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Research article

Generalized Stević-Sharma type operators from derivative Hardy spaces into Zygmund-type spaces

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Abstract: Let u, v be two analytic functions on the open unit disk $\mathbb{D}$ in the complex plane, φ an analytic self-map of $\mathbb{D}$, and m, n nonnegative integers such that $m < n$. In this paper, we consider the generalized Stević-Sharma type operator $T^{m,n}_{u,v,\varphi}f(z) = u(z)f^{(m)}(\varphi(z)) + v(z)f^{(n)}(\varphi(z))$ acting from the derivative Hardy spaces into Zygmund-type spaces, and investigate its boundedness, essential norm and compactness.

Keywords: generalized Stević-Sharma operator; derivative Hardy space; Zygmund-type space; essential norm
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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$. Denoted by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $0 < p < \infty$, Hardy space H^p, consists of all $f \in H(\mathbb{D})$ such that (see [1])

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

The derivative Hardy space, which is denoted by S^p, is the set of all $f \in H(\mathbb{D})$ whose derivative $f' \in H^p$. For $p > 1$, the space S^p, which is contained in the disk algebra, becomes a Banach space under the norm

$$\|f\|^p_{S^p} = |f(0)| + \|f'\|^p_{H^p}.$$

See [2–5] and the references therein for the study of (weighted) composition operators, which are described below, on the space S^p.

Suppose that μ is a radial weight, that is, a strictly positive continuous function on $\mathbb{D}$ which is radial (i.e., $\mu(z) = \mu(|z|)$ for any $z \in \mathbb{D}$). The Zygmund-type space $\mathcal{Z}_\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Under the norm $\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$, $\mathcal{Z}_\mu$ becomes a Banach space. When $\mu(z) = 1 - |z|^2$, the induced space $\mathcal{Z}_\mu$ reduce to the classical Zygmund space. For some results on $\mathcal{Z}_\mu$ and operators on them see for instance [6–16].

Let $\varphi \in S(\mathbb{D})$ and $\psi \in H(\mathbb{D})$, the composition and multiplication operator are defined respectively on $H(\mathbb{D})$ by

$$C_\varphi f(z) = f(\varphi(z)) \quad \text{and} \quad M_\psi f(z) = \psi(z)f(z),$$

where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The product of C_φ and M_ψ is known as the weighted composition operator $W_{\psi,\varphi} = \psi(z)f(\varphi(z))$. It is important to provide function theoretic characterizations when φ and ψ induce a bounded or compact weighted composition operator on various analytic function spaces, and one can consult [17] for more research about this topic. The differentiation operator D, which is defined by $Df(z) = f'(z)$ for $f \in H(\mathbb{D})$, plays an important role in operator theory and many other different areas of mathematics.

The first papers on product-type operators, which included the differentiation operator dealt with the products of differentiation and composition operators (see, for example, [14, 18–21]). In [22, 23], Stević et al. introduced the following so-called Stević-Sharma operator:

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $u, v \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. By taking some specific choices of the involving symbols, we can easily get the general product-type operators:

$$\begin{aligned}
&M_uC_\varphi = T_{u,0,\varphi}, \quad C_\varphi M_u = T_{u\circ\varphi,0,\varphi}, \quad M_uD = T_{0,u,id}, \quad DM_u = T_{u',u,id}, \quad C_\varphi D = T_{0,1,\varphi},\\
&DC_\varphi = T_{0,\varphi',\varphi}, \quad M_uC_\varphi D = T_{0,u,\varphi}, \quad M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad C_\varphi M_uD = T_{0,u\circ\varphi,\varphi},\\
&DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u = T_{u'\circ\varphi,u\circ\varphi,\varphi}, \quad DC_\varphi M_u = T_{\varphi'(u'\circ\varphi),\varphi'(u\circ\varphi),\varphi}.
\end{aligned}$$

There has been an increasing interest in studying the Stević-Sharma operator between various spaces of analytic function recently. For instance, Stević et al. in [22, 23] characterized the boundedness, compactness and essential norm of $T_{u,v,\varphi}$ on the weighted Bergman space under some assumptions. Liu et al. [15, 24] investigated the boundedness and compactness of $T_{u,v,\varphi}$ from Hardy space to the Bloch-type space or Zygmund-type space. Wang et al. in [25] considered the difference of two Stević-Sharma operators and studied its boundedness, compactness and order boundedness between Banach spaces of analytic functions. Some more related results can be found (see, e.g., [9, 26–29] and the references therein).

Quite recently, Abbasi et al. in [30] generalized the Stević-Sharma operator as follows

$$T^m_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f^{(m)}(\varphi(z)), \quad m \in \mathbb{N},$$

and investigated its boundedness, essential norm and compactness from Hardy space into the nth weighted-type space, which was introduced by Stević in [31] (see also [32]). In [33], Stević et al. introduced the following product-type operator:

$$(T^n_{u,v,\varphi}f)(z) = u(z)f^{(n)}(\varphi(z)) + v(z)f^{(n+1)}(\varphi(z)), \quad n \in \mathbb{N}_0,$$

and characterize the boundedness and compactness of $T^n_{u,v,\varphi}$ from a general space to Bloch-type space. Subsequently, Abbasi and Zhu et al. in [6, 16] characterized the boundedness, compactness and essential norm of $T^n_{u,v,\varphi}$ from or to Zygmund-type space. In [34], Abbasi investigated the boundedness, compactness and essential norm of $T^n_{u,v,\varphi}$ from Hardy space to nth weighted-type space. The first author et al. studied the boundedness and compactness of $T^n_{u,v,\varphi}$ from Hardy space [8] and $Q_k(p, q)$ space [35] to Zygmund-type space or Bloch-type space.

