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Research article

A stochastic linear-quadratic optimal control problem with jumps in an infinite horizon

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Abstract: In this paper, a stochastic linear-quadratic (LQ, for short) optimal control problem with jumps in an infinite horizon is studied, where the state system is a controlled linear stochastic differential equation containing affine term driven by a one-dimensional Brownian motion and a Poisson stochastic martingale measure, and the cost functional with respect to the state process and control process is quadratic and contains cross terms. Firstly, in order to ensure the well-posedness of our stochastic optimal control of infinite horizon with jumps, the L^2-stabilizability of our control system with jump is introduced. Secondly, it is proved that the L^2-stabilizability of our control system with jump is equivalent to the non-emptiness of the admissible control set for all initial state and is also equivalent to the existence of a positive solution to some integral algebraic Riccati equation (ARE, for short). Thirdly, the equivalence of the open-loop and closed-loop solvability of our infinite horizon optimal control problem with jumps is systematically studied. The corresponding equivalence is established by the existence of a *stabilizing solution* of the associated generalized algebraic Riccati equation, which is different from the finite horizon case. Moreover, any open-loop optimal control for the initial state x admiting a closed-loop representation is obatined.

Keywords: stochastic linear quadratic optimal control; stabilizability; open-loop solvability; closed-loop solvability; algrbraic Riccati equation; stabilizing solution; closed-loop representation
Mathematics Subject Classification: 60H10, 93E24

1. Introduction

LQ optimal control is an important branch of control theory. The state equation of the control system is a linear equation, the performance index is a quadratic index and the optimal control can be given in the form of linear feedback. Getting the feedback form of optimal control is the most basic issue of LQ optimal control problems. The research on LQ optimal control problems has a long history. It can be traced back to the works of Bellman-Glicksberg-Gross [1] in 1958, and they first attempted to solve deterministic LQ optimal control problem. In 1960, Kalman [10] solved the deterministic LQ optimal control problem in the form of linear state feedback, and introduced Riccati equation into the control theory. The above-mentioned works were concerned with deterministic cases, i.e., the state equation is a linear ordinary differential equation (ODE, for short), and all the involved functions are deterministic. Compared with LQ optimal control problems, the control system of stochastic LQ optimal control probelms is stochastic. In 1962, Kushner [11] first study a stochastic LQ optimal control probelm driven by Brownian motion with stochastic differential equation of Itô-type by dynamic programming. In 1968, Wonham [21, 22] first extended the deterministic LQ optimal control problem to a stochastic LQ optimal control problem that contained Riccati differential equation, followed by several researchers (see, for example, Davis [27] and Bensoussan [28]). Kohlmann and Zhou [29] discussed the relationship between a stochastic control problem and a backward stochastic differential equation (BSDE, for short). Based on [29], Lim and Zhou [12] firstly solved a general LQ optimal control problem of BSDE and gave an explicit form of optimal control. Li, Sun and Yong [17] studied the open-loop and closed-loop solvability of stochastic LQ optimal control.

In classical optimal control problems, the termination time is a real number. But among many dynamic optimization problems in economics, the termination time of the optimal control problem might be infinite. We call the mentioned case the optimal control problems with infinite horizon. In 1974, Halkin [6] introduced the necessary conditions for optimal control problems with infinite horizon. In 2000, Rami, Zhou, and Moore [15] discussed well-posedness and attainability of indefinite stochastic linear-quadratic control problems over an infinite time horizon. In 2003, Wu and Li [23] studied an infinite horizon LQ problem with unbounded controls in a Hilbert space. In 2005, Guatteri and Tessitore [3] studied the backward stochastic Riccati equation in infinite dimensions, and in 2008, Guatteri and Tessitore [4] studied LQ optimal control problems with stochastic coefficients over an infinite horizon. In 2009, Guatteri and Masiero [5] discussed ergodic optimal quadratic control problems for an affine equation with stochastic coefficients over an infinite horizon. Then, Hu [7] studied the optimal quadratic control for an affine equation driven by Lévy processes over an infinite horizon in 2013. In 2015, Huang, Li and Yong [9] discussed a LQ optimal control problem for mean-field stochastic differential equations over an infinite horizon. In 2016, Sun-Li-Yong [17] put forward the concepts of open-loop and closed-loop solvabilities, and it was shown that the closed-loop solvability is equivalent to the existence of a regular solution of the Riccati equation. Different from finite-horizon, Sun-Yong [18] found that for infinite-horizon LQ optimal control problems, both the open-loop and closed-loop solvabilities are equivalent to the existence of a *static stabilizing solution* to the associated generalized ARE. Also, every open-loop optimal control admits a closed-loop representation.

In fact, many random phenomena presented discontinuous motion characteristics with jumps. Therefore, the Poisson jump process which describes such discontinuous random phenomena came into. The following are the research status of the system with jumps. In [2], Boel Varaiya et al. discussed the optimal control problem of processes with jumps for the first time. In 1994, Tang and Li [20] first proved the necessary condition of stochastic optimal control with jumps and first discussed the BSDE with a Poisson process. In 1997, Situ [16] made a further research on the solutions of BSDE with jumps. In 2003, Wu and Wang [24] studied the stochastic LQ optimal control problem with state equation driven by Brownian motion and Poisson jump process, and obtained the existence and uniqueness result of solutions of the deterministic Riccati equation. In 2008, Hu and Øksendal [8] discussed the partial information linear quadratic control for jump diffusions. Then in 2009, Oksendal and Sulem [14] studied the maximum principles for optimal control of forward-backward stochastic differential equations with jumps. In 2014, Meng [13] discussed the existence and uniqueness of solutions to the backward stochastic Riccati equation with jumps. In 2018, Li-Wu-Yu [30] studied the stochastic LQ optimal control problem with Poisson processes under the indefinite case.

Different from the research on the optimal control problem of the jump diffusion system mentioned above, in this paper we will generalize the results of the infinite horizon stochastic LQ problem of Sun-Yong [18] to the jump diffusion system and our goal is to establish the corresponding stability theory and the optimal state feedback representing of the optimal control for jump diffusion system. Firstly, we introduce the concept of L^2-stabilizability of a jump-diffusion system over an infinite horizon, and find that the existence of the admissible control set is equivalent to the L^2-stabilizability of a jump-diffusion system over an infinite horizon. Secondly, we introduce the definition of the ARE of a jump-diffusion system over an infinite horizon and prove the equivalent among the existence of the admissible control set, the L^2-stabilizability of the system and the positive solvalibity of the ARE. Thirdly, we give the concepts of the open-loop and closed-loop solvability of the stochastic LQ optimal control problem with jumps over an infinite horizon. We then introduce the concept of the stabilizing solution of the associated generalized ARE and find that both the open-loop and closed-loop solvability of the problem are equivalent to the existence of a stabilizing solution of the associated generalized ARE, which is different from the finite horizon case. Finally, we find that any open-loop optimal control for the initial state x admits a closed-loop representation. In addition, Our results are generalizations of a recently published paper, Sun and Yong [18], on the similar topic to jump diffusion systems. The state equation in our case is a stochastic differential equation driven by a one-dimensional standard Brownian motion and a Poisson stochastic martingale measure, which is more general. It is well-known that processes arising by Poisson random measures play an increasing role in modeling stochastic dynamical systems. And it helps us deal with unexpected situations in financial problems, so it is necessary to use a jump system to characterize.

The paper is organized as follows. In Section 2, we introduce some basic notions and lemmas used throughout this paper. In Section 3, we state the problem. Section 4 aims to give the definition of L^2-stable and L^2-stabilizable, further we describe the structure of admissible control set and prove the equivalence of the non-emptiness of the admissible control set, the L^2-stabilizability of the control system and the existence of a positive solution to an algebraic Riccati equation. In Section 5, we introduce the notions of open-loop and closed-loop solvabilities as well as the algebraic Riccati equation, and finally we state the main result of the paper.

2. Preliminaries

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space on which a one-dimensional standard Brownian motion $W = \{W(t); t \geq 0\}$ is defined. Denote by $\mathcal{P}$ the $\mathcal{F}_t$-predictable σ-field on $[0, \infty) \times \Omega$ and by $\mathcal{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ. Let $(Z, \mathcal{B}(Z), \nu)$ be a measurable space with $\nu(Z) < \infty$ and $\eta : \Omega \times D_\eta \to Z$ be an $\mathcal{F}_t$-adapted stationary Poisson point process with characteristic measure ν, where D_η is a countable subset of $(0, \infty)$. Then the counting measure induced by η is

$$\mu((0,t]\times A) := \#\{s\in D_\eta; s\le t, \eta(s)\in A\},\quad for\ \ t>0, A\in\mathcal{B}(Z).$$

And $\tilde{\mu}(dt,d\theta) = \mu(dt,d\theta) - dt\nu(d\theta)$ is a compensated Poisson random martingle measure which is assumed to be independent of the Brownian motion W. In the following, we introduce the basic notations used throughout this paper.

2.1. Notations

- $\mathbb{H}$: The Hilbert space with norm $\|\cdot\|_{\mathbb{H}}$.
- $\mathbb{R}^n$: The n-dimensional Euclidean space.
- $\mathbb{R}^{n\times m}$: The space of all $(n\times m)$ matrices.
- $\langle\alpha,\beta\rangle$: The inner product in $\mathbb{R}^n$, $\forall\alpha,\ \beta\in\mathbb{R}^n$.
- $|\alpha| = \sqrt{\langle\alpha,\alpha\rangle}$: The norm of α, $\forall\alpha\in\mathbb{R}^n$.
- $M^\top$: The transpose of matrix M.
- $M^\dagger$: The Moore-Penrose pseudoinverse of a matrix M.
- $\langle M,N\rangle = tr(M^\top N)$: The inner product in $\mathbb{R}^{n\times m}$, $\forall M,\ N\in\mathbb{R}^{n\times m}$.
- $|M| = \sqrt{tr(M^\top M)}$: The norm of M, $\forall M\in\mathbb{R}^{n\times m}$.
- $\mathscr{R}(M)$: The range of a matrix or an operator M.
- $\mathbb{S}^n\in\mathbb{R}^{n\times n}$: The set of all $(n\times n)$ symmetric matrices.
- $\mathbb{S}^n_+\in\mathbb{S}^n$: The set of all $(n\times n)$ non-negative definite symmetric matrices.
- $\mathbb{F}=\{\mathcal{F}_t\}_{t\ge0}$: The natural filtration. In other words, the flow of information generated by all market noises.
- $(\Omega,\mathcal{F},P)$: A complete probability space.
- $(\Omega,\mathcal{F},\mathbb{F},P)$: The complete filtered probability space.
- $C([0,\infty);\mathbb{R})$: The space of all continuous functions $\varphi:[0,\infty)\to\mathbb{R}$.
- $L^2_{\mathbb{F}}(\mathbb{H})$: The space of all $\mathbb{H}$-valued and $\mathbb{F}$-progressively measurable processes $g(\cdot)$ satisfying $g:[0,\infty)\times\Omega\to\mathbb{H}$ and

$$\mathbb{E}\int_0^\infty \|g(t)\|^2_{\mathbb{H}}dt<\infty.$$

- $\mathcal{X}_{loc}[0,\infty)$: The space of all $\mathcal{F}_t$-adapted and càdlàg processes $g(\cdot)$ satisfying $g:[0,\infty)\times\Omega\to\mathbb{H}$ and

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|g(s)\|^2_{\mathbb{H}}\Big)<\infty$$

for every $T>0$.
- $\mathcal{X}[0,\infty)$: The space of all $\mathcal{F}_t$-adapted and càdlàg processes $g(\cdot)$ satisfying $g\in\mathcal{X}_{loc}[0,\infty)$ and

$$\mathbb{E}\int_0^\infty \|g(t)\|^2_{\mathbb{H}}dt<\infty.$$

- $L^{\nu,2}(Z;\mathbb{H})$: The space of all $\mathbb{H}$-valued measurable function $r=\{r(\theta),\ \theta\in Z\}$ defined on the measurable space $(Z,\mathcal{B}(E);\nu)$ satisfying

$$\int_z \|r(\theta)\|^2_{\mathbb{H}}\nu(d\theta)<\infty.$$

- $L_{\mathbb{F}}^{\nu,2}([0,\infty)\times Z;\mathbb{H})$: The space of all $L^{\nu,2}(\mathbb{H})$-valued and $\mathcal{F}_t$ - predictable processes $r = \{r(t,\omega,\theta),\ (t,\omega,\theta)\in[0,\infty)\times\Omega\times Z\}$ satisfying

$$\mathbb{E}\int_0^\infty \|r(s,\cdot)\|_{\mathbb{H}}^2 ds < \infty.$$

2.2. Pseudoinverse

We recall some properties of the reference [19].

Lemma 2.1. *For any $M\in\mathbb{R}^{m\times n}$, there exists a unique matrix $M^\dagger\in\mathbb{R}^{n\times m}$ such that*

$$MM^\dagger M = M,\quad (MM^\dagger)^\top = MM^\dagger,\quad M^\dagger MM^\dagger = M^\dagger,\quad (M^\dagger M)^\top = M^\dagger M.$$

In addition, if $M\in\mathbb{S}^n$, then $M^\dagger\in\mathbb{S}^n$, $MM^\dagger = M^\dagger M$, and $M\geq 0$ if and only if $M^\dagger\geq 0$.

Lemma 2.2. *Let $L\in\mathbb{R}^{n\times k}$ and $N\in\mathbb{R}^{n\times m}$. The matrix equation NX=L has a solution if and only if*

$$\mathscr{R}(L)\subseteq\mathscr{R}(N), \tag{2.1}$$

in which case the general solution is given by

$$X = N^\dagger L + (I - N^\dagger N)Y, \tag{2.2}$$

while $Y\in\mathbb{R}^{m\times k}$ is arbitrary.

The matrix $M^\dagger$ above is called the *Moore-Penrose pseudoinverse* of M.

Remark 2.1. (i) Clearly, condition (2.1) is equivalent to $NN^\dagger L = L$.
(ii) By Lemma 2.2, if $N\in\mathbb{S}^n$, and $NX = L$, then $X^\top NX = L^\top N^\dagger L$.

Lemma 2.3. (Extended Schur's lemma) *Let $L\in\mathbb{R}^{n\times m}$, $M\in\mathbb{S}^n$, and $N\in\mathbb{S}^m$. The following conditions are equivalent:*
(i) $M - LN^\dagger L^\top\geq 0$, $N\geq 0$, and $\mathscr{R}(L^\top)\subseteq\mathscr{R}(N)$;
(ii) $\begin{pmatrix} M & L \\ L^\top & N\end{pmatrix}\geq 0$.

3. Formulation of problem

Now we consider a controlled linear stochastic system on the infinite horizon $[0,\infty)$, the state equation is as follows:

$$\begin{cases} dX(t) = [AX(t) + Bu(t) + b(t)]dt + [CX(t) + Du(t) + \sigma(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta)X(t-) + F(\theta)u(t) + h(t,\theta)]\tilde{\mu}(dt,d\theta),\quad t\in[0,\infty), \\ X(0) = x\in\mathbb{R}^n, \end{cases} \tag{3.1}$$

and the quadratic cost functional is given by:

$$\begin{aligned} J(x;u) \triangleq & \mathbb{E}\int_0^\infty \Big[\langle QX(t),X(t)\rangle + 2\langle SX(t),u(t)\rangle + \langle Ru(t),u(t)\rangle \\ & \quad + 2\langle q(t),X(t)\rangle + 2\langle \rho(t),u(t)\rangle\Big]dt \\ = & \mathbb{E}\int_0^\infty \left[\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}\right\rangle + 2\left\langle \begin{pmatrix} q(t) \\ \rho(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}\right\rangle\right]dt, \end{aligned} \tag{3.2}$$

where $A, C \in \mathbb{R}^{n\times n}$; $E \in L^{\nu,2}(Z;\mathbb{R}^{n\times n})$; $B, D, \in \mathbb{R}^{n\times m}$; $F \in L^{\nu,2}(Z;\mathbb{R}^{n\times m})$; $Q \in \mathbb{S}^n$; $S \in \mathbb{R}^{m\times n}$; $R \in \mathbb{S}^m$ are given constant matrices, and $b(\cdot), \sigma(\cdot), q(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^n); h(\cdot,\cdot) \in L^{\nu,2}_{\mathbb{F}}([0,\infty)\times Z;\mathbb{R}^n); \rho(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ are given vector-valued $\mathcal{F}_t$-measurable processes. In the above, $u(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called the *control process*, x, which belongs to $\mathbb{R}^n$ is called the *initial state*, and the $X(\cdot) \equiv X(\cdot; x, u(\cdot)) \in \mathbb{R}^n$, which is the solution of SDE (3.1) is called the *state process* corresponding to the initial state x and the control $u(\cdot)$.

