}_{k}\right)$ [Chu97]. Hence, one would expect that, when $G$ consists of $k$ densely connected components (clusters) connected by sparse cuts, the bottom eigenvectors $\left\{\boldsymbol{f}_{i}\right\}_{i=1}^{k}$ of $\mathbf{N}$ are close to $\left\{\overline{\boldsymbol{g}}_{i}\right\}_{i=1}^{k}$.

In their work, Peng et al. [PSZ17] assume $\Upsilon(k)=\Omega\left(k^{2}\right)$, and prove that the space spanned by $\left\{\boldsymbol{f}_{i}\right\}_{i=1}^{k}$ and the one spanned by $\left\{\overline{\boldsymbol{g}}_{i}\right\}_{i=1}^{k}$ are indeed close to each other. Specifically, they show that

1. for every $\overline{\boldsymbol{g}}_{i}$, there is some linear combination of $\left\{\boldsymbol{f}_{i}\right\}_{i=1}^{k}$, denoted by $\widehat{\boldsymbol{f}}_{i}$, such that $\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2} \leq 1 / \Upsilon(k)$; and
2. for every $\boldsymbol{f}_{i}$ there is some linear combination of $\left\{\overline{\boldsymbol{g}}_{i}\right\}_{i=1}^{k}$, denoted by $\widehat{\boldsymbol{g}}_{i}$, such that $\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq 1.1 k / \Upsilon(k)$.

See Theorem 3.4 for the formal statement. In essence, their structure theorem gives a quantitative explanation on why spectral methods work for graph clustering when there is a clear cluster-structure in $G$ characterised by $\Upsilon(k)$. As it holds for graphs with clusters of different sizes and edge densities, this structure theorem has been shown to be a powerful tool in analysing clustering algorithms, and inspired many subsequent works (e.g., [CPS15, [CSWZ16, KM16, KUK17, [V19, Miz21, Pen20, PY20, SZ19]).

We show that a stronger statement of the original structure theorem holds under a much weaker assumption, and this result is summarised as follows:

Theorem 4.1 (The Stronger Structure Theorem). The following statements hold:

1. For any $i \in[k]$, there is a vector $\widehat{\boldsymbol{f}}_{i} \in \mathbb{R}^{n}$ which is a linear combination of $\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{k}$, such that $\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}\right\|^{2} \leq 1 / \Upsilon(k)$.
2. There are vectors $\widehat{\boldsymbol{g}}_{1}, \ldots, \widehat{\boldsymbol{g}}_{k}$, each of which is a linear combination of $\overline{\boldsymbol{g}}_{1}, \ldots, \overline{\boldsymbol{g}}_{k}$, such that $\sum_{i=1}^{k}\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq k / \Upsilon(k)$.

To examine the significance of Theorem 4.1, notice that the two statements hold for any $\Upsilon(k)$, while the original structure theorem relies on the assumption that $\Upsilon(k)=\Omega\left(k^{2}\right)$. Since $\Upsilon(k)=\Omega\left(k^{2}\right)$ is a strong and even questionable assumption when $k$ is large, obtaining these statements for general $\Upsilon(k)$ is important. Secondly, the second statement of Theorem 4.1
significantly improves Theorem 3.4. Specifically, instead of stating $\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq 1.1 k / \Upsilon(k)$ for any $i \in[k]$, the second statement shows that $\sum_{i=1}^{k}\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq k / \Upsilon(k)$; hence, it holds in expectation that $\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq 1 / \Upsilon(k)$, the upper bound of which matches the first statement. This implies that the vectors $\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{k}$ and $\overline{\boldsymbol{g}}_{1}, \ldots, \overline{\boldsymbol{g}}_{k}$ can be linearly approximated by each other with roughly the same approximation guarantee. Thirdly, rather than employing the machinery from matrix analysis used by Peng et al. [PSZ17] to prove the original theorem, our new proof is simple and purely linear-algebraic. Therefore, both the stronger theorem and its much simplified proof are significant, and could have further applications in graph clustering and related problems.

Proof of Theorem 4.1. Let $\widehat{\boldsymbol{f}}_{i}=\sum_{j=1}^{k}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}$, and we write $\overline{\boldsymbol{g}}_{i}$ as a linear combination of the vectors $\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{n}$ by $\overline{\boldsymbol{g}}_{i}=\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}$. Since $\widehat{\boldsymbol{f}}_{i}$ is a projection of $\overline{\boldsymbol{g}}_{i}$, we have that $\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}$ is perpendicular to $\widehat{\boldsymbol{f}}_{i}$ and

$$
\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2}=\left\|\overline{\boldsymbol{g}}_{i}\right\|^{2}-\left\|\hat{\boldsymbol{f}}_{i}\right\|^{2}=\left(\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2}\right)-\left(\sum_{j=1}^{k}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2}\right)=\sum_{j=k+1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2} .
$$

Now, let us consider the quadratic form

$$
\begin{align*}
\overline{\boldsymbol{g}}_{i}^{\top} \mathbf{N} \overline{\boldsymbol{g}}_{i} & =\left(\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}^{\top}\right) \mathbf{N}\left(\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}\right) \\
& =\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2} \lambda_{j} \geq \lambda_{k+1}\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2} \tag{4.2}
\end{align*}
$$

where the last inequality follows by the fact that $\lambda_{i} \geq 0$ holds for any $1 \leq i \leq n$. This gives us that

$$
\begin{align*}
\overline{\boldsymbol{g}}_{i}^{\top} \mathbf{N} \overline{\boldsymbol{g}}_{i} & =\sum_{(u, v) \in E} w(u, v)\left(\frac{\overline{\boldsymbol{g}}_{i}(u)}{\sqrt{d(u)}}-\frac{\overline{\boldsymbol{g}}_{i}(v)}{\sqrt{d(v)}}\right)^{2} \\
& =\sum_{(u, v) \in E} w(u, v)\left(\frac{\boldsymbol{\chi}_{i}(u)}{\sqrt{\operatorname{vol}\left(S_{i}\right)}}-\frac{\boldsymbol{\chi}_{i}(v)}{\sqrt{\operatorname{vol}\left(S_{i}\right)}}\right)^{2} \\
& =\frac{w\left(S_{i}, \overline{S_{i}}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq \rho(k) . \tag{4.3}
\end{align*}
$$

Combining (4.2) with (4.3), we have that

$$
\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2} \leq \frac{\overline{\boldsymbol{g}}_{i}^{\top} \mathbf{N} \overline{\boldsymbol{g}}_{i}}{\lambda_{k+1}} \leq \frac{\rho(k)}{\lambda_{k+1}} \leq \frac{1}{\Upsilon(k)},
$$

which proves the first statement of the theorem.
Now we prove the second statement. We define for any $1 \leq i \leq k$ that $\widehat{\boldsymbol{g}}_{i}=\sum_{j=1}^{k}\left\langle\boldsymbol{f}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle \overline{\boldsymbol{g}}_{j}$
and have that, since $\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}$ is perpendicular to $\widehat{\boldsymbol{g}}_{i}$,

$$
\begin{aligned}
\sum_{i=1}^{k}\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} & =\sum_{i=1}^{k}\left(\left\|\boldsymbol{f}_{i}\right\|^{2}-\left\|\widehat{\boldsymbol{g}}_{i}\right\|^{2}\right) \\
& =\sum_{i=1}^{k}\left(1-\sum_{j=1}^{k}\left\langle\overline{\boldsymbol{g}}_{j}, \boldsymbol{f}_{i}\right\rangle^{2}\right) \\
& =k-\sum_{i=1}^{k} \sum_{j=1}^{k}\left\langle\overline{\boldsymbol{g}}_{j}, \boldsymbol{f}_{i}\right\rangle^{2} \\
& =\sum_{j=1}^{k}\left(1-\sum_{i=1}^{k}\left\langle\overline{\boldsymbol{g}}_{j}, \boldsymbol{f}_{i}\right\rangle^{2}\right) \\
& =\sum_{j=1}^{k}\left(\left\|\overline{\boldsymbol{g}}_{j}\right\|^{2}-\left\|\widehat{\boldsymbol{f}}_{j}\right\|^{2}\right) \\
& =\sum_{j=1}^{k}\left\|\overline{\boldsymbol{g}}_{j}-\widehat{\boldsymbol{f}}_{j}\right\|^{2} \\
& \leq \sum_{j=1}^{k} \frac{1}{\Upsilon(k)} \\
& =\frac{k}{\Upsilon(k)},
\end{aligned}
$$

where the inequality follows by the first statement of Theorem 4.1.

### 4.2 Analysis of Spectral Clustering

For any input graph $G=(V, E)$ and $k \in[n]$, we study the following spectral clustering algorithm:

1. compute the eigenvectors $\boldsymbol{f}_{1}, \ldots \boldsymbol{f}_{k}$ of $\mathbf{N}$, and embed each $u \in V$ to the point $F(u) \in \mathbb{R}^{k}$ defined by

$$
\begin{equation*}
F(u) \triangleq \frac{1}{\sqrt{d(u)}}\left(\boldsymbol{f}_{1}(u), \ldots, \boldsymbol{f}_{k}(u)\right)^{\top} \tag{4.4}
\end{equation*}
$$

2. apply $k$-means on the embedded points $\{F(u)\}_{u \in V}$;
3. partition $V$ into $k$ clusters based on the output of $k$-means.

In this rest of this section, we prove the following theorem, where APT is the approximation ratio of the $k$-means algorithm used in spectral clustering. Recall that we can take APT to be some small constant [CGKR21, KSS04].

Theorem 4.2. Let $G=(V, E)$ be a graph with $k$ clusters $S_{1}, \ldots, S_{k}$ of almost-balanced size, and $\Upsilon(k) \geq 2176(1+A P T)$. Let $\left\{A_{i}\right\}_{i=1}^{k}$ be the output of spectral clustering and
assume, without loss of generality, that the optimal correspondent of $A_{i}$ is $S_{i}$. Then, it holds that

$$
\sum_{i=1}^{k} \operatorname{vol}\left(A_{i} \triangle S_{i}\right) \leq 2176(1+A P T) \frac{\operatorname{vol}(V)}{\Upsilon(k)}
$$

The most significant feature of Theorem 4.2 is that the lower bound of $\Upsilon(k)$ is independent of the number of clusters $k$. The theorem gives a non-trivial guarantee on the performance of spectral clustering so long as the spectral gap, as measured by $\Upsilon(k)$, is at least a constant. This improves on the previous analysis of graph clustering algorithms which make stronger assumptions on the input graphs [PSZ17, Miz21, LOGT14].

### 4.2.1 Properties of Spectral Embedding

Let us first study the properties of the spectral embedding defined in (4.4), and in the next subsection we use these properties to prove Theorem 4.2. For every cluster $S_{i}$, define the vector $\boldsymbol{p}^{(i)} \in \mathbb{R}^{k}$ by

$$
\boldsymbol{p}^{(i)}(j)=\frac{1}{\sqrt{\operatorname{vol}\left(S_{i}\right)}}\left\langle\boldsymbol{f}_{j}, \overline{\boldsymbol{g}}_{i}\right\rangle,
$$

and we can view these $\left\{\boldsymbol{p}^{(i)}\right\}_{i=1}^{k}$ as the approximate centres of the embedded points from the optimal clusters $\left\{S_{i}\right\}_{i=1}^{k}$. We prove that the total $k$-means cost of the embedded points can be upper bounded as follows:

Lemma 4.1. It holds that

$$
\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|F(u)-\boldsymbol{p}^{(i)}\right\|^{2} \leq \frac{k}{\Upsilon(k)}
$$

Proof. We have

$$
\begin{aligned}
\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|F(u)-\boldsymbol{p}^{(i)}\right\|^{2} & =\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left[\sum_{j=1}^{k}\left(\frac{\boldsymbol{f}_{j}(u)}{\sqrt{d(u)}}-\frac{\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle}{\sqrt{\operatorname{vol}\left(S_{i}\right)}}\right)^{2}\right] \\
& =\sum_{i=1}^{k} \sum_{u \in S_{i}} \sum_{j=1}^{k}\left(\boldsymbol{f}_{j}(u)-\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \overline{\boldsymbol{g}}_{i}(u)\right)^{2} \\
& =\sum_{i=1}^{k} \sum_{u \in S_{i}} \sum_{j=1}^{k}\left(\boldsymbol{f}_{j}(u)-\widehat{\boldsymbol{g}}_{j}(u)\right)^{2} \\
& =\sum_{j=1}^{k}\left\|\boldsymbol{f}_{j}-\widehat{\boldsymbol{g}}_{j}\right\|^{2} \leq \frac{k}{\Upsilon(k)},
\end{aligned}
$$

where we use that for $u \in S_{x}$ it holds that $\widehat{\boldsymbol{g}}_{i}(u)=\sum_{j=1}^{k}\left\langle\boldsymbol{f}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle \overline{\boldsymbol{g}}_{j}(u)=\left\langle\boldsymbol{f}_{i}, \overline{\boldsymbol{g}}_{x}\right\rangle \overline{\boldsymbol{g}}_{x}(u)$, and the final inequality follows by the second statement of Theorem 4.1.

The importance of Lemma 4.1 is that, although the optimal centres for $k$-means are unknown, the existence of $\left\{\boldsymbol{p}^{(i)}\right\}_{i=1}^{k}$ is sufficient to show that the cost of an optimal $k$-means clustering on $\{F(u)\}_{u \in V}$ is at most $k / \Upsilon(k)$. Since one can always use an $O(1)$-approximate $k$-means algorithm for spectral clustering (e.g., [KMN $\left.{ }^{+} 04, ~ K K S 04\right]$ ), the cost of the output of $k$-means on $\{F(u)\}_{u \in V}$ is $O(k / \Upsilon(k))$. Next, we show that the length of $\boldsymbol{p}^{(i)}$ is approximately equal to $1 / \operatorname{vol}\left(S_{i}\right)$, which is used in our later analysis.

Lemma 4.2. It holds for any $i \in[k]$ that

$$
\frac{1}{\operatorname{vol}\left(S_{i}\right)}\left(1-\frac{1}{\Upsilon(k)}\right) \leq\left\|\boldsymbol{p}^{(i)}\right\|^{2} \leq \frac{1}{\operatorname{vol}\left(S_{i}\right)}
$$

Proof. By definition, we have

$$
\operatorname{vol}\left(S_{i}\right)\left\|\boldsymbol{p}^{(i)}\right\|^{2}=\sum_{j=1}^{k}\left\langle\boldsymbol{f}_{j}, \overline{\boldsymbol{g}}_{i}\right\rangle^{2}=\left\|\widehat{\boldsymbol{f}}_{i}\right\|^{2}=1-\left\|\widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}\right\|^{2} \geq 1-\frac{1}{\Upsilon(k)},
$$

where the inequality follows by Theorem 4.1. The other direction of the inequality follows similarly.

In the remainder of this subsection, we prove a sequence of lemmas showing that any pair of $\boldsymbol{p}^{(i)}$ and $\boldsymbol{p}^{(j)}$ for $i \neq j$ are well separated. Moreover, their distance is essentially independent of $k$ and $\Upsilon(k)$, as long as $\Upsilon(k) \geq 20$.

Lemma 4.3. It holds for any different $i, j \in[k]$ that

$$
\left\|\sqrt{\operatorname{vol}\left(S_{i}\right)} \cdot \boldsymbol{p}^{(i)}-\sqrt{\operatorname{vol}\left(S_{j}\right)} \cdot \boldsymbol{p}^{(j)}\right\|^{2} \geq 2-\frac{8}{\Upsilon(k)} .
$$

Proof. By the definitions of $\boldsymbol{p}^{(i)}$ and $\widehat{\boldsymbol{f}}$, we have

$$
\begin{aligned}
& \left\|\sqrt{\operatorname{vol}\left(S_{i}\right)} \cdot \boldsymbol{p}^{(i)}-\sqrt{\operatorname{vol}\left(S_{j}\right)} \cdot \boldsymbol{p}^{(j)}\right\|^{2} \\
& =\sum_{x=1}^{k}\left(\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{i}\right\rangle-\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{j}\right\rangle\right)^{2} \\
& =\left(\sum_{x=1}^{k}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{\boldsymbol{i}}\right\rangle^{2}\right)+\left(\sum_{x=1}^{k}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{j}\right\rangle^{2}\right)-2 \sum_{x=1}^{k}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{i}\right\rangle\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{j}\right\rangle \\
& \geq\left\|\hat{\boldsymbol{f}}_{i}\right\|^{2}+\left\|\hat{\boldsymbol{f}}_{j}\right\|^{2}-2\left|\hat{\boldsymbol{f}}_{i}^{\top} \hat{\boldsymbol{f}}_{j}\right| .
\end{aligned}
$$

Then, considering only the final term, we have

$$
\left.\begin{array}{rl}
\left|\widehat{\boldsymbol{f}}_{i}^{\top} \widehat{\boldsymbol{f}}_{j}\right| & =\left|\left(\overline{\boldsymbol{g}}_{i}+\widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}\right)^{\top}\left(\overline{\boldsymbol{g}}_{j}+\widehat{\boldsymbol{f}}_{j}-\overline{\boldsymbol{g}}_{j}\right)\right| \\
& =\left|\left\langle\overline{\boldsymbol{g}}_{i}, \widehat{\boldsymbol{f}}_{j}-\overline{\boldsymbol{g}}_{j}\right\rangle+\left\langle\overline{\boldsymbol{g}}_{j}, \widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}\right\rangle+\left\langle\widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}, \widehat{\boldsymbol{f}}_{j}-\overline{\boldsymbol{g}}_{j}\right\rangle\right| \\
& =\mid\left\langle\widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}, \widehat{\boldsymbol{f}_{j}}-\overline{\boldsymbol{g}}_{j}\right\rangle+\left\langle\widehat{\boldsymbol{f}}_{j}-\overline{\boldsymbol{g}}_{j}, \widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}\right\rangle+\left\langle\widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}, \widehat{\boldsymbol{f}} j\right.
\end{array} \overline{\boldsymbol{g}}_{j}\right\rangle \left\lvert\, \leq \frac{3}{\Upsilon(k)}\right., ~ .
$$

where we use the fact that $\left\langle\overline{\boldsymbol{g}}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle=0$ for $i \neq j$ and $\left\|\widehat{\boldsymbol{f}}_{i}-\overline{\boldsymbol{g}}_{i}\right\|^{2} \leq 1 / \Upsilon(k)$. Then, since $\left\|\widehat{\boldsymbol{f}}_{i}\right\|^{2} \geq 1-1 / \Upsilon(k)$, we have

$$
\begin{aligned}
\left\|\sqrt{\operatorname{vol}\left(S_{i}\right)} \cdot \boldsymbol{p}^{(i)}-\sqrt{\operatorname{vol}\left(S_{j}\right)} \cdot \boldsymbol{p}^{(j)}\right\|^{2} & \geq 2\left(1-\frac{1}{\Upsilon(k)}\right)-\frac{6}{\Upsilon(k)} \\
& =2-\frac{8}{\Upsilon(k)}
\end{aligned}
$$

Lemma 4.4. It holds for any different $i, j \in[k]$ that

$$
\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\|^{2} \geq 2-\frac{20}{\Upsilon(k)}
$$

Proof. Assume without loss of generality that $\sqrt{\operatorname{vol}\left(S_{i}\right)}\left\|\boldsymbol{p}^{(i)}\right\| \leq \sqrt{\operatorname{vol}\left(S_{j}\right)}\left\|\boldsymbol{p}^{(j)}\right\|$. Let $\boldsymbol{a}_{i}=\sqrt{\operatorname{vol}\left(S_{i}\right)} \boldsymbol{p}^{(i)}$ and $\boldsymbol{a}_{j}=\sqrt{\operatorname{vol}\left(S_{j}\right)} \boldsymbol{p}^{(j)}$ and notice that $\left\|\boldsymbol{a}_{i}\right\| \leq\left\|\boldsymbol{a}_{j}\right\| \leq 1$ by Lemma 4.2. Then, since $\boldsymbol{a}_{i}$ and $\left(\left\|\boldsymbol{a}_{i}\right\| /\left\|\boldsymbol{a}_{j}\right\|\right) \boldsymbol{a}_{j}$ are scaled versions of $\boldsymbol{p}^{(i)}$ and $\boldsymbol{p}^{(j)}$ with smaller norm, we have

$$
\begin{aligned}
\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\| & \geq\left\|\boldsymbol{a}_{i}-\frac{\left\|\boldsymbol{a}_{i}\right\|}{\left\|\boldsymbol{a}_{j}\right\|} \boldsymbol{a}_{j}\right\| \\
& \geq\left\|\boldsymbol{a}_{i}-\boldsymbol{a}_{j}\right\|-\left(\left\|\boldsymbol{a}_{j}\right\|-\left\|\boldsymbol{a}_{i}\right\|\right) \\
& \geq \sqrt{2-\frac{8}{\Upsilon(k)}}-\left(\sqrt{\operatorname{vol}\left(S_{j}\right)} \cdot\left\|\boldsymbol{p}^{(j)}\right\|-\sqrt{\operatorname{vol}\left(S_{i}\right)}\left\|\boldsymbol{p}^{(i)}\right\|\right) \\
& \geq \sqrt{2}\left(1-\frac{4}{\Upsilon(k)}\right)+\sqrt{1-\frac{1}{\Upsilon(k)}}-1 \\
& \geq \sqrt{2}-\frac{4 \sqrt{2}}{\Upsilon(k)}-\frac{1}{\Upsilon(k)} \geq \sqrt{2}-\frac{7}{\Upsilon(k)}
\end{aligned}
$$

where the second inequality follows by the triangle inequality, and the third and fourth use Lemma 4.3 and Lemma 4.2. We also use the fact that it holds for $x \leq 1$ that $\sqrt{1-x} \geq 1-x$. This gives

$$
\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\|^{2} \geq 2-\frac{14 \sqrt{2}}{\Upsilon(k)} \geq 2-\frac{20}{\Upsilon(k)}
$$

which proves the lemma.

Lemma 4.5. It holds for any different $i, j \in[k]$ that

$$
\left\|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(j)}\right\|^{2} \geq \frac{1}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}\left(\frac{1}{2}-\frac{8}{\Upsilon(k)}\right) .
$$

Proof. Assume without loss of generality that $\left\|\boldsymbol{p}^{(i)}\right\| \geq\left\|\boldsymbol{p}^{(j)}\right\|$. Then, let $\left\|\boldsymbol{p}^{(j)}\right\|=\alpha\left\|\boldsymbol{p}^{(i)}\right\|$ for some $\alpha \in[0,1]$. By Lemma 4.2 it holds that

$$
\left\|\boldsymbol{p}^{(i)}\right\|^{2} \geq \frac{1}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}\left(1-\frac{1}{\Upsilon(k)}\right) .
$$

Additionally, notice that by the proof of Lemma 4.4

$$
\left\langle\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}, \frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\rangle \leq \sqrt{2}-\frac{1}{2}\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\| \leq \frac{\sqrt{2}}{2}+\frac{7}{2 \Upsilon(k)},
$$

where we use the fact that if $x^{2}+y^{2}=1$, then $x+y \leq \sqrt{2}$. One can understand the equation above by considering the right-angled triangle with one edge given by $\boldsymbol{p}^{(i)} /\left\|\boldsymbol{p}^{(i)}\right\|$ and another edge given by $\left(\boldsymbol{p}^{(i)} /\left\|\boldsymbol{p}^{(i)}\right\|\right) \cdot\left(\boldsymbol{p}^{(j)} /\left\|\boldsymbol{p}^{(j)}\right\|\right)$. Then,

$$
\begin{aligned}
\left\|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(j)}\right\|^{2} & =\left\|\boldsymbol{p}^{(i)}\right\|^{2}+\left\|\boldsymbol{p}^{(j)}\right\|^{2}-2\left\langle\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}, \frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\rangle\left\|\boldsymbol{p}^{(i)}\right\|\left\|\boldsymbol{p}^{(j)}\right\| \\
& \geq(1+\alpha)\left\|\boldsymbol{p}^{(i)}\right\|^{2}-\left(\sqrt{2}+\frac{7}{\Upsilon(k)}\right) \alpha\left\|\boldsymbol{p}^{(i)}\right\|^{2} \\
& \geq\left(1-(\sqrt{2}-1) \alpha-\frac{7}{\Upsilon(k)}\right)\left\|\boldsymbol{p}^{(i)}\right\|^{2} \\
& \geq \frac{1}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}\left(\frac{1}{2}-\frac{7}{\Upsilon(k)}\right)\left(1-\frac{1}{\Upsilon(k)}\right) \\
& \geq \frac{1}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}\left(\frac{1}{2}-\frac{8}{\Upsilon(k)}\right)
\end{aligned}
$$

which completes the proof.
It is worth noting that, despite the similarity in their formulation, the technical lemmas presented here are stronger than the ones in Peng et al. [PSZ17]. These results are obtained through the stronger structure theorem (Theorem 4.1), and are crucial for us to prove Theorem 4.2.

### 4.2.2 Bounding the Misclassified Vertices

Now we prove Theorem 4.2 and see why a mild condition like $\Upsilon(k) \geq C$ for some constant $C \in \mathbb{N}$ suffices for spectral clustering to perform well in practice. Let $\left\{A_{i}\right\}_{i=1}^{k}$ be the output of spectral clustering, and we denote the centre of the embedded points $\{F(u)\}_{u \in A_{i}}$ for any $A_{i}$ by $\boldsymbol{c}_{i}$. As the starting point of the analysis, we show that every $\boldsymbol{c}_{i}$ is close to its 'optimal' correspondent $\boldsymbol{p}^{(\sigma(i))}$ for some $\sigma(i) \in[k]$. That is, the actual centre of embedded points from every $A_{i}$ is close to the approximate centre of the embedded points from some optimal $S_{i}$. To formalise this, we define the function $\sigma:[k] \rightarrow[k]$ by

$$
\begin{equation*}
\sigma(i)=\underset{j \in[k]}{\arg \min }\left\|\boldsymbol{p}^{(j)}-\boldsymbol{c}_{i}\right\| ; \tag{4.5}
\end{equation*}
$$

that is, cluster $A_{i}$ should correspond to $S_{\sigma(i)}$ in which the value of $\left\|\boldsymbol{p}^{(\sigma(i))}-\boldsymbol{c}_{i}\right\|$ is the lowest among all the distances between $\boldsymbol{c}_{i}$ and all of the $\boldsymbol{p}^{(j)}$ for $j \in[k]$. However, one needs to be cautious as (4.5) doesn't necessarily define a permutation, and there might exist different $i, i^{\prime} \in[k]$ such that both of $A_{i}$ and $A_{i^{\prime}}$ map to the same $S_{\sigma(i)}$. Taking this into account, for any fixed $\sigma:[k] \rightarrow[k]$ and $i \in[k]$, we further define $M_{\sigma, i}$ by

$$
\begin{equation*}
M_{\sigma, i} \triangleq \bigcup_{j: \sigma(j)=i} A_{j} . \tag{4.6}
\end{equation*}
$$

The following lemma shows that, when mapping every output $A_{i}$ to $S_{\sigma(i)}$, the total ratio of misclassified volume with respect to each cluster can be upper bounded:

Lemma 4.6. Let $\left\{A_{i}\right\}_{i=1}^{k}$ be the output of spectral clustering, and $\sigma$ and $M_{\sigma, i}$ be defined as in (4.5) and (4.6). If $\Upsilon(k) \geq 32$, then

$$
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq 64(1+A P T) \frac{k}{\Upsilon(k)}
$$

Proof. Let us define $B_{i j}=A_{i} \cap S_{j}$ to be the vertices in $A_{i}$ which belong to the ground-truth cluster $S_{j}$. Then, we have that

$$
\begin{align*}
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} & =\sum_{i=1}^{k} \sum_{\substack{j=1 \\
j \neq \sigma(i)}}^{k} \operatorname{vol}\left(B_{i j}\right)\left(\frac{1}{\operatorname{vol}\left(S_{\sigma(i)}\right)}+\frac{1}{\operatorname{vol}\left(S_{j}\right)}\right) \\
& \leq 2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\
j \neq \sigma(i)}}^{k} \frac{\operatorname{vol}\left(B_{i j}\right)}{\min \left\{\operatorname{vol}\left(S_{\sigma(i)}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \tag{4.7}
\end{align*}
$$

and that

$$
\begin{aligned}
& \operatorname{COST}\left(A_{1}, \ldots A_{k}\right) \\
& =\sum_{i=1}^{k} \sum_{u \in A_{i}} d(u)\left\|F(u)-\boldsymbol{c}_{i}\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} d(u)\left\|F(u)-\boldsymbol{c}_{i}\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq(i) \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} d(u)\left(\frac{\left\|\boldsymbol{p}^{(j)}-\boldsymbol{c}_{i}\right\|^{2}}{2}-\left\|\boldsymbol{p}^{(j)}-F(u)\right\|^{2}\right) \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} \frac{d(u)\left\|\boldsymbol{p}^{(j)}-\boldsymbol{p}^{(\sigma(i))}\right\|^{2}}{8}-\sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} d(u)\left\|\boldsymbol{p}^{(j)}-F(u)\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \operatorname{vol}\left(B_{i j}\right) \frac{\left\|\boldsymbol{p}^{(j)}-\boldsymbol{p}^{(\sigma(i))}\right\|^{2}}{8}-\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|\boldsymbol{p}^{(i)}-F(u)\right\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \frac{\operatorname{vol}\left(B_{i j}\right)}{8 \min \left\{\operatorname{vol}\left(S_{\sigma(i)}\right), \operatorname{vol}\left(S_{j}\right)\right\}}\left(\frac{1}{2}-\frac{8}{\Upsilon(k)}\right)-\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|\boldsymbol{p}^{(i)}-F(u)\right\|^{2} \\
& \geq \frac{1}{16} \cdot\left(\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}\right)\left(\frac{1}{2}-\frac{8}{\Upsilon(k)}\right)-\frac{k}{\Upsilon(k)},
\end{aligned}
$$

where the second inequality follows by the inequality

$$
\|\boldsymbol{a}-\boldsymbol{b}\|^{2} \geq \frac{\|\boldsymbol{b}-\boldsymbol{c}\|^{2}}{2}-\|\boldsymbol{a}-\boldsymbol{c}\|^{2}
$$

the third inequality follows since $\boldsymbol{c}_{i}$ is closer to $\boldsymbol{p}^{(\sigma(i))}$ than $\boldsymbol{p}^{(j)}$, the fifth one follows from Lemma 4.5, and the last one follows by (4.7).

On the other side, since $\operatorname{COST}\left(A_{1}, \ldots, A_{k}\right) \leq \mathrm{APT} \cdot k / \Upsilon(k)$, we have that

$$
\frac{1}{16} \cdot\left(\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \Delta S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}\right)\left(\frac{1}{2}-\frac{8}{\Upsilon(k)}\right)-\frac{k}{\Upsilon(k)} \leq \mathrm{APT} \cdot \frac{k}{\Upsilon(k)} .
$$

This implies that

$$
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \Delta S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq 16 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)} \cdot\left(\frac{1}{2}-\frac{8}{\Upsilon(k)}\right)^{-1} \leq 64 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}
$$

where the last inequality holds by the assumption that $\Upsilon(k) \geq 32$.
It remains to study the case in which $\sigma$ isn't a permutation. Notice that, if this occurs, there is some $i \in[k]$ such that $M_{\sigma, i}=\emptyset$, and different values of $x, y \in[k]$ such that $\sigma(x)=\sigma(y)=j$ for some $j \neq i$. Based on this, we construct the function $\sigma^{\prime}:[k] \rightarrow[k]$ from $\sigma$ based on the following procedure:

- set $\sigma^{\prime}(z)=i$ if $z=x$;
- set $\sigma^{\prime}(z)=\sigma(z)$ for any other $z \in[k] \backslash\{x\}$.

Notice that one can construct $\sigma^{\prime}$ in this way as long as $\sigma$ isn't a permutation, and this constructed $\sigma^{\prime}$ reduces the number of $M_{\sigma, i}$ which are equal to $\emptyset$ by one. We show that one only needs to construct such $\sigma^{\prime}$ at most $64(1+\mathrm{APT})(k / \Upsilon(k))$ times to obtain the final permutation called $\sigma^{\star}$, and it holds for $\sigma^{\star}$ that

$$
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma^{\star}, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq 1088 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}
$$

Combining this with the fact that $\operatorname{vol}\left(S_{i}\right)=\Theta\left(\operatorname{vol}\left(V_{G}\right) / k\right)$ for any $i \in[k]$ proves Theorem 4.2.
Proof of Theorem 4.2. By Lemma 4.1, we have

$$
\operatorname{CosT}\left(S_{1}, \ldots, S_{k}\right) \leq \frac{k}{\Upsilon(k)}
$$

Combining this with the fact that one can apply an approximate $k$-means clustering algorithm with approximation ratio APT for spectral clustering, we have that

$$
\operatorname{CosT}\left(A_{1}, \ldots, A_{k}\right) \leq \mathrm{APT} \cdot \frac{k}{\Upsilon(k)}
$$

Then, let $\sigma:[1, k] \rightarrow[1, k]$ be the function which assigns each cluster $A_{i}$ to some groundtruth cluster $S_{\sigma(i)}$ such that $\sigma(i)=\arg \min _{j \in[k]}\left\|\boldsymbol{p}^{(j)}-\boldsymbol{c}_{i}\right\|$. Then, it holds by Lemma 4.6 that

$$
\begin{equation*}
\epsilon(\sigma) \triangleq \sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq 64 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)} . \tag{4.8}
\end{equation*}
$$

Now, assume that $\sigma$ is not a permutation, and we'll apply the following procedure inductively to construct a permutation from $\sigma$. Since $\sigma$ isn't a permutation, there is $i \in[k]$ such that $M_{\sigma, i}=\emptyset$. Therefore, there are different values of $x, y \in[k]$ such that $\sigma(x)=\sigma(y)=j$ for some $j \neq i$. Based on this, we construct the function $\sigma^{\prime}:[k] \rightarrow[k]$ such that $\sigma^{\prime}(z)=i$ if $z=x$, and $\sigma^{\prime}(z)=\sigma(z)$ for any other $z \in[k] \backslash\{x\}$. Notice that we can construct $\sigma^{\prime}$ in this way as long as $\sigma$ isn't a permutation. By the definition of $\epsilon(\cdot)$ and function $\sigma^{\prime}$, the difference between $\epsilon\left(\sigma^{\prime}\right)$ and $\epsilon(\sigma)$ can be written as

$$
\begin{align*}
& \epsilon\left(\sigma^{\prime}\right)-\epsilon(\sigma) \\
& =\underbrace{\left(\frac{\operatorname{vol}\left(M_{\sigma^{\prime}, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}-\frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}\right)}_{\alpha}+\underbrace{\left(\frac{\operatorname{vol}\left(M_{\sigma^{\prime}, j} \triangle S_{j}\right)}{\operatorname{vol}\left(S_{j}\right)}-\frac{\operatorname{vol}\left(M_{\sigma, j} \triangle S_{j}\right)}{\operatorname{vol}\left(S_{j}\right)}\right)}_{\beta} . \tag{4.9}
\end{align*}
$$

Let us consider 4 cases based on the sign of $\alpha, \beta$ defined above. In each case, we bound the cost introduced by the change from $\sigma$ to $\sigma^{\prime}$, and then consider the total cost introduced throughout the entire procedure of constructing a permutation.

Case 1: $\alpha<0, \beta<0$. In this case, it is clear that $\epsilon\left(\sigma^{\prime}\right)-\epsilon(\sigma) \leq 0$, and hence the total introduced cost is at most 0 .

Case 2: $\alpha>0, \beta<0$. In this case, we have

$$
\begin{aligned}
& \epsilon\left(\sigma^{\prime}\right)-\epsilon(\sigma) \leq \frac{1}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}( \operatorname{vol}\left(M_{\sigma^{\prime}, i} \triangle S_{i}\right)-\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right) \\
&\left.+\left|\operatorname{vol}\left(M_{\sigma^{\prime}, j} \triangle S_{j}\right)-\operatorname{vol}\left(M_{\sigma, j} \triangle S_{j}\right)\right|\right) \\
&=\frac{1}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}( \operatorname{vol}\left(M_{\sigma^{\prime}, i} \triangle S_{i}\right)-\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right) \\
&\left.+\operatorname{vol}\left(M_{\sigma, j} \triangle S_{j}\right)-\operatorname{vol}\left(M_{\sigma^{\prime}, j} \triangle S_{j}\right)\right) \\
&=\frac{1}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}\left(\operatorname{vol}\left(A_{x} \backslash S_{i}\right)-\operatorname{vol}\left(A_{x} \cap S_{i}\right)\right. \\
&\left.+\operatorname{vol}\left(A_{x} \backslash S_{j}\right)-\operatorname{vol}\left(A_{x} \cap S_{j}\right)\right) \\
& \leq \frac{2 \cdot \operatorname{vol}\left(A_{x} \backslash S_{j}\right)}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}
\end{aligned}
$$

$$
\leq \frac{8 \cdot \operatorname{vol}\left(A_{x} \backslash S_{j}\right)}{\operatorname{vol}\left(S_{j}\right)}
$$

where the last inequality follows by the fact that the clusters are almost balanced. Since each set $A_{x}$ is moved at most once in order to construct a permutation, the total introduced cost due to this case is at most

$$
\sum_{j=1}^{k} \sum_{A_{x} \in M_{\sigma, j}} \frac{8 \cdot \operatorname{vol}\left(A_{x} \backslash S_{j}\right)}{\operatorname{vol}\left(S_{j}\right)} \leq 8 \cdot \sum_{j=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, j} \triangle S_{j}\right)}{\operatorname{vol}\left(S_{j}\right)} \leq 512 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}
$$

Case 3: $\alpha>0, \beta>0$. In this case, we have

$$
\begin{aligned}
& \epsilon\left(\sigma^{\prime}\right)-\epsilon(\sigma) \leq \frac{1}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}\left(\operatorname{vol}\left(M_{\sigma^{\prime}, i} \triangle S_{i}\right)-\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)\right. \\
&\left.\quad+\operatorname{vol}\left(M_{\sigma^{\prime}, j} \triangle S_{j}\right)-\operatorname{vol}\left(M_{\sigma, j} \triangle S_{j}\right)\right) \\
&= \frac{1}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}\left(\operatorname{vol}\left(A_{x} \backslash S_{i}\right)-\operatorname{vol}\left(A_{x} \cap S_{i}\right)\right. \\
&\left.\quad+\operatorname{vol}\left(A_{x} \cap S_{j}\right)-\operatorname{vol}\left(A_{x} \backslash S_{j}\right)\right) \\
& \leq \frac{1}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)}\left(2 \cdot \operatorname{vol}\left(S_{x} \cap S_{j}\right)\right) \\
& \leq \frac{2 \cdot \operatorname{vol}\left(S_{j}\right)}{\min \left(\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right)} \\
& \leq 8
\end{aligned}
$$

where the last inequality follows by the fact that the clusters are almost balanced. We will consider the total introduced cost due to this case and Case 4 together, and so let's first examine Case 4.

Case 4: $\alpha<0, \beta>0$. In this case, we have

$$
\begin{aligned}
\epsilon\left(\sigma^{\prime}\right)-\epsilon(\sigma) & \leq \frac{1}{\operatorname{vol}\left(S_{j}\right)}\left(\operatorname{vol}\left(M_{\sigma^{\prime}, j} \triangle S_{j}\right)-\operatorname{vol}\left(M_{\sigma, j} \triangle S_{j}\right)\right) \\
& \leq \frac{1}{\operatorname{vol}\left(S_{j}\right)}\left(\operatorname{vol}\left(A_{x} \cap S_{j}\right)-\operatorname{vol}\left(A_{x} \backslash S_{j}\right)\right) \\
& \leq \frac{\operatorname{vol}\left(S_{j}\right)}{\operatorname{vol}\left(S_{j}\right)} \\
& =1
\end{aligned}
$$

Now, let us bound the total number of times we need to construct $\sigma^{\prime}$ in order to obtain a permutation. For any $i$ with $M_{\sigma, i}=\emptyset$, we have

$$
\frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}=\frac{\operatorname{vol}\left(S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}=1
$$

so the total number of required iterations is upper bounded by

$$
\left|\left\{i: M_{\sigma, i}=\emptyset\right\}\right| \leq \sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq 64 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}
$$

As such, the total introduced cost due to Cases 3 and 4 is at most

$$
8 \cdot 64 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}=512 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}
$$

Putting everything together, we have that

$$
\epsilon\left(\sigma^{\star}\right) \leq \epsilon(\sigma)+1024 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)} \leq 1088 \cdot(1+\mathrm{APT}) \cdot \frac{k}{\Upsilon(k)}
$$

This implies that

$$
\sum_{i=1}^{k} \operatorname{vol}\left(M_{\sigma^{\star}, i} \Delta S_{i}\right) \leq 2176 \cdot(1+\mathrm{APT}) \cdot \frac{\operatorname{vol}\left(V_{G}\right)}{\Upsilon(k)}
$$

and completes the proof.

It is worth noting that this method of upper bounding the ratio of misclassified vertices is very different from the ones used in previous references, e.g., [DPRS19, Miz21, PSZ17]. In particular, instead of examining all the possible mappings between $\left\{A_{i}\right\}_{i=1}^{k}$ and $\left\{S_{i}\right\}_{i=1}^{k}$, in this proof we directly work with some specifically defined function $\sigma$, and constructed our desired mapping $\sigma^{\star}$ from $\sigma$. This is another key for us to obtain stronger results than the previous work.

### 4.3 Future Work

Notice that the analysis presented in this chapter requires that the optimal clusters have almost-balanced size, which is the only technical assumption of our result. This raises a natural question on the necessity of such a condition.

Open Question 1. Is it possible for spectral clustering to theoretically achieve the same approximation guarantee as Theorem 4.2 when the optimal clusters of an input graph have arbitrary sizes? Alternatively, is there a hard instance showing that our presented analysis is tight up to some constant factor?

Secondly, it would be very interesting to see whether one can theoretically analyse the performance of spectral clustering with respect to $\lambda_{k+1} / \lambda_{k}$, instead of $\Upsilon(k)$.

Open Question 2. Is it possible to upper bound the number (or volume) of misclassified vertices from spectral clustering with respect to the ratio between $\lambda_{k+1}$ and $\lambda_{k}$ ?

Notice that, in contrast to $\Upsilon(k)$, the value of $\lambda_{k+1} / \lambda_{k}$ can be computed in polynomial time; therefore, an affirmative answer to the question will have more practical impact than our current presented result. In addition, such an answer would further justify the eigen-gap heuristic [vL07], helping to close the gap between the theoretical analysis of spectral clustering and its empirical performance.

## Chapter 5

## a

## Spectral Clustering with Meta-Graphs

In this chapter we build on the analysis of the last chapter and develop a variant of spectral clustering which employs fewer than $k$ eigenvectors to find $k$ clusters. We prove the surprising result that, when the structure among the optimal clusters in an input graph satisfies certain conditions, spectral clustering with fewer eigenvectors is able to produce better results than classical spectral clustering. This presents the opportunity to significantly speed up the runtime of spectral clustering in practice, since fewer eigenvectors are used to construct the embedding.

The starting point of our new analysis is the following observation: in many graphs arising from real-world data, the structure of clusters is not symmetric. For example, let us consider the well-known MNIST dataset of handwritten digits [Hul94]. This dataset consists of 70,000 images of handwritten digits 0 to 9 . Each image has $28 \times 28$ pixels, and each pixel has a greyscale value between 0 and 255. Figure 5.1 shows some example images from the MNIST dataset.

Then, we can represent each image in the dataset by a vector $\boldsymbol{x}_{i} \in \mathbb{R}^{28 \times 28}$ and construct


Figure 5.1: Example images from the MNIST dataset.


