α_s from an improved τ vector isovector spectral function

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Abstract. After discussing difficulties in determining α_s from tau decay due to the existence of Duality Violations and the associated asymptotic nature of the OPE, we describe a new determination based on an improved vector isovector spectral function, now based solely on experimental input, obtained by (i) combining ALEPH and OPAL results for $2\pi + 4\pi$ and (ii) replacing K^-K^0 and higher-multiplicity exclusive-mode contributions, both previously estimated using Monte Carlo, with new experimental BaBar results for K^-K^0 and results implied by e^+e^- cross sections and CVC for the higher-multiplicity modes. We find $\alpha_s(m_\tau) = 0.3077 \pm 0.0075$, which corresponds to $\alpha_s(m_Z) = 0.1171 \pm 0.0010$. Finally, we comment on some of the shortcomings in the criticism of our approach by Pich and Rodriguez-Sanchez.

It has been clear since the pioneering work of Ref. [1] (see also Ref. [2] for other pre-1992 references and a then-up-to-date implementation of the approach) that Finite Energy Sum Rules (FESRs) provide a potentially useful tool for extracting α_s from hadronic τ decay data. Subsequent increases in the precision of both the experimental data [3, 4] and the order to which the dominant perturbative contribution is known [5] have significantly reduced the resulting uncertainty on α_s, to the extent that assumptions/approximations which might have been reasonable in 1992 now need to be revisited.

In our context, the FESR consists of the identity

$$\underbrace{\int_{s_{th}}^{s_0} \frac{ds}{s_0}\, w\!\left(\frac{s}{s_0}\right) \frac{1}{\pi}\mathrm{Im}\Pi(s)}_{I_w^{exp}(s_0)} \;=\; \underbrace{-\frac{1}{2i\pi} \oint_{|z|=s_0} \frac{dz}{s_0}\, \hat{w}\!\left(\frac{z}{s_0}\right) D(z)}_{I_w^{th}(s_0)}\,, \qquad (1)$$

with $\Pi(s)$ the correlator associated with the $I = 1$ vector current $\bar{u}\gamma^\mu d$, w and $\hat{w}$ polynomials related by an integration by parts, and $D(z)$ the Adler function, $D(z) = -z\frac{d\Pi(z)}{dz}$.[1]

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[1]In general, the axial current can also be considered.

At large enough s_0, it is reasonable to approximate the RHS using the Operator Product Expansion (OPE)[2] and extract α_s using experimental data for $\mathrm{Im}\,\Pi(z)$ as input on the LHS. The presence of a cut in $\Pi(z)$, however, ensures that the OPE is at best an asymptotic expansion in $1/z$.[3] The RHS must, then, in general, contain an additional, non-OPE ("duality violating", or DV) contribution to compensate for the lack of convergence on the circle $|z| = s_0$ [6]. Explicitly,

$$-\frac{1}{2i\pi}\oint_{|z|=s_0}\frac{dz}{s_0}\hat{w}\left(\frac{z}{s_0}\right)\,D(z) = -\frac{1}{2i\pi}\oint_{|z|=s_0}\frac{dz}{s_0}\,\hat{w}\left(\frac{z}{s_0}\right)\,D_{OPE}(z) - \int_{s_0}^{\infty}\frac{ds}{s_0}w\left(\frac{s}{s_0}\right)\mathrm{Im}\Pi_{DV}(s)\,, \tag{2}$$

where the DV contribution, $\mathrm{Im}\,\Pi_{DV}(s)$, accounts for the oscillatory behavior seen at lower s in the spectrum, before perturbative dominance sets in. This oscillatory behavior, which cannot be accounted for by any power behavior in the OPE, led Ref. [7] to conclude that the OPE should not be used "too close" to the cut. For a number of years this stricture was implemented by "pinching", i.e., by choosing for $\hat{w}(s/s_0)$ polynomials having a higher order zero at $s = s_0$, thus suppressing contributions from near the cut. One, however, faces two potential problems. First, it is not known *a priori* how much pinching is needed for a determination of α_s free from DV contamination. Second, a polynomial with a higher-degree zero is necessarily higher degree in s, and generates higher-dimension, D, OPE contributions on the RHS of Eq. (2). This is potentially problematic, not only because the relevant higher-D condensates are not known, but also because an asymptotic expansion like the OPE ceases to be valid at high orders. This leads us to our first message:

It is not possible to simultaneously suppress the contribution from DVs and high-order condensates. One should restrict oneself to low orders of the OPE, but in a consistent manner.