Different from the finite case, the solution $X(\cdot) \equiv X(\cdot; x, u(\cdot))$ of (3.1) might not always be square-integrable for $(x, u(\cdot)) \in \mathbb{R}^n \times L^2_{\mathbb{F}}(\mathbb{R}^m)$ over an infinite time horizon. To make sure the cost functional $J(x;u)$ is well defined, we introduce:

$$\mathscr{U} \triangleq \left\{u \in L^2_{\mathbb{F}}(\mathbb{R}^m) \mid \mathbb{E}\int_0^\infty |X(t;x,u)|^2 dt < \infty\right\}, \qquad x \in \mathbb{R}^n.$$

The element $u \in \mathscr{U}$ is called an *admissible control* associated with x, and the linear quadratic optimal control problem over an infinite time horizon now can be stated as follows.

Problem 3.1. *For any given initial state $x \in \mathbb{R}^n$, find an admissible control $u^* \in \mathscr{U}$ such that*

$$J(x;u^*) = \inf_{u\in\mathscr{U}} J(x;u) = V(x). \tag{3.3}$$

$u^* \in \mathscr{U}$ is called an *open-loop optimal control* of Problem $(\mathrm{SLQ})_\infty$ for the initial state x if it satisfies (3.3), and the corresponding state process $X^*(\cdot) \equiv X(\cdot; x, u^*)$ is called an *optimal state process*. The function $V(\cdot)$ is called the *value function* of Problem $(\mathrm{SLQ})_\infty$. A special case when $b, \sigma, h, q, \rho = 0$, we use $(\mathrm{SLQ})^0_\infty$, $J^0(x;u)$, and $V^0(x)$ to denote the Problem, the cost functional, and the value function corresponding to Problem 3.1.

The following assumptions on the coefficients will be in force throughout this paper.

Assumption 3.1. The coefficients of the state equation satisfy the following: $A, C \in \mathbb{R}^{n\times n}$; $B, D \in \mathbb{R}^{n\times m}$ are given constant matrices; $E(\cdot) \in L^{\nu,2}(Z;\mathbb{R}^{n\times n})$; $F(\cdot) \in L^{\nu,2}(Z;\mathbb{R}^{n\times m})$ are given deterministic matrix-valued function, $b(\cdot), \sigma(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^n); h(\cdot,\cdot) \in L^{\nu,2}_{\mathbb{F}}([0,\infty)\times Z;\mathbb{R}^n)$ are given vector-valued $\mathcal{F}_t$-measurable processes.

Assumption 3.2. The coefficients of the cost functional satisfy the following: $Q \in \mathbb{S}^n$; $S \in \mathbb{R}^{m\times n}$; $R \in \mathbb{S}^m$ are given constant matrices, $q(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^n), \rho(\cdot) \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ are given vector-valued $\mathcal{F}_t$-measurable processes.

Assumption 3.3. For all $\theta \in Z$, there exists a constant $\lambda > 0$ such that

$$I + E(\theta) \geq \lambda I.$$

As we mentioned before, admissible controls might not exist. To solve this question, the concept of *stability* is introduced as follows. Let us consider the following uncontrolled linear system:

$$dX(t) = AX(t)dt + CX(t)dW(t) + \int_Z E(\theta)X(t-)\widetilde{\mu}(dt, d\theta), \quad t \geq 0,$$

and we use $[A, C, E]$ to denote this system.

Definition 3.1. System $[A, C, E]$ is said to be L^2-*stable* if for any initial state $x \in \mathbb{R}^n$, its solution $X(\cdot; x)$ satisfies

$$\mathbb{E}\int_0^\infty |X(t; x)|^2 dt < \infty, \quad \forall x \in \mathbb{R}^n,$$

i.e., $X(\cdot; x) \in \mathcal{X}[0, \infty)$.

Consider the following BSDE over an infinite horizon $[0,\infty)$:

$$dY(t) = -[A^\top Y(t) + C^\top Z(t) + \int_Z E(\theta)^\top r(t, \theta)\nu(d\theta) + \varphi(t)]dt + Z(t)dW(t) + \int_Z r(t, \theta)\tilde{\mu}(dt, d\theta), \quad (3.4)$$

where Assumption 3.1 holds, and $\{\varphi(t); 0 \leq t < \infty\}$ is a given $\mathbb{F}$-progressively measurable, $\mathbb{R}^n$-valued process. We call the solution $(Y(\cdot), Z(\cdot), r(\cdot, \cdot))$ the adjoint processes corresponding to $X(\cdot)$.

Definition 3.2. If $(Y(\cdot), Z(\cdot), r(\cdot, \cdot)) \in \mathcal{X}[0, \infty)\times L^2_{\mathbb{F}}(\mathbb{R}^n)\times L^{\nu,2}_{\mathbb{F}}([0, \infty)\times Z; \mathbb{R}^n)$ satisfies the integral version of (3.4):

$$\begin{aligned} Y(t) = Y(0) - &\int_0^t [A^\top Y(s) + C^\top Z(s) + \int_Z E(\theta)^\top r(s, \theta)\nu(d\theta) + \varphi(s)]ds \\ &+ \int_0^t Z(s)dW(s) + \int_0^t \int_Z r(s, \theta)\tilde{\mu}(ds, d\theta), \quad t \geq 0, \quad a.s, \end{aligned} \quad (3.5)$$

then we call it an L^2-stable adapted solution of (3.4).

Lemma 3.2. *Let Assumptions 3.1–3.3 be satisfied. Suppose that [A,C,E] is L^2-stable, then for any $\varphi \in L^2_{\mathbb{F}}(\mathbb{R}^n)$, Eq* (3.4) *admits a unique L^2-stable adapted solution (Y,Z,r).*

The proof is similar to proof of Theorem A.2.2. of [19].

Lemma 3.3. *Under Assumptions 3.1–3.3, the strong solution $X(\cdot)$ of the SDE* (3.1) *with $u(\cdot) = 0$ has the explicit expression:*

$$\begin{aligned} X(t) = &\Lambda(t)\Lambda(0)^{-1}x \\ &+ \Lambda(t)\int_0^t \Lambda(s)^{-1}\left[b(s) - C\sigma(s) - \int_Z E(\theta)[I + E(\theta)]^{-1}h(s, \theta)\nu(d\theta)\right]ds \\ &+ \Lambda(t)\int_0^t \Lambda(s)^{-1}\sigma(s)dW(s) \\ &+ \Lambda(t)\int_0^t \int_Z \Lambda(s-)^{-1}[I + E(\theta)]^{-1}h(s, \theta)\tilde{\mu}(ds, d\theta), \end{aligned} \quad (3.6)$$

where $\Lambda(\cdot)$ is the unique solution of the following matrix-valued SDE:

$$\begin{cases} d\Lambda(t) = A\Lambda(t)dt + C\Lambda(t)dW(t) + \int_Z E(\theta)\Lambda(t-)\tilde{\mu}(dt,d\theta), \quad t \geq 0, \\ \Lambda(0) = I_n, \end{cases} \tag{3.7}$$

and $\Lambda(t)^{-1}$ exists, satisfying

$$\begin{cases} d\Lambda(t)^{-1} = \Lambda(t)^{-1}\left[-A + C^2 + \int_Z E^2(\theta)[I + E(\theta)]^{-1}\nu(d\theta)\right]dt \\ \qquad\qquad - \Lambda(t)^{-1}CdW(t) - \int_Z \Lambda(t-)^{-1}E(\theta)[I + E(\theta)]^{-1}\tilde{\mu}(dt,d\theta), \\ \Lambda(0)^{-1} = I_n. \end{cases} \tag{3.8}$$

The proof is similar to the proof of Theorem 6.14 of [26] .

4. Admissible control sets and stabilizability

4.1. Stability

Regarding L^2-*stability* of the system $[A, C, E]$ the following result holds:

Theorem 4.1. *Let Assumptions 3.1–3.3 be satisfied. The system [A,C,E] is L^2-stable if and only if there exists a $K \in \mathbb{S}^n_+$ such that*

$$KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta) < 0. \tag{4.1}$$

In this case, the Lyapunov equation

$$KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta) + \Psi = 0$$

admits a unique solution $K \in \mathbb{S}^n$ for any $\Psi \in \mathbb{S}^n$, which is given by

$$K = \mathbb{E}\Big[\int_0^\infty \Lambda(t)^\top \Psi \Lambda(t)dt\Big],$$

where Λ is the solution of (3.7).

Proof. Necessity. Consider the following linear ODE on $[0,\infty)$ for any fixed $\Psi \in \mathbb{S}^n$:

$$\begin{cases} \dot{\Phi}(t) = \Phi(t)A + A^\top\Phi(t) + C^\top\Phi(t)C + \int_Z E(\theta)^\top\Phi(t)E(\theta)\nu(d\theta) + \Psi; \\ \Phi(0) = 0. \end{cases} \tag{4.2}$$

It is clear that (4.2) is uniquely solvable on $[0,\infty)$. And to the solution $\Phi(t)$, we define a function

$$\Phi_\tau(s) = \Phi(\tau - s), \quad s \in [0, \tau],$$

which is a solution of the following equation

$$\begin{cases} \dot{\Phi}_\tau(s) + \Phi_\tau(s)A + A^\top\Phi_\tau(s) + C^\top\Phi_\tau(s)C + \int_Z E(\theta)^\top\Phi_\tau(s)E(\theta)\nu(d\theta) + \Psi = 0; \\ \Phi_\tau(\tau) = 0 \end{cases}$$

on the interval $[0,\tau]$ for any fixed $\tau > 0$. Clearly, we have $X(s) = \Lambda(s)x$, and $X(\cdot)$ is the solution of $[A, C, E]$ with initial state x. Applying Itô's formula to $s \mapsto \langle\Phi_\tau(s)X(s), X(s)\rangle$, we have

$$\begin{aligned} -\langle\Phi_\tau(0)x, x\rangle =& \mathbb{E}[\langle\Phi_\tau(\tau)X(\tau), X(\tau)\rangle - \langle\Phi_\tau(0)x, x\rangle] \\ =& \mathbb{E}\int_0^\tau \langle(\dot{\Phi}_\tau + \Phi_\tau A + A^\top\Phi_\tau + C^\top\Phi_\tau C + \int_Z E(\theta)^\top\Phi_\tau E(\theta)\nu(d\theta))X, X\rangle(s)ds \\ =& -\mathbb{E}\int_0^\tau \langle\Psi X(s), X(s)\rangle ds \\ =& -x^\top\Big[\mathbb{E}\int_0^\tau \Lambda(s)^\top\Psi\Lambda(s)ds\Big]x. \end{aligned}$$

Thus,

$$\Phi(\tau) = \Phi_\tau(0) = \mathbb{E}\int_0^\tau \Lambda(s)^\top\Psi\Lambda(s)ds, \quad \tau \geq 0.$$

Since the system $[A, C, E]$ is L^2-stable, it is easy to check that

$$\lim_{\tau\mapsto\infty} \Phi(\tau) = \mathbb{E}\int_0^\infty \Lambda(s)^\top\Psi\Lambda(s)ds \equiv K.$$

Because $\Phi(t)$ is the solution to (4.2), we have for any $t > 0$,

$$\begin{aligned} \Phi(t+1) - \Phi(t) =& \Big(\int_t^{t+1} \Phi(s)ds\Big)A + A^\top\Big(\int_t^{t+1} \Phi(s)ds\Big) + C^\top\Big(\int_t^{t+1} \Phi(s)ds\Big)C \\ &+ \int_Z E(\theta)^\top\Big(\int_t^{t+1} \Phi(s)ds\Big)E(\theta)\nu(d\theta) + \Psi. \end{aligned}$$

Letting $t \to \infty$, we have

$$KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta) + \Psi = 0.$$

What's more, (4.1) holds when we take $\Psi = I_n$, and the corresponding $K \in \mathbb{S}^n_+$.

Sufficiency. Now we suppose that $X(\cdot) \equiv X(\cdot; x)$ is a solution of $[A, C, E]$ with initial state x, and suppose $K \in \mathbb{S}^n_+$ satisfies (4.1). Applying Itô's formula to $s \mapsto \langle KX(s), X(s)\rangle$, we have for any $t > 0$,

$$\mathbb{E}\langle KX(t), X(t)\rangle - \langle Kx, x\rangle = \mathbb{E}\int_0^t \Big\langle\Big[KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta)\Big]X(s), X(s)\Big\rangle ds.$$

Let $\lambda > 0$ be the smallest eigenvalue of $-(KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta))$. We get that

$$\begin{aligned} \lambda\mathbb{E}\int_0^t |X(s)|^2 ds \leq& -\mathbb{E}\int_0^t \Big\langle\Big[KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta)\Big]X(s).X(s)\Big\rangle ds \\ =& \langle Kx, x\rangle - \mathbb{E}\big\langle KX(t), X(t)\big\rangle \\ \leq& \langle Kx, x\rangle, \quad \forall t > 0, \end{aligned}$$

which implies the L^2-stability of $[A, C, E]$. □

With regard to the nonhomogeneous system:

$$dX(t) = [AX(t) + \varphi(t)]dt + [CX(t) + \rho(t)]dW(t) + \int_Z \Big[E(\theta)X(t-) + \omega(t,\theta)\Big]\widetilde{\mu}(dt, d\theta), \quad t \geq 0, \tag{4.3}$$

we have the following result.

Proposition 4.2. *Let Assumptions 3.1–3.3 hold. If $[A, C, E]$ is L^2-stable, then the solution $X(\cdot) \equiv X(\cdot; x, \varphi, \rho, \omega)$ of (4.3) is in $\mathcal{X}[0,\infty)$ for any $\varphi, \rho \in L^2_{\mathbb{F}}(\mathbb{R}^n)$, $\omega \in L^{\nu,2}_{\mathbb{F}}([0,\infty)\times Z; \mathbb{R}^n)$, and any initial state $x \in \mathbb{R}^n$. Moreover, there exists a positive constant M, which is independent of x, φ, ρ and ω, such that*

$$\mathbb{E}\int_0^\infty |X(t)|^2 dt \leq M\Big\{|x|^2 + \mathbb{E}\int_0^\infty \Big[|\varphi(t)|^2 + |\rho(t)|^2 + \int_Z |\omega(t,\theta)|^2 \nu(d\theta)\Big]dt\Big\}.$$

Proof. Since $[A, C, E]$ is L^2-stable, there exists a $K \in \mathbb{S}^n_+$ such that

$$KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta) + I_n = 0$$

because of Theorem 4.1. Applying Itô's formula to $s \mapsto \langle KX(s), X(s)\rangle$, we have

$$\begin{aligned}
&\mathbb{E}\langle KX(t), X(t)\rangle - \langle Kx, x\rangle \\
=&\mathbb{E}\int_0^t \Big[\Big\langle\Big(KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta)\Big)X(s), X(s)\Big\rangle \\
&\quad + 2\Big\langle K\varphi(s) + C^\top K\rho(s) + \int_Z E(\theta)^\top K\omega(s,\theta)\nu(d\theta), X(s)\Big\rangle \\
&\quad + \Big\langle K\rho(s), \rho(s)\Big\rangle + \int_Z \Big\langle K\omega(s,\theta), \omega(s,\theta)\Big\rangle\nu(d\theta)\Big]ds \\
=&\mathbb{E}\int_0^t \Big[- |X(s)|^2 + 2\Big\langle K\varphi(s) + C^\top K\rho(s) + \int_Z E(\theta)^\top K\omega(s,\theta)\nu(d\theta), X(s)\Big\rangle \\
&\quad + \Big\langle K\rho(s), \rho(s)\Big\rangle + \int_Z \Big\langle K\omega(s,\theta), \omega(s,\theta)\Big\rangle\nu(d\theta)\Big]ds
\end{aligned}$$

for all $t > 0$. We define

$$\begin{aligned}
\chi(s) =&K\varphi(s) + C^\top K\rho(s) + \int_Z E(\theta)^\top K\omega(s,\theta)\nu(d\theta), \\
\gamma(s) =&\Big\langle K\rho(s), \rho(s)\Big\rangle + \int_Z \Big\langle K\omega(s,\theta), \omega(s,\theta)\Big\rangle\nu(d\theta); \quad s > 0.
\end{aligned}$$