Motivated by these, now we consider the more general operator

$$T^{m,n}_{u,v,\varphi}f(z) = u(z)f^{(m)}(\varphi(z)) + v(z)f^{(n)}(\varphi(z)), \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N},$$

and without loss of generality, we can assume that $m < n$. Note that when $m = 0, n = 1$, we get the classical Stević-Sharma operator. In this paper, we mainly investigate the boundedness and essential norm of the generalized Stević-Sharma type operators $T^{m,n}_{u,v,\varphi}$ from the derivative Hardy spaces $\mathcal{S}^p$ into Zygmund-type spaces $\mathcal{Z}_\mu$. As corollaries, we give the characterizations of their compactness.

Recall that for two Banach spaces X and Y, the essential norm of a bounded linear operator $T : X \to Y$ is the distance from T to the compact operators $K : X \to Y$, namely

$$\|T\|_{e,X\to Y} = \inf \{\|T - K\|_{X\to Y} : K \text{ is compact}\}.$$

It is well-known that $\|T\|_{e,X\to Y} = 0$ if and only if $T : X \to Y$ is compact.

Throughout this paper, for nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ if there exists a positive constant C independent of X and Y such that $X \le CY$. Moreover, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2. Boundedness

In this section, we give some necessary and sufficient conditions for the generalized Stević-Sharma type operators $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ to be bounded in different cases involving m and n. For this purpose, we need the following known lemma which follows from known estimates for the point evaluation functional on the Hardy space. For example, for the second one see some lemmas in [36].

Lemma 1. *Suppose* $1 < p < \infty$ *and* $k \in \mathbb{N}$*, then*

$$\|f\|_\infty \lesssim \|f\|_{\mathcal{S}^p} \quad \textit{and} \quad |f^{(k)}(z)| \lesssim \frac{\|f\|_{\mathcal{S}^p}}{(1 - |z|^2)^{\frac{1}{p}+k-1}}$$

for each $f \in \mathcal{S}^p$.

For any $w \in \mathbb{D}$ and $j \in \mathbb{N}$, set

$$f_{j,w}(z) = \frac{(1 - |w|^2)^j}{(1 - \overline{w}z)^{\frac{1}{p}+j-1}}, \quad z \in \mathbb{D}. \tag{2.1}$$

It can be shown that $f_{j,w} \in \mathcal{S}^p$ and $\sup_{w\in\mathbb{D}} \|f_{j,w}\|_{\mathcal{S}^p} \lesssim 1$ for every $j \in \mathbb{N}$. Moreover, it is evident that $f_{j,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$.

Lemma 2. *Let $1 < p < \infty$, $m,n \in \mathbb{N}$ and $m+2 < n$. For any $w \in \mathbb{D} \setminus \{0\}$ and $i,k \in \{m, m+1, m+2, n, n+1, n+2\}$, there exists a function $g_{i,w} \in \mathcal{S}^p$ such that*

$$g_{i,w}^{(k)}(w) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^{\frac{1}{p}+k-1}},$$

where δ_{ik} is Kronecker delta.

Proof. We use here the method for constructing the test functions given in [31,36]. For any $w \in \mathbb{D} \setminus \{0\}$ and constants $c_1, c_2, c_3, c_4, c_5, c_6$, let

$$g_w(z) = \sum_{j=1}^{6} c_j f_{j,w}(z),$$

where $f_{j,w}$ is defined in (2.1). For each $i \in \{m, m+1, m+2, n, n+1, n+2\}$, the system of linear equations

$$\begin{cases} g_w^{(m)}(w) = \frac{\overline{w}^m}{(1-|w|^2)^{\frac{1}{p}+m-1}} \sum\limits_{j=1}^{6} c_j(\frac{1}{p}+j-1)(\frac{1}{p}+j)\cdots(\frac{1}{p}+j+m-2) = \frac{\overline{w}^m \delta_{im}}{(1-|w|^2)^{\frac{1}{p}+m-1}} \\ g_w^{(m+1)}(w) = \frac{\overline{w}^{m+1}}{(1-|w|^2)^{\frac{1}{p}+m}} \sum\limits_{j=1}^{6} c_j(\frac{1}{p}+j-1)(\frac{1}{p}+j)\cdots(\frac{1}{p}+j+m-1) = \frac{\overline{w}^{m+1} \delta_{i(m+1)}}{(1-|w|^2)^{\frac{1}{p}+m}} \\ g_w^{(m+2)}(w) = \frac{\overline{w}^{m+2}}{(1-|w|^2)^{\frac{1}{p}+m+1}} \sum\limits_{j=1}^{6} c_j(\frac{1}{p}+j-1)(\frac{1}{p}+j)\cdots(\frac{1}{p}+j+m) = \frac{\overline{w}^{m+2} \delta_{i(m+2)}}{(1-|w|^2)^{\frac{1}{p}+m+1}} \\ g_w^{(n)}(w) = \frac{\overline{w}^n}{(1-|w|^2)^{\frac{1}{p}+n-1}} \sum\limits_{j=1}^{6} c_j(\frac{1}{p}+j-1)(\frac{1}{p}+j)\cdots(\frac{1}{p}+j+n-2) = \frac{\overline{w}^n \delta_{in}}{(1-|w|^2)^{\frac{1}{p}+n-1}} \\ g_w^{(n+1)}(w) = \frac{\overline{w}^{n+1}}{(1-|w|^2)^{\frac{1}{p}+n}} \sum\limits_{j=1}^{6} c_j(\frac{1}{p}+j-1)(\frac{1}{p}+j)\cdots(\frac{1}{p}+j+n-1) = \frac{\overline{w}^{n+1} \delta_{i(n+1)}}{(1-|w|^2)^{\frac{1}{p}+n}} \\ g_w^{(n+2)}(w) = \frac{\overline{w}^{n+2}}{(1-|w|^2)^{\frac{1}{p}+n+1}} \sum\limits_{j=1}^{6} c_j(\frac{1}{p}+j-1)(\frac{1}{p}+j)\cdots(\frac{1}{p}+j+n) = \frac{\overline{w}^{n+2} \delta_{i(n+2)}}{(1-|w|^2)^{\frac{1}{p}+n+1}} \end{cases}$$