Figure 5.2: The $k$ nearest neighbour graph of a random subset of the MNIST dataset, for $k=3$. The graph has a cluster corresponding to each digit.
the $\boldsymbol{k}$ nearest neighbours graph, $G=(V, E)$, in the following way:

- for each data point $\boldsymbol{x}_{i}$ in the dataset, there is a corresponding vertex $v_{i} \in V$;
- for each $v_{i}$ there is an undirected edge to the $k$ vertices representing the data points with the smallest distances to $\boldsymbol{x}_{i}$.

Figure 5.2 illustrates the $k$ nearest neighbour graph for a randomly chosen subset of the data. Notice that some of the clusters in this graph are more similar than others. For example, the cluster representing the digit 4 has more connections with the cluster representing 9 than the one representing 6 . The central result of this chapter is that this asymmetry can be used to improve the performance of spectral clustering, and allow us to recover all 10 clusters while using fewer than 10 eigenvectors of the graph Laplacian.

Main Result 3 (See Theorem 5.2 for the formal statement). Given a graph $G$, let $S_{1}, \ldots, S_{k}$ be the clusters which achieve $\rho(k)$ and suppose that $S_{1}, \ldots, S_{k}$ are almost-balanced. Additionally, suppose that the clusters $S_{1}, \ldots, S_{k}$ exhibit some significant high-level structure. Then, spectral clustering with fewer than $k$ eigenvectors performs better than spectral clustering with $k$ eigenvectors.

The significance of this result is further demonstrated by experiments on the BSDS image segmentation dataset. It is well known that spectral methods are an effective technique for image segmentation [SM00, TWC10, ZZL ${ }^{+}$21], and in this chapter we see that spectral clustering is effective for image segmentation, and that using fewer eigenvectors often results in a better segmentation. Figure 5.3 shows examples in which, in order to find 6 and 45 clusters, spectral clustering with 3 and 7 eigenvectors produce better results than the ones with 6 and 45 eigenvectors. Section 5.3 describes the full experimental details.


Figure 5.3: Examples of image segmentation using spectral clustering; the original images are from the BSDS dataset. The Rand Index of segmentation (b) is 0.83 , while (c) has Rand Index 0.78 . Segmentation (e) has Rand Index 0.92, and (f) has Rand Index 0.80 . Hence, it's clear that spectral clustering with fewer than $k$ eigenvectors suffices to produce comparable or better output.

The result in this chapter is related to previous work on efficient spectral algorithms to find cluster-structures. While it's known that flow and path structures of clusters in directed graphs can be uncovered with complex-valued Hermitian matrices [CLSZ20, LS20], our result shows that one can apply real-valued Laplacians of undirected graphs, and find more general patterns of clusters.

### 5.1 Encoding the Cluster-Structure into Meta-Graphs

Suppose that $\left\{S_{i}\right\}_{i=1}^{k}$ is a partition of $V_{G}$ that corresponds to the $k$-way expansion $\rho(k)$ for an input graph $G$. We define the matrix $\mathbf{A}_{M} \in \mathbb{R}^{k \times k}$ by

$$
\mathbf{A}_{M}(i, j)= \begin{cases}w\left(S_{i}, S_{j}\right) & \text { if } i \neq j \\ 2 w\left(S_{i}, S_{j}\right) & \text { if } i=j\end{cases}
$$

and, taking $\mathbf{A}_{M}$ to be the adjacency matrix, this defines a graph $M=\left(V_{M}, E_{M}, w_{M}\right)$ which we refer to as the meta-graph of the clusters. Informally, the meta-graph is constructed by viewing every cluster $S_{i}$ as a 'giant vertex', and captures the structure of the optimal clusters. Then, define the normalised adjacency matrix of $M$ by $\mathbf{B}_{M} \triangleq \mathbf{D}_{M}^{-\frac{1}{2}} \mathbf{A}_{M} \mathbf{D}_{M}^{-\frac{1}{2}}$, and the normalised Laplacian matrix of $M$ by $\mathbf{N}_{M} \triangleq \mathbf{I}-\mathbf{B}_{M}$. Let the eigenvalues of $\mathbf{N}_{M}$ be $\gamma_{1} \leq \ldots \leq \gamma_{k}$, and $\boldsymbol{g}_{i} \in \mathbb{R}^{k}$ be the eigenvector corresponding to $\gamma_{i}$ for any $i \in[k]$.

The starting point of our novel approach is to look at the structure of this meta-graph $M$,
and study how the spectral information of $\mathbf{N}_{M} \in \mathbb{R}^{k \times k}$ is encoded in the bottom eigenvectors of $\mathbf{N}_{G}$. To achieve this, for any $\ell \in[k]$ and vertex $i \in V_{M}$, let

$$
\begin{equation*}
\overline{\boldsymbol{x}}^{(i)} \triangleq\left(\boldsymbol{g}_{1}(i), \ldots, \boldsymbol{g}_{\ell}(i)\right)^{\top} \tag{5.1}
\end{equation*}
$$

notice that $\overline{\boldsymbol{x}}^{(i)} \in \mathbb{R}^{\ell}$ defines the spectral embedding of $i \in V_{M}$ with the bottom $\ell$ eigenvectors of $\mathbf{N}_{M}$.

Definition $5.1((\theta, \ell)$-distinguishable graph $)$. For any $M=\left(V_{M}, E_{M}, w_{M}\right)$ with $k$ vertices, $\ell \in[k]$, and $\theta \in \mathbb{R}_{\geq 0}$, we say that $M$ is $(\theta, \ell)$-distinguishable if

- it holds for any $i \in[k]$ that $\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2} \geq \theta$, and
- it holds for any different $i, j \in[k]$ that $\left\|\frac{\bar{x}^{(i)}}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\overline{\boldsymbol{x}}^{(j)}}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right\|^{2} \geq \theta$.

In other words, graph $M$ is $(\theta, \ell)$-distinguishable if (i) every embedded point $\overline{\boldsymbol{x}}^{(i)}$ has squared length at least $\theta$, and (ii) any pair of different embedded points with normalisation are separated by a distance of at least $\theta$. By definition, it is easy to see that, if $M$ is $(\theta, \ell)$ distinguishable for some large value of $\theta$, then the embedded points $\left\{\overline{\boldsymbol{x}}^{(i)}\right\}_{i \in V_{M}}$ can be easily separated even if $\ell<k$. Examples 5.1 and 5.2 below demonstrate that this is indeed the case and, since the meta-graph $M$ is constructed from $\left\{S_{i}\right\}_{i=1}^{k}$, this well-separation property for $\left\{\overline{\boldsymbol{x}}^{(i)}\right\}_{i \in V_{M}}$ usually implies that the clusters $\left\{S_{i}\right\}_{i=1}^{k}$ are also well separated when the vertices are mapped to the points $\{F(u)\}_{u \in V_{G}}$, in which

$$
\begin{equation*}
F(u) \triangleq \frac{1}{\sqrt{d(u)}} \cdot\left(\boldsymbol{f}_{1}(u), \ldots \boldsymbol{f}_{\ell}(u)\right)^{\top} \tag{5.2}
\end{equation*}
$$

Example 5.1. Suppose the meta-graph is $C_{6}$, the cycle on 6 vertices. Figure 5.4(a) shows that the vertices of $C_{6}$ are well separated by the second and third eigenvectors of $\mathbf{N}_{C_{6}}{ }^{1}$ Since the minimum distance between any pair of vertices in this embedding is $2 / 3$, we say that $C_{6}$ is (2/3,3)-distinguishable. Figure 5.4(b) shows that, when using $\boldsymbol{f}_{2}, \boldsymbol{f}_{3}$ of $\mathbf{N}_{G}$ to embed the vertices of a 600 -vertex graph with a cyclical cluster pattern, the embedded points closely match the ones from the meta-graph. 2

Example 5.2. Suppose the meta-graph is $P_{4,4}$, which is the $4 \times 4$ grid graph. Figure 5.5(a) shows that the vertices are well separated using the second and third eigenvectors of $\mathbf{N}_{P_{4,4}}$. The minimum distance between any pair of vertices in this embedding is roughly 0.1 , and so $P_{4,4}$ is ( $0.1,3$ )-distinguishable. Figure 5.5(b) demonstrates that, when using $\boldsymbol{f}_{2}, \boldsymbol{f}_{3}$ of $\mathbf{N}_{G}$ to construct the embedding, the embedded points closely match the ones from the meta-graph.

[^4]

Figure 5.4: The spectral embedding of a large graph $G$ whose clusters exhibit a cyclical structure closely matches the embedding of the meta-graph $C_{6}$.


Figure 5.5: The spectral embedding of a large 1600-vertex graph $G$ whose clusters exhibit a grid structure closely matches the embedding of the meta-graph $P_{4,4}$.

From these examples, it is clear to see that there is a close connection between the embedding $\left\{\overline{\boldsymbol{x}}^{(i)}\right\}$ defined in (5.1) and the embedding $\{F(u)\}$ defined in (5.2). To formally analyse this connection, we develop a new structure theorem with meta-graphs.

We define vectors $\overline{\boldsymbol{g}}_{1}, \ldots, \overline{\boldsymbol{g}}_{k} \in \mathbb{R}^{n}$ which represent the eigenvectors of $\mathbf{N}_{M}$ 'blown-up' to $n$ dimensions. Formally, we define $\overline{\boldsymbol{g}}_{i}$ such that

$$
\overline{\boldsymbol{g}}_{i}=\sum_{j=1}^{k} \frac{\mathbf{D}^{\frac{1}{2}} \boldsymbol{\chi}_{j}}{\left\|\mathbf{D}^{\frac{1}{2}} \boldsymbol{\chi}_{j}\right\|} \cdot \boldsymbol{g}_{i}(j)
$$

where $\chi_{j} \in \mathbb{R}^{n}$ is the indicator vector of cluster $S_{j}$ defined in Section 2.3. By definition, it holds for any $u \in S_{j}$ that

$$
\overline{\boldsymbol{g}}_{i}(u)=\sqrt{\frac{d(u)}{\operatorname{vol}\left(S_{j}\right)}} \cdot \boldsymbol{g}_{i}(j)
$$

The following lemma shows that $\left\{\overline{\boldsymbol{g}}_{i}\right\}_{i=1}^{k}$ form an orthonormal basis.
Lemma 5.1. The following statements hold:

1. it holds for any $i \in[k]$ that $\left\|\overline{\boldsymbol{g}}_{i}\right\|=1$;
2. it holds for any different $i, j \in[k]$ that $\left\langle\overline{\boldsymbol{g}}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle=0$.

Proof. By definition, we have that

$$
\begin{aligned}
\left\|\overline{\boldsymbol{g}}_{i}\right\|^{2}=\overline{\boldsymbol{g}}_{i}^{\top} \overline{\boldsymbol{g}}_{i} & =\sum_{j=1}^{k} \sum_{u \in S_{j}} \overline{\boldsymbol{g}}_{i}(u)^{2} \\
& =\sum_{j=1}^{k} \sum_{u \in S_{j}}\left(\frac{\sqrt{d(u)}}{\sqrt{\operatorname{vol}\left(S_{j}\right)}} \cdot \boldsymbol{g}_{i}(j)\right)^{2} \\
& =\sum_{j=1}^{k} \boldsymbol{g}_{i}(j)^{2}=\left\|\boldsymbol{g}_{i}\right\|^{2}=1,
\end{aligned}
$$

which proves the first statement.
To prove the second statement, we have for any $i \neq j$ that

$$
\begin{aligned}
\left\langle\overline{\boldsymbol{g}}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle & =\sum_{x=1}^{k} \sum_{u \in S_{x}} \overline{\boldsymbol{g}}_{i}(u) \overline{\boldsymbol{g}}_{j}(u) \\
& =\sum_{x=1}^{k} \sum_{u \in S_{x}} \frac{d(u)}{\operatorname{vol}\left(S_{x}\right)} \cdot \boldsymbol{g}_{i}(x) \boldsymbol{g}_{j}(x) \\
& =\sum_{x=1}^{k} \boldsymbol{g}_{i}(x) \boldsymbol{g}_{j}(x) \\
& =\boldsymbol{g}_{i}^{\top} \boldsymbol{g}_{j}=0,
\end{aligned}
$$

which completes the proof.

The following lemma presents an important relationship between the eigenvalues of $\mathbf{N}_{M}$ and those of $\mathbf{N}_{G}$.

Lemma 5.2. It holds for any $i \in[k]$ that $\lambda_{i} \leq \gamma_{i}$.

Proof. Notice that we have for any $j \leq k$ that

$$
\begin{aligned}
\overline{\boldsymbol{g}}_{j}^{\top} \mathbf{N}_{G} \overline{\boldsymbol{g}}_{j} & =\sum_{(u, v) \in E_{G}} w_{G}(u, v)\left(\frac{\overline{\boldsymbol{g}}_{j}(u)}{\sqrt{d(u)}}-\frac{\overline{\boldsymbol{g}}_{j}(v)}{\sqrt{d(v)}}\right)^{2} \\
& =\sum_{x=1}^{k-1} \sum_{y=x+1}^{k} \sum_{a \in S_{x}} \sum_{b \in S_{y}} w(a, b)\left(\frac{\overline{\boldsymbol{g}}_{j}(a)}{\sqrt{d(a)}}-\frac{\overline{\boldsymbol{g}}_{j}(b)}{\sqrt{d(b)}}\right)^{2} \\
& =\sum_{x=1}^{k-1} \sum_{y=x+1}^{k} w\left(S_{x}, S_{y}\right)\left(\frac{\boldsymbol{g}_{j}(x)}{\sqrt{\operatorname{vol}\left(S_{x}\right)}}-\frac{\boldsymbol{g}_{j}(y)}{\sqrt{\operatorname{vol}\left(S_{y}\right)}}\right)^{2} \\
& =\boldsymbol{g}_{j}^{\top} \mathbf{N}_{M} \boldsymbol{g}_{j}=\gamma_{j} .
\end{aligned}
$$

By Lemma 5.1, we have an $i$-dimensional subspace $S_{i} \subset \mathbb{R}^{n}$ such that

$$
\max _{\boldsymbol{x} \in S_{i}} \frac{\boldsymbol{x}^{\top} \mathbf{N}_{G} \boldsymbol{x}}{\boldsymbol{x}^{\top} \boldsymbol{x}}=\gamma_{i},
$$

from which the lemma follows by the Courant-Fischer theorem (Theorem 2.1).

Next, similar to the function $\Upsilon(k)$ defined in (4.1), for any input graph $G=\left(V_{G}, E_{G}, w_{G}\right)$ and $(\theta, \ell)$-distinguishable meta-graph $M$, we define the function $\Psi(\ell)$ by

$$
\Psi(\ell) \triangleq \sum_{i=1}^{\ell} \frac{\gamma_{i}}{\lambda_{\ell+1}}
$$

Notice that we have by the higher-order Cheeger inequality that $\gamma_{i} / 2 \leq \rho_{M}(i)$ holds for any $i \in[\ell]$, and $\rho_{M}(i) \leq \rho_{G}(k)$ by the construction of matrix $\mathbf{A}_{M}$. Hence, one can view $\Psi(\ell)$ as a refined version of $l / \Upsilon(k)$.

We now see that the vectors $\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{\ell}$ and $\overline{\boldsymbol{g}}_{1}, \ldots, \overline{\boldsymbol{g}}_{\ell}$ are well approximated by each other. In order to show this, we define for any $i \in[\ell]$ the vectors

$$
\widehat{\boldsymbol{f}}_{i}=\sum_{j=1}^{\ell}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j} \quad \text { and } \quad \widehat{\boldsymbol{g}}_{i}=\sum_{j=1}^{\ell}\left\langle\boldsymbol{f}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle \overline{\boldsymbol{g}}_{j},
$$

and can prove the following structure theorem with meta-graphs.
Theorem 5.1 (The Structure Theorem with Meta-Graphs). The following statements hold:

1. it holds for any $i \in[\ell]$ that $\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2} \leq \gamma_{i} / \lambda_{\ell+1}$;
2. it holds for any $\ell \in[k]$ that

$$
\sum_{i=1}^{\ell}\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} \leq \Psi(\ell)
$$

Proof. For the first statement, we write $\overline{\boldsymbol{g}}_{i}$ as a linear combination of the vectors $\boldsymbol{f}_{1}, \ldots, \boldsymbol{f}_{n}$, i.e., $\overline{\boldsymbol{g}}_{i}=\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}$. Since $\widehat{\boldsymbol{f}}_{i}$ is a projection of $\overline{\boldsymbol{g}}_{i}$, we have that $\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}$ is perpendicular to $\widehat{\boldsymbol{f}_{i}}$, and that

$$
\begin{aligned}
\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2} & =\left\|\overline{\boldsymbol{g}}_{i}\right\|^{2}-\left\|\widehat{\boldsymbol{f}}_{i}\right\|^{2} \\
& =\left(\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2}\right)-\left(\sum_{j=1}^{\ell}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2}\right) \\
& =\sum_{j=\ell+1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2} .
\end{aligned}
$$

Now, we study the quadratic form $\overline{\boldsymbol{g}}_{i}^{\top} \mathbf{N}_{G} \overline{\boldsymbol{g}}_{i}$ and have that

$$
\begin{aligned}
\overline{\boldsymbol{g}}_{i}^{\top} \mathbf{N}_{G} \overline{\boldsymbol{g}}_{i} & =\left(\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}^{\top}\right) \mathbf{N}_{G}\left(\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle \boldsymbol{f}_{j}\right) \\
& =\sum_{j=1}^{n}\left\langle\overline{\boldsymbol{g}}_{i}, \boldsymbol{f}_{j}\right\rangle^{2} \lambda_{j} \\
& \geq \lambda_{\ell+1}\left\|\overline{\boldsymbol{g}}_{i}-\widehat{\boldsymbol{f}}_{i}\right\|^{2}
\end{aligned}
$$

By the proof of Lemma 5.2, we have that $\overline{\boldsymbol{g}}_{i}^{\top} \mathbf{N}_{G} \overline{\boldsymbol{g}}_{i} \leq \gamma_{i}$, from which the first statement follows.

Now we prove the second statement. We define the vectors $\overline{\boldsymbol{g}}_{k+1}, \ldots, \overline{\boldsymbol{g}}_{n}$ to be an arbitrary orthonormal basis of the space orthogonal to the space spanned by $\overline{\boldsymbol{g}}_{1}, \ldots, \overline{\boldsymbol{g}}_{k}$. Then, we can write any $\boldsymbol{f}_{i}$ as $\boldsymbol{f}_{i}=\sum_{j=1}^{n}\left\langle\boldsymbol{f}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle \overline{\boldsymbol{g}}_{j}$, and have that

$$
\begin{aligned}
\sum_{i=1}^{\ell}\left\|\boldsymbol{f}_{i}-\widehat{\boldsymbol{g}}_{i}\right\|^{2} & =\sum_{i=1}^{\ell}\left(\left\|\boldsymbol{f}_{i}\right\|^{2}-\left\|\widehat{\boldsymbol{g}}_{i}\right\|^{2}\right) \\
& =\ell-\sum_{i=1}^{\ell} \sum_{j=1}^{\ell}\left\langle\boldsymbol{f}_{i}, \overline{\boldsymbol{g}}_{j}\right\rangle^{2} \\
& =\sum_{j=1}^{\ell}\left(1-\sum_{i=1}^{\ell}\left\langle\overline{\boldsymbol{g}}_{j}, \boldsymbol{f}_{i}\right\rangle^{2}\right) \\
& =\sum_{j=1}^{\ell}\left(\left\|\overline{\boldsymbol{g}}_{j}\right\|^{2}-\left\|\widehat{\boldsymbol{f}}_{j}\right\|^{2}\right) \\
& =\sum_{j=1}^{\ell}\left\|\overline{\boldsymbol{g}}_{j}-\widehat{\boldsymbol{f}}_{j}\right\|^{2} \leq \sum_{j=1}^{\ell} \frac{\gamma_{j}}{\lambda_{\ell+1}},
\end{aligned}
$$

where the final inequality follows by the first statement of the theorem.

### 5.2 Spectral Clustering with Fewer Eigenvectors

In this section, we analyse spectral clustering with fewer eigenvectors. The algorithm is essentially the same as the standard spectral clustering described in Section 4.2, with the only difference that every $u \in V_{G}$ is embedded into a point in $\mathbb{R}^{\ell}$ by the mapping defined in (5.2). The analysis follows from the one in Section 4.2 at a very high level. However, since we require that $\{F(u)\}_{u \in V_{G}}$ are well separated in $\mathbb{R}^{\ell}$ for some $\ell<k$, the proof is more involved. For any $i \in[k]$, we define the approximate centre $\boldsymbol{p}^{(i)} \in \mathbb{R}^{\ell}$ of every cluster $S_{i}$ by

$$
\boldsymbol{p}^{(i)}(j)=\frac{1}{\sqrt{\operatorname{vol}\left(S_{i}\right)}} \cdot \sum_{x=1}^{\ell}\left\langle\boldsymbol{f}_{j}, \overline{\boldsymbol{g}}_{x}\right\rangle \cdot \boldsymbol{g}_{x}(i),
$$

and prove that the total $k$-means cost for the points $\{F(u)\}_{u \in V_{G}}$ can be upper bounded with respect to $\Psi(\ell)$.

Lemma 5.3. It holds that

$$
\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|F(u)-\boldsymbol{p}^{(i)}\right\|^{2} \leq \Psi(\ell)
$$

Proof. By definition, it holds that

$$
\begin{aligned}
& \sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|F(u)-\boldsymbol{p}^{(i)}\right\|^{2} \\
& =\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left[\sum_{j=1}^{\ell}\left(\frac{\boldsymbol{f}_{j}(u)}{\sqrt{d(u)}}-\left(\sum_{x=1}^{\ell}\left\langle\boldsymbol{f}_{j}, \overline{\boldsymbol{g}}_{x}\right\rangle \frac{\boldsymbol{g}_{x}(i)}{\sqrt{\operatorname{vol}\left(S_{i}\right)}}\right)\right)^{2}\right] \\
& =\sum_{i=1}^{k} \sum_{u \in S_{i}} \sum_{j=1}^{\ell}\left(\boldsymbol{f}_{j}(u)-\left(\sum_{x=1}^{\ell}\left\langle\boldsymbol{f}_{j}, \overline{\boldsymbol{g}}_{x}\right\rangle \overline{\boldsymbol{g}}_{x}(u)\right)\right)^{2} \\
& =\sum_{i=1}^{k} \sum_{u \in S_{i}} \sum_{j=1}^{\ell}\left(\boldsymbol{f}_{j}(u)-\widehat{\boldsymbol{g}}_{j}(u)\right)^{2} \\
& =\sum_{j=1}^{\ell}\left\|\boldsymbol{f}_{j}-\widehat{\boldsymbol{g}}_{j}\right\|^{2} \leq \Psi(\ell)
\end{aligned}
$$

where the final inequality follows from the second statement of Theorem 5.1 .

We now prove a sequence of lemmas establishing that the distance between different $\boldsymbol{p}^{(i)}$ and $\boldsymbol{p}^{(j)}$ can be bounded with respect to $\theta$ and $\Psi(\ell)$.

Lemma 5.4. It holds for $i \in[k]$ that

$$
\left(1-\frac{4 \sqrt{\Psi(\ell)}}{\theta}\right) \frac{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}}{\operatorname{vol}\left(S_{i}\right)} \leq\left\|\boldsymbol{p}^{(i)}\right\|^{2} \leq \frac{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}}{\operatorname{vol}\left(S_{i}\right)}\left(1+\frac{2 \sqrt{\Psi(\ell)}}{\theta}\right)
$$

Proof. It holds by definition that

$$
\begin{aligned}
& \operatorname{vol}\left(S_{i}\right) \cdot\left\|\boldsymbol{p}^{(i)}\right\|^{2} \\
& =\sum_{x=1}^{\ell}\left(\sum_{y=1}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle \boldsymbol{g}_{y}(i)\right)^{2} \\
& =\sum_{x=1}^{\ell} \sum_{y=1}^{\ell} \sum_{z=1}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{z}\right\rangle \boldsymbol{g}_{y}(i) \boldsymbol{g}_{z}(i) \\
& =\sum_{x=1}^{\ell} \sum_{y=1}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle^{2} \boldsymbol{g}_{y}(i)^{2}+\sum_{x=1}^{\ell} \sum_{y=1}^{\ell} \sum_{\substack{z=1 \\
z \neq y}}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{z}\right\rangle \boldsymbol{g}_{y}(i) \boldsymbol{g}_{z}(i)
\end{aligned}
$$

$$
\begin{equation*}
=\sum_{x=1}^{\ell} \sum_{y=1}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle^{2} \boldsymbol{g}_{y}(i)^{2}+\sum_{y=1}^{\ell} \sum_{\substack{z=1 \\ z \neq y}}^{\ell} \boldsymbol{g}_{y}(i) \boldsymbol{g}_{z}(i) \cdot\left(\widehat{\boldsymbol{f}}_{y}^{\top} \widehat{\boldsymbol{f}}_{z}\right) . \tag{5.3}
\end{equation*}
$$

We study the two terms of (5.3) separately. For the second term, we have that

$$
\begin{aligned}
& \sum_{y=1}^{\ell} \sum_{\substack{z=1 \\
z \neq y}}^{\ell} \boldsymbol{g}_{y}(i) \boldsymbol{g}_{z}(i) \cdot\left(\widehat{\boldsymbol{f}}_{y}^{\top} \widehat{\boldsymbol{f}_{z}}\right) \\
& \leq \sum_{y=1}^{\ell}\left|\boldsymbol{g}_{y}(i)\right| \sum_{\substack{z=1 \\
z \neq y}}^{\ell}\left|\boldsymbol{g}_{z}(i)\right|\left|\widehat{\boldsymbol{f}}_{y}^{\top} \widehat{\boldsymbol{f}}_{z}\right| \\
& =\sum_{y=1}^{\ell}\left|\boldsymbol{g}_{y}(i)\right| \sum_{\substack{z=1 \\
z \neq y}}^{\ell}\left|\boldsymbol{g}_{z}(i)\right|\left|\left(\overline{\boldsymbol{g}}_{y}+\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}\right)^{\top}\left(\overline{\boldsymbol{g}}_{z}+\widehat{\boldsymbol{f}}_{z}-\overline{\boldsymbol{g}}_{z}\right)\right| \\
& =\sum_{y=1}^{\ell}\left|\boldsymbol{g}_{y}(i)\right| \sum_{\substack{z=1 \\
z \neq y}}^{\ell}\left|\boldsymbol{g}_{z}(i)\right|\left|\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle+\left\langle\widehat{\boldsymbol{f}}_{z}-\overline{\boldsymbol{g}}_{z}, \overline{\boldsymbol{g}}_{y}\right\rangle+\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \widehat{\boldsymbol{f}}_{z}-\overline{\boldsymbol{g}}_{z}\right\rangle\right| \\
& =\sum_{y=1}^{\ell}\left|\boldsymbol{g}_{y}(i)\right| \sum_{\substack{z=1 \\
z \neq y}}^{\ell}\left|\boldsymbol{g}_{z}(i)\right|\left|\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle\right| \\
& \leq \sqrt{\left(\sum_{y=1}^{\ell} \boldsymbol{g}_{y}(i)^{2}\right) \sum_{y=1}^{\ell}\left(\sum_{\substack{1 \leq z \leq \ell \\
z \neq y}}\left|\boldsymbol{g}_{z}(i)\right| \mid\left\langle\widehat{\boldsymbol{f}_{y}}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle\right)^{2}} \\
& \leq \sqrt{\sum_{y=1}^{\ell}\left(\sum_{\substack{1 \leq z \leq \ell \\
z \neq y}} \boldsymbol{g}_{z}(i)^{2}\right)\left(\sum_{\substack{1 \leq \leq \leq \ell \\
z \neq y}}\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle^{2}\right)} \\
& \leq \sqrt{\sum_{y=1}^{\ell} \sum_{\substack{1 \leq z \leq \ell \\
z \neq y}}\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle^{2}} \\
& \leq \sqrt{\sum_{y=1}^{\ell}\left\|\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}\right\|^{2}} \leq \sqrt{\Psi(\ell)},
\end{aligned}
$$

where we used the fact that $\sum_{y=1}^{k} \boldsymbol{g}_{y}(i)^{2}=1$ for all $i \in[k]$. Therefore, we have that

$$
\begin{aligned}
\operatorname{vol}\left(S_{i}\right)\left\|\boldsymbol{p}^{(i)}\right\|^{2} & \leq \sum_{y=1}^{\ell}\left(\sum_{x=1}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle^{2}\right) \boldsymbol{g}_{y}(i)^{2}+\sqrt{\Psi(\ell)} \\
& \leq \sum_{y=1}^{\ell} \boldsymbol{g}_{y}(i)^{2}+\sqrt{\Psi(\ell)} \\
& \leq\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}+\sqrt{\Psi(\ell)} \leq\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}\left(1+\frac{2 \sqrt{\Psi(\ell)}}{\theta}\right) .
\end{aligned}
$$

On the other hand, we have that

$$
\begin{aligned}
\operatorname{vol}\left(S_{i}\right)\left\|\boldsymbol{p}^{(i)}\right\|^{2} & \geq \sum_{y=1}^{\ell}\left(\sum_{x=1}^{\ell}\left\langle\boldsymbol{f}_{x}, \overline{\boldsymbol{g}}_{y}\right\rangle^{2}\right) \boldsymbol{g}_{y}(i)^{2}-2 \sqrt{\Psi(\ell)} \\
& \geq \sum_{y=1}^{\ell}\left\|\widehat{\boldsymbol{f}}_{y}\right\|^{2} \boldsymbol{g}_{y}(i)^{2}-2 \sqrt{\Psi(\ell)} \\
& \geq(1-\Psi(\ell))\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}-2 \sqrt{\Psi(\ell)} \\
& \geq\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}\left(1-\frac{4 \sqrt{\Psi(\ell)}}{\theta}\right)
\end{aligned}
$$

where the last inequality holds by the fact that $\theta \leq\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2} \leq 1$ and $\Psi(\ell)<1$. Hence, the statement holds.

Lemma 5.5. It holds for any different $i, j \in[k]$ that

$$
\left\|\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot \boldsymbol{p}^{(i)}-\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot \boldsymbol{p}^{(j)}\right\|^{2} \geq \theta-3 \sqrt{\Psi(\ell)}
$$

Proof. By definition, it holds that

$$
\begin{aligned}
& \left\|\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot \boldsymbol{p}^{(i)}-\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot \boldsymbol{p}^{(j)}\right\|^{2} \\
& =\sum_{t=1}^{\ell}\left(\sum_{y=1}^{\ell}\left\langle\boldsymbol{f}_{t}, \overline{\boldsymbol{g}}_{y}\right\rangle\left(\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right)\right)^{2} \\
& =\sum_{t=1}^{\ell} \sum_{y=1}^{\ell}\left\langle\boldsymbol{f}_{t}, \overline{\boldsymbol{g}}_{y}\right\rangle^{2}\left(\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right)^{2} \\
& \quad+\sum_{t=1}^{\ell} \sum_{y=1}^{\ell} \sum_{\substack{\leq z \leq \ell \\
z \neq y}}\left\langle\boldsymbol{f}_{t}, \overline{\boldsymbol{g}}_{y}\right\rangle\left\langle\boldsymbol{f}_{t}, \overline{\boldsymbol{g}}_{z}\right\rangle\left(\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right)\left(\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right) .
\end{aligned}
$$

We upper bound the second term by

$$
\begin{aligned}
& \sum_{y=1}^{\ell}\left|\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right| \sum_{\substack{1 \leq z \leq \ell \\
z \neq y}}\left|\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right| \sum_{t=1}^{\ell}\left|\left\langle\boldsymbol{f}_{t}, \overline{\boldsymbol{g}}_{y}\right\rangle\left\langle\boldsymbol{f}_{t}, \overline{\boldsymbol{g}}_{x}\right\rangle\right| \\
= & \sum_{y=1}^{\ell}\left|\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right| \sum_{\substack{1 \leq z \leq \ell \\
z \neq y}}\left|\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right|\left|\widehat{\boldsymbol{f}}_{y}^{\top} \widehat{\boldsymbol{f}}_{z}\right| \\
= & \sum_{y=1}^{\ell}\left|\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right| \sum_{\substack{1 \leq z \leq \ell \\
z \neq y}}\left|\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right|\left|\left(\overline{\boldsymbol{g}}_{y}+\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}\right)^{\top}\left(\overline{\boldsymbol{g}}_{z}+\widehat{\boldsymbol{f}}_{z}-\overline{\boldsymbol{g}}_{z}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{y=1}^{\ell}\left|\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right| \sum_{\substack{\leq z \leq \ell \\
z \neq y}}\left|\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right|\left|\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle\right| \\
& \leq \sqrt{\left(\sum_{y=1}^{\ell}\left(\frac{\boldsymbol{g}_{y}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{y}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right)^{2}\right) \sum_{y=1}^{\ell}\left(\sum_{\substack{1 \leq z \leq \ell \\
z \neq y}}\left|\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right|\left|\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle\right|\right)^{2}} \\
& \leq \sqrt{2 \sum_{y=1}^{\ell}\left(\sum_{\substack{\leq z \leq \ell \\
z \neq y}}\left(\frac{\boldsymbol{g}_{z}(i)}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|}-\frac{\boldsymbol{g}_{z}(j)}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|}\right)^{2}\right)\left(\sum_{\substack{\leq z \leq \ell \\
z \neq y}}\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle^{2}\right)} \\
& \leq 2 \\
& \sqrt{\sum_{y=1}^{\ell} \sum_{1 \leq z \leq \ell}^{z \neq y}\left\langle\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}, \overline{\boldsymbol{g}}_{z}\right\rangle^{2}} \\
& \leq 2 \sqrt{\sum_{y=1}^{\ell}\left\|\widehat{\boldsymbol{f}}_{y}-\overline{\boldsymbol{g}}_{y}\right\|^{2}} \\
& \leq 2 \sqrt{\Psi(\ell)}
\end{aligned}
$$

from which we conclude that

$$
\begin{aligned}
\left\|\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot \boldsymbol{p}^{(i)}-\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot \boldsymbol{p}^{(j)}\right\|^{2} & \geq(1-\Psi(\ell)) \theta-2 \sqrt{\Psi(\ell)} \\
& \geq \theta-3 \sqrt{\Psi(\ell)}
\end{aligned}
$$

Lemma 5.6. It holds for any different $i, j \in[k]$ that

$$
\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\|^{2} \geq \frac{\theta}{4}-8 \sqrt{\frac{\Psi}{\theta}} .
$$

Proof. To follow the proof, it may help to refer to the illustration in Figure 5.6 .
We set the parameter $\epsilon=4 \sqrt{\Psi} / \theta$, and define

$$
\boldsymbol{a}_{i}=\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{(1+\epsilon)\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot \boldsymbol{p}^{(i)}, \quad \quad \boldsymbol{a}_{j}=\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{(1+\epsilon)\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot \boldsymbol{p}^{(j)}
$$

By the definition of $\epsilon$ and Lemma 5.4, it holds that $\left\|\boldsymbol{a}_{i}\right\| \leq 1$, and $\left\|\boldsymbol{a}_{j}\right\| \leq 1$. We can also assume without loss of generality that $\left\|\boldsymbol{a}_{i}\right\| \leq\left\|\boldsymbol{a}_{j}\right\|$. Then, as illustrated in Figure 5.6, we have

$$
\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\|^{2} \geq\left\|\boldsymbol{a}_{i}-\frac{\left\|\boldsymbol{a}_{i}\right\|}{\left\|\boldsymbol{a}_{j}\right\|} \boldsymbol{a}_{j}\right\|^{2}
$$



Figure 5.6: Illustration of the proof of Lemma 5.6. Our goal is to give a lower bound on the length of $\left(\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right)$, which is the blue dashed line in the figure. We instead calculate a lower bound on the length of $\left(\boldsymbol{a}_{i}-\frac{\left\|\boldsymbol{a}_{i}\right\|}{\left\|\boldsymbol{a}_{j}\right\|} \boldsymbol{a}_{j}\right)$, which is the red dashed line, and use the fact that by construction $\left\|\boldsymbol{a}_{i}\right\| \leq 1$ and $\left\|\boldsymbol{a}_{j}\right\| \leq 1$.
and so it suffices to lower bound the right-hand side of the inequality above. By the triangle inequality, we have

$$
\begin{aligned}
\left\|\boldsymbol{a}_{i}-\frac{\left\|\boldsymbol{a}_{i}\right\|}{\left\|\boldsymbol{a}_{j}\right\|} \boldsymbol{a}_{j}\right\| & \geq\left\|\boldsymbol{a}_{i}-\boldsymbol{a}_{j}\right\|-\left\|\boldsymbol{a}_{j}-\frac{\left\|\boldsymbol{a}_{i}\right\|}{\left\|\boldsymbol{a}_{j}\right\|} \boldsymbol{a}_{j}\right\| \\
& =\frac{1}{1+\epsilon}\left\|\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot \boldsymbol{p}^{(i)}-\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot \boldsymbol{p}^{(j)}\right\|-\left(\left\|\boldsymbol{a}_{j}\right\|-\left\|\boldsymbol{a}_{i}\right\|\right) .
\end{aligned}
$$

Now, we have that

$$
\begin{aligned}
\left\|\boldsymbol{a}_{j}\right\|-\left\|\boldsymbol{a}_{i}\right\| & =\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{(1+\epsilon)\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot\left\|\boldsymbol{p}^{(j)}\right\|-\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{(1+\epsilon)\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot\left\|\boldsymbol{p}^{(i)}\right\| \\
& \leq 1-\frac{1-\epsilon}{1+\epsilon}=\frac{2 \epsilon}{1+\epsilon}
\end{aligned}
$$

and have by Lemma 5.5 that

$$
\left\|\frac{\sqrt{\operatorname{vol}\left(S_{i}\right)}}{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|} \cdot \boldsymbol{p}^{(i)}-\frac{\sqrt{\operatorname{vol}\left(S_{j}\right)}}{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|} \cdot \boldsymbol{p}^{(j)}\right\| \geq \sqrt{\theta-3 \sqrt{\Psi(\ell)}} \geq \sqrt{\theta}-\sqrt{2 \epsilon} \geq \sqrt{\theta}-2 \epsilon
$$

since $\epsilon=4 \sqrt{\Psi} / \theta<1$ by the assumption on $\Psi$. This gives us

$$
\left\|\boldsymbol{a}_{i}-\frac{\left\|\boldsymbol{a}_{i}\right\|}{\left\|\boldsymbol{a}_{j}\right\|} \boldsymbol{a}_{j}\right\| \geq \frac{\sqrt{\theta}-2 \epsilon}{1+\epsilon}-\frac{2 \epsilon}{1+\epsilon} \geq \frac{1}{2}(\sqrt{\theta}-4 \epsilon) .
$$

Finally, we have that

$$
\left\|\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}-\frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\|^{2} \geq \frac{1}{4}\left(\sqrt{\theta}-16 \frac{\sqrt{\Psi(\ell)}}{\theta}\right)^{2} \geq \frac{\theta}{4}-8 \sqrt{\frac{\Psi(\ell)}{\theta}}
$$

which completes the proof.

Lemma 5.7. It holds for different $i, j \in[k]$ that

$$
\left\|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(j)}\right\|^{2} \geq \frac{\theta^{2}-20 \sqrt{\theta \cdot \Psi(\ell)}}{16 \min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}
$$

Proof. We assume without loss of generality that $\left\|\boldsymbol{p}^{(i)}\right\|^{2} \geq\left\|\boldsymbol{p}^{(j)}\right\|^{2}$. Then, by Lemma 5.4 and the fact that $\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2} \geq \theta$ holds for any $i \in[k]$, we have

$$
\left\|\boldsymbol{p}^{(i)}\right\|^{2} \geq\left(1-\frac{4 \sqrt{\Psi(\ell)}}{\theta}\right) \cdot \frac{\left\|\overline{\boldsymbol{x}}^{(i)}\right\|^{2}}{\operatorname{vol}\left(S_{i}\right)}
$$

and

$$
\left\|\boldsymbol{p}^{(j)}\right\|^{2} \geq\left(1-\frac{4 \sqrt{\Psi(\ell)}}{\theta}\right) \cdot \frac{\left\|\overline{\boldsymbol{x}}^{(j)}\right\|^{2}}{\operatorname{vol}\left(S_{j}\right)}
$$

which implies that

$$
\left\|\boldsymbol{p}^{(i)}\right\|^{2} \geq \frac{\theta-4 \sqrt{\Psi(\ell)}}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}
$$

Now, we proceed by case distinction.
Case 1: $\left\|\boldsymbol{p}^{(i)}\right\| \geq 4\left\|\boldsymbol{p}^{(j)}\right\|$. In this case, we have

$$
\left\|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(j)}\right\| \geq\left\|\boldsymbol{p}^{(i)}\right\|-\left\|\boldsymbol{p}^{(j)}\right\| \geq \frac{3}{4}\left\|\boldsymbol{p}^{(i)}\right\|
$$

and

$$
\begin{aligned}
\left\|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(j)}\right\|^{2} & \geq \frac{9}{16} \cdot \frac{\theta-4 \sqrt{\Psi(\ell)}}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \\
& \geq \frac{\theta(\theta-20 \sqrt{\Psi(\ell) / \theta})}{16 \min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \\
& =\frac{\theta^{2}-20 \sqrt{\theta \cdot \Psi(\ell)}}{16 \min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}
\end{aligned}
$$

since $\theta<1$.
Case 2: $\left\|\boldsymbol{p}^{(j)}\right\|=\alpha\left\|\boldsymbol{p}^{(i)}\right\|$ for some $\alpha \in\left(\frac{1}{4}, 1\right]$. By Lemma 5.6, we have

$$
\left\langle\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|}, \frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\rangle \leq 1-\frac{1}{2}\left(\frac{\theta}{4}-8 \sqrt{\frac{\Psi}{\theta}}\right) \leq 1-\frac{\theta}{8}+2 \sqrt{\frac{\Psi}{\theta}} .
$$

Then, it holds that

$$
\begin{aligned}
& \left\|\boldsymbol{p}^{(i)}-\boldsymbol{p}^{(j)}\right\|^{2} \\
& =\left\|\boldsymbol{p}^{(i)}\right\|^{2}+\left\|\boldsymbol{p}^{(j)}\right\|^{2}-2\left\langle\frac{\boldsymbol{p}^{(i)}}{\left\|\boldsymbol{p}^{(i)}\right\|} \frac{\boldsymbol{p}^{(j)}}{\left\|\boldsymbol{p}^{(j)}\right\|}\right\rangle\left\|\boldsymbol{p}^{(i)}\right\|\left\|\boldsymbol{p}^{(j)}\right\| \\
& \geq\left(1+\alpha^{2}\right)\left\|\boldsymbol{p}^{(i)}\right\|^{2}-2\left(1-\frac{\theta}{8}+2 \sqrt{\frac{\Psi(\ell)}{\theta}}\right) \alpha\left\|\boldsymbol{p}^{(i)}\right\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \geq\left(1+\alpha^{2}-2 \alpha+\frac{\theta}{4} \alpha-4 \sqrt{\frac{\Psi(\ell)}{\theta}} \alpha\right)\left\|\boldsymbol{p}^{(i)}\right\|^{2} \\
& \geq\left(\frac{\theta}{4}-4 \sqrt{\frac{\Psi(\ell)}{\theta}}\right) \cdot \alpha \cdot \frac{\theta-4 \sqrt{\Psi(\ell)}}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \\
& \geq\left(\frac{\theta}{16}-\sqrt{\frac{\Psi(\ell)}{\theta}}\right)(\theta-4 \sqrt{\Psi(\ell)}) \cdot \frac{1}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \\
& \geq\left(\frac{\theta^{2}}{16}-\frac{5}{4} \sqrt{\theta \Psi(\ell)}\right) \cdot \frac{1}{\min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \\
& =\frac{\theta^{2}-20 \sqrt{\theta \Psi(\ell)}}{16 \min \left\{\operatorname{vol}\left(S_{i}\right), \operatorname{vol}\left(S_{j}\right)\right\}}
\end{aligned}
$$

which completes the proof.