According to Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\lambda\mathbb{E}|X(t)|^2 \leq&\mathbb{E}\langle KX(t), X(t)\rangle \\
\leq&\langle Kx, x\rangle + \mathbb{E}\int_0^t \Big[- |X(s)|^2 + 2\langle\chi(s), X(s)\rangle + \gamma(s)\Big]ds \\
\leq&\langle Kx, x\rangle + \mathbb{E}\int_0^t \Big[- \frac{1}{2}|X(s)|^2 + 2|\chi(s)|^2 + \gamma(s)\Big]ds \\
=&\langle Kx, x\rangle + \int_0^t \Big[- \frac{1}{2}\mathbb{E}|X(s)|^2 + 2\mathbb{E}|\chi(s)|^2 + \mathbb{E}\gamma(s)\Big]ds,
\end{aligned}$$

where the $\lambda > 0$ is the smallest eigenvalue of K. Finally, by Gronwall's inequality, we get that

$$\lambda \mathbb{E}|X(t)|^2 \leq \langle Kx, x\rangle e^{-(2\lambda)^{-1}t} + \int_0^t e^{-(2\lambda)^{-1}(t-s)}\Big[2\mathbb{E}|\chi(s)|^2 + \mathbb{E}\gamma(s)\Big]ds.$$

In summary, the conclusion of the integrability of $\mathbb{E}|X(t)|^2$ over $[0,\infty)$ can be obtained together with Young's inequality. □

4.2. Stabilizability

According to Proposition 4.2, the admissible control set $\mathscr{U}$ is nonempty (actually it equals to $L^2_{\mathbb{F}}(\mathbb{R}^n)$) for all $x \in \mathbb{R}^n$ if the system $[A, C, E]$ is L^2-stable. Now we introduce the concept of *stabilizability* to characterize the admissible control set. Denote by $[A, C, E; B, D, F]$ the following controlled linear system:

$$dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t) + \int_Z [E(\theta)X(t-) + F(\theta)u(t)]\tilde{\mu}(dt, d\theta), \quad t \geq 0.$$

Definition 4.1. $[A, C, E; B, D, F]$ is said to be L^2-*stabilizability* if there exists a matrix $\Phi \in \mathbb{R}^{m\times n}$ such that $[A + B\Phi, C + D\Phi, E + F\Phi]$ is L^2-stable, and we call Φ a *stabilizer* of $[A, C, E; B, D, F]$. $\mathscr{T} \equiv \mathscr{T}[A, C, E; B, D, F]$ is the set of all stabilizers of $[A, C, E; B, D, F]$.

Proposition 4.3. *Let the Assumptions 3.1–3.3 be satisfied. Suppose that $\Phi \in \mathscr{T}[A, C, E; B, D, F]$. Then for any $x \in \mathbb{R}^n$,*

$$\mathscr{U} = \{\Phi X_\Phi(\cdot; x, v) + v : v \in L^2_{\mathbb{F}}(\mathbb{R}^m)\},$$

where $X_\Phi(\cdot; x, v)$ is the solution to the SDE

$$\begin{cases} dX_\Phi(t) = [(A + B\Phi)X_\Phi(t) + Bv(t) + b(t)]dt \\ \qquad\qquad + [(C + D\Phi)X_\Phi(t) + Dv(t) + \sigma(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z \Big[\Big(E(\theta) + F(\theta)\Phi\Big)X_\Phi(t) + F(\theta)v(t) + h(t,\theta)\Big]\tilde{\mu}(dt, d\theta), \quad t \in [0,\infty), \\ X_\Phi(0) = x \in \mathbb{R}^n. \end{cases} \tag{4.4}$$

Proof. Since $\Phi \in \mathscr{T}[A, C, E; B, D, F]$, the system $[A + B\Phi, C + D\Phi, E + F\Phi]$ is L^2-stable. Let $X_\Phi(\cdot) \equiv X_\Phi(\cdot; x, v)$ be the corresponding solution to (4.4) and let $v \in L^2_{\mathbb{F}}(\mathbb{R}^m)$. By Proposition 4.2, $X_\Phi \in \mathcal{X}[0,\infty)$. We set

$$u = \Phi X_\Phi + v \in L^2_{\mathbb{F}}(\mathbb{R}^m),$$

and let $X \in \mathcal{X}_{loc}[0,\infty)$ be the solution of

$$\begin{cases} dX(t) = [AX(t) + Bu(t) + b(t)]dt + [CX(t) + Du(t) + \sigma(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta)X(t-) + F(\theta)u(t) + h(t,\theta)]\tilde{\mu}(dt, d\theta), \quad t \in [0,\infty), \\ X(0) = x \in \mathbb{R}^n. \end{cases} \tag{4.5}$$

Then we have $X = X_\Phi \in \mathcal{X}[0,\infty)$. For the proof the reader is referred to Theorem 1.19 in [31]. And hence $u \in \mathscr{U}$.

On the flip side, let $X \in \mathcal{X}[0,\infty)$ be the corresponding solution to (4.5) and suppose that $u \in \mathscr{U}$. Then we denote the control v by

$$v \triangleq u - \Phi X \in L^2_{\mathbb{F}}(\mathbb{R}^m).$$

Again by the uniqueness of solutions, the solution X_Φ of (4.4) coincides with X, and we get that u admits a representation of the form $\Phi X_\Phi(\cdot; x, v) + v$. $\square$

The above result shows that the L^2-stabilizability is sufficient for the existence of an admissible control and gives an explicit description of the admissible control set. The following we will show a further result that the L^2-stabilizability is not only sufficient, but also necessary, for the non-emptiness of $\mathscr{U}$ for all $x \in \mathbb{R}^n$. We define

$$\begin{cases} \mathcal{L}(K) = KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta) + Q, \\ \mathcal{M}(K) = B^\top K + D^\top KC + \int_Z F(\theta)^\top KE(\theta)\nu(d\theta) + S, \\ \mathcal{N}(K) = R + D^\top KD + \int_Z F(\theta)^\top KF(\theta)\nu(d\theta), \end{cases} \tag{4.6}$$

for $K \in \mathbb{S}^n$. We first present the following lemma.

Lemma 4.4. *Let Assumptions 3.1–3.3 hold. Suppose that for each $T > 0$, the following differential Riccati equation*

$$\begin{cases} \dot{K}_T(s) + \mathcal{L}(K_T(s)) - \mathcal{M}(K_T(s))^\top \mathcal{N}(K_T(s))^{-1}\mathcal{M}(K_T(s)) = 0, \\ K_T(T) = G \end{cases} \tag{4.7}$$

admits a solution $K_T \in C([0,T];\mathbb{S}^n)$, which satisfies

$$\mathcal{N}(K_T(s)) > 0, \quad \forall s \in [0,T].$$

Then the following algebraic Riccati equation:

$$\mathcal{L}(K) - \mathcal{M}(K)^\top \mathcal{N}(K)^{-1}\mathcal{M}(K) = 0 \tag{4.8}$$

has a solution K if $K_T(0)$ converges to K as $T \to \infty$ and $\mathcal{N}(K)$ is invertible.

Proof. We denote

$$\begin{cases} K_1(s) = K_{T_1}(T_1 - s), & 0 \le s \le T_1, \\ K_2(s) = K_{T_2}(T_2 - s), & 0 \le s \le T_2 \end{cases}$$

for any fixed T_1 and T_2, which satisfy $0 < T_1 < T_2 < \infty$. Then we define

$$\Phi_i(s) = \mathcal{N}(K_i(s))^{-1}\mathcal{M}(K_i(s)), \quad i = 1, 2.$$

It is easy to conclude that both K_1 and K_2 are solutions of the following equatiion:

$$\begin{cases} \dot{\Sigma}(s) - \mathcal{L}(\Sigma(s)) + \mathcal{M}(\Sigma(s))^\top \mathcal{N}(\Sigma(s))^{-1}\mathcal{M}(\Sigma(s)) = 0, \\ \Sigma(0) = G \end{cases} \tag{4.9}$$

on the interval $[0,T_1]$. Thus, for any $s \in [0, T_1]$, it follows that

$$K_1(s) = K_2(s)$$

because of the uniqueness of solutions to ODEs, and the difference $\Delta = K_1 - K_2$ satisfies $\Delta(0) = 0$. To interpret it, we get that

$$\begin{aligned}\dot{\Delta} =& \Delta A + A^\top \Delta + C^\top \Delta C + \int_Z E(\theta)^\top \Delta E(\theta)\nu(d\theta)\\ &- [\Delta B + C^\top \Delta D + \int_Z E(\theta)^\top \Delta F(\theta)\nu(d\theta)]\Phi_1\\ &+ \Phi_2^\top [D^\top \Delta D + \int_Z F(\theta)^\top \Delta F(\theta)\nu(d\theta)]\Phi_1\\ &- \Phi_2^\top [B^\top \Delta + D^\top \Delta C + \int_Z F(\theta)^\top \Delta E(\theta)\nu(d\theta)], \qquad s \in [0, T_1].\end{aligned}$$

By assumption, Φ_1 and Φ_2 are continuous and hence bounded. Thus, for some positive constant M which is independent of Δ, we have

$$\begin{aligned}|\Delta(t)| \leq& \int_0^t \Big| \Delta A + A^\top \Delta + C^\top \Delta C + \int_Z E(\theta)^\top \Delta E(\theta)\nu(d\theta)\\ &- [\Delta B + C^\top \Delta D + \int_Z E(\theta)^\top \Delta F(\theta)\nu(d\theta)]\Phi_1\\ &+ \Phi_2^\top [D^\top \Delta D + \int_Z F(\theta)^\top \Delta F(\theta)\nu(d\theta)]\Phi_1\\ &- \Phi_2^\top [B^\top \Delta + D^\top \Delta C + \int_Z F(\theta)^\top \Delta E(\theta)\nu(d\theta)]\Big| ds\\ \leq& M \int_0^t |\Delta(s)| ds, \quad \forall t \in [0, T_1].\end{aligned}$$

Then we get $\Delta(s) = 0$ for all $s \in [0, T_1]$ by Gronwall's inequality. Thus, the function $\Sigma : [0, \infty) \to \mathbb{S}^n$ can be defined by

$$\Sigma(s) = K_T(T - s), \qquad 0 \leq s \leq T.$$

Assume $\mathcal{N}(K)$ is invertible, since we define

$$\begin{aligned}\Theta(s) &\triangleq \mathcal{L}(\Sigma(s)) - \mathcal{M}(\Sigma(s))^\top \mathcal{N}(\Sigma(s))^{-1} \mathcal{M}(\Sigma(s)),\\ \Theta_\infty &\triangleq \mathcal{L}(K) - \mathcal{M}(K)^\top \mathcal{N}(K)^{-1} \mathcal{M}(K),\end{aligned}$$

it is easy to get that $\lim_{s\to\infty} \Theta(s) = \Theta_\infty$ if $\Sigma(T) = K_T(0)$ converges to K as $T \to \infty$. On the other side, we get that

$$\Sigma(T + 1) - \Sigma(T) = \int_T^{T+1} \Theta(t) dt, \qquad \forall T > 0$$

on the whole interval $[0,\infty)$ since Σ satisfies (4.9). Thus, we have

$$\begin{aligned}|\Theta_\infty| &= \left|\int_T^{T+1} \Theta_\infty dt\right| \\ &= \left|\int_T^{T+1} [\Theta(t) + \Theta_\infty - \Theta(t)]\, dt\right| \\ &\leq \left|\int_T^{T+1} \Theta(t)dt\right| + \left|\int_T^{T+1} [\Theta_\infty - \Theta(t)]dt\right| \\ &\leq |\Sigma(T+1) - \Sigma(T)| + \int_T^{T+1} |\Theta_\infty - \Theta(t)|dt.\end{aligned}$$

Letting $T \to \infty$, the desired result can be proved. □

Theorem 4.5. *Under Assumptions 3.1–3.3, the following statements are equivalent:*
(i) $\mathscr{U} \neq \emptyset$ *for all* $x \in \mathbb{R}^n$*;*
(ii)$\mathscr{T}[A, C, E; B, D, F] \neq \emptyset$*;*
(iii)The following algebraic Riccati equation (ARE, for short) admits a positive solution $K \in \mathbb{S}^n_+$*:*

$$\begin{aligned}&KA{+}A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta) + I \\ &\quad -[KB + C^\top KD + \int_Z E(\theta)^\top KF(\theta)\nu(d\theta)][I + D^\top KD \\ &\quad + \int_Z F(\theta)^\top KF(\theta)\nu(d\theta)]^{-1}[B^\top K + D^\top KC + \int_Z F(\theta)^\top KE(\theta)\nu(d\theta)] = 0.\end{aligned} \tag{4.10}$$

We define

$$\begin{aligned}\Pi \triangleq &- [I + D^\top KD + \int_Z F(\theta)^\top KF(\theta)\nu(d\theta)]^{-1} \\ &[B^\top K + D^\top KC + \int_Z F(\theta)^\top KE(\theta)\nu(d\theta)].\end{aligned} \tag{4.11}$$

If the above are satisfied and K is a positive solution of (4.10)*, then we have* $\Pi \in \mathscr{T}[A, C, E; B, D, F]$.

Proof. The implication (ii) ⇒ (i) has been proved in Proposition 4.3. Then we show the implication (iii) ⇒ (ii). We assume $K \in \mathbb{S}^n_+$ is a solution of (4.10) and Π has the expression (4.11), then we have

$$\begin{aligned}&K(A + B\Pi){+}(A + B\Pi)^\top K + (C + D\Pi)^\top K(C + D\Pi) \\ &\qquad + \int_Z [E(\theta) + F(\theta)\Pi]^\top K[E(\theta) + F(\theta)\Pi]\nu(d\theta) = -I - \Pi^\top\Pi < 0.\end{aligned}$$

Thus, Π is a stabilizer of $[A, C, E; B, D, F]$ according to Theorem 4.1 and Definition 4.1.