has a unique solution c_j^i, $j \in \{1,2,3,4,5,6\}$ that is independent of w, since the value of the determinant of coefficient matrix is

$$\begin{aligned} &2\Big(\frac{1}{p}+m-1\Big)\Big(\frac{1}{p}+m\Big)\Big(\frac{1}{p}+m+1\Big)\Big(\frac{1}{p}+m+2\Big)\Big(\frac{1}{p}+m+3\Big)^2\Big(\frac{1}{p}+m+4\Big)^3 \\ &\cdot\Big(\frac{1}{p}+m+5\Big)^3\Big(\frac{1}{p}+m+6\Big)^3\cdots\Big(\frac{1}{p}+n-1\Big)^3\Big(\frac{1}{p}+n\Big)^2\Big(\frac{1}{p}+n+1\Big) \\ &\cdot(n-m-2)(n-m-1)^2(n-m)^3(n-m+1)^2(n-m+2), \end{aligned}$$

which is not equal to zero. For such chosen numbers c_j^i, $j \in \{1,2,3,4,5,6\}$ the function

$$g_{i,w}(z) := \sum_{j=1}^{6} c_j^i f_{j,w}(z)$$

satisfies the desired conditions. □

Now, we state and prove our main results. For simplicity of the expressions, we write

$$\begin{aligned}
A_m(z) &= u''(z), \\
A_{m+1}(z) &= 2u'(z)\varphi'(z) + u(z)\varphi''(z), \\
A_{m+2}(z) &= u(z)\varphi'(z)^2, \\
A_n(z) &= v''(z), \\
A_{n+1}(z) &= 2v'(z)\varphi'(z) + v(z)\varphi''(z), \\
A_{n+2}(z) &= v(z)\varphi'(z)^2.
\end{aligned}$$

Theorem 1. *Let $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$, $m + 2 < n$, I denotes the set $\{m, m+1, m+2, n, n+1, n+2\}$ and μ be a radial weight. Then, the following statements are equivalent.*

(i) *The operator $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded.*

(ii)

$$\sum_{j=1}^{6} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty, \quad \textit{and} \quad \sum_{i\in I} \sup_{z\in\mathbb{D}} \mu(z)|A_i(z)| < \infty,$$

where $f_{j,w}$ are defined in (2.1).

(iii)

$$\sum_{i\in I} \sup_{z\in\mathbb{D}} \frac{\mu(z)|A_i(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+i-1}} < \infty.$$

Proof. (i)⇒(ii). Suppose that $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded. For each $w \in \mathbb{D}$ and $j \in \{1, 2, 3, 4, 5, 6\}$, $\|f_{j,w}\|_{\mathcal{S}^p} \lesssim 1$ and hence by the boundedness of $T^{m,n}_{u,v,\varphi}$ we have $\|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty$. Therefore,

$$\sum_{j=1}^{6} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty.$$

Taking $f_m(z) = z^m \in \mathcal{S}^p$, by the boundedness of $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ we get

$$\infty > \|T^{m,n}_{u,v,\varphi} f_m\|_{\mathcal{Z}_\mu} \geq \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_m)''(z)| = \sup_{z\in\mathbb{D}} \mu(z)|A_m(z)|m!,$$

then we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|A_m(z)| < \infty. \tag{2.2}$$

Applying the operator $T^{m,n}_{u,v,\varphi}$ to $f_{m+1}(z) = z^{m+1} \in \mathcal{S}^p$ we have

$$\begin{aligned}
\infty > \|T^{m,n}_{u,v,\varphi} f_{m+1}\|_{\mathcal{Z}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{m+1})''(z)| \\
&= \sup_{z\in\mathbb{D}} \mu(z)|A_m(z)\varphi(z)(m+1)! + A_{m+1}(z)(m+1)!| \\
&\geq \sup_{z\in\mathbb{D}} \mu(z)|A_{m+1}(z)|(m+1)! - \sup_{z\in\mathbb{D}} \mu(z)|A_m(z)\varphi(z)|(m+1)!,
\end{aligned}$$

from which along with (2.2) and the fact that $|\varphi(z)| < 1$ it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)|A_{m+1}(z)| < \infty. \tag{2.3}$$

Similarly, taking $f_{m+2}(z) = z^{m+2} \in \mathcal{S}^p$, we get

$$\begin{aligned}\infty > \|T^{m,n}_{u,v,\varphi} f_{m+2}\|_{\mathcal{Z}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{m+2})''(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)\Big|A_m(z)\varphi(z)^2\frac{(m+2)!}{2} + A_{m+1}(z)\varphi(z)(m+2)! + A_{m+2}(z)(m+2)!\Big|,\end{aligned}$$

from which along with (2.2), (2.3), the triangle inequality and the fact that $|\varphi(z)| < 1$ yields

$$\sup_{z\in\mathbb{D}} \mu(z)|A_{m+2}(z)| < \infty. \tag{2.4}$$

By using the function $f_n(z) = z^n \in \mathcal{S}^p$, we obtain

$$\begin{aligned}\infty > \|T^{m,n}_{u,v,\varphi} f_n\|_{\mathcal{Z}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_n)''(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)\Big|A_m(z)\varphi(z)^{n-m}\frac{n!}{(n-m)!} + A_{m+1}(z)\varphi(z)^{n-m-1}\frac{n!}{(n-m-1)!} \\ &\quad + A_{m+2}(z)\varphi(z)^{n-m-2}\frac{n!}{(n-m-2)!} + A_n(z)n!\Big|,\end{aligned}$$

from which along with (2.2)–(2.4), the triangle inequality and the fact that $|\varphi(z)| < 1$ it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)|A_n(z)| < \infty. \tag{2.5}$$