Using a high-degree polynomial with strong pinching, but truncating the OPE at low D, when unsuppressed higher-D contributions are, in principle, present, is thus a dangerous practice and leads, not surprisingly to inconsistencies [8]. We comment further on this practice, which we refer to as the "truncated OPE" (tOPE) approach [4, 9, 10], in Sec. 3.1.

The above discussion makes it clear it is not safe to ignore DV contributions without further investigation. While no first-principles derivation of the form of $\mathrm{Im}\,\Pi_{DV}(s)$ exists, some of its general properties are known. As for the asymptotic expansion in powers of the coupling g^2, where terms missed in the expansion are known to behave as e^{-const/g^2},[4] so terms missed in the OPE expansion of $\mathrm{Im}\,\Pi(s)$ are expected to behave as $e^{-const\cdot s}\times$ (oscillation). This expectation was confirmed in Refs. [11, 12], where the combination of a Regge-like spectrum ($M_n^2 \sim n$) asymptotically and a stringy relation ($\Gamma_n \sim M_n/N_c$) between resonance masses and widths at large (but finite) number of colors, N_c, and large resonance excitation number, n,[5] was shown to lead to the large-s expectation

$$\frac{1}{\pi}\mathrm{Im}\Pi_{DV}(s) = e^{-\delta-\gamma s}\sin\left(\alpha+\beta s+O(\log s)\right)\left(1+O\left(\frac{1}{N_c},\frac{1}{\log s},\frac{1}{s}\right)\right)\,. \tag{3}$$

We will use Eq. (3) in conjunction with Eq. (2), and comment on the impact of possible subleading corrections in Sec. 3.2.

[2]In what follows, we consider the perturbative series as the contribution from the identity operator.
[3]A convergent expansion in inverse powers of z must have a disc of convergence.
[4]Recall the case of renormalons and the perturbative series.
[5]These are properties of QCD in 2 dimensions in the large-N_c limit and also born out phenomenologically in the real world [13–15]

1 Analysis

A determination of $\alpha_s(m_\tau)$ requires one to specify (i) the treatment of the perturbative series; (ii) the choice of weight $w(s)$ in (1) and whether the tOPE is used; (iii) whether or not DVs are included; and (iv) last, but not least, the experimental data to be used.

The perturbative series is currently known to order α_s^4 [5], with a generous guesstimate of the α_s^5 coefficient. The existence of the contour in Eq. (1), however, leaves open the possibility of playing with the dependence on the renomalization scale μ. Two major choices have been most popular. In Contour Improved Perturbation Theory (CIPT) [16, 17], the perturbative scale μ^2 is set equal to the variable, complex, local value $s_0 e^{i\phi}$ at each point on the contour. In Fixed Order Perturbation Theory (FOPT), in contrast, the same fixed-scale choice $\mu^2 = s_0$ is used at all points on the contour. With $s_0 > 0$ a euclidean quantity, the FOPT choice naturally matches the usual $\overline{\text{MS}}$ scheme. For many years $\alpha_s(m_\tau)$ has been obtained by averaging CIPT and FOPT results, with a systematic error component given by half their difference. Recently, however, Hoang and Regner [18] have shown that CIPT gives rise to a perturbative series incompatible with the OPE. Pending a complete analysis where this fundamental flaw is fixed [19, 20], all determinations of $\alpha_s(m_\tau)$ employing CIPT to date should be discarded. In what follows, therefore, we use only FOPT. This leads us to our second message:

Previous CIPT-based analyses should be discarded. Unless CIPT is properly mended, averaging CIPT and FOPT values for $\alpha_s(m_\tau)$ should be avoided.

We next turn to the choice of weights for use in Eq. (1). In Ref. [21] we chose a set of 4 linearly independent polynomials of degree ≤ 4 having no linear term: $w_0(y) = 1$, $w_2(y) = 1 - y^2$, $w_3(y) = (1 - y)^2(1 + 2y)$, $w_4(y) = (1 - y^2)^2$. The linear term is avoided because renormalon model analyses show FESRs with such a term to be problematic [22]. This choice has the advantage of variable sensitivity to DVs: w_0 has no pinching, while $w_{2,3,4}$ are singly, doubly and doubly pinched, respectively. It also produces variable sensitivity to the OPE condensates: up to α_s-suppressed logarithmic corrections, the w_0 FESR is free of condensate contributions and sensitive only to perturbation theory while the $w_{2,3,4}$ FESRs receive contributions also from the $D = 6$ condensate, both $D = 6$ and 8 condensates, and both $D = 6$ and 10 condensates, respectively, with $D = 6$ contributions in the ratios $1 : 3 : 2$. The consistency (or lack thereof) of results for this condensate obtained from combined w_0 and w_2, combined w_0 and w_3 and combined w_0 and w_4 analyses provides a non-trivial check on our analysis framework. Fits to Eq. (2) determine, in addition to α_s, the relevant OPE condensates and the DV parameters α, β, γ and δ, also the value of s, s_{min}, above which the parametrization (3) admits fits compatible with the underlying weighted spectral integrals.