We next show that (i) ⇒ (iii). Yet the general, we may assume $b = \sigma = 0$. Now take a series of $u_i \in \mathscr{U}(e_i)$, $i = 1, ..., n$, and define $U = (u_1, ..., u_n)$ while $e_1, ..., e_n$ is the standard basis for $\mathbb{R}^n$. Then, we have $Ux \in \mathscr{U}$ for all $x \in \mathbb{R}^n$ because of the linearity of the state equation. Consider the following cost functional

$$\hat{J}(x; u) = \mathbb{E}\int_0^\infty [|\mathbb{X}(t)x|^2 + |U(t)x|^2]dt,$$

while $\mathbb{X} \in L^2_{\mathbb{F}}(\mathbb{R}^{n\times n})$ solves the following matrix SDE:

$$\begin{cases} d\mathbb{X}(t) = [A\mathbb{X}(t) + BU(t)]dt + [C\mathbb{X}(t) + DU(t)]dW(t) \\ \qquad\qquad + \int_Z [E(\theta)\mathbb{X}(t) + F(\theta)U(t)]\tilde{\mu}(dt, d\theta), \qquad t \in [0, \infty), \\ \mathbb{X}(0) = I_n. \end{cases}$$

We have for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \inf_{u\in\mathscr{U}} \hat{J}(x; u) \leq & \mathbb{E} \int_0^\infty \Big[|\mathbb{X}(t)x|^2 + |U(t)x|^2\Big]dt \\ = & \Big\langle\Big(\mathbb{E} \int_0^\infty [\mathbb{X}(t)^\top \mathbb{X}(t) + U(t)^\top U(t)]dt\Big)x, x\Big\rangle. \end{aligned} \tag{4.12}$$

Now for a fixed but arbitrary $T > 0$, let us consider the optimal control problem in the finite time horizon $[0,T]$ with state equation

$$\begin{cases} dX_T(t) = [AX_T(t) + Bu(t)]dt + [CX_T(t) + Du(t)]dW(t) \\ \qquad\qquad + \int_Z [E(\theta)X_T(t) + F(\theta)u(t)]\tilde{\mu}(dt, d\theta), \quad t \in [0, T], \\ X_T(0) = x, \end{cases}$$

and cost functional

$$\hat{J}_T(x; u) = \mathbb{E} \int_0^T [|X_T(t)|^2 + |u(t)|^2]dt.$$

According to Proposition 4.2 of the reference [25], the differential Riccati equation:

$$\begin{cases} \dot{K}_T(t) + K_T(t)A + A^\top K_T(t) + C^\top K_T(t)C + \int_Z E(\theta)^\top K_T(t)E(\theta)\nu(d\theta) + I \\ \qquad - [K_T(t)B + C^\top K_T(t)D + \int_Z E(\theta)^\top K_T(t)F(\theta)\nu(d\theta)][I + D^\top K_T(t)D \\ \qquad + \int_Z F(\theta)^\top K_T(t)F(\theta)\nu(d\theta)]^{-1}[B^\top K_T(t) + D^\top K_T(t)C \\ \qquad + \int_Z F(\theta)^\top K_T(t)E(\theta)\nu(d\theta)] = 0, \qquad t \in [0, T], \\ K_T(T) = 0 \end{cases}$$

admits a unique solution $K_T \in C([0, T]; \mathbb{S}^n_+)$ such that

$$\langle K_T(0)x, x\rangle = \inf_{u\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} \hat{J}_T(x; u), \quad \forall x \in \mathbb{R}^n.$$

Because of the restriction $u|_{[0,T]}$ of u to $[0,T]$ belongs to $\mathscr{U}[0, T] \equiv L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ for any $u \in \mathscr{U}$, we have

$$\langle K_T(0)x, x\rangle \leq \hat{J}_T(x; u|_{[0,T]}) \leq \hat{J}(x; u), \quad \forall u \in \mathscr{U}.$$

And along with (4.12), it implies that

$$\langle K_T(0)x, x\rangle \leq \langle \Psi x, x\rangle, \quad \forall x \in \mathbb{R}^n, \tag{4.13}$$

where $\Psi \triangleq \int_0^\infty [\mathbb{X}(t)^\top \mathbb{X}(t) + U(t)^\top U(t)]dt$. On the flip side, the restriction $u|_{[0,T]}$ of $u \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ also belongs to $L^2_{\mathbb{F}}(0,T';\mathbb{R}^m)$ for any fixed $T' > T > 0$. Thus, we have

$$\langle K_T(0)x, x\rangle \le \hat{J}_T(x; u|_{[0,T]}) \le \hat{J}_{T'}(x;u), \quad \forall u \in L^2_{\mathbb{F}}(0,T';\mathbb{R}^m),$$

which in turn gives

$$\langle K_T(0)x, x\rangle \le \langle K_{T'}(0)x, x\rangle, \quad \forall x \in \mathbb{R}^n. \tag{4.14}$$

Combining (4.13) and (4.14), since $K_T \in C([0,T];\mathbb{S}^n_+)$, it is easy to get that

$$0 < K_T(0) \le K_{T'}(0) \le \Psi, \quad \forall 0 < T < T' < \infty.$$

Thus, it follows that $K_T(0)$ converges increasingly to some $K \in \mathbb{S}^n_+$ as $T \nearrow \infty$. By Lemma 4.4, the limit matrix K is a solution of the ARE (4.10). □

5. Solvability and the algebraic Riccati equation

According to Theorem 4.5, we have proved that $\mathscr{U} \neq \emptyset$ for all $x \in \mathbb{R}^n$ is equivalent to $\mathscr{T}[A,C,E;B,D,F] \neq \emptyset$. Thus, it is reasonable to give the following assumption:

(A) System $[A,C,E;B,D,F]$ is L^2-stabilizable, i.e., $\mathscr{T}[A,C,E;B,D,F] \neq \emptyset$.

Definition 5.1. For an initial state $x \in \mathbb{R}^n$, an element $u^* \in \mathscr{U}$ is called an *open-loop optimal control* of (SLQ)$_\infty$ if

$$J(x;u^*) \le J(x;u), \quad \forall u \in \mathscr{U}.$$

Problem (SLQ)$_\infty$ is said to be *(uniquely) open-loop solvable at x* if an open-loop optimal control (uniquely) exists for x. If it is (uniquely) open-loop solvable at all $x \in \mathbb{R}^n$, Problem (SLQ)$_\infty$ is said to be *(uniquely) open-loop solvable*.

Definition 5.2. A pair $(\Phi^*, v^*) \in \mathscr{T}[A,C,E;B,D,F] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$ is called a *closed-loop optimal strategy* of Problem (SLQ)$_\infty$ if

$$J(x;\Phi^* X^* + v^*) \le J(x;\Phi X + v)$$

for all $(x,\Phi,v) \in \mathbb{R}^n \times \mathscr{T}[A,C,E;B,D,F] \times L^2_{\mathbb{F}}(\mathbb{R}^m)$, where X^* and X are the closed-loop state processes corresponding to (x,Φ^*,v^*) and (x,Φ,v), respectively. Problem (SLQ)$_\infty$ is said to be (*uniquely*) *closed-loop solvable* if a closed-loop optimal strategy (uniquely) exists. The outcome

$$u \triangleq \Phi X + v$$

of a closed-loop strategy (Φ, v) is called a *closed-loop control* for the initial state x, where X is the closed-loop state process corresponding to :

$$\begin{cases} dX(t) = [(A + B\Phi)X(t) + Bv(t) + b(t)]dt \\ \qquad\quad + [(C + D\Phi)X(t) + Dv(t) + \sigma(t)]dW(t) \\ \qquad\quad + \displaystyle\int_Z \Big[(E(\theta) + F(\theta)\Phi)X(t) + F(\theta)v(t) + h(t,\theta)\Big]\tilde{\mu}(dt,d\theta), \quad t \ge 0, \\ X(0) = x. \end{cases}$$

Remark 5.1. From Proposition 4.3, $\mathscr{U}$ is composed of closed-loop controls for all x when (A) holds. In Definition 5.2, the outcome $u^* \equiv \Phi^* X^* + v^*$ of a closed-loop optimal strategy (Φ^*, v^*) is an open-loop optimal control for the initial state $X^*(0)$, which implies that closed-loop solvability is sufficient to open-loop solvability.

According to the definition of the open-loop and closed-loop solvabilities in [17], it is easy to get that for finite horizon cases, closed-loop solvability implies open-loop solvability, whereas open-loop solvability does not necessarily imply closed-loop solvability. However, when it comes to infinite horizon, the open-loop and closed-loop solvabilities are equivalent, and both are equivalent to the existence of a *stabilizing solution* to a *generalized algebraic Riccati equation* which we will prove later.

Definition 5.3. Let Assumptions 3.1–3.3 be satisfied. The following constrained a generalized algebraic Riccati equation (ARE)

$$\begin{cases} \mathcal{L}(K) - \mathcal{M}(K)^\top \mathcal{N}(K)^\dagger \mathcal{M}(K) = 0, \\ \mathscr{R}(\mathcal{M}(K)) \subseteq \mathscr{R}(\mathcal{N}(K)), \\ \mathcal{N}(K) \geq 0. \end{cases} \tag{5.1}$$

If there exists a $\Theta \in \mathbb{R}^{m\times n}$ such that the matrix

$$\Phi \triangleq -\mathcal{N}(K)^\dagger \mathcal{M}(K) + [I - \mathcal{N}(K)^\dagger \mathcal{N}(K)]\Theta \tag{5.2}$$

is a stabilizer of $[A, C, E; B, D, F]$, then the solution $K \in \mathbb{S}^n$ of (5.1) is said to be *stabilizing*.

Remark 5.2. Because of the properties of the Moore-Penrose pseudoinverse (see Lemma 2.2 and Remark 2.1), one has

$$\mathcal{N}(K)\Phi = -\mathcal{M}(K), \quad \mathcal{M}(K)^\top \Phi = -\Phi^\top \mathcal{N}(K)\Phi = -\mathcal{M}(K)^\top \mathcal{N}(K)^\dagger \mathcal{M}(K)$$

if K is a solution (not necessarily stabilizing) to the ARE (5.1) and Φ is defined by (5.2).

5.1. A study of Problem $(\text{SLQ})^0_\infty$

In order to solve Problem $(\text{SLQ})_\infty$, we first discuss Problem $(\text{SLQ})^0_\infty$ in this section. Assume that the nonhomogeneous terms b, σ, h, q, and ρ are all zero, and we assume that the system $[A, C, E]$ is L^2-stable (i.e., $0 \in \mathscr{S}[A, C, E; B, D, F]$) for the sake of simplicify.

Proposition 5.1. *Let Assumptions 3.1–3.3 hold. Suppose that $[A, C, E]$ is L^2-stable. Then there exist a bounded self-adjoint linear operator $H_2 : L^2_{\mathbb{F}}(\mathbb{R}^m) \to L^2_{\mathbb{F}}(\mathbb{R}^m)$, a bounded linear operator $H_1 : \mathbb{R}^n \to L^2_{\mathbb{F}}(\mathbb{R}^m)$, a matrix $H_0 \in \mathbb{S}^n$, and $\tilde{u} \in L^2_{\mathbb{F}}(\mathbb{R}^m)$, $\tilde{x} \in \mathbb{R}^n$, $g \in \mathbb{R}$ such that for any $(x, u) \in \mathbb{R}^n \times L^2_{\mathbb{F}}(\mathbb{R}^m)$,*

$$J(x; u) = \langle H_2 u, u\rangle + 2\langle H_1 x, u\rangle + \langle H_0 x, x\rangle + 2\langle u, \tilde{u}\rangle + 2\langle x, \tilde{x}\rangle + g.$$

In particular, in the case of Problem $(\text{SLQ})^0_\infty$*(i.e., $b, \sigma, h, q, \rho = 0$),*

$$J^0(x; u) = \langle H_2 u, u\rangle + 2\langle H_1 x, u\rangle + \langle H_0 x, x\rangle,$$

where

$$\begin{cases} H_2 = M_0^* Q M_0 + 2S M_0 + R, \\ H_1 = M_0^* Q L_0 + S L_0, \\ H_0 = L_0^* Q L_0, \\ \tilde{u} = M_0^* Q N_0 + M_0^* q + S N_0 + \rho, \\ \tilde{x} = L_0^* q + L_0^* Q N_0, \\ g = \langle Q N_0, N_0 \rangle + 2 \langle q, N_0 \rangle, \end{cases}$$

and

$$\begin{cases} L_0(t) \;= \Lambda(t), \\ M_0(t) = \Lambda(t) \displaystyle\int_0^t \Lambda(s)^{-1} \left[B - CD - \int_Z E(\theta)(I + E(\theta))^{-1} F(\theta) \nu(d\theta) \right] ds \\ \qquad + \Lambda(t) \displaystyle\int_0^t \Lambda(s)^{-1} D dW_s + \Lambda(t) \int_0^t \int_Z F(\theta)(I + E(\theta))^{-1} \tilde{\mu}(ds, d\theta), \\ N_0(t) = \Lambda(t) \displaystyle\int_0^t \Lambda(s)^{-1} \left[b - C\sigma - \int_Z E(\theta)(I + E(\theta))^{-1} h(s, \theta) \nu(d\theta) \right] ds \\ \qquad + \Lambda(t) \displaystyle\int_0^t \Lambda(s)^{-1} \sigma dW_s + \Lambda(t) \int_0^t \int_Z h(s, \theta)(I + E(\theta))^{-1} \tilde{\mu}(ds, d\theta). \end{cases}$$

Proposition 5.2. *Under Assumptions 3.1–3.3, if $[A, C, E]$ is L^2-stable, then we have the following conclusion:*
(i) Problem $(SLQ)_\infty$ is open-loop solvable at x if and only if $H_2 \geq 0$ (i.e.,H_2 is a positive operator) and $H_1 x + \tilde{u} \in \mathscr{R}(H_2)$. In this instance, u^* is an open-loop optimal control for the initial state x if and only if

$$H_2 u^* + H_1 x + \tilde{u} = 0.$$

(ii) If Problem $(SLQ)_\infty$ is open-loop solvable, then so is Problem $(SLQ)_\infty^0$.
(iii) If Problem $(SLQ)_\infty^0$ is open-loop solvable, then there exists a $U^* \in L_{\mathbb{F}}^2(\mathbb{R}^{m\times n})$ such that for any $x \in \mathbb{R}^n$, $U^* x$ is an open-loop optimal control of Problem $(SLQ)_\infty^0$ for the initial state x.

Proof. (i) By definition, a process $u^* \in L_{\mathbb{F}}^2(\mathbb{R}^m)$ is an open-loop optimal control for the initial state x if and only if

$$J(x; u^* + \gamma w) - J(x; u^*) \geq 0, \quad \forall w \in L_{\mathbb{F}}^2(\mathbb{R}^m), \ \ \forall \gamma \in \mathbb{R}. \tag{5.3}$$

According to Proposition 5.1, we have

$$\begin{aligned} J(x; u^* + \gamma w) &= \langle H_2(u^* + \gamma w), u^* + \gamma w \rangle + 2 \langle H_1 x, u^* + \gamma w \rangle \\ &\quad + \langle H_0 x, x \rangle + 2 \langle u^* + \gamma w, \tilde{u} \rangle + 2 \langle x, \tilde{x} \rangle + g \\ &= J(x; u^*) + \gamma^2 \langle H_2 w, w \rangle + 2\gamma \langle H_2 u^* + H_1 x + \tilde{u}, w \rangle . \end{aligned}$$

Thus, (5.3) is equivalent to

$$\gamma^2 \langle H_2 w, w \rangle + 2\gamma \langle H_2 u^* + H_1 x + \tilde{u}, w \rangle \geq 0, \quad \forall w \in L_{\mathbb{F}}^2(\mathbb{R}^m), \ \ \forall \gamma \in \mathbb{R},$$

which in turn is equivalent to

$$\langle H_2 w, w \rangle \geq 0, \ \ \forall w \in L_{\mathbb{F}}^2(\mathbb{R}^m) \text{ and } H_2 u^* + H_1 x + \tilde{u} = 0.$$

So (i) is proved.

For conclusion (ii), since Problem $(\mathrm{SLQ})_\infty$ is open-loop solvable, it is easy to get that $H_2 \geq 0$ and $H_1 x + \tilde{u} \in \mathscr{R}(H_2)$ for all $x \in \mathbb{R}^n$ according to (i). Then we take $x = 0$, it is tempting to conclude that $\tilde{u} \in \mathscr{R}(H_2)$, and hence $H_1 x \in \mathscr{R}(H_2)$ for all $x \in \mathbb{R}^n$. Finally, we obtain the open-loop solvability of Problem $(\mathrm{SLQ})^0_\infty$ by using (i) again.

For conclusion (iii), assume $e_1, ..., e_n$ is the standard basis for $\mathbb{R}^n$. It is easy to get that $U^* \triangleq (u^*_1, ..., u^*_n)$ has the desired properties since u^*_i ia an open-loop optimal control of Problem $(\mathrm{SLQ})^0_\infty$ for the initial state e_i. □

5.1.1. A finite horizon approach

Let Λ be the solution of (3.7), and the matrix

$$G \triangleq \mathbb{E}\int_0^\infty \Lambda(t)^\top Q\Lambda(t)dt$$

is well-defined if [A,C,E] is L^2-stable. Thus, we can consider the following LQ problem over the finite time horizon $[0, T]$ for any $T > 0$.

Problem $(\mathrm{SLQ})^0_T$. On the finite time horizon $[0, T]$, we assume $X_T(t)$ is a solution of the state equation:

$$\begin{cases} dX_T(t) = [AX_T(t) + Bu(t)]dt + [CX_T(t) + Du(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta)X_T(t) + F(\theta)u(t)]\tilde{\mu}(dt, d\theta), \\ X_T(0) = x, \end{cases} \tag{5.4}$$

then for any given $x \in \mathbb{R}^n$, find a $u^* \in \mathscr{U}[0, T]$ such that the cost functional

$$J^0_T(x; u) \triangleq \mathbb{E}\Big\{\langle GX_T(T), X_T(T)\rangle + \int_0^T \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X_T(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X_T(t) \\ u(t) \end{pmatrix}\Big\rangle dt\Big\}$$

is minimized over $\mathscr{U}[0, T]$.