It is important to recognise that the lower bound in Lemma 5.7 implies a condition on $\theta$ and $\Psi(\ell)$ under which $\boldsymbol{p}^{(i)}$ and $\boldsymbol{p}^{(j)}$ are well separated. With this, we analyse the performance of spectral clustering when fewer eigenvector are employed to construct the embedding and show that it works when the optimal clusters present a noticeable pattern.

Lemma 5.8. Let $A_{1}, \ldots A_{k}$ be the output of spectral clustering with $\ell$ eigenvectors, and $\sigma$ and $M_{\sigma, i}$ be defined as in (4.5) and (4.6). If $\Psi(\ell) \leq \theta^{3} / 40^{2}$, then

$$
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} \leq 64(1+A P T) \frac{\Psi(\ell)}{\theta^{2}}
$$

Proof. Let us define $B_{i j}=A_{i} \cap S_{j}$ to be the vertices in $A_{i}$ which belong to the ground-truth cluster $S_{j}$. Then, we have that

$$
\begin{align*}
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} & =\sum_{i=1}^{k} \sum_{\substack{j=1 \\
j \neq \sigma(i)}}^{k} \operatorname{vol}\left(B_{i j}\right)\left(\frac{1}{\operatorname{vol}\left(S_{\sigma(i)}\right)}+\frac{1}{\operatorname{vol}\left(S_{j}\right)}\right) \\
& \leq 2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\
j \neq \sigma(i)}}^{k} \frac{\operatorname{vol}\left(B_{i j}\right)}{\min \left\{\operatorname{vol}\left(S_{\sigma(i)}\right), \operatorname{vol}\left(S_{j}\right)\right\}} \tag{5.4}
\end{align*}
$$

and that

$$
\begin{aligned}
\operatorname{COST}\left(A_{1}, \ldots A_{k}\right) & =\sum_{i=1}^{k} \sum_{u \in A_{i}} d(u)\left\|F(u)-\boldsymbol{c}_{i}\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} d(u)\left\|F(u)-\boldsymbol{c}_{i}\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} d(u)\left(\frac{\left\|\boldsymbol{p}^{(j)}-\boldsymbol{c}_{i}\right\|^{2}}{2}-\left\|\boldsymbol{p}^{(j)}-F(u)\right\|^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \sum_{u \in B_{i j}} \frac{d(u)\left\|\boldsymbol{p}^{(j)}-\boldsymbol{p}^{(\sigma(i))}\right\|^{2}}{8} \\
& \quad-\sum_{i=1}^{k} \sum_{\substack{\leq j \leq k \\
j \neq i}} \sum_{u \in B_{i j}} d(u)\left\|\boldsymbol{p}^{(j)}-F(u)\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \operatorname{vol}\left(B_{i j}\right) \frac{\left\|\boldsymbol{p}^{(j)}-\boldsymbol{p}^{(\sigma(i))}\right\|^{2}}{8}-\sum_{i=1}^{k} \sum_{u \in S_{i}} d(u)\left\|\boldsymbol{p}^{(i)}-F(u)\right\|^{2} \\
& \geq \sum_{i=1}^{k} \sum_{\substack{1 \leq j \leq k \\
j \neq \sigma(i)}} \frac{\operatorname{vol}\left(B_{i j}\right)}{16 \cdot \min \left\{\operatorname{vol}\left(S_{\sigma(i)}\right), \operatorname{vol}\left(S_{j}\right)\right\}}\left(\theta^{2}-20 \sqrt{\theta \cdot \Psi(\ell)}\right) \\
& \geq \frac{1}{32} \cdot\left(\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \Delta S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)}\right)\left(\theta^{2}-20 \sqrt{\theta \cdot \Psi(\ell)}\right)-\Psi(\ell),
\end{aligned}
$$

where the second inequality follows by the inequality $\|\boldsymbol{a}-\boldsymbol{b}\|^{2} \geq \frac{\|\boldsymbol{b} \boldsymbol{c}\|^{2}}{2}-\|\boldsymbol{a}-\boldsymbol{c}\|^{2}$, the third inequality follows since $\boldsymbol{c}_{i}$ is closer to $\boldsymbol{p}^{(\sigma(i))}$ than $\boldsymbol{p}^{(j)}$, the fifth one follows from Lemma 5.7. and the last one follows by (5.4).

On the other hand, since $\operatorname{COST}\left(A_{1}, \ldots, A_{k}\right) \leq \operatorname{APT} \cdot \Psi(\ell)$ by Lemma 5.3, we have that

$$
\begin{aligned}
\sum_{i=1}^{k} \frac{\operatorname{vol}\left(M_{\sigma, i} \triangle S_{i}\right)}{\operatorname{vol}\left(S_{i}\right)} & \leq 32 \cdot(1+\mathrm{APT}) \cdot \Psi(\ell) \cdot\left(\theta^{2}-20 \sqrt{\theta \cdot \Psi(\ell)}\right)^{-1} \\
& \leq 64 \cdot(1+\mathrm{APT}) \cdot \Psi(\ell)
\end{aligned}
$$

where the last inequality follows by the assumption that $\Psi(\ell) \leq \theta^{3} / 1600$. Therefore, the statement follows.

Combining this with other technical ingredients, including the new technique for constructing the desired mapping $\sigma^{*}$ described in Section 4.2, we obtain the performance guarantee of the designed algorithm, which is summarised as follows:

Theorem 5.2. Let $G=(V, E)$ be a graph with $k$ clusters $S_{1}, \ldots, S_{k}$ of almost balanced size, and a $(\theta, \ell)$-distinguishable meta-graph that satisfies $\Psi(\ell) \leq(2176(1+A P T))^{-1} \theta^{3}$. Let $A_{1}, \ldots, A_{k}$ be the output of spectral clustering with $\ell$ eigenvectors, and without loss of generality let the optimal correspondent of $A_{i}$ be $S_{i}$. Then, it holds that

$$
\sum_{i=1}^{k} \operatorname{vol}\left(A_{i} \triangle S_{i}\right) \leq 2176(1+A P T) \frac{\Psi(\ell) \cdot \operatorname{vol}(V)}{k \cdot \theta^{2}} .
$$

Proof. This result can be obtained by using the same technique as the one used in the proof of Theorem 4.2, but applying Lemma 5.3 instead of Lemma 4.1 and Lemma 5.8 instead of Lemma 4.6 in the analysis.

Notice that if we take $\ell=k$, then we have that $\theta=1$ and $\Psi(\ell) \leq k / \Upsilon(k)$ which makes the guarantee in Theorem 5.2 the same as the one in Theorem 4.2. However, if the meta-graph corresponding to the optimal clusters is $(\theta, \ell)$-distinguishable for large $\theta$ and $\ell \ll k$, then it holds that $\Psi(\ell) \ll k / \Upsilon(k)$ and Theorem 5.2 gives a stronger guarantee than the one from Theorem 4.2.

### 5.3 Experimental Results

In this section we empirically evaluate the performance of spectral clustering for finding $k$ clusters while using fewer than $k$ eigenvectors. Our results on synthetic data demonstrate that, for graphs with a clear pattern of clusters, spectral clustering with fewer than $k$ eigenvectors performs better. This is further confirmed on real-world datasets including BSDS, MNIST, and USPS. The code used to produce all experimental results is available at
https://github.com/pmacg/spectral-clustering-meta-graphs.

We implement spectral clustering in Python, using the scipy library for computing eigenvectors, and the $k$-means algorithm from the sklearn library. Our experiments on synthetic data are performed on a desktop computer with an Intel(R) Core(TM) i5-8500 CPU @ 3.00 GHz processor and 16 GB RAM. The experiments on the BSDS, MNIST, and USPS datasets are performed on a compute server with 64 AMD EPYC 7302 16-Core Processors.

### 5.3.1 Results on Synthetic Data

We first study the performance of spectral clustering on random graphs whose clusters exhibit a clear pattern. Given the parameters $n \in \mathbb{Z}^{+}, 0 \leq q \leq p \leq 1$, and some meta-graph $M=\left(V_{M}, E_{M}\right)$ with $k$ vertices, we generate a graph with clusters $\left\{S_{i}\right\}_{i=1}^{k}$, each of size $n$, as follows. For each pair of vertices $u \in S_{i}$ and $v \in S_{j}$, we add the edge $(u, v)$ with probability $p$ if $i=j$, and with probability $q$ if $i \neq j$ and $(i, j) \in E_{M}$. The metric used for our evaluation is defined by $\frac{1}{n k} \sum_{i=1}^{k}\left|S_{i} \cap A_{i}\right|$, for the optimal matching between the output $\left\{A_{i}\right\}_{i=1}^{k}$ and the ground truth $\left\{S_{i}\right\}_{i=1}^{k}$.

In our experiments, we fix $n=1000, p=0.01$, and consider the meta-graphs $C_{10}$ and $P_{4,4}$, similar to those illustrated in Examples 5.1 and 5.2, this results in graphs with 10,000 and 16,000 vertices respectively. We vary the ratio $p / q$ and the number of eigenvectors used to find the clusters. Our experimental result, which is reported as the average score over


Figure 5.7: A comparison on the output of spectral clustering with meta-graphs $C_{10}$ and $P_{4,4}$, when different number of eigenvectors are used. Note that classical spectral clustering uses 10 and 16 eigenvectors respectively.

10 trials and shown in Figure 5.7, clearly shows that spectral clustering with fewer than $k$ eigenvectors performs better. This is particularly the case when $p$ and $q$ are close, which corresponds to the more challenging regime in the model.

### 5.3.2 Results on the BSDS Dataset

In this experiment, we study the performance of spectral clustering for image segmentation when using different numbers of eigenvectors. We consider the Berkeley Segmentation Data Set (BSDS) [AMFM11], which consists of 500 images along with their ground-truth segmentations. For each image, we construct a similarity graph on the pixels and take $k$ to be the number of clusters in the ground-truth segmentation ${ }^{3}$. Given a particular image in the dataset, we first downsample the image to have at most 20,000 pixels. Then, we represent each pixel by the point $(r, g, b, x, y)^{\top} \in \mathbb{R}^{5}$, where $r, g, b \in[255]$ are the RGB values of the pixel and $x$ and $y$ are the coordinates of the pixel in the downsampled image. We construct the similarity graph by taking each pixel to be a vertex in the graph, and for every pair of pixels $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{5}$, we add an edge with weight $\exp \left(-\|\boldsymbol{u}-\boldsymbol{v}\|^{2} / 2 \sigma^{2}\right)$ where $\sigma=20$. Then we apply spectral clustering, varying the number of eigenvectors used. We evaluate each segmentation produced with spectral clustering using the Rand Index [Ran71] as implemented in the benchmarking code provided along with the BSDS dataset. For each image, this computes the average Rand Index across all of the provided ground-truth segmentations for the image. Figure 5.3 shows two images from the dataset along with the segmentations produced with spectral clustering, and additional examples are given in Figures 5.9 and 5.10 . These examples demonstrate that spectral clustering with fewer eigenvectors performs better.

[^5]Table 5.1: The average Rand Index across the BSDS dataset for different numbers of eigenvectors. Optimal refers to the algorithm which runs spectral clustering with $\ell$ eigenvectors for all possible $\ell \in[k]$ and returns the output with the highest Rand Index.

| Number of Eigenvectors | Average Rand Index |
| :---: | :---: |
| $k$ | 0.71 |
| $k / 2$ | 0.74 |
| OptimaL | 0.76 |

We conduct the experiments on the entire BSDS dataset, and the average Rand Index of the algorithm's output is reported in Table 5.1. it is clear that spectral clustering with $k / 2$ eigenvectors consistently out-performs the one with $k$ eigenvectors. We further notice that, on $89 \%$ of the images across the entire dataset, using fewer than $k$ eigenvectors gives a better result than using $k$ eigenvectors.

### 5.3.3 Results on the MNIST and USPS Datasets

We further demonstrate the real-world applicability of our results on the MNIST and USPS datasets [Hul94, LBBH98], which consist of images of hand-written digits. In both the MNIST and USPS datasets, each image is represented as an array of greyscale pixels with values between 0 and 255. The MNIST dataset has 60,000 images with dimensions $28 \times 28$ and the USPS dataset has 7,291 images with dimensions $16 \times 16$. In each dataset, we consider each image to be a single data point in $\mathbb{R}^{\left(d^{2}\right)}$ where $d$ is the dimension of the images and construct the $k$ nearest neighbour graph for $k=3$. For the MNIST dataset, this gives a graph with 60,000 vertices and 138,563 edges, and for the USPS dataset this gives a graph with 7,291 vertices and 16,715 edges. We use spectral clustering to partition the graphs into 10 clusters. We measure the similarity between the found clusters and the ground truth using the Adjusted Rand Index [GA17] and accuracy [Ran71], and plot the results in Figure 5.8 . Our experiments show that spectral clustering with just 7 eigenvectors gives the best performance on both datasets.

### 5.4 Future Work

This work is merely the starting point in the study of spectral clustering with fewer than $k$ eigenvectors, and leaves many open questions. For example, although one can enumerate the number of used eigenvectors from 1 to $k$ and take the best clustering result with respect to some objective function (such as graph conductance or the normalised cut value), it would be interesting to know whether the 'optimal' number of eigenvectors can be computed directly.


Figure 5.8: Experimental results on the MNIST and USPS datasets. (a) and (b): Adjusted Rand Index and accuracy of spectral clustering on the MNIST dataset; (c) and (d): Adjusted Rand Index and accuracy of spectral clustering on the USPS dataset. These experiments show that spectral clustering with 7 eigenvectors gives the best partition of the input into 10 clusters.

Open Question 3. Can the 'optimal' number of eigenvectors for spectral clustering be computed directly from the Laplacian spectrum?

Additionally, more empirical studies could give further insight into the structure of the meta-graphs of real-world datasets.

Open Question 4. What are the typical meta-graph structures appearing in real-world datasets? Do they share any common properties that can be exploited to improve spectral clustering in practice?


Figure 5.9: Examples of the segmentations produced with spectral clustering on the BSDS dataset.


Figure 5.10: Examples of the segmentations produced with spectral clustering on the BSDS dataset.

## Chapter 6

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## Finding Densely Connected Clusters Locally

In this chapter we turn our attention from learning the arbitrary structure of clusters to the problem of finding clusters with a specific known structure. Moreover, we would like to solve the problem locally so that we can learn the structure of a graph close to some particular vertex of interest.

To begin, let us recall the definition of a local clustering algorithm. Given an arbitrary vertex $u$ of graph $G=(V, E)$ as input, a local graph clustering algorithm finds some lowconductance set $S \subset V$ containing $u$, while the algorithm runs in time proportional to the size of the target cluster and independent of the size of the graph $G$. Because of the increasing size of available datasets, which makes centralised computation too expensive, local graph clustering has become an important learning technique for analysing a number of large-scale graphs and has been applied to solve many other learning and combinatorial optimisation problems [ALM13, And10, AOGPT16, FWY20, LG20, TMIY20, WFH ${ }^{+} 17$, YBLG17]. In this chapter, we study local graph clustering for learning the structure of clusters that are defined by their inter-connections, and develop local algorithms to achieve this objective in both undirected and directed graphs. The design and analysis of the algorithms in this chapter build on the personalised Pagerank and evolving set process techniques for local clustering which are described in Section 3.3.

The first result of this chapter is a local algorithm for finding densely connected clusters in an undirected graph $G=(V, E)$.

Main Result 4 (See Theorem 6.1 for the formal statement). Given an undirected graph $G=(V, E)$ and a seed vertex $u$, our local algorithm finds two clusters $L, R$ around $u$, which are densely connected to each other and are loosely connected to $V \backslash(L \cup R)$.

The design of the algorithm is based on a new reduction that allows us to relate the connections between $L, R$ and $V \backslash(L \cup R)$ to a single cluster in the resulting graph $H$, and a


Figure 6.1: Clusters found by LocBipartDC on the interstate dispute network, using the USA as the seed vertex. In each case, countries in the yellow cluster tend to enter conflicts with countries in the blue cluster and vice versa.
generalised analysis of Pagerank-based algorithms for local graph clustering. The significance of the designed algorithm is demonstrated by experimental results on the Interstate Dispute Network from 1816 to 2010. By connecting two vertices (countries) with an undirected edge weighted according to the severity of their military disputes and using the USA as the seed vertex, the algorithm recovers two groups of countries that tend to have conflicts with each other, and shows how the two groups evolve over time. In particular, as shown in Figure 6.1, the algorithm not only identifies the changing roles of Russia, Japan, and eastern Europe in line with 20th century geopolitics, but also the reunification of east and west Germany around 1990 .

The second result of this chapter is a local algorithm for finding densely connected clusters in a directed graph.

Main Result 5 (See Theorem 6.2 for the formal statement). Given a directed graph $G=(V, E)$ and a seed vertex $u$, our local algorithm finds two clusters $L$ and $R$, such that (i) there are many edges from $L$ to $R$, and (ii) $L \cup R$ is loosely connected to $V \backslash(L \cup R)$.

The design of this algorithm is based on the following two techniques: (1) a new reduction that allows us to relate the edge weight from $L$ to $R$, as well as the edge connections between $L \cup R$ and $V \backslash(L \cup R)$, to a single vertex set in the resulting undirected graph $H$; (2) a refined analysis of the ESP-based algorithm for local graph clustering. We demonstrate that the algorithm is able to recover local densely connected clusters in the US migration dataset, in which two vertex sets $L$ and $R$ defined as above could represent a higher-order migration


Figure 6.2: Clusters found by EvoCutDirected in the US migration network, in which four different counties are used as starting vertices of the local algorithm. In each case, people tend to migrate from the red counties to the green counties.
trend. In particular, as shown in Figure 6.2, by using different counties as starting vertices, the algorithm uncovers refined and more localised migration patterns than the previous work on the same dataset [CLSZ20]. This algorithm is the first local clustering algorithm that achieves a similar goal on directed graphs.

### 6.1 Algorithm for Undirected Graphs

We first develop and analyse a local algorithm for finding two clusters in an undirected graph with a dense cut between them. To formalise this notion, for any undirected graph $G=(V, E)$ and disjoint $L, R \subset V$, we recall the definition of bipartiteness given in Section 3.2.

$$
\beta(L, R) \triangleq 1-\frac{2 e(L, R)}{\operatorname{vol}(L \cup R)}
$$

Notice that a low $\beta(L, R)$ value means that there is a dense cut between $L$ and $R$, and there is a sparse cut between $L \cup R$ and $V \backslash(L \cup R)$. In particular, $\beta(L, R)=0$ implies that $(L, R)$ forms a bipartite and connected component of $G$. We describe a local algorithm for finding almost-bipartite sets $L$ and $R$ with a low value of $\beta(L, R)$.

### 6.1.1 Reduction by Double Cover

The design of most local algorithms for finding a target set $S \subset V$ of low conductance is based on analysing the behaviour of random walks starting from vertices in $S$. In particular, when the conductance $\Phi_{G}(S)$ is low, a random walk starting from most vertices in $S$ leaves


Figure 6.3: An example of the construction of the double cover
$S$ with low probability. However, for our setting, the target is a pair of sets $L, R$ with many connections between them. As such, a random walk starting in either $L$ or $R$ is very likely to leave the starting set. To address this, we use a novel technique based on the double cover of $G$ to reduce the problem of finding two sets of high conductance to the problem of finding one of low conductance. Formally, for any undirected graph $G=\left(V_{G}, E_{G}\right)$, its double cover is the bipartite graph $H=\left(V_{H}, E_{H}\right)$ defined as follows: (1) every vertex $v \in V_{G}$ has two corresponding vertices $v_{1}, v_{2} \in V_{H}$; (2) for every edge $\{u, v\} \in E_{G}$, there are edges $\left\{u_{1}, v_{2}\right\}$ and $\left\{u_{2}, v_{1}\right\}$ in $E_{H}$. See Figure 6.3 for an illustration.

Now we show a tight connection between the value of $\beta(L, R)$ for any disjoint sets $L, R \subset V_{G}$ and the conductance of a single set in the double cover of $G$. To this end, for any $S \subset V_{G}$, we define $S_{1} \subset V_{H}$ and $S_{2} \subset V_{H}$ by $S_{1} \triangleq\left\{v_{1} \mid v \in S\right\}$ and $S_{2} \triangleq\left\{v_{2} \mid v \in S\right\}$. We formalise the connection in the following lemma.

Lemma 6.1. Let $G$ be an undirected graph, $S \subset V$ with partitioning $(L, R)$, and $H$ be the double cover of $G$. Then, it holds that $\Phi_{H}\left(L_{1} \cup R_{2}\right)=\beta_{G}(L, R)$.

Proof. Let $S^{\prime}=L_{1} \cup R_{2}$, and by definition we have that $\operatorname{vol}_{G}(S)=\operatorname{vol}_{H}\left(S^{\prime}\right)$. On the other hand, it holds that

$$
\operatorname{vol}_{H}\left(S^{\prime}\right)=w_{H}\left(S^{\prime}, V \backslash S^{\prime}\right)+2 w_{H}\left(S^{\prime}, S^{\prime}\right)=w_{H}\left(S^{\prime}, V \backslash S^{\prime}\right)+2 w_{H}\left(L_{1}, R_{2}\right)
$$

This implies that

$$
\Phi_{H}\left(S^{\prime}\right)=\frac{w_{H}\left(S^{\prime}, V \backslash S^{\prime}\right)}{\operatorname{vol}_{H}\left(S^{\prime}\right)}=\frac{\operatorname{vol}_{G}(S)-2 w_{H}\left(L_{1}, R_{2}\right)}{\operatorname{vol}_{G}(S)}=1-\frac{2 w_{G}(L, R)}{\operatorname{vol}_{G}(S)}=\beta_{G}(L, R),
$$

which proves the statement.

Next we look at the other direction of this correspondence. Specifically, given any $S \subset V_{H}$ in the double cover of a graph $G$, we would like to find two disjoint sets $L \subset V_{G}$ and $R \subset V_{G}$ such that $\beta_{G}(L, R)=\Phi_{H}(S)$. However, such a connection does not hold in general. To overcome this, we restrict our attention to those subsets of $V_{H}$ which can be unambiguously interpreted as two disjoint sets in $V_{G}$.


Figure 6.4: The indicated vertex set $S \subset V_{H}$ has low conductance. However, since $S$ contains many pairs of vertices which correspond to the same vertex in $G, S$ gives us little information for finding disjoint $L, R \subset V_{G}$ with a low $\beta(L, R)$ value.

Definition 6.1. (Simple set) We call $S \subset V_{H}$ simple if $\left|\left\{v_{1}, v_{2}\right\} \cap S\right| \leq 1$ holds for all $v \in V_{G}$.

Given this definition, the following corollary follows directly from Lemma 6.1.
Corollary 6.1. For any simple set $S \subset V_{H}$, let $L=\left\{u: u_{1} \in S\right\}$ and $R=\left\{u: u_{2} \in S\right\}$. Then, $\beta_{G}(L, R)=\Phi_{H}(S)$.

### 6.1.2 Design of the Algorithm

So far we have shown that the problem of finding densely connected sets $L, R \subset V_{G}$ can be reduced to finding $S \subset V_{H}$ of low conductance in the double cover $H$, and this reduction raises the natural question of whether existing local algorithms can be directly employed to find $L$ and $R$ in $G$. However, this is not the case: even though a set $S \subset V_{H}$ returned by most local algorithms is guaranteed to have low conductance, vertices of $G$ could be included in $S$ twice, and as such $S$ doesn't necessarily give us disjoint sets $L, R \subset V_{G}$ with low value of $\beta(L, R)$. See Figure 6.4 for an illustration.

### 6.1.2.1 The Simplify Operator

Taking the example shown in Figure 6.4 into account, our objective is to design a local algorithm for finding some set $S \subset V_{H}$ of low conductance which is also simple. To ensure this, we introduce the simplify operator, and analyse its properties.

Definition 6.2 (Simplify operator). Let $H$ be the double cover of $G$, where the two corresponding vertices of any $u \in V_{G}$ are defined as $u_{1}, u_{2} \in V_{H}$. Then, for any $\boldsymbol{p} \in \mathbb{R}_{\geq 0}^{2 n}$, the simplify operator is a function $\sigma: \mathbb{R}_{\geq 0}^{2 n} \rightarrow \mathbb{R}_{\geq 0}^{2 n}$ defined by

$$
\begin{aligned}
& (\sigma \circ \boldsymbol{p})\left(u_{1}\right) \triangleq \max \left(0, \boldsymbol{p}\left(u_{1}\right)-\boldsymbol{p}\left(u_{2}\right)\right), \\
& (\sigma \circ \boldsymbol{p})\left(u_{2}\right) \triangleq \max \left(0, \boldsymbol{p}\left(u_{2}\right)-\boldsymbol{p}\left(u_{1}\right)\right)
\end{aligned}
$$

for every $u \in V_{G}$.

Notice that, for any vector $\boldsymbol{p}$ and any $u \in V_{G}$, at most one of $u_{1}$ and $u_{2}$ is in the support of $\sigma \circ \boldsymbol{p}$; hence, the support of $\sigma \circ \boldsymbol{p}$ is always simple. To explain the meaning of $\sigma$, for a vector $\boldsymbol{p}$ one could view $\boldsymbol{p}\left(u_{1}\right)$ as our 'confidence' that $u \in L$, and $\boldsymbol{p}\left(u_{2}\right)$ as our 'confidence' that $u \in R$. Hence, when $\boldsymbol{p}\left(u_{1}\right) \approx \boldsymbol{p}\left(u_{2}\right)$, both $(\sigma \circ \boldsymbol{p})\left(u_{1}\right)$ and $(\sigma \circ \boldsymbol{p})\left(u_{2}\right)$ are low which captures the fact that we would not prefer to include $u$ in either $L$ or $R$. On the other hand, when $\boldsymbol{p}\left(u_{1}\right) \gg \boldsymbol{p}\left(u_{2}\right)$, we have $(\sigma \circ \boldsymbol{p})\left(u_{1}\right) \approx \boldsymbol{p}\left(u_{1}\right)$, which captures our confidence that $u$ should belong to $L$. The following lemma summaries some key properties of $\sigma$.

Lemma 6.2. The following holds for the $\sigma$-operator:

- $\sigma \circ(c \cdot \boldsymbol{p})=c \cdot(\sigma \circ \boldsymbol{p})$ for $\boldsymbol{p} \in \mathbb{R}_{\geq 0}^{2 n}$ and any $c \in \mathbb{R}_{\geq 0}$;
- $\sigma \circ(\boldsymbol{a}+\boldsymbol{b}) \preceq \sigma \circ \boldsymbol{a}+\sigma \circ \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}_{\geq 0}^{2 n}$;
- $\sigma \circ(\mathbf{W} \boldsymbol{p}) \preceq \mathbf{W}(\sigma \circ \boldsymbol{p})$ for $\boldsymbol{p} \in \mathbb{R}_{\geq 0}^{2 n}$.

Proof. For any $u_{1} \in V_{H}$, we have by definition that

$$
\begin{aligned}
\sigma \circ(c \cdot \boldsymbol{p})\left(u_{1}\right) & =\max \left(0, c \cdot \boldsymbol{p}\left(u_{1}\right)-c \cdot \boldsymbol{p}\left(u_{2}\right)\right) \\
& =c \cdot \max \left(0, \boldsymbol{p}\left(u_{1}\right)-\boldsymbol{p}\left(u_{2}\right)\right) \\
& =c \cdot \sigma \circ \boldsymbol{p}\left(u_{1}\right),
\end{aligned}
$$

and similarly we have $\sigma \circ(c \cdot \boldsymbol{p})\left(u_{2}\right)=c \cdot \sigma \circ \boldsymbol{p}\left(u_{2}\right)$. Therefore, the first statement holds.
Now we prove the second statement. Let $u_{1}, u_{2} \in V_{H}$ be the vertices that correspond to $u \in V_{G}$. We have by definition that

$$
\begin{aligned}
\sigma \circ(\boldsymbol{a}+\boldsymbol{b})\left(u_{1}\right) & =\max \left(0, \boldsymbol{a}\left(u_{1}\right)+\boldsymbol{b}\left(u_{1}\right)-\boldsymbol{a}\left(u_{2}\right)-\boldsymbol{b}\left(u_{2}\right)\right) \\
& =\max \left(0, \boldsymbol{a}\left(u_{1}\right)-\boldsymbol{a}\left(u_{2}\right)+\boldsymbol{b}\left(u_{1}\right)-\boldsymbol{b}\left(u_{2}\right)\right) \\
& \leq \max \left(0, \boldsymbol{a}\left(u_{1}\right)-\boldsymbol{a}\left(u_{2}\right)\right)+\max \left(0, \boldsymbol{b}\left(u_{1}\right)-\boldsymbol{b}\left(u_{2}\right)\right) \\
& =\sigma \circ \boldsymbol{a}\left(u_{1}\right)+\sigma \circ \boldsymbol{b}\left(u_{1}\right),
\end{aligned}
$$

where the last inequality holds by the fact that $\max (0, x+y) \leq \max (0, x)+\max (0, y)$ for any $x, y \in \mathbb{R}$. For the same reason, we have that

$$
\sigma \circ(\boldsymbol{a}+\boldsymbol{b})\left(u_{2}\right) \leq \sigma \circ \boldsymbol{a}\left(u_{2}\right)+\sigma \circ \boldsymbol{b}\left(u_{2}\right) .
$$

Combining these proves the second statement.
Finally, we prove the third statement. Recall that the lazy random walk matrix for a graph is defined to be $\mathbf{W}=\frac{1}{2}\left(\mathbf{I}+\mathbf{A D} \mathbf{D}^{-1}\right)$. By this definition, we have that

$$
(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)=\frac{1}{2} \boldsymbol{p}\left(u_{1}\right)+\frac{1}{2} \sum_{v \in N_{G}(u)} \frac{\boldsymbol{p}\left(v_{2}\right)}{d_{H}\left(v_{2}\right)},
$$

and

$$
(\mathbf{W} \boldsymbol{p})\left(u_{2}\right)=\frac{1}{2} \boldsymbol{p}\left(u_{2}\right)+\frac{1}{2} \sum_{v \in N_{G}(u)} \frac{\boldsymbol{p}\left(v_{1}\right)}{d_{H}\left(v_{1}\right)} .
$$

Without loss of generality, we assume that $(\mathbf{W} \boldsymbol{p})\left(u_{1}\right) \geq(\mathbf{W} \boldsymbol{p})\left(u_{2}\right)$. This implies that $(\sigma \circ(\mathbf{W} \boldsymbol{p}))\left(u_{2}\right)=0 \leq(\mathbf{W}(\sigma \circ \boldsymbol{p}))\left(u_{2}\right)$, hence it suffices to prove that $(\sigma \circ(\mathbf{W} \boldsymbol{p}))\left(u_{1}\right) \leq$ $(\mathbf{W}(\sigma \circ \boldsymbol{p}))\left(u_{1}\right)$. By definition, we have that

$$
\begin{align*}
(\sigma \circ(\mathbf{W} \boldsymbol{p}))\left(u_{1}\right) & =\max \left(0,(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)-(\mathbf{W} \boldsymbol{p})\left(u_{2}\right)\right) \\
& =(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)-(\mathbf{W} \boldsymbol{p})\left(u_{2}\right) \\
& =\frac{1}{2}\left(\boldsymbol{p}\left(u_{1}\right)-\boldsymbol{p}\left(u_{2}\right)\right)+\frac{1}{2} \sum_{v \in N_{G}(u)}\left(\frac{\boldsymbol{p}\left(v_{2}\right)}{d_{H}\left(v_{2}\right)}-\frac{\boldsymbol{p}\left(v_{1}\right)}{d_{H}\left(v_{1}\right)}\right) \\
& =\frac{1}{2}\left(\boldsymbol{p}\left(u_{1}\right)-\boldsymbol{p}\left(u_{2}\right)\right)+\frac{1}{2} \sum_{v \in N_{G}(u)} \frac{\boldsymbol{p}\left(v_{2}\right)-\boldsymbol{p}\left(v_{1}\right)}{d_{G}(v)}, \tag{6.1}
\end{align*}
$$

where the last equality follows by the fact that $d_{H}\left(v_{1}\right)=d_{H}\left(v_{2}\right)=d_{G}(v)$ for any $v \in V_{G}$. To analyse (6.1), notice that $\boldsymbol{p}\left(v_{1}\right)-\boldsymbol{p}\left(v_{2}\right)=(\sigma \circ \boldsymbol{p})\left(v_{1}\right)-(\sigma \circ \boldsymbol{p})\left(v_{2}\right)$ for any $v \in V_{G}$ for the following reasons:

- When $\boldsymbol{p}\left(v_{1}\right) \geq \boldsymbol{p}\left(v_{2}\right)$, we have that $(\sigma \circ \boldsymbol{p})\left(v_{1}\right)-(\sigma \circ \boldsymbol{p})\left(v_{2}\right)=\left(\boldsymbol{p}\left(v_{1}\right)-\boldsymbol{p}\left(v_{2}\right)\right)-0=$ $\boldsymbol{p}\left(v_{1}\right)-\boldsymbol{p}\left(v_{2}\right)$;
- Otherwise, we have $\boldsymbol{p}\left(v_{1}\right)<\boldsymbol{p}\left(v_{2}\right)$ and $(\sigma \circ \boldsymbol{p})\left(v_{1}\right)-(\sigma \circ \boldsymbol{p})\left(v_{2}\right)=0-\left(\boldsymbol{p}\left(v_{2}\right)-\boldsymbol{p}\left(v_{1}\right)\right)=$ $\boldsymbol{p}\left(v_{1}\right)-\boldsymbol{p}\left(v_{2}\right)$.

Therefore, we can rewrite (6.1) as

$$
\begin{aligned}
(\sigma \circ(\mathbf{W} \boldsymbol{p}))\left(u_{1}\right) & =\frac{1}{2}\left((\sigma \circ \boldsymbol{p})\left(u_{1}\right)-(\sigma \circ \boldsymbol{p})\left(u_{2}\right)\right)+\frac{1}{2} \sum_{v \in N(u)} \frac{(\sigma \circ \boldsymbol{p})\left(v_{2}\right)-(\sigma \circ \boldsymbol{p})\left(v_{1}\right)}{d_{G}(v)} \\
& \leq \frac{1}{2} \cdot(\sigma \circ \boldsymbol{p})\left(u_{1}\right)+\frac{1}{2} \sum_{v \in N(u)} \frac{(\sigma \circ \boldsymbol{p})\left(v_{2}\right)}{d_{G}(v)} \\
& =(\mathbf{W}(\sigma \circ \boldsymbol{p}))\left(u_{1}\right),
\end{aligned}
$$

where the inequality holds by the fact that $(\sigma \circ \boldsymbol{p})\left(u_{1}\right) \geq 0$ and $(\sigma \circ \boldsymbol{p})\left(u_{2}\right) \geq 0$ for any $u \in V_{G}$. Since the analysis above holds for any $v \in V_{G}$, we have that $\sigma \circ(\mathbf{W} \boldsymbol{p}) \preceq \mathbf{W}(\sigma \circ \boldsymbol{p})$, which finishes the proof of the lemma.

While our analysis is based on all the three properties in Lemma 6.2 the third is of particular importance: it implies that, if $\boldsymbol{p}$ is the probability distribution of a random walk in $H$, applying $\sigma$ before taking a one-step random walk would never result in lower probability mass than applying $\sigma$ after taking a one-step random walk. This means that no probability mass would be lost when the $\sigma$-operator is applied between every step of a random walk, in comparison with applying $\sigma$ at the end of an entire random walk process.

### 6.1.2.2 Description of the Algorithm

Our proposed algorithm is conceptually simple: every vertex $u$ of the input graph $G$ maintains two copies $u_{1}, u_{2}$ of itself, and these two 'virtual' vertices are used to simulate $u$ 's corresponding vertices in the double cover $H$ of $G$. Then, as the neighbours of $u_{1}$ and $u_{2}$ in $H$ are entirely determined by $u$ 's neighbours in $G$ and can be constructed locally, a random walk process in $H$ is simulated in $G$. This allows us to apply a local algorithm similar to the one by Anderson et al. [ACL06] on this 'virtual' graph $H$. Finally, since all the required information about $u_{1}, u_{2} \in V_{H}$ is maintained by $u \in V_{G}$, the $\sigma$-operator can be applied locally.

The formal description of the algorithm is given in Algorithm 6.1, which invokes Algorithm 6.2 as the key component to compute the approximate Pagerank $\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{u_{1}}, \boldsymbol{r}\right)$. Specifically, Algorithm 6.2 maintains, for every vertex $u \in V_{G}$, tuples $\left(\boldsymbol{p}\left(u_{1}\right), \boldsymbol{p}\left(u_{2}\right)\right)$ and $\left(\boldsymbol{r}\left(u_{1}\right), \boldsymbol{r}\left(u_{2}\right)\right)$ to keep track of the values of $\boldsymbol{p}$ and $\boldsymbol{r}$ in $G^{\prime}$ s double cover. For a given vertex $u$, the entries in these tuples are expressed by $\boldsymbol{p}_{1}(u), \boldsymbol{p}_{2}(u), \boldsymbol{r}_{1}(u)$, and $\boldsymbol{r}_{2}(u)$ respectively. Every DCPUSH operation (Algorithm 6.3) preserves the invariant $\boldsymbol{p}+\operatorname{ppr}_{H}(\alpha, \boldsymbol{r})=\operatorname{ppr}_{H}\left(\alpha, \boldsymbol{\chi}_{v_{1}}\right)$, which ensures that the final output of Algorithm 6.2 is an approximate Pagerank vector. It is worth noting that, although the presentation of the ApproximatePagerankDC procedure is similar to the one in Andersen et al. [ACL06], in the DCPuSh procedure the update of the residual vector $\boldsymbol{r}$ is slightly more involved: specifically, for every vertex $u \in V_{G}$, both $\boldsymbol{r}_{1}(u)$ and $\boldsymbol{r}_{2}(u)$ are needed in order to update $\boldsymbol{r}_{1}(u)$ (or $\boldsymbol{r}_{2}(u)$ ). That is one of the reasons that the performance of the algorithm in Andersen et al. [ACL06] cannot be directly applied for this algorithm, and a more technical analysis, some of which is parallel to theirs, is needed in order to analyse the correctness and performance of the algorithm.

```
Algorithm 6.1: LocBipartDC \((G, u, \gamma, \beta)\)
    Input : A graph \(G\), starting vertex \(u\), target volume \(\gamma\), and target bipartiteness \(\beta\)
    Output: Two disjoint sets \(L\) and \(R\)
    Set \(\alpha=\frac{\beta^{2}}{378}\), and \(\epsilon=\frac{1}{20 \gamma}\)
    2 Compute \(\boldsymbol{p}^{\prime}=\) ApproximatePagerankDC \((u, \alpha, \epsilon)\)
    Compute \(\boldsymbol{p}=\sigma \circ \boldsymbol{p}^{\prime}\)
    4 for \(j \in[1,|\operatorname{supp}(p)|]\) do
        if \(\Phi\left(S_{j}^{\boldsymbol{p}}\right) \leq \beta\) then
            Set \(L=\left\{u: u_{1} \in S_{j}^{p}\right\}\), and \(R=\left\{u: u_{2} \in S_{j}^{\boldsymbol{p}}\right\}\)
            return \((L, R)\)
        end
    end
```

```
Algorithm 6.2: APPROXIMATEPAGERANKDC \((v, \alpha, \epsilon)\)
    Input : Starting vertex \(v\), parameters \(\alpha\) and \(\epsilon\)
    Output: Approximate Pagerank vector \(\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{v_{1}}, \boldsymbol{r}\right)\)
    Set \(\boldsymbol{p}_{1}=\boldsymbol{p}_{2}=\boldsymbol{r}_{2}=\mathbf{0}\); set \(\boldsymbol{r}_{1}=\chi_{v}\)
    while \(\max _{(u, i) \in V \times\{1,2\}} \frac{r_{i}(u)}{d(u)} \geq \epsilon\) do
        Choose any \(u\) and \(i \in\{1,2\}\) such that \(\frac{r_{i}(u)}{d(u)} \geq \epsilon\)
        \(\left(\boldsymbol{p}_{1}, \boldsymbol{p}_{2}, \boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)=\operatorname{DCPUSH}\left(\alpha,(u, i), \boldsymbol{p}_{1}, \boldsymbol{p}_{2}, \boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)\)
    end
    6 return \(\boldsymbol{p}=\left[\boldsymbol{p}_{1}, \boldsymbol{p}_{2}\right]\)
```

```
Algorithm 6.3: \(\mathrm{DCPUSH}\left(\alpha,(u, i), \boldsymbol{p}_{1}, \boldsymbol{p}_{2}, \boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)\)
    Input : \(\alpha,(u, i), \boldsymbol{p}_{1}, \boldsymbol{p}_{2}, \boldsymbol{r}_{1}, \boldsymbol{r}_{2}\)
    Output: \(\left(\boldsymbol{p}_{1}^{\prime}, \boldsymbol{p}_{2}^{\prime}, \boldsymbol{r}_{1}^{\prime}, \boldsymbol{r}_{2}^{\prime}\right)\)
    \(1 \operatorname{Set}\left(\boldsymbol{p}_{1}^{\prime}, \boldsymbol{p}_{2}^{\prime}, \boldsymbol{r}_{1}^{\prime}, \boldsymbol{r}_{2}^{\prime}\right)=\left(\boldsymbol{p}_{1}, \boldsymbol{p}_{2}, \boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)\)
    2 Set \(\boldsymbol{p}_{i}^{\prime}(u)=\boldsymbol{p}_{i}(u)+\alpha \boldsymbol{r}_{i}(u)\)
    3 Set \(\boldsymbol{r}_{i}^{\prime}(u)=(1-\alpha) \frac{\boldsymbol{r}_{i}(u)}{2}\)
    4 for \(v \in N_{G}(u)\) do
        Set \(\boldsymbol{r}_{3-i}^{\prime}(v)=\boldsymbol{r}_{3-i}(v)+(1-\alpha) \frac{\boldsymbol{r}_{i}(u)}{2 d(u)}\)
    6 end
    7 return \(\left(\boldsymbol{p}_{1}^{\prime}, \boldsymbol{p}_{2}^{\prime}, \boldsymbol{r}_{1}^{\prime}, \boldsymbol{r}_{2}^{\prime}\right)\)
```


### 6.1.3 Analysis of ApproximatePagerankDC

We begin by considering the ApproximatePagerankDC algorithm, and show that it computes an approximate Pagerank vector on the double cover of the input graph. The following lemma shows that DCPUSH maintains that throughout the algorithm $\left[\boldsymbol{p}_{1}^{\top}, \boldsymbol{p}_{2}^{\top}\right]^{\top}$ is an approximate Pagerank vector on the double cover with residual $\left[\boldsymbol{r}_{1}^{\top}, \boldsymbol{r}_{2}^{\top}\right]^{\top}$, where $\left[\boldsymbol{a}^{\top}, \boldsymbol{b}^{\top}\right]^{\top}$ for $\boldsymbol{a} \in \mathbb{R}^{n_{1}}$ and $\boldsymbol{b} \in \mathbb{R}^{n_{2}}$ is the vector in $\mathbb{R}^{n_{1}+n_{2}}$ obtained by concatenating $\boldsymbol{a}$ and $\boldsymbol{b}$.