We now turn to the experimental data. Two major compilations of τ data have been used in previous $\alpha_s(m_\tau)$ [3, 4] determinations. While this data is on average very precise, contributions from $K\bar{K}$ and several of the multiparticle states relevant in the higher-s region were based on Monte Carlo rather than experimental data. Fortunately, (i) BaBar results now exist for the $\tau \to K^- K_s \nu_\tau$ distribution [23], and (ii) e^+e^- data[6] combined with the Conserved Vector Current relation allow the multiparticle-mode contributions to be replaced with experimental input.[7] Adding the contributions obtained using updated branching fractions and combining ALEPH and OPAL $2\pi + 4\pi$ results following the algorithm of Ref. [24], used for the R-data analysis of the muon $g-2$, we obtained the updated $I = 1$ vector spectral function shown in Fig. 1 [21], which also shows the corresponding original ALEPH and OPAL results. The gain in precision near the end point of the spectrum is striking.

[6] See Ref. [21] for a full list of references.

[7] Isospin-breaking corrections to the CVC relation are safely negligible for contributions from this part of the spectrum.

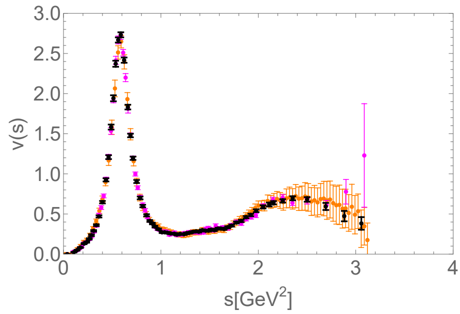
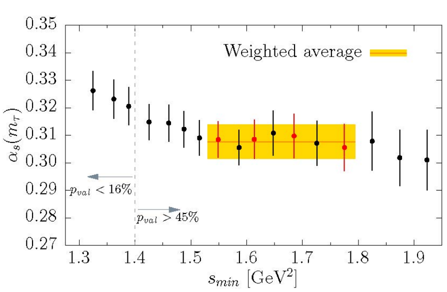


Figure 1. ALEPH (magenta) and OPAL (orange) vector spectral functions, $v(s) = 2\pi \mathrm{Im}\Pi(s)$, together with the improved combined result of Ref. [21] described in the text (black).

2 Results

We have performed a set of single- and multi-polynomial fits with $s_0 \in [s_{min}, s_{max}]$, where $s_{max} = 3.0574\mathrm{GeV}^2$ is the maximum s of the new spectral function combination. Fig. 2 (left panel) shows the s_{min} dependence for the simplest fit, using $w_0(x) = 1$. We see that $\alpha_s(m_\tau)$ drifts at low s_{min}, where the large-s DV parametrization (3) no longer describes the data well for all $s > s_{min}$, but reaches a plateau (in the region of the yellow band) as s_{min} is increased, indicating that above such s_{min} the form (3) is compatible with the underlying data. The value of $\alpha_s(m_\tau)$ remains stable at larger s_{min}, with an error that grows as the fit runs out of experimental data.



mom.	α_s	$c_6[\mathrm{GeV}^6]$
w_0	0.3077(65)	$--$
$w_0\&w_2$	0.3091(69)	$-0.0059(13)$
$w_0\&w_3$	0.3080(70)	$-0.0070(12)$
$w_0\&w_4$	0.3079(70)	$-0.0068(12)$

Figure 2. Left panel: Single-weight $w_0 = 1$ fit results for $\alpha_s(m_\tau)$ vs. s_{min}. P-values are $< 16\%$ to the left of the dashed vertical line and $> 45\%$ to the right. Right panel: results for $\alpha_s(m_\tau)$ and the $D = 6$ condensate, c_6, from fits using the weights shown in the first column.