Proposition 5.3. *Let Assumptions 3.1–3.3 be satisfied. If [A,C,E] is L^2-stable, since we use $V^0_T(x)$ to denote the value function of Problem $(SLQ)^0_T$, and use $V^0(x)$ to denote the value function of Problem $(SLQ)^0_\infty$, it is easy to get the following conclusions:*
(i) For any $x \in \mathbb{R}^n$ and $u \in \mathscr{U}[0, T]$,

$$J^0_T(x; u) = J^0(x; u_e),$$

where $u_e \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ is the zero-extension of u:

$$u_e(t) = u(t) \quad if\ t \in [0, T]; \quad u_e(t) = 0 \quad if\ t \in (T, \infty).$$

(ii) If there exists a $\varepsilon > 0$ such that

$$\langle H_2 v, v\rangle \geq \varepsilon ||v||^2, \quad \forall v \in L^2_{\mathbb{F}}(\mathbb{R}^m), \tag{5.5}$$

then

$$J^0_T(0; u) \geq \varepsilon\mathbb{E}\int_0^T |u(t)|^2 dt, \quad \forall u \in \mathscr{U}[0, T]. \tag{5.6}$$

(iii) $\lim_{T\to\infty} V^0_T(x) = V^0(x)$ *for all* $x \in \mathbb{R}^n$.

Proof. (i) Let X_T be the solution of (5.4) and X be the solution of

$$\begin{cases} dX(t) = [AX(t) + Bu_e(t)]dt + [CX(t) + Du_e(t)]dW(t) \\ \qquad\qquad + \int_Z [E(\theta)X(t) + F(\theta)u_e(t)]\tilde{\mu}(dt, d\theta), \quad t \geq 0, \\ X(0) = x, \end{cases}$$

for a fixed $x \in \mathbb{R}^n$ and an arbitrary $u \in \mathscr{U}[0, T]$. Thus, we have that

$$X(t) = \begin{cases} X_T(t), & t \in [0, T], \\ \Lambda(t)\Lambda(T)^{-1}X_T(T), & t \in (T, \infty). \end{cases}$$

Noting that for $t \geq T$, $\Lambda(t)\Lambda(T)^{-1}$ and $\Lambda(t - T)$ have the same distribution and are independent of $\mathcal{F}_T$, we have

$$\begin{aligned} &\mathbb{E}\left\langle\left(\mathbb{E}\int_0^\infty \Lambda(t)^\top Q\Lambda(t)dt\right)X_T(T), X_T(T)\right\rangle \\ &= \mathbb{E}\left\langle\left(\mathbb{E}\int_T^\infty [\Lambda(t-T)]^\top Q[\Lambda(t-T)]dt\right)X_T(T), X_T(T)\right\rangle \\ &= \mathbb{E}\left\langle\left(\mathbb{E}\int_T^\infty [\Lambda(t)\Lambda(T)^{-1}]^\top Q[\Lambda(t)\Lambda(T)^{-1}]dt\right)X_T(T), X_T(T)\right\rangle \\ &= \mathbb{E}\int_T^\infty \left\langle Q\Lambda(t)\Lambda(T)^{-1}X_T(T), \Lambda(t)\Lambda(T)^{-1}X_T(T)\right\rangle dt \\ &= \mathbb{E}\int_T^\infty \langle QX(t), X(t)\rangle\, dt. \end{aligned}$$

It follows that

$$\begin{aligned} J_T^0(x; u) &= \mathbb{E}\Bigg\{\left\langle\left(\mathbb{E}\int_o^\infty \Lambda(t)^\top Q\Lambda(t)dt\right)X_T(T), X_T(T)\right\rangle \\ &\qquad + \int_0^T \left\langle\begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X_T(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X_T(t) \\ u(t) \end{pmatrix}\right\rangle dt\Bigg\} \\ &= \mathbb{E}\Bigg\{\int_T^\infty \langle QX, X\rangle\, dt + \int_0^T \left\langle\begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X \\ u_e \end{pmatrix}, \begin{pmatrix} X \\ u_e \end{pmatrix}\right\rangle dt\Bigg\} \\ &= J^0(x; u_e). \end{aligned} \tag{5.7}$$

(ii) Taking $x = 0$ in (5.7), we obtain

$$\begin{aligned} J_T^0(0; u) &= J^0(0; u_e) = \langle H_2 u_e, u_e\rangle \\ &\geq \varepsilon\mathbb{E}\int_0^\infty |u_e(t)|^2 dt = \varepsilon\mathbb{E}\int_0^T |u(t)|^2 dt, \end{aligned}$$

which proves (ii).

Finally let us prove (iii). According to (5.7), it is easy to obtain

$$V^0(x) \leq J^0(x; u_e) = J_T^0(x; u), \quad \forall u \in \mathscr{U}[0, T].$$

Let $V_T^0(x)$ be the infimum of $J_T^0(x; u)$ over $u \in \mathscr{U}[0, T]$, thus

$$V^0(x) \leq V_T^0(x), \quad \forall T > 0. \tag{5.8}$$

On the flip side, when $V^0(x) > -\infty$, for any given $\varsigma > 0$, we can find a $u^\varsigma \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ such that

$$\mathbb{E}\int_0^\infty \left\langle\!\!\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X^\varsigma(t) \\ u^\varsigma(t) \end{pmatrix}, \begin{pmatrix} X^\varsigma(t) \\ u^\varsigma(t) \end{pmatrix} \right\rangle\!\!\right\rangle dt = J^0(x; u^\varsigma) \leq V^0(x) + \varsigma, \tag{5.9}$$

where X^ς is the solution of

$$\begin{cases} dX^\varsigma(t) = [AX^\varsigma(t) + Bu^\varsigma(t)]dt + [CX^\varsigma(t) + Du^\varsigma(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta)X^\varsigma(t) + F(\theta)u^\varsigma(t)]\tilde{\mu}(dt, d\theta), \quad t \geq 0, \\ X^\varsigma(0) = x. \end{cases}$$

Since by Proposition 4.2 $X^\varsigma \in \mathcal{X}[0, \infty)$, it is easy to get that for large $T > 0$,

$$|\mathbb{E}\langle GX^\varsigma(T), X^\varsigma(T)\rangle| + \left|\mathbb{E}\int_T^\infty \left\langle\!\!\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X^\varsigma(t) \\ u^\varsigma(t) \end{pmatrix}, \begin{pmatrix} X^\varsigma(t) \\ u^\varsigma(t) \end{pmatrix} \right\rangle\!\!\right\rangle dt\right| \leq \varsigma.$$

Now we use u_T^ς to be the restriction of u^ς to $[0, T]$, thus we have

$$\begin{aligned} J^0(x; u^\varsigma) &= J_T^0(x; u_T^\varsigma) - \mathbb{E}\langle GX^\varsigma(T), X^\varsigma(T)\rangle \\ &\quad + \mathbb{E}\int_T^\infty \left\langle\!\!\left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X^\varsigma(t) \\ u^\varsigma(t) \end{pmatrix}, \begin{pmatrix} X^\varsigma(t) \\ u^\varsigma(t) \end{pmatrix} \right\rangle\!\!\right\rangle dt \\ &\geq V_T^0(x) - \varsigma. \end{aligned} \tag{5.10}$$

Since we combine (5.9) and (5.10), it is not hard to get that for large $T > 0$,

$$V_T^0(x) \leq V^0(x) + 2\varsigma.$$

To summarize, we have that $V_T^0(x) \to V^0(x)$ as $T \to \infty$. The case when $V^0(x) = -\infty$ can also be proved by a similar method. □

Proposition 5.2(i) implies that if [A,C,E] is L^2-stable and the operator H_2 is uniformly positive, then Problem $(\text{SLQ})^0_\infty$ is uniquely open-loop solvable and the unique optimal control is given by

$$u_x^* = -H_2^{-1}H_1 x.$$

And if we substitute the optimal control u_x^* into the cost functional, it is easy to get

$$V^0(x) = \left\langle (H_0 - H_1^* H_2^{-1} H_1)x, x \right\rangle, \quad x \in \mathbb{R}^n.$$

Notice that $H_0 - H_1^* H_2^{-1} H_1$ is a matrix.

Theorem 5.4. *Let Assumptions 3.1–3.3 hold. If [A,C,E] is L^2-stable and* (5.5) *holds for some $\varepsilon > 0$, then we have the following conclusions*
(i) the matrix $K \triangleq H_0 - H_1^* H_2^{-1} H_1$ solves the ARE

$$\begin{cases} \mathcal{L}(K) - \mathcal{M}(K)^\top \mathcal{N}(K)^{-1}\mathcal{M}(K) = 0, \\ \mathcal{N}(K) > 0; \end{cases}$$

(ii) the matrix $\Phi \triangleq -\mathcal{N}(K)^{-1}\mathcal{M}(K)$ is a stabilizer of [A,C,E; B,D,F];
(iii) the unique open-loop optimal control of Problem $(SLQ)^0_\infty$ for the initial state x is given by

$$u_x^*(t) = \Phi X_\Phi(t; x), \quad t \in [0, \infty),$$

where $X_\Phi(\cdot; x)$ is the solution to the following equation:

$$\begin{cases} dX_\Phi(t) = (A + B\Phi)X_\Phi(t)dt + (C + D\Phi)X_\Phi(t)dW(t) + \displaystyle\int_Z [E(\theta) + F(\theta)\Phi]X_\Phi(t)\tilde{\mu}(dt, d\theta), \ t \in [0, \infty), \\ X_\Phi(0) = x. \end{cases}$$

Proof. By Proposition 5.3(ii), (5.6) holds. According to Proposition 4.2 of [25], it is easy to get that for any $T > 0$, the differential Riccati equation

$$\begin{cases} \dot{K}_T(t) + \mathcal{L}(K_T(t)) - \mathcal{M}(K_T(t))^\top \mathcal{N}(K_T(t))^{-1}\mathcal{M}(K_T(t)) = 0, \quad t \in [0, T], \\ K_T(T) = G \end{cases}$$

admits a unique solution $K_T \in C([0, T]; \mathbb{S}^n)$ such that

$$\mathcal{N}(K_T(t)) \geq \varepsilon I, \quad \forall t \in [0, T]; \quad V_T^0(x) = \langle K_T(0)x, x\rangle, \quad \forall x \in \mathbb{R}^n.$$

From Proposition 5.3(iii), we see that

$$\lim_{T\to\infty} K_T(0) = K, \quad \mathcal{N}(K) > 0.$$

Thus the conclusion (i) can be proved by Lemma 4.4.

Then we will prove the conclusions (ii) and (iii). Firstly, we fix an $x \in \mathbb{R}^n$ and let (X_x^*, u_x^*) be the corresponding optimal pair of Problem $(\mathrm{SLQ})^0_\infty$. Applying Itô's formula to $t \to \langle KX_x^*(t), X_x^*(t)\rangle$ and noting that $\lim_{t\to\infty} \langle KX_x^*(t), X_x^*(t)\rangle = 0$, we have

$$\begin{aligned} -\langle Kx, x\rangle &= \mathbb{E}\int_0^\infty \Big\{ 2\langle K[AX_x^*(t) + Bu_x^*(t)], X_x^*(t)\rangle \\ &\quad + \langle K[CX_x^*(t) + Du_x^*(t)], CX_x^*(t) + Du_x^*(t)\rangle \\ &\quad + \int_Z \langle K[E(\theta)X_x^*(t) + F(\theta)u_x^*(t)], E(\theta)X_x^*(t) + F(\theta)u_x^*(t)\rangle \nu(d\theta)\Big\} dt \\ &= \mathbb{E}\int_0^\infty \bigg\{ \bigg\langle [KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta)]X_x^*(t), X_x^*(t)\bigg\rangle \\ &\quad + 2\bigg\langle [B^\top K + D^\top KC + \int_Z F(\theta)^\top KE(\theta)\nu(d\theta)]X_x^*(t), u_x^*(t)\bigg\rangle \\ &\quad + \bigg\langle [D^\top KD + \int_Z F(\theta)^\top KF(\theta)\nu(d\theta)]u_x^*(t), u_x^*(t)\bigg\rangle \bigg\} dt. \end{aligned}$$

On the flip side, we have

$$\langle Kx, x\rangle = J^0(x; u_x^*) = \mathbb{E}\int_0^\infty \Big[\langle QX_x^*, X_x^*\rangle + 2\langle SX_x^*, u_x^*\rangle + \langle Ru_x^*, u_x^*\rangle\Big]dt.$$

Adding the last two equations yields

$$\begin{aligned}
0 &= \mathbb{E}\int_0^\infty \Big[\langle \mathcal{L}(K)X_x^*, X_x^*\rangle + 2\langle \mathcal{M}(K)X_x^*, u_x^*\rangle + \langle \mathcal{N}(K)u_x^*, u_x^*\rangle\Big]dt \\
&= \mathbb{E}\int_0^\infty \Big[\Big\langle \mathcal{M}(K)^\top \mathcal{N}(K)^{-1}\mathcal{M}(K)X_x^*, X_x^*\Big\rangle + 2\langle \mathcal{M}(K)X_x^*, u_x^*\rangle + \langle \mathcal{N}(K)u_x^*, u_x^*\rangle\Big]dt \\
&= \mathbb{E}\int_0^\infty \Big[\langle \Phi^\top \mathcal{N}(K)\Phi X_x^*, X_x^*\rangle - 2\langle \mathcal{N}(K)\Phi X_x^*, u_x^*\rangle + \langle \mathcal{N}(K)u_x^*, u_x^*\rangle\Big]dt \\
&= \mathbb{E}\int_0^\infty \langle \mathcal{N}(K)[u_x^*(t) - \Phi X_x^*(t)], u_x^*(t) - \Phi X_x^*(t)\rangle\, dt.
\end{aligned}$$

Since $\mathcal{N}(K) = R + D^\top KD + \int_Z F(\theta)^\top KF(\theta)\nu(d\theta) > 0$, it is easy to get that

$$u_x^*(t) = \Phi X_x^*(t), \quad t \in [0, \infty),$$

and hence X_x^* is the solution of

$$\begin{cases} dX_x^*(t) = (A + B\Phi)X_x^*(t)dt + (C + D\Phi)X_x^*(t)dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta) + F(\theta)\Phi]X_x^*(t)\tilde{\mu}(dt, d\theta), \qquad t \in [0, \infty), \\ X_x^*(0) = x. \end{cases}$$

Since $X_x^* \in \mathcal{X}[0, \infty)$ and x is arbitrary, it is not hard to get that Φ is a stabilizer of $[A, C, E; B, D, F]$, and the rest of the proof is clear. □

5.1.2. Open-loop and closed-loop solvability

As we have proved in Proposition 5.2, the condition $H_2 \geq 0$ is merely necessary for the existence of an open-loop optimal control, and according to Theorem 5.4, the uniform positivity condition (5.5) is only sufficient. In order to find the connection between the above, let us consider the following cost functional for $\varsigma > 0$:

$$\begin{aligned}
J_\varsigma^0(x; u) &\triangleq \mathbb{E}\int_0^\infty \left\langle\!\!\left\langle \begin{pmatrix} Q & S^\top \\ S & R + \varsigma I \end{pmatrix}\begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle\!\!\right\rangle dt \\
&= J^0(x; u) + \varsigma\mathbb{E}\int_0^\infty |u(t)|^2 dt \\
&= \langle (H_2 + \varsigma I)u, u\rangle + 2\langle H_1 x, u\rangle + \langle H_0 x.x\rangle.
\end{aligned}$$

Now we denote by Problem $(\mathrm{SLQ})_\infty^{0,\varsigma}$ the problem of minimizing $J_\varsigma^0(x; u)$ subject to the state equation

$$\begin{cases} dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta)X(t) + F(\theta)u(t)]\tilde{\mu}(dt, d\theta), \quad t \in [0, \infty), \\ X(0) = x, \end{cases}$$

and by $V_\varsigma^0(x)$ the corresponding value function. Since we assume $H_2 \geq 0$, and then the operator $H_2+\varsigma I$ is uniformly positive for all $\varsigma > 0$, it is easy to apply Theorem 5.4 to Problem $(\mathrm{SLQ})_\infty^{0,\varsigma}$. Thus we will obtain a characterization for the value function $V^0(x)$ of Problem $(\mathrm{SLQ})_\infty^0$ while $\varsigma \to 0$.