Taking $f_{n+1}(z) = z^{n+1} \in \mathcal{S}^p$, by the boundedness of $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ we get

$$\begin{aligned}\infty > \|T^{m,n}_{u,v,\varphi} f_{n+1}\|_{\mathcal{Z}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{n+1})''(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)\Big|A_m(z)\varphi(z)^{n-m+1}\frac{(n+1)!}{(n-m+1)!} + A_{m+1}(z)\varphi(z)^{n-m}\frac{(n+1)!}{(n-m)!} \\ &\quad + A_{m+2}(z)\varphi(z)^{n-m-1}\frac{(n+1)!}{(n-m-1)!} + A_n(z)\varphi(z)(n+1)! + A_{n+1}(z)(n+1)!\Big|,\end{aligned}$$

from which along with (2.2)–(2.5), the triangle inequality and the fact that $|\varphi(z)| < 1$ gives

$$\sup_{z\in\mathbb{D}} \mu(z)|A_{n+1}(z)| < \infty. \tag{2.6}$$

Finally, using the function $f_{n+2}(z) = z^{n+2} \in \mathcal{S}^p$ we get

$$\begin{aligned}\infty > \|T^{m,n}_{u,v,\varphi} f_{n+2}\|_{\mathcal{Z}_\mu} &\geq \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f_{n+2})''(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)\Big|A_m(z)\varphi(z)^{n-m+2}\frac{(n+2)!}{(n-m+2)!} + A_{m+1}(z)\varphi(z)^{n-m+1}\frac{(n+2)!}{(n-m+1)!}\end{aligned}$$

$$+ A_{m+2}(z)\varphi(z)^{n-m}\frac{(n+2)!}{(n-m)!} + A_n(z)\varphi(z)^2\frac{(n+2)!}{2}$$
$$+ A_{n+1}(z)\varphi(z)(n+2)! + A_{n+2}(z)(n+2)!\Big|,$$

from which along with (2.2)–(2.6), the triangle inequality and the fact that $|\varphi(z)| < 1$ it follows that

$$\sup_{z\in\mathbb{D}}\mu(z)|A_{n+2}(z)| < \infty. \tag{2.7}$$

Combining (2.2)–(2.7) we deduce that

$$\sum_{i\in I}\sup_{z\in\mathbb{D}}\mu(z)|A_i(z)| < \infty.$$

(ii)⇒(iii). Assume that (ii) holds. By Lemma 2, for each $i \in I$ and $\varphi(w) \neq 0$, there exist constants c^i_j, $j \in \{1, 2, 3, 4, 5, 6\}$ such that

$$g_{i,\varphi(w)}(z) = \sum_{j=1}^{6} c^i_j f_{j,\varphi(w)}(z) \in \mathcal{S}^p, \tag{2.8}$$

and

$$g^{(k)}_{i,\varphi(w)}(z) = \frac{\overline{\varphi(w)}^k\delta_{ik}}{(1-|\varphi(w)|^2)^{\frac{1}{p}+k-1}},$$

where $f_{j,\varphi(w)}$ are defined in (2.1) and $k \in I$. Then we have

$$\infty > \sum_{j=1}^{6}\sup_{w\in\mathbb{D}}\|T^{m,n}_{u,v,\varphi}f_{j,\varphi(w)}\|_{\mathcal{Z}_\mu} \gtrsim \sup_{w\in\mathbb{D}}\|T^{m,n}_{u,v,\varphi}g_{i,\varphi(w)}\|_{\mathcal{Z}_\mu}$$
$$\geq \mu(w)\left|(T^{m,n}_{u,v,\varphi}g_{i,\varphi(w)})''(w)\right| = \frac{\mu(w)|A_i(w)||\varphi(w)|^i}{(1-|\varphi(w)|^2)^{\frac{1}{p}+i-1}}. \tag{2.9}$$

From (2.9) and (ii), for each $i \in I$, we have

$$\sup_{w\in\mathbb{D}}\frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^{\frac{1}{p}+i-1}} \leq \sup_{|\varphi(w)|>\frac{1}{2}}\frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^{\frac{1}{p}+i-1}} + \sup_{|\varphi(w)|\leq\frac{1}{2}}\frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^{\frac{1}{p}+i-1}}$$
$$\leq 2^i\sup_{|\varphi(w)|>\frac{1}{2}}\frac{\mu(w)|A_i(w)||\varphi(w)|^i}{(1-|\varphi(w)|^2)^{\frac{1}{p}+i-1}} + \Big(\frac{4}{3}\Big)^{\frac{1}{p}+i-1}\sup_{|\varphi(w)|\leq\frac{1}{2}}\mu(w)|A_i(w)|$$
$$< \infty.$$

Therefore,

$$\sum_{i\in I}\sup_{z\in\mathbb{D}}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} < \infty.$$

(iii)⇒(i). Suppose that (iii) holds. For any $f \in \mathcal{S}^p$, by Lemma 1 we have

$$\mu(z)|(T^{m,n}_{u,v,\varphi}f)''(z)| \leq \sum_{i\in I}\mu(z)|A_i(z)||f^{(i)}(\varphi(z))| \lesssim \|f\|_{\mathcal{S}^p}\sum_{i\in I}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.$$