The right panel of Fig. 2 shows results for $\alpha_s(m_\tau)$ and the $D = 6$ condensate, c_6, obtained from one- or two-weight fits using the weights listed in column 1. The values of both $\alpha_s(m_\tau)$ and c_6 are seen to be consistent across fits. Other fits were carried out, always showing consistent results for $\alpha_s(m_\tau)$, the condensates, and the DV parameters α, β, γ and δ [21]. Combining this information, we find, in FOPT,

$$\alpha_s(m_\tau) = 0.3077 \pm 0.0065_{stat} \pm 0.0038_{pert.th.+DVs} \Leftrightarrow \alpha_s(M_Z) = 0.1171 \pm 0.0010\,(\overline{MS}_{n_f=5})\,. \quad (4)$$

where the first error is statistical and the second reflects perturbative and DV uncertainties [21]. The result is in excellent agreement with the PDG world average $\alpha_s(M_Z) = 0.1179 \pm 0.0009$ [25].

3 The criticism of Pich and Rodriguez Sanchez

In a 25th-anniversary special issue of the JHEP journal, the authors of Ref. [26] took the opportunity to express what they consider to be the drawbacks of the approach presented here, while arguing for the alleged virtues of their own truncated OPE (tOPE) approach. Since the limited space available here precludes refuting and/or responding in full to all the claims made there, we restrict ourselves to a couple of important clarifying remarks.[8]

3.1 Comparing theory with experiment: don't ignore correlations!

The approach followed in Ref. [26] fits α_s and lower-D OPE condensates using sets of higher-degree, at least doubly pinched, polynomial FESRs, neglecting DVs, and, more importantly, neglecting higher-D OPE condensate contributions in principle present for the weights included in the analysis (cfr. our first message). The latter neglect is often argued to be justified by the expectation that OPE contributions will behave as $\sim C_D\left(\Lambda^2_{QCD}/m^2_\tau\right)^D$, where Λ_{QCD} is a typical QCD scale, of the order a few hundred MeV, and $C_D \sim 1$. This argument, however, tacitly treats the OPE as if it were convergent. In fact, the expansion is at best asymptotic, and the C_D are expected to eventually grow close to factorially with increasing D. Still, even if not justified, it might be that the tOPE assumption happens to work at the τ scale.

Let us look at some results purporting to support this suggestion, obtained from analyses of V+A τ spectral data in Ref. [26]. Table 2 of that reference quotes the CIPT results $\alpha_s(m_\tau) = 0.314^{+0.013}_{-0.009}$ and $\alpha_s(m_\tau) = 0.348^{+0.014}_{-0.012}$, obtained from the $w(x) = 1-2x+x^2$ and $w(x) = 1-x^6$ FESRs, in both cases setting *all* non-perturbative (NP) OPE and DV contributions to zero, even though the former receives OPE contributions up to $D = 6$ and the latter up to $D = 14$. These results (together with other entries in the table) are characterized by the authors as showing "amazing stability". One should, however, bear in mind that these values are extracted from the same τ data and are *very* strongly correlated. Declaring two results A and B for the same observable to be compatible based on the overlap of the two errors, δA and δB, is extremely misleading when the results are strongly correlated. A proper compatibility assessment requires determining whether the difference $\Delta_{AB} = A - B$ is compatible with zero within error, taking all correlations into account. When A and B are essentially 100% correlated, the error on Δ_{AB} is $\delta_{AB} = |\delta A - \delta B|$ and, to claim compatibility, one should find $\Delta_{AB} = 0$ to within this correlated error δ_{AB}, not within the sum $\delta A + \delta B$, which determines whether the two errors overlap. The claimed compatibility of the two α_s values above is thus highly questionable (at best).[9]

Similar comments pertain to conclusions drawn from some of the figures in Ref. [26], e.g., the left panel of Fig. 3, which shows the s_0-dependence of the LHSs of the $w^{(2,0)}(x) = (1-x)^2$ FESRs, Eq. (1), for the V (purple), A(green) and $\frac{1}{2}(V+A)$ (red) channels, together with predictions obtained including perturbative CIPT and FOPT contributions only, with fitted α_s as input, once more setting all NP contributions to zero. The CIPT and FOPT results are shown, respectively, by the orange and light-blue bands. From these curves the authors conclude that this OPE truncation is confirmed by the data, since "above 2.2 GeV^2 all experimental curves remain within the 1σ perturbative bands". It is, however, clear that the "theory error", represented by the orange and light-blue bands, results largely from the errors in the data also being shown. The declared agreement between theory and experimental integrals involves a dangerous double counting of errors which hides the impact of the very strong correlations between theory results at different s_0, experimental results at different s_0,

[8] We will respond in full detail to all claims in a future regular publication.