Theorem 5.5. *Let Assumptions 3.1–3.3 be satisfied. If [A,C,E] is L^2-stable, and Problem $(SLQ)_\infty^0$ is open-loop solvable, then the generalized ARE* (5.1) *admits a stabilizing solution $K \in \mathbb{S}^n$. Moreover, $V^0(x) = \langle Kx, x\rangle$ for all $x \in \mathbb{R}^n$.*

Proof. According to Proposition 5.2, the open-loop solvability of Problem $(\mathrm{SLQ})_\infty^0$ implies $H_2 \geq 0$. Thus, for any $\varsigma > 0$, the ARE:

$$\begin{cases} \mathcal{L}(K_\varsigma) - \mathcal{M}(K_\varsigma)^\top[\mathcal{N}(K_\varsigma) + \varsigma I]^{-1}\mathcal{M}(K_\varsigma) = 0, \\ \mathcal{N}(K_\varsigma) + \varsigma I > 0 \end{cases} \tag{5.11}$$

admits a unique solution $K_\varsigma \in \mathbb{S}^n$ such that $V_\varsigma^0(x) = \langle K_\varsigma x, x\rangle$ for all $x \in \mathbb{R}^n$ according to Theorem 5.4. Then we define a stabilizer of [A,C,E; B,D,F]:

$$\Phi_\varsigma \triangleq -[\mathcal{N}(K_\varsigma) + \varsigma I]^{-1}\mathcal{M}(K_\varsigma). \tag{5.12}$$

Thus, it follows that the unique open-loop optimal control $u_\varsigma^*(\cdot; x)$ of Problem $(\mathrm{SLQ})_\infty^{0,\varsigma}$ for the initial state x is given by

$$u_\varsigma^*(t; x) = \Phi_\varsigma\Pi_\varsigma(t)x, \quad t \geq 0,$$

since Π_ς is a solution of the matrix SDE:

$$\begin{cases} d\Pi_\varsigma(t) = (A + B\Phi_\varsigma)\Pi_\varsigma(t)dt + (C + D\Phi_\varsigma)\Pi_\varsigma(t)dW(t) \\ \qquad\qquad + \displaystyle\int_z [E(\theta) + F(\theta)\Phi_\varsigma]\Pi_\varsigma(t)\tilde{\mu}(dt, d\theta), \quad t \geq 0, \\ \Pi_\varsigma(0) = I. \end{cases}$$

Then we denote $U^* \in L_{\mathbb{F}}^2(\mathbb{R}^{m\times n})$ to be a process which is defined in Proposition 5.2(iii). According to the definition of the value function, it is easy to get that for any $x \in \mathbb{R}^n$ and $\varsigma > 0$,

$$\begin{aligned} V^0(x) + \varsigma\mathbb{E}\int_0^\infty |\Phi_\varsigma\Pi_\varsigma(t)x|^2dt &\leq J^0(x; \Phi_\varsigma\Pi_\varsigma(t)x) + \varsigma\mathbb{E}\int_0^\infty |\Phi_\varsigma\Pi_\varsigma(t)x|^2dt \\ &= J_\varsigma^0(x; \Phi_\varsigma\Pi_\varsigma(t)x) = V_\varsigma^0(x) = \langle K_\varsigma x, x\rangle \leq J_\varsigma^0(x; U^*x) \\ &= V^0(x) + \varsigma\mathbb{E}\int_0^\infty |U^*(t)x|^2dt. \end{aligned} \tag{5.13}$$

Equation (5.13) implies that for any $x \in \mathbb{R}^n$ and $\varsigma > 0$,

$$V^0(x) \leq \langle K_\varsigma x, x\rangle \leq V^0(x) + \varsigma\mathbb{E}\int_0^\infty |U^*(t)x|^2dt, \tag{5.14}$$

$$0 \leq \mathbb{E}\int_0^\infty |\Phi_\varsigma\Pi_\varsigma(t)x|^2dt \leq \mathbb{E}\int_0^\infty |U^*(t)x|^2dt. \tag{5.15}$$

From (5.14)) we see that $K \equiv \lim_{\varsigma\to 0} K_\varsigma$ exists and $V^0(x) = \langle Kx, x\rangle$ for all $x \in \mathbb{R}^n$.
From (5.15) we see that the family of positive semi-definite matrices:

$$\Theta_\varsigma = \mathbb{E}\int_0^\infty \Pi_\varsigma(t)^\top \Phi_\varsigma^\top \Phi_\varsigma \Pi_\varsigma(t)dt, \quad \varsigma > 0$$

is bounded, and the system $[A + B\Phi_\varsigma, C + D\Phi_\varsigma, E + F\Phi_\varsigma]$ is L^2-stable for Φ_ς is a stabilizer of $[A, C, E; B, D, F]$. According to Theorem 4.1, we have

$$\begin{aligned}&\Theta_\varsigma(A + B\Phi_\varsigma) + (A + B\Phi_\varsigma)^\top\Theta_\varsigma + (C + D\Phi_\varsigma)^\top\Theta_\varsigma(C + D\Phi_\varsigma)\\ &\qquad + \int_Z [E(\theta) + F(\theta)\Phi_\varsigma]^\top \Theta_\varsigma [E(\theta) + F(\theta)\Phi_\varsigma]\,\nu(d\theta) + \Phi_\varsigma^\top\Phi_\varsigma = 0.\end{aligned}$$

It follows that

$$0 \le \Phi_\varsigma^\top\Phi_\varsigma \le -\left[\Theta_\varsigma(A + B\Phi_\varsigma) + (A + B\Phi_\varsigma)^\top\Theta_\varsigma\right], \quad \forall\varsigma > 0.$$

The above, together with the boundedness of $\{\Theta_\varsigma\}_{\varsigma>0}$, shows that

$$|\Phi_\varsigma|^2 \le J(1 + |\Phi_\varsigma|), \quad \forall\varsigma > 0, \tag{5.16}$$

for some constant $J > 0$. Noting that (5.16) implies the boundedness of $\{\Phi_\varsigma\}_{\varsigma>0}$, we may choose a sequence $\{\varsigma_k\}_{k=1}^\infty \subseteq (0, \infty)$ with $\lim_{k\to\infty} \varsigma_k = 0$ such that $\Phi \equiv \lim_{k\to\infty} \Phi_{\varsigma_k}$ exists. Observe that

$$\mathcal{N}(K)\Phi = \lim_{k\to\infty}[\mathcal{N}(K_{\varsigma_k}) + \varsigma_k I]\Phi_{\varsigma_k} = -\lim_{k\to\infty} \mathcal{M}(K_{\varsigma_k}) = -\mathcal{M}(K).$$

Thus, we have by Lemma 2.2 that

$$\mathscr{R}(\mathcal{M}(K)) \subseteq \mathscr{R}(\mathcal{N}(K)), \tag{5.17}$$

$$\Phi = -\mathcal{N}(K)^\dagger\mathcal{M}(K) + [I - \mathcal{N}(K)^\dagger\mathcal{N}(K)]\Theta, \tag{5.18}$$

for some $\Theta \in \mathbb{R}^{m\times n}$. Notice that by (5.12), $\mathcal{M}(K_\varsigma)^\top = -\Phi_\varsigma^\top[\mathcal{N}(K_\varsigma) + \varsigma I]$. Thus (5.11) can be written as

$$\begin{cases} \mathcal{L}(K_\varsigma) - \Phi_\varsigma^\top[\mathcal{N}(K_\varsigma) + \varsigma I]\Phi_\varsigma = 0,\\ \mathcal{N}(K_\varsigma) + \varsigma I > 0.\end{cases}$$

Now let $k \to \infty$ along $\{\varsigma_k\}_{k=1}^\infty$, then $\varsigma_k \to 0$ in the above. Thus we have

$$\begin{cases} \mathcal{L}(K) - \Phi^\top\mathcal{N}(K)\Phi = 0,\\ \mathcal{N}(K) \ge 0,\end{cases}$$

which, together with (5.17) and (5.18), implies that K solves the generalized ARE (5.1). Following we will show that K is a stabilizing solution, and we need only to show $\Phi \in \mathscr{T}[A, C, E; B, D, F]$. To prove it, we define Π as a solution of the matrix SDE:

$$\begin{cases} d\Pi(t) = (A + B\Phi)\Pi(t)dt + (C + D\Phi)\Pi(t)dW(t)\\ \qquad\qquad + \displaystyle\int_Z [E(\theta) + F(\theta)\Phi]\,\Pi(t)\tilde{\mu}(dt, d\theta), \quad t \ge 0,\\ \Pi(0) = I.\end{cases}$$

Since $\Phi_{\varsigma_k} \to \Phi$ as $k \to \infty$, we get $\Pi_{\varsigma_k}(t) \to \Pi(t)$, a.s.for all $t \geq 0$. By Fatou's lemma and (5.15), we get that

$$\begin{aligned}\mathbb{E}\int_0^\infty |\Phi\Pi(t)x|^2 dt &\leq \liminf_{k\to\infty} \mathbb{E}\int_0^\infty |\Phi_{\varsigma_k}\Pi_{\varsigma_k}(t)x|^2 dt \\ &\leq \mathbb{E}\int_0^\infty |U^*(t)x|^2 dt < \infty, \quad \forall x \in \mathbb{R}^n.\end{aligned}$$

This implies $\Phi\Pi \in L^2_{\mathbb{F}}(\mathbb{R}^{m\times n})$. Thus, by Proposition 4.2, it is easy to get that $\Pi \in L^2_{\mathbb{F}}(\mathbb{R}^{m\times n})$. Consequently, $\Phi \in \mathscr{T}[A, C, E; B, D, F]$. □

According to Theorem 5.5, if $[A, C, E]$ is L^2-stable, then the existence of a stabilizing solution to the generalized ARE is necessary for the open-loop solvability of Problem $(\text{SLQ})^0_\infty$. Later we will show that the converse is also true.

Proposition 5.6. *Let Assumptions 3.1–3.3 hold. Suppose that the generalized ARE* (5.1) *admits a stabilizing solution $K \in \mathbb{S}^n$. Then Problem* $(\text{SLQ})^0_\infty$ *is closed-loop solvable.*

Proof. Let $X(\cdot) \equiv X(\cdot; x, u)$ be the solution of the following equation:

$$\begin{cases} dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta)X(t) + F(\theta)u(t)]\tilde{\mu}(dt, d\theta), \quad t \in [0, \infty), \\ X(0) = x, \end{cases}$$

for arbitrary fixed initial state x and admissible control $u \in \mathscr{U}$. Applying Itô's formula to $t \mapsto \langle KX(t), X(t)\rangle$, it is easy to get that

$$\begin{aligned} -\langle Kx, x\rangle = \mathbb{E}\int_0^\infty \Big[&\Big\langle \Big(KA + A^\top K + C^\top KC + \int_Z E(\theta)^\top KE(\theta)\nu(d\theta)\Big)X(t), X(t)\Big\rangle \\ &+ 2\Big\langle \Big(B^\top K + D^\top KC + \int_Z F(\theta)^\top KE(\theta)\nu(d\theta)\Big)X(t), u(t)\Big\rangle \\ &+ \Big\langle \Big(D^\top KD + \int_Z F(\theta)^\top KF(\theta)\nu(d\theta)\Big)u(t), u(t)\Big\rangle \Big] dt. \end{aligned}$$

Thus, we have

$$J^0(x; u) - \langle Kx, x\rangle = E\int_0^\infty \left\langle \begin{pmatrix} \mathcal{L}(K) & \mathcal{M}(K)^\top \\ \mathcal{M}(K) & \mathcal{N}(K) \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \right\rangle dt.$$

By the extended Schur's lemma (Lemma 2.3), we obtain

$$\begin{pmatrix} \mathcal{L}(K) & \mathcal{M}(K)^\top \\ \mathcal{M}(K) & \mathcal{N}(K) \end{pmatrix} \geq 0.$$

Thus,

$$J^0(x; u) \geq \langle Kx, x\rangle, \quad \forall u \in \mathscr{U}. \tag{5.19}$$

On the flip side, we can choose a $\Theta \in \mathbb{R}^{m\times n}$ such that the matrix

$$\Phi^* \triangleq -\mathcal{N}(K)^\dagger \mathcal{M}(K) + [I - \mathcal{N}(K)^\dagger \mathcal{N}(K)]\Theta$$

is a stabilizer of $[A, C, E; B, D, F]$ since K is stabilizing. By Remark 5.2,

$$\mathcal{N}(K)\Phi^* = -\mathcal{M}(K), \quad \mathcal{M}(K)^\top \Phi^* = -(\Phi^*)^\top \mathcal{N}(K)\Phi^* = -\mathcal{M}(K)^\top \mathcal{N}(K)^\dagger \mathcal{M}(K).$$

Thus, for any $x^* \in \mathbb{R}^n$,

$$\begin{aligned}
&\left\langle\!\!\left\langle \begin{pmatrix} \mathcal{L}(K) & \mathcal{M}(K)^\top \\ \mathcal{M}(K) & \mathcal{N}(K) \end{pmatrix} \begin{pmatrix} x^* \\ \Phi^* x^* \end{pmatrix}, \begin{pmatrix} x^* \\ \Phi^* x^* \end{pmatrix} \right\rangle\!\!\right\rangle \\
&= \langle [\mathcal{L}(K) + 2\mathcal{M}(K)^\top \Phi^* + (\Phi^*)^\top \mathcal{N}(K)\Phi^*] x^*, x^* \rangle \\
&= \left\langle [\mathcal{L}(K) - \mathcal{M}(K)^\top \mathcal{N}(K)^\dagger \mathcal{M}(K)] x^*, x^* \right\rangle = 0.
\end{aligned} \tag{5.20}$$

Following we will show that $(\Phi^*, 0)$ is a closed-loop optimal strategy of Problem (SLQ)$^0_\infty$. Firstly, let X^* be the closed-loop state process corresponding to $(x, \Phi^*, 0)$:

$$\begin{cases} dX^*(t) = (A + B\Phi^*)X^*(t)dt + (C + D\Phi^*)X^*(t)dW(t) \\ \qquad\qquad + \displaystyle\int_Z [E(\theta) + F(\theta)\Phi^*]X^*(t)\tilde{\mu}(dt, d\theta), \quad t \in [0, \infty), \\ X^*(0) = x. \end{cases}$$

Then, applying Itô's formula to $t \mapsto \langle KX^*(t), X^*(t)\rangle$ and using (5.20), we obtain

$$\begin{aligned}
&J^0(x; \Phi^* X^*) - \langle Kx, x\rangle \\
&= \mathbb{E}\int_0^\infty \left\langle\!\!\left\langle \begin{pmatrix} \mathcal{L}(K) & \mathcal{M}(K)^\top \\ \mathcal{M}(K) & \mathcal{N}(K) \end{pmatrix} \begin{pmatrix} X^*(t) \\ \Phi^* X^*(t) \end{pmatrix}, \begin{pmatrix} X^*(t) \\ \Phi^* X^*(t) \end{pmatrix} \right\rangle\!\!\right\rangle dt = 0.
\end{aligned}$$

Combining the last equation with (5.19), the conclusion is proved. □

Remark 5.3. According to the proof of Proposition 5.6, it is clear that if K is a stabilizing solution to the generalized ARE (5.1), then $V^0(x) = \langle Kx, x\rangle$ for all $x \in \mathbb{R}^n$.

Combining Remark 5.1, Theorem 5.5, and Proposition 5.6, we have the following result.

Theorem 5.7. *Let Assumptions 3.1–3.3 be satisfied. Suppose that $[A.C, E]$ is L^2-stable. Then the following statements are equivalent:*
(i) Problem (SLQ)$^0_\infty$ is open-loop solvable;
(ii) Problem (SLQ)$^0_\infty$ is closed-loop solvable;
(iii) The generalized ARE (5.1) *admits a unique stabilizing solution.*

5.1.3. Nonhomogeneous problems

Following we will focus our attention to Problem (SLQ)$_\infty$, and show a result that is similar to Theorem 5.7.