Moreover,

$$\begin{aligned}
&|(T^{m,n}_{u,v,\varphi}f)(0)| + |(T^{m,n}_{u,v,\varphi}f)'(0)| \\
\leq&(|u(0)| + |u'(0)|)|f^{(m)}(\varphi(0))| + |u(0)\varphi'(0)||f^{(m+1)}(\varphi(0))| \\
&+ (|v(0)| + |v'(0)|)|f^{(n)}(\varphi(0))| + |v(0)\varphi'(0)||f^{(n+1)}(\varphi(0))| \\
\lesssim&\Big(\frac{|u(0)| + |u'(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+m-1}} + \frac{|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+m}} + \frac{|v(0)| + |v'(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+n-1}} + \frac{|v(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+n}}\Big)\|f\|_{\mathcal{S}^p}.
\end{aligned}$$

Consequently, $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded. The proof is completed. □

When $m + 2 = n$, as in the proof of Lemma 2, we have for any $0 \neq w \in \mathbb{D}$ and $i, k \in \{m, m+1, n, n+1, n+2\}$, there exist constants d^i_j, $j \in \{1,2,3,4,5\}$ such that the function $h_{i,w} = \sum_{j=1}^{5} d^i_j f_{j,w}(z) \in \mathcal{S}^p$ satisfying

$$h^{(k)}_{i,w}(w) = \frac{\overline{w}^k\delta_{ik}}{(1-|w|^2)^{\frac{1}{p}+k-1}}.$$

By this and analysis similar to that in the proof of Theorem 1, we can get the following result.

Theorem 2. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$, $m + 2 = n$, I_1 *denotes the set* $\{m, m+1, n+1, n+2\}$ *and* μ *be a radial weight. Then, the following statements are equivalent.*

(i) *The operator* $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded.*

(ii)

$$\sum_{j=1}^{5}\sup_{w\in\mathbb{D}}\|T^{m,n}_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu} < \infty,$$

and

$$\sum_{i\in I_1}\sup_{z\in\mathbb{D}}\mu(z)|A_i(z)| + \sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'(z)^2 + v''(z)| < \infty,$$

where $f_{j,w}$ *are defined in* (2.1).

(iii)

$$\sum_{i\in I_1}\sup_{z\in\mathbb{D}}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} + \sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)^2 + v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n-1}} < \infty.$$

For the case $m + 1 = n$, similar to Lemma 2, for any $0 \neq w \in \mathbb{D}$ and $i, k \in \{m, n, n+1, n+2\}$, there exist constants e^i_j, $j \in \{1,2,3,4\}$ and the function $q_{i,w} = \sum_{j=1}^{4} e^i_j f_{j,w}(z) \in \mathcal{S}^p$ such that

$$q^{(k)}_{i,w}(w) = \frac{\overline{w}^k\delta_{ik}}{(1-|w|^2)^{\frac{1}{p}+k-1}},$$

which along with the same method as in the proof of Theorem 1, we obtain the following theorem.

Theorem 3. *Let $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$, $m + 1 = n$, I_2 denotes the set $\{m, n+2\}$ and μ be a radial weight. Then, the following statements are equivalent.*

(i) *The operator $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded.*

(ii)

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty,$$

and

$$\sum_{i\in I_2} \sup_{z\in\mathbb{D}} \mu(z)|A_i(z)| + \sup_{z\in\mathbb{D}} \mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)| + \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)| < \infty,$$

where $f_{j,w}$ are defined in (2.1).

(iii)

$$\sum_{i\in I_2} \sup_{z\in\mathbb{D}} \frac{\mu(z)|A_i(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+i-1}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+n-1}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+n}} < \infty.$$

For the case $m = 0$, we need to break the problem into three different cases: $n = 1$, $n = 2$ and $n > 2$. In the same manner as before we have the following theorems.

Theorem 4. *Let $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight. Then, the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded.*

(ii) *$u \in \mathcal{Z}_\mu$,*

$$\sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty,$$

and

$$\sup_{z\in\mathbb{D}} \mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)| + \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)| + \sup_{z\in\mathbb{D}} \mu(z)|v(z)||\varphi'(z)|^2 < \infty,$$

where $f_{j,w}$ are defined in (2.1).

(iii) *$u \in \mathcal{Z}_\mu$,*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z) + v''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)^2 + 2v'(z)\varphi'(z) + v(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+1}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|v(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{\frac{1}{p}+2}} < \infty.$$

Theorem 5. *Let $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight. Then, the following statements are equivalent.*

(i) *The operator $T^{0,2}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded.*

(ii) *$u \in \mathcal{Z}_\mu$,*

$$\sum_{j=1}^{5} \sup_{w\in\mathbb{D}} \|T^{0,2}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty,$$

and

$$\begin{aligned}&\sup_{z\in\mathbb{D}} \mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)| + \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)^2 + v''(z)| \\ &+ \sup_{z\in\mathbb{D}} \mu(z)|2v'(z)\varphi'(z) + v(z)\varphi''(z)| + \sup_{z\in\mathbb{D}} \mu(z)|v(z)||\varphi'(z)|^2 < \infty,\end{aligned}$$

where $f_{j,w}$ are defined in (2.1).