[9] Assuming, e.g., a 90% correlation, the above two values for α_s are 6 sigma apart.

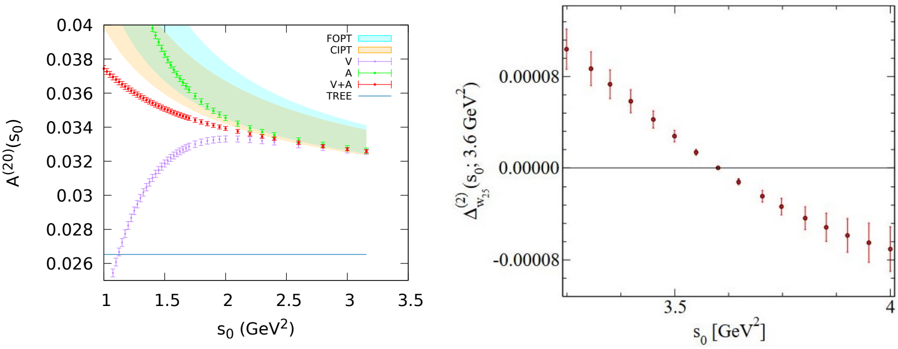


Figure 3. Left panel: s_0 dependence of the LHS of the $w^{(2,0)}(x) = (1-x)^2$ FESR for the V (purple), A (green) and $\frac{1}{2}(V+A)$ (red) channels. The orange and light-blue regions are the CIPT and FOPT perturbative predictions for $\alpha_s(m_\tau) = 0.329^{+0.020}_{-0.018}$. The blue horizontal line at the bottom indicates the parton-model prediction. Taken from Ref. [26]. Right panel: Double difference $\Delta^{(2)}_w(s_0, s_0^*)$ using e^+e^- data (see text) .

as well as between the theory curves and underlying experimental data. As in our previous example, the difference, in this case *theory − data*, employing an error assessment which takes into account all of these correlations, would provide a more reliable means of assessing the supposed compatibility. Regrettably, such checks are not shown in Ref. [26].

The authors of Ref. [26] have also argued that the existence of acceptable fits for 4 OPE parameters (α_s and either the $D = 4, 6$ and 8 or $D = 6, 8$ and 10 condensates) using 5 independent $s_0 = m_\tau^2$ FESRs with at-least-doubly-pinched weights of degree up to 7 (and hence, in principle, OPE contributions up to $D = 16$) establishes the validity of the neglect of in-principle-present $D = 10, 12, 14$ and 16 or $D = 12, 14$ and 16 contributions. They have, moreover, argued against using the s_0 dependence of the resulting fits at lower s_0 to test this assumption, on the grounds that considering such lower-s_0 input enhances neglected DV contributions. While kinematic restrictions preclude carrying out τ-based analyses at higher s_0, where this rational for avoiding s_0-dependence tests would not apply, one can perform such higher-s_0, tOPE assumption tests by shifting to analyses of e^+e^- data, where s_0 is kinematically unrestricted. According to the arguments of Ref. [26], if a 5-weight, 4-OPE-parameter tOPE fit to e^+e^- hadroproduction data is successful at some sufficiently large s_0, s_0^*, then, since NP OPE condensate and DV contributions are supposedly negligible at $s_0 = s_0^*$, they will be even more negligible at even higher s_0 and the tOPE fit results will continue to provide a good representation of the experimental spectral integrals of the analysis at $s_0 > s_0^*$. As noted above, to draw statistically meaningful conclusions, it is necessary to take all correlations into account, including those between data and the fitted OPE parameters. In Ref. [27] we carried out this test, first establishing that an acceptable 5-weight, 4-OPE-parameter tOPE fit indeed existed, for $s_0^* = 3.6$ GeV^2, and then evaluating the double differences between the LHSs and RHSs of Eq. (1), $\Delta^{(2)}_w(s_0, s_0^*) = \left[I^{th}_w - I^{exp}_w\right](s_0) - \left[I^{th}_w - I^{exp}_w\right](s_0^*)$, produced by this fit at $s_0 > s_0^*$. The right panel in Fig. 3 shows an example of the results obtained, in this case for one of the weights, $w_{25}(x) = 1 - 7x^6 + 6x^7$, which forms part of the "optimal weight" set of Ref. [9]. It is clear that, away from $s_0 = s_0^*$, $\Delta^{(2)}_w(s_0, s_0^*)$ is nowhere near zero within errors. See Ref. [27] for more details and further examples. It is clear that the tOPE assumption fails badly, and this leads us to our third message:

To establish agreement between two strongly correlated observables A and B, the result for $A - B$ should be shown to be 0 within errors obtained taking all correlations into account.