Denote by Ω a stabilizer of $[A, C, E; B, D, F]$, and define

$$\begin{cases} \hat{A} = A + B\Omega, \quad \hat{C} = C + D\Omega, \quad \hat{E} = E + F\Omega, \\ \hat{S} = S + R\Omega, \quad \hat{Q} = Q + S^\top\Omega + \Omega^\top S + \Omega^\top R\Omega, \\ \hat{q} = q + \Omega^\top\rho. \end{cases} \tag{5.21}$$

Then we use $\hat{X}(\cdot; x, v)$ to denote the solution of the following state equation

$$\begin{cases} d\hat{X}(t) = [\hat{A}\hat{X}(t) + Bv(t) + b(t)]dt \\ \qquad\quad + [\hat{C}\hat{X}(t) + Dv(t) + \sigma(t)]dW(t) \\ \qquad\quad + \int_Z [\hat{E}(\theta)\hat{X}(t) + F(\theta)v(t) + h(\theta)]\tilde{\mu}(dt, d\theta), \quad t \geq 0, \\ \hat{X}(0) = x \end{cases} \tag{5.22}$$

with regard to x and v. And we have the cost functional

$$\begin{aligned} \hat{J}(x; v) \triangleq & J(x; \Omega\hat{X} + v) \\ = & \mathbb{E}\int_0^\infty \Big[\Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} \hat{X}(t) \\ \Omega\hat{X}(t) + v(t) \end{pmatrix}, \begin{pmatrix} \hat{X}(t) \\ \Omega\hat{X}(t) + v(t) \end{pmatrix} \Big\rangle \\ & + 2\Big\langle \begin{pmatrix} q(t) \\ \rho(t) \end{pmatrix}, \begin{pmatrix} \hat{X}(t) \\ \Omega\hat{X}(t) + v(t) \end{pmatrix} \Big\rangle \Big]dt \\ = & \mathbb{E}\int_0^\infty \Big[\Big\langle \begin{pmatrix} \hat{Q} & \hat{S}^\top \\ \hat{S} & R \end{pmatrix} \begin{pmatrix} \hat{X} \\ v \end{pmatrix}, \begin{pmatrix} \hat{X} \\ v \end{pmatrix} \Big\rangle + 2\Big\langle \begin{pmatrix} \hat{q} \\ \rho \end{pmatrix}, \begin{pmatrix} \hat{X} \\ v \end{pmatrix} \Big\rangle \Big]dt. \end{aligned} \tag{5.23}$$

It is easy to find that $[\hat{A}, \hat{C}, \hat{E}]$ is L^2-stable. And we denote by $(\text{SLQ})_\infty'$ the problem of minimizing (5.23) subject to (5.22). According to Proposition 4.3, we have the following obvious conclusions about Problem $(\text{SLQ})_\infty'$.

Proposition 5.8. *Under Assumptions 3.1–3.3, let Ω be a stabilizer of [A,C,E; B,D,F]. Then*
(i) Problem $(SLQ)_\infty'$ is open-loop solvable at $x \in \mathbb{R}^n$ if and only if Problem $(SLQ)_\infty$ is so. In this case, v^* is an open-loop optimal control of Problem $(SLQ)_\infty'$ if and only if $u^* \triangleq v^* + \Omega\hat{X}(\cdot; x, v^*)$ is an open -loop optimal control of Problem $(SLQ)_\infty$;
(ii) Problem $(SLQ)_\infty'$ is closed-loop solvable if and only if Problem $(SLQ)_\infty$ is so. In this case, (Ω^*, v^*) is a closed-loop optimal strategy of Problem $(SLQ)_\infty'$ if and only if $(\Omega^* + \Omega, v^*)$ is a closed-loop optimal strategy of Problem $(SLQ)_\infty$.

Following we will show the main result of this section.

Theorem 5.9. *Under Assumptions 3.1–3.3 and (A), we have the following equivalent statements:*
(i) Problem $(SLQ)_\infty$ is open-loop solvable;
(ii) Problem $(SLQ)_\infty$ is closed-loop solvable;
(iii) The generalized ARE (5.1) *admits a stabilizing solution $K \in \mathbb{S}^n$, and the BSDE*

$$\begin{aligned} d\eta = -\Big\{ & [A - B\mathcal{N}(K)^\dagger\mathcal{M}(K)]^\top\eta + [C - D\mathcal{N}(K)^\dagger\mathcal{M}(K)]^\top(\zeta_1 + K\sigma) \\ & + \int_Z [E(\theta) - F(\theta)\mathcal{N}(K)^\dagger\mathcal{M}(K)]^\top [\zeta_2 + Kh(t,\theta)]\,\nu(d\theta) - \mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\rho \\ & + Kb + q\Big\}dt + \zeta_1 dW(t) + \int_Z \zeta_2\tilde{\mu}(dt, d\theta), \quad t \geq 0, \end{aligned} \tag{5.24}$$

admits an L^2-stable adapted solution (η, ζ_1, ζ_2) such that

$$\varpi(t) \triangleq B^\top \eta(t) + D^\top [\zeta_1(t) + K\sigma(t)] + \int_Z F(\theta)^\top [\zeta_2(t) + Kh(t,\theta)]\nu(d\theta) + \rho(t) \in \mathscr{R}(\mathcal{N}(K)), \quad a.e.\ t \in [0,\infty), \quad a.s. \tag{5.25}$$

In the above case, all closed-loop optimal strategies (Φ^*, v^*) are given by

$$\begin{cases} \Phi^* = -\mathcal{N}(K)^\dagger \mathcal{M}(K) + [I - \mathcal{N}(K)^\dagger \mathcal{N}(K)]\Theta, \\ v^* \ = -\mathcal{N}(K)^\dagger \varpi + [I - \mathcal{N}(K)^\dagger \mathcal{N}(K)]w, \end{cases} \tag{5.26}$$

where $\Theta \in \mathbb{R}^{m\times n}$ and $w \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ are arbitrary. And it turns out that $\Phi^* \in \mathscr{T}[A, C, E; B, D, F]$. Then every open-loop optimal control u^* for the initial state x admits a closed-loop representation:

$$u^*(t) = \Phi^* X^*(t) + v^*(t), \quad t \in [0,\infty), \tag{5.27}$$

where (Φ^*, v^*) is a closed-loop optimal strategy of Problem (SLQ)$_\infty$ and X^* is the corresponding closed-loop state process. Moreover,

$$\begin{aligned} V(x) &= \langle Kx, x\rangle + 2\mathbb{E}\langle \eta(0), x\rangle \\ &+ \mathbb{E}\int_0^\infty \Big[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle \nu(d\theta) + 2\langle \eta, b\rangle + 2\langle \zeta_1, \sigma\rangle \\ &+ 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle - \left\langle \mathcal{N}(K)^\dagger \varpi, \varpi\right\rangle \Big] dt. \end{aligned}$$

Proof. Since Remark 5.1 holds, it is obvious that the implication (ii) $\Rightarrow$ (i) follows.

To prove the implication (i) $\Rightarrow$ (iii), we will consider Problem (SLQ)$'_\infty$ firstly. According to Proposition 5.8(i), it is open-loop solvable. Since the system $[\hat{A}, \hat{C}, \hat{E}]$ is L^2-stable, it is easy to get that by Proposition 5.2 and Theorem 5.5, the ARE

$$\begin{cases} K\hat{A} + \hat{A}^\top K + \hat{C}^\top K\hat{C} + \int_Z \hat{E}(t,\theta)^\top K\hat{E}(t,\theta)\nu(d\theta) + \hat{Q} \\ -\Big[KB + \hat{C}^\top KD + \int_Z \hat{E}(t,\theta)^\top KF(t,\theta)\nu(d\theta) + \hat{S}^\top\Big] \\ \times\Big[R + D^\top KD + \int_Z F(t,\theta)^\top KF(t,\theta)\nu(d\theta)\Big]^\dagger \\ \times\Big[B^\top K + D^\top K\hat{C} + \int_Z F(t,\theta)^\top K\hat{E}(t,\theta)\nu(d\theta) + \hat{S}\Big] = 0, \\ \mathscr{R}\Big[B^\top K + D^\top K\hat{C} + \int_Z F(t,\theta)^\top K\hat{E}(t,\theta)\nu(d\theta) + \hat{S}\Big] \subseteq \\ \mathscr{R}\Big[R + D^\top KD + \int_Z F(t,\theta)^\top KF(t,\theta)\nu(d\theta)\Big], \\ \mathcal{N}(K) = R + D^\top KD + \int_Z F(t,\theta)^\top KF(t,\theta)\nu(d\theta) \geq 0 \end{cases} \tag{5.28}$$

admits a (unique) stabilizing solution $K \in \mathbb{S}^n$. Choose a $\Gamma \in \mathbb{R}^{m\times n}$ such that

$$\Omega^* \triangleq -\mathcal{N}(K)^\dagger \Big[B^\top K + D^\top K\hat{C} + \int_Z F(\theta)^\top K\hat{E}(\theta)\nu(d\theta) + \hat{S}\Big] + [I - \mathcal{N}(K)^\dagger \mathcal{N}(K)]\Gamma$$

is a stabilizer of $[\hat{A},\hat{C},\hat{E};B,D,F]$. According to Remark 5.2 and (5.21),

$$\begin{aligned}\mathcal{N}(K)(\Omega^*+\Omega)&=-\left[B^\top K+D^\top K\hat{C}+\int_Z F(\theta)^\top K\hat{E}(\theta)\nu(d\theta)+\hat{S}\right]+\mathcal{N}(K)\Omega\\&=-\left[B^\top K+D^\top KC+\int_Z F(\theta)^\top KE(\theta)\nu(d\theta)+S\right]=-\mathcal{M}(K).\end{aligned}\tag{5.29}$$

It follows that $\mathscr{R}(\mathcal{M}(K))\subseteq\mathscr{R}(\mathcal{N}(K))$. Substituting (5.21) into the first equation of (5.28) gives

$$\begin{aligned}0=&\mathcal{L}(K)+\mathcal{M}(K)^\top\Omega+\Omega^\top\mathcal{M}(K)-\mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\mathcal{M}(K)\\&-\mathcal{M}(K)^\top N(K)^\dagger\mathcal{N}(K)\Omega-\Omega^\top\mathcal{N}(K)N(K)^\dagger\mathcal{M}(K)\\=&\mathcal{L}(K)-\mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\mathcal{M}(K)+\mathcal{M}(K)^\top[I-\mathcal{N}(K)^\dagger\mathcal{N}(K)]\Omega\\&+\Omega^\top[I-\mathcal{N}(K)\mathcal{N}(K)^\dagger]\mathcal{M}(K)\\=&\mathcal{L}(K)-\mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\mathcal{M}(K)-(\Omega^*+\Omega)^\top\mathcal{N}(K)[I-\mathcal{N}(K)^\dagger\mathcal{N}(K)]\Omega\\&-\Omega^\top[I-\mathcal{N}(K)\mathcal{N}(K)^\dagger]\mathcal{N}(K)(\Omega^*+\Omega)\\=&\mathcal{L}(K)-\mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\mathcal{M}(K).\end{aligned}$$

Therefore, K solves the ARE (5.1). Since $\Omega^*+\Omega$ is a stabilizer of $[A,C,E;B,D,F]$, it is easy to get that K is stabilizing according to (5.29) and Lemma 2.2.

Now choose $\Theta\in\mathbb{R}^{m\times n}$ such that the matrix

$$\Phi\triangleq-\mathcal{N}(K)^\dagger\mathcal{M}(K)+[I-\mathcal{N}(K)^\dagger\mathcal{N}(K)]\Theta$$

is a stabilizer of $[A,C,E;B,D,F]$, and think over the following BSDE on $[0,\infty)$:

$$\begin{aligned}d\eta(t)=&-\Big[(A+B\Phi)^\top\eta+(C+D\Phi)^\top(\zeta_1+K\sigma)+\int_Z(E(\theta)+F(\theta)\Phi)^\top\\&\cdot(\zeta_2+Kh(t,\theta))\nu(d\theta)+\Phi^\top\rho+Kb+q\Big]dt+\zeta_1dW(t)+\int_Z\zeta_2\tilde{\mu}(dt,d\theta).\end{aligned}\tag{5.30}$$

Since $[A+B\Phi,C+D\Phi,E+F\Phi]$ is L^2-stable, it is not hard to get the conclusion that (5.30) admits a unique L^2-stable adapted solution (η,ζ_1,ζ_2) by Lemma 3.2. Now we let $X(\cdot)\equiv X(\cdot;x,u)$ be the corresponding state process for fixed but arbitrary x and $u\in\mathscr{U}$. Applying Itô's formula to $t\mapsto\langle KX(t),X(t)\rangle$, then it follows that

$$\begin{aligned}-\langle Kx,x\rangle=\mathbb{E}\int_0^\infty\Bigg[&\left\langle\left(KA+A^\top K+C^\top KC+\int_Z E(\theta)^\top KE(\theta)\nu(d\theta)\right)X,X\right\rangle\\&+2\left\langle\left(B^\top K+D^\top KC+\int_Z F(\theta)^\top KE(\theta)\nu(d\theta)\right)X,u\right\rangle\\&+\left\langle\left(D^\top KD+\int_Z F(\theta)^\top KF(\theta)\nu(d\theta)\right)u,u\right\rangle\\&+2\left\langle C^\top K\sigma+Kb+\int_Z E(\theta)^\top Kh(t,\theta)\nu(d\theta),X\right\rangle\\&+2\left\langle D^\top K\sigma+\int_Z F(\theta)^\top Kh(t,\theta)\nu(d\theta),u\right\rangle\\&+\langle K\sigma,\sigma\rangle+\int_Z\langle Kh(t,\theta),h(t,\theta)\rangle\,\nu(d\theta)\Bigg]dt.\end{aligned}$$

Applying Itô's formula to $t \mapsto \langle \eta(t), X(t) \rangle$ yields

$$\begin{aligned}
\mathbb{E}\langle \eta(0), x\rangle = \mathbb{E}\int_0^\infty \Bigg[& \left\langle \Phi^\top \left(B^\top \eta + D^\top \zeta_1 + D^\top K\sigma + \rho + \int_Z F(\theta)^\top (\zeta_2 + Kh(t,\theta))\nu(d\theta) \right), X \right\rangle \\
& + \left\langle C^\top K\sigma + Kb + q + \int_Z E(\theta)^\top Kh(t,\theta)\nu(d\theta), X \right\rangle \\
& - \left\langle B^\top \eta + D^\top \zeta_1 + \int_Z F(\theta)^\top \zeta_2 \nu(d\theta), u \right\rangle \\
& - \langle \eta, b\rangle - \langle \zeta_1, \sigma\rangle - \left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta) \right\rangle \Bigg] dt.
\end{aligned}$$

Denote $\varpi(t) = B^\top \eta(t) + D^\top[\zeta_1(t) + K\sigma(t)] + \int_Z F(t,\theta)^\top [\zeta_2(t) + Kh(t,\theta)]\nu(d\theta) + \rho(t)$, then we have

$$\begin{aligned}
& J(x;u) - \langle Kx, x\rangle - 2\mathbb{E}\langle \eta(0), x\rangle \\
&\quad = \mathbb{E}\int_0^\infty \Big[\langle \mathcal{L}(K)X, X\rangle + 2\langle \mathcal{M}(K)X, u\rangle + \langle \mathcal{N}(K)u, u\rangle - 2\langle \Phi^\top \varpi, X\rangle + 2\langle \varpi, u\rangle + \langle K\sigma, \sigma\rangle \\
&\qquad + 2\langle \eta, b\rangle + 2\langle \zeta_1, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle \nu(d\theta) + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle \Big] dt \\
&\quad = \mathbb{E}\int_0^\infty \Big[\langle \mathcal{N}(K)(u - \Phi X), u - \Phi X\rangle + 2\langle \varpi, u - \Phi X\rangle + \langle K\sigma, \sigma\rangle + 2\langle \eta, b\rangle + 2\langle \zeta_1, \sigma\rangle \\
&\qquad + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle \nu(d\theta) + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle \Big] dt.
\end{aligned} \tag{5.31}$$

Now we assume that u^* is an open-loop optimal control of Problem $(\mathrm{SLQ})_\infty$ for the initial state x, and $X_\Phi(\cdot; x, v)$ is a solution to the following SDE:

$$\begin{cases}
dX_\Phi(t) = [(A + B\Phi)X_\Phi(t) + Bv(t) + b(t)]dt \\
\qquad\qquad + [(C + D\Phi)X_\Phi(t) + Dv(t) + \sigma(t)]dW(t) \\
\qquad\qquad + \int_Z [(E(\theta) + F(\theta)\Phi)X_\Phi(t) + Fv(t) + h(t,\theta)]\tilde{\mu}(dt, d\theta), \quad t \in [0, \infty), \\
X_\Phi(0) = x \in \mathbb{R}^n.
\end{cases}$$