(iii) *$u \in \mathcal{Z}_\mu$,*

$$\begin{aligned}&\sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)^2 + v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}} \\ &+ \sup_{z\in\mathbb{D}} \frac{\mu(z)|2v'(z)\varphi'(z) + v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+2}} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|v(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{1}{p}+3}} < \infty.\end{aligned}$$

Theorem 6. *Let $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$, $n > 2$, I_3 denotes the set $\{1, 2, n, n+1, n+2\}$ and μ be a radial weight. Then, the following statements are equivalent.*

(i) *The operator $T^{0,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded.*

(ii) *$u \in \mathcal{Z}_\mu$,*

$$\sum_{j=1}^{6} \sup_{w\in\mathbb{D}} \|T^{0,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} < \infty, \quad \textit{and} \quad \sum_{i\in I_3} \sup_{z\in\mathbb{D}} \mu(z)|A_i(z)| < \infty,$$

where $f_{j,w}$ are defined in (2.1).

(iii) *$u \in \mathcal{Z}_\mu$,*

$$\sum_{i\in I_3} \sup_{z\in\mathbb{D}} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} < \infty.$$

Note that $A_1(z) = 2u'(z)\varphi'(z) + u(z)\varphi''(z)$ and $A_2(z) = u(z)\varphi'(z)^2$ in Theorem 6.

3. Essential norm

In order to estimate the essential norm of $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$, we need the following lemma, which is a direct consequence of Lemmas 3.2 and 3.3 in [37].

Lemma 3. *Let $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$ and μ be a radial weight such that the operator $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded. Then $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is compact if and only if $\|T^{m,n}_{u,v,\varphi} f_k\|_{\mathcal{Z}_\mu} \to 0$ as $k \to \infty$ for each norm-bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}^p$ which converges to zero uniformly in $\overline{\mathbb{D}}$.*

Theorem 7. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$, $m + 2 < n$, *I denotes the set* $\{m, m+1, m+2, n, n+1, n+2\}$ *and* μ *be a radial weight such that* $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded. Then*

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \approx \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} \approx \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}},$$

where $f_{j,w}$ *are defined in* (2.1).

Proof. First, we prove that

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \gtrsim \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}.$$

It is immediate that for each $j \in \{1,2,3,4,5,6\}$ and $w \in \mathbb{D}$, $\|f_{j,w}\|_{\mathcal{S}^p} \lesssim 1$. For any compact operator K from $\mathcal{S}^p$ into $\mathcal{Z}_\mu$, we have

$$\begin{aligned}\|T^{m,n}_{u,v,\varphi} - K\|_{\mathcal{S}^p\to\mathcal{Z}_\mu} &\gtrsim \limsup_{|w|\to 1} \|(T^{m,n}_{u,v,\varphi} - K) f_{j,w}\|_{\mathcal{Z}_\mu} \\ &\geq \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} - \limsup_{|w|\to 1} \|K f_{j,w}\|_{\mathcal{Z}_\mu}.\end{aligned}$$

Since $f_{j,w}$ converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$, by using some standard arguments (see, e.g., [38, 39]) we have

$$\lim_{|w|\to 1} \|K f_{j,w}\|_{\mathcal{Z}_\mu} = 0.$$

By the definition of the essential norm, we obtain

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} = \inf_K \|T^{m,n}_{u,v,\varphi} - K\|_{\mathcal{S}^p\to\mathcal{Z}_\mu} \gtrsim \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}. \quad (3.1)$$

Next, we show that

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \gtrsim \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.$$

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Since $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is bounded, for any compact operator $K : \mathcal{S}^p \to \mathcal{Z}_\mu$ and $i \in I$, applying Lemma 3 and (2.9) we get

$$\begin{aligned}\|T^{m,n}_{u,v,\varphi} - K\|_{\mathcal{S}^p\to\mathcal{Z}_\mu} &\gtrsim \limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} g_{i,\varphi(z_j)}\|_{\mathcal{Z}_\mu} - \limsup_{j\to\infty} \|K g_{i,\varphi(z_j)}\|_{\mathcal{Z}_\mu} \\ &\gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)|A_i(z_j)||\varphi(z_j)|^i}{(1-|\varphi(z_j)|^2)^{\frac{1}{p}+i-1}} \\ &= \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}},\end{aligned}$$

where $g_{i,\varphi(z_j)}$ are defined in (2.8). Consequently,

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \gtrsim \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}. \tag{3.2}$$

Combining (3.1) and (3.2), we see that it is sufficient to show

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \lesssim \min\Big\{ \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}, \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \Big\}.$$

Define $K_r f(z) = f_r(z) = f(rz)$, where $0 \le r < 1$. Then K_r is a compact operator on $\mathcal{S}^p$ with $\|K_r\| \le 1$ and $f_r \to f$ on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for every $j \in \mathbb{N}$, $T^{m,n}_{u,v,\varphi} K_{r_j} : \mathcal{S}^p \to \mathcal{Z}_\mu$ is compact, and so

$$\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \le \limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}\|_{\mathcal{S}^p\to\mathcal{Z}_\mu}.$$

Therefore, we only need to prove that

$$\begin{aligned} &\limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}\|_{\mathcal{S}^p\to\mathcal{Z}_\mu} \\ &\lesssim \min\Big\{ \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}, \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \Big\}. \end{aligned} \tag{3.3}$$