3.2 Duality Violations

Though the DV parametrization (3) is supposed to be valid up to corrections falling off at large s, the authors of Ref. [9] have performed analyses with this expression multiplied by a factor s^k, with k a *positive* integer as large as 8, and concluded that the result for $\alpha_s(m_\tau)$ is very sensitive to k and hence to the form chosen for the DV contribution to the spectral function.[10] Interestingly, the same authors made no attempt to study the effect of adding such an s^k factor in a whole series of papers analyzing the $V - A$ correlator using precisely the DV parametrization shown in Eq. (3) (see Ref. [28] and references therein). Such a factor, in any case, makes little sense given the functional dependence of the corrections to the large-s limit indicated in Eq. (3).

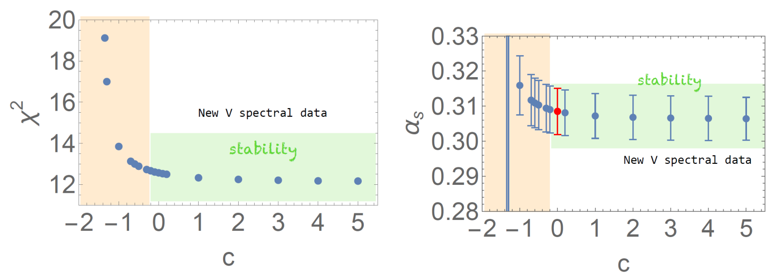


Figure 4. χ^2 (left panel) and $\alpha_s(m_\tau)$ (right panel) as functions of c (see text) from a fit to the new V-channel tau data. The red point is our result for $w_0 = 1$ in the RHS panel of Fig. 2.

A variation which does make sense is obtained by multiplying the expression (3) by a correction $(1 + c/s)$, with c a constant [11], which the authors of Ref. [26] also explored, obtaining the results $\alpha_s(m_\tau) = 0.319$ and $\alpha_s(m_\tau) = 0.260$ for the choices $c = -1.35$ and $c = -2$ GeV^2, in fits with $s_{min} = 1.55\,\mathrm{GeV}^2$ to the $w_0 = 1$, ALEPH V-spectral-function-based [4] FESR. Looking at the difference between the two central values, the authors of this reference concluded that the DV parametrization (3) induces a large systematic uncertainty in the determination of α_s. Although not quoted in Ref. [26], the errors on these results turn out to be ± 0.016 and ± 0.089, respectively. The ALEPH data employed is thus, in fact, insufficiently precise to allow such a conclusion to be drawn. Here, the new improved V channel data can help [21]. The left panel of Fig. 4 shows the χ^2 results for the same fit, now using the improved V data, as a function of the correction parameter c. One sees a very steep increase in χ^2 for negative c, in particular for the values $c = -1.35$ and $c = -2$ GeV^2 chosen in Ref. [26]. These two values, therefore, should be considered unphysical. In addition, even with the improved data, the flatness of the χ^2 at $c > 0$ means the fit is unable to determine a preferred value for c. The right panel of Fig. 4 shows the c-dependence of the $\alpha_s(m_\tau)$ obtained from this fit. Although one expects a small value of c to be most reasonable, even for large $c > 0$, where the term $(1 + c/s)$ could no longer be considered a correction, e.g., in the region $s \sim 2\,\mathrm{GeV}^2$, the result for $\alpha_s(m_\tau)$ is very insensitive to this term because its effect gets absorbed in the DV parameters. The conclusion is that the error quoted for our determination of $\alpha_s(m_\tau)$ in Eq. (4) is reliable, and already includes a sensible estimate

[10]Even though a similar spread in α_s values is found in tests of the tOPE strategy.

for the systematic error coming from the DV parametrization, contrary to the claim made in Ref. [26]. We finally conclude with our last message:

Don't ignore Duality Violations if you want to learn about their impact on α_s from FESRs.

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