By Proposition 4.3, we have the conclusion that any admissible control with respect to the initial state x is of the form

$$\Phi X_\Phi(\cdot; x, v) + v, \quad v \in L^2_{\mathbb{F}}(\mathbb{R}^m).$$

Thus $u^* = \Phi X_\Phi(\cdot; x, v^*) + v^*$ for some $v^* \in L^2_{\mathbb{F}}(\mathbb{R}^m)$. Then we have

$$\begin{aligned}
& J(x; \Phi X_\Phi(\cdot; x, v^*) + v^*) = J(x; u^*) \\
& \quad \le J(x; \Phi X_\Phi(\cdot; x, v) + v), \quad \forall v \in L^2_{\mathbb{F}}(\mathbb{R}^m).
\end{aligned} \tag{5.32}$$

Since we take $u = \Phi X_\Phi(\cdot; x, v) + v$, it is easy to get that

$$X(\cdot; x, u^*) = X_\Phi(\cdot; x, v^*), \quad X(\cdot; x, u) = X_\Phi(\cdot; x, v),$$

thus, according to (5.31) and (5.32), it follows that for any $v \in L^2_{\mathbb{F}}(\mathbb{R}^m)$,

$$
\begin{aligned}
&\mathbb{E}\int_0^\infty \left[\langle \mathcal{N}(K)v^*, v^*\rangle + 2\langle \varpi, v^*\rangle\right] dt \\
&\quad = J(x;u^*) - \langle Kx, x\rangle - 2\mathbb{E}\langle \eta(0), x\rangle \\
&\qquad - \mathbb{E}\int_0^\infty \left[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle \nu(d\theta) + 2\langle \eta, b\rangle + 2\langle \zeta_1, \sigma\rangle + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle\right] dt \\
&\quad \le J(x;u) - \langle Kx, x\rangle - 2\mathbb{E}\langle \eta(0), x\rangle \\
&\qquad - \mathbb{E}\int_0^\infty \left[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle \nu(d\theta) + 2\langle \eta, b\rangle + 2\langle \zeta_1, \sigma\rangle + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle\right] dt \\
&\quad = \mathbb{E}\int_0^\infty \left[\langle \mathcal{N}(K)v, v\rangle + 2\langle \varpi, v\rangle\right] dt.
\end{aligned}
$$

The above inequality implies that v^* is a minimizer of the functional

$$H(v) = \mathbb{E}\int_0^\infty \left[\langle \mathcal{N}(K)v, v\rangle + 2\langle \varpi, v\rangle\right] dt, \quad v \in L^2_{\mathbb{F}}(\mathbb{R}^m).$$

Thus, it follows that

$$\mathcal{N}(K)v^* + \varpi = 0, \quad a.e.\ t \ge 0, \ \ a.s.$$

By Lemma 2.2, we have

$$
\begin{cases}
\varpi \in \mathscr{R}(\mathcal{N}(K)), \ \text{ and} \\
v^* = -\mathcal{N}(K)^\dagger \varpi + \left[I - \mathcal{N}(K)^\dagger \mathcal{N}(K)\right]\vartheta \text{ for some } \vartheta \in \mathrm{L}^2_{\mathbb{F}}(\mathbb{R}^{\mathrm{m}}).
\end{cases}
$$

Observing that

$$\left[\Phi^\top + \mathcal{M}(K)^\top \mathcal{N}(K)^\dagger\right]\varpi = -\Theta^\top\left[I - \mathcal{N}(K)\mathcal{N}(K)^\dagger\right]\mathcal{N}(K)v^* = 0,$$

we obtain

$$
\begin{aligned}
&(A + B\Phi)^\top \eta + (C + D\Phi)^\top(\zeta_1 + K\sigma) + \int_Z [E(\theta) + F(\theta)\Phi]^\top [\zeta_2 + Kh(t,\theta)]\,\nu(d\theta) + \Phi^\top\rho + Kb + q \\
&\quad = A^\top\eta + C^\top(\zeta_1 + K\sigma) + \int_Z E(\theta)^\top[\zeta_2 + Kh(t,\theta)]\,\nu(d\theta) + Kb + q + \Phi^\top\varpi \\
&\quad = A^\top\eta + C^\top(\zeta_1 + K\sigma) + \int_Z E(\theta)^\top[\zeta_2 + Kh(t,\theta)]\,\nu(d\theta) + Kb + q - \mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\varpi \\
&\quad = \left[A - B\mathcal{N}(K)^\dagger\mathcal{M}(K)\right]^\top \eta + \left[C - D\mathcal{N}(K)^\dagger\mathcal{M}(K)\right]^\top(\zeta_1 + K\sigma) \\
&\qquad + \int_Z \left[E(\theta) - F(\theta)\mathcal{N}(K)^\dagger\mathcal{M}(K)\right]^\top[\zeta_2 + Kh(t,\theta)]\,\nu(d\theta) - \mathcal{M}(K)^\top\mathcal{N}(K)^\dagger\rho + Kb + q.
\end{aligned}
$$

Thus, it follows that (η, ζ_1, ζ_2) is an L^2-stable adapted solution of the BSDE (5.24). And from Remark 2.1, we have

$$\langle \varpi, v^*\rangle = -\langle \mathcal{N}(K)v^*, v^*\rangle = -\left\langle \mathcal{N}(K)^\dagger\varpi, \varpi\right\rangle.$$

Therefore, since we replace u by $u^* = \Phi X_\Phi(\cdot; x, v^*) + v^*$ in (5.31), it is not hard to get that

$$
\begin{aligned}
V(x) &= J(x; u^*) \\
&= \langle Kx, x\rangle + 2\mathbb{E}\,\langle \eta(0), x\rangle \\
&\quad + \mathbb{E}\int_0^\infty \left[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle\, \nu(d\theta) + 2\,\langle \eta, b\rangle + 2\,\langle \zeta_1, \sigma\rangle + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle\right] dt \\
&\quad + \mathbb{E}\int_0^\infty \left[\langle \mathcal{N}(K)v^*, v^*\rangle + 2\,\langle \varpi, v^*\rangle\right] dt \\
&= \langle Kx, x\rangle + 2\mathbb{E}\,\langle \eta(0), x\rangle \\
&\quad + \mathbb{E}\int_0^\infty \left[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle\, \nu(d\theta) + 2\,\langle \eta, b\rangle + 2\,\langle \zeta_1, \sigma\rangle + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle\right] dt \\
&\quad - \mathbb{E}\int_0^\infty \left\langle \mathcal{N}(K)^\dagger \varpi, \varpi\right\rangle dt.
\end{aligned}
$$

Finally, to prove the implication (iii)⇒(ii), we firstly take an arbitrary $(x, u) \in \mathbb{R}^n \times \mathscr{U}$ and let $X(\cdot) \equiv X(\cdot; x, u)$ be the corresponding state process. According to (5.31), we have

$$
\begin{aligned}
J(x; u) &= \langle Kx, x\rangle + 2\mathbb{E}\,\langle \eta(0), x\rangle \\
&\quad + \mathbb{E}\int_0^\infty \Big[\,\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle\, \nu(d\theta) \\
&\quad + 2\,\langle \eta, b\rangle + 2\,\langle \zeta_1, \sigma\rangle + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle\Big] dt \\
&\quad + \mathbb{E}\int_0^\infty \Big[\,\langle \mathcal{L}(K)X, X\rangle + 2\,\langle \mathcal{M}(K)X, u\rangle \\
&\quad + \langle \mathcal{N}(K)u, u\rangle + 2\left\langle \varpi, u + \mathcal{N}(K)^\dagger \mathcal{M}(K)X\right\rangle\Big] dt.
\end{aligned}
\tag{5.33}
$$

Let (Φ^*, v^*) be defined by (5.26). Then by Lemma 2.2 and Remark 2.1, we have

$$
\begin{aligned}
&\mathcal{M}(K) = -\mathcal{N}(K)\Phi^*, \quad \mathcal{L}(K) = \mathcal{M}(K)^\top \mathcal{N}(K)^\dagger \mathcal{M}(K) = (\Phi^*)^\top \mathcal{N}(K)\Phi^*, \\
&\varpi = -\mathcal{N}(K)v^*, \quad \mathcal{N}(K)\mathcal{N}(K)^\dagger \mathcal{M}(K) = -\mathcal{N}(K)\Phi^*.
\end{aligned}
$$

Then we substitute the above into (5.33) and complete the square, it follows that

$$
\begin{aligned}
J(x; u) &= \langle Kx, x\rangle + 2\mathbb{E}\,\langle \eta(0), x\rangle \\
&\quad + \mathbb{E}\int_0^\infty \Big[\,\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle\, \nu(d\theta) + 2\,\langle \eta, b\rangle + 2\,\langle \zeta_1, \sigma\rangle \\
&\quad + 2\left\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\right\rangle - \langle \mathcal{N}(K)v^*, v^*\rangle\Big] dt \\
&\quad + \mathbb{E}\int_0^\infty \langle \mathcal{N}(K)(u - \Phi^* X - v^*), u - \Phi^* X - v^*\rangle\, dt.
\end{aligned}
\tag{5.34}
$$

Thus, owing to $\mathcal{N}(K) \geq 0$ and Φ^* is a stabilizer of $[A, C, E; B, D, F]$, it is not hard to get that

$$\begin{aligned}
J(x; u) \geq\ & \langle Kx, x\rangle + 2\mathbb{E}\,\langle \eta(0), x\rangle \\
& + \mathbb{E}\int_0^\infty \Big[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle\, \nu(d\theta) + 2\,\langle \eta, b\rangle + 2\,\langle \zeta_1, \sigma\rangle \\
& + 2\Big\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\Big\rangle - \langle \mathcal{N}(K)v^*, v^*\rangle\Big]dt \\
& = J(x; \Phi^* X^* + v^*), \quad \forall x \in \mathbb{R}^n, \quad \forall u \in \mathscr{U},
\end{aligned} \tag{5.35}$$

which shows that (Φ^*, v^*) is a closed-loop optimal strategy of Problem $(\mathrm{SLQ})_\infty$.

Finally, assume that $(\check{\Phi}, \check{v})$ is an another closed-loop optimal strategy, and $\check{X}$ is the solution of the following closed-loop system

$$\begin{cases}
d\check{X}(t) = [(A + B\check{\Phi})\check{X}(t) + B\check{v}(t) + b(t)]dt \\
\qquad\qquad + [(C + D\check{\Phi})\check{X}(t) + D\check{v}(t) + \sigma(t)]dW(t) \\
\qquad\qquad + \int_Z [(E(\theta) + F(\theta)\check{\Phi})\check{X}(t) + F(\theta)\check{v}(t) + h(t,\theta)]\tilde{\mu}(dt, d\theta), \quad t \in [0, \infty), \\
\check{X}(0) = x \in \mathbb{R}^n.
\end{cases}$$

Since we denote by $\check{u} = \check{\Phi}\check{X} + \check{v}$ the outcome of $(\check{\Phi}, \check{v})$, it is obvious that

$$X(t; x, \check{u}) = \check{X}(t), \quad t \geq 0.$$

Now (5.34) and (5.35) imply that

$$\begin{aligned}
V(x) = J(x; \check{u}) =\ & \langle Kx, x\rangle + 2\mathbb{E}\,\langle \eta(0), x\rangle \\
& + \mathbb{E}\int_0^\infty \Big[\langle K\sigma, \sigma\rangle + \int_Z \langle Kh(t,\theta), h(t,\theta)\rangle\, \nu(d\theta) + 2\,\langle \eta, b\rangle + 2\,\langle \zeta_1, \sigma\rangle + 2\Big\langle \zeta_2, \int_Z h(t,\theta)\nu(d\theta)\Big\rangle \\
& - \langle \mathcal{N}(K)v^*, v^*\rangle\Big]dt \\
& + \mathbb{E}\int_0^\infty \Big\langle \mathcal{N}(K)(\check{u} - \Phi^*\check{X} - v^*), \check{u} - \Phi^*\check{X} - v^*\Big\rangle\, dt \\
& = V(x) + \mathbb{E}\int_0^\infty |\mathcal{N}(K)^{\frac{1}{2}}(\check{\Phi}\check{X} + \check{v} - \Phi^*\check{X} - v^*)|^2 dt,
\end{aligned}$$

from which we have that

$$\mathcal{N}(K)^{\frac{1}{2}}(\check{\Phi}\check{X} + \check{v} - \Phi^*\check{X} - v^*) = 0, \quad \forall x \in \mathbb{R}^n.$$

Multiplying the above by $\mathcal{N}(K)^{\frac{1}{2}}$, we obtain

$$\mathcal{N}(K)(\check{\Phi} - \Phi^*)\check{X} + \mathcal{N}(K)(\check{v} - v^*) = 0, \quad \forall x \in \mathbb{R}^n. \tag{5.36}$$

Since $\check{\Phi}, \Phi^*, \check{v}$, and v^* are independent of x, and (5.36) holds for all $x \in \mathbb{R}^n$, it is easy to get that for any $x \in \mathbb{R}^n$, the solution X_0 of

$$\begin{cases}
dX_0(t) = (A + B\check{\Phi})X_0(t)dt + (C + D\check{\Phi})X_0(t)dW(t) \\
\qquad\qquad + \int_Z [E(\theta) + F(\theta)\check{\Phi}]X_0(t)\tilde{\mu}(dt, d\theta), \quad t \in [0, \infty), \\
X_0(0) = x,
\end{cases}$$

satisfies $\mathcal{N}(K)(\check{\Phi} - \Phi^*)X_0 = 0$. It follows that $\mathcal{N}(K)(\check{\Phi} - \Phi^*) = 0$ and hence $\mathcal{N}(K)(\check{v} - v^*) = 0$. Now we have

$$\mathcal{N}(K)\check{\Phi} = \mathcal{N}(K)\Phi^* = -\mathcal{M}(K), \quad \mathcal{N}(K)\check{v} = \mathcal{N}(K)v^* = -\varpi.$$

According to Lemma 2.2, $(\check{\Phi}, \check{v})$ must be of the form (5.26). Since we denote $\bar{X}$ the corresponding optimal state process, if $\bar{u}$ is an open-loop optimal control for the initial state x, it is not hard to get that

$$\mathcal{N}(K)(\bar{u} - \Phi^*\bar{X} - v^*) = 0,$$

i.e.,

$$\mathcal{N}(K)\bar{u} = \mathcal{N}(K)\Phi^*\bar{X} + \mathcal{N}(K)v^* = -\mathcal{M}(K)\bar{X} - \varpi.$$

According to Lemma 2.2, there exists a $w \in L^2_{\mathbb{F}}(\mathbb{R}^m)$ such that

$$\begin{aligned}\bar{u} &= -\mathcal{N}(K)^\dagger\mathcal{M}(K)\bar{X} - \mathcal{N}(K)^\dagger\varpi + \left[I - \mathcal{N}(K)^\dagger\mathcal{N}(K)\right]w \\ &= \left\{-\mathcal{N}(K)^\dagger\mathcal{M}(K) + \left[I - \mathcal{N}(K)^\dagger\mathcal{N}(K)\right]\Theta\right\}\bar{X} \\ &\quad - \mathcal{N}(K)^\dagger\varpi + \left[I - \mathcal{N}(K)^\dagger\mathcal{N}(K)\right](w - \Theta\bar{X}),\end{aligned}$$

where $\Theta \in \mathbb{R}^{m\times n}$ satisfies

$$-\mathcal{N}(K)^\dagger\mathcal{M}(K) + \left[I - \mathcal{N}(K)^\dagger\mathcal{N}(K)\right]\Theta \in \mathscr{T}[A, C, E; B, D, F],$$

which shows that $\bar{u}$ has the closed-loop representation (5.27). □

6. Conclusions

This paper mainly studies a kind of stochastic LQ optimal control problem with jumps in an infinite horizon. Firstly, we prove that the L^2-stabilizability of our control system is equivalent to the non-emptiness of the admissible control set for all initial state and is also equivalent to the existence of a positive solution to some integral ARE. Then we conclude the equivalence of the open-loop and closed-loop solvability and the existence of a stabilizing solution of the associated generalized ARE. Finally, we explore that any open-loop optimal control for the initial state admits a closed-loop representation, and the representation is obtained.

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Conflict of interest

The authors declare that they have no competing interests.

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