For each $f \in \mathcal{S}^p$ satisfying $\|f\|_{\mathcal{S}^p} \le 1$, we have

$$\begin{aligned} &\|(T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}) f\|_{\mathcal{Z}_\mu} \\ =&|(T^{m,n}_{u,v,\varphi} f - T^{m,n}_{u,v,\varphi} f_{r_j})(0)| + |(T^{m,n}_{u,v,\varphi} f - T^{m,n}_{u,v,\varphi} f_{r_j})'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|(T^{m,n}_{u,v,\varphi} f - T^{m,n}_{u,v,\varphi} f_{r_j})''(z)| \\ \le& \underbrace{(|u(0)| + |u'(0)|)|(f - f_{r_j})^{(m)}(\varphi(0))| + |u(0)\varphi'(0)(f - f_{r_j})^{(m+1)}(\varphi(0))|}_{\Phi_0} \\ &+ \underbrace{(|v(0)| + |v'(0)|)|(f - f_{r_j})^{(n)}(\varphi(0))| + |v(0)\varphi'(0)(f - f_{r_j})^{(n+1)}(\varphi(0))|}_{\Phi_1} \\ &+ \underbrace{\sup_{|\varphi(z)|\le r_N} \sum_{i\in I} \mu(z)|A_i(z)(f - f_{r_j})^{(i)}(\varphi(z))|}_{\Phi_2} \\ &+ \underbrace{\sup_{|\varphi(z)|> r_N} \sum_{i\in I} \mu(z)|A_i(z)(f - f_{r_j})^{(i)}(\varphi(z))|}_{\Phi_3}, \end{aligned} \tag{3.4}$$

where $N \in \mathbb{N}$ such that $r_j \ge \frac{2}{3}$ for all $j \ge N$. Furthermore, we have $(f - f_{r_j})^{(t)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$ for any $t \in \mathbb{N}_0$. Thus

$$\limsup_{j\to\infty} \Phi_0 = \limsup_{j\to\infty} \Phi_1 = \limsup_{j\to\infty} \Phi_2 = 0. \tag{3.5}$$

Finally, we estimate Φ_3. Obviously,

$$\Phi_3 \le \sum_{i\in I} \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z)|A_i(z)f^{(i)}(\varphi(z))|}_{\Psi_i} + \sum_{i\in I} \underbrace{\sup_{|\varphi(z)|>r_N} \mu(z)|A_i(z)r_j^i f^{(i)}(r_j\varphi(z))|}_{\Omega_i}. \tag{3.6}$$

For each $i \in I$, using Lemma 1, (2.8) and (2.9) we obtain

$$\begin{aligned}\Psi_i &= \sup_{|\varphi(z)|>r_N} \frac{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}|f^{(i)}(\varphi(z))|}{|\varphi(z)|^i} \frac{\mu(z)|A_i(z)||\varphi(z)|^i}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \\ &\lesssim \|f\|_{\mathcal{S}^p} \sup_{|\varphi(z)|>r_N} \|T^{m,n}_{u,v,\varphi} g_{i,\varphi(z)}\|_{\mathcal{Z}_\mu} \\ &\lesssim \sum_{j=1}^{6} \sup_{|w|>r_N} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}.\end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}\Psi_i &= \sup_{|\varphi(z)|>r_N} (1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}|f^{(i)}(\varphi(z))| \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \\ &\lesssim \|f\|_{\mathcal{S}^p} \sup_{|\varphi(z)|>r_N} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.\end{aligned} \tag{3.8}$$

Taking the limits as $N \to \infty$ in (3.7) and (3.8) we get

$$\limsup_{j\to\infty} \Psi_i \lesssim \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}, \tag{3.9}$$

and

$$\limsup_{j\to\infty} \Psi_i \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}. \tag{3.10}$$

Similarly, we have

$$\limsup_{j\to\infty} \Omega_i \lesssim \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} \quad \text{and} \quad \limsup_{j\to\infty} \Omega_i \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}. \tag{3.11}$$

Therefore, by (3.4)–(3.6), (3.9)–(3.11), we obtain

$$\begin{aligned}\limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}\|_{\mathcal{S}^p\to\mathcal{Z}_\mu} &= \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{S}^p}\le 1} \|(T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}) f\|_{\mathcal{Z}_\mu} \\ &\lesssim \sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu},\end{aligned}$$

and

$$\limsup_{j\to\infty} \|T^{m,n}_{u,v,\varphi} - T^{m,n}_{u,v,\varphi} K_{r_j}\|_{\mathcal{S}^p\to\mathcal{Z}_\mu} \lesssim \sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.$$

From the last two inequalities we get (3.3) and the proof is completed. □

Similar arguments apply to the case $m+2=n$ or $m+1=n$, which along with Theorems 2 and 3, we obtain the following results.

Theorem 8. *Let* $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m,n\in\mathbb{N}$, $m+2=n$, I_1 *denotes the set* $\{m,m+1,n+1,n+2\}$ *and* μ *be a radial weight such that* $T^{m,n}_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ *is bounded. Then*

$$\begin{aligned}\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} &\approx \sum_{j=1}^{5}\limsup_{|w|\to1}\|T^{m,n}_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu}\\ &\approx \sum_{i\in I_1}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)^2+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n-1}},\end{aligned}$$

where $f_{j,w}$ *are defined in* (2.1).

Theorem 9. *Let* $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m,n\in\mathbb{N}$, $m+1=n$, I_2 *denotes the set* $\{m,n+2\}$ *and* μ *be a radial weight such that* $T^{m,n}_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ *is bounded. Then*

$$\begin{aligned}\|T^{m,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} &\approx \sum_{j=1}^{4}\limsup_{|w|\to1}\|T^{m,n}_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu}\\ &\approx \sum_{i\in I_2}\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n-1}}\\ &\quad+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n}},\end{aligned}$$

where $f_{j,w}$ *are defined in* (2.1).

For the case $m=0$, note that every sequence in $\mathcal{S}^p$ bounded in norm has a subsequence which converges uniformly in $\overline{\mathbb{D}}$ to a function in $\mathcal{S}^p$ (see [37, Lemma 3.2]), which along with the similar arguments as in the proof of Theorem 7 yields the following theorems.