Lemma 6.3. Let $\boldsymbol{p}_{1}^{\prime}, \boldsymbol{p}_{2}^{\prime}, \boldsymbol{r}_{1}^{\prime}$ and $\boldsymbol{r}_{2}^{\prime}$ be the result of $\operatorname{DCPush}\left((u, i), \boldsymbol{p}_{1}, \boldsymbol{p}_{2}, \boldsymbol{r}_{1}, \boldsymbol{r}_{2}\right)$. Let $\boldsymbol{p}=\left[\boldsymbol{p}_{1}^{\top}, \boldsymbol{p}_{2}^{\top}\right]^{\top}, \boldsymbol{r}=\left[\boldsymbol{r}_{1}^{\top}, \boldsymbol{r}_{2}^{\top}\right]^{\top}, \boldsymbol{p}^{\prime}=\left[\left(\boldsymbol{p}_{1}^{\prime}\right)^{\top},\left(\boldsymbol{p}_{2}^{\prime}\right)^{\top}\right]^{\top}$ and $\boldsymbol{r}^{\prime}=\left[\left(\boldsymbol{r}_{1}^{\prime}\right)^{\top},\left(\boldsymbol{r}_{2}^{\prime}\right)^{\top}\right]^{\top}$. Then, we have that

$$
\boldsymbol{p}^{\prime}+\operatorname{ppr}_{H}\left(\alpha, \boldsymbol{r}^{\prime}\right)=\boldsymbol{p}+\operatorname{ppr}_{H}(\alpha, \boldsymbol{r})
$$

Proof. After the push operation, we have

$$
\begin{aligned}
\boldsymbol{p}^{\prime} & =\boldsymbol{p}+\alpha \boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}} \\
\boldsymbol{r}^{\prime} & =\boldsymbol{r}-\boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}}+(1-\alpha) \boldsymbol{r}\left(u_{i}\right) \mathbf{W}_{H} \boldsymbol{\chi}_{u_{i}}
\end{aligned}
$$

where $\mathbf{W}_{H}$ is the lazy random walk matrix on the double cover of the input graph. Then, we have

$$
\begin{aligned}
& \boldsymbol{p}+\operatorname{ppr}_{H}(\alpha, \boldsymbol{r}) \\
& =\boldsymbol{p}+\operatorname{ppr}_{H}\left(\alpha, \boldsymbol{r}-\boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}}\right)+\operatorname{ppr}_{H}\left(\alpha, \boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}}\right) \\
& =\boldsymbol{p}+\operatorname{ppr}_{H}\left(\alpha, \boldsymbol{r}-\boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}}\right)+\left[\alpha \boldsymbol{r}(u) \boldsymbol{\chi}_{u_{i}}+(1-\alpha) \operatorname{ppr}_{H}\left(\alpha, \boldsymbol{r}(u) \mathbf{W}_{H} \boldsymbol{\chi}_{u_{i}}\right)\right] \\
& =\left[\boldsymbol{p}+\alpha \boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}}\right]+\operatorname{ppr}_{H}\left(\alpha,\left[\boldsymbol{r}-\boldsymbol{r}\left(u_{i}\right) \boldsymbol{\chi}_{u_{i}}+(1-\alpha) \boldsymbol{r}\left(u_{i}\right) \mathbf{W}_{H} \boldsymbol{\chi}_{u_{i}}\right]\right) \\
& =\boldsymbol{p}^{\prime}+\operatorname{ppr}_{H}\left(\alpha, \boldsymbol{r}^{\prime}\right) .
\end{aligned}
$$

Here, we use the fact that $\operatorname{ppr}(\alpha, \cdot)$ is linear and

$$
\operatorname{ppr}(\alpha, \boldsymbol{s})=\alpha \boldsymbol{s}+(1-\alpha) \operatorname{ppr}\left(\alpha, \mathbf{W}_{H} \boldsymbol{s}\right),
$$

which follows from the fact that $\operatorname{ppr}(\alpha, \boldsymbol{s})=\alpha \sum_{t=0}^{\infty}(1-\alpha)^{t} \mathbf{W}_{H}^{t} \boldsymbol{s}$.
Now we can show the correctness and performance guarantee of the ApproximatePagerANKDC algorithm.

Lemma 6.4. For a graph $G$ and any $v \in V_{G}$, ApproximatePagerankDC $(v, \alpha, \epsilon)$ has running time $O\left(\frac{1}{\epsilon \alpha}\right)$ and computes $\boldsymbol{p}=\operatorname{apr}_{H}\left(\alpha, \chi_{v_{1}}, \boldsymbol{r}\right)$ such that $\max _{u \in V_{H}} \frac{r(u)}{d(u)}<\epsilon$ and $\operatorname{vol}(\operatorname{supp}(\boldsymbol{p}))) \leq \frac{1}{\epsilon \alpha}$, where $H$ is the double cover of $G$.

Proof. By Lemma 6.3, we have that $\boldsymbol{p}=\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{v_{1}}, \boldsymbol{r}\right)$ throughout the algorithm and so this clearly holds for the output vector. By the stopping condition of the algorithm, it holds that $\max _{u \in V_{H}} \frac{r(u)}{d(u)}<\epsilon$.
Let $T$ be the total number of push operations performed by ApproximatePagerankDC, and $d_{j}$ be the degree of the vertex $u_{i}$ used in the $j$-th push. The amount of probability mass on $u_{i}$ at time $j$ is at least $\epsilon d_{j}$ and so the amount of probability mass that moves from $\boldsymbol{r}_{i}$ to $\boldsymbol{p}_{i}$ at step $j$ is at least $\alpha \epsilon d_{i}$. Since $\left\|\boldsymbol{r}_{1}\right\|_{1}+\left\|\boldsymbol{r}_{2}\right\|_{1}=1$ at the beginning of the algorithm, we have that $\alpha \epsilon \sum_{j=1}^{T} d_{j} \leq 1$, which implies that

$$
\begin{equation*}
\sum_{j=1}^{T} d_{j} \leq \frac{1}{\alpha \epsilon} \tag{6.2}
\end{equation*}
$$

Now, note that for every vertex in $\operatorname{supp}(\boldsymbol{p})$, there must be at least one push operation on that vertex. So,

$$
\operatorname{vol}(\operatorname{supp}(\boldsymbol{p}))=\sum_{v \in \operatorname{supp}(\boldsymbol{p})} d(v) \leq \sum_{j=1}^{T} d_{j} \leq \frac{1}{\epsilon \alpha} .
$$

Finally, to bound the running time, we can implement the algorithm by maintaining a queue of vertices satisfying $\frac{r(u)}{d(u)} \geq \epsilon$, and repeat the push operation on the first item in the queue at every iteration, updating the queue as appropriate. The push operation and queue updates can be performed in time proportional to $d(u)$ and so the running time follows from (6.2).

### 6.1.4 Analysis of LocBipartDC

We now come to the analysis of the main algorithm and the core technical result of this chapter. To prove the correctness of the LOCBIPARTDC algorithm, we show two complementary facts which we state informally here:

1. If there is a simple set $S \subset V_{H}$ with low conductance, then for most $u_{1} \in S$ the simplified approximate Pagerank vector $\boldsymbol{p}=\sigma \circ \operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{u_{1}}, \boldsymbol{r}\right)$ would have a lot of probability mass on a small set of vertices.
2. If $\boldsymbol{p}=\sigma \circ \operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{u_{1}}, \boldsymbol{r}\right)$ contains a lot of probability mass on some small set of vertices, then there is a sweep set of $\boldsymbol{p}$ with low conductance.
As we see in Section 6.1.1, there is a direct correspondence between almost-bipartite sets in $G$, and low-conductance and simple sets in $H$. This means that the two facts above are exactly what we need to prove that Algorithm 6.1 can find densely connected sets in $G$.

We first show in Lemma 6.5 how the $\sigma$-operator affects some standard mixing properties of Pagerank vectors in order to establish the first fact promised above. This lemma relies on the fact that $S \subset V_{G}$ corresponds to a simple set in $V_{H}$. This allows us to apply the $\sigma$-operator to the approximate Pagerank vector $\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{u_{1}}, \boldsymbol{r}\right)$ while preserving a large probability mass on the target set. We make use of the following fact proved by Andersen et al. [ACL06] which shows that the approximate Pagerank vector has a large probability mass on a set of vertices if they have low conductance.

Proposition 6.1 ( ACL 06$]$, Theorem 4). For any vertex set $S$ and any constant $\alpha \in[0,1]$, there is a subset $S_{\alpha} \subseteq S$ with $\operatorname{vol}\left(S_{\alpha}\right) \geq \operatorname{vol}(S) / 2$ such that for any vertex $v \in S_{\alpha}$, the approximate Pagerank vector $\operatorname{apr}\left(\alpha, \chi_{v}, r\right)$ satisfies

$$
\operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)(S) \geq 1-\frac{\Phi(S)}{\alpha}-\operatorname{vol}(S) \max _{u \in V} \frac{\boldsymbol{r}(u)}{d(u)}
$$

The following lemma generalises this result to show that applying the $\sigma$-operator to the approximate Pagerank vector maintains this large probability mass up to a constant factor.

Lemma 6.5. For any set $S \subset V_{G}$ with partitioning ( $L, R$ ) and any constant $\alpha \in[0,1]$, there is a subset $S_{\alpha} \subseteq S$ with $\operatorname{vol}\left(S_{\alpha}\right) \geq \operatorname{vol}(S) / 2$ such that, for any vertex $v \in S_{\alpha}$, the simplified approximate Pagerank on the double cover $\boldsymbol{p}=\sigma \circ\left(\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{v_{1}}, \boldsymbol{r}\right)\right)$ satisfies

$$
\boldsymbol{p}\left(L_{1} \cup R_{2}\right) \geq 1-\frac{2 \beta(L, R)}{\alpha}-2 \operatorname{vol}(S) \max _{u \in V} \frac{\boldsymbol{r}(u)}{d(u)} .
$$

Proof. By Lemma 6.1 and Proposition 6.1, there is a set $S_{\alpha} \subseteq E$ with $\operatorname{vol}\left(S_{\alpha}\right) \geq \operatorname{vol}(S) / 2$ such that it holds for $v \in S_{\alpha}$ that

$$
\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{1} \cup R_{2}\right) \geq 1-\frac{\beta(L, R)}{\alpha}-\operatorname{vol}(S) \max _{u \in V} \frac{\boldsymbol{r}(u)}{d(u)}
$$

Therefore, by the definition of $p$ we have that

$$
\begin{aligned}
\boldsymbol{p}\left(L_{1} \cup R_{2}\right) & \geq \operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{1} \cup R_{2}\right)-\operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{2} \cup R_{1}\right) \\
& \geq \operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{1} \cup R_{2}\right)-\operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(\overline{L_{1} \cup R_{2}}\right) \\
& \geq \operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{1} \cup R_{2}\right)-\left(1-\operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{1} \cup R_{2}\right)\right) \\
& =2 \cdot \operatorname{apr}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\left(L_{1} \cup R_{2}\right)-1 \\
& \geq 1-\frac{2 \beta(L, R)}{\alpha}-2 \operatorname{vol}(S) \max _{u \in V} \frac{\boldsymbol{r}(u)}{d(u)},
\end{aligned}
$$

which proves the statement of the lemma.

To prove the second fact promised at the beginning of this subsection, we show as an intermediate lemma that the value of $\boldsymbol{p}\left(u_{1}\right)$ can be bounded with respect to its value after taking a step of the random walk: $(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)$. In order to prove this, we need to carefully analyse the effect of the $\sigma$-operator on the approximate Pagerank vector, and in particular examine the effect of the residual vector $r$.

Lemma 6.6. Let $G$ be a graph with double cover $H$, and $\operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r})$ be the approximate Pagerank vector defined with respect to $H$. Then, $\boldsymbol{p}=\sigma \circ(\operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))$ satisfies that $\boldsymbol{p}\left(u_{1}\right) \leq$ $\alpha\left(\boldsymbol{s}\left(u_{1}\right)+\boldsymbol{r}\left(u_{2}\right)\right)+(1-\alpha)(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)$, and $\boldsymbol{p}\left(u_{2}\right) \leq \alpha\left(\boldsymbol{s}\left(u_{2}\right)+\boldsymbol{r}\left(u_{1}\right)\right)+(1-\alpha)(\mathbf{W} \boldsymbol{p})\left(u_{2}\right)$ for any $u \in V_{G}$.

Proof. Let $u \in V_{G}$ be an arbitrary vertex. We have that

$$
\begin{aligned}
\boldsymbol{p}\left(u_{1}\right)= & \sigma \circ(\operatorname{ppr}(\alpha, \boldsymbol{s})-\operatorname{ppr}(\alpha, \boldsymbol{r}))\left(u_{1}\right) \\
= & \sigma \circ(\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{ppr}(\alpha, \boldsymbol{s})-\alpha \boldsymbol{r}-(1-\alpha) \mathbf{W} \operatorname{ppr}(\alpha, \boldsymbol{r}))\left(u_{1}\right) \\
= & \sigma \circ(\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{prg}(\alpha, \boldsymbol{s}, \boldsymbol{r})-\alpha \boldsymbol{r})\left(u_{1}\right) \\
= & \max \left(0,(\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))\left(u_{1}\right)\right. \\
& \left.-(\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))\left(u_{2}\right)+\alpha \boldsymbol{r}\left(u_{2}\right)-\alpha \boldsymbol{r}\left(u_{1}\right)\right) \\
\leq & \sigma \circ(\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))\left(u_{1}\right)+\max \left(0, \alpha \boldsymbol{r}\left(u_{2}\right)-\alpha \boldsymbol{r}\left(u_{1}\right)\right) \\
\leq & \sigma \circ(\alpha \boldsymbol{s}+(1-\alpha) \mathbf{W} \operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))\left(u_{1}\right)+\alpha \boldsymbol{r}\left(u_{2}\right) \\
\leq & \sigma \circ(\alpha \boldsymbol{s})\left(u_{1}\right)+(1-\alpha) \sigma \circ(\mathbf{W} \operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))\left(u_{1}\right)+\alpha \boldsymbol{r}\left(u_{2}\right) \\
\leq & \alpha \boldsymbol{s}\left(u_{1}\right)+\alpha \boldsymbol{r}\left(u_{2}\right)+(1-\alpha)(\mathbf{W} \boldsymbol{p})\left(u_{1}\right),
\end{aligned}
$$

where the first equality follows by the definition of the approximate Pagerank, the second equality follows by the definition of the Pagerank, and the last two inequalities follow by Lemma 6.2. This proves the first statement, and the second statement follows by symmetry.

Notice that applying the $\sigma$-operator for any vertex $u_{1}$ introduces a new dependency on the value of the residual vector $r$ at $u_{2}$. This subtle observation demonstrates the additional complexity introduced by the $\sigma$-operator when compared with previous analysis of Pagerankbased local algorithms ACL06]. Taking account of the $\sigma$-operator, let us further analyse the Lovász-Simonovits curve defined by $\boldsymbol{p}$, which is a common technique in the analysis of random walks on graphs [LS90]. Our goal is to show that if there is a set $S$ with a large value of $\boldsymbol{p}(S)$, then there must be a sweep set $S_{j}^{\boldsymbol{p}}$ with low conductance. We first bound the curve at some point by the conductance of the corresponding sweep set.

Lemma 6.7. Let $\boldsymbol{p}=\sigma \circ(\operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r}))$. It holds for any $j \in[n-1]$ that

$$
\begin{aligned}
\boldsymbol{p}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right] \leq & \alpha\left(s\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right]+\boldsymbol{r}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right]\right) \\
& +(1-\alpha) \frac{1}{2}\left(\boldsymbol{p}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)-\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|\right]\right) \\
& +(1-\alpha) \frac{1}{2}\left(\boldsymbol{p}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)+\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|\right]\right)
\end{aligned}
$$

Since $\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right) \pm\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|=\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\left(1 \pm \Phi\left(S_{j}^{\boldsymbol{p}}\right)\right)$, this tells us that when the conductance of the sweep set $S_{j}^{\boldsymbol{p}}$ is high, the Lovász-Simonovits curve around the point $\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)$ is close to a straight line. For the proof of Lemma 6.7, we view the undirected graph $H$ instead as a directed graph, and each edge $\{u, v\} \in E_{H}$ as a pair of directed edges $(u, v)$ and $(v, u)$. For any edge $(u, v)$ and vector $\boldsymbol{p} \in \mathbb{R}_{\geq 0}^{2 n}$, let

$$
p(u, v)=\frac{\boldsymbol{p}(u)}{d(u)}
$$

and for a set of directed edges $A$, let

$$
p(A)=\sum_{(u, v) \in A} p(u, v) .
$$

Now for any set of vertices $S \subset V_{H}$, let $\operatorname{in}(S)=\left\{(u, v) \in E_{H}: v \in S\right\}$ and out $(S)=$ $\left\{(u, v) \in V_{H}: u \in S\right\}$. Additionally, for any simple set $S \subset V_{H}$, we define the simple complement of the set $S$ by

$$
\widehat{S}=\left\{u_{2}: u_{1} \in S\right\} \cup\left\{u_{1}: u_{2} \in S\right\} .
$$

We need the following property of the Lovász-Simonovitz curve with regard to sets of edges.
Lemma 6.8. For any set of edges $A$, it holds that

$$
p(A) \leq \boldsymbol{p}[|A|] .
$$

Proof. For each $u \in V$, let $x_{u}=|\{(u, v) \in A\}|$ be the number of edges in $A$ with the tail at $u$. Then, we have by definition that

$$
p(A)=\sum_{(u, v) \in A} \frac{\boldsymbol{p}(u)}{d(u)}=\sum_{u \in V} \frac{x_{u}}{d(u)} \boldsymbol{p}(u) .
$$

Since $x_{u} / d(u) \in[0,1]$ and

$$
\sum_{u \in V} \frac{x_{u}}{d(u)} d(u)=\sum_{u \in V} x_{u}=|A|
$$

the statement follows from (3.1).
We also make use of the following lemma from [ACL06].
Lemma 6.9 ( ACL 06$]$, Lemma 4). For any distribution $\boldsymbol{p}$ and any set of vertices $A$,

$$
(\mathbf{W} \boldsymbol{p})(A) \leq \frac{1}{2} \cdot(p(\operatorname{in}(A) \cup \operatorname{out}(A))+p(\operatorname{in}(A) \cap \operatorname{out}(A)))
$$

These two facts give us everything we need to prove the behaviour of the Lovász-Simonovits curve described in Lemma 6.7.

Proof of Lemma 6.7. By definition, we have that

$$
\begin{aligned}
& \boldsymbol{p}(S) \\
& =\sum_{u_{1} \in S} \boldsymbol{p}\left(u_{1}\right)+\sum_{u_{2} \in S} \boldsymbol{p}\left(u_{2}\right) \\
& \leq \sum_{u_{1} \in S}\left(\alpha\left(\boldsymbol{s}\left(u_{1}\right)+\boldsymbol{r}\left(u_{2}\right)\right)+(1-\alpha)(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)\right) \\
& +\sum_{u_{2} \in S}\left(\alpha\left(\boldsymbol{s}\left(u_{2}\right)+\boldsymbol{r}\left(u_{1}\right)\right)+(1-\alpha)(\mathbf{W} \boldsymbol{p})\left(u_{2}\right)\right) \\
& =\alpha \cdot\left(\sum_{u_{1} \in S} \boldsymbol{s}\left(u_{1}\right)+\sum_{u_{2} \in S} \boldsymbol{s}\left(u_{2}\right)+\sum_{u_{1} \in S} \boldsymbol{r}\left(u_{2}\right)+\sum_{u_{2} \in S} \boldsymbol{r}\left(u_{1}\right)\right) \\
& +(1-\alpha) \cdot\left(\sum_{u_{1} \in S}(\mathbf{W} \boldsymbol{p})\left(u_{1}\right)+\sum_{u_{2} \in S}(\mathbf{W} \boldsymbol{p})\left(u_{2}\right)\right) \\
& =\alpha \cdot(s(S)+\boldsymbol{r}(\widehat{S}))+(1-\alpha) \cdot(\mathbf{W} \boldsymbol{p})(S) \\
& \leq \alpha \cdot(s(S)+\boldsymbol{r}(\widehat{S}))+(1-\alpha) \cdot \frac{1}{2} \cdot(p(\operatorname{in}(S) \cup \operatorname{out}(S))+p(\operatorname{in}(S) \cap \operatorname{out}(S))),
\end{aligned}
$$

where the first inequality follows by Lemma 6.6 and the last one follows by Lemma 6.9 .
Now, recall that $\boldsymbol{p}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right]=\boldsymbol{p}\left(S_{j}^{\boldsymbol{p}}\right)$ and notice that $\operatorname{vol}\left(\widehat{S_{j}^{\boldsymbol{p}}}\right)=\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)$ holds by the definition of $\widehat{S}$. Hence, using Lemma 6.8, we have that

$$
\begin{aligned}
& \boldsymbol{p}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right] \\
& =\boldsymbol{p}\left(S_{j}^{\boldsymbol{p}}\right) \\
& \leq \alpha\left(s\left(S_{j}^{\boldsymbol{p}}\right)+\boldsymbol{r}\left(\widehat{S_{j}^{\boldsymbol{p}}}\right)\right)+(1-\alpha) \cdot \frac{1}{2}\left(p\left(\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cup \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right)+p\left(\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cap \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right)\right) \\
& \leq \alpha\left(s\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right]+\boldsymbol{r}\left[\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)\right]\right) \\
& \quad+(1-\alpha) \cdot \frac{1}{2}\left(\boldsymbol{p}\left[\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cup \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|\right]+\boldsymbol{p}\left[\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cap \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|\right]\right) .
\end{aligned}
$$

It remains to show that

$$
\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cap \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|=\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)-\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|,
$$

and

$$
\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cup \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|=\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)+\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right| .
$$

The first follows by noticing that $\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cap \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|$ counts precisely twice the number of edges with both endpoints inside $S_{j}^{p}$. The second follows since

$$
\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cup \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|=\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \cap \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|+\left|\operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right) \backslash \operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right)\right|+\left|\operatorname{out}\left(S_{j}^{\boldsymbol{p}}\right) \backslash \operatorname{in}\left(S_{j}^{\boldsymbol{p}}\right)\right|
$$

and both the second and third terms on the right hand side are equal to $\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|$.
We now show another technical lemma which bounds the Lovász-Simonovits curve with respect to the conductance of the sweep sets $S_{j}^{p}$. To understand this lemma, recall from the discussion in Section 3.3 that the Lovász-Simonovits curve of the stationary distribution is linear: $\boldsymbol{\pi}[k]=k / \operatorname{vol}\left(V_{H}\right)$. This lemma bounds how far the Lovász-Simonovits curve deviates from that of the stationary distribution. In particular, if there is no sweep set with low conductance, then the deviation must be small.

Lemma 6.10. Let $\boldsymbol{p}=\sigma \circ\left(\operatorname{apr}_{H}(\alpha, s, \boldsymbol{r})\right)$ such that $\max _{u \in V_{H}} \frac{r(u)}{d(u)} \leq \epsilon$, and $\phi$ be any constant in $[0,1]$. If $\Phi\left(S_{j}^{\boldsymbol{p}}\right) \geq \phi$ for all $j \in[|\operatorname{supp}(\boldsymbol{p})|]$, then

$$
\boldsymbol{p}[k]-\frac{k}{\operatorname{vol}\left(V_{H}\right)} \leq \alpha t+\alpha \epsilon k t+\sqrt{\min \left(k, \operatorname{vol}\left(V_{H}\right)-k\right)}\left(1-\frac{\phi^{2}}{8}\right)^{t}
$$

for all $k \in\left[0, \operatorname{vol}\left(V_{H}\right)\right]$ and integer $t \geq 0$.
Proof. For ease of discussion we write $k_{j}=\operatorname{vol}\left(S_{j}^{\boldsymbol{p}}\right)$ and let $\overline{k_{j}}=\min \left(k_{j}, \operatorname{vol}\left(V_{H}\right)-k_{j}\right)$. For convenience we write

$$
f_{t}(k)=\alpha t+\alpha \epsilon k t+\sqrt{\min \left(k, \operatorname{vol}\left(V_{H}\right)-k\right)}\left(1-\frac{\phi^{2}}{8}\right)^{t}
$$

We prove by induction that for all $t \geq 0$,

$$
\boldsymbol{p}[k]-\frac{k}{\operatorname{vol}\left(V_{H}\right)} \leq f_{t}(k) .
$$

For the base case, this holds for $t=0$ for any $\phi$. Our proof is by case distinction: (1) for any integer $k \in\left[\operatorname{vol}\left(V_{H}\right)-1\right]$,

$$
\boldsymbol{p}[k]-\frac{k}{\operatorname{vol}\left(V_{H}\right)} \leq 1 \leq \sqrt{\min \left(k, \operatorname{vol}\left(V_{H}\right)-k\right)} \leq f_{0}(k) ;
$$

(2) for $k=0$ or $k=\operatorname{vol}\left(V_{H}\right), \boldsymbol{p}[k]-\frac{k}{\operatorname{vol}\left(V_{H}\right)}=0 \leq f_{0}(k)$. The base case follows, since $f_{0}$ is concave and $\boldsymbol{p}[k]$ is linear between integer values.

Now for the inductive case, we assume that the claim holds for $t$ and show that it holds for $t+1$. It suffices to show that the following holds for every $j \in[|\operatorname{supp}(\boldsymbol{p})|]$ :

$$
\boldsymbol{p}\left[k_{j}\right]-\frac{k_{j}}{\operatorname{vol}\left(V_{H}\right)} \leq f_{t+1}\left(k_{j}\right) .
$$

The lemma follows because this inequality holds trivially for $j=0$ and $j=n$ and $\boldsymbol{p}[k]$ is linear between $k_{j}$ for $j \in\{0, n\} \cup[1,|\operatorname{supp}(\boldsymbol{p})|]$.

For any $j \in[1,|\operatorname{supp}(\boldsymbol{p})|]$, it holds by Lemma 6.7 that

$$
\begin{aligned}
\boldsymbol{p}\left[k_{j}\right] & \leq \alpha\left(\boldsymbol{s}\left[k_{j}\right]+\boldsymbol{r}\left[k_{j}\right]\right)+(1-\alpha) \frac{1}{2}\left(\boldsymbol{p}\left[k_{j}-\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|\right]+\boldsymbol{p}\left[k_{j}+\left|\partial\left(S_{j}^{\boldsymbol{p}}\right)\right|\right]\right) \\
& \leq \alpha+\alpha \epsilon k_{j}+\frac{1}{2}\left(\boldsymbol{p}\left[k_{j}-\Phi\left(S_{j}^{\boldsymbol{p}}\right) \cdot \overline{k_{j}}\right]+\boldsymbol{p}\left[k_{j}+\Phi\left(S_{j}^{\boldsymbol{p}}\right) \cdot \overline{k_{j}}\right]\right) \\
& \leq \alpha+\alpha \epsilon k_{j}+\frac{1}{2}\left(\boldsymbol{p}\left[k_{j}-\phi \overline{k_{j}}\right]+\boldsymbol{p}\left[k_{j}+\phi \overline{k_{j}}\right]\right)
\end{aligned}
$$

where the second inequality uses the fact that $\boldsymbol{r}\left[k_{j}\right] \leq \epsilon k_{j}$ and the last inequality uses the concavity of $\boldsymbol{p}[k]$ and the fact that $\Phi\left(S_{j}^{\boldsymbol{p}}\right) \geq \phi$. By the induction hypothesis, we have that

$$
\begin{aligned}
& \boldsymbol{p}\left[k_{j}\right] \\
& \leq \alpha+\alpha \epsilon k_{j}+\frac{1}{2}\left(f_{t}\left(k_{j}-\phi \overline{k_{j}}\right)+\frac{k_{j}-\phi \overline{k_{j}}}{\operatorname{vol}\left(V_{H}\right)}+f_{t}\left(k_{j}+\phi \overline{k_{j}}\right)+\frac{k_{j}+\phi \overline{k_{j}}}{\operatorname{vol}\left(V_{H}\right)}\right) \\
& =\alpha+\alpha \epsilon k_{j}+\frac{k_{j}}{\operatorname{vol}\left(V_{H}\right)}+\frac{1}{2}\left(f_{t}\left(k_{j}+\phi \overline{k_{j}}\right)+f_{t}\left(k_{j}-\phi \overline{k_{j}}\right)\right) \\
& =\alpha+\alpha \epsilon k_{j}+\alpha t+\alpha \epsilon k_{j} t+\frac{k_{j}}{\operatorname{vol}\left(V_{H}\right)}+\frac{1}{2}\left(\sqrt{\min \left(k_{j}-\phi \overline{k_{j}}, \operatorname{vol}\left(V_{H}\right)-k_{j}+\phi \overline{k_{j}}\right)}\right. \\
& \left.\quad+\sqrt{\min \left(k_{j}+\phi \overline{k_{j}}, \operatorname{vol}\left(V_{H}\right)-k_{j}-\phi \overline{k_{j}}\right)}\right)\left(1-\frac{\phi^{2}}{8}\right)^{t} \\
& \leq \alpha(t+1)+\alpha \epsilon k_{j}(t+1)+\frac{k_{j}}{\operatorname{vol}\left(V_{H}\right)}+\frac{1}{2}\left(\sqrt{\overline{k_{j}}-\phi \overline{k_{j}}}+\sqrt{\overline{k_{j}}+\phi \overline{k_{j}}}\right)\left(1-\frac{\phi^{2}}{8}\right)^{t}
\end{aligned}
$$

Now, using the Taylor series of $\sqrt{1+\phi}$ at $\phi=0$, we see that for any $x \geq 0$ and $\phi \in[0,1]$,

$$
\begin{aligned}
\frac{1}{2}(\sqrt{x-\phi x}+\sqrt{x+\phi x}) & \leq \frac{\sqrt{x}}{2}\left(\left(1-\frac{\phi}{2}-\frac{\phi^{2}}{8}-\frac{\phi^{3}}{16}\right)+\left(1+\frac{\phi}{2}-\frac{\phi^{2}}{8}+\frac{\phi^{3}}{16}\right)\right) \\
& \leq \sqrt{x}\left(1-\frac{\phi^{2}}{8}\right)
\end{aligned}
$$

Applying this with $x=\overline{k_{j}}$, we have

$$
\begin{aligned}
\boldsymbol{p}\left[k_{j}\right]-\frac{k_{j}}{\operatorname{vol}\left(V_{H}\right)} & \leq \alpha(t+1)+\alpha \epsilon k_{j}(t+1)+\sqrt{\overline{k_{j}}}\left(1-\frac{\phi^{2}}{8}\right)\left(1-\frac{\phi^{2}}{8}\right)^{t} \\
& =f_{t+1}\left(k_{j}\right),
\end{aligned}
$$

which completes the proof.

We can now use this bound to show the other direction: if the Lovász-Simonovits curve is particularly large at some point, then there must be a sweep set with low conductance.

Lemma 6.11. Let $G$ be a graph with double cover $H$, and let $\boldsymbol{p}=\sigma \circ\left(\operatorname{apr}_{H}(\alpha, \boldsymbol{s}, \boldsymbol{r})\right)$ such that $\max _{u \in V_{H}} \frac{r(u)}{d(u)} \leq \epsilon$. If there is a set of vertices $S \subset V_{H}$ and a constant $\delta$ such that $\boldsymbol{p}(S)-\frac{\operatorname{vol}(S)}{\operatorname{vol}\left(V_{H}\right)} \geq \delta$, then there is some $j \in[1,|\operatorname{supp}(\boldsymbol{p})|]$ such that

$$
\Phi_{H}\left(S_{j}^{\boldsymbol{p}}\right)<6 \sqrt{(1+\epsilon \operatorname{vol}(S)) \alpha \ln \left(\frac{4}{\delta}\right) / \delta}
$$

Proof. Let $H_{c}$ be a weighted version of the double cover with the weight of every edge equal to some constant $c$. Let $k=\operatorname{vol}_{H_{c}}(S)$ and $\bar{k}=\min \left(k, \operatorname{vol}\left(V_{H_{c}}\right)-k\right)$. Notice that the behaviour of random walks and the conductance of vertex sets are unchanged by re-weighting the graph by a constant factor. As such, $\boldsymbol{p}=\sigma \circ \operatorname{apr}_{H_{c}}(\alpha, \boldsymbol{s}, \boldsymbol{r})$ and we let $\phi=\min _{j} \Phi_{H_{c}}\left(S_{j}^{\boldsymbol{p}}\right)$. Hence, by Lemma 6.10 it holds for any $t>0$ that

$$
\delta<\boldsymbol{p}[k]-\frac{k}{\operatorname{vol}\left(V_{H_{c}}\right)}<\alpha t(1+\epsilon k)+\sqrt{\bar{k}}\left(1-\frac{\phi^{2}}{8}\right)^{t}
$$

Letting $v=\operatorname{vol}\left(V_{H_{c}}\right)$ we set $t=\left\lceil\frac{8}{\phi^{2}} \ln \frac{2 \sqrt{v / 2}}{\delta}\right\rceil$. Now, assuming that

$$
\begin{equation*}
\delta \geq \frac{2}{\sqrt{v / 2}} \tag{6.3}
\end{equation*}
$$

we have

$$
\delta<(1+\epsilon k) \alpha\left\lceil\frac{8}{\phi^{2}} \ln \frac{2 \sqrt{v / 2}}{\delta}\right\rceil+\frac{\delta}{2 \sqrt{v / 2}} \sqrt{\bar{k}}<\frac{9(1+\epsilon k) \alpha \ln (v / 2)}{\phi^{2}}+\frac{\delta}{2}
$$

We set $c$ such that

$$
v=c \operatorname{vol}\left(V_{H}\right)=\operatorname{vol}\left(V_{H_{c}}\right)=\left(\frac{4}{\delta}\right)^{2}
$$

which satisfies 6.3) and implies that

$$
\phi<\sqrt{\frac{36(1+\epsilon k) \alpha \ln (4 / \delta)}{\delta}}
$$

By the definition of $\phi$, there must be some $S_{j}^{\boldsymbol{p}}$ with the desired property.

We have now shown the two facts promised at the beginning of this subsection. Putting these together, if there is a simple set $S \subset V_{H}$ with low conductance, then we can find a sweep set of $\sigma \circ \operatorname{apr}(\alpha, \boldsymbol{s}, \boldsymbol{r})$ with low conductance. By the reduction from almost-bipartite sets in $G$ to low-conductance simple sets in $H$, our target set corresponds to a simple set $S \subset V_{H}$ which leads to Algorithm 6.1 for finding almost-bipartite sets. Our result is summarised as follows.

Theorem 6.1. Let $G$ be an n-vertex undirected graph, and $L, R \subset V_{G}$ be disjoint sets such that $\beta(L, R) \leq \beta$ and $\operatorname{vol}(L \cup R) \leq \gamma$. Then, there is a set $C \subseteq L \cup R$ with $\operatorname{vol}(C) \geq \operatorname{vol}(L \cup R) / 2$ such that, for any $v \in C, \operatorname{LocBipartDC}(G, v, \gamma, \sqrt{7560 \beta})$ returns $\left(L^{\prime}, R^{\prime}\right)$ with $\beta\left(L^{\prime}, R^{\prime}\right)=O(\sqrt{\beta})$ and $\operatorname{vol}\left(L^{\prime} \cup R^{\prime}\right)=O\left(\beta^{-1} \gamma\right)$. Moreover, the algorithm has running time $O\left(\beta^{-1} \gamma \log n\right)$.

Proof. We set the parameters of Algorithm 6.1 to be

- $\alpha=\widehat{\beta}^{2} / 378=20 \beta$, and
- $\epsilon=1 /(20 \gamma)$.

Let $\boldsymbol{p}=\sigma \circ\left(\operatorname{apr}_{H}\left(\alpha, \boldsymbol{\chi}_{v}, \boldsymbol{r}\right)\right)$ be the simplified approximate Pagerank vector computed in the algorithm, which satisfies that $\max _{v \in V_{H}} \frac{r(v)}{d(v)} \leq \epsilon$, and $C^{\prime}=L_{1} \cup R_{2}$ be the target set in the double cover. Hence, by Lemma 6.5 we have that

$$
\boldsymbol{p}\left(C^{\prime}\right) \geq 1-\frac{2 \beta}{\alpha}-2 \epsilon \cdot \operatorname{vol}(C) \geq 1-\frac{1}{10}-\frac{1}{10}=\frac{4}{5} .
$$

Notice that, since $C^{\prime}$ is simple, we have that $\frac{\operatorname{vol}\left(C^{\prime}\right)}{\operatorname{vol}\left(V_{H}\right)} \leq \frac{1}{2}$ and

$$
\boldsymbol{p}\left(C^{\prime}\right)-\frac{\operatorname{vol}\left(C^{\prime}\right)}{\operatorname{vol}\left(V_{H}\right)} \geq \frac{4}{5}-\frac{1}{2}=\frac{3}{10} .
$$

Therefore, by Lemma 6.11 and choosing $\delta=3 / 10$, there is some $j \in[1,|\operatorname{supp}(\boldsymbol{p})|]$ such that

$$
\Phi\left(S_{j}^{\boldsymbol{p}}\right)<\sqrt{\frac{360}{3}(1+\epsilon \operatorname{vol}(C)) \alpha \ln \left(\frac{40}{3}\right)} \leq \sqrt{\left(1+\frac{1}{20}\right) 7200 \beta}=\widehat{\beta}
$$

Since such a $S_{j}^{\boldsymbol{p}}$ is a simple set, by Lemma 6.1, we know that the sets $L^{\prime}=\left\{u \in V_{G}\right.$ : $\left.u_{1} \in S_{j}^{\boldsymbol{p}}\right\}$ and $R^{\prime}=\left\{u \in V_{G}: u_{2} \in S_{j}^{\boldsymbol{p}}\right\}$ satisfy $\beta\left(L^{\prime}, R^{\prime}\right) \leq \widehat{\beta}$. For the volume guarantee, Lemma 6.4 gives that

$$
\operatorname{vol}(\operatorname{supp}(\boldsymbol{p})) \leq \frac{1}{\epsilon \alpha}=\frac{\gamma}{\beta}
$$

Since the sweep set procedure in the algorithm is over the support of $p$, we have that

$$
\operatorname{vol}\left(L^{\prime} \cup R^{\prime}\right) \leq \operatorname{vol}(\operatorname{supp}(\boldsymbol{p})) \leq \beta^{-1} \gamma
$$

Now, we analyse the running time of the algorithm. By Lemma 6.4, computing the approximate Pagerank vector $p^{\prime}$ takes time

$$
O\left(\frac{1}{\epsilon \alpha}\right)=O\left(\beta^{-1} \gamma\right)
$$

Since $\operatorname{vol}\left(\operatorname{supp}\left(\boldsymbol{p}^{\prime}\right)\right)=O\left(\frac{1}{\epsilon \alpha}\right)$, computing $\sigma \circ\left(\boldsymbol{p}^{\prime}\right)$ can also be done in time $O\left(\beta^{-1} \gamma\right)$. The cost of checking each conductance in the sweep set loop is $O(\operatorname{vol}(\operatorname{supp}(\boldsymbol{p})) \log (n))=$ $O\left(\beta^{-1} \gamma \log (n)\right)$. This dominates the running time of the algorithm, which completes the proof of the theorem.


Figure 6.5: An example of a directed graph and its semi-double cover

We remark that the quadratic approximation guarantee in Theorem 6.1 matches the state-of-the-art local algorithm for finding a single set with low conductance [AOGPT16]. Furthermore, this result presents a significant improvement over the previous state-of-the-art by Li and Peng [LP13], whose design is based on an entirely different technique. For any $\epsilon \in[0,1 / 2]$, their algorithm has running time $O\left(\epsilon^{2} \beta^{-2} \gamma^{1+\epsilon} \log ^{3} \gamma\right)$ and returns a set with volume $O\left(\gamma^{1+\epsilon}\right)$ and bipartiteness $O(\sqrt{\beta / \epsilon})$. In particular, their algorithm requires much higher time complexity in order to guarantee the same bipartiteness $O(\sqrt{\beta})$.

### 6.2 Algorithm for Directed Graphs

We now turn our attention to local algorithms for finding densely-connected clusters in directed graphs. In comparison with undirected graphs, we are interested in finding disjoint $L, R \subset V$ of an input directed graph $G=(V, E)$ such that most of the edges adjacent to $L \cup R$ are from $L$ to $R$. To formalise this, we define the flow ratio from $L$ to $R$ as

$$
F(L, R) \triangleq 1-\frac{2 w(L, R)}{\operatorname{vol}_{\mathrm{out}}(L)+\operatorname{vol}_{\mathrm{in}}(R)}
$$

where $w(L, R)$ is the number of directed edges from $L$ to $R$. Notice that we take not only edge densities but also edge directions into account: a low $F(L, R)$-value tells us that almost all edges with their tail in $L$ have their head in $R$, and conversely almost all edges with their head in $R$ have their tail in $L$. One could also see $F(L, R)$ as a generalisation of $\beta(L, R)$. In particular, if we view an undirected graph as a directed graph by replacing each edge with two directed edges, then $\beta(L, R)=F(L, R)$. In this section, we develop a local algorithm for finding such vertex sets in a directed graph, and analyse the algorithm's performance.

### 6.2.1 Reduction by Semi-Double Cover

Given a directed graph $G=\left(V_{G}, E_{G}\right)$, we construct its semi-double cover $H=\left(V_{H}, E_{H}\right)$ as follows: (1) every vertex $v \in V_{G}$ has two corresponding vertices $v_{1}, v_{2} \in V_{H}$; (2) for every edge $(u, v) \in E_{G}$, we add the edge $\left\{u_{1}, v_{2}\right\}$ in $E_{H}$, see Figure 6.5 for illustration.

It is worth comparing this reduction with the one for undirected graphs:

- For undirected graphs, we apply the standard double cover and every undirected edge in $G$ corresponds to two edges in the double cover $H$;
- For directed graphs, every directed edge in $G$ corresponds to one undirected edge in $H$. This asymmetry would allow us to 'recover' the direction of any edge in $G$.
We follow the use of $S_{1}, S_{2}$ from Section 6.1 for any $S \subset V_{G}$, we define $S_{1} \subset V_{H}$ and $S_{2} \subset V_{H}$ by $S_{1} \triangleq\left\{v_{1} \mid v \in S\right\}$ and $S_{2} \triangleq\left\{v_{2} \mid v \in S\right\}$. The lemma below shows the connection between the value of $F_{G}(L, R)$ for any $L, R$ and $\Phi_{H}\left(L_{1} \cup R_{2}\right)$.

Lemma 6.12. Let $G$ be a directed graph with semi-double cover $H$. Then, it holds for any $L, R \subset V_{G}$ that $F_{G}(L, R)=\Phi_{H}\left(L_{1} \cup R_{2}\right)$. Similarly, for any simple set $S \subset V_{H}$, let $L=\left\{u: u_{1} \in S\right\}$ and $R=\left\{u: u_{2} \in S\right\}$. Then, it holds that $F_{G}(L, R)=\Phi_{H}(S)$.

Proof. We define $S_{H}=L_{1} \cup R_{2}$, and have that

$$
\begin{aligned}
\Phi_{H}\left(S_{H}\right) & =\frac{w\left(S_{H}, V_{H} \backslash S_{H}\right)}{\operatorname{vol}_{H}\left(S_{H}\right)} \\
& =\frac{\operatorname{vol}_{H}\left(S_{H}\right)-2 w_{H}\left(S_{H}, S_{H}\right)}{\operatorname{vol}_{H}\left(S_{H}\right)} \\
& =1-\frac{2 w_{H}\left(S_{H}, S_{H}\right)}{\operatorname{vol}_{H}\left(S_{H}\right)} \\
& =1-\frac{2 w_{G}(L, R)}{\operatorname{vol}_{\text {out }}(L)+\operatorname{vol}_{\text {in }}(R)} \\
& =F_{G}(L, R) .
\end{aligned}
$$

This proves the first statement of the lemma. The second statement of the lemma follows by the same argument.

### 6.2.2 Design and Analysis of the Algorithm

The new algorithm is a modification of the algorithm by Andersen and Peres [AP09] described in Section 3.3.2. Given a directed graph $G$ as input, the algorithm simulates the volume-biased ESP on $G^{\prime}$ 's semi-double cover $H$. Notice that the graph $H$ can be constructed locally in the same way as the local construction of the double cover. However, as the output set $S$ of an ESP-based algorithm is not necessarily simple, the algorithm only returns vertices $u \in V_{G}$ in which exactly one of $u_{1}$ and $u_{2}$ are included in $S$. The key procedure for the algorithm is given in Algorithm 6.4, in which the GenerateSample procedure is the one described at the end of Section 3.3.2.

Notice that in the constructed graph $H, \Phi\left(L_{1} \cup R_{2}\right) \leq \phi$ does not imply that $\Phi\left(L_{2} \cup R_{1}\right) \leq \phi$. Due to this asymmetry, Algorithm 6.4 takes a parameter $i \in\{1,2\}$ to indicate whether the starting vertex is in $L$ or $R$. If it is not known whether $u$ is in $L$ or $R$, two copies can be run in parallel, one with $i=1$ and the other with $i=2$. Once one of them terminates with the

```
Algorithm 6.4: EvoCutDirected \((u, i, \phi)\)
    Input : Starting vertex \(u, i \in\{1,2\}\), target flow ratio \(\phi\)
    Output: A pair of sets \(L, R \subset V_{G}\)
    1 Set \(T=\left\lfloor\left(100 \phi^{\frac{2}{3}}\right)^{-1}\right\rfloor\)
    2 Compute \(S=\operatorname{GenerateSample}_{H}\left(u_{i}, T\right)\)
    3 Let \(L=\left\{u \in V_{G}: u_{1} \in S\right.\) and \(\left.u_{2} \notin S\right\}\)
    4 Let \(R=\left\{u \in V_{G}: u_{2} \in S\right.\) and \(\left.u_{1} \notin S\right\}\)
    5 return \(L\) and \(R\)
```

performance guaranteed in Theorem 6.2, the other can be terminated. Hence, we can always assume that it is known whether the starting vertex $u$ is in $L$ or $R$.

We now analyse the algorithm. Notice that, since the evolving set process gives us an arbitrary set on the semi-double cover, Algorithm 6.4 converts this into a simple set by removing any vertices $u$ where $u_{1} \in S$ and $u_{2} \in S$. The following definition allows us to discuss sets which are close to being simple.

Definition 6.3 ( $\epsilon$-simple set). For any set $S \subset V_{H}$, let $P=\left\{u_{1}, u_{2}: u \in V_{G}, u_{1} \in\right.$ $S$ and $\left.u_{2} \in S\right\}$. We call set $S \epsilon$-simple if it holds that $\frac{\operatorname{vol}(P)}{\operatorname{vol}(S)} \leq \epsilon$.

The notion of $\epsilon$-simple sets measures the ratio of vertices in which both $u_{1}$ and $u_{2}$ are in $S$. In particular, any simple set defined in Definition 6.1 is 0 -simple. We show that, if $S \subset V_{H}$ is an $\epsilon$-simple set for small $\epsilon$, we can construct a simple set $S^{\prime}$ with low conductance.

Lemma 6.13. For any $\epsilon$-simple set $S \subset V_{H}$, let $P=\left\{u_{1}, u_{2}: u \in V_{G}, u_{1} \in S\right.$ and $\left.u_{2} \in S\right\}$ and set $S^{\prime}=S \backslash P$. Then, $\Phi\left(S^{\prime}\right) \leq \frac{1}{1-\epsilon}(\Phi(S)+\epsilon)$.

Proof. By the definition of conductance, we have

$$
\begin{aligned}
\Phi\left(S^{\prime}\right) & =\frac{w\left(S^{\prime}, V \backslash S^{\prime}\right)}{\operatorname{vol}\left(S^{\prime}\right)} \\
& \leq \frac{w(S, V \backslash S)+\operatorname{vol}(P)}{\operatorname{vol}(S)-\operatorname{vol}(P)} \\
& \leq \frac{1}{1-\epsilon} \frac{w(S, V \backslash S)+\operatorname{vol}(P)}{\operatorname{vol}(S)} \\
& \leq \frac{1}{1-\epsilon}\left(\frac{w(S, V \backslash S)}{\operatorname{vol}(S)}+\epsilon\right) \\
& =\frac{1}{1-\epsilon}(\Phi(S)+\epsilon)
\end{aligned}
$$

which proves the lemma.

Given this, in order to guarantee that $\Phi\left(S^{\prime}\right)$ is low, we need to construct $S$ such that $\Phi(S)$ is low and $S$ is $\epsilon$-simple for small $\epsilon$. The following lemma shows that, by choosing the number
of steps of the evolving set process carefully, we can very tightly control the overlap of the output of the evolving set process with the target set $S$, which we know to be simple.

Lemma 6.14. Let $C \subset V_{H}$ be a set with conductance $\Phi(C)=\phi$, and let $T=\left(100 \phi^{\frac{2}{3}}\right)^{-1}$. There exists a set $C_{g} \subseteq C$ with volume at least $\operatorname{vol}(C) / 2$ such that for any $x \in C_{g}$, with probability at least $7 / 9$, a sample path $\left(S_{0}, S_{1}, \ldots, S_{T}\right)$ from the volume-biased ESP started from $S_{0}=\{x\}$ satisfies the following:

- $\Phi\left(S_{t}\right)<60 \phi^{\frac{1}{3}} \sqrt{\ln (\operatorname{vol}(V))}$ for some $t \in[0, T]$;
- $\operatorname{vol}\left(S_{t} \cap C\right) \geq\left(1-\frac{1}{10} \phi^{\frac{1}{3}}\right) \operatorname{vol}\left(S_{t}\right)$ for all $t \in[0, T]$.

Proof. By Proposition 3.3. with probability at least $(1-1 / 9)$ there is some $t<T$ such that

$$
\Phi\left(S_{t}\right) \leq 3 \sqrt{4 T^{-1} \ln \operatorname{vol}\left(V_{H}\right)}=3 \sqrt{400 \phi^{\frac{2}{3}} \ln \operatorname{vol}\left(V_{H}\right)}=60 \phi^{\frac{1}{3}} \sqrt{\ln \left(\operatorname{vol}\left(V_{H}\right)\right)} .
$$

Then, by Proposition 3.1 we have that $\operatorname{esc}(x, T, C) \leq T \phi=\frac{1}{100} \phi^{1 / 3}$. Hence, by Proposition 3.4 with probability at least $9 / 10$,

$$
\max _{t \leq T} \frac{\operatorname{vol}\left(S_{t} \backslash C\right)}{\operatorname{vol}\left(S_{t}\right)} \leq 10 \cdot \operatorname{esc}(x, T, C) \leq \frac{1}{10} \cdot \phi^{1 / 3}
$$

Applying the union bound on these two events proves the statement of the lemma.
This allows us to analyse the performance of Algorithm 6.4, which is summarised in Theorem 6.2.

Theorem 6.2. Let $G$ be an $n$-vertex directed graph, and $L, R \subset V_{G}$ be disjoint sets such that $F(L, R) \leq \phi$ and $\operatorname{vol}(L \cup R) \leq \gamma$. There is a set $C \subseteq L \cup R$ with $\operatorname{vol}(C) \geq \operatorname{vol}(L \cup R) / 2$ such that, for any $v \in C$ and some $i \in\{1,2\}$, EvoCutDirected $(G, v, i, \phi)$ returns $\left(L^{\prime}, R^{\prime}\right)$ such that $F\left(L^{\prime}, R^{\prime}\right)=O\left(\phi^{\frac{1}{3}} \log ^{\frac{1}{2}} n\right)$ and $\operatorname{vol}\left(L^{\prime} \cup R^{\prime}\right)=O\left(\left(1-\phi^{\frac{1}{3}}\right)^{-1} \gamma\right)$. Moreover, the algorithm has running time $O\left(\phi^{-\frac{1}{2}} \gamma \log ^{\frac{3}{2}} n\right)$.

Proof. By Lemma 6.14, we know that, with probability at least $7 / 9$, the set $S$ computed by Algorithm 6.4 satisfies

$$
\operatorname{vol}(S \cap C) \geq\left(1-\frac{1}{10} \phi^{\frac{1}{3}}\right) \operatorname{vol}(S)
$$

Let $P=\left\{u_{1}, u_{2}: u_{1} \in S\right.$ and $\left.u_{2} \in S\right\}$. Then, since $C$ is simple, we have that

$$
\operatorname{vol}(P) \leq 2 \cdot(\operatorname{vol}(S)-\operatorname{vol}(S \cap C)) \leq 2 \cdot\left(1-\left(1-\frac{1}{10} \phi^{\frac{1}{3}}\right)\right) \cdot \operatorname{vol}(S)=\frac{1}{5} \cdot \phi^{\frac{1}{3}} \operatorname{vol}(S)
$$

which implies that $S$ is $(1 / 5) \cdot \phi^{\frac{1}{3}}$-simple. Therefore, by letting $S^{\prime}=S \backslash P$, we have by Lemma 6.13 that

$$
\Phi\left(S^{\prime}\right) \leq \frac{1}{1-\epsilon} \cdot(\Phi(S)+\epsilon) \leq \frac{1}{1-\frac{1}{5}}\left(\Phi(S)+\frac{1}{5} \phi^{\frac{1}{3}}\right) \leq \frac{5}{4} \cdot \Phi(S)+\frac{1}{4} \cdot \phi^{\frac{1}{3}} .
$$

Since $\Phi(S)=O\left(\phi^{\frac{1}{3}} \ln ^{\frac{1}{2}}(n)\right)$ by Lemma 6.14. we have that $\Phi\left(S^{\prime}\right)=O\left(\phi^{\frac{1}{3}} \ln ^{\frac{1}{2}}(n)\right)$. For the volume guarantee, direct calculation shows that

$$
\operatorname{vol}\left(S^{\prime}\right) \leq \operatorname{vol}(S) \leq \frac{\operatorname{vol}(S \cap C)}{1-\frac{1}{10} \phi^{\frac{1}{3}}} \leq \frac{\operatorname{vol}(C)}{1-\phi^{\frac{1}{3}}}
$$

Finally, the running time of the algorithm is dominated by the GenerateSample method, which is shown in [AP09] to have running time $O\left(\operatorname{vol}(C) \phi^{-\frac{1}{2}} \ln ^{\frac{3}{2}}(n)\right)$.

Algorithm 6.4 is the first local algorithm for directed graphs that approximates a pair of densely connected clusters, and demonstrates that finding such a pair appears to be much easier than finding a low-conductance set in a directed graph; in particular, existing local algorithms for finding a low-conductance set require the stationary distribution of the random walk in the directed graph [ACL07], the sublinear-time computation of which is unknown [CKP $\left.{ }^{+} 17\right]$. However, the knowledge of the stationary distribution is not needed for our algorithm.

### 6.3 Experimental Results

In this section we evaluate the performance of the newly proposed algorithms on both synthetic and real-world datasets. For undirected graphs, we compare the performance of LOCBIPARTDC against the previous state-of-the-art [LP13] through synthetic datasets with various parameters, and apply a real-world dataset to demonstrate the significance of the new algorithm. For directed graphs, we compare the performance of EvoCutDirected with the state-of-the-art non-local algorithm since our presented local algorithm for directed graphs is the first such algorithm in the literature. All experiments are performed on a Lenovo Yoga 2 Pro with an Intel(R) Core(TM) i7-4510U CPU @ 2.00 GHz processor and 8GB of RAM. The code can be downloaded from
https://github.com/pmacg/local-densely-connected-clusters.

### 6.3.1 Results for Undirected Graphs

We first compare the performance of LOCBIPARTDC with the previous state-of-the-art given by Li and Peng [LP13], which we can refer to as LP, on synthetic graphs generated from the stochastic block model (SBM). Specifically, we assume that the graph has $k=3$ clusters $\left\{C_{j}\right\}_{j=1}^{3}$, and the number of vertices in each cluster, denoted by $n_{1}, n_{2}$ and $n_{3}$ respectively, satisfy $n_{1}=n_{2}=0.1 n_{3}$. Moreover, any pair of vertices $u \in C_{i}$ and $v \in C_{j}$ is connected with probability $P_{i, j}$. We assume that $P_{1,1}=P_{2,2}=p_{1}, P_{3,3}=p_{2}, P_{1,2}=q_{1}$, and $P_{1,3}=P_{2,3}=q_{2}$. Throughout the experiments, we maintain the ratios $p_{2}=2 p_{1}$ and $q_{2}=0.1 p_{1}$, leaving the parameters $n_{1}, p_{1}$ and $q_{1}$ free. Notice that the different values of


Figure 6.6: (a) A comparison of the runtimes of the LocBipartDC algorithm and the LP algorithm with various values of $\epsilon_{L P}$. (b) Setting $\epsilon_{L P}=0.01$ results in a significantly worse Adjusted Rand Index (ARI) than the LocBipartDC algorithm.
$q_{1}$ and $q_{2}$ guarantee that $C_{1}$ and $C_{2}$ are the ones optimising the $\beta$-value, which is why the proposed model is slightly more involved than the standard SBM described in Section 2.3.2.

We evaluate the quality of the output $(L, R)$ returned by each algorithm with respect to its $\beta$-value, the Adjusted Rand Index (ARI) [GA17], as well as the ratio of the misclassified vertices defined by $\left(\left|L \triangle C_{1}\right|+\left|R \triangle C_{2}\right|\right) /\left(\left|L \cup C_{1}\right|+\left|R \cup C_{2}\right|\right)$. All the reported results are the average performance of each algorithm over 10 runs, in which a random vertex from $C_{1} \cup C_{2}$ is chosen as the starting vertex of the algorithm.

Setting the $\epsilon_{L P}$ parameter in the LP algorithm. The LP algorithm has an additional parameter over LocBipartDC, which we refer to as $\epsilon_{L P}$. This parameter influences the runtime and performance of the algorithm and must be in the range $[0,1 / 2]$. In order to choose a fair value of $\epsilon_{L P}$ for comparison with LOCBIPARTDC, we consider several values on graphs with a range of target volumes.

We generate graphs from the SBM such that $p_{1}=1 / n_{1}$ and $q_{1}=100 p_{1}$ and vary the size of the target set by varying $n_{1}$ between 100 and 10,000 . For values of $\epsilon_{L P}$ in $\{0.01,0.02,0.1,0.5\}$, Figure 6.6(a) shows how the runtime of the LP algorithm compares to the LocBipartDC algorithm for a range of target volumes. In this experiment, the runtime of the LOCBIPARTDC algorithm lies between the runtimes of the LP algorithm for $\epsilon_{L P}=0.01$ and $\epsilon_{L P}=0.02$. However, Figure 6.6 (b) shows that the performance of the LP algorithm with $\epsilon_{L P}=0.01$ is significantly worse than the performance of the LOCBIPARTDC algorithm, and so for a fair comparison we set $\epsilon_{L P}=0.02$ for the remainder of the experiments.

Table 6.1: Full comparison between Algorithm 6.1 (LBDC) and the previous state-of-the-art [LP13] (LP). For clarity we report the target bipartiteness $\beta=\beta\left(C_{1}, C_{2}\right)$ and target volume $\gamma=\operatorname{vol}\left(C_{1} \cup C_{2}\right)$ along with the SBM parameters. The final column shows the ratio between the number of misclassified vertices and the total number of vertices in the target clusters.

| Input GRaph parameters | Algo. | RUNTIME | $\beta$-value | ARI | Misclass. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $n_{1}=10^{3}, p_{1}=10^{-3}, q_{1}=0.018$ | LBDC | $\mathbf{0 . 0 9}$ | $\mathbf{0 . 1 5 4}$ | $\mathbf{0 . 9 6 8}$ | $\mathbf{0 . 0 7 3}$ |
| $\beta \approx 0.1, \gamma \approx 4 \cdot 10^{4}$ | LP | 0.146 | 0.202 | 0.909 | 0.138 |
| $n_{1}=10^{4}, p_{1}=10^{-4}, q_{1}=0.0018$ | LBDC | $\mathbf{0 . 9 9 2}$ | $\mathbf{0 . 2 1 5}$ | $\mathbf{0 . 9 4 0}$ | $\mathbf{0 . 1 4 5}$ |
| $\beta \approx 0.1, \gamma \approx 4 \cdot 10^{5}$ | LP | 1.327 | 0.297 | 0.857 | 0.256 |
| $n_{1}=10^{5}, p_{1}=10^{-5}, q_{1}=0.00018$ | LBDC | $\mathbf{1 9 . 5 8 5}$ | $\mathbf{0 . 2 5 0}$ | $\mathbf{0 . 9 5 0}$ | $\mathbf{0 . 1 6 6}$ |
| $\beta \approx 0.1, \gamma \approx 4 \cdot 10^{6}$ | LP | 30.285 | 0.300 | 0.865 | 0.225 |
| $n_{1}=10^{3}, p_{1}=4 \cdot 10^{-3}, q_{1}=0.012$ | LBDC | $\mathbf{1 . 2 4 9}$ | $\mathbf{0 . 5 0 6}$ | $\mathbf{0 . 5 0 3}$ | $\mathbf{0 . 7 6 3}$ |
| $\beta \approx 0.4, \gamma \approx 4 \cdot 10^{4}$ | LP | 1.329 | 0.597 | 0.445 | 0.785 |

Comparison on the synthetic data. We now compare the LOcBipartDC and LP algorithms' performance on graphs generated from the SBM with different values of $n_{1}, p_{1}$ and $q_{1}$. As shown in Table 6.1, LocBipartDC not only runs faster, but also produces better clusters with respect to all three metrics. Secondly, since the clustering task becomes more challenging when the target clusters have higher $\beta$-value, we compare the algorithms' performance on a sequence of instances with increasing value of $\beta$. Since $q_{1} / p_{1}=2(1-\beta) / \beta$, we simply fix the values of $n_{1}, p_{1}$ as $n_{1}=1,000, p_{1}=0.001$, and generate graphs with increasing value of $q_{1} / p_{1}$; this gives us graphs with monotone values of $\beta$. As shown in Figure 6.7 (a), the performance of our algorithm is always better than the previous state-of-the-art.

Thirdly, notice that both algorithms use some parameters to control the algorithm's runtime and the output's approximation ratio, which are naturally influenced by each other. To study this dependency, we generate graphs according to $n_{1}=100,000, p_{1}=0.000015$, and $q_{1}=0.00027$ which results in target sets with $\beta \approx 0.1$ and volume $\gamma \approx 6,000,000$. Figure 6.7(b) shows that, in comparison with the previous state-of-the-art, LOcBIPARTDC takes much less time to produce output with the same ARI value.

Experiments on the real-world dataset. We further demonstrate the significance of our algorithm on the Interstate Disputes Dataset (v3.1) [MJK $\left.{ }^{+19}\right]$, which records every interstate dispute during 1816-2010, including the level of hostility resulting from the dispute and the number of casualties, and has been widely studied in the social and political sciences [MMR02, MMT08] as well as the machine learning community [HRC17, ME11, TB09]. For a given


Figure 6.7: (a) The ARI value of each algorithm when varying the target $\beta$-value. A higher $q_{1} / p_{1}$ ratio corresponds to a lower $\beta$-value. (b) The ARI value of each algorithm when bounding the runtime of the algorithm.
time period, we construct a graph from the data by representing each country with a vertex and adding an edge between each pair of countries weighted according to the severity of any military disputes between those countries. Specifically, if there's a war ${ }^{11}$ between the two countries, the corresponding two vertices are connected by an edge with weight 30 ; for any other dispute that is not part of an interstate war, the two corresponding vertices are connected by an edge with weight 1 . We always use the USA as the starting vertex of the algorithm, and the algorithm's output, as visualised in Figure 6.1, can be well explained by geopolitics. The $\beta$-values of the pairs of clusters in Figures 6.1 are $0.361,0.356,0.170$ and 0.191 respectively.

### 6.3.2 Results for Directed Graphs

Next we evaluate the performance of our algorithm for directed graphs on synthetic and real-world datasets. Since there are no previous local directed graph clustering algorithms that achieve similar objectives, we compare the output of Algorithm 6.4 (ECD) with the state-of-the-art non-local algorithm proposed by Cucuringu et al. [CLSZ20], and we refer this to as CLSZ in the following.

Synthetic dataset. Let us first look at the cyclic block model (CBM) described in Cucuringu et al. [CLSZ20] with parameters $k, n, p, q$, and $\eta$. In this model, we generate a directed graph with $k$ clusters $C_{1}, \ldots, C_{k}$ of size $n$, and for $u, v \in C_{i}$, there is an edge between $u$ and $v$ with probability $p$ and the edge direction is chosen uniformly at random. For $u \in C_{i}$ and $v \in C_{i+1} \bmod k$, there is an edge between $u$ and $v$ with probability $q$, and the edge is directed from $u$ to $v$ with probability $\eta$ and from $v$ to $u$ with probability $1-\eta$. Throughout our experiments, we fix $p=0.001, q=0.01$, and $\eta=0.9$.

[^6]Table 6.2: Comparison of ECD with CLSZ on synthetic data.

| Model | $n^{\prime}$ | $n$ | $k$ | Time |  | ARI |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | ECD | CLSZ | ECD | CLSZ |
| CBM | - | $10^{3}$ | 5 | 1.59 | 3.99 | 0.92 | 1.00 |
| CBM | - | $10^{3}$ | 50 | 3.81 | 156.24 | 0.99 | 0.99 |
| CBM + | $10^{2}$ | $10^{3}$ | 3 | 0.24 | 6.12 | 0.98 | 0.35 |
| CBM + | $10^{2}$ | $10^{4}$ | 3 | 0.32 | 45.17 | 0.99 | 0.01 |

Secondly, since the goal of our algorithm is to find local structure in a graph, we extend the cyclic block model with additional local clusters and refer to this model as CBM+. In addition to the parameters of the CBM, we introduce the parameters $n^{\prime}, q_{1}^{\prime}, q_{2}^{\prime}$, and $\eta^{\prime}$. In this model, the clusters $C_{1}$ to $C_{k}$ are generated as in the CBM, and there are two additional clusters $C_{k+1}$ and $C_{k+2}$ of size $n^{\prime}$. For $u, v \in C_{k+i}$ for $i \in\{1,2\}$, there is an edge between $u$ and $v$ with probability $p$ and for $u \in C_{k+1}$ and $v \in C_{k+2}$, there is an edge with probability $q_{1}^{\prime}$; the edge directions are chosen uniformly at random. For $u \in C_{k+1} \cup C_{k+2}$ and $v \in C_{1}$, there is an edge with probability $q_{2}^{\prime}$. If $u \in C_{k+1}$, the orientation is from $v$ to $u$ with probability $\eta^{\prime}$ and from $u$ to $v$ with probability $1-\eta^{\prime}$ and if $u \in C_{k+2}$, the orientation is from $u$ to $v$ with probability $\eta^{\prime}$ and from $v$ to $u$ with probability $1-\eta^{\prime}$. We always fix $q_{1}^{\prime}=0.5, q_{2}^{\prime}=0.005$, and $\eta^{\prime}=1$. Notice that the clusters $C_{k+1}$ and $C_{k+2}$ form a 'local' cycle with the cluster $C_{1}$.

Table 6.2 reports the average performance over 10 runs with a variety of parameters. We can see that CLSZ uncovers the global structure in the CBM more accurately than ECD. On the other hand, CLSZ fails to identify the local cycle in the CBM+ model, while ECD succeeds.

Real-world dataset. Now we evaluate the algorithms' performance on the US Migration Dataset [CB]. For fair comparison, we follow Cucuringu et al. [CLSZ20] and construct the directed graph as follows: every county in the mainland USA is represented by a vertex; for any vertices $j, \ell$, the edge weight of $(j, \ell)$ is given by

$$
\left|\frac{\mathbf{M}_{j, \ell}-\mathbf{M}_{\ell, j}}{\mathbf{M}_{j, \ell}+\mathbf{M}_{\ell, j}}\right|
$$

where $\mathbf{M}_{j, \ell}$ is the number of people who migrated from county $j$ to county $\ell$ between 1995 and 2000; in addition, the direction of $(j, \ell)$ is set to be from $j$ to $\ell$ if $\mathbf{M}_{j, \ell}>\mathbf{M}_{\ell, j}$, otherwise the direction is set to be the opposite.

For CLSZ, we follow their suggestion on the same dataset and set $k=10$. Both algorithms' performance is evaluated with respect to the flow ratio, as well as the cut imbalance ratio


Figure 6.8: The top 4 pairs of clusters found by CLSZ on the US Migration dataset.
used in their work. For any vertex sets $L$ and $R$, the cut imbalance ratio is defined by

$$
\mathrm{CI}(L, R)=\frac{1}{2} \cdot\left|\frac{w(L, R)-w(R, L)}{w(L, R)+w(R, L)}\right|,
$$

and a higher $\mathrm{CI}(L, R)$ value indicates the connection between $L$ and $R$ is more significant. Using counties in Ohio, New York, California, and Florida as the starting vertices, the outputs of EvoCutDirected are visualised in Figures 6.2, and in Table 6.3 we compare them to the top 4 pairs returned by CLSZ, which are shown in Figure 6.8. The new algorithm produces better outputs with respect to the two metrics.

These experiments suggest that local algorithms are not only more efficient, but can also be much more effective than global algorithms when learning certain structures in graphs. In particular, some localised structure might be hidden when applying the objective function

Table 6.3: Comparison of EvoCutDirected with CLSZ on the US migration dataset.

| Figure | Algorithm | Cluster | Cut Imbalance | Flow Ratio |
| :---: | :---: | :---: | :---: | :---: |
| 6.8 (A) | CLSZ | Pair 1 | 0.41 | 0.80 |
| 6.8 (в) | CLSZ | Pair 2 | 0.35 | 0.83 |
| 6.8 (c) | CLSZ | Pair 3 | 0.32 | 0.84 |
| 6.8 (D) | CLSZ | Pair 4 | 0.29 | 0.84 |
| 6.2 (A) | EvoCutDirected | Ohio Seed | 0.50 | 0.56 |
| 6.2 (B) | EvoCutDirected | New York Seed | 0.49 | 0.58 |
| 6.2 (c) | EvoCutDirected | California Seed | 0.49 | 0.67 |
| 6.2(D) | EvoCutDirected | Florida Seed | 0.42 | 0.79 |



Figure 6.9: Consider the directed graph and its semi-double cover above. Suppose $\boldsymbol{p}\left(a_{1}\right)=\boldsymbol{p}\left(a_{2}\right)=$ 0.5 and $\boldsymbol{p}\left(b_{1}\right)=\boldsymbol{p}\left(b_{2}\right)=0$. It is straightforward to check that $(\sigma \circ(\mathbf{W} \boldsymbol{p}))\left(b_{2}\right)=0.25$ and $(\mathbf{W}(\sigma \circ \boldsymbol{p}))\left(b_{2}\right)=0$.
over the entire graph.

### 6.4 Future Work

This chapter presents a novel graph reduction technique, and applies this to design two local algorithms for learning a certain structure of clusters in both undirected and directed graphs. Our experimental results further demonstrate their potential wide applications in learning meaningful structures of real-world datasets. Our work leaves several open questions for future work.

Firstly, there is a gap between the approximation guarantee of the algorithm for directed graphs when compared with undirected graphs, and it looks very challenging to close this gap with the techniques presented in this chapter. To understand this, it is insightful to discuss why Algorithm 6.1 cannot be applied for directed graphs, although the input graph is translated into an undirected graph by the semi-double cover reduction. This is because, when translating a directed graph into a bipartite undirected graph, the third property of the $\sigma$-operator in Lemma 6.2 no longer holds, since the existence of the edge $\left\{u_{1}, v_{2}\right\} \in E_{H}$ does not necessarily imply that $\left\{u_{2}, v_{1}\right\} \in E_{H}$. Indeed, Figure 6.9 gives a counterexample in which $(\sigma \circ(\mathbf{W} \boldsymbol{p}))(u) \not \leq(\mathbf{W}(\sigma \circ \boldsymbol{p}))(u)$. This means that the typical analysis of a Pagerank vector with the Lovász-Simonovitz curve cannot be applied anymore. In addition, it is interesting to note that one cannot apply the tighter analysis of the ESP process given by Andersen et al. AOGPT16] to the new algorithm in this chapter. The key to their analysis is an improved bound on the probability that a random walk escapes from the target cluster. This bound given a better guarantee for the conductance of the output set at the cost of a relaxed guarantee on the overlap between the output set and the target set. Since the analysis for directed graphs in this chapter relies on a very tight guarantee on the overlap of the output set with the target set, we cannot use their improvement in this setting. These difficulties leave the following important open question.

Open Question 5. Suppose we are given a directed graph $G=(V, E)$ containing clusters $L, R \subset V$ with $F(L, R)=\phi$. Is there a local algorithm which is guaranteed to return
$L^{\prime}, R^{\prime} \subset V$ with $F\left(L^{\prime}, R^{\prime}\right)=\widetilde{O}(\sqrt{\phi}) ?$
Secondly, although this chapter shows that two densely connected clusters can be approximately recovered with local algorithms, it would be very interesting to know whether our presented technique can be generalised to find clusters whose structure is defined by some meta-graph as described in Chapter 5

Open Question 6. Suppose we are given a graph $G=(V, E)$, and $S_{1}, \ldots, S_{k} \subset V$ are clusters defined with respect to some meta-graph. Is there a local algorithm that approximately recovers these $S_{1}, \ldots, S_{k}$ ?

## Chapter 7

## Finding Bipartite Components in Hypergraphs

Hypergraphs are important objects to model ternary or higher-order relations of objects, and have a number of applications in analysing many complex datasets occurring in practice. In this chapter we study a new heat diffusion process in hypergraphs, and employ this process to design a polynomial-time algorithm that approximately finds bipartite components in a hypergraph. We theoretically study the performance of the proposed algorithm, and compare it against the previous state-of-the-art through extensive experimental analysis on both synthetic and real-world datasets. We see that the new algorithm consistently and significantly outperforms the previous state-of-the-art across a wide range of hypergraphs. The problem studied in this chapter can be seen as a generalisation of the problem studied in the previous chapter, although we study algorithms in the global setting rather than the local one.

A unifying theme of this thesis is the design of efficient algorithms based on spectral methods, which study the graph Laplacian matrix in particular. The results of the present chapter form part of a sequence of work to generalise many of the results of spectral graph theory to hypergraphs. We introduce and study the non-linear Laplacian-type operator $\mathbf{J}_{H}$ for any hypergraph $H$. While we formally define the operator $\mathbf{J}_{H}$ in Section 7.3, one can informally think about $\mathbf{J}_{H}$ as a variant of the non-linear hypergraph Laplacian $\mathbf{L}_{H}$ studied in [CLTZ18, LM18, TMIY20] which generalises the signless Laplacian operator of graphs. We develop a polynomial-time algorithm that finds an eigenvalue $\lambda$ and its associated eigenvector of $\mathbf{J}_{H}$.

Main Result 6 (See Theorem 7.2 for the formal statement). Given a hypergraph $H$, there is a non-linear operator $\mathbf{J}_{H}$ which generalises the signless graph Laplacian operator for hypergraphs. Moreover, there is a polynomial-time algorithm that finds an eigenvalue of $\mathbf{J}_{H}$ along with its associated eigenvector.


Figure 7.1: Illustration of the proposed diffusion process. In each time step, we construct a graph $G$ based on the current vector $\boldsymbol{f}_{t}$, and update $\boldsymbol{f}_{t}$ with the $\mathbf{J}_{G}$ operator. Notice that the graph $G$ changes throughout the execution of the algorithm, and that the final $\boldsymbol{f}_{t}$ vector can be used to partition the vertices of $H$ into two well-connected sets (all the edges are adjacent to both sets), by splitting according to positive and negative entries.

The algorithm is based on the following heat diffusion process: starting from an arbitrary vector $\boldsymbol{f}_{0} \in \mathbb{R}^{n}$ that describes the initial heat distribution among the vertices, we use $\boldsymbol{f}_{0}$ to construct some grap $\sqrt{1} G_{0}$, and use the diffusion process in $G_{0}$ to represent the one in the original hypergraph $H$ and update $\boldsymbol{f}_{t}$; this process continues until the time at which $G_{0}$ cannot be used to appropriately simulate the diffusion process in $H$ any more. At this point, we use the currently maintained $\boldsymbol{f}_{t}$ to construct another graph $G_{t}$ to simulate the diffusion process in $H$, and update $\boldsymbol{f}_{t}$. This process continues until the vector $\boldsymbol{f}_{t}$ converges; see Figure 7.1 for illustration. We prove that this heat diffusion process is unique, well-defined, and the maintained vector $\boldsymbol{f}_{t}$ converges to an eigenvector of $\mathbf{J}_{H}$.

The second result of this chapter is a polynomial-time algorithm that, given a hypergraph $H=\left(V_{H}, E_{H}, w\right)$ as input, finds disjoint subsets $L, R \subset V_{H}$ that are highly connected with each other.

Main Result 7 (See Theorem 7.1 for the formal statement). Suppose we are given a hypergraph $H=\left(V_{H}, E_{H}, w\right)$, and an eigenvalue $\lambda$ of the operator $\mathbf{J}_{H}$ with its corresponding eigenvector. There is an algorithm that finds two clusters $L, R \subset V_{H}$ such that $\beta_{H}(L, R) \leq$

[^7]$\sqrt{2 \lambda}$, where $\beta_{H}$ is the hypergraph bipartiteness.
The key to this algorithm is a Cheeger-type inequality for hypergraphs that relates the spectrum of $\mathbf{J}_{H}$ and the hypergraph bipartiteness of $H$, an analogue of the graph bipartiteness studied in Chapter 6. Both the design and analysis of the algorithm is inspired by Trevisan [Tre12], however the analysis is much more involved because of the non-linear operator $\mathbf{J}_{H}$ and hyperedges of different ranks. This result alone answers an open question posed by Yoshida [Yos19], who asks whether there is a hypergraph operator which satisfies a Cheeger-type inequality for bipartiteness.

The significance of our algorithmic result is demonstrated by extensive experimental studies of the algorithm on both synthetic and real-world datasets. In particular, on the well-known Penn Treebank corpus that contains 49,208 sentences and over 1 million words, our purely unsupervised algorithm is able to identify a significant fraction of verbs from non-verbs in its two output clusters. Hence, this work could potentially have many applications in unsupervised learning for hypergraphs.

Finally, we study the spectrum of the non-linear Laplacian-type operators for hypergraphs, and show that, for any $n \in \mathbb{Z}_{\geq 0}$, there is a hypergraph on $n$ vertices such that $\mathbf{L}_{H}$ and $\mathbf{J}_{H}$ have more than $n$ eigenvectors. This answers an open question posed by Chan et al. [CLTZ18], who ask whether $\mathbf{L}_{H}$ could have more than 2 eigenvectors ${ }^{2}$.

### 7.1 Additional Notation

In order to study densely connected clusters in hypergraphs, we need some new notation and definitions. Suppose, we have a hypergraph $H=(V, E, w)$, then for any $A, B \subset V$ we can overload the weight function to define the cut value between $A$ and $B$ by

$$
w(A, B) \triangleq \sum_{e \in E} w(e) \cdot[e \cap A \neq \emptyset \wedge e \cap B \neq \emptyset]
$$

Sometimes, we would like to analyse the weights of edges that intersect some vertex sets and not others. To this end, for any $A, B, C \subseteq V$, let

$$
w(A, B \mid C) \triangleq \sum_{e \in E} w(e) \cdot[e \cap A \neq \emptyset \wedge e \cap B \neq \emptyset \wedge e \cap C=\emptyset]
$$

and we can also define $w(A \mid C) \triangleq w(A, A \mid C)$ for simplicity. Generalising the notion of the bipartiteness of a graph, the hypergraph bipartiteness of sets $L, R \subset V$ is defined by

$$
\beta_{H}(L, R) \triangleq \frac{2 w(L \mid \bar{L})+2 w(R \mid \bar{R})+w(L, \bar{L} \mid R)+w(R, \bar{R} \mid L)}{\operatorname{vol}(L \cup R)}
$$

[^8]and we define
$$
\beta_{H} \triangleq \min _{S \subset V_{H}} \beta_{H}\left(S, V_{H} \backslash S\right)
$$

This definition generalises the graph bipartiteness (Definition 3.3) in that, if every edge in $H$ contains 2 vertices, then $H$ is a (non-hyper) graph and $\beta_{H}(L, R)$ is equivalent to the graph bipartiteness. Informally, a low value of $\beta_{H}(L, R)$ indicates that most of the edges which intersect either $L$ or $R$ connect $L$ and $R$ together. The goal of this chapter is to develop an algorithm to find such sets $L$ and $R$ in a hypergraph.

For any hypergraph $H$ and $f \in \mathbb{R}^{n}$, we define the discrepancy of an edge $e \in E_{H}$ with respect to $f$ as

$$
\Delta_{\boldsymbol{f}}(e) \triangleq \max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v) .
$$

Moreover, the weighted discrepancy of $e \in E_{H}$ is

$$
c_{\boldsymbol{f}}(e) \triangleq w(e)\left|\Delta_{\boldsymbol{f}}(e)\right|,
$$

and for any set $S \subseteq E_{H}$ we have

$$
c_{\boldsymbol{f}}(S) \triangleq \sum_{e \in S} c_{\boldsymbol{f}}(e) .
$$

The discrepancy of $H$ with respect to $f$ is defined by

$$
D(\boldsymbol{f}) \triangleq \frac{\sum_{e \in E_{H}} w(e) \cdot \Delta_{\boldsymbol{f}}(e)^{2}}{\sum_{v \in V_{H}} d_{H}(v) \cdot \boldsymbol{f}(v)^{2}} .
$$

For any $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^{n}$, the weighted inner product between $\boldsymbol{f}$ and $\boldsymbol{g}$ is defined by

$$
\langle\boldsymbol{f}, \boldsymbol{g}\rangle_{w} \triangleq \boldsymbol{f}^{\top} \mathbf{D}_{H} \boldsymbol{g}
$$

where $\mathbf{D}_{H} \in \mathbb{R}^{n \times n}$ is the diagonal matrix consisting of the degrees of all the vertices of $H$. The weighted norm of $\boldsymbol{f}$ is defined by $\|\boldsymbol{f}\|_{w}^{2} \triangleq\langle\boldsymbol{f}, \boldsymbol{f}\rangle_{w}$.

For any non-linear operator $\mathbf{J}: \mathbb{R}^{n} \mapsto \mathbb{R}^{n}$, we say that $(\lambda, \boldsymbol{f})$ is an eigen-pair if and only if $\mathbf{J} \boldsymbol{f}=\lambda \boldsymbol{f}$. Note that in contrast with linear operators, a non-linear operator does not necessarily have $n$ eigenvalues. Additionally, we define the weighted Rayleigh quotient of the operator $\mathbf{J}$ and vector $f$ to be

$$
R_{\mathbf{J}}(\boldsymbol{f}) \triangleq \frac{\boldsymbol{f}^{\boldsymbol{\top}} \mathbf{J} \boldsymbol{f}}{\|\boldsymbol{f}\|_{w}^{2}}
$$

It is important to remember that throughout this chapter the letter $H$ is always used to represent a hypergraph, and $G$ to represent a graph.


Figure 7.2: Hypergraphs illustrating the downside of simple reductions

### 7.2 Baseline Algorithms

Before we present the new algorithm, let us first consider the baseline algorithms for finding bipartite components in hypergraphs. One natural idea is to construct a graph $G$ from the original hypergraph $H$ and apply a graph algorithm on $G$ to find a good approximation of the cut structure of $H$ [CR19, LM17, ZHS06]. However, most graph reductions would introduce a factor of $r$, which relates to the rank of hyperedges in $H$, into the approximation guarantee. To see this, consider the following two natural graph reductions:

1. Random Reduction: From $H=\left(V, E_{H}\right)$, we construct $G=\left(V, E_{G}\right)$ in which every hyperedge $e \in E_{H}$ is replaced by an edge connecting two randomly chosen vertices in $e$;
2. Clique Reduction: From $H=\left(V, E_{H}\right)$, we construct $G=\left(V, E_{G}\right)$ in which every hyperedge $e \in E_{H}$ is replaced by a clique of $\operatorname{rank}(e)$ vertices in $G$.

To discuss the drawback of both reductions, we study the following two $r$-uniform hypergraphs $H_{1}$ and $H_{2}$, in each of which the vertex set is $L \cup R$ and the edge set is defined as follows:

1. in $H_{1}$, every edge contains exactly one vertex from $L$, and $r-1$ vertices from $R$;
2. in $H_{2}$, every edge contains exactly $r / 2$ vertices from $L$ and $r / 2$ vertices from $R$.

As such, we have that $w_{H_{1}}(L, R)=\left|E_{H_{1}}\right|$, and $w_{H_{2}}(L, R)=\left|E_{H_{2}}\right|$. See Figure 7.2 for illustration.

Now we analyse the size of the cut $(L, R)$ in the reduced graphs.

- Let $w_{G}(L, R)$ be the cut value of $(L, R)$ in the random graph $G$ constructed from $H$ by the random reduction. We have for $H_{1}$ that $\mathbb{E}\left[w_{G}(L, R)\right]=\Theta(1 / r) \cdot w_{H_{1}}(L, R)$ and for $H_{2}$ that $\mathbb{E}\left[w_{G}(L, R)\right]=\Theta(1) \cdot w_{H_{2}}(L, R)$.
- Similarly, by setting $w_{G}(L, R)$ to be the cut value of $(L, R)$ in the reduced graph $G$ constructed from $H$ by the clique reduction, we have for $H_{1}$ that $w_{G}(L, R)=$

$$
\Theta(r) \cdot w_{H_{1}}(L, R) \text { and for } H_{2} \text { that } w_{G}(L, R)=\Theta\left(r^{2}\right) \cdot w_{H_{2}}(L, R) .
$$

Notice that these two approximation ratios differ by a factor of $r$ in the two reductions. This suggests that reducing an $r$-uniform hypergraph $H$ to a graph with some simple reduction would always introduce a factor of $r$ into the approximation guarantee. This is one of the main reasons to develop spectral theory for hypergraphs through heat diffusion processes [CLTZ18, TMIY20, Yos19].

### 7.3 Diffusion Process and Algorithm

In this section, we introduce a new diffusion process in hypergraphs, and use it to design a polynomial-time algorithm for finding bipartite components in hypergraphs. We first study the heat diffusion process in graphs to give some intuition. Then, we generalise it to hypergraphs and describe the new algorithm. Finally, we sketch some of the analysis which proves that the diffusion process is well defined. The full analysis is given in Sections 7.4 and 7.5

### 7.3.1 Diffusion Process in Graphs

To discuss the intuition behind our designed diffusion process, let us look at the case of graphs. Let $G=(V, E, w)$ be a graph, and we have for any $\boldsymbol{x} \in \mathbb{R}^{n}$ that

$$
\frac{\boldsymbol{x}^{\top} \mathbf{Z}_{G} \boldsymbol{x}}{\boldsymbol{x}^{\top} \boldsymbol{x}}=\frac{\boldsymbol{x}^{\top}\left(\mathbf{I}+\mathbf{D}_{G}^{-\frac{1}{2}} \mathbf{A}_{G} \mathbf{D}_{G}^{-\frac{1}{2}}\right) \boldsymbol{x}}{\boldsymbol{x}^{\top} \boldsymbol{x}}
$$

By setting $\boldsymbol{x}=\mathbf{D}_{G}^{\frac{1}{2}} \boldsymbol{y}$, we have that

$$
\begin{align*}
\frac{\boldsymbol{x}^{\top} \mathbf{Z}_{G} \boldsymbol{x}}{\boldsymbol{x}^{\top} \boldsymbol{x}} & =\frac{\boldsymbol{y}^{\top} \mathbf{D}_{G}^{\frac{1}{2}} \mathbf{Z}_{G} \mathbf{D}_{G}^{\frac{1}{2}} \boldsymbol{y}}{\boldsymbol{y}^{\top} \mathbf{D}_{G} \boldsymbol{y}} \\
& =\frac{\boldsymbol{y}^{\top}\left(\mathbf{D}_{G}+\mathbf{A}_{G}\right) \boldsymbol{y}}{\boldsymbol{y}^{\top} \mathbf{D}_{G} \boldsymbol{y}} \\
& =\frac{\sum_{\{u, v\} \in E_{G}} w(u, v) \cdot(\boldsymbol{y}(u)+\boldsymbol{y}(v))^{2}}{\sum_{u \in V_{G}} d_{G}(u) \cdot \boldsymbol{y}(u)^{2}} . \tag{7.1}
\end{align*}
$$

It is easy to see that $\lambda_{1}\left(\mathbf{Z}_{G}\right)=0$ if $G$ is bipartite, and it is known that $\lambda_{1}\left(\mathbf{Z}_{G}\right)$ and its corresponding eigenvector $f_{1}\left(\mathbf{Z}_{G}\right)$ are closely related to two densely connected components of $G$ [Tre12]. Moreover, similar to the heat equation for graph Laplacian $\mathbf{L}_{G}$, suppose $\mathbf{D}_{G} \boldsymbol{f}_{t} \in \mathbb{R}^{n}$ is some measure on the vertices of $G$, then a diffusion process defined by the differential equation

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{f}_{t}}{\mathrm{~d} t}=-\mathbf{D}_{G}^{-1} \mathbf{J}_{G} \boldsymbol{f}_{t} \tag{7.2}
\end{equation*}
$$

converges to the eigenvector of $\mathbf{D}_{G}^{-1} \mathbf{J}_{G}$ with minimum eigenvalue and can be employed to find two densely connected components of the underlying graph 3

[^9]
### 7.3.2 Hypergraph Diffusion and our Algorithm

Now we study whether one can construct a new hypergraph operator $\mathbf{J}_{H}$ that generalises the diffusion in graphs to hypergraphs. First of all, we focus on a fixed time $t$ with measure vector $\mathbf{D}_{H} \boldsymbol{f}_{t} \in \mathbb{R}^{n}$. We follow (7.2) and define the rate of change

$$
\frac{\mathrm{d} \boldsymbol{f}_{t}}{\mathrm{~d} t}=-\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}
$$

so that the diffusion can proceed for an infinitesimal time step. Our intuition is that the rate of change due to some edge $e \in E_{H}$ should involve only the vertices in $e$ with the maximum or minimum value in the normalised measure $\boldsymbol{f}_{t}$. To formalise this, for any edge $e \in E_{H}$ we define

$$
S_{\boldsymbol{f}}(e) \triangleq\left\{v \in e: \boldsymbol{f}_{t}(v)=\max _{u \in e} \boldsymbol{f}_{t}(u)\right\} \quad \text { and } \quad I_{\boldsymbol{f}}(e) \triangleq\left\{v \in e: \boldsymbol{f}_{t}(v)=\min _{u \in e} \boldsymbol{f}_{t}(u)\right\} .
$$

That is, for any edge $e$ and normalised measure $\boldsymbol{f}_{t}, S_{\boldsymbol{f}}(e) \subseteq e$ consists of the vertices $v$ adjacent to $e$ whose $\boldsymbol{f}_{t}(v)$ values are maximum, and $I_{\boldsymbol{f}}(e) \subseteq e$ consists of the vertices $v$ adjacent to $e$ whose $\boldsymbol{f}_{t}(v)$ values are minimum. See Figure 7.3 for an example. Then, applying the $\mathbf{J}_{H}$ operator to a vector $\boldsymbol{f}_{t}$ should be equivalent to applying the operator $\mathbf{J}_{G}$ for some graph $G$ which we construct by splitting the weight of each hyperedge $e \in E_{H}$ between the edges in $S_{\boldsymbol{f}}(e) \times I_{\boldsymbol{f}}(e)$. Similar to the case for graphs and (7.1), for any $\boldsymbol{x}=\mathbf{D}_{H}^{\frac{1}{2}} \boldsymbol{f}_{t}$ this gives us the quadratic form

$$
\begin{aligned}
\frac{\boldsymbol{x}^{\top} \mathbf{D}_{H}^{-\frac{1}{2}} \mathbf{J}_{H} \mathbf{D}_{H}^{-\frac{1}{2}} \boldsymbol{x}}{\boldsymbol{x}^{\top} \boldsymbol{x}}=\frac{\boldsymbol{f}_{t}^{\top} \mathbf{J}_{G} \boldsymbol{f}_{t}}{\boldsymbol{f}_{t}^{\top} \mathbf{D}_{H} \boldsymbol{f}_{t}} & =\frac{\sum_{\{u, v\} \in E_{G}} w_{G}(u, v) \cdot\left(\boldsymbol{f}_{t}(u)+\boldsymbol{f}_{t}(v)\right)^{2}}{\sum_{u \in V_{G}} d_{H}(u) \cdot \boldsymbol{f}_{t}(u)^{2}} \\
& =\frac{\sum_{e \in E_{H}} w_{H}(e)\left(\max _{u \in e} \boldsymbol{f}_{t}(u)+\min _{v \in e} \boldsymbol{f}_{t}(v)\right)^{2}}{\sum_{u \in V_{H}} d_{H}(u) \cdot \boldsymbol{f}_{t}(u)^{2}}
\end{aligned}
$$

where $w_{G}(u, v)$ is the weight of the edge $\{u, v\}$ in $G$, and $w_{H}(e)$ is the weight of the edge $e$ in $H$. We see later in this chapter that $\mathbf{J}_{H}$ has an eigenvalue of 0 if the hypergraph is 2-colourabl $\AA_{4}^{4}$, and the spectrum of $\mathbf{J}_{H}$ is closely related to the hypergraph bipartiteness.

For this reason, we would expect that the diffusion process based on the operator $\mathbf{J}_{H}$ can be used to find sets with small hypergraph bipartiteness. However, one needs to be very cautious here as, by the nature of the diffusion process, the values $\boldsymbol{f}_{t}(v)$ of all the vertices $v$ change over time and, as a result, the sets $S_{\boldsymbol{f}}(e)$ and $I_{\boldsymbol{f}}(e)$ that consist of the vertices with the maximum and minimum $\boldsymbol{f}_{t}$-value might change after an infinitesimal time step; this prevents the process from continuing. We discuss this issue in detail through the so-called diffusion continuity condition in Section 7.3.3. In essence, the diffusion continuity condition ensures
heat diffusion: through $\mathbf{J}_{G}$, the heat diffusion can be used to find two densely connected components of $G$.
${ }^{4}$ Hypergraph $H$ is 2-colourable if there are disjoint sets $L, R \subset V_{H}$ such that every edge intersects $L$ and $R$.


Hyperedge $e$
Figure 7.3: Illustration of $S_{\boldsymbol{f}}(e)$ and $I_{\boldsymbol{f}}(e)$. Vertices are labelled with their value in $\boldsymbol{f}_{\boldsymbol{t}}$.
that one can always construct a graph $G$ by allocating the weight of each hyperedge $e$ to the edges in $S_{\boldsymbol{f}}(e) \times I_{\boldsymbol{f}}(e)$ such that the sets $S_{\boldsymbol{f}}(e)$ and $I_{\boldsymbol{f}}(e)$ don't change in infinitesimal time although $\boldsymbol{f}_{t}$ changes according to $\left(\mathrm{d} \boldsymbol{f}_{t}\right) /(\mathrm{d} t)=-\mathbf{D}_{H}^{-1} \mathbf{J}_{G} \boldsymbol{f}_{t}$. We also develop an efficient procedure in Section 7.3 .3 to compute the weights of edges in $S_{\boldsymbol{f}}(e) \times I_{\boldsymbol{f}}(e)$. All of these guarantee that (i) every graph that corresponds to the hypergraph diffusion process at any time step can be efficiently constructed; (ii) with this sequence of constructed graphs, the diffusion process defined by $\mathbf{J}_{H}$ is able to continue until the heat distribution converges. With this, we can summarise the main idea of the new algorithm as follows:

- First of all, we introduce some arbitrary $f_{0} \in \mathbb{R}^{n}$ as the initial diffusion vector, and a step size parameter $\epsilon>0$ to discretise the diffusion process. At each step, the algorithm constructs the graph $G$ guaranteed by the diffusion continuity condition, and updates $\boldsymbol{f}_{t} \in \mathbb{R}^{n}$ according to the rate of change $\left(\mathrm{d} \boldsymbol{f}_{t}\right) /(\mathrm{d} t)=-\mathbf{D}_{H}^{-1} \mathbf{J}_{G} \boldsymbol{f}_{t}$. The algorithm terminates when $\boldsymbol{f}_{t}$ has converged, i.e., the ratio between the current Rayleigh quotient $\left(\boldsymbol{f}_{t}^{\top} \mathbf{J}_{G} \boldsymbol{f}_{t}\right) /\left(\boldsymbol{f}_{t}^{\top} \mathbf{D}_{H} \boldsymbol{f}_{t}\right)$ and the one in the previous time step is bounded by some predefined constant.
- Secondly, similar to many previous spectral graph clustering algorithms (e.g. ACL06, TMIY20, Tre12]), the algorithm constructs the sweep sets defined by $f_{t}$ and returns the two sets with minimum $\beta_{H}$-value among all the constructed sweep sets. Specifically, for every $i \in[n]$, the algorithm constructs $L_{j}=\left\{v_{i}:\left|\boldsymbol{f}_{t}\left(v_{i}\right)\right| \geq\left|\boldsymbol{f}_{t}\left(v_{j}\right)\right| \wedge \boldsymbol{f}_{t}\left(v_{i}\right)<0\right\}$ and $R_{j}=\left\{v_{i}:\left|\boldsymbol{f}_{t}\left(v_{i}\right)\right| \geq\left|\boldsymbol{f}_{t}\left(v_{j}\right)\right| \wedge \boldsymbol{f}_{t}\left(v_{i}\right) \geq 0\right\}$. Then, among the $n$ pairs $\left(L_{j}, R_{j}\right)$, the algorithm returns the one with the minimum $\beta_{H}$-value.

See Algorithm 7.1 for the formal description, and its performance is summarised in Theorem 7.1. We prove the theorem in Section 7.5.

Theorem 7.1. Given a hypergraph $H=\left(V_{H}, E_{H}, w\right)$ and parameter $\epsilon>0$, the following holds:

1. There is an algorithm that finds an eigen-pair $(\lambda, \boldsymbol{f})$ of the operator $\mathbf{J}_{H}$ such that $\lambda \leq \lambda_{1}\left(\mathbf{J}_{G}\right)$, where $G$ is the clique reduction of $H$ and the inequality is strict if
$\min _{e \in E_{H}} \operatorname{rank}(e)>2$. The algorithm has running time poly $\left(\left|V_{H}\right|,\left|E_{H}\right|, 1 / \epsilon\right)$.
2. Given an eigen-pair $(\lambda, \boldsymbol{f})$ of the operator $\mathbf{J}_{H}$, there is an algorithm that constructs the two-sided sweep sets defined on $\boldsymbol{f}$, and finds sets $L$ and $R$ such that $\beta_{H}(L, R) \leq \sqrt{2 \lambda}$. The algorithm has running time poly $\left(\left|V_{H}\right|,\left|E_{H}\right|\right)$.
```
Algorithm 7.1: FindBipartiteComponents \(\left(H, \boldsymbol{f}_{0}, \epsilon\right)\)
    Input : Hypergraph \(H\), starting vector \(f_{0} \in \mathbb{R}^{n}\), step size \(\epsilon>0\)
    Output: Sets \(L\) and \(R\)
    \(t:=0\)
    while \(f_{t}\) has not converged do
        Use \(\boldsymbol{f}_{t}\) to construct graph \(G\) satisfying the diffusion continuity condition
        \(\boldsymbol{f}_{t+\epsilon}:=\boldsymbol{f}_{t}-\epsilon \mathbf{D}_{H}^{-1} \mathbf{J}_{G} \boldsymbol{f}_{t}\)
        \(t:=t+\epsilon\)
    end
    7 Set \(j:=\arg \min _{1 \leq i \leq n} \beta_{H}\left(L_{i}, R_{i}\right)\)
    8 return \(\left(L_{j}, R_{j}\right)\)
```


### 7.3.3 Dealing with the Diffusion Continuity Condition

It remains for us to discuss the diffusion continuity condition, which guarantees that $S_{\boldsymbol{f}}(e)$ and $I_{\boldsymbol{f}}(e)$ don't change in infinitesimal time and the diffusion process eventually converges to some stable distribution. Formally, let $\boldsymbol{f}_{t}$ be the normalised measure on the vertices of $H$, and

$$
\boldsymbol{r} \triangleq \frac{\mathrm{d} \boldsymbol{f}_{t}}{\mathrm{~d} t}=-\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}
$$

be the derivative of $f_{t}$, which describes the rate of change for every vertex at the current time $t$. We write $\boldsymbol{r}(v)$ for any $v \in V_{H}$ as $\boldsymbol{r}(v)=\sum_{e \in E_{H}} \boldsymbol{r}_{e}(v)$, where $\boldsymbol{r}_{e}(v)$ is the contribution of edge $e$ towards the rate of change of $\boldsymbol{f}_{t}(v)$. Now we discuss three rules that we expect the diffusion process to satisfy, and later prove that these three rules uniquely define the rate of change $\boldsymbol{r}$.

First of all, as we mentioned in Section 7.3.2, we expect that only the vertices in $S_{\boldsymbol{f}}(e) \cup I_{\boldsymbol{f}}(e)$ participate in the diffusion process, i.e., $\boldsymbol{r}_{e}(u)=0$ unless $u \in S_{\boldsymbol{f}}(e) \cup I_{\boldsymbol{f}}(e)$. Moreover, any vertex $u$ participating in the diffusion process must satisfy the following:

- Rule (0a): if $\left|\boldsymbol{r}_{e}(u)\right|>0$ and $u \in S_{\boldsymbol{f}}(e)$, then $\boldsymbol{r}(u)=\max _{v \in S_{\boldsymbol{f}}(e)}\{\boldsymbol{r}(v)\}$.
- Rule (Ob): if $\left|\boldsymbol{r}_{e}(u)\right|>0$ and $u \in I_{\boldsymbol{f}}(e)$, then $\boldsymbol{r}(u)=\min _{v \in I_{\boldsymbol{f}}(e)}\{\boldsymbol{r}(v)\}$.

To explain Rule (0), notice that for an infinitesimal time, $\boldsymbol{f}_{t}(u)$ is increased according to $\left(\mathrm{d} \boldsymbol{f}_{t} / \mathrm{d} t\right)(u)=\boldsymbol{r}(u)$. Hence, by Rule (0) we know that, if $u \in S_{\boldsymbol{f}}(e)$ (resp. $\left.u \in I_{\boldsymbol{f}}(e)\right)$
participates in the diffusion process in edge $e$, then in an infinitesimal time $\boldsymbol{f}(u)$ remains the maximum (resp. minimum) among the vertices in $e$. Such a rule is necessary to ensure that the vertices involved in the diffusion in edge $e$ do not change in infinitesimal time, and the diffusion process is able to continue.

Our next rule states that the total rate of change of the measure due to edge $e$ is equal to $-w(e) \cdot \Delta_{\boldsymbol{f}}(e)$ :

- Rule (1): $\sum_{v \in S_{f}(e)} d(v) \boldsymbol{r}_{e}(v)=\sum_{v \in I_{f}(e)} d(v) \boldsymbol{r}_{e}(v)=-w(e) \cdot \Delta_{\boldsymbol{f}}(e)$ for all $e \in E_{H}$.

This rule is a generalisation from the operator $\mathbf{J}_{G}$ in graphs. In particular, since

$$
\mathbf{D}_{G}^{-1} \mathbf{J}_{G} \boldsymbol{f}_{t}(u)=\sum_{\{u, v\} \in E_{G}} w_{G}(u, v)\left(\boldsymbol{f}_{t}(u)+\boldsymbol{f}_{t}(v)\right) / d_{G}(u),
$$

the rate of change of $\boldsymbol{f}_{t}(u)$ due to the edge $\{u, v\} \in E_{G}$ is $-w_{G}(u, v)\left(\boldsymbol{f}_{t}(u)+\boldsymbol{f}_{t}(v)\right) / d_{G}(u)$. Rule (1) states that in the hypergraph case the rate of change of the vertices in $S_{f}(e)$ and $I_{\boldsymbol{f}}(e)$ together behave like the rate of change of $u$ and $v$ in the graph case.

One might have expected that Rules (0) and (1) together would define a unique process. Unfortunately, this isn't the case. For example, let us define an unweighted hypergraph $H=\left(V_{H}, E_{H}\right)$, where $V_{H}=\{u, v, w\}$ and $E_{H}=\{\{u, v, w\}\}$. By setting the measure to be $\boldsymbol{f}_{t}=(1,1,-2)^{\top}$ and $e=\{u, v, w\}$, we have $\Delta_{\boldsymbol{f}_{t}}(e)=-1$, and $\boldsymbol{r}(u)+\boldsymbol{r}(v)=\boldsymbol{r}(w)=1$ by Rule (1). In such a scenario, either $\{u, w\}$ or $\{v, w\}$ can participate in the diffusion and satisfy Rule (0), which makes the process not uniquely defined. To overcome this, we introduce the following stronger rule to replace Rule (0):

- Rule (2a): Assume that $\left|\boldsymbol{r}_{e}(u)\right|>0$ and $u \in S_{\boldsymbol{f}}(e)$ :
- if $\Delta_{\boldsymbol{f}}(e)>0$, then $\boldsymbol{r}(u)=\max _{\left.v \in S_{f}(e)\right)}\{\boldsymbol{r}(v)\}$;
- if $\Delta_{\boldsymbol{f}}(e)<0$, then $\boldsymbol{r}(u)=\boldsymbol{r}(v)$ for all $v \in S_{\boldsymbol{f}}(e)$.
- Rule (2b): Assume that $\left|\boldsymbol{r}_{e}(u)\right|>0$ and $u \in I_{\boldsymbol{f}}(e)$ :
- if $\Delta_{\boldsymbol{f}}(e)<0$, then $\boldsymbol{r}(u)=\min _{\left.v \in I_{f}(e)\right)}\{\boldsymbol{r}(v)\}$;
- if $\Delta_{\boldsymbol{f}}(e)>0$, then $\boldsymbol{r}(u)=\boldsymbol{r}(v)$ for all $v \in I_{\boldsymbol{f}}(e)$.

Notice that the first conditions of Rules (2a) and (2b) correspond to Rules (0a) and (0b) respectively; the second conditions are introduced for purely technical reasons: they state that, if the discrepancy of $e$ is negative (resp. positive), then all the vertices $u \in S_{\boldsymbol{f}}(e)$ (resp. $u \in I_{\boldsymbol{f}}(e)$ ) have the same value of $\boldsymbol{r}(u)$. Lemma 7.1 shows that there is a unique $\boldsymbol{r} \in \mathbb{R}^{n}$ that satisfies Rules (1) and (2), and $r$ can be computed in polynomial time. Therefore, our two rules uniquely define a diffusion process, and we can use the computed $r$ to simulate the continuous diffusion process with a discretised version. ${ }^{5}$

[^10]Lemma 7.1 (Existence of diffusion process). For any given $f_{t} \in \mathbb{R}^{n}$, there is a unique $\boldsymbol{r}=\mathrm{d} \boldsymbol{f}_{t} / \mathrm{d} t$ and associated $\left\{\boldsymbol{r}_{e}(v)\right\}_{e \in E, v \in V}$ that satisfy Rule (1) and (2), and $\boldsymbol{r}$ can be computed in polynomial time by linear programming.

It is worth emphasising that our defined rules and the proof of Lemma 7.1 are more involved than those used in [CLTZ18] to define the hypergraph Laplacian operator. In particular, in contrast to [CLTZ18], in our case the discrepancy $\Delta_{\boldsymbol{f}}(e)$ within a hyperedge $e$ can be either positive or negative. This results in the four different cases in Rule (2) which have to be carefully considered throughout the proof of Lemma 7.1 .

### 7.4 Existence of Diffusion Process

In this section we prove Lemma 7.1 which shows that the diffusion process introduced in Section 7.3 is well-defined. The proof consists of two parts. First, we construct a linear program which can compute the rate of change $\boldsymbol{r}$ satisfying the rules of the diffusion process. Then we analyse the new linear program which establishes Lemma 7.1.

### 7.4.1 Computing $r$ by a Linear Program

Now we present an algorithm that computes the vector $\boldsymbol{r}=\mathrm{d} \boldsymbol{f}_{t} / \mathrm{d} t$ for any $\boldsymbol{f}_{t} \in \mathbb{R}^{n}$. Without loss of generality, let us fix $f_{t} \in \mathbb{R}^{n}$, and define a set of equivalence classes $\mathcal{U}$ on $V$ such that vertices $u, v \in V_{H}$ are in the same equivalence class if $\boldsymbol{f}_{t}(u)=\boldsymbol{f}_{t}(v)$. Next we study every equivalence class $U \in \mathcal{U}$ in turn, and set the $\boldsymbol{r}$-value of the vertices in $U$ recursively. In each iteration, we fix the $\boldsymbol{r}$-value of some subset $P \subseteq U$ and recurse on $U \backslash P$. As we prove later, it's important to highlight that the recursive procedure ensures that the $r$-values assigned to the vertices always decrease after each recursion. Notice that it suffices to consider the edges $e$ in which $\left(S_{\boldsymbol{f}_{t}}(e) \cup I_{\boldsymbol{f}_{t}}(e)\right) \cap U \neq \emptyset$, since the diffusion process induced by other edges $e$ has no impact on $\boldsymbol{r}(u)$ for any $u \in U$. Hence, we introduce the sets

$$
\begin{aligned}
& \mathcal{S}_{U} \triangleq\left\{e \in E_{U}: S_{\boldsymbol{f}_{t}}(e) \cap U \neq \emptyset\right\} \\
& \mathcal{I}_{U} \triangleq\left\{e \in E_{U}: I_{\boldsymbol{f}_{t}}(e) \cap U \neq \emptyset\right\}
\end{aligned}
$$

where $E_{U}$ consists of the edges adjacent to some vertex in $U$. To work with the four cases listed in Rule (2a) and (2b), we define

$$
\begin{aligned}
& \mathcal{S}_{U}^{+} \triangleq\left\{e \in \mathcal{S}_{U}: \Delta_{\boldsymbol{f}_{t}}(e)<0\right\} \\
& \mathcal{S}_{U}^{-} \triangleq\left\{e \in \mathcal{S}_{U}: \Delta_{\boldsymbol{f}_{t}}(e)>0\right\} \\
& \mathcal{I}_{U}^{+} \triangleq\left\{e \in \mathcal{I}_{U}: \Delta_{\boldsymbol{f}_{t}}(e)<0\right\} \\
& \mathcal{I}_{U}^{-} \triangleq\left\{e \in \mathcal{I}_{U}: \Delta_{\boldsymbol{f}_{t}}(e)>0\right\}
\end{aligned}
$$

Our objective is to find some $P \subseteq U$ and assign the same $r$-value to every vertex in $P$. To this end, for any $P \subseteq U$ we define

$$
\begin{aligned}
& \mathcal{S}_{U, P}^{+} \triangleq\left\{e \in \mathcal{S}_{U}^{+}: S_{f_{t}}(e) \subseteq P\right\}, \\
& \mathcal{I}_{U, P}^{+} \triangleq\left\{e \in \mathcal{I}_{U}^{+}: I_{f_{t}}(e) \subseteq P\right\}, \\
& \mathcal{S}_{U, P}^{-} \triangleq\left\{e \in \mathcal{S}_{U}^{-}: S_{f_{t}}(e) \cap P \neq \emptyset\right\}, \\
& \mathcal{I}_{U, P}^{-} \triangleq\left\{e \in \mathcal{I}_{U}^{-}: I_{f_{t}}(e) \subseteq P\right\} .
\end{aligned}
$$

These are the edges contributing to the rate of change of the vertices in $P$. Before continuing the analysis, we briefly explain the intuition behind these four definitions:

1. For $\mathcal{S}_{U, P}^{+}$, since every $e \in \mathcal{S}_{U}^{+}$satisfies $\Delta_{\boldsymbol{f}_{t}}(e)<0$ and all the vertices in $S_{\boldsymbol{f}_{t}}(e)$ must have the same value by Rule (2a), all such $e \in \mathcal{S}_{U, P}^{+}$must satisfy that $S_{f_{t}}(e) \subseteq P$, since the unassigned vertices would receive lower values of $r$ in the remaining part of the recursion process.
2. For $\mathcal{I}_{U, P}^{+}$, since every $e \in \mathcal{I}_{U}^{+}$satisfies $\Delta_{\boldsymbol{f}_{t}}(e)<0$, Rule (2b) implies that if $\boldsymbol{r}_{e}(u) \neq 0$ then $\boldsymbol{r}(u) \leq \boldsymbol{r}(v)$ for all $v \in I_{\boldsymbol{f}_{t}}(e)$. Since unassigned vertices would receive lower values of $\boldsymbol{r}$ later, such $e \in \mathcal{I}_{U, P}^{+}$must satisfy $I_{\boldsymbol{f}_{t}}(e) \subseteq P$.
3. For $\mathcal{S}_{U, P}^{-}$, since every $e \in \mathcal{S}_{U}^{-}$satisfies $\Delta_{f_{t}}(e)>0$, by Rule (2a) it suffices that some vertex in $S_{f_{t}}(e)$ receives the assignment in the current iteration, i.e., every such $e$ must satisfy $S_{f_{t}}(e) \cap P \neq \emptyset$.
4. The case for $\mathcal{I}_{U, P}^{-}$is the same as $\mathcal{S}_{U, P}^{+}$.

As we expect all the vertices $u \in S_{f_{t}}(e)$ to have the same $r$-value for every $e$ as long as $\Delta_{f_{t}}(e)<0$ by Rule (2a) and at the moment we are only considering the assignment of the vertices in $P$, we expect that

$$
\begin{equation*}
\left\{e \in \mathcal{S}_{U}^{+} \backslash \mathcal{S}_{U, P}^{+}: S_{f_{t}}(e) \cap P \neq \emptyset\right\}=\emptyset, \tag{7.3}
\end{equation*}
$$

and this ensures that, as long as $\Delta_{f_{t}}(e)<0$ and some $u \in S_{f_{t}}(e)$ gets its $r$-value, all the other vertices in $S_{f_{t}}(e)$ would be assigned the same value as $u$. Similarly, by Rule (2b), we expect all the vertices $u \in I_{\boldsymbol{f}_{t}}(e)$ to have the same $\boldsymbol{r}$-value for every $e$ as long as $\Delta_{\boldsymbol{f}_{t}}(e)>0$, and so we expect that

$$
\begin{equation*}
\left\{e \in \mathcal{I}_{U}^{-} \backslash \mathcal{I}_{U, P}^{-}: I_{f_{t}}(e) \cap P \neq \emptyset\right\}=\emptyset \tag{7.4}
\end{equation*}
$$

We set the $\boldsymbol{r}$-value by dividing the total discrepancy of the edges in $\mathcal{I}_{U, P}^{+} \cup \mathcal{S}_{U, P}^{+} \cup \mathcal{I}_{U, P}^{-} \cup \mathcal{S}_{U, P}^{-}$ between the vertices in $P$. As such, we would like to find some $P \subseteq U$ that maximises the value of

$$
\frac{1}{\operatorname{vol}(P)} \cdot\left(\sum_{e \in \mathcal{S}_{U, P}^{+} \cup \mathcal{I}_{U, P}^{+}} c_{f_{t}}(e)-\sum_{e \in \mathcal{S}_{U, P}^{-} \cup \mathcal{I}_{U, P}^{-}} c_{f_{t}}(e)\right)
$$

Taking all of these requirements into account, we show that, for any equivalence class $U$, we can find the desired set $P$ by solving the following linear program:

$$
\begin{array}{cc}
\operatorname{maximise} & c(\boldsymbol{x})=\sum_{e \in \mathcal{S}_{U}^{+} \cup \mathcal{I}_{U}^{+}} c_{\boldsymbol{f}_{t}}(e) \cdot x_{e}-\sum_{e \in \mathcal{S}_{U}^{-} \cup \mathcal{I}_{U}^{-}} c_{\boldsymbol{f}_{t}}(e) \cdot x_{e}  \tag{7.5}\\
\text { subject to } \quad \sum_{v \in U} d(v) y_{v}=1 \\
x_{e}=y_{u} \quad e \in \mathcal{S}_{U}^{+}, u \in S_{f_{t}}(e), \\
x_{e} \leq y_{u} & e \in \mathcal{I}_{U}^{+}, u \in I_{\boldsymbol{f}_{t}}(e), \\
x_{e} \geq y_{u} & e \in \mathcal{S}_{U}^{-}, u \in S_{\boldsymbol{f}_{t}}(e), \\
x_{e}=y_{u} & e \in \mathcal{I}_{U}^{-}, u \in I_{\boldsymbol{f}_{t}}(e), \\
x_{e}, y_{v} \geq 0 & \forall v \in U, e \in E_{U} .
\end{array}
$$

Since the linear program only gives partial assignment to the vertices' $\boldsymbol{r}$-values, we solve the same linear program on the reduced instance given by the set $U \backslash P$. The formal description of our algorithm is given in Algorithm 7.2 .

```
Algorithm 7.2: ComputeChangeRate \(\left(U, E_{U}\right)\)
    Input : vertex set \(U \subseteq V\), and edge set \(E_{U}\)
    Output: Values of \(\{\boldsymbol{r}(v)\}_{v \in U}\)
    1 Construct sets \(\mathcal{S}_{U}^{+}, \mathcal{S}_{U}^{-}, \mathcal{I}_{U}^{+}\), and \(\mathcal{I}_{U}^{-}\)
    2 Solve the linear program defined by (7.5), and define \(P:=\{v \in U: y(v)>0\}\)
    3 Construct sets \(\mathcal{S}_{U, P}^{+}, \mathcal{S}_{U, P}^{-}, \mathcal{I}_{U, P}^{+}\), and \(\mathcal{I}_{U, P}^{-}\)
    4 Set \(C(P):=c_{\boldsymbol{f}_{t}}\left(\mathcal{S}_{U, P}^{+}\right)+c_{\boldsymbol{f}_{t}}\left(\mathcal{I}_{U, P}^{+}\right)-c_{\boldsymbol{f}_{t}}\left(\mathcal{S}_{U, P}^{-}\right)-c_{\boldsymbol{f}_{t}}\left(\mathcal{I}_{U, P}^{-}\right)\)
    5 Set \(\delta(P):=C(P) / \operatorname{vol}(P)\)
    6 Set \(\boldsymbol{r}(u):=\delta(P)\) for every \(u \in P\)
    7 ComputeChangeRate \(\left(U \backslash P, E_{U} \backslash\left(\mathcal{S}_{U, P}^{+} \cup \mathcal{I}_{U, P}^{+} \cup \mathcal{S}_{U, P}^{-} \cup \mathcal{I}_{U, P}^{-}\right)\right)\)
```


### 7.4.2 Analysis of the Linear Program

Now we analyse Algorithm 7.2, and the properties of the $\boldsymbol{r}$-values it computes. Specifically, we show the following facts which together allow us to establish Lemma 7.1.

1. Algorithm 7.2 always produces a unique vector $\boldsymbol{r}$, no matter which optimal result is returned when computing the linear program (7.5).
2. If there is any vector $\boldsymbol{r}$ which is consistent with Rules (1) and (2), then it must be equal to the output of Algorithm 7.2
3. The vector $\boldsymbol{r}$ produced by Algorithm 7.2 is consistent with Rules (1) and (2).

### 7.4.2.1 Output of Algorithm 7.2 is Unique

First of all, for any $P \subseteq U$ that satisfies (7.3) and (7.4), we define vectors $\boldsymbol{x}_{P}$ and $\boldsymbol{y}_{P}$ by

$$
\begin{gathered}
\boldsymbol{x}_{P}(e)= \begin{cases}\frac{1}{\operatorname{vol}(P)} & \text { if } e \in \mathcal{S}_{U, P}^{+} \cup \mathcal{I}_{U, P}^{+} \cup \mathcal{S}_{U, P}^{-} \cup \mathcal{I}_{U, P}^{-} \\
0 & \text { otherwise }\end{cases} \\
\boldsymbol{y}_{P}(v)= \begin{cases}\frac{1}{\operatorname{vol}(P)} & \text { if } v \in P \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

and $z_{P}=\left(\boldsymbol{x}_{P}, \boldsymbol{y}_{P}\right)$. It is easy to verify that $\left(\boldsymbol{x}_{P}, \boldsymbol{y}_{P}\right)$ is a feasible solution to (7.5) with the objective value $c\left(\boldsymbol{x}_{P}\right)=\delta(P)$. We prove that ComputeChangeRate (Algorithm 7.2) computes a unique vector $\boldsymbol{r}$ regardless of how ties are broken when computing the subsets $P$.

For any feasible solution $z=(\boldsymbol{x}, \boldsymbol{y})$, we say that a non-empty set $Q$ is a level set of $z$ if there is some $t>0$ such that $Q=\left\{u \in U: y_{u} \geq t\right\}$. We first show that any non-empty level set of an optimal solution $z$ also corresponds to an optimal solution.

Lemma 7.2. Suppose that $z^{\star}=\left(\boldsymbol{x}^{\star}, \boldsymbol{y}^{\star}\right)$ is an optimal solution of the linear program (7.5). Then, any non-empty level set $Q$ of $z^{\star}$ corresponds to an optimal solution of (7.5) as well.

Proof. Let $P=\left\{v \in V_{H}: \boldsymbol{y}^{\star}(v)>0\right\}$. The proof is by case distinction. We first look at the case in which all the vertices in $P$ have the same value of $\boldsymbol{y}^{\star}(v)$ for any $v \in P$. Then, it must be that $z^{\star}=z_{P}$ and every non-empty level set $Q$ of $z^{\star}$ is equal to $P$; as such, the statement holds trivially.

Secondly, we assume that the vertices in $P$ have at least two different $\boldsymbol{y}^{\star}$-values. We define $\alpha=\min \left\{y_{v}^{\star}: v \in P\right\}$, and have

$$
\alpha \cdot \operatorname{vol}(P)<\sum_{u \in U} d(u) \cdot y_{u}^{\star}=1 .
$$

We introduce $\widehat{z}=(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ defined by

$$
\widehat{\boldsymbol{x}}(e)=\left\{\begin{array}{ll}
\frac{x^{\star}(e)-\alpha}{1-\alpha v o l}(P) & \text { if } \boldsymbol{x}^{\star}(e) \geq 0 \\
0 & \text { otherwise }
\end{array},\right.
$$

and

$$
\widehat{\boldsymbol{y}}(v)=\left\{\begin{array}{ll}
\frac{y^{\star}(v)-\alpha}{1-\alpha \operatorname{vol}(P)} & \text { if } v \in P \\
0 & \text { otherwise }
\end{array} .\right.
$$

This implies that

$$
\begin{equation*}
\boldsymbol{x}^{\star}=(1-\alpha \operatorname{vol}(P)) \widehat{\boldsymbol{x}}+\alpha \cdot \mathbf{1}_{P}=(1-\alpha \operatorname{vol}(P)) \widehat{\boldsymbol{x}}+\alpha \cdot \operatorname{vol}(P) \cdot \boldsymbol{x}_{P}, \tag{7.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{y}^{\star}=(1-\alpha \operatorname{vol}(P)) \widehat{\boldsymbol{y}}+\alpha \cdot \mathbf{1}_{P}=(1-\alpha \operatorname{vol}(P)) \widehat{\boldsymbol{y}}+\alpha \cdot \operatorname{vol}(P) \cdot \boldsymbol{y}_{P} \tag{7.7}
\end{equation*}
$$

where $\mathbf{1}_{P}$ is the indicator vector of the set $P$. Notice that $\widehat{z}$ preserves the relative ordering of the vertices and edges with respect to $\boldsymbol{x}^{\star}$ and $\boldsymbol{y}^{\star}$, and all the constraints in (7.5) hold for $\widehat{z}$. These imply that $\widehat{z}$ is a feasible solution to $(7.5$ as well. Moreover, it's not difficult to see that $\widehat{z}$ is an optimal solution of (7.5), since otherwise by the linearity of (7.6) and (7.7) $z_{P}$ would have a higher objective value than $z^{\star}$, contradicting the fact that $z^{\star}$ is an optimal solution. Hence, the non-empty level set defined by $\widehat{z}$ corresponds to an optimal solution. Finally, by applying the second case inductively, we prove the claimed statement of the lemma.

By applying the lemma above and the linearity of the objective function of (7.5), we obtain the following corollary.

## Corollary 7.1. The following statements hold:

- Suppose that $P_{1}$ and $P_{2}$ are optimal subsets of $U$. Then, $P_{1} \cup P_{2}$, as well as $P_{1} \cap P_{2}$ satisfying $P_{1} \cap P_{2} \neq \emptyset$, is an optimal subset of $U$.
- The optimal set of maximum size is unique, and contains all optimal subsets.

Now we are ready to show that the procedure ComputeChangeRate (Algorithm 7.2) and the linear program (7.5) together always give us the same set of $\boldsymbol{r}$-values regardless of which optimal solution of $(7.5)$ is used for the recursive construction of the entire vector $\boldsymbol{r}$.

Lemma 7.3. Let $\left(U, E_{U}\right)$ be the input to ComputeChangeRate, and $P \subset U$ be the set returned by (7.5). Moreover, let $\left(U^{\prime}=U \backslash P, E_{U^{\prime}}\right)$ be the input to the recursive call ComputeChangeRate $\left(U^{\prime}, E_{U^{\prime}}\right)$. Then, it holds for any $P^{\prime} \subseteq U^{\prime}$ that $\delta\left(P^{\prime}\right) \leq \delta(P)$, where the equality holds if and only if $\delta\left(P \cup P^{\prime}\right)=\delta(P)$.

Proof. By the definition of the function $c$ and sets $\mathcal{S}^{+}, \mathcal{S}^{-}, \mathcal{I}^{+}, \mathcal{I}^{-}$, it holds that

$$
c\left(\mathcal{S}_{U^{\prime}, P^{\prime}}^{+}\right)=c\left(\mathcal{S}_{U, P \cup P^{\prime}}^{+}\right)-c\left(\mathcal{S}_{U, P}^{+}\right)
$$

and the same equality holds for sets $\mathcal{S}^{-}, \mathcal{I}^{+}$and $\mathcal{I}^{-}$. We have that

$$
\begin{aligned}
\delta\left(P^{\prime}\right) & =\frac{c\left(\mathcal{S}_{U^{\prime}, P^{\prime}}^{+}\right)+c\left(\mathcal{I}_{U^{\prime}, P^{\prime}}^{+}\right)-c\left(\mathcal{S}_{U^{\prime}, P^{\prime}}^{-}\right)-c\left(\mathcal{I}_{U^{\prime}, P^{\prime}}^{-}\right)}{\operatorname{vol}\left(P^{\prime}\right)} \\
& =\frac{\delta\left(P \cup P^{\prime}\right) \cdot \operatorname{vol}\left(P \cup P^{\prime}\right)-\delta(P) \cdot \operatorname{vol}(P)}{\operatorname{vol}\left(P \cup P^{\prime}\right)-\operatorname{vol}(P)}
\end{aligned}
$$

Therefore, it holds for any operator $\bowtie \in\{<,=,>\}$ that

$$
\begin{array}{lr} 
& \delta\left(P^{\prime}\right) \bowtie \delta(P) \\
\Longleftrightarrow & \frac{\delta\left(P \cup P^{\prime}\right) \cdot \operatorname{vol}\left(P \cup P^{\prime}\right)-\delta(P) \cdot \operatorname{vol}(P)}{\operatorname{vol}\left(P \cup P^{\prime}\right)-\operatorname{vol}(P)} \bowtie \delta(P) \\
\Longleftrightarrow & \delta\left(P \cup P^{\prime}\right) \bowtie \delta(P),
\end{array}
$$

which implies that $\delta\left(P^{\prime}\right) \leq \delta(P)$ if and only if $\delta\left(P \cup P^{\prime}\right) \leq \delta(P)$ with equality if and only if $\delta\left(P \cup P^{\prime}\right)=\delta(P)$. Since $P$ is optimal, it cannot be the case that $\delta\left(P \cup P^{\prime}\right)>\delta(P)$, and therefore the lemma follows.

Combining everything together, we have that for any input instance $\left(U, E_{U}\right)$, Algorithm 7.2 always returns the same output $r \in \mathbb{R}^{|U|}$ no matter which optimal sets are returned by solving the linear program (7.5). In particular, Algorithm 7.2 always finds the unique optimal set $P \subseteq U$ of maximum size, and assigns $\boldsymbol{r}(u)=\delta(P)$ to every $u \in P$. After removing the computed $P \subset U$, the computed $\boldsymbol{r}(v)=\delta\left(P^{\prime}\right)$ for some $P^{\prime} \subseteq U \backslash P$ and any $v \in P^{\prime}$ is always strictly less than $\boldsymbol{r}(u)=\delta(P)$ for any $u \in P$.

### 7.4.2.2 Any $r$ Satisfying Rules (1) and (2) is Computed by Algorithm 7.2

Next we show that if there is any vector $\boldsymbol{r}$ satisfying Rules (1) and (2), it must be equal to the output of Algorithm 7.2 .

Lemma 7.4. For any hypergraph $H=\left(V_{H}, E_{H}, w\right)$ and $\boldsymbol{f}_{t} \in \mathbb{R}^{n}$, if there is a vector $\boldsymbol{r}=\mathrm{d} \boldsymbol{f}_{t} / \mathrm{d} t$ with an associated $\left\{\boldsymbol{r}_{e}(v)\right\}_{e \in E_{H}, v \in V_{H}}$ satisfying Rules (1) and (2), then $\boldsymbol{r}$ is equal to the output of Algorithm 7.2 .

Proof. We focus our attention on a single equivalence class $U \subset V$ where for any $u, v \in U$, $\boldsymbol{f}(u)=\boldsymbol{f}(v)$. Recall that for each $e \in E_{U}, c_{\boldsymbol{f}_{t}}(e)=w(e)\left|\Delta_{f_{t}}(e)\right|$, which is the rate of flow due to $e$ into $U$ (if $e \in \mathcal{S}_{U}^{+} \cup \mathcal{I}_{U}^{+}$) or out of $U$ (if $e \in \mathcal{S}_{U}^{-} \cup \mathcal{I}_{U}^{-}$). Let $r \in \mathbb{R}^{n}$ be the vector supposed to satisfy Rules (1) and (2). We assume that $U \subseteq V$ is an arbitrary equivalence class, and define

$$
T \triangleq\left\{u \in U: \boldsymbol{r}(u)=\max _{v \in U} \boldsymbol{r}(v)\right\} .
$$

Let us study which properties $\boldsymbol{r}$ must satisfy according to Rules (1) and (2).

- Assume that $e \in \mathcal{S}_{U}^{-}$, i.e., it holds that $S_{f_{t}}(e) \cap U \neq \emptyset$ and $\Delta_{f_{t}}(e)>0$. To satisfy Rule (2a), it suffices to have that

$$
c_{\boldsymbol{f}_{t}}(e)=w(e) \cdot \Delta_{f_{t}}(e)=-\sum_{v \in S_{\boldsymbol{f}}(e)} d(v) \boldsymbol{r}_{e}(v)=-\sum_{v \in T} d(v) \boldsymbol{r}_{e}(v)
$$

if $S_{\boldsymbol{f}_{t}}(e) \cap T \neq \emptyset$, and $\boldsymbol{r}_{e}(v)=0$ for all $v \in T$ otherwise.

- Assume that $e \in \mathcal{S}_{U}^{+}$, i.e., it holds that $S_{f_{t}}(e) \cap U \neq \emptyset$ and $\Delta_{f_{t}}(e)<0$. To satisfy Rule (2a), it suffices to have $S_{f_{t}}(e) \subseteq T$, or $S_{f_{t}}(e) \cap T=\emptyset$.
- Assume that $e \in \mathcal{I}_{U}^{+}$, i.e., it holds that $I_{\boldsymbol{f}}(e) \cap U \neq \emptyset$ and $\Delta_{\boldsymbol{f}_{\boldsymbol{t}}}(e)<0$. To satisfy Rule (2b), it suffices to have that

$$
c_{\boldsymbol{f}_{t}}(e)=\sum_{v \in I_{f}(e)} d(v) \boldsymbol{r}_{e}(v)=\sum_{v \in T} d(v) \boldsymbol{r}_{e}(v)
$$

if $I_{\boldsymbol{f}}(e) \subseteq T$, and $\boldsymbol{r}_{e}(v)=0$ for all $v \in T$ otherwise.

- Assume that $e \in \mathcal{I}_{U}^{-}$, i.e., it holds that $I_{\boldsymbol{f}}(e) \cap U \neq \emptyset$ and $\Delta_{\boldsymbol{f}_{t}}(e)>0$. To satisfy Rule (2b), it suffices to have $I_{\boldsymbol{f}}(e) \subseteq T$, or $I_{\boldsymbol{f}}(e) \cap T=\emptyset$.

Notice that the four conditions above needed to satisfy Rule (2) naturally reflect our definitions of the sets $\mathcal{S}_{U, P}^{+}, \mathcal{I}_{U, P}^{+}, \mathcal{S}_{U, P}^{-}$, and $\mathcal{I}_{U, P}^{-}$and for all $u \in T$, it must be that $\boldsymbol{r}(u)=\delta(T)$.

Now we show that the output set $P$ returned by solving the linear program (7.5) is the set $T$. To prove this, notice that on one hand, by Corollary 7.1 the linear program gives us the unique maximal optimal subset $P \subseteq U$, and every $v \in P$ satisfies that $\boldsymbol{r}(v)=\delta(P) \leq \boldsymbol{r}(u)=\delta(T)$ for any $u \in T$ as every vertex in $T$ has the maximum $r$-value. On the other side, we have that $\delta(T) \leq \delta(P)$ since $P$ is the set returned by the linear program, and therefore $T=P$. We can apply this argument recursively, and this proves that Algorithm 7.2 must return the vector $r$.

### 7.4.2.3 Output of Algorithm 7.2 Satisfies Rules (1) and (2)

Now we show that the output of Algorithm 7.2 does indeed satisfy Rules (1) and (2) which, together with Lemma 7.4 implies that there is exactly one such vector satisfying the rules.

Lemma 7.5. For any hypergraph $H=\left(V_{H}, E_{H}, w\right)$ and vector $\boldsymbol{f}_{t} \in \mathbb{R}^{n}$, the vector $\boldsymbol{r}$ constructed by Algorithm 7.2 has corresponding $\left\{\boldsymbol{r}_{e}(v)\right\}_{e \in E_{H}, v \in V_{H}}$ which satisfies Rules (1) and (2). Moreover, the $\left\{\boldsymbol{r}_{e}(v)\right\}_{e \in E_{H}, v \in V_{H}}$ values can be computed in polynomial time using the vector $\boldsymbol{r}$.

Proof. We focus on a single iteration of the algorithm, in which $r(v)$ is assigned for the vertices in some set $T \subset V_{H}$. We use the notation

$$
\mathcal{E}_{T}^{+}=\mathcal{I}_{U, T}^{+} \cup \mathcal{S}_{U, T}^{+}, \quad \mathcal{E}_{T}^{-}=\mathcal{I}_{U, T}^{-} \cup \mathcal{S}_{U, T}^{-}
$$

and show that the values of $\boldsymbol{r}_{e}(v)$ for $e \in \mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-}$can be computed and satisfy Rules (1) and (2). Therefore, by applying this argument to each recursive call of the algorithm, we establish the lemma. Given the set $T$, construct the following undirected flow graph, which is illustrated in Figure 7.4 .


Figure 7.4: (a) An illustration of the constructed max-flow graph, when $\delta(T) \geq 0$. The minimum cut is given by $\{s\}$. (b) A cut induced by $T^{\prime} \subset T$. We can assume that every $e \in \mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-}$connected to $T^{\prime}$ is on the same side of the cut as $T^{\prime}$. Otherwise, there would be an edge with infinite capacity crossing the cut.

- The vertex set is $\mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-} \cup T \cup\{s, t\}$.
- For all $e \in \mathcal{E}_{T}^{+}$, there is an edge $(s, e)$ with capacity $c_{\boldsymbol{f}_{t}}(e)$.
- For all $e \in \mathcal{E}_{T}^{-}$, there is an edge $(e, t)$ with capacity $c_{\boldsymbol{f}_{t}}(e)$.
- For all $v \in T$, if $\delta(T) \geq 0$, there is an edge $(v, t)$ with capacity $d(v) \delta(T)$. Otherwise, there is an edge $(s, v)$ with capacity $d(v)|\delta(T)|$.
- For each $e \in \mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-}$, and each $v \in T \cap\left(S_{\boldsymbol{f}_{t}}(e) \cup I_{\boldsymbol{f}_{t}}(e)\right)$, there is an edge $(e, v)$ with capacity $\infty$.

We use $\operatorname{cut}(A)$ to denote the weight of the cut defined by the set $A$ in this constructed graph, and note that $\operatorname{cut}(\{s\})=\operatorname{cut}(\{t\})$ since

$$
\operatorname{cut}(\{s\})-\operatorname{cut}(\{t\})=\sum_{e \in \mathcal{E}_{T}^{+}} c_{\boldsymbol{f}}(e)-\sum_{e \in \mathcal{E}_{T}^{-}} c_{\boldsymbol{f}}(e)-\operatorname{vol}(T) \delta(T)=0
$$

by the definition of $\delta(T)$.
Now, suppose that the maximum flow value on this graph is $\operatorname{cut}(\{s\})$, and let the corresponding flow from $u$ to $v$ be given by $\Theta(u, v)=-\Theta(v, u)$. Then, set $d(v) \boldsymbol{r}_{e}(v)=\Theta(e, v)$ for any $e \in \mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-}$and $v \in T \cap e$. This configuration of the values $\boldsymbol{r}_{e}(v)$ would be compatible with the vector $\boldsymbol{r}$ computed by Algorithm 7.2, and would satisfy the rules of the diffusion process for the following reasons:

- For all $v \in T$, the edge $(v, t)$ or $(s, v)$ is saturated and so

$$
\sum_{e \in E} d(v) \boldsymbol{r}_{e}(v)=d(v) \delta(T)=d(v) \boldsymbol{r}(v)
$$

- For any $e \in \mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-}$, the edge $(s, e)$ or $(e, t)$ is saturated and so we have

$$
\sum_{v \in T \cap e} d(v) \boldsymbol{r}_{v}(e)=-w(e) \Delta(e)
$$

Since $\mathcal{E}_{T}^{+} \cup \mathcal{E}_{T}^{-}$is removed in the recursive step of the algorithm, $\boldsymbol{r}_{e}(v)=0$ for all $v \in U \backslash T$ and so $\sum_{v \in U \cap e} d(v) \boldsymbol{r}_{e}(v)=-w(e) \Delta(e)$. This establishes Rule (1) since $U \cap e$ is equal to either $S_{\boldsymbol{f}}(e)$ or $I_{\boldsymbol{f}}(e)$.

- For edges in $\mathcal{S}_{U, T}^{+}\left(\right.$resp. $\left.\mathcal{I}_{U, T}^{+}, \mathcal{I}_{U, T}^{-}\right)$, since $S_{\boldsymbol{f}_{t}}(e)$ (resp. $\left.I_{\boldsymbol{f}_{t}}(e), I_{\boldsymbol{f}_{t}}(e)\right)$ is a subset of $T$ and every $v \in T$ has the same value of $\boldsymbol{r}(v)$, Rule (2) is satisfied. For edges in $\mathcal{S}_{U, T}^{-}$, for any $v \notin T$, we have $\boldsymbol{r}_{e}(v)=0$ and $\boldsymbol{r}(v)<\delta(T)$ which satisfies Rule (2).

We now show that every cut separating $s$ and $t$ has weight at least $\operatorname{cut}(\{s\})$ which would establish that the maximum flow on this graph is $\operatorname{cut}(\{s\})$ by the max-flow min-cut theorem.

Consider some arbitrary cut given by $X=\{s\} \cup T^{\prime} \cup \mathcal{E}_{T^{\prime}}^{+} \cup \mathcal{E}_{T^{\prime}}^{-}$where $T^{\prime}$ (resp. $\mathcal{E}_{T^{\prime}}^{+}, \mathcal{E}_{T^{\prime}}^{-}$) is a subset of $T$ (resp. $\mathcal{E}_{T}^{+}, \mathcal{E}_{T}^{-}$). Figure 7.4 illustrates this cut. Since all of the edges not connected to $s$ or $t$ have infinite capacity, we can assume that no such edge crosses the cut which implies that

- for all $e \in \mathcal{E}_{T^{\prime}}^{+}, e \cap\left(T \backslash T^{\prime}\right)=\emptyset$;
- for all $e \in \mathcal{E}_{T^{\prime}}^{-}, e \cap\left(T \backslash T^{\prime}\right)=\emptyset$;
- for all $e \in\left(\mathcal{E}_{T}^{+} \backslash \mathcal{E}_{T^{\prime}}^{+}\right)$, $e \cap T^{\prime}=\emptyset$;
- for all $e \in\left(\mathcal{E}_{T}^{-} \backslash \mathcal{E}_{T^{\prime}}^{-}\right), e \cap T^{\prime}=\emptyset$.

These conditions, along with the definition of $\mathcal{E}_{T}^{+}$and $\mathcal{E}_{T}^{-}$, allow us to assume that $\mathcal{E}_{T^{\prime}}^{+}=$ $\mathcal{I}_{U, T^{\prime}}^{+} \cup \mathcal{S}_{U, T^{\prime}}^{+}$and $\mathcal{E}_{T^{\prime}}^{-}=\mathcal{I}_{U, T^{\prime}}^{-} \cup \mathcal{S}_{U, T^{\prime}}^{-}$. The size of this arbitrary cut is

$$
\operatorname{cut}(X)=\operatorname{cut}(\{s\})-\sum_{e \in \mathcal{E}_{T^{\prime}}^{+}} c(e)+\sum_{e \in \mathcal{E}_{T^{\prime}}^{-}} c(e)+\sum_{v \in T^{\prime}} d(v) \delta(T)
$$

Since $T$ maximises the objective function $\delta$, we have

$$
\sum_{e \in \mathcal{E}_{T^{\prime}}^{+}} c(e)-\sum_{e \in \mathcal{E}_{T^{\prime}}^{-}} c(e)=\operatorname{vol}\left(T^{\prime}\right) \delta\left(T^{\prime}\right) \leq \operatorname{vol}\left(T^{\prime}\right) \delta(T)=\sum_{v \in T^{\prime}} d(v) \delta(T)
$$

and can conclude that $\operatorname{cut}(X) \geq \operatorname{cut}(\{s\})$. This completes the proof.

We can now combine the results in Lemmas 7.4 and 7.5 to prove Lemma 7.1 .

Proof of Lemma 7.1. Lemma 7.4 and Lemma 7.5 together imply that there is a unique vector $\boldsymbol{r}$ and corresponding $\left\{\boldsymbol{r}_{e}(v)\right\}_{e \in E_{H}, v \in V_{H}}$ which satisfies Rules (1) and (2). Lemma 7.4 further shows that Algorithm 7.2 computes this vector $\boldsymbol{r}$, and the proof of Lemma 7.5 gives
a polynomial-time algorithm for computing the $\left\{\boldsymbol{r}_{e}(v)\right\}$ values by solving a sequence of max-flow problems.

### 7.5 Analysis of Algorithm 7.1

In this section we analyse the FindBipartiteComponents algorithm and prove Theorem 7.1. This section consists of two parts corresponding to the two statements in Theorem 7.1. First, we show that the diffusion process converges to an eigenvector of $\mathbf{J}_{H}$. We then show that this allows us to find sets $L, R \subset V_{H}$ with low hypergraph bipartiteness.

### 7.5.1 Convergence of the Diffusion Process

We now show that the diffusion process determined by the operator $\mathbf{J}_{H}$ converges in polynomial time to an eigenvector of $\mathbf{J}_{H}$.

Theorem 7.2. For any $\epsilon>0$, there is some $t=O\left(1 / \epsilon^{3}\right)$ such that for any starting vector $f_{0}$, there is an interval $[c, c+2 \epsilon]$ such that

$$
\frac{1}{\left\|\boldsymbol{f}_{t}\right\|_{w}} \sum_{\boldsymbol{u}_{i}: \lambda_{i} \in[c, c+2 \epsilon]}\left\langle\boldsymbol{f}_{t}, \boldsymbol{u}_{i}\right\rangle_{w} \geq 1-\epsilon,
$$

where $\left(\boldsymbol{u}_{i}, \lambda_{i}\right)$ are the eigen-pairs of $\mathbf{D}_{H}^{-1} \mathbf{J}_{t}=\mathbf{D}_{H}^{-1}\left(\mathbf{D}_{G_{t}}+\mathbf{A}_{G_{t}}\right)$ and $G_{t}$ is the graph constructed to represent the diffusion operator $\mathbf{J}_{H}$ at time $t$.

By taking $\epsilon$ to be some small constant, this shows that the vector $f_{t}$ converges to an eigenvector of the hypergraph operator in polynomial time.

Proof. We show that the Rayleigh quotient $R_{\mathbf{J}_{H}}\left(\boldsymbol{f}_{t}\right)$ is always decreasing at a rate of at least $\epsilon^{3}$ whenever the conclusion of Theorem 7.2 does not hold. Since $R_{\mathbf{J}_{H}}\left(\boldsymbol{f}_{t}\right)$ can only decrease by a constant amount, the theorem follows.

First, we derive an expression for the rate of change of the Rayleigh quotient $R_{\mathbf{J}_{H}}\left(\boldsymbol{f}_{t}\right)$. Let $\boldsymbol{x}_{t} \triangleq \mathbf{D}_{H}^{\frac{1}{2}} \boldsymbol{f}_{t}$, and

$$
\mathbf{Z}_{t}=\mathbf{D}_{H}^{-\frac{1}{2}}\left(\mathbf{D}_{G_{t}}+\mathbf{A}_{G_{t}}\right) \mathbf{D}_{H}^{-\frac{1}{2}}
$$

Then, we have

$$
R_{\mathbf{J}_{H}}\left(\boldsymbol{f}_{t}\right)=\frac{\boldsymbol{f}_{t}^{\top}\left(\mathbf{D}_{G_{t}}+\mathbf{A}_{G_{t}}\right) \boldsymbol{f}_{t}}{\boldsymbol{f}_{t}^{\top} \mathbf{D}_{H} \boldsymbol{f}_{t}}=\frac{\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t} \boldsymbol{x}_{t}}{\boldsymbol{x}_{t}^{\top} \boldsymbol{x}_{t}}
$$

For every eigenvector $\boldsymbol{u}_{j}$ of $\mathbf{D}_{H}^{-1} \mathbf{J}_{H}$, the vector $\boldsymbol{v}_{j}=\mathbf{D}_{H}^{\frac{1}{2}} \boldsymbol{u}_{j}$ is an eigenvector of $\mathbf{Z}$ with the same eigenvalue $\lambda_{i}$. Since $\mathbf{Z}$ is symmetric, the vectors $\boldsymbol{v}_{1}, \ldots, \boldsymbol{v}_{n}$ are orthogonal. Additionally, notice that

$$
\left\langle\boldsymbol{x}_{t}, \boldsymbol{v}_{j}\right\rangle=\boldsymbol{f}_{t}^{\top} \mathbf{D}_{H} \boldsymbol{u}_{j}=\left\langle\boldsymbol{f}_{t}, \boldsymbol{u}_{j}\right\rangle_{w}
$$

and we define $\alpha_{j}=\left\langle\boldsymbol{f}_{t}, \boldsymbol{u}_{j}\right\rangle_{w}$ so we can write $\boldsymbol{x}_{t}=\sum_{j=1}^{n} \alpha_{j} \boldsymbol{v}_{j}$ and

$$
\boldsymbol{f}_{t}=\mathbf{D}_{H}^{-\frac{1}{2}} \boldsymbol{x}_{t}=\sum_{j=1}^{n} \alpha_{j} \boldsymbol{u}_{j} .
$$

Now, we have that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}\left\langle\boldsymbol{x}_{t}, \mathbf{Z}_{t} \boldsymbol{x}_{t}\right\rangle & =\left\langle\frac{\mathrm{d}}{\mathrm{~d} t} \boldsymbol{x}_{t}, \mathbf{Z}_{t} \boldsymbol{x}_{t}\right\rangle+\left\langle\boldsymbol{x}_{t}, \frac{\mathrm{~d}}{\mathrm{~d} t} \mathbf{Z}_{t} \boldsymbol{x}_{t}\right\rangle \\
& =-\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t}^{2} \boldsymbol{x}_{t}-\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t}^{2} \boldsymbol{x}_{t} \\
& =-2 \sum_{j=1}^{n} \alpha_{j}^{2} \lambda_{j}^{2} .
\end{aligned}
$$

Additionally, we have

$$
\frac{\mathrm{d}}{\mathrm{~d} t} x_{t}^{\top} \boldsymbol{x}_{t}=-\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t} x_{t}-\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t} x_{t}=-2 x_{t}^{\top} \mathbf{Z}_{t} x_{t}
$$

Recalling that $\boldsymbol{x}_{t}^{\top} \boldsymbol{x}_{t}=\sum_{j=1}^{n} \alpha_{j}^{2}$, this gives

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} R\left(\boldsymbol{f}_{t}\right) & =\frac{1}{\left(\boldsymbol{x}_{t}^{\top} \boldsymbol{x}_{t}\right)^{2}}\left(\left(\frac{\mathrm{~d}}{\mathrm{~d} t}\left(\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t} \boldsymbol{x}_{t}\right)\right)\left(\boldsymbol{x}_{t}^{\top} \boldsymbol{x}_{t}\right)-\left(\frac{\mathrm{d}}{\mathrm{~d} t}\left(\boldsymbol{x}_{t}^{\top} \boldsymbol{x}_{t}\right)\right)\left(\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t} \boldsymbol{x}_{t}\right)\right) \\
& =\frac{1}{\boldsymbol{x}_{t}^{\top} \boldsymbol{x}_{t}}\left(\frac{\mathrm{~d}}{\mathrm{~d} t}\left(\boldsymbol{x}_{t}^{\top} \mathbf{Z}_{t} \boldsymbol{x}_{t}\right)\right)+2 R\left(\boldsymbol{f}_{t}\right)^{2} \\
& =2\left(R\left(\boldsymbol{f}_{t}\right)^{2}-\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j=1}^{n} \alpha_{j}^{2} \lambda_{j}^{2}\right) \tag{7.8}
\end{align*}
$$

We now show that at any time $t$, if the conclusion of the theorem does not hold, then

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} R\left(\boldsymbol{f}_{t}\right) \leq-\epsilon^{3} . \tag{7.9}
\end{equation*}
$$

Assuming that this holds and using the fact that $R\left(\boldsymbol{f}_{0}\right) \leq 2$, when $t=2 / \epsilon^{3}$, either $R\left(\boldsymbol{f}_{t}\right)=0$ or there is some $t^{\prime}<t$ when $\left(\mathrm{d} / \mathrm{d} t^{\prime}\right) R\left(\boldsymbol{f}_{t^{\prime}}\right)>-\epsilon^{3}$ and the conclusion of the theorem holds. Now, to show (7.9), consider the partial derivative

$$
\begin{align*}
\frac{\partial}{\partial \lambda_{i}} 2\left(R\left(\boldsymbol{f}_{t}\right)^{2}-\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j=1}^{n} \alpha_{j}^{2} \lambda_{j}^{2}\right) & =2\left(\frac{2 \alpha_{i}^{2}}{\sum_{j=1}^{n} \alpha_{j}^{2}} R\left(\boldsymbol{f}_{t}\right)-\frac{2 \alpha_{i}^{2}}{\sum_{j=1}^{n} \alpha_{j}^{2}} \lambda_{i}\right) \\
& =\frac{4 \alpha_{i}^{2}}{\sum_{j=1}^{n} \alpha_{j}^{2}}\left(R\left(\boldsymbol{f}_{t}\right)-\lambda_{i}\right), \tag{7.10}
\end{align*}
$$

where we use the fact that $R\left(\boldsymbol{f}_{t}\right)=\left(\sum_{j=1}^{n} \alpha_{j}^{2} \lambda_{j}\right) /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)$. Notice that the derivative in (7.10) is greater than 0 if $\lambda_{i}<R\left(\boldsymbol{f}_{t}\right)$, and the derivative is less than 0 if $\lambda_{i}>R\left(\boldsymbol{f}_{t}\right)$. This means that, in order to establish an upper-bound of $(\mathrm{d} / \mathrm{d} t) R\left(\boldsymbol{f}_{t}\right)$, we can assume that the eigenvalues $\lambda_{j}$ are as close to the value of $R\left(\boldsymbol{f}_{t}\right)$ as possible.

Now, we assume that at time $t$ the conclusion of the theorem does not hold. Then, one of the following cases must hold:

1. $\left(\sum_{j: \lambda_{j}>R\left(f_{t}\right)+\epsilon} \alpha_{j}^{2}\right) /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)>\epsilon / 2$;
2. $\left(\sum_{j: \lambda_{j}<R\left(f_{t}\right)-\epsilon} \alpha_{j}^{2}\right) /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)>\epsilon / 2$.

Suppose the first case holds. By the conclusions we draw from (7.10), we can assume that there is an eigenvalue $\lambda_{i}=R\left(\boldsymbol{f}_{t}\right)+\epsilon$ such that $\alpha_{i}^{2} /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)=\epsilon / 2$ and that

$$
\left(\sum_{j: \lambda_{j}<R\left(f_{t}\right)} \alpha_{j}^{2}\right) /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)=1-\epsilon / 2 .
$$

Then, since

$$
R\left(\boldsymbol{f}_{t}\right)=\left(\sum_{j=1}^{n} \alpha_{j}^{2} \lambda_{j}\right) /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right),
$$

we have

$$
\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}<R\left(\boldsymbol{f}_{t}\right)} \alpha_{j}^{2} \lambda_{j}=R\left(\boldsymbol{f}_{t}\right)-\frac{\epsilon}{2}\left(R\left(\boldsymbol{f}_{t}\right)+\epsilon\right),
$$

which is equivalent to

$$
\begin{equation*}
\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}<R\left(\boldsymbol{f}_{t}\right)} \alpha_{j}^{2}\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right)=\frac{\epsilon}{2} \cdot R\left(\boldsymbol{f}_{t}\right)-\frac{\epsilon}{2} \cdot\left(R\left(\boldsymbol{f}_{t}\right)+\epsilon\right)=-\frac{\epsilon^{2}}{2} . \tag{7.11}
\end{equation*}
$$

Now, notice that for any $\lambda_{j}<R\left(\boldsymbol{f}_{t}\right)$ we have

$$
\begin{aligned}
\left(\lambda_{j}^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right) & =\left(\lambda_{j}+R\left(\boldsymbol{f}_{t}\right)\right)\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right) \\
& \geq 2 R\left(\boldsymbol{f}_{t}\right)\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right),
\end{aligned}
$$

since $\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)<0$. As such, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} R\left(\boldsymbol{f}_{t}\right) & =2\left(R\left(\boldsymbol{f}_{t}\right)^{2}-\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j=1}^{n} \alpha_{j}^{2} \lambda_{j}^{2}\right) \\
& =-\frac{2}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j=1}^{n} \alpha_{j}^{2}\left(\lambda_{j}^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right) \\
& =-\epsilon\left(\left(R\left(\boldsymbol{f}_{t}\right)+\epsilon\right)^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right)-\frac{2}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}<R\left(\boldsymbol{f}_{t}\right)} \alpha_{j}^{2}\left(\lambda_{j}^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right) \\
& \leq-\epsilon\left(2 \epsilon R\left(\boldsymbol{f}_{t}\right)+\epsilon^{2}\right)-\frac{2}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}<R\left(\boldsymbol{f}_{t}\right)} 2 \alpha_{j}^{2} R\left(\boldsymbol{f}_{t}\right)\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right) \\
& =-2 \epsilon^{2} R\left(\boldsymbol{f}_{t}\right)-\epsilon^{3}+2 \epsilon^{2} R\left(\boldsymbol{f}_{t}\right) \\
& =-\epsilon^{3},
\end{aligned}
$$

where the second last equality follows by (7.11). We now consider the second case. We assume that there is an eigenvalue $\lambda_{i}=R\left(\boldsymbol{f}_{t}\right)-\epsilon$ such that $\alpha_{i}^{2} /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)=\epsilon / 2$, and that
$\left(\sum_{j: \lambda_{j}>R\left(f_{t}\right)} \alpha_{j}^{2}\right) /\left(\sum_{j=1}^{n} \alpha_{j}^{2}\right)=1-\epsilon / 2$. Then, we have

$$
\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}>R\left(\boldsymbol{f}_{t}\right)} \alpha_{j}^{2} \lambda_{j}=R\left(\boldsymbol{f}_{t}\right)-\frac{\epsilon}{2}\left(R\left(\boldsymbol{f}_{t}\right)-\epsilon\right),
$$

which is equivalent to

$$
\begin{equation*}
\frac{1}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}>R\left(\boldsymbol{f}_{t}\right)} \alpha_{j}^{2}\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right)=\frac{\epsilon}{2} R\left(\boldsymbol{f}_{t}\right)-\frac{\epsilon}{2}\left(R\left(\boldsymbol{f}_{t}\right)-\epsilon\right)=\frac{\epsilon^{2}}{2} . \tag{7.12}
\end{equation*}
$$

Now, notice that for any $\lambda_{j}>R\left(\boldsymbol{f}_{t}\right)$ we have

$$
\begin{aligned}
\lambda_{j}^{2}-R\left(\boldsymbol{f}_{t}\right)^{2} & =\left(\lambda_{j}+R\left(\boldsymbol{f}_{t}\right)\right)\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right) \\
& \geq 2 R\left(\boldsymbol{f}_{t}\right) \cdot\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right) .
\end{aligned}
$$

As such, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} R\left(\boldsymbol{f}_{t}\right) & =-\frac{2}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j=1}^{n} \alpha_{j}^{2}\left(\lambda_{j}^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right) \\
& =-\epsilon\left(\left(R\left(\boldsymbol{f}_{t}\right)-\epsilon\right)^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right)-\frac{2}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}>R\left(\boldsymbol{f}_{t}\right)} \alpha_{j}^{2}\left(\lambda_{j}^{2}-R\left(\boldsymbol{f}_{t}\right)^{2}\right) \\
& \leq-\epsilon\left(\epsilon^{2}-2 \epsilon R\left(\boldsymbol{f}_{t}\right)\right)-\frac{2}{\sum_{j=1}^{n} \alpha_{j}^{2}} \sum_{j: \lambda_{j}<R\left(\boldsymbol{f}_{t}\right)} 2 \alpha_{j}^{2} R\left(\boldsymbol{f}_{t}\right)\left(\lambda_{j}-R\left(\boldsymbol{f}_{t}\right)\right) \\
& =2 \epsilon^{2} g(t)-\epsilon^{3}-2 \epsilon^{2} R\left(\boldsymbol{f}_{t}\right) \\
& =-\epsilon^{3}
\end{aligned}
$$

where the second last equality follows by (7.12). These two cases establish (7.9) and complete the proof of the theorem.

### 7.5.1.1 Relationship of the Eigenvalue to the Clique Graph

We now show that the eigenvalue corresponding to the eigenvector to which the algorithm converges is at most the minimum eigenvalue of $\mathbf{J}_{G}$ where $G$ is the clique reduction of $H$. We start by showing the following technical lemma.

Lemma 7.6. For any hypergraph $H=\left(V_{H}, E_{H}, w\right)$, vector $\boldsymbol{f} \in \mathbb{R}^{n}$, and edge $e \in E_{H}$, it holds that

$$
\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2} \leq \sum_{u, v \in e} \frac{1}{\operatorname{rank}(e)-1}(\boldsymbol{f}(u)+\boldsymbol{f}(v))^{2} .
$$

The equality holds if and only if there is exactly one vertex $v \in e$ with $\boldsymbol{f}(v) \neq 0$ or $\operatorname{rank}(e)=2$.

Proof. We consider some ordering of the vertices in $e$,

$$
u_{1}, u_{2}, \ldots, u_{\operatorname{rank}(e)},
$$

such that $u_{1}=\arg \max _{u \in e} \boldsymbol{f}(u)$ and $u_{2}=\arg \min _{u \in e} \boldsymbol{f}(u)$ and the remaining vertices are ordered arbitrarily. Then, for any $2 \leq k \leq \operatorname{rank}(e)$, we define

$$
C_{k}=\sum_{u, v \in\left\{u_{1}, \ldots, u_{k}\right\}} \frac{1}{k-1}(\boldsymbol{f}(u)+\boldsymbol{f}(v))^{2} .
$$

We show by induction on $k$ that

$$
\begin{equation*}
C_{k} \geq\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2} \tag{7.13}
\end{equation*}
$$

for all $2 \leq k \leq \operatorname{rank}(e)$, with equality if and only if $k=2$ or there is exactly one vertex $u_{i} \in e$ with $\boldsymbol{f}(v) \neq 0$. The lemma follows by setting $k=\operatorname{rank}(e)$.

The base case of $k=2$ follows trivially by the definitions and the choice of $u_{1}$ and $u_{2}$.
For the inductive step, we assume that (7.13) holds for some $k$, and show that it holds for $k+1$. We have that

$$
C_{k+1}=\sum_{u, v \in\left\{u_{1}, \ldots, u_{k+1}\right\}} \frac{1}{k} \cdot(\boldsymbol{f}(u)+\boldsymbol{f}(v))^{2},
$$

which is equivalent to

$$
\begin{aligned}
C_{k+1} & =\frac{1}{k} \sum_{i=1}^{k}\left(\boldsymbol{f}\left(u_{i}\right)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2}+\frac{1}{k} \sum_{u, v \in\left\{u_{1}, \ldots, u_{k}\right\}}(\boldsymbol{f}(u)+\boldsymbol{f}(v))^{2} \\
& =\frac{1}{k} \sum_{i=1}^{k}\left(\boldsymbol{f}\left(u_{i}\right)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2}+\frac{k-1}{k} C_{k} \\
& \geq\left(1-\frac{1}{k}\right)\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2}+\frac{1}{k} \sum_{i=1}^{k}\left(\boldsymbol{f}\left(u_{1}\right)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2}
\end{aligned}
$$

where the final inequality holds by the induction hypothesis. Therefore, it is sufficient to show that

$$
\sum_{i=1}^{k}\left(\boldsymbol{f}\left(u_{i}\right)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2} \geq\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2}
$$

We instead show the stronger fact that

$$
\begin{equation*}
\left(\max _{v \in e} \boldsymbol{f}(v)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2}+\left(\min _{v \in e} \boldsymbol{f}(v)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2} \geq\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2} \tag{7.14}
\end{equation*}
$$

The proof is by case distinction. The first case is when

$$
\operatorname{sign}\left(\max _{u \in e} f(v)\right)=\operatorname{sign}\left(\min _{u \in e} f(u)\right) .
$$

Assume without loss of generality that the sign is positive. Then, since $\boldsymbol{f}\left(u_{k+1}\right) \geq$ $\min _{v \in e} \boldsymbol{f}(v)$, we have

$$
\left(\max _{v \in e} \boldsymbol{f}(v)+\boldsymbol{f}\left(u_{k+1}\right)\right)^{2} \geq\left(\max _{v \in e} \boldsymbol{f}(v)+\min _{u \in e} \boldsymbol{f}(v)\right)^{2}
$$

and (7.14) holds. Moreover, the inequality is strict if $\left|\min _{u \in e} \boldsymbol{f}(u)\right|>0$ or $\left|\boldsymbol{f}\left(u_{k+1}\right)\right|>0$.
The second case is when

$$
\operatorname{sign}\left(\min _{u \in e} \boldsymbol{f}(u)\right) \neq \operatorname{sign}\left(\max _{v \in e} \boldsymbol{f}(v)\right) .
$$

Expanding (7.14), we would like to show

$$
\begin{aligned}
& \left(\max _{u \in e} \boldsymbol{f}(u)\right)^{2}+\left(\min _{v \in e} \boldsymbol{f}(v)\right)^{2}+2 \boldsymbol{f}\left(u_{k+1}\right)\left(\max _{u \in e} \boldsymbol{f}(u)\right)+2 \boldsymbol{f}\left(u_{k+1}\right)\left(\min _{v \in e} \boldsymbol{f}(v)\right) \\
& \quad+2 \boldsymbol{f}\left(u_{k+1}\right)^{2} \geq\left(\max _{u \in e} \boldsymbol{f}(u)\right)^{2}+\left(\min _{u \in e} \boldsymbol{f}(u)\right)^{2}-2\left(\max _{u \in e} \boldsymbol{f}(u)\right)\left|\min _{u \in e} \boldsymbol{f}(u)\right|,
\end{aligned}
$$

which is equivalent to
$2 \boldsymbol{f}\left(u_{k+1}\right)^{2}+2 \boldsymbol{f}\left(u_{k+1}\right)\left(\max _{u \in e} \boldsymbol{f}(u)\right)+2 \boldsymbol{f}\left(u_{k+1}\right)\left(\min _{v \in e} \boldsymbol{f}(v)\right) \geq-2\left(\max _{u \in e} \boldsymbol{f}(u)\right)\left|\min _{v \in e} \boldsymbol{f}(v)\right|$.
Notice that exactly one of the terms on the left hand side is negative. Recalling that $\min _{u \in e} \boldsymbol{f}(u) \leq \boldsymbol{f}\left(u_{k+1}\right) \leq \max _{v \in e} \boldsymbol{f}(v)$, it is clear that

- if $\boldsymbol{f}\left(u_{k+1}\right)<0$, then

$$
-2\left(\max _{u \in e} \boldsymbol{f}(u)\right)\left|\min _{v \in e} \boldsymbol{f}(v)\right| \leq 2 \boldsymbol{f}\left(u_{k+1}\right)\left(\max _{v \in e} \boldsymbol{f}(v)\right) \leq 0
$$

and the inequality holds.

- if $\boldsymbol{f}\left(u_{k+1}\right) \geq 0$, then

$$
-2\left(\max _{u \in e} \boldsymbol{f}(u)\right)\left|\min _{v \in e} \boldsymbol{f}(v)\right| \leq 2 \boldsymbol{f}\left(u_{k+1}\right)\left(\min _{u \in e} \boldsymbol{f}(u)\right) \leq 0
$$

and the inequality holds.
Moreover, in both cases the inequality is strict if $-2\left(\max _{v \in e} \boldsymbol{f}(v)\right)\left|\min _{u \in e} \boldsymbol{f}(u)\right|<0$ or $\left|\boldsymbol{f}\left(u_{k_{1}}\right)\right|>0$.

Now, we show that one always finds an eigenvector whose corresponding eigenvalue is at most the minimum eigenvalue of the clique reduction.

Lemma 7.7. For any hypergraph $H=\left(V_{H}, E_{H}, w\right)$ with clique reduction $G$, if $\boldsymbol{f}$ is the eigenvector corresponding to $\lambda_{1}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right)$, then

$$
\frac{\boldsymbol{f}^{\top} \mathbf{J}_{H} \boldsymbol{f}}{\boldsymbol{f}^{\top} \mathbf{D}_{H} \boldsymbol{f}} \leq \lambda_{1}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right)
$$

and the inequality is strict if $\min _{e \in E_{H}} \operatorname{rank}(e)>2$.

Proof. Since $\lambda_{1}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right)=\left(\boldsymbol{f}^{\top} \mathbf{J}_{G} \boldsymbol{f}\right) /\left(\boldsymbol{f}^{\top} \mathbf{D}_{G} \boldsymbol{f}\right)$ and $\left(\boldsymbol{f}^{\top} \mathbf{D}_{G} \boldsymbol{f}\right)=\left(\boldsymbol{f}^{\top} \mathbf{D}_{H} \boldsymbol{f}\right)$ by the construction of the clique graph, it suffices to show that

$$
\boldsymbol{f}^{\top} \mathbf{J}_{H} \boldsymbol{f} \leq \boldsymbol{f}^{\top} \mathbf{J}_{G} \boldsymbol{f}
$$

This is equivalent to

$$
\begin{aligned}
\sum_{e \in E_{H}} w(e)\left(\max _{v \in e} \boldsymbol{f}(v)+\min _{u \in e} \boldsymbol{f}(u)\right)^{2} & \leq \sum_{(u, v) \in E_{G}} w_{G}(u, v)(\boldsymbol{f}(u)+\boldsymbol{f}(v))^{2} \\
& =\sum_{e \in E_{H}} w(e) \sum_{u, v \in e} \frac{1}{\operatorname{rank}(e)-1}(\boldsymbol{f}(u)+\boldsymbol{f}(v))^{2}
\end{aligned}
$$

which holds by Lemma 7.6 .
Furthermore, if $\min _{e \in E_{H}} \operatorname{rank}(e)>2$, then by Lemma 7.6 the inequality is strict unless every edge $e \in E_{H}$ contains at most one $v \in e$ with $\boldsymbol{f}(v) \neq 0$. Suppose the inequality is not strict, then it must be that

$$
\begin{aligned}
\lambda_{1}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right) & =\frac{\sum_{(u, v) \in E_{G}} w_{G}(u, v)(\boldsymbol{f}(v)+\boldsymbol{f}(u))^{2}}{\sum_{v \in V_{G}} d_{G}(v) \boldsymbol{f}(v)^{2}} \\
& =\frac{\sum_{v \in V_{G}} d_{G}(v) \boldsymbol{f}(v)^{2}}{\sum_{v \in V_{G}} d_{G}(v) \boldsymbol{f}(v)^{2}} \\
& =1
\end{aligned}
$$

since for every edge $(u, v) \in E_{G}$, at most one of $\boldsymbol{f}(u)$ or $\boldsymbol{f}(v)$ is not equal to 0 . This cannot be the case, since it is well-known that the maximum eigenvalue $\lambda_{n}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right)=2$ and so $\sum_{i=1}^{n} \lambda_{i}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right) \geq(n-1)+2=n+1$ which contradicts the fact that the trace $\operatorname{tr}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right)$ is equal to $n$. This proves the final statement of the lemma.

### 7.5.2 Cheeger-type Inequality for Hypergraph Bipartiteness

We now prove some intermediate facts about the new hypergraph operator $\mathbf{J}_{H}$. These allow us to show that the operator $\mathbf{J}_{H}$ has a well-defined minimum eigenvector, and the corresponding eigenvalue satisfies a Cheeger-type inequality for hypergraph bipartiteness. Given a hypergraph $H=\left(V_{H}, E_{H}, w\right)$, any edge $e \in E_{H}$ and weighted measure vector $\boldsymbol{f}_{t}$, let $r_{e}^{S} \triangleq \max _{v \in S_{f_{t}}(e)}\{\boldsymbol{r}(v)\}$ and $r_{e}^{I} \triangleq \min _{v \in I_{f_{t}}(e)}\{\boldsymbol{r}(v)\}$; recall that $c_{\boldsymbol{f}_{t}}(e)=w(e)\left|\Delta_{\boldsymbol{f}_{t}}(e)\right|$.

Lemma 7.8. Given a hypergraph $H=\left(V_{H}, E_{H}, w\right)$ and normalised measure vector $\boldsymbol{f}_{t}$, let

$$
\boldsymbol{r}=\frac{\mathrm{d} \boldsymbol{f}_{t}}{\mathrm{~d} t}=-\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t} .
$$

Then, it holds that

$$
\|\boldsymbol{r}\|_{w}^{2}=-\sum_{e \in E} w(e) \Delta_{\boldsymbol{f}_{t}}(e)\left(r_{e}^{S}+r_{e}^{I}\right)
$$

Proof. Let $P \subset V$ be one of the densest vertex sets defined with respect to the solution of the linear program (7.5). By the description of Algorithm 7.2, we have $\boldsymbol{r}(u)=\delta(P)$ for every $u \in P$, and therefore

$$
\begin{aligned}
& \sum_{u \in P} d(u) \boldsymbol{r}(u)^{2} \\
& =\operatorname{vol}(P) \cdot \delta(P)^{2} \\
& =\left(c_{f_{t}}\left(\mathcal{S}_{U, P}^{+}\right)+c_{f_{t}}\left(\mathcal{I}_{U, P}^{+}\right)-c_{f_{t}}\left(\mathcal{S}_{U, P}^{-}\right)-c_{f_{t}}\left(\mathcal{I}_{U, P}^{-}\right)\right) \cdot \delta(P) \\
& =\left(\sum_{e \in \mathcal{S}_{U, P}^{+}} c_{f_{t}}(e)+\sum_{e \in \mathcal{I}_{U, P}^{+}} c_{f_{t}}(e)-\sum_{e \in \mathcal{S}_{U, P}^{-}} c_{f_{t}}(e)-\sum_{e \in \mathcal{I}_{U, P}^{-}} c_{f_{t}}(e)\right) \cdot \delta(P) \\
& =\sum_{e \in \mathcal{S}_{U, P}^{+}} c_{f_{t}}(e) \cdot r_{e}^{S}+\sum_{e \in \mathcal{I}_{U, P}^{+}} c_{f_{t}}(e) \cdot r_{e}^{I}-\sum_{e \in \mathcal{S}_{U, P}^{-}} c_{f_{t}}(e) \cdot r_{e}^{S}-\sum_{e \in \mathcal{I}_{U, P}^{-}} c_{f_{t}}(e) \cdot r_{e}^{I} .
\end{aligned}
$$

Since each vertex is included in exactly one set $P$ and each edge appears either in one each of $\mathcal{S}_{U, P}^{+}$and $\mathcal{I}_{U, P}^{+}$or in one each of $\mathcal{S}_{U, P}^{-}$and $\mathcal{I}_{U, P}^{-}$, it holds that

$$
\begin{aligned}
\|\boldsymbol{r}\|_{w}^{2} & =\sum_{v \in V} d(v) \boldsymbol{r}(v)^{2} \\
& =\sum_{P} \sum_{v \in P} d(v) \boldsymbol{r}(v)^{2} \\
& =\sum_{P}\left(\sum_{e \in \mathcal{S}_{U, P}^{+}} c_{\boldsymbol{f}_{t}}(e) \cdot r_{e}^{S}+\sum_{e \in \mathcal{I}_{U, P}^{+}} c_{\boldsymbol{f}_{t}}(e) \cdot r_{e}^{I}-\sum_{e \in \mathcal{S}_{U, P}^{-}} c_{f_{t}}(e) \cdot r_{e}^{S}-\sum_{e \in \mathcal{I}_{U, P}^{I}} c_{\boldsymbol{f}_{t}}(e) \cdot r_{e}^{I}\right) \\
& =-\sum_{e \in E} w(e) \Delta_{\boldsymbol{f}_{t}}(e)\left(r_{e}^{S}+r_{e}^{I}\right),
\end{aligned}
$$

which proves the lemma.

Next, we define $\gamma_{1}=\min _{\boldsymbol{f}} D(\boldsymbol{f})$ and show that any vector $\boldsymbol{f}$ that satisfies $\gamma_{1}=D(\boldsymbol{f})$ is an eigenvector of $\mathbf{J}_{H}$ with eigenvalue $\gamma_{1}$. We start by showing that the Raleigh quotient of the new operator is equivalent to the discrepancy ratio of the hypergraph.

Lemma 7.9. For any hypergraph $H$ and vector $\boldsymbol{f}_{t} \in \mathbb{R}^{n}$, it holds that $D\left(\boldsymbol{f}_{t}\right)=R_{\mathbf{J}_{H}}\left(\boldsymbol{f}_{t}\right)$.

Proof. Since $\boldsymbol{f}_{t}^{\top} \mathbf{D}_{H} \boldsymbol{f}_{t}=\sum_{v \in V} d(v) \boldsymbol{f}_{t}(v)^{2}$, it is sufficient to show that

$$
\boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t}=\sum_{e \in E_{H}} w(e)\left(\max _{u \in e} \boldsymbol{f}_{t}(u)+\min _{v \in e} \boldsymbol{f}_{t}(v)\right)^{2}
$$

Recall that for some graph $G_{t}, \mathbf{J}_{H}=\mathbf{D}_{G_{t}}+\mathbf{A}_{G_{t}}$. Then, we have that

$$
\begin{aligned}
\boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t} & =\boldsymbol{f}_{t}^{\top}\left(\mathbf{D}_{G_{t}}+\mathbf{A}_{G_{t}}\right) \boldsymbol{f}_{t} \\
& =\sum_{(u, v) \in E_{G}} w_{G}(u, v)\left(\boldsymbol{f}_{t}(u)+\boldsymbol{f}_{t}(v)\right)^{2} \\
& =\sum_{e \in E_{H}}\left(\sum_{(u, v) \in S_{f_{t}}(e) \times I_{f_{t}}(e)} w_{G_{t}}(u, v)\left(\boldsymbol{f}_{t}(u)+\boldsymbol{f}_{t}(v)\right)^{2}\right) \\
& =\sum_{e \in E_{H}} w(e)\left(\max _{u \in e} \boldsymbol{f}_{t}(u)+\min _{v \in e} \boldsymbol{f}_{t}(v)\right)^{2},
\end{aligned}
$$

which follows since the graph $G_{t}$ is constructed by splitting the weight of each hyperedge $e \in E_{H}$ between the edges $S_{f_{t}}(e) \times I_{f_{t}}(e)$.

Lemma 7.10. For a hypergraph $H$, operator $\mathbf{J}_{H}$, and vector $\boldsymbol{f}_{t}$, the following statements hold:

1. $\frac{\mathrm{d}}{\mathrm{d} t}\left\|\boldsymbol{f}_{t}\right\|_{w}^{2}=-2 \boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t}$;
2. $\frac{\mathrm{d}}{\mathrm{d} t}\left(\boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t}\right)=-2\left\|\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\|_{w}^{2}$;
3. $\frac{\mathrm{d}}{\mathrm{d} t} R\left(\boldsymbol{f}_{t}\right) \leq 0$ with equality if and only if $\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t} \in \operatorname{span}\left(\boldsymbol{f}_{t}\right)$.

Proof. By definition, we have that

$$
\begin{aligned}
\frac{\mathrm{d}\left\|\boldsymbol{f}_{t}\right\|_{w}^{2}}{\mathrm{~d} t} & =\frac{\mathrm{d}}{\mathrm{~d} t} \sum_{v \in V} d(v) \boldsymbol{f}_{t}(v)^{2} \\
& =\sum_{v \in V} d(v) \cdot \frac{\mathrm{d} \boldsymbol{f}_{t}(v)^{2}}{\mathrm{~d} t} \\
& =\sum_{v \in V} d(v) \cdot \frac{\mathrm{d} \boldsymbol{f}_{t}(v)^{2}}{\mathrm{~d} \boldsymbol{f}_{t}(v)} \frac{\mathrm{d} \boldsymbol{f}_{t}(v)}{\mathrm{d} t} \\
& =2 \sum_{v \in V} d(v) \boldsymbol{f}_{t}(v) \cdot \frac{\mathrm{d} \boldsymbol{f}_{t}(v)}{\mathrm{d} t} \\
& =2\left\langle\boldsymbol{f}_{t}, \frac{\mathrm{~d} \boldsymbol{f}_{t}}{\mathrm{~d} t}\right\rangle_{w}=-2\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w}
\end{aligned}
$$

which proves the first statement.
For the second statement, by Lemma 7.9 we have

$$
\boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t}=\sum_{e \in E} w(e)\left(\max _{u \in e} \boldsymbol{f}_{t}(u)+\min _{v \in e} \boldsymbol{f}_{t}(v)\right)^{2}
$$

and therefore

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} \boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t} & =\frac{\mathrm{d}}{\mathrm{~d} t} \sum_{e \in E} w(e)\left(\max _{u \in e} \boldsymbol{f}_{t}(u)+\min _{v \in e} \boldsymbol{f}_{t}(v)\right)^{2} \\
& =2 \sum_{e \in E} w(e) \Delta_{f_{t}}(e) \cdot \frac{\mathrm{d}}{\mathrm{~d} t}\left(\max _{u \in e} \boldsymbol{f}_{t}(u)+\min _{v \in e} \boldsymbol{f}_{t}(v)\right) \\
& =2 \sum_{e \in E} w(e) \Delta_{f_{t}}(e) \cdot\left(r_{e}^{S}+r_{e}^{I}\right), \tag{7.15}
\end{align*}
$$

where the last equality holds by the way that all the vertices receive their $r$-values by the algorithm and the definitions of $r_{e}^{S}$ and $r_{e}^{I}$. On the other side, by definition $\boldsymbol{r}=-\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}$ and so by Lemma 7.8 ,

$$
\begin{equation*}
\left\|\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\|_{w}^{2}=\|\boldsymbol{r}\|_{w}^{2}=-\sum_{e \in E} w(e) \Delta_{f_{t}}(e)\left(r_{e}^{S}+r_{e}^{I}\right) . \tag{7.16}
\end{equation*}
$$

By combining (7.15) with (7.16), we have the second statement.
For the third statement, notice that we can write $\boldsymbol{f}_{t}^{\top} \mathbf{J}_{H} \boldsymbol{f}_{t}$ as $\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w}$. Then, we have that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t} \frac{\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w}}{\left\|\boldsymbol{f}_{t}\right\|_{w}^{2}} \\
& =\frac{1}{\left\|\boldsymbol{f}_{t}\right\|_{w}^{2}} \cdot \frac{\mathrm{~d}\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w}}{\mathrm{~d} t}-\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w} \cdot \frac{1}{\left\|\boldsymbol{f}_{t}\right\|_{w}^{4}} \frac{\mathrm{~d}\left\|\boldsymbol{f}_{t}\right\|_{w}^{2}}{\mathrm{~d} t} \\
& =-\frac{1}{\left\|\boldsymbol{f}_{t}\right\|_{w}^{4}} \cdot\left(2 \cdot\left\|\boldsymbol{f}_{t}\right\|_{w}^{2} \cdot\left\|\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\|_{w}^{2}+\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w} \cdot \frac{\mathrm{~d}\left\|\boldsymbol{f}_{t}\right\|_{w}^{2}}{\mathrm{~d} t}\right) \\
& =-\frac{2}{\left\|\boldsymbol{f}_{t}\right\|_{w}^{4}} \cdot\left(\left\|\boldsymbol{f}_{t}\right\|_{w}^{2} \cdot\left\|\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\|_{w}^{2}-\left\langle\boldsymbol{f}_{t}, \mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t}\right\rangle_{w}^{2}\right) \\
& \leq 0,
\end{aligned}
$$

where the last inequality holds by the Cauchy-Schwarz inequality on the inner product $\langle\cdot, \cdot\rangle_{w}$ with the equality if and only if $\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}_{t} \in \operatorname{span}\left(\boldsymbol{f}_{t}\right)$.

This allows us to establish the following key lemma.
Lemma 7.11. For any hypergraph $H, \gamma_{1}=\min _{\boldsymbol{f}} D(\boldsymbol{f})$ is an eigenvalue of $\mathbf{J}_{H}$ and any minimiser $f$ is its corresponding eigenvector.

Proof. By Lemma 7.9 it holds that $R(\boldsymbol{f})=D(\boldsymbol{f})$ for any $\boldsymbol{f} \in \mathbb{R}^{n}$. When $\boldsymbol{f}$ is a minimiser of $D(\boldsymbol{f})$, it holds that

$$
\frac{\mathrm{d} R(\boldsymbol{f})}{\mathrm{d} t}=0
$$

which implies by Lemma 7.10 that $\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f} \in \operatorname{span}(\boldsymbol{f})$ and proves that $\boldsymbol{f}$ is an eigenvector.

We are now ready to prove a Cheeger-type inequality for our operator and the hypergraph bipartiteness. We first show that the minimum eigenvalue of $\mathbf{J}_{H}$ is at most twice the hypergraph bipartiteness.

Lemma 7.12. Given a hypergraph $H=\left(V_{H}, E_{H}\right)$ with sets $L, R \subset V_{H}$ such that $\beta(L, R)=$ $\beta$, it holds that

$$
\gamma_{1} \leq 2 \beta
$$

where $\gamma_{1}$ is the smallest eigenvalue of $\mathbf{J}_{H}$.

Proof. Let $\chi_{L, R} \in\{-1,0,1\}^{n}$ be the indicator vector of the cut $L, R$ such that

$$
\chi_{L, R}(u)= \begin{cases}1 & \text { if } u \in L \\ -1 & \text { if } u \in R \\ 0 & \text { otherwise }\end{cases}
$$

Then, by Lemma 7.9 the Rayleigh quotient is given by

$$
\begin{aligned}
R_{\mathbf{J}_{H}}\left(\chi_{L, R}\right) & =\frac{\sum_{e \in E} w(e)\left(\max _{u \in e} \chi_{L, R}(u)+\min _{v \in e} \chi_{L, R}(v)\right)^{2}}{\sum_{v \in V} d(v) \chi_{L, R}(v)^{2}} \\
& =\frac{4 w(L \mid \bar{L})+4 w(R \mid \bar{R})+w(L, \overline{L \cup R} \mid R)+w(R, \overline{L \cup R} \mid L)}{\operatorname{vol}(L \cup R)} \\
& \leq 2 \beta(L, R)
\end{aligned}
$$

which completes the proof since $\gamma_{1}=\min _{\boldsymbol{f}} R_{\mathbf{J}_{H}}(\boldsymbol{f})$ by Lemma 7.11.

The proof of the other direction of the Cheeger-type inequality is more involved, and forms the basis of the second claim in Theorem 7.1.

Lemma 7.13. For any hypergraph $H=(V, E, w)$, let $\mathbf{J}_{H}$ be the operator defined with respect to $H$, and $\gamma_{1}=\min _{\boldsymbol{f}} D(\boldsymbol{f})=\min _{\boldsymbol{f}} R(\boldsymbol{f})$ be the minimum eigenvalue of $\mathbf{D}_{H}^{-1} \mathbf{J}_{H}$. Then, there are disjoint sets $L, R \subset V$ such that

$$
\beta(L, R) \leq \sqrt{2 \gamma_{1}}
$$

Proof. Let $\boldsymbol{f} \in \mathbb{R}^{n}$ be the vector such that $D(\boldsymbol{f})=\gamma_{1}$. For any threshold $t \in\left[0, \max _{u} \boldsymbol{f}(u)^{2}\right]$, define $\boldsymbol{x}_{t}$ such that

$$
\boldsymbol{x}_{t}(u)= \begin{cases}1 & \text { if } \boldsymbol{f}(u) \geq \sqrt{t} \\ -1 & \text { if } \boldsymbol{f}(u) \leq-\sqrt{t} \\ 0 & \text { otherwise }\end{cases}
$$

We show that choosing $t \in\left[0, \max _{u} \boldsymbol{f}(u)^{2}\right]$ uniformly at random gives

$$
\begin{equation*}
\mathbb{E}\left[\sum_{e \in E_{H}} w(e)\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|\right] \leq \sqrt{2 \gamma_{1}} \cdot \mathbb{E}\left[\sum_{v \in V_{H}} d(v)\left|\boldsymbol{x}_{t}(v)\right|\right] \tag{7.17}
\end{equation*}
$$

Notice that every such $\boldsymbol{x}_{t}$ defines disjoints vertex sets $L$ and $R$ that satisfy

$$
\beta(L, R)=\frac{\sum_{e \in E_{H}} w(e)\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|}{\sum_{v \in V_{H}} d(v)\left|\boldsymbol{x}_{t}(v)\right|}
$$

Hence, (7.17) would imply that there is some $\boldsymbol{x}_{t}$ such that the disjoint $L, R$ defined by $\boldsymbol{x}_{t}$ would satisfy

$$
\sum_{e \in E_{H}} w(e)\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right| \leq \sqrt{2 \gamma_{1}} \cdot \sum_{v \in V_{H}} d(v)\left|\boldsymbol{x}_{t}(v)\right|
$$

which implies that

$$
\beta(L, R) \leq \sqrt{2 \gamma_{1}}
$$

Hence, it suffices to prove (7.17). We assume without loss of generality that $\max _{u}\left\{\boldsymbol{f}(u)^{2}\right\}=$ 1 , so $t$ is chosen uniformly from $[0,1]$. First of all, we have that

$$
\mathbb{E}\left[\sum_{v \in V_{H}} d(v)\left|\boldsymbol{x}_{t}(v)\right|\right]=\sum_{v \in V_{H}} d(v) \mathbb{E}\left[\left|\boldsymbol{x}_{t}(v)\right|\right]=\sum_{v \in V_{H}} d(v) \boldsymbol{f}(v)^{2}
$$

To analyse the left-hand side of (7.17), we focus on a particular edge $e \in E_{H}$. Let $a_{e}=\max _{u \in e} \boldsymbol{f}(u)$ and $b_{e}=\min _{v \in e} \boldsymbol{f}(v)$. We drop the subscript $e$ when it is clear from context. We show that

$$
\begin{equation*}
\mathbb{E}\left[\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|\right] \leq\left|a_{e}+b_{e}\right|\left(\left|a_{e}\right|+\left|b_{e}\right|\right) \tag{7.18}
\end{equation*}
$$

1. Suppose $\operatorname{sign}\left(a_{e}\right)=\operatorname{sign}\left(b_{e}\right)$. Our analysis is by case distinction:

- $\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|=2$ with probability $\min \left(a_{e}^{2}, b_{e}^{2}\right)$;
- $\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|=1$ with probability $\left|a_{e}^{2}-b_{e}^{2}\right|$;
- $\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|=0$ with probability $1-\max \left(a_{e}^{2}, b_{e}^{2}\right)$.

Assume without loss of generality that $a_{e}^{2}=\min \left(a_{e}^{2}, b_{e}^{2}\right)$. Then, it holds hat

$$
\mathbb{E}\left[\left|\max _{u \in e} \boldsymbol{x}(u)+\min _{v \in e} \boldsymbol{x}(v)\right|\right]=2 a_{e}^{2}+\left|a_{e}^{2}-b_{e}^{2}\right|=a_{e}^{2}+b_{e}^{2} \leq\left|a_{e}+b_{e}\right|\left(\left|a_{e}\right|+\left|b_{e}\right|\right)
$$

2. Suppose $\operatorname{sign}\left(a_{e}\right) \neq \operatorname{sign}\left(b_{e}\right)$. Our analysis is by case distinction:

- $\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|=2$ with probability 0 ;
- $\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|=1$ with probability $\left|a_{e}^{2}-b_{e}^{2}\right|$;
- $\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|=0$ with probability $\min \left(a_{e}^{2}, b_{e}^{2}\right)$.

Assume without loss of generality that $a_{e}^{2}=\min \left(a_{e}^{2}, b_{e}^{2}\right)$. Then, it holds that

$$
\begin{aligned}
\mathbb{E}\left[\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|\right] & =\left|a_{e}^{2}-b_{e}^{2}\right| \\
& =\left(\left|a_{e}\right|-\left|b_{e}\right|\right)\left(\left|a_{e}\right|+\left|b_{e}\right|\right) \\
& =\left|a_{e}+b_{e}\right|\left(\left|a_{e}\right|+\left|b_{e}\right|\right),
\end{aligned}
$$

where the final equality follows because $a_{e}$ and $b_{e}$ have different signs.
These two cases establish (7.18). Now, we have that

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{e \in E_{H}} w(e)\left|\max _{u \in e} \boldsymbol{x}_{t}(u)+\min _{v \in e} \boldsymbol{x}_{t}(v)\right|\right] \\
& \leq \sum_{e \in E_{H}} w(e)\left|a_{e}+b_{e}\right|\left(\left|a_{e}\right|+\left|b_{e}\right|\right) \\
& \leq \sqrt{\sum_{e \in E_{H}} w(e)\left|a_{e}+b_{e}\right|^{2}} \sqrt{\sum_{e \in E_{H}} w(e)\left(\left|a_{e}\right|+\left|b_{e}\right|\right)^{2}} \\
& =\sqrt{\sum_{e \in E_{H}} w(e)\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2}} \sqrt{\sum_{e \in E_{H}} w(e)\left(\left|a_{e}\right|+\left|b_{e}\right|\right)^{2}} .
\end{aligned}
$$

By our assumption that $\boldsymbol{f}$ is the eigenvector corresponding to the eigenvalue $\gamma_{1}$, it holds that

$$
\sum_{e \in E_{H}} w(e)\left(\max _{u \in e} \boldsymbol{f}(u)+\min _{v \in e} \boldsymbol{f}(v)\right)^{2} \leq \gamma_{1} \sum_{v \in V_{H}} d(v) \boldsymbol{f}(v)^{2} .
$$

On the other side, we have that

$$
\sum_{e \in E_{H}} w(e)\left(\left|a_{e}\right|+\left|b_{e}\right|\right)^{2} \leq 2 \sum_{e \in E_{H}} w(e)\left(\left|a_{e}\right|^{2}+\left|b_{e}\right|^{2}\right) \leq 2 \sum_{v \in V_{H}} d(v) \boldsymbol{f}(v)^{2} .
$$

This gives us that

$$
\begin{aligned}
\mathbb{E}\left[\sum_{e \in E_{H}} w(e)\left|\max _{u \in e} \boldsymbol{x}(u)+\min _{v \in e} \boldsymbol{x}(v)\right|\right] & \leq \sqrt{2 \gamma_{1}} \sum_{v \in V_{H}} d(v) \boldsymbol{f}(v)^{2} \\
& =\sqrt{2 \gamma_{1}} \cdot \mathbb{E}\left[\sum_{v \in V_{H}} d(v)|\boldsymbol{x}(v)|\right]
\end{aligned}
$$

which proves the statement.
Now, we are able to combine these results to prove Theorem 7.1 .

Proof of Theorem 7.1. The first statement of the theorem follows by setting the starting vector $f_{0}$ of the diffusion to be the minimum eigenvector of the clique graph $G$. By Lemma 7.7, we have that $R_{\mathbf{J}_{H}}\left(\boldsymbol{f}_{0}\right) \leq \lambda_{1}\left(\mathbf{D}_{G}^{-1} \mathbf{J}_{G}\right)$ and the inequality is strict if $\min _{e \in E_{H}} \operatorname{rank}(e)>2$.

Then, Theorem 7.2 shows that the diffusion process converges to an eigenvector and the Rayleigh quotient only decreases during convergence, and so the inequality holds. The algorithm runs in polynomial time, since by Theorem 7.2 the diffusion process converges in polynomial time and each step of Algorithm 7.1 can be computed in polynomial time using a standard algorithm for solving linear programs.

The second statement is a restatement of Lemma 7.13. The sweep set algorithm runs in polynomial time, since there are $n$ different sweep sets and computing the hypergraph bipartiteness for each one also takes polynomial time.

### 7.6 Further Study of the Hypergraph Laplacian Spectrum

In this section, we further study the spectrum of the operators $\mathbf{L}_{H}$ and $\mathbf{J}_{H}$. We first demonstrate that there exist infinite families of hypergraphs $\{H\}$ for which the number of eigenvectors of $\mathbf{L}_{H}$ and $\mathbf{J}_{H}$ exceeds the number of vertices. We then show that finding the eigenvector of $\mathbf{J}_{H}$ corresponding to its smallest eigenvalue is NP-hard.

### 7.6.1 Discussion on the Number of Eigenvectors of $\mathbf{J}_{H}$ and $\mathbf{L}_{H}$

We now investigate the spectrum of the non-linear hypergraph Laplacian operator and our new operator $\mathbf{J}_{H}$ by considering some example hypergraphs. In particular, we show that the hypergraph Laplacian $\mathbf{L}_{H}$ can have more than 2 eigenvalues, which answers the following open question posed by Chan et al. [CLTZ18].
$\ldots$ apart from $f_{1}$, the Laplacian has another eigenvector $f_{2} \ldots$ it is not clear if $\mathbf{L}$ has any other eigenvalues. We leave as an open problem the task of investigating if other eigenvalues exist. (Chan et al. [CLTZ18])

Furthermore, we show that the number of eigenvectors of the new operator $\mathbf{J}_{H}$ can be exponential in the number of vertices.

### 7.6.1.1 $\quad \mathbf{L}_{H}$ Can Have More Than 2 Eigenvalues

Similar to our new operator $\mathbf{J}_{H}$, for some vector $\boldsymbol{f}$, the operator $\mathbf{L}_{H}$ behaves like the graph Laplacian $\mathbf{L}_{G}=\mathbf{D}_{G}-\mathbf{A}_{G}$ for some graph $G$ constructed by splitting the weight of each hyperedge $e$ between the edges in $S_{\boldsymbol{f}}(e) \times I_{\boldsymbol{f}}(e)$. We refer the reader to [CLTZ18] for the full details of this construction.

Now, with reference to Figure 7.5, we consider the simple hypergraph $H$ where $V_{H}=$ $\left\{v_{1}, v_{2}, v_{3}\right\}$ and $E_{H}=\left\{\left\{v_{1}, v_{2}, v_{3}\right\}\right\}$. Letting $\boldsymbol{f}_{1}=[1,1,1]^{\top}$, notice that the graph $G_{1}$ in


Figure 7.5: Given the hypergraph $H$, there are three graphs $G_{1}, G_{2}$, and $G_{3}$ which correspond to eigenvalues of the hypergraph Laplacian $\mathbf{L}_{H}$.

Figure 7.5 is the graph for which $\mathbf{L}_{H} \boldsymbol{f}_{1}$ is equivalent to $\mathbf{L}_{G_{1}} \boldsymbol{f}_{1}$. In this case,

$$
\mathbf{L}_{G_{1}}=\left[\begin{array}{ccc}
\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}\right]
$$

and we have

$$
\mathbf{L}_{H} \boldsymbol{f}_{1}=\mathbf{L}_{G_{1}} \boldsymbol{f}_{1}=[0,0,0]^{\top}
$$

which shows that $\boldsymbol{f}_{1}$ is the trivial eigenvector of $\mathbf{L}_{H}$ with eigenvalue 0 .
Now, consider $\boldsymbol{f}_{2}=[1,-2,1]^{\top}$. In this case, $G_{2}$ shown in Figure 7.5 is the graph for which $\mathbf{L}_{H} f_{2}$ is equivalent to $\mathbf{L}_{G_{2}} f_{2}$. Then, we have

$$
\mathbf{L}_{G_{2}}=\left[\begin{array}{ccc}
\frac{1}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{2} \\
0 & -\frac{1}{2} & \frac{1}{2}
\end{array}\right]
$$

and

$$
\mathbf{L}_{H} \boldsymbol{f}_{2}=\mathbf{L}_{G_{2}} \boldsymbol{f}_{2}=\left[\frac{3}{2},-3, \frac{3}{2}\right]^{\top}
$$

which shows that $\boldsymbol{f}_{2}$ is an eigenvector of $\mathbf{L}_{H}$ with eigenvalue $3 / 2$.
Finally, we consider $\boldsymbol{f}_{3}=[1,-1,0]$ and notice that $G_{3}$ is the graph such that $\mathbf{L}_{H} \boldsymbol{f}_{3}=\mathbf{L}_{G} \boldsymbol{f}_{3}$. Then,

$$
\mathbf{L}_{G_{3}}=\left[\begin{array}{ccc}
\frac{1}{2} & -\frac{1}{2} & 0 \\
-\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0
\end{array}\right]
$$



Figure 7.6: Given the hypergraph $H$, there are $2^{n / 3}$ possible graphs $G$, each of which corresponds to a different eigenvector of the hypergraph operator $\mathbf{J}_{H}$.
and

$$
\mathbf{L}_{H} \boldsymbol{f}_{3}=\mathbf{L}_{G_{3}} \boldsymbol{f}_{3}=[2,-2,0]^{\top}
$$

which shows that $\boldsymbol{f}_{3}$ is an eigenvector of $\mathbf{L}_{H}$ with eigenvalue 2 .

Through an exhaustive search of the other possible constructed graphs on $\left\{v_{1}, v_{2}, v_{3}\right\}$, we find that these are the only eigenvalues of $\mathbf{L}_{H}$. By the symmetries of $\boldsymbol{f}_{2}$ and $\boldsymbol{f}_{3}$, this means that the operator $\mathbf{L}_{H}$ has a total of 7 different eigenvectors and 3 distinct eigenvalues. It is useful to point out that, since the $\mathbf{L}_{H}$ operator is non-linear, a linear combination of eigenvectors with the same eigenvalue is not, in general, an eigenvector. Additionally, notice that the constructions of $\boldsymbol{f}_{1}$ and $\boldsymbol{f}_{2}$ can be generalised to hypergraphs with more than 3 vertices, demonstrating that the number of eigenvectors of $\mathbf{L}_{H}$ can grow faster than the number of vertices.

### 7.6.1.2 $\mathbf{J}_{H}$ Can Have an Exponential Number of Eigenvectors

To study the spectrum of our new operator $\mathbf{J}_{H}$, we construct a hypergraph $H$ in the following way:

- There are $n$ vertices split into two clusters $L$ and $R$ of size $n / 2$. There is a clique on each cluster.
- The cliques are joined by $n / 3$ edges of rank 3 , such that every vertex is a member of exactly one such edge.

Now, we can construct a vector $f$ as follows: for each edge $e$ of rank 3 in the hypergraph, let $u \in e$ be the vertex alone in one of the cliques, and $v, w \in e$ be the two vertices in the other clique. Then, we set $\boldsymbol{f}(u)=1$ and one of $\boldsymbol{f}(v)$ or $\boldsymbol{f}(w)$ to be -1 and the other to be 0 . Notice that there are $2^{n / 3}$ such vectors. Each one corresponds to a different graph $G$, as illustrated in Figure 7.6, which is the graph constructed such that $\mathbf{J}_{H}$ is equivalent to $\mathbf{J}_{G}$ when applied to the vector $f$.

Notice that, by the construction of the graph $H$, half of the edges of rank 3 must have one vertex in $L$ and two vertices in $R$, and half of the rank- 3 edges must have two vertices in $L$ and one vertex in $R$. This means that, within each cluster $L$ or $R$, one third of the vertices have $f$-value 1 , one third have $f$-value -1 , and one third have $f$-value 0 .

Now, we have that

$$
\left(\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}\right)(u)=\left(\mathbf{D}_{H}^{-1} \mathbf{J}_{G} \boldsymbol{f}\right)(u)=\frac{2}{n} \sum_{\{u, v\} \in E_{G}}(\boldsymbol{f}(u)+\boldsymbol{f}(v))
$$

Suppose that $\boldsymbol{f}(u)=0$, meaning that it does not have an adjacent edge in $G$ from its adjacent rank-3 edge in $H$. In this case,

$$
\begin{aligned}
\left(\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}\right)(u) & =\frac{2}{n} \sum_{u \sim_{G} v} \boldsymbol{f}(v) \\
& =\frac{2}{n}\left[\frac{n}{3 \cdot 2} \cdot 1+\frac{n}{3 \cdot 2} \cdot(-1)\right] \\
& =0
\end{aligned}
$$

Now, suppose that $\boldsymbol{f}(u)=1$, and so it has an adjacent edge in $G$ from its adjacent rank- 3 edge in $H$. Then,

$$
\begin{aligned}
\left(\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}\right)(u) & =\frac{2}{n} \sum_{u \sim_{G} v}(1+\boldsymbol{f}(v)) \\
& =\frac{2}{n}\left[\frac{n}{2}+\frac{n}{3 \cdot 2} \cdot(-1)+\left(\frac{n}{3 \cdot 2}-1\right) \cdot 1-1\right] \\
& =1-\frac{4}{n}
\end{aligned}
$$

Similarly, if $\boldsymbol{f}(u)=-1$, then we find that

$$
\left(\mathbf{D}_{H}^{-1} \mathbf{J}_{H} \boldsymbol{f}\right)(u)=\frac{4}{n}-1
$$

and so we can conclude that $\boldsymbol{f}$ is an eigenvector of the operator $\mathbf{J}_{H}$ with eigenvalue $(n-4) / n$. Since there are $2^{n / 3}$ possible such vectors $\boldsymbol{f}$, there are an exponential number of eigenvectors with this eigenvalue. Once again, we highlight that due to the non-linearity of $\mathbf{J}_{H}$, a linear combination of these eigenvectors is in general not an eigenvector of the operator.

### 7.6.2 Hardness of Finding the Minimum Eigenvalue of $\mathbf{J}_{H}$

The second statement of Theorem 7.1 answers an open question posed by Yoshida [Yos19] by showing that our constructed hypergraph operator satisfies a Cheeger-type inequality for hypergraph bipartiteness. However, the following theorem shows that there is no polynomialtime algorithm to compute the eigenvector corresponding to the minimum eigenvalue of any such operator unless $P=N P$. This means that the problem we consider in this work is fundamentally more difficult than the equivalent problem for graphs, as well as the problem of finding a sparse cut in a hypergraph.

Theorem 7.3. Given any operator that satisfies a Cheeger-type inequality for hypergraph bipartiteness, there is no polynomial-time algorithm that computes a multiplicative-factor approximation of the minimum eigenvalue or eigenvector unless $\mathrm{P}=\mathrm{NP}$.

Proof. The proof is based on considering the following Max Set Splitting problem: For any given hypergraph, the problem looks for a partition $L, R$ with $L \cup R=V_{H}$ and $L \cap R=\emptyset$, such that it holds for any $e \in E_{H}$ that $e \cap L \neq \emptyset$ and $e \cap R \neq \emptyset$. This problem is known to be NP-complete [GJ79]. This is also referred to as Hypergraph 2-Colourability and we can consider the problem of colouring every vertex in the hypergraph either red or blue such that every edge contains at least one vertex of each colour.

We assume that there is some operator $\mathbf{J}$ which satisfies the Cheeger-type inequality

$$
\begin{equation*}
\frac{\lambda_{1}(\mathbf{J})}{2} \leq \beta_{H} \leq \sqrt{2 \lambda_{1}(\mathbf{J})} \tag{7.19}
\end{equation*}
$$

and that there is an algorithm which can compute the minimum eigen-pair of the operator in polynomial time. We show that this would allow us to solve the Max Set Splitting problem in polynomial time.

Given a 2-colourable hypergraph $H$ with colouring $(L, R)$, we use the eigenvector of the operator $\mathbf{J}_{H}$ to find a valid colouring. By definition, we have that $\beta(L, R)=0$ and $\gamma_{1}=0$ by (7.19). Furthermore, by the second statement of Theorem 7.1, we can compute disjoint sets $L^{\prime}, R^{\prime}$ such that $\beta\left(L^{\prime}, R^{\prime}\right)=0$. Note that in general $L^{\prime}$ and $R^{\prime}$ are different from $L$ and R.

Now, let $E^{\prime}=\left\{e \in E_{H}: e \cap\left(L^{\prime} \cup R^{\prime}\right) \neq \emptyset\right\}$. Then, by the definition of bipartiteness, for all $e \in E^{\prime}$ we have $e \cap L^{\prime} \neq \emptyset$ and $e \cap R^{\prime} \neq \emptyset$. Therefore, if we colour every vertex in $L^{\prime}$ blue and every vertex in $R^{\prime}$ red, then every $e \in E^{\prime}$ is satisfied. That is, they contain at least one vertex of each colour.

It remains to colour the vertices in $V \backslash\left(L^{\prime} \cup R^{\prime}\right)$ such that the edges in $E_{H} \backslash E^{\prime}$ are satisfied. The hypergraph $H^{\prime}=\left(V \backslash\left(L^{\prime} \cup R^{\prime}\right), E_{H} \backslash E^{\prime}\right)$ must also be 2-colourable, and so we can recursively apply our algorithm until every vertex is coloured. This algorithm runs in polynomial time, since there are at most $O(n)$ iterations and each iteration runs in polynomial time by our assumption.

### 7.7 Experimental Results

In this section, we evaluate the performance of Algorithm 7.1 on synthetic and real-world datasets. All algorithms are implemented in Python 3.6, using the scipy library for sparse matrix representations and linear programs. The experiments are performed using an Intel(R)

Core(TM) i5-8500 CPU @ 3.00 GHz processor, with 16 GB RAM. Code to reproduce these experiments can be downloaded from
https://github.com/pmacg/hypergraph-bipartite-components.
Since this is the first proposed algorithm for approximating hypergraph bipartiteness, we compare it to a simple and natural baseline algorithm, which we call CliqueCut (CC). In this algorithm, we construct the clique reduction of the hypergraph, and use the two-sided sweep-set algorithm described in [Tre12] to find a set with low bipartiteness in the clique reduction $\sqrt{6}$

Additionally, we compare two versions of our proposed algorithm. FindBipartiteCompoNENTS (FBC) is the new algorithm described in Algorithm 7.1 and FBCApprox (FBCA) is an approximate version in which we do not solve the linear programs in Lemma 7.1 to compute the graph $G$. Instead, at each step of the algorithm, we construct $G$ by splitting the weight of each hyperedge $e$ evenly between the edges in $S(e) \times I(e)$.

We always set the parameter $\epsilon=1$ for FBC and FBCA, and set the starting vector $f_{0} \in \mathbb{R}^{n}$ for the diffusion to be the eigenvector corresponding to the minimum eigenvalue of $\mathbf{J}_{G}$, where $G$ is the clique reduction of the hypergraph $H$.

### 7.7.1 Synthetic Datasets

We first evaluate the algorithms using a random hypergraph model. Given the parameters $n$, $r, p$, and $q$, we generate an $n$-vertex $r$-uniform hypergraph in the following way: the vertex set $V$ is divided into two clusters $L$ and $R$ of size $n / 2$. For every set $S \subset V$ with $|S|=r$, we add the hyperedge $S$ with probability $p$ if $S \subset L$ or $S \subset R$, and we add the hyperedge with probability $q$ otherwise. We remark that this is a special case of the hypergraph stochastic block model (e.g., [CLW18]). We limit the number of free parameters for simplicity while maintaining enough flexibility to generate random hypergraphs with a wide range of optimal $\beta_{H}$-values.

We compare the algorithms' performance using four metrics: the hypergraph bipartiteness $\beta_{H}(L, R)$, the clique graph bipartiteness $\beta_{G}(L, R)$, the $F_{1}$-score [vR04] of the returned clustering, and the runtime of the algorithm. We always report the average result on 10 hypergraphs randomly generated with each parameter configuration.

[^11]

Figure 7.7: The average performance and standard error of each algorithm, when $n=200, r=3$ and $p=10^{-4}$.

Comparison of FBC and FBCA. We first fix the values $n=200, r=3, p=10^{-4}$, and vary the ratio of $q / p$ from 2 to 6 which produces hypergraphs with 250 to 650 edges. The performance of each algorithm on these hypergraphs is shown in Figure 7.7, from which we can make the following observations.

- From Figure 7.7(a) we observe that FBC and FBCA find sets with very similar bipartiteness and they perform better than the CliqueCut baseline.
- Figure 7.7(b) shows that the FBCA algorithm is much faster than FBC.

From this, we conclude that in practice it is sufficient to use the much faster FBCA algorithm in place of the FBC algorithm.

Experiments on larger graphs. We now compare only the FBCA and CliqueCut algorithms, which allows us to run on hypergraphs with higher rank and number of vertices. We fix the parameters $n=2,000, r=5$, and $p=10^{-11}$, producing hypergraphs with between 5,000 and 75, 000 edges 7 and show the results in Figure 7.8. The new algorithm consistently and significantly outperforms the baseline on every metric and across a wide variety of input hypergraphs.

To compare the algorithms' runtime, we fix the parameter $n=2,000$, the ratio $q=2 p$, and report the runtime of the FBCA and CC algorithms on a variety of hypergraphs in Table 7.1. Our new algorithm takes more time than the baseline CC algorithm, but both algorithms appear to scale linearly in the size of the input hypergraph ${ }^{8}$, this suggests that our algorithm's runtime is roughly a constant factor multiple of the baseline.

[^12]

Figure 7.8: The average performance of each algorithm, when $n=2,000, r=5$, and $p=10^{-11}$.

### 7.7.2 Real-world Datasets

Next, we demonstrate the broad utility of our new algorithm on complex real-world datasets with higher-order relationships which are most naturally represented by hypergraphs. Moreover, the hypergraphs are inhomogeneous, meaning that they contain vertices of different types, although this information is not available to the algorithm and so an algorithm has to treat every vertex identically. We see that Algorithm 7.1 is able to find clusters which correspond to the vertices of different types. Table 7.2 shows the $F_{1}$-score of the clustering produced by the algorithm on each dataset, and demonstrates that it consistently outperforms the CliqueCut algorithm.

Penn Treebank. The Penn Treebank dataset is an English-language corpus with examples of written American English from several sources, including fiction and journalism [MMS93]. The dataset contains 49,208 sentences and over 1 million words, which are labelled with their part of speech. We construct a hypergraph in the following way: the

Table 7.1: The runtime in seconds of the FBCA and CC algorithms

|  |  |  | Avg. Runtime |  |
| :--- | :--- | :--- | :--- | :--- |
| $r$ | $p$ | Avg. $\left\|E_{H}\right\|$ | FBCA | CC |
|  | $10^{-9}$ | 1239 | 1.15 | 0.12 |
| 4 | $10^{-8}$ | 12479 | 10.14 | 0.86 |
|  | $10^{-7}$ | 124717 | 89.92 | 9.08 |
|  | $10^{-11}$ | 5177 | 3.99 | 0.62 |
| 5 | $10^{-10}$ | 51385 | 44.10 | 6.50 |
|  | $10^{-9}$ | 514375 | 368.48 | 69.25 |

Table 7.2: The performance of the FBCA and CC algorithms on real-world datasets

|  |  | $F_{1}$-Score |  |
| :--- | :--- | :--- | :--- |
| Dataset | Cluster | FBCA | CC |
| Penn | Verbs | $\mathbf{0 . 7 3}$ | 0.69 |
| Treebank | Non-Verbs | $\mathbf{0 . 5 9}$ | 0.56 |
| DBLP | Conferences | $\mathbf{1 . 0 0}$ | 0.25 |
|  | Authors | $\mathbf{1 . 0 0}$ | 0.98 |

vertex set consists of all the verbs, adverbs, and adjectives which occur at least 10 times in the corpus, and for every 4-gram (a sequence of 4 words) we add a hyperedge containing the co-occurring words. This results in a hypergraph with 4,686 vertices and 176,286 edges. The clustering returned by the new algorithm correctly distinguishes between verbs and non-verbs with an accuracy of $67 \%$. This experiment demonstrates that our unsupervised general purpose algorithm can recover non-trivial structure in a dataset which would ordinarily be clustered using domain knowledge and a complex pre-trained model [ABV18, GG07].

DBLP. We construct a hypergraph from a subset of the DBLP network consisting of 14,376 papers published in artificial intelligence and machine learning conferences [DBL, WJS ${ }^{+} 19$. For each paper, we include a hyperedge linking the authors of the paper with the conference in which it is published, giving a hypergraph with 14,495 vertices and 14,376 edges. The clusters returned by our algorithm successfully separate the authors from the conferences with an accuracy of $100 \%$.

### 7.8 Future Work

In this chapter, we introduce a new Laplacian-type operator which generalises the signless Laplacian operator from graphs to hypergraphs. We show that the resulting algorithm is effective for recovering the structure of complex datasets, and this further justifies the recent effort to develop a spectral theory of hypergraphs. The applicability of spectral hypergraph theory for clustering remains an important area for future research. For example, as demonstrated in this chapter, the Laplacian operator of Chan et al. [CLTZ18] can have more than two eigenvectors. It would be interesting to investigate whether these other eigenvectors can be applied for learning clusters in hypergraphs.

Open Question 7. Can the eigenvectors of the non-linear hypergraph Laplacian be used to develop a general algorithm for learning clusters in hypergraphs?

Such an algorithm could avoid the downsides of using graph reductions for hypergraph clustering and make full use of the higher-order information encoded in the hypergraph edges.

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[^0]:    ${ }^{1}$ For completeness, the opposite of heterophily is homophily, in which vertices are more likely to share edges with other vertices in the same cluster.

[^1]:    ${ }^{2}$ Notice that this is not guaranteed when using $\mathbf{X}$. For example, if $G$ is bipartite such that every edge connects $L \subset V$ to $R \subset V$, then $x_{t}$ always alternates between $L$ and $R$.

[^2]:    ${ }^{3}$ Note that they use a slightly different definition of conductance: $\Phi(S)=w(S, \bar{S}) / \operatorname{vol}(S)$. The difference in the denominator does not impact our use of this theorem, since we guarantee that the volume of $S$ is small.

[^3]:    ${ }^{1}$ Note that we can take any constant approximation in Definition 4.1 with a different corresponding constant in the main result.

[^4]:    ${ }^{1}$ Notice that the first eigenvector is always the constant vector and gives no useful information. This is why we visualise the second and third eigenvectors only.
    ${ }^{2}$ In Examples 5.1 and 5.2 the graphs $G$ are generated according to a generalised stochastic block model based on the defined meta-graph. The model is formally described in Section 5.3

[^5]:    ${ }^{3}$ The BSDS dataset provides several human-generated ground truth segmentations for each image. Since there are different numbers of ground truth clusterings associated with each image, in our experiments we take the target number of clusters for a given image to be the one closest to the median.

[^6]:    ${ }^{1} \mathrm{~A}$ war is defined by the maintainers of the dataset as a series of battles resulting in at least 1,000 deaths.

[^7]:    ${ }^{1}$ Note that the word 'graph' always refers to a non-hyper graph. Similarly, we always use $\mathbf{L}_{H}$ to refer to the non-linear hypergraph Laplacian operator, and use $\mathbf{L}_{G}$ as the standard graph Laplacian.

[^8]:    ${ }^{2}$ Notice that, while the operator $\mathbf{L}_{G}$ of a graph $G$ has $n$ eigenvalues, the number of eigenvalues of $\mathbf{L}_{H}$ is unknown because of its non-linearity.

[^9]:    ${ }^{3}$ For the reader familiar with the heat diffusion process of graphs (e.g., [Chu07 KG14]), notice that the above-defined process essentially employs the operator $\mathbf{J}_{G}$ to replace the Laplacian $\mathbf{L}_{G}$ when defining the

[^10]:    ${ }^{5}$ Note that the graph $G$ used for the diffusion at time $t$ can be easily computed from the $\left\{\boldsymbol{r}_{e}(v)\right\}$ values, although in practice this is not actually needed since the $\boldsymbol{r}(u)$ values can be used to update the diffusion directly.

[^11]:    ${ }^{6}$ We choose to use the algorithm in [Tre12] here since, as far as we know, this is the only non-SDP based algorithm for solving the MAX-CUT problem for graphs. Notice that, although SDP-based algorithms achieve a better approximation ratio for the MAX-CUT problem, they are not practical even for hypergraphs of medium sizes.

[^12]:    ${ }^{7}$ In the random hypergraph model, small values of $p$ and $q$ are needed, since in an $n$-vertex, $r$-uniform hypergraph there are $\binom{n}{r}$ possible edges which can be a very large number. In this case, $\binom{2000}{5} \approx 2.6 \times 10^{14}$.
    ${ }^{8}$ Although $n$ is fixed, the CliqueCut algorithm's runtime is not constant since the time to compute an eigenvalue of the sparse adjacency matrix scales with the number and rank of the hyperedges.