Theorem 10. *Let* $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ *and* μ *be a radial weight such that* $T_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ *is bounded. Then*

$$\begin{aligned}\|T_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} &\approx \sum_{j=1}^{4}\limsup_{|w|\to1}\|T_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu}\\ &\approx \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}}\\ &\quad+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}}\\ &\quad+\limsup_{|\varphi(z)|\to1}\frac{\mu(z)|v(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{1}{p}+2}},\end{aligned}$$

where $f_{j,w}$ *are defined in* (2.1).

Theorem 11. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ *and* μ *be a radial weight such that* $T^{0,2}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded. Then*

$$\begin{aligned}\|T^{0,2}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} &\approx \sum_{j=1}^{5} \limsup_{|w|\to 1} \|T^{0,2}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu}\\ &\approx \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z)^2 + v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}}\\ &\quad + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|2v'(z)\varphi'(z) + v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+2}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|v(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{1}{p}+3}},\end{aligned}$$

where $f_{j,w}$ *are defined in* (2.1).

Theorem 12. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$, $n > 2$, I_3 *denotes the set* $\{1, 2, n, n+1, n+2\}$ *and* μ *be a radial weight such that* $T^{0,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded. Then*

$$\|T^{0,n}_{u,v,\varphi}\|_{e,\mathcal{S}^p\to\mathcal{Z}_\mu} \approx \sum_{j=1}^{5} \limsup_{|w|\to 1} \|T^{0,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} \approx \sum_{i\in I_3} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}},$$

where $f_{j,w}$ *are defined in* (2.1).

From Theorems 7–12 and the fact that $\|T\|_{e,X\to Y} = 0$ if and only if $T : X \to Y$ is compact, we can get the following corollaries, which characterize the compactness of $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$.

Corollary 1. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$, $m + 2 < n$, I *denotes the set* $\{m, m+1, m+2, n, n+1, n+2\}$ *and* μ *be a radial weight. Suppose that* $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded, Then, the following statements are equivalent.*

(i) *The operator* $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is compact.*

(ii)

$$\sum_{j=1}^{6} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} = 0.$$

(iii)

$$\sum_{i\in I} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} = 0.$$

Corollary 2. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m, n \in \mathbb{N}$, $m + 2 = n$, I_1 *denotes the set* $\{m, m+1, n+1, n+2\}$ *and* μ *be a radial weight. Suppose that* $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded, Then, the following statements are equivalent.*

(i) *The operator* $T^{m,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is compact.*

(ii)

$$\sum_{j=1}^{5} \limsup_{|w|\to 1} \|T^{m,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} = 0.$$

(iii)

$$\sum_{i\in I_1}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}+\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)\varphi'(z)^2+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n-1}}=0.$$

Corollary 3. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $m,n\in\mathbb{N}$, $m+1=n$, I_2 denotes the set $\{m,n+2\}$ and μ be a radial weight. Suppose that $T^{m,n}_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ is bounded, Then, the following statements are equivalent.*

(i) *The operator $T^{m,n}_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{4}\limsup_{|w|\to 1}\|T^{m,n}_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu}=0.$$

(iii)

$$\sum_{i\in I_2}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}+\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n-1}}$$
$$+\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)\varphi'(z)^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+n}}=0.$$

Corollary 4. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ and μ be a radial weight. Suppose that $T_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ is bounded, Then, the following statements are equivalent.*

(i) *The operator $T_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{4}\limsup_{|w|\to 1}\|T_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu}=0.$$

(iii)

$$\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}}$$
$$+\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)\varphi'(z)^2+2v'(z)\varphi'(z)+v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}}$$
$$+\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|v(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{1}{p}+2}}=0.$$

Corollary 5. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ and μ be a radial weight. Suppose that $T^{0,2}_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ is bounded, Then, the following statements are equivalent.*

(i) *The operator $T^{0,2}_{u,v,\varphi}:\mathcal{S}^p\to\mathcal{Z}_\mu$ is compact.*

(ii)

$$\sum_{j=1}^{5}\limsup_{|w|\to 1}\|T^{0,2}_{u,v,\varphi}f_{j,w}\|_{\mathcal{Z}_\mu}=0.$$

(iii)

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z)^2+v''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|2v'(z)\varphi'(z)+v(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+2}} + \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|v(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\frac{1}{p}+3}} = 0.$$

Corollary 6. *Let* $1 < p < \infty$, $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$, $n > 2$, I_3 *denotes the set* $\{1, 2, n, n+1, n+2\}$ *and* μ *be a radial weight. Suppose that* $T^{0,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is bounded, Then, the following statements are equivalent.*

(i) *The operator* $T^{0,n}_{u,v,\varphi} : \mathcal{S}^p \to \mathcal{Z}_\mu$ *is compact.*

(ii)

$$\sum_{j=1}^{5} \limsup_{|w|\to 1} \|T^{0,n}_{u,v,\varphi} f_{j,w}\|_{\mathcal{Z}_\mu} = 0.$$

(iii)

$$\sum_{i\in I_3} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} = 0.$$

4. Conclusions

In this paper, we generalize the Stević-Sharma operator by $T^{m,n}_{u,v,\varphi} f(z) = u(z)f^{(m)}(\varphi(z)) + v(z)f^{(n)}(\varphi(z))$, where m, n are nonnegative integers such that $m < n$, and investigate the boundedness and essential norm of $T^{m,n}_{u,v,\varphi}$ acting from the derivative Hardy spaces $\mathcal{S}^p$ into Zygmund-type spaces $\mathcal{Z}_\mu$ in different cases. As an application, we also give the characterizations of their compactness.

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Conflict of interest

The authors state no conflicts of interest.